Existence and uniqueness for dislocation dynamics with nonnegative velocity

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Abstract

We study the problem of large time existence of solutions for a mathematical model describing dislocation dynamics in crystals. The mathematical model is a geometric and non local eikonal equation which does not preserve the inclusion. Under the assumption that the dislocation line is expanding, we prove existence and uniqueness of the solution in the framework of discontinuous viscosity solutions. We also show that this solution satisfies some variational properties, which allows to prove that the energy associated to the dislocation dynamics is non increasing.

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1 Introduction

In this paper, we study a simple model for dislocation dynamics. Dislocations are line defects in crystals that can be observed by elecron microscopy.

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The typical length of these dislocation lines in metallic alloys is of the order of 10^{-6} m.

The concept of dislocations in crystals has been introduced and developed in the XXth century, as the main microscopic explanation of the macroscopic plastic behaviour of metallic crystals (see for instance the physical monographs Nabarro [20], Hirth and Lothe [16], or Lardner [18] for a mathematical presentation). Since the beginning of the 90's, the research field of dislocations has enjoyed a new boom based on the increasing power of computers, allowing simulations with a large number of dislocations (see for instance Kubin et al. [17]). This simultaneously motivated new theoretical developments for the modelling of dislocations. Recently Rodney, Le Bouar and Finel introduced in [21] a new model, called the *phase field model of dislocations*, that we study in this paper.

In this model, the dislocation line in the crystal moves in its slip plane with a normal velocity which is proportional to the Peach-Koehler force acting on this line. This force may have two possible contributions: the first one is the self-force created by the elastic field generated by the dislocation line itself; the second one is the force created by everything exterior to the dislocation line, like the exterior stress applied on the material, or the force created by other defects.

Mathematically, a dislocation is formalized by a closed curve $\Gamma(t)$ in $\mathbb{R}^2$ moving with a normal velocity $V_{t,x}$ given at each time t and at each point $x \in \Gamma(t)$ by the following nonlocal law:

$$V_{t,x} = \bar{c}_0 \star \mathbf{1}_{K(t)}(t,x) + \bar{c}_1(t,x) \tag{1}$$

In the above equality, $K(t)$ denotes the compact set enclosed by the curve $\Gamma(t)$, the function $\bar{c}_0(t,x)$ is a kernel associated to the equations of linearized elasticity and the function $\bar{c}_1(t,x)$ describes some external field. The convolution is done in space for $x \in \mathbb{R}^2$. Here the term $\bar{c}_0 \star \mathbf{1}_{K(t)}(t,x)$ corresponds to the part of the velocity created by the self-force, and the term $\bar{c}_1(t,x)$ is associated to the exterior forces acting on the dislocation line.

If we set

$$\rho(t,x) = \mathbf{1}_{K(t)}(x) = \begin{cases} 1 & \text{if } x \in K(t) \\ 0 & \text{otherwise} \end{cases}$$

then equation (1) is equivalent to saying that ρ is a discontinuous viscosity solution of the following nonlocal Hamilon-Jacobi equation (for the definition of discontinuous viscosity solution, see [6], [5]):

$$\frac{\partial \rho}{\partial t} = (\bar{c}_0 \star \rho(t,\cdot) + \bar{c}_1)|D\rho| \; . \tag{2}$$

Such a nonlocal equation has been poorly investigated until now: If $\bar{c}_0 \geq 0$, then the equation satisfies the inclusion principle, and existence and (generic) uniqueness of generalized solutions can be obtained as application of [11]. Unfortunately, for dislocation dynamics, the kernel $\bar{c}_0$ has a zero mean, which implies in particular that it changes sign. In [2, 3] short time existence and uniqueness of the solution is proved under the assumption that the initial position of the dislocation is a Lipschitz graph.

In this paper we consider the existence and uniqueness generalized solutions for arbitrary time interval, provided that the initial curve is sufficiently smooth and the external field $\bar{c}_1$ is large with respect to the kernel $\bar{c}_0$: Namely we assume that

$$\bar{c}_1(t,x) \geq \|\bar{c}_0(t,\cdot)\|_{L^1(\mathbb{R}^2)} \qquad \forall (t,x) \in [0,+\infty) \times \mathbb{R}^2 \ . \tag{3}$$

This condition ensures that the dislocation is expanding because it implies that, for any Borel subset K of $\mathbb{R}^2$, one has

$$\bar{c}_0 \star \mathbf{1}_K(t,x) + \bar{c}_1(t,x) \geq \bar{c}_1(t,x) - \|\bar{c}_0(t,\cdot)\|_{L^1(\mathbb{R}^2)} \geq 0 \qquad \forall (t,x) \in [0,+\infty) \times \mathbb{R}^2 \ .$$

As for the regularity of the initial curve, we assume that the compact set K_0 enclosed by this curve satisfies *an interior ball condition.* This means that there is some $r > 0$ such that, for any point $x \in K_0$, there is some unit vector $p \in \mathbb{R}^2$ with $B(x - rp, r) \subset K_0$, where $B(y, r)$ is the closed ball centered at y and of radius r:

$$\exists r > 0, \ \forall x \in K_0, \ \exists p \in \mathbb{R}^N, \ |p| = 1 \text{ and } B(x - rp, r) \subset K_0 \ . \tag{4}$$

For instance, if K_0 is the closure of some open bounded set with a $\mathcal{C}^2$ boundary, then it satisfies the interior ball condition for some radius r.

Under these two assumptions, we prove (in any dimension N) that the problem of dislocation dynamics has a unique solution ρ, and that this solution depends in a Lipschitz way on the initial condition. We also show that this solution ρ is a variational solution, in the sense that

$$\begin{aligned} &\int_{\mathbb{R}^N} \varphi(t,x)\rho(t,x)dx - \int_{\mathbb{R}^N} \varphi(0,x)\rho(0,x)dx \\ &\qquad = \int_0^t \left[\int_{\mathbb{R}^N} \frac{\partial \varphi}{\partial t}(s,x)\rho(s,x)dx + \int_{\partial\{\rho(s,\cdot)=1\}} \varphi(s,y)c(s,y)d\mathcal{H}^{N-1}(y) \right] ds \end{aligned}$$

for any $\varphi \in \mathcal{C}^1([0,+\infty) \times \mathbb{R}^N)$. As a consequence, we prove that, when the data do not depend on time, the energy $E(t)$ naturally associated to the dislocation

$$E(t) = \int_{R^N} -\frac{1}{2}(\bar{c}_0 \star \rho)\rho - \bar{c}_1 \rho$$

is non increasing:

$$\frac{d}{dt}E(t) = -\int_{\partial\{\rho(t,\cdot)=1\}} c^2 d\mathcal{H}^{N-1} ,$$

where $c = \bar{c}_0 \star \rho + \bar{c}_1$.

In order to explain the role played by our two main assumptions (3) and (4), a description of the method of proof is now in order. As in [3] and in Alibaud [1] we use a Banach fixed point argument. We consider the mapping Φ which associates to any $\rho^0 \in \mathcal{C}^0([0,T], L^1(I\!\!R^N))$, with $0 \le \rho^0 \le 1$, the unique discontinuous viscosity solution $\rho = \Phi(\rho^0)$ to

$$\left\{ \begin{array}{l} \dfrac{\partial \rho}{\partial t} = c_{\rho^0}(t,x)|D\rho| \\ \rho(0,\cdot) = \mathbf{1}_{K_0} \end{array} \right. \tag{5}$$

where we have set

$$c_{\rho^0}(t,x) = \bar{c}_0 \star \rho^0(t,x) + \bar{c}_1(t,x) .$$

The solution of our problem problem is clearly a fixed point of Φ. In order to prove that Φ is a contraction (for the some adequate norm, which here turns out to be $\sup_{t\in[0,T]} \|\rho(t,\cdot)\|_{L^1(R^N)}$), we are lead to combine three types of arguments.

