Convergence of a first order scheme for a non local eikonal equation *

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Abstract

We prove the convergence of a first order finite difference scheme approximating a non local eikonal Hamilton-Jacobi equation. The non local character of the problem makes the scheme not monotone in general. However, by using in a convenient manner the convergence result for monotone scheme of Crandall Lions, we obtain the same bound $\sqrt{|\Delta X| + \Delta t}$ for the rate of convergence.

Introduction

The paper is concerned with the convergence of a first order finite difference scheme that approximates the solution of a non local Hamilton Jacobi equation of the form

$$u_t = c[u]\,|\nabla u| \quad \text{in } \mathbb{R}^2 \times (0, \overline{T}), \qquad u(\cdot, 0) = u^0 \quad \text{in } \mathbb{R}^2. \tag{1}$$

The non-local mapping c enjoys suitable regularity assumptions and the initial condition u^0 is globally Lipschitz continuous, possibly unbounded.

A typical example of mapping $c[u]$ we have in mind is

$$c[u] = c^0 \star [u]$$

where $[u]$ is the characteristic function of the set $\{u \geq 0\}$, defined by

$$[u] = \begin{cases} 1 & \text{if } u \geq 0, \\ 0 & \text{if } u < 0. \end{cases} \tag{2}$$

Here the kernel c^0, which depends only on the space variables, is integrable and has bounded variation and $\star$ denotes the convolution in space. The zero level set of the solution of the resulting equation

$$\begin{cases} u_t(x,y,t) = \big(c^0 \star [u](x,y,t)\big)\,|\nabla u(x,y,t)| & \mathbb{R}^2 \times (0, \overline{T}) \\ u(x,y,0) = u^0(x,y) & \mathbb{R}^2, \end{cases} \tag{3}$$

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models the evolution of a dislocation line in a 2D plane (see [AHLM1] and [AHLM2] for a physical presentation of the model for dislocation dynamics).

We approximate the non-local equation (1) by a first order finite difference scheme that uses a monotone numerical Hamiltonian for the norm of the spatial gradient, the forward Euler scheme for the time derivative and a suitable abstract discrete approximation of the nonlocal operator c. The non local character of the problem makes the scheme not monotone in general. However, by using in a convenient manner the convergence result for monotone scheme of Crandall Lions, we are able to obtain the same bound $\sqrt{|\Delta X| + \Delta t}$ for the rate of convergence provided the terminal horizon $\overline{T} > 0$ is small enough.

The present paper is fairly abstract. Its main objective is to find out general assumptions on the discrete approximation of the non-local operator $c[u]$ that guarantee the convergence of the scheme. A companion paper to this article is [ACMR] where we apply this convergence result to the study of the dislocation dynamics equation (3).

The paper is organized as follows. The precise assumptions on the non-local velocity that guarantee the solvability of the nonlocal Hamilton-Jacobi equation (1) are given in section 1. In section 2, we recall the classical finite-difference scheme for the approximation of local eikonal equations. The extension of the scheme to non-local equations is given in section 4. Section 3 recalls the Crandall-Lions [CL1] estimate of the rate of convergence for local Hamilton-Jacobi equations. We give an updated proof of the result that tackles the non-classical assumptions we make (in particular, the non-boundedness of the initial condition). Finally, we state and prove in section 5 our main convergence result.

1 The continuous problem

We are interested in the nonlocal Hamilton-Jacobi equation

$$u_t = c[u]\,|\nabla u| \quad \text{in } \mathbb{R}^2 \times (0, \overline{T}), \qquad u(\cdot, 0) = u^0 \quad \text{in } \mathbb{R}^2,$$

where the mapping c enjoys suitable regularity assumptions to be specified in a moment and where the initial condition u^0 is (globally) Lipschitz continuous, possibly unbounded.

First, we consider the eikonal equation

$$u_t = c(x, y, t)|\nabla u| \quad \text{in } \mathbb{R}^2 \times (0, \overline{T}), \qquad u(\cdot, 0) = u^0 \quad \text{in } \mathbb{R}^2 \tag{4}$$

to set a few notations. We assume that the velocity c is bounded and Lipschitz continuous with respect to all the variables.