- **A representation formula :** Since $0 \le \rho^0 \le 1$, $c_{\rho^0}(t,x)$ is non negative and the set $\{\Phi(\rho^0)(t,\cdot) = 1\}$ can be represented as the reachable set of an associated control problem: namely $\{\Phi(\rho^0)(t,\cdot) = 1\}$ is equal to the set of points $z \in I\!\!R^2$ for which there is some initial position $x_0 \in K_0$ and some measurable map $u : [0,t] \to I\!\!R^2$ (the control), with $|u| \le 1$ a.e. in $[0,t]$, and such that the solution to

$$\left\{ \begin{array}{l} x'(s) = c_{\rho^0}(t,x)u(s) \\ x(0) = x_0 \end{array} \right.$$

satisfies $x(t) = z$. Then, using Grownall Lemma, one can easily show that Φ has the following contraction property: Let $\rho^{0,1}$ and $\rho^{0,2}$ belong to $\mathcal{C}^0([0,T], L^1(\mathbb{R}^N))$ with $0 \leq \rho^{0,i} \leq 1$ ($i = 1, 2$) and let us set $K^1(t) = \{\Phi(\rho^{0,1})(t,\cdot) = 1\}$ and $K^2(t) = \{\Phi(\rho^{0,2})(t,\cdot) = 1\}$. We have

$$\sup_{t\in[0,T]} d^{\mathcal{H}}(K^1(t), K^2(t)) \leq C\,T \sup_{t\in[0,T]} \|\rho^{0,1}(t,\cdot) - \rho^{0,2}(t,\cdot)\|_{L^1(R^2)}$$

where $d^{\mathcal{H}}(K^1(t), K^2(t))$ denotes the Hausdorff distance between the sets $K^1(t)$ and $K^2(t)$, and C is some given constant (independant of $\rho^{0,1}$ and $\rho^{0,2}$ and T).

In order to prove that Φ is indeed a contraction, it remains to show an inequality of the form

$$\sup_{t\in[0,T]} \|\Phi(\rho^{0,1})(t,\cdot) - \Phi(\rho^{0,2})(t,\cdot)\|_{L^1(R^2)} \leq C \sup_{t\in[0,T]} d^{\mathcal{H}}(K^1(t), K^2(t)) \,. \tag{6}$$

This amounts to estimate the volume of the symmetric difference between two sets by their Haudorff distance. In general, such an estimate is hopeless, as simple examples show. This is here that the interior ball condition plays a role.

- **Propagation of the interior ball condition :** A remarkable property of Hamilton-Jacobi equations of the form (5) is the fact that it preserves the interior ball condition: If the initial set K_0 satisfies the interior ball condition of radius $r > 0$ and if we denote by $\rho(t,x) = \mathbf{1}_{K(t)}(x)$ the solution to (5), then $K(t)$ still satisfies the interior ball condition for some other (but controlled) radius. This result, which has also been noticed in [19], is strongly inspired from [12] and [9]. Let us also point out that [9] contains the much stronger assertion that, when the velocity is positive, the set $K(t)$ develops immediately an interior ball for any compact initial condition K_0.

- **Perimeter and volume estimate of enlarged sets :** From the interior ball condition, we can get an inequality of the form (6). Indeed, if a set K satisfies the interior ball condition for some radius $r > 0$, then, for any set K_1, the volume of the difference $K\backslash K_1$ can be controlled in terms of the Haudorff distance between K and K_1.

  This result is an consequence of the following monotonicity formula for the perimeter of an enlarged set: If K is a compact subset of $\mathbb{R}^N$, and if we denote by $K + tB$ the set of points which are at a distance less than t of K, then the map $t \to \mathcal{H}^{N-1}(\partial(K+tB))/t^{N-1}$ is nonincreasing.

Let us now explain how this paper is organized: section 2 is devoted to the monotony formula described above and to its applications, among which the fact that the Hausdorff distance controls the volume of the symmetric difference between sets satisfying the interior ball condition. In section 3 we recall some results on the propagation of the interior ball condition and derive the main estimates needed for proving that the map Φ has a fixed point. Statement and proof of the existence and uniqueness for (2) are given in section 4. In section 5 we give a variational formulation of the problem and show that the energy of the dislocation decreases. We also consider the case of several dynamics. We prove in appendix the result on the propagation of interior ball condition.

Let us finally underline that throughout the paper, we work in $\mathbb{R}^N$, for $N \geq 2$, although the physical problem has a meaning only for $N = 2$.

Some notation : We complete this introduction by collecting some notations used throughout the paper. We denote by $|\cdot|$ the euclidean norm of $\mathbb{R}^N$, by $B(x,r)$ the closed ball of radius r centered at the point x. If K is a subset of $\mathbb{R}^N$, $d_K(x)$ denotes the distance of the point x to the set K: $d_K(x) = \inf_{y\in K} |y-x|$. For $r > 0$, we note by $K + rB$ the set of points $x \in \mathbb{R}^N$ such that $d_K(x) \leq r$ and $B = B(0,1)$. Finally, for any function f, we denote the gradient of f by Df.

2 Sets with interior ball condition

We say that a closed set $K \subset \mathbb{R}^N$ satisfies *an interior ball condition* of radius $r > 0$ if, for any point $x \in K$, there is some unit vector $p \in \mathbb{R}^N$ with $B(x - rp, r) \subset K$. Then we have the following result (see also [8])

Lemma 2.1 *If a closed set $K \subset \mathbb{R}^N$ has the interior ball property of radius $r > 0$, there is some closed subset K_0 of K such that $K = K_0 + rB$.*

From this Lemma, it can be seen easily that a closed set $K \subset \mathbb{R}^N$ satisfies *an interior ball condition* of radius $r > 0$, if and only if there is a closed set $K_0 \subset K$ such that K is the set of points $x \in \mathbb{R}^N$ with $d_{K_0}(x) \leq r$. Namely, $K = K_0 + rB$.

Proof of Lemma 2.1: Let us set $K_0 = \{x \in K \mid d_{\partial K}(x) \geq r\}$. Then $K_0 + rB \subset K$, from the definition of K_0.

Conversely, let $x \in K$. There is some $p \in \mathbb{R}^N$ with $|p| = 1$, such that $B(x - rp, r) \subset K$. Hence $x - rp \in K_0$ and $d_{K_0}(x) \leq r$.

QED

In this section we give some estimates of the volume and perimeter of sets satisfying the interior ball condition.

Let us start with an elementary result.

Lemma 2.2 *Let K be a closed subset of $\mathbb{R}^N$, y_1 and y_2 be points of ∂K at which K has the interior ball property of radius $r > 0$: Namely, there exists p_1, p_2 unit vectors such that $B(y_i - rp_i, r) \subset K$ for $i = 1, 2$.*

Then $\langle p_1 - p_2, y_1 - y_2\rangle \leq \frac{1}{r}|y_1 - y_2|^2$.

Proof of Lemma 2.2 : Since y_2 does not belong to the interior of the ball $B(y_1 - rp_1, r)$, we have

$$|y_2 - (y_1 - rp_1)|^2 \geq r^2 \ , \quad \text{whence} \quad |y_2 - y_1|^2 + 2r\langle p_1, y_2 - y_1\rangle \geq 0 \ .$$

In the same way, since y_1 does not belong to the interior of the ball $B(y_2 - rp_2, r)$, we have $|y_2 - y_1|^2 + 2r\langle p_2, y_1 - y_2\rangle \geq 0$. Putting the two inequalities together gives the desired result.

QED

The next Lemma plays a crucial role in our study.

Lemma 2.3 (A monotonicity formula (I)) *Let K be a compact subset of $\mathbb{R}^N$. Then the function $t \to \mathcal{H}^{N-1}(\partial(K + tB))/t^{N-1}$ is nonincreasing.*

Proof : We start with a preliminary result. Let $0 < t_0 < t_1$, y_1, y_2 belong to $\partial(K + t_1B)$, y_1', y_2' be a projection of y_1, y_2 onto $K + t_0B$. We claim that

$$|y_1 - y_2| \leq \frac{t_1}{t_0}|y_1' - y_2'| \ . \tag{7}$$

Proof of the claim : Let z_1 and z_2 be a projection of y_1 and y_2 respectively onto K, and let us set

$$p_1 = \frac{y_1 - z_1}{t_1} \quad \text{and} \quad p_2 = \frac{y_2 - z_2}{t_1} \ .$$

Let us finally set for $j = 1, 2$ and $t \in [0, t_1]$, the maps $y_j(t) = z_j + tp_j$. We note that $y_j(t) \in \partial(K + tB)$ for any $t \in [0, t_1]$ and that $y_j(t_0) = y_j'$.

Let $\rho(t) = \frac{1}{2}|y_1(t) - y_2(t)|^2$. Then

$$\rho'(t) = \langle y_1(t) - y_2(t), p_1 - p_2\rangle \ .$$

Since $y_j(t)$ belong to $\partial(K+tB)$ for $t\in[0,t_1]$ and since the set $K+tB$ satisfies the interior ball condition of radius t, we get from Lemma 2.2 that

$$\rho'(t)\le \frac{1}{t}|y_1(t)-y_2(t)|^2=\frac{2}{t}\rho(t)\ .$$

Integrating this inequality between t_0 and t_1 gives our claim (7).

Next we note that, since d_K is a Lipschitz continuous function, with compact level sets, the co-area formula states that almost all level sets of d_K have finite $\mathcal{H}^{N-1}$ Hausdorff measure. Let us chose $t_0\in(0,t_1)$ a level for which $\mathcal{H}^{N-1}(\partial(K+t_0B))<+\infty$.

Let $\epsilon>0$ and $r_i\in(0,\dfrac{t_0\epsilon}{2t_1})$ be such that

$$\partial(K+t_0B)\subset\bigcup_{i=0}^{\infty}A_i \quad\text{and}\quad \mathcal{H}^{N-1}(\partial(K+t_0B))\ge \mathcal{H}^{N-1}(B^{N-1}(0,1))\sum_{i=0}^{\infty}r_i^{N-1}-\epsilon$$

for some sets A_i of diameter less or equal to $2r_i$, and where $B^{N-1}(0,1)$ is the unit ball of $\mathbb{R}^{N-1}$. We denote by K_i the subset of points of $\partial(K+t_1B)$ for which a projection onto $K+t_0B$ belongs to A_i. Then

$$\partial(K+t_1B)\subset\bigcup_{i=0}^{\infty}K_i\ .$$

We now estimate the diameter $diam(K_i)$ of K_i. Let y_1, y_2 belong to K_i, y_1', y_2' be projections of y_1, y_2 onto $(K+t_0B)$ which belong to A_i. Then from (7), we have

$$|y_1-y_2|\le\frac{t_1}{t_0}|y_1'-y_2'|\le\frac{t_1}{t_0}(2r_i)\ .$$

Hence $diam(K_i)\le\frac{t_1}{t_0}(2r_i)\le\epsilon$. Therefore

$$\begin{aligned}\mathcal{H}^{N-1}_{\epsilon}(\partial(K+t_1B)) &\le \mathcal{H}^{N-1}(B^{N-1}(0,1))\textstyle\sum_{i=0}^{\infty}\left(\frac{diam(K_i)}{2}\right)^{N-1}\\ &\le (t_1/t_0)^{N-1}\mathcal{H}^{N-1}(B^{N-1}(0,1))\textstyle\sum_{i=0}^{\infty}r_i^{N-1}\\ &\le (t_1/t_0)^{N-1}(\mathcal{H}^{N-1}(\partial(K+t_0B))+\epsilon)\ .\end{aligned}$$

Letting $\epsilon\to0^+$ gives

$$\mathcal{H}^{N-1}(\partial(K+t_1B))\le\left(\frac{t_1}{t_0}\right)^{N-1}\mathcal{H}^{N-1}(\partial(K+t_0B))\ .$$

Hence the $\mathcal{H}^{N-1}$ Hausdorff measure of $\partial(K+tB)$ is finite for any level $t>0$, and the map $\mathcal{H}^{N-1}(\partial(K+tB))/t^{N-1}$ is nonincreasing.

QED

For $t > 0$, we always have $\partial(K+tB) \subset \{d_K(x) = t\}$, but the inclusion is not an equality in general. This is why we introduce the following variant of the previous monotonicity formula:

Lemma 2.4 (A monotonicity formula (II)) *Let K be a compact subset of $\mathbb{R}^N$, and d_K the distance function to the set K. Then for any $t_1 > t_0 > 0$, we have*

$$\frac{1}{t_1^{N-1}}\mathcal{H}^{N-1}(\{d_K(x) = t_1\}) \leq \frac{1}{t_0^{N-1}}\mathcal{H}^{N-1}(\partial(K+t_0B))$$

Proof : The proof is similar to the proof of lemma 2.3. We only have to replace $\partial(K+t_1B)$ by $\{d_K(x) = t_1\}$ everywhere in the proof.

QED

As an application we have the following perimeter estimate for bounded sets which satisfy some interior ball condition.

Lemma 2.5 *Let $0 < r < R$. Then, for any compact subset K of $\mathbb{R}^N$ such that $K \subset B(0,R)$ and such that K satisfies the interior ball condition of radius r, we have*

$$\mathcal{H}^{N-1}(\partial K) \leq N|B|\frac{R^N}{r} ,$$

where $|B|$ denotes the volume of the unit ball of $\mathbb{R}^N$.

Proof : Since K satisfies the interior ball condition of radius r, there is some compact set K_0 such that $K = K_0 + rB$. Let us set $K_t = K_0 + tB$. Note that $K_r = K$. From Lemma 2.3 we have

$$\mathcal{H}^{N-1}(\partial K) \leq \left(\frac{r}{t}\right)^{N-1}\mathcal{H}^{N-1}(\partial K_t) \qquad \forall t \in (0,r] .$$

Let us fix for $\theta \in (0,r]$. We now apply the coarea formula (see for instance [4]) to the Lipschitz function d_{K_θ} (the distance function from K_θ): since $|Dd_{K_\theta}| = 1$ a.e., we have

$$\begin{aligned} |K\backslash K_\theta| &= \int_\theta^r dt \int_{\{d_{K_0}(x)=t\}} d\mathcal{H}^{N-1} \\ &\geq \int_\theta^r \mathcal{H}^{N-1}(\partial K_t)dt \\ &\geq \mathcal{H}^{N-1}(\partial K)\int_\theta^r \left(\frac{t}{r}\right)^{N-1} dt \\ &\geq \frac{\mathcal{H}^{N-1}(\partial K)}{N}\left(\frac{r^N-\theta^N}{r^{N-1}}\right) \end{aligned}$$

Next we note that $|K\backslash K_\theta| \leq |K| \leq R^N|B|$. Whence

$$\mathcal{H}^{N-1}(\partial K) \leq \frac{NR^N|B|}{r - \theta^N/r^{N-1}} \ .$$

Letting $\theta \to 0^+$ gives the result.

QED

Finally we show that, under the interior ball condition, it is possible to estimate the Hausdorff distance between sets by the difference of their volume:

Lemma 2.6 *Let K be a compact subset of $I\!R^N$ satisfying the interior ball condition of radius $\sigma > 0$. Then for any $r > 0$ we have*

$$|(K+rB)\backslash K| \leq \frac{\sigma \mathcal{H}^{N-1}(\partial K)}{N}\Big((1+\frac{r}{\sigma})^N - 1\Big) \ .$$

Proof : Let K_0 be a compact subset of $I\!R^N$ such that $K_0 + \sigma B = K$. Then, using Lemma 2.4, we get

$$\begin{aligned} |(K+rB)\backslash K| &= \int_\sigma^{\sigma+r} \mathcal{H}^{N-1}(\{d_{K_0}(x) = t\})dt \\ &\leq \mathcal{H}^{N-1}(\partial K)\int_\sigma^{\sigma+r}\left(\frac{t}{\sigma}\right)^{N-1} dt \\ &\leq \sigma\frac{\mathcal{H}^{N-1}(\partial K)}{N}\Big((1+\frac{r}{\sigma})^N - 1\Big) \end{aligned}$$

QED

3 Estimates of the reachable set of a controlled system

In this section we provide our main estimates in order to prove that the map Φ defined in the introduction is a contraction.

For this, we investigate the propagation of the interior ball for the reachable set of the control system

$$y'(t) = c(t, y(t))u(t), \qquad u(t) \in B(0,1) \tag{8}$$

For any initial position $x_0 \in \mathbb{R}^N$ and any measurable control $u : [0,+\infty) \to B(0,1)$, we denote by $y[y_0,u]$ the solution to (8) with initial position $y[y_0,u](0) = y_0$. We denote by $\mathcal{R}(K,t)$ the reachable set at time t when starting from some closed set K:

$$\mathcal{R}(K,t) = \{z \in \mathbb{R}^N \mid \exists y_0 \in K,\ \exists u : [0,+\infty) \to B(0,1) \text{ measurable },\ y[y_0,u](t) = z\}\ .$$

From now on, we assume that the velocity c satisfies the following regularity properties:

$$\begin{cases} i) & c \text{ is nonnegative, continuous,} \\ & \text{derivable with respect to the second variable} \\ ii) & |c(t,y)| \leq M \qquad \forall (t,y) \in \mathbb{R} \times \mathbb{R}^N \\ iii) & |c(t,y_1) - c(t,y_2)| \leq L_0|y_1 - y_2| \qquad \forall (t,y_1,y_2) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \\ iv) & |D_x c(t,y_1) - D_x c(t,y_2)| \leq L_1|y_1 - y_2| \quad \forall (t,y_1,y_2) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \end{cases} \tag{9}$$

where $M, L_0, L_1 \geq 0$ are fixed constants.