The classical theory of viscosity solution ensures that (4) as a unique continuous viscosity solution with at most linear growth in space. Moreover, the solution u is Lipschitz continuous in space and time, with a Lipschitz constant that depends only on the velocity c and on u^0. We denote by $G : W^{1,\infty}(\mathbb{R}^2 \times [0, \overline{T})) \to \mathrm{Lip}(\mathbb{R}^2 \times [0, \overline{T}))$ the operator that associates to c the solution u of (4), i.e.

$$G(c) = u. \tag{5}$$

Here, $\mathrm{Lip}(\mathbb{R}^2 \times [0, \overline{T}))$ denotes the set of the globally Lipschitz functions in space and time, possibly unbounded.

Next, we consider two sets $U \subset \mathrm{Lip}(\mathbb{R}^2 \times [0, \overline{T}))$ and $V \subset W^{1,\infty}(\mathbb{R}^2 \times [0, \overline{T}))$. We assume that V is bounded in $W^{1,\infty}(\mathbb{R}^2 \times [0, \overline{T}))$, i.e. that there is a constant K_0 such that

$$|w|_{W^{1,\infty}(\mathbb{R}^2 \times (0, \overline{T}))} \le K_0, \qquad \text{for all } w \in V. \tag{6}$$

For any $0 \le T \le \overline{T}$, we set $U_T = U \cap \mathrm{Lip}(\mathbb{R}^2 \times [0,T))$ and $V_T = V \cap W^{1,\infty}(\mathbb{R}^2 \times [0,T))$, i.e. U_T and V_T are the restrictions to $[0,T)$ of the functions in U and V respectively. We suppose that, for all T, U_T and V_T are closed for the uniform convergence and that $G(V_T) \subset U_T$. The non local velocity we consider is a mapping $c : U \to V$ so that $c(U_T) \subset V_T$ for all $T \in [0, \overline{T})$.

We make the following stability assumptions on the operators G and c.

Stability property of the operator G on V. There is a constant K_1 so that, for all $0 \le T \le \overline{T}$ and all $c_1, c_2 \in V_T$,

$$|G(c_2) - G(c_1)|_{L^\infty(\mathbb{R}^2 \times (0,T))} \le K_1 T |c_2 - c_1|_{L^\infty(\mathbb{R}^2 \times (0,T))} \tag{7}$$

By the comparison principle, the stability of G is ensured if for instance all the functions in U satisfy $|\nabla u| \le K_1$ a.e. in $\mathbb{R}^2 \times [0, \overline{T})$.

Stability property of the velocity c. There are constants K_2 and K_3 so that, for all $0 \le T \le \overline{T}$ and all $u_1, u_2 \in U_T$,

$$|c[u_2] - c[u_1]|_{L^\infty(\mathbb{R}^2 \times (0,T))} \le K_2 \, |u_2 - u_1|_{L^\infty(\mathbb{R}^2 \times (0,T))} \wedge K_3. \tag{8}$$

Here, $a \wedge b = \min(a,b)$. Note that the quantity $|u_2 - u_1|_{L^\infty(\mathbb{R}^2 \times (0,T))}$ may be infinite.

Theorem 1 (Existence and uniqueness of a solution for short time)
Assume that the operator $G : V \to U$ defined in (5) verifies (7) and that the mapping $c : U \to V$ satisfies (8). Then there exists $0 < T^* \le \overline{T}$ such that the operator $G \circ c$ has a unique fixed point in U_{T^*}.

In other words, the non-local eikonal equation

$$u_t = c[u] \, |\nabla u| \quad \text{in } \mathbb{R}^2 \times (0, T^*), \qquad u(\cdot, 0) = u^0 \quad \text{in } \mathbb{R}^2. \tag{9}$$

has a unique solution in U_{T^*}.