Let us also fix some closed set K. When there is no ambiguity, we simply drop K in the notation of the reachable set: $\mathcal{R}(t) := \mathcal{R}(K,t)$. If a point z belongs to the boundary of the set $\mathcal{R}(T)$ for some $T > 0$, then there exists a measurable control $b : [0,+\infty) \to B(0,1)$ and an initial position $y_0 \in K$ such that $y[y_0,b](T) = z$. It follows from this property that $\mathcal{R}(T)$ is closed. We call such a trajectory $y[y_0,b]$ an extremal trajectory on the time interval $[0,T]$. It is well-known that $y[y_0,b](t) \in \partial\mathcal{R}(t)$ for all $t \in [0,T]$.

Lemma 3.1 *We assume that the set $K \subset \mathbb{R}^N$ is compact and satisfies the interior ball condition of radius $r \in (0,1]$. Then the set $\mathcal{R}(t)$ satisfies the interior ball condition of radius $re^{-\kappa t}$ for any $t \geq 0$, where $\kappa = 3L_0 + L_1$.*

Proof : If $z_0 \in \mathcal{R}(t)$, then there is a time measurable control $b_0 : [0,t] \to B(0,1)$ and some $y_0 \in K$ such that $y[y_0,b_0](t) = z_0$. We now apply Lemma 6.1 in appendix to the differential equation with dynamics $f(t,y) = c(t,y)b_0(t)$: the reachable set for this dynamics f starting from K has the interior ball condition with radius $re^{-\kappa t}$. But this reachable set is contained in $\mathcal{R}(t)$. Hence $\mathcal{R}(t)$ itself satisfies the interior ball condition with radius $re^{-\kappa t}$.

QED

In particular,

Corollary 3.2 *Under the assumption of Lemma 3.1, the map*

$$t \to \rho(t) = \mathbf{1}_{\mathcal{R}(t)} \qquad \forall t \geq 0$$

is continuous in $L^1(I\!\!R^N)$.

Proof : From Lemma 3.1, $\mathcal{R}(t)$ satisfies the interior ball condition for any $t \geq 0$. So the boundary $\partial\mathcal{R}(t)$ has a zero Lebesgue measure. Since moreover $t \to \mathcal{R}(t)$ is increasing and Hausdorff continuous, the desired result follows.

QED

Let $\mathcal{R}_i(t)$ (for $i = 1, 2$) be the reachable set at time t for the controlled system with dynamics

$$\left\{ \begin{array}{ll} y'(t) = c_i(t, y(t))b(t), & |b(t)| \leq 1 \quad a.e.\ t \geq 0 \\ y(0) \in K_i \end{array} \right.$$

We assume that the K_i are closed subsets of $I\!\!R^N$, with $K_1, K_2 \subset B(0, R)$, and satisfy the interior ball condition of radius $r > 0$. We also assume that the c_i satisfy assumption (9) for $i = 1, 2$. We fix some $T > 0$ and we suppose that

$$\|c_2 - c_1\|_\infty := \|c_2 - c_1\|_{L^\infty([0,T]\times R^N)} < +\infty\ .$$

Let $\gamma_0 > 0$ be the Hausdorff distance between K_1 and K_2. Recall that $\gamma_0 \geq 0$ is the smallest real number such that

$$K_2 \subset K_1 + \gamma_0 B \quad \text{and} \quad K_1 \subset K_2 + \gamma_0 B\ .$$

Our aim is estimate the volume of the symmetric difference $\mathcal{R}_1(t)\Delta\mathcal{R}_2(t)$.

Proposition 3.3 *Under the previous assumptions, we have for any* $t \in [0, T]$,

$$|\mathcal{R}_1(t)\Delta\mathcal{R}_2(t)| \leq C\left[\gamma_0 + \|c_1 - c_2\|_\infty t\right]$$

whenever γ_0 *and* $\|c_1 - c_2\|_\infty$ *are so small that*

$$\left[\gamma_0 + \|c_1 - c_2\|_\infty T\right] \leq r e^{-(L_0+\kappa)T}\ , \tag{10}$$

where $C = C(N, T, M, L_0, L_1, r, R)$ *and* $\kappa = 3L_0 + L_1$.

For proving the Proposition we need a preliminary Lemma:

Lemma 3.4 *Under the previous assumptions,*

$$\mathcal{R}_2(t) \subset \mathcal{R}_1(t) + \gamma(t)B$$

where

$$\gamma(t) = \gamma_0 e^{L_0 t} + \|c_1 - c_2\|_\infty \left(\frac{e^{L_0 t} - 1}{L_0} \right) .$$

Proof : Let $z \in \mathcal{R}_2(t)$. There is a time-measurable control $b : [0,t] \to B(0,1)$ and a solution z to

$$\begin{cases} z'(s) = c_2(s, z(s))b(s), & a.e.\ s \geq 0 \\ z(0) \in K_2 \end{cases}$$

such that $z(t) = z$. Let $y_0 \in K_1$ be such that $|y_0 - z(0)| \leq \gamma_0$ and let y be the solution to

$$\begin{cases} y'(s) = c_1(s, y(s))b(s) \\ y(0) = y_0 \end{cases}$$

Then

$$\begin{array}{rcl} |y(s) - z(s)| & \leq & |y_0 - z(0)| + \int_0^s |c_1(\tau, y(\tau)) - c_2(\tau, z(\tau))| d\tau \\ & \leq & \gamma_0 + \|c_1 - c_2\|_\infty s + L_0 \int_0^s |y(\tau) - z(\tau)| d\tau \end{array}$$

since c_1 is L_0-Lipschitz continuous. From Gronwall Lemma we get

$$|y(t) - z(t)| \leq \gamma(t) ,$$

which implies the desired inclusion.

QED

Proof of Proposition 3.3 : It is enough to estimate the difference $|\mathcal{R}_2(t) \backslash \mathcal{R}_1(t)|$. From Lemma 3.4 we have

$$|\mathcal{R}_2(t) \backslash \mathcal{R}_1(t)| \leq |(\mathcal{R}_1(t) + \gamma(t)B) \backslash \mathcal{R}_1(t)| .$$

Following Lemma 3.1 we know that the reachable set $\mathcal{R}_1(t)$ satisfies the interior ball property of radius $\sigma(t) > 0$ for any $t \geq 0$, with $\sigma(t) = re^{-\kappa t}$ and $\kappa = 3L_0 + L_1$. Then Lemma 2.6 states that

$$|(\mathcal{R}_1(t) + \gamma(t)B) \backslash \mathcal{R}_1(t)| \leq \frac{\sigma(t)\mathcal{H}^{N-1}(\partial \mathcal{R}_1(t))}{N} \left((1 + \frac{\gamma(t)}{\sigma(t)})^N - 1 \right) .$$

From assumption (10) we know that

$$\begin{aligned}\frac{\gamma(t)}{\sigma(t)} &= \left[\gamma_0 e^{L_0 t} + \|c_1 - c_2\|_\infty \left(\frac{e^{L_0 t}-1}{L_0}\right)\right] \frac{e^{\kappa t}}{r} \\ &\leq [\gamma_0 + \|c_1 - c_2\|_\infty T] \frac{e^{(L_0+\kappa)T}}{r} \\ &\leq 1 .\end{aligned}$$

Hence $\left((1 + \frac{\gamma(t)}{\sigma(t)})^N - 1\right) \leq N 2^{N-1} \frac{\gamma(t)}{\sigma(t)}$ and we get

$$|\mathcal{R}_2(t) \backslash \mathcal{R}_1(t)| \leq C_N \mathcal{H}^{N-1}(\partial \mathcal{R}_1(t)) \gamma(t)$$

for some constant C_N which only depends on N.

We now note that $\mathcal{R}_1(t) \subset B(0, R + Mt)$ because $K \subset B(0, R)$ and $\|c_1\|_\infty \leq M$. Then Lemma 2.5 together with the interior ball estimate give

$$\mathcal{H}^{N-1}(\partial \mathcal{R}_1(t)) \leq N|B| \frac{(R + Mt)^N e^{\kappa t}}{r} ,$$

from which one gets:

$$|\mathcal{R}_2(t) \backslash \mathcal{R}_1(t)| \leq C_N N|B| \frac{(R + Mt)^N e^{\kappa t}}{r} \left[\gamma_0 e^{L_0 t} + \|c_1 - c_2\|_\infty \left(\frac{e^{L_0 t} - 1}{L_0}\right)\right]$$

Whence the result for a suitable constant $C = C(N, T, M, L_0, L_1, r, R)$.

QED

4 Application to dislocation dynamics

We are now ready to investigate the nonlocal equation arising in dislocation dynamics:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} = (\bar{c}_1 + \bar{c}_0 \star \rho)|D\rho| \\ \rho(0, x) = \rho_0(x) \end{cases} \tag{11}$$

where $\rho_0(x) = \mathbf{1}_{K_0}(x)$.