Proof Fix $T^* < 1/K_1 K_2$ so that $T^* \le \overline{T}$. Endow U_{T^*} with the topology of uniform convergence, say for the distance

$$d(u_1, u_2) = |u_2 - u_1|_{L^\infty(\mathbb{R}^2 \times (0,T^*))} \wedge K_3.$$

Since U_{T^*} is closed under the uniform convergence, it is a complete metric space. On the other hand, the stability assumptions on G and c ensures that the operator $G \circ c : U_{T^*} \to U_{T^*}$ is a contraction for the distance d. We can thus apply the classical fixed point theorem. □

2 Approximation of the local eikonal equation

In this section, we briefly recall the finite difference scheme associated to the eikonal equation (4).

We first need a few notations. We approximate the underlying space $\mathbb{R}^2 \times [0,T)$ by a lattice

$$Q_T^\Delta = (\Delta x)\mathbb{Z} \times (\Delta y)\mathbb{Z} \times \{0, \ldots, (\Delta t)N_T\}$$

where N_T is the integer part of $T/\Delta t$. We refer generically to the lattice by Δ in the sequel. The discrete running point is (X_I, t_n) with $X_I = (x_i, y_j)$ for $I = (i,j) \in \mathbb{Z}^2$, $x_i = i(\Delta x)$, $y_j = j(\Delta y)$ and $t_n = n(\Delta t)$. We set $\Delta X = (\Delta x, \Delta y)$ and we define its Euclidean norm $|\Delta X|$ as the space mesh size. We shall assume throughout that $|\Delta X| \leq 1$ and $\Delta t \leq 1$. The approximation of the solution u at the node (X_I, t_n) is written indifferently as $v(X_I, t_n)$ or v_I^n according to whether we view it as a function defined on the lattice or as a sequence.

Given a discrete velocity c^Δ, the discrete solution v that approximates the solution of the eikonal equation (4) is computed iteratively by the explicit scheme

$$v_I^0 = \tilde{u}^0(X_I), \qquad v_I^{n+1} = v_I^n + \Delta t\, c^\Delta(X_I, t_n)\, E_d^{\mathrm{sign}(c^\Delta(X_I,t_n))}(D^+v^n, D^-v^n), \tag{10}$$

where $\tilde{u}^0(X_I)$ is an approximation of $u^0(X_I)$ (which can be chosen equal to $u^0(X_I)$), where $E_d^\pm$ is a suitable approximation of the Euclidean norm and D^+v^n, D^-v^n are discrete gradients whose precise definition is recalled in the next paragraph. Denoting by S the operator on the right-hand side of (10) acting on discrete functions, we can write the scheme more compactly as

$$v_I^0 = \tilde{u}^0(X_I), \qquad v^{n+1} = Sv^n.$$

Defining $E_T^\Delta = \mathbb{R}^{\mathbb{Z}^2 \times \{0,\dots,N_T\}}$ the set of the discrete functions defined on the mesh Q_T^Δ and $E^\Delta = E_T^\Delta$, we denote by $G^\Delta : E^\Delta \to E^\Delta$ the operator that gives the discrete solution v of problem (10) for a given discrete velocity $c^\Delta \in E^\Delta$, i.e.

$$G^\Delta(c^\Delta) = v. \tag{11}$$

The discrete gradients are $D^\pm v^n(X_I) = \left(D_x^\pm v^n(X_I), D_y^\pm v^n(X_I)\right)$, where $D_x^\pm v^n(X_I)$ and $D_y^\pm v^n(X_I)$ are the standard forward and backward first order differences, i.e. for a general function $f(X_I)$:

$$\begin{aligned} D_x^+ f(X_I) &= \frac{f(x_{i+1}, y_j) - f(x_i, y_j)}{\Delta x}, \\ D_x^- f(X_I) &= \frac{f(x_i, y_j) - f(x_{i-1}, y_j)}{\Delta x}, \\ D_y^+ f(X_I) &= \frac{f(x_i, y_{j+1}) - f(x_i, y_j)}{\Delta y}, \\ D_y^- f(X_I) &= \frac{f(x_i, y_j) - f(x_i, y_{j-1})}{\Delta y}. \end{aligned}$$