We assume that $\bar{c}_0$ and $\bar{c}_1$ are such that

$$\bar{c}_1(t, x) \geq \|\bar{c}_0(t, \cdot)\|_{L^1(R^N)} \qquad \forall (t, x) \in [0, +\infty) \times \mathbb{R}^N \tag{12}$$

and satisfy for $i = 0, 1$,

$$\begin{cases} i) & \bar{c}_i \text{ is uniformly continuous with respect to all the variables} \\ & \quad \text{and derivable with respect to the second variable} \\ ii) & |\bar{c}_i(t, y)| \leq M \qquad \forall (t, y) \in \mathbb{R} \times \mathbb{R}^N \\ iii) & |\bar{c}_i(t, y_1) - \bar{c}_i(t, y_2)| \leq L_0 |y_1 - y_2| \qquad \forall (t, y_1, y_2) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \\ iv) & |D_x \bar{c}_i(t, y_1) - D_x \bar{c}_i(t, y_2)| \leq L_1 |y_1 - y_2| \quad \forall (t, y_1, y_2) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \end{cases} \tag{13}$$

where $M, L_0, L_1 \geq 0$ are fixed constants. Let us recall that assumption (12) implies that

$$\bar{c}_1(t,x) + \bar{c}_0 \star \mathbf{1}_K(t,x) \geq 0$$

for any $(t,x) \in I\!R \times I\!R^N$ and any Borel measurable set K.

In order to explain what we mean by a solution to (11), we have to recall some existence and uniqueness results for the (discontinuous) solutions to Hamilton-Jacobi equation of the form

$$\begin{cases} \dfrac{\partial \rho}{\partial t} = c(t,x)|D\rho| \\ \rho(0,x) = \mathbf{1}_{K_0}(x) \end{cases} \tag{14}$$

and the link with the reachable set of the control system (8). In the sequel, we denote by ρ_* the lower semicontinuous envelope of some function ρ and by ρ^* its upper semicontinuous envelope. We recall that uniqueness for (14) means that all discontinuous solutions have the same lower semicontinuous envelope and the same upper semicontinuous envelope.

Lemma 4.1 *Let us assume that c is continuous with respect to all variables and satisfies (9), and that K_0 is equal to the closure of its interior. Then (14) has a unique discontinuous viscosity solution ρ. Moreover, for any time $t > 0$,*

$$\{\rho^*(t,\cdot) = 1\} = \mathcal{R}(K_0, t) \qquad \forall t \geq 0\ ,$$

where, as in section 3, $\mathcal{R}(K_0,t)$ is the reachable set at time t from K_0 for the controlled system (8).

Proof : The uniqueness result for the geometric evolution equation (14) comes from ([7]), Theorem 4.1. In order to show the link between the level set $\{\rho^*(t,\cdot) = 1\}$ and the reachable set $\mathcal{R}(K_0,t)$, let us introduce a new control problem: The value function $u = u(t,x)$ is defined by

$$u(t,x) = \max_b \mathbf{1}_{K_0}(y(0))$$

where y is the solution to the backward differential equation

$$\begin{cases} y'(s) = c(s, y(s))b(s) \quad \text{a.e. in } [0,t] \\ y(t) = x \end{cases}$$

and where the maximum is taken over the measurable maps $b : [0,t] \to B(0,1)$. Following [5] or [6], a routine verification shows that u is a discontinuous viscosity solution to (14), hence it is the unique discontinuous viscosity solution. To complete the proof of the assertion, it suffices to notice that u is upper semicontinuous and that, by definition, $\mathcal{R}(K_0,t) = \{x \in \mathbb{R}^N \mid u(t,x) = 1\}$.

QED

Let us now explain what we mean by a viscosity solution to (11) (see also the discussion in [3]).

Definition 4.2 *We say that $\rho : [0,+\infty) \times \mathbb{R}^N \to \mathbb{R}$ is a viscosity solution to (11) if $\rho \in \mathcal{C}^0([0,+\infty), L^1(\mathbb{R}^N))$ and if ρ is the unique discontinuous viscosity solution to*

$$\begin{cases} \dfrac{\partial \rho}{\partial t} = c_\rho(t,x)|D\rho| \\ \rho(0,x) = \rho_0(x) \end{cases} \tag{15}$$

where $c_\rho(t,x) := (\bar{c}_1(t,x) + (\bar{c}_0 \star \rho)(t,x))$.

Remark : Since $\rho \in \mathcal{C}^0([0,+\infty), L^1(\mathbb{R}^N))$ and from assumption (13), the function c_ρ is continuous with respect to all the variables and satisfies (9), with new constants $M' = M \min\left(1 + ||\rho||_{L^\infty((0,T);L^1(R^N))}, 1 + ||\rho||_{L^\infty((0,T)\times R^N)}\right)$, $L'_0 = L_0\left(1 + ||\rho||_{L^\infty((0,T);L^1(R^N))}\right)$, $L'_1 = L_1\left(1 + ||\rho||_{L^\infty((0,T);L^1(R^N))}\right)$.
In the proof of Theorem 4.3, we will have $||\rho||_{L^\infty((0,T);L^1(R^N))} \leq 1$, $||\rho||_{L^\infty((0,T)\times R^N)} \leq |K_0| + 1$, which gives the following choice: $M' = 2M$, $L'_0 = L_0(2 + |K_0|)$, $L'_1 = L_1(2 + |K_0|)$. From the uniform bound $M' = 2M$ on the velocity, we see in particular that if supp $(\rho_0) \subset B(0, R-\delta)$, the minimal time for the solution to exit the ball $B(0,R)$ is bounded from below by $\delta/(2M)$, which explains what there will be no blow-up of $||\rho(t,\cdot)||_{L^1(R^N)}$ in finite time for this equation.
In the proof of Theorem 4.3, up to redefine M, L_0, L_1 by M', L'_0, L'_1, we will keep the notation M, L_0, L_1.

Theorem 4.3 *Let us assume that the compact set K_0 satisfies the interior ball condition of radius $r > 0$. Then, under assumption (12), the Cauchy problem (11) has a unique discontinuous viscosity solution ρ defined on $[0,+\infty)$.*

Moreover, the solution ρ depends in a Lipschitz way on the initial set K_0 in the following sense: For any $T > 0$ and $R > 0$, there are constants $\epsilon > 0$ and $C > 0$, such that, for any compact sets K_0^i which satisfy the interior *ball condition of radius r, and such that $K_0^i \subset B(0,R)$ (for $i = 1,2$), if we denote by ρ^i the unique solution to (11) with initial condition $\mathbf{1}_{K_0^i}$, then*

$$d^{\mathcal{H}}(K_0^1, K_0^2) \le \epsilon \quad \Rightarrow \quad \sup_{t\in[0,T]} \|\rho^1(t,\cdot) - \rho^2(t,\cdot)\|_{L^1(\mathbb{R}^N)} \le C d^{\mathcal{H}}(K_0^1, K_0^2) \ ,$$

where $d^{\mathcal{H}}(K_0^1, K_0^2)$ denotes the Hausdorff distance between the sets K_0^1 and K_0^2.

Remark: With slight modifications of the proofs, it is possible to prove a similar result when K_0 is the closure of the exterior of a compact set, with K_0 still satisfying the interior sphere condition.

Proof of Theorem 4.3 : We first prove the local existence and uniqueness of the solution. Up to reduce r, we can assume that $r \in (0,1]$. Let $R > 0$ be such that $K_0 \subset B(0,R)$, $T > 0$ and let us set

$$\mathcal{E}_T = \left\{ \rho \in \mathcal{C}^0([0,T], L^1(\mathbb{R}^N)) \mid \rho(0) = \rho_0,\ 0 \le \rho \le 1,\ \sup_{t\in[0,T]} \|\rho(t)\|_1 \le |K_0| + 1 \right\} .$$

We fix $T \in (0,1)$ such that

$$[\ \|\bar{c}_1\|_\infty + 2\|\bar{c}_0\|_\infty(|K_0|+1)\ ]\, T \frac{e^{(L_0+\kappa)T}}{r} \le 1 \tag{16}$$

where $\kappa = 3L_0 + L_1$, and such that

$$C\,[\|\bar{c}_1\|_\infty + \|\bar{c}_0\|_\infty(|K_0|+1)]\ T \le 1 \tag{17}$$

where $C = C(N,1,M,L_0,L_1,r,R)$ is the constant given in Proposition 3.3 for $T = 1$, and

$$\theta := C\|\bar{c}_0\|_\infty T < 1\ . \tag{18}$$

Note that T only depends—besides the data—on the radius $r > 0$ and on the volume $|K_0|$ of K_0, and is bounded from below by some positive constant as long as r is bounded from below by a positive constant and that $|K_0|$ remain bounded.

Let us define the map Φ which associates to any $\rho^0 \in \mathcal{E}_T$ the unique viscosity solution ρ to

$$\left\{ \begin{array}{l} \dfrac{\partial \rho}{\partial t} = (\bar{c}_1 + \bar{c}_0 \star \rho^0)|D\rho| \\ \rho(0,x) = \rho_0(x) \end{array} \right. \tag{19}$$

We first claim that $\Phi(\mathcal{E}_T) \subset \mathcal{E}_T$. Indeed, from assumption (12) and (13), the velocity $c_1(t,x) = \bar{c}_1(t,x) + (\bar{c}_0 \star \rho^0)(t,x)$ satisfies assumptions (9). Corollary 3.2 then states that $\rho \in \mathcal{C}^0([0,T], L^1(I\!R^N))$.

We want to apply Proposition 3.3 to the velocity c_1 and the velocity $c_2 = 0$ (for which $\rho_2(t) = \mathbf{1}_{K_0}$ for all $t \geq 0$). For this we first check that (10) holds: Indeed

$$\begin{array}{rcl} \|c_1 - c_2\|_\infty T & \leq & \left(\|\bar{c}_1\|_\infty + \|\bar{c}_0\|_\infty \sup_{t\in[0,T]} \|\rho^0(t)\|_1\right) T \\ & \leq & \left(\|\bar{c}_1\|_\infty + \|\bar{c}_0\|_\infty(|K_0|+1)\right) T \leq r e^{-(L_0+\kappa)T} \end{array}$$

from the choice of T in (16). Proposition 3.3 then states that (recall that $\rho_2(t) = \mathbf{1}_{K_0}$)

$$\begin{aligned} \|\rho(t)\|_1 &= \|\rho(t) - \rho_2(t)\|_1 + |K_0| \\ &\leq C \left[\|\bar{c}_1\|_\infty + \|\bar{c}_0\|_\infty \sup_{t\in[0,T]} \|\rho^0(t,\cdot)\|_1\right] T + |K_0| \\ &\leq |K_0| + 1 \end{aligned}$$

from (17). Hence $\rho \in \mathcal{E}_T$.

Finally we want to prove that Φ is a contraction. Let ρ_1^0 and ρ_2^0 belong to $\mathcal{E}_T$, $c_1 = \bar{c}_1 + \bar{c}_0 \star \rho_1^0$ and $c_2 = \bar{c}_1 + \bar{c}_0 \star \rho_2^0$, $\rho_1 = \Phi(\rho_1^0)$ and $\rho_2 = \Phi(\rho_2^0)$. We first check that c_1 and c_2 satisfy condition (10). Indeed,

$$\begin{array}{rcl} \|c_1 - c_2\|_\infty T & \leq & \|\bar{c}_0\|_\infty \sup_{t\in[0,T]} \|\rho_1^0(t,\cdot) - \rho_2^0(t,\cdot)\|_1 T \\ & \leq & 2\|\bar{c}_0\|_\infty(|K_0|+1)T \\ & \leq & r e^{-(L_0+\kappa)T} \; . \end{array}$$

from the definition of $\mathcal{E}_T$ and the choice of T in (16). Then using Proposition 3.3 again, we get

$$\|\rho_1(t) - \rho_2(t)\|_1 \leq C\|c_1 - c_2\|_\infty \, t \; ,$$

which finally gives, from the choice of T in (18)

$$\begin{array}{rcl} \|\rho_1(t) - \rho_2(t)\|_1 & \leq & (C\|\bar{c}_0\|_\infty T) \sup_{t\in[0,T]} \|\rho_1^0(t,\cdot) - \rho_2^0(t,\cdot)\|_1 \\ & \leq & \theta \sup_{t\in[0,T]} \|\rho_1^0(t,\cdot) - \rho_2^0(t,\cdot)\|_1 \end{array}$$

with $\theta < 1$.

Since Φ is a contraction on $\mathcal{E}_T$, it has a unique fixed point. So we have proved that the problem has a unique solution $\rho(t,\cdot) = 1_{K(t)}$ at least on the time interval $[0,T]$, where T depends on the volume of K_0, on R (where $K_0 \subset B(0,R)$) and on the radius of the interior ball r for K_0. Using Lemma

3.1, we know that the set $K(t)$ satisfies the interior ball condition of radius $re^{-\kappa t}$, where κ depends only on L_0 and L_1. Moreover, the volume of $K(t)$ and the radius R' such that $K(t) \subset B(0, R')$ are bounded for bounded times because of the finite speed of propagation. Therefore we can extend the solution in a unique way on $[0, +\infty)$.

The proof of the Lipschitz continuity of the solution with respect to the initial set is based on similar arguments as for the local existence and uniqueness, and the use of Proposition 3.3 with $\gamma_0 = d^{\mathcal{H}}(K_0^1, K_0^2)$.

QED

5 More on dislocation dynamics

5.1 The notion of variational solution

Our aim is to investigate a notion of weak solutions which implies that the natural energy associated to dislocation dynamics is non-increasing in time. For this reason, these weak solutions will be called variational solutions. In particular, this allows to prove energy estimates for the generalized evolution. Towards this aim, we first need the following:

Lemma 5.1 *Let us assume that $c = c(t, x)$ satisfies (9) and moreover that*

$$c(t, x) > 0 \qquad \forall (t, x) \in [0, +\infty) \times \mathbb{R}^N . \tag{20}$$

Let $\mathcal{R}(t)$ be the reachable set (defined in Section 3) at time t starting from some fixed compact set $K \subset \mathbb{R}^N$ which satisfies the interior ball property. Then, for any map $\varphi \in \mathcal{C}^1([0, +\infty) \times \mathbb{R}^N)$, the map $t \to \int_{\mathcal{R}(t)} \varphi(t, x)dx$ is absolutely continuous and

$$\frac{d}{dt} \int_{\mathcal{R}(t)} \varphi(t, x)dx = \int_{\mathcal{R}(t)} \frac{\partial \varphi}{\partial t}(t, x)dx + \int_{\partial \mathcal{R}(t)} \varphi(t, y)c(t, y)d\mathcal{H}^{N-1}(y) \tag{21}$$

Proof of Lemma 5.1 : Let us introduce the minimal time $\tau : \mathbb{R}^N \to \mathbb{R}$

$$\tau(x) = \min\{t \geq 0 \mid x \in \mathcal{R}(t)\} \qquad \forall x \in \mathbb{R}^N .$$

Under assumption (9) and (20), the map τ is locally Lipschitz continuous and satisfies

$$c(\tau(x), x)|D\tau(x)| = 1 \qquad \text{for almost all } x \in \mathbb{R}^N \backslash K . \tag{22}$$

In particular assumption (20) implies that, for any $R > 0$, there is a constant $\alpha = \alpha(R) > 0$ such that

$$|D\tau(x)| \geq \alpha \qquad \text{for almost all } x \in B(0,R)\backslash K\ . \tag{23}$$

Moreover, $\{\tau \leq t\} = \mathcal{R}(t)$ for any $t \geq 0$.

Step 1 : Let us first prove that (21) holds for $\varphi = \varphi(x) \in \mathcal{C}_c^\infty(I\!R^N)$. From the coarea formula, that we can apply under this form thanks to (23) and the fact that φ has a compact support, we have

$$\int_{\{\tau>t\}} \varphi(x)dx = \int_t^{+\infty} \int_{\{\tau=s\}} \frac{\varphi(y)}{|D\tau(y)|} d\mathcal{H}^{N-1}(y)ds\ . \tag{24}$$

In order to proceed we need to show that

$$\mathcal{H}^{N-1}(\{\tau = s\}\backslash\partial\mathcal{R}(s)) = 0 \quad \text{for almost all } s > 0\ . \tag{25}$$

For getting this we first note that

$$\partial^*\{\tau > s\}\ \subset\ \partial\mathcal{R}(s)\ \subset\ \{\tau = s\}\ ,$$

where $\partial^*\{\tau > s\}$ denotes the reduced boundary of the set $\{\tau > s\}$. Set $U = \{\tau < t\}$. Using the coarea formula for Lipschitz continuous functions on the one hand and for BV functions (see [14]) on another hand gives

$$\int_U |D\tau(x)|dx = \int_0^t \mathcal{H}^{N-1}(\{\tau = s\}) = \int_0^t \mathcal{H}^{N-1}(\partial^*\{\tau > s\})$$