We suppose that the functions $E_d^\pm$ are Lipschitz continuous with respect to the discrete gradients, i.e.

$$\left|E_d^\pm(P,Q) - E_d^\pm(P',Q')\right| \leq L_1 \left(|P - P'| + |Q - Q'|\right) \tag{12}$$

and that they are consistent with the Euclidean norm

$$E_d^\pm(P,P) = |P|. \tag{13}$$

We also suppose that $E_d^\pm(p_x^+, p_y^+, p_x^-, p_y^-)$ enjoy suitable monotonicity with respect to each variable

$$\begin{aligned} &\frac{\partial E_d^+}{\partial p_x^+} \geq 0, \quad \frac{\partial E_d^+}{\partial p_y^+} \geq 0, \quad \frac{\partial E_d^+}{\partial p_x^-} \leq 0, \quad \frac{\partial E_d^+}{\partial p_y^-} \leq 0, \\ &\frac{\partial E_d^-}{\partial p_x^+} \leq 0, \quad \frac{\partial E_d^-}{\partial p_y^+} \leq 0, \quad \frac{\partial E_d^-}{\partial p_x^-} \geq 0, \quad \frac{\partial E_d^-}{\partial p_y^-} \geq 0. \end{aligned} \tag{14}$$

Finally, we assume that the mesh satisfies the following CFL (Courant, Friedrichs, Levy) condition

$$\Delta t \leq \frac{L_0}{\|c^\Delta\|_{L^\infty}}\ \Delta x, \qquad \Delta t \leq \frac{L_0}{\|c^\Delta\|_{L^\infty}}\ \Delta y \tag{15}$$

for

$$L_0^{-1} = 2\max\Big(|\frac{\partial E_d^\pm}{\partial p_x^+}|_{L^\infty} + |\frac{\partial E_d^\pm}{\partial p_x^-}|_{L^\infty}, |\frac{\partial E_d^\pm}{\partial p_y^+}|_{L^\infty} + |\frac{\partial E_d^\pm}{\partial p_y^-}|_{L^\infty}\Big),$$

where we assume that the max is also taken on the signs $\pm$. Under these assumptions, one checks classically that the scheme S is monotone in the sense that

$$\text{if } v \geq w, \qquad \text{then } Sv \geq Sw.$$

An example of scheme, satisfying the previous assumptions, is given by the following expression:

$$\begin{cases} v_I^{n+1} = & v_I^n + \Delta t \times \begin{cases} c(X_I,t_n)E_d^+ & \text{if } c(X_I,t_n) \geq 0 \\ c(X_I,t_n)E_d^- & \text{if } c(X_I,t_n) < 0 \end{cases} \\ v_{i,j}^0 = & u^0(x_i,y_j), \end{cases}$$

where E_d^+, E_d^- are the numerical monotone Hamiltonians proposed by Osher and Sethian in [OS]:

$$E_d^+ = \left\{\max(D_x^+v^n(x_I),0)^2 + \max(D_y^+v^n(x_I),0)^2 + \min(D_x^-v^n(x_I),0)^2 + \min(D_y^-v^n(x_I),0)^2\right\}^{\frac{1}{2}}$$
$$E_d^- = \left\{\min(D_x^+v^n(x_I),0)^2 + \min(D_y^+v^n(x_I),0)^2 + \max(D_x^-v^n(x_I),0)^2 + \max(D_y^-v^n(x_I),0)^2\right\}^{\frac{1}{2}}.$$

A second example is given by choosing the numerical Hamiltonians proposed by Rouy and Tourin in [RT]:

$$E_d^+ = \left\{\max(D_x^+v^n(x_I), -D_x^-v^n(x_I),0)^2 + \max(D_y^+v^n(x_I), -D_y^-v^n(x_I),0)^2\right\}^{\frac{1}{2}}$$
$$E_d^- = \left\{\min(D_x^+v^n(x_I), -D_x^-v^n(x_I),0)^2 + \min(D_y^+v^n(x_I), -D_y^-v^n(x_I),0)^2\right\}^{\frac{1}{2}}.$$