Hence

$$\mathcal{H}^{N-1}(\{\tau = s\}\backslash\partial^*\{\tau > s\}) = 0 \quad \text{for almost all } s > 0\ ,$$

and therefore (25) holds. Coming back to (24), using first (22) and then (25) now gives

$$\int_{\{\tau>t\}} \varphi(x)dx = \int_t^{+\infty} \int_{\partial\mathcal{R}(s)} \varphi(y)c(s,y)d\mathcal{H}^{N-1}(y)ds\ .$$

In particular, the map

$$t \to \int_{\mathcal{R}(t)} \varphi(x)dx = \int_{R^N} \varphi(x)dx - \int_{\{\tau>t\}} \varphi(x)dx$$

is absolutely continuous and

$$\frac{d}{dt}\int_{\mathcal{R}(t)}\varphi(x)dx=\int_{\partial\mathcal{R}(t)}\varphi(y)c(t,y)d\mathcal{H}^{N-1}(y)\ .$$

Step 2 : We now prove that (21) holds for any map $\varphi\in\mathcal{C}^2_c([0,+\infty)\times\mathbb{R}^N)$. For this let us fix $n\geq 1$ and let us define the partition (t_k) of $[0,t]$ by $t_k=\frac{kt}{n}$ for $k=0,\ldots,n$. Then

$$\begin{aligned}\int_{\mathcal{R}(t)}\varphi(t,x)dx&-\int_{\mathcal{R}(0)}\varphi(0,x)dx\\&=\sum_{k=0}^{n-1}\left(\int_{\mathcal{R}(t_{k+1})}\varphi(t_{k+1},x)dx-\int_{\mathcal{R}(t_k)}\varphi(t_k,x)dx\right)\ .\end{aligned}$$

We have

$$\begin{aligned}\int_{\mathcal{R}(t_{k+1})}&\varphi(t_{k+1},x)dx-\int_{\mathcal{R}(t_k)}\varphi(t_k,x)dx\\&=\int_{\mathcal{R}(t_{k+1})\setminus\mathcal{R}(t_k)}\varphi(t_{k+1},x)dx+\frac{t}{n}\int_{\mathcal{R}(t_k)}\frac{\partial\varphi}{\partial t}(t_k,x)dx+\epsilon(t_k)\end{aligned}$$

where

$$|\epsilon(t_k)|\leq\int_{\mathcal{R}(t_k)}\left|\varphi(t_{k+1},x)-\varphi(t_k,x)-\frac{t}{n}\frac{\partial\varphi}{\partial t}(t_k,x)\right|dx\leq\frac{1}{2}\left(\frac{t}{n}\right)^2\|\frac{\partial^2\varphi}{\partial t^2}\|_\infty\,|\mathcal{R}(t)|$$

and where, from the first step of the proof,

$$\int_{\mathcal{R}(t_{k+1})\setminus\mathcal{R}(t_k)}\varphi(t_{k+1},x)dx=\int_{t_k}^{t_{k+1}}\int_{\partial\mathcal{R}(s)}\varphi(t_{k+1},y)c(s,y)d\mathcal{H}^{N-1}(y)ds\ .$$

Therefore

$$\int_{\mathcal{R}(t)}\varphi(t,x)dx-\int_{\mathcal{R}(0)}\varphi(0,x)dx\ =$$
$$\sum_{k=0}^{n-1}\left(\int_{t_k}^{t_{k+1}}\int_{\partial\mathcal{R}(s)}\varphi(t_{k+1},y)c(s,y)d\mathcal{H}^{N-1}(y)ds+\frac{t}{n}\int_{\mathcal{R}(t_k)}\frac{\partial\varphi}{\partial t}(t_k,x)dx+\epsilon(t_k)\right)$$

Our aim is to let $n\to+\infty$ in the above formula. For this, we note that $\mathcal{R}(s)$ is bounded for bounded times and satisfies the interior ball property with a locally uniform radius (Lemma 3.1). Therefore Lemma 2.5 states that $\mathcal{H}^{N-1}(\partial\mathcal{R}(s))$ is locally uniformly bounded. Thus

$$\lim_n \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \int_{\partial\mathcal{R}(s)} \varphi(t_{k+1},y)c(s,y)d\mathcal{H}^{N-1}(y)ds \\ = \int_0^t \int_{\partial\mathcal{R}(s)} \varphi(s,y)c(s,y)d\mathcal{H}^{N-1}(y)ds$$

by Lebesgue Theorem. We also have

$$\lim_{n\to+\infty} \frac{t}{n} \sum_{k=0}^{n-1} \int_{\mathcal{R}(t_k)} \frac{\partial\varphi}{\partial t}(t_k,x)dx = \int_0^t \int_{\mathcal{R}(s)} \frac{\partial\varphi}{\partial t}(s,x)dx$$

because it is a Riemann sum and the map $s \to \int_{\mathcal{R}(s)} \frac{\partial\varphi}{\partial t}(s,x)dx$ is continuous since $s \to \mathbf{1}_{\mathcal{R}(s)}$ is continuous in $L^1(\mathbb{R}^N)$ from Lemma (3.2). So we have proved that

$$\int_{\mathcal{R}(t)} \varphi(t,x)dx - \int_{\mathcal{R}(0)} \varphi(0,x)dx = \\ \int_0^t \left(\int_{\partial\mathcal{R}(s)} \varphi(s,y)c(s,y)d\mathcal{H}^{N-1}(y)ds + \int_{\mathcal{R}(s)} \frac{\partial\varphi}{\partial t}(s,x)dx \right) ds$$

which is the desired result for $\varphi \in \mathcal{C}^2_c([0,+\infty) \times \mathbb{R}^N)$. We complete the proof of the Lemma by density arguments.

QED

A straightforward application of Lemma 5.1 gives:

Corollary 5.2 *Let us assume that $K_0 \subset \mathbb{R}^N$ is compact and satisfies the interior ball condition. Let us assume that $\bar{c}_0$ and $\bar{c}_1$ satisfy (13) and that*

$$\bar{c}_1(t,x) > \|\bar{c}_0\|_{L^1(\mathbb{R}^N)} \qquad \forall x \in \mathbb{R}^N,\ \forall t \geq 0\ .$$

Let ρ be the unique solution to the dislocation dynamic problem (11). Then ρ also satisfies the following: for any $\varphi \in \mathcal{C}^1([0,+\infty) \times \mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} \varphi(t,x)\rho(t,x)dx - \int_{\mathbb{R}^N} \varphi(0,x)\rho(0,x)dx \\ = \int_0^t \left[\int_{\mathbb{R}^N} \frac{\partial\varphi}{\partial t}(s,x)\rho(s,x)dx + \int_{\partial\{\rho(s,\cdot)=1\}} \varphi(s,y)c(s,y)d\mathcal{H}^{N-1}(y) \right] ds \tag{26}$$

where $c = \bar{c}_0 \star \rho + \bar{c}_1$.

Remarks:

1. This equation allows to define a notion of variational solution for the problem of dislocation dynamics.

2. Equation (26) also holds if φ is continuous, and such that its time derivative $\frac{\partial \varphi}{\partial t}$ in the sense of distribution is in $L^1_{loc}([0,+\infty)\times I\!R^N)$.

When the data do not depend on time, namely $\bar c_0 = \bar c_0(x)$ and $\bar c_1 = \bar c_1(x)$, and when the kernel $\bar c_0$ is symmetric, the energy naturally associated to the dislocation is

$$E(t) = \int_{R^N} -\frac{1}{2}(\bar c_0 \star \rho)\rho - \bar c_1 \rho \ .$$

This energy is non increasing:

Proposition 5.3 *Under the assumptions and notations of Corollary (5.2), let us suppose that $\bar c_0 = \bar c_0(x)$ and $\bar c_1 = \bar c_1(x)$, and that $\bar c_0(-x) = \bar c_0(x)$ for any $x \in I\!R^N$. Then the energy $t \to E(t)$ is a locally Lipschitz continuous and*

$$\frac{d}{dt}E(t) = -\int_{\partial\{\rho(t,\cdot)=1\}} c^2 d\mathcal{H}^{N-1} \ ,$$

where $c = \bar c_0 \star \rho + \bar c_1$.

Proof : Let $\varphi(t,x) = \frac{1}{2}\bar c_0 \star \rho + \bar c_1$. We note that φ is continuous and that $t \to \varphi(t,x)$ is absolutely continuous thanks to Corollary 5.2, with

$$\frac{d}{dt}\varphi(t,x) = \frac{1}{2}\frac{d}{dt}\int_{R^N} \bar c_0(y-x)\rho(t,y)dy = \frac{1}{2}\int_{\partial\{\rho(t,\cdot)=1\}} \bar c_0(x-y)c(t,y)d\mathcal{H}^{N-1}(y) \ .$$

Let us recall (see the proof of Lemma 5.1), that $\mathcal{H}^{N-1}(\partial\{\rho(t,\cdot)=1\})$ is locally bounded. Therefore $t \to \varphi(t,x)$ is locally Lipschitz continuous as well as $t \to E(t)$. So, using Corollary 5.2 again, we have

$$\frac{d}{dt}E(t) = -\int_{R^N} \frac{\partial \varphi}{\partial t}\rho dx - \int_{\partial\{\rho(t,\cdot)=1\}} \varphi c d\mathcal{H}^{N-1}$$

where

$$\begin{aligned}\int_{R^N} \frac{\partial \varphi}{\partial t}\rho dx &= \frac{1}{2}\int_{\partial\{\rho(t,\cdot)=1\}}\int_{R^N} \bar c_0(x-y)\rho(t,x)c(t,y)dx d\mathcal{H}^{N-1}(y)\\ &= \frac{1}{2}\int_{\partial\{\rho(t,\cdot)=1\}} (\bar c_0 \star \rho) c d\mathcal{H}^{N-1}\end{aligned}$$

Therefore

$$\frac{d}{dt}E(t) = -\int_{\partial\{\rho(t,\cdot)=1\}} (\varphi + \frac{1}{2}(\bar c_0 \star \rho)) c d\mathcal{H}^{N-1} = -\int_{\partial\{\rho(t,\cdot)=1\}} c^2 d\mathcal{H}^{N-1} \ .$$

QED

5.2 Dynamics with several dislocations

Let $M > 0$ be an integer. We will assume

$$\bar{c}_1(t,x) \geq M\|\bar{c}_0(t,\cdot)\|_{L^1(\mathbb{R}^N)} \qquad \forall (t,x) \in [0,+\infty) \times \mathbb{R}^N \tag{27}$$

When we consider the dynamics of M dislocations of the same type (same Burgers vector, and in the same slip plane), it is possible to state a result similar to Theorem 4.3.

We have the

Theorem 5.4 *We consider M compact sets K_0^m, $m = 1, ..., M$, such that $K_0^1 \supset K_0^2 \supset ... \supset K_0^m$. We assume that each compact K_0^m satisfies the interior ball condition of radius $r > 0$. Then, under assumption (27), the Cauchy problem (11), with initial condition*

$$\rho_0(x) = \sum_{m=1}^{M} \mathbf{1}_{K_0^m}$$

has a unique discontinuous viscosity solution ρ defined on $[0,+\infty)$.

The proof is an adaptation of the proof of Theorem 4.3 and is left to the reader.

Remark: If $K_0^m \supset\supset K_0^{m+1}$, then for every time $t > 0$, we have $\{\rho \geq m\} \supset\supset \{\rho \geq m+1\}$. (This is an easy consequence of the representation of each set $\{\rho \geq m\}$ as the reachable set for the controlled system (8) with $c(t,x) = \bar{c}_0 \star \rho + \bar{c}_1$.)

6 Appendix : Propagation of the interior ball condition

In this section, we consider a differential equation

$$y'(t) = f(t, y(t)) \; . \tag{28}$$

The reachable set for f when starting from an initial closed set K is defined in the usual way and denoted as before $\mathcal{R}(t)$. Our aim is to show that this reachable set satisfies the interior ball condition provided the initial set does. The computations below are strongly inspired by those of ([9]).

For this we assume that f enjoys the following regularity:

$$\begin{cases} i) & f \text{ is Borel measurable, derivable with respect to the second variable} \\ & \quad \text{for almost every } t \\ ii) & |f(t,y_1) - f(t,y_2)| \le L_0|y_1 - y_2| \qquad \forall (t,y_1,y_2) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \\ iii) & |D_x f(t,y_1) - D_x f(t,y_2)| \le L_1|y_1 - y_2| \quad \forall (t,y_1,y_2) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \end{cases} \tag{29}$$

where $L_0, L_1 \ge 0$ are fixed constants.

Lemma 6.1 (Propagation of the interior ball condition) *We assume that the closed set K satisfies the interior ball condition of radius $r \in (0,1]$. Then the set $\mathcal{R}(t)$ satisfies the interior ball condition of radius $re^{-\kappa t}$ for any $t \ge 0$, where $\kappa = 3L_0 + L_1$.*

More precisely, if y is a solution of (28) with $y(0) \in K$, if p_0 is a unit vector such that $B(y(0)-rp_0, r) \subset K$, and if $p : [0,T] \to \mathbb{R}^N$ is an absolutely continuous map satisfying

$$\begin{cases} -p'(t) = D_x f(t,y(t))^* p(t) \\ p(0) = p_0 \end{cases} \tag{30}$$

(where $D_x f(t,y(t))^*$ denotes the transpose of the matrix $D_x f(t,y(t))$), then $B(y(t) - re^{-\kappa t}\frac{p(t)}{|p(t)|}, re^{-\kappa t})$ is contained in $\mathcal{R}(t)$ for any $t \in [0,T]$.

Proof of Lemma 6.1: Let y and p as in the Lemma. Note that $p(t) \neq 0$ for any $t \in [0,T]$.

For any $\theta \in B(0, |p(T)|)$ we consider the solution y_θ of the (backward) differential equation

$$\begin{cases} y_\theta'(t) = f(t, y_\theta(t)) \\ y_\theta(T) = y(T) - re^{-kT}(p(T) - \theta) \end{cases}$$

where

$$k = 2L_0 + L_1 \ .$$

We are first going to prove that $y_\theta(0) \in K$. For this, let us consider the function

$$\phi(t) = \frac{1}{2}|y_\theta(t) - y(t)|^2 + re^{-kt}\langle y_\theta(t) - y(t), p(t)\rangle \ .$$

Note for later use that

$$\phi(T) = \frac{1}{2}r^2 e^{-2kT}|p(T) - \theta|^2 - r^2 e^{-2kT}\langle p(T) - \theta, p(T)\rangle \le 0 \ , \tag{31}$$

since $\theta \in B(0, |p(T)|)$. Then

$$\begin{aligned} \phi'(t) \;=\; & \langle y_\theta(t) - y(t), f(t, y_\theta(t)) - f(t, y(t))\rangle \\ & + re^{-kt}\langle f(t, y_\theta(t)) - f(t, y(t)), p(t)\rangle \\ & - re^{-kt}\langle y_\theta(t) - y(t), D_x f(t, y(t))^* p(t)\rangle \\ & - rke^{-kt}\langle y_\theta(t) - y(t), p(t)\rangle \ . \end{aligned}$$

From (29(ii)),

$$\langle y_\theta(t) - y(t), f(t, y_\theta(t)) - f(t, y(t))\rangle \geq -L_0|y_\theta(t) - y(t)|^2 \ .$$

Since

$$f(t, y_\theta(t)) - f(t, y(t)) = \int_0^1 D_x f(t, sy_\theta(t) + (1-s)y(t))(y_\theta(t) - y(t))ds$$

we have

$$\begin{aligned} & \langle f(t, y_\theta(t)) - f(t, y(t)), p(t)\rangle - \langle y_\theta(t) - y(t), D_x f(t, y(t))^* p(t)\rangle \\ & = \int_0^1 \langle (D_x f(t, sy_\theta(t) + (1-s)y(t)) - D_x f(t, y(t)))(y_\theta(t) - y(t)), p(t)\rangle ds \\ & \geq -\frac{L_1}{2}|y_\theta(t) - y(t)|^2|p(t)| \end{aligned}$$

thanks to (29(iii)). Since $|p(t)| \leq e^{L_0 t}$, $k = 2L_0 + L_1$ and $r \in (0,1]$, we have $2L_0 + re^{-kt}L_1|p(t)| \leq k$ for any $t \in [0,T]$. Hence we get

$$\phi'(t) \geq -k\phi(t) \ ,$$

which gives $\phi(0) \leq e^{kT}\phi(T) \leq 0$ from (31). Therefore

$$\phi(0) = \frac{1}{2}|y_\theta(0) - y(0) + rp_0|^2 - \frac{1}{2}r^2 \leq 0 \ ,$$

which proves that $y_\theta(0) \in K$ because $B(y_0 - rp_0, r) \subset K$.

Since $y_\theta(0) \in K$, we also have $y_\theta(T) = y(T) - re^{-kT}(p(T) - \theta) \in \mathcal{R}(T)$ for any $\theta \in B(0, |p(T)|)$. Hence $\mathcal{R}(T)$ satisfies the interior ball condition of radius $re^{-kT}|p(T)|$. Since $|p(T)| \geq e^{-L_0 T}$, we have finally proved our claim with $\kappa = k + L_0 = 3L_0 + L_1$.

QED

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