3 The Crandall-Lions rate of convergence for the local eikonal equation

The Crandall-Lions [CL1] estimate provides a bound for the rate of convergence of finite difference schemes for solving numerically Hamilton-Jacobi equations. The scheme is of the form

$$v_I^0 = \tilde{u}^0(X_I), \qquad v_I^{n+1} = v_I^n + \Delta t\ H_d(X_I,t_n,D^+v^n,D^-v^n). \tag{16}$$

where $\tilde{u}^0(X_I)$ is an approximation of $u^0(X_I)$. To simplify the presentation and to be consistent with the notations of the preceding sections, we shall limit ourselves to the eikonal equation (4) with velocity $c \in W^{1,\infty}(\mathbb{R}^2 \times [0,T))$. In this case, the discrete Hamiltonian is of the form

$$H_d(X_I,t_n,P,Q) = c(X_I,t_n)E_d^{\mathrm{sign}(c(X_I,t_n))}(P,Q).$$

We give an updated proof of the result that concentrates on (globally) Lipschitz solutions of the eikonal equation (4). One technicality is that the functions may be unbounded.

Theorem 2 (Crandall-Lions rate of convergence) *Assume that $T \le 1$ and $|\Delta X| + \Delta t \le 1$. Assume that $c \in W^{1,\infty}(\mathbb{R}^2 \times [0,T))$ and that the discrete approximation of the Euclidean norm E_d satisfies the assumptions* (12), (13), (14). *Assume also that the CFL condition* (15) *holds.*

Then there exists a constant $K > 0$ such that the error estimate between the solution u of the eikonal equation (4) *and the discrete solution v of the finite difference scheme* (16) *is given by*

$$\sup_{Q_T^\Delta} |u-v| \le K\big(T(|\Delta X|+\Delta t)\big)^{1/2} + \sup_{Q_0^\Delta} |u^0-v^0|, \quad \textit{if} \quad |\Delta X|+\Delta t \le \frac{T}{K} \quad \textit{and} \quad \sup_{Q_0^\Delta} |u^0-v^0| \le 1$$

for a constant K depending only on $\left|\nabla u^0\right|_{L^\infty(\mathbb{R}^2)}$, $|c|_{W^{1,\infty}(\mathbb{R}^2\times(0,T))}$ and the constants L_0 and L_1.

Proof The proof splits into several steps. We denote throughout by K various constants depending only on $\left|\nabla u^0\right|_{L^\infty(\mathbb{R}^2)}$, $|c|_{W^{1,\infty}(\mathbb{R}^2\times(0,T))}$ and the constants L_0 and L_1. We first assume that

$$u^0(X_I) \ge v_I^0 = \tilde{u}^0(X_I) \tag{17}$$

and we set

$$\mu^0 = \sup_{Q_0^\Delta} |u^0 - v^0| \ge 0.$$

Step 1 We have the estimate for the discrete solution $Kt_n + \mu^0 \ge u^0(X_I) - v(X_I, t_n) \ge -Kt_n$.

To show this, we note that the function $w(X_I, t_n) = u^0(X_I) - Kt_n - \mu^0$ is a discrete subsolution of (10) provided the constant K is conveniently chosen. Indeed, $w(\cdot, 0) = u^0$ and, using the Lipschitz continuity of H_d, its consistency and the Lipschitz continuity of u^0, we get

$$\begin{aligned} w_I^{n+1} - (Sw^n)_I &= -K\Delta t - \Delta t\, H_d(X_I, t_n, D^+u^0, D^-u^0) \\ &\le -\Delta t\big(K - L_1|c|_{L^\infty}\,|D^+u^0| - L_1|c|_{L^\infty}\,|D^-u^0| + H_d(X_I, t_n, 0, 0)\big) \le 0 \end{aligned}$$

for $K = 2\sqrt{2}L_1|c|_{L^\infty}\,|\nabla u^0|_{L^\infty(\mathbb{R}^2)}$. By the monotonicity of the scheme, we deduce by induction that $w^n \le v^n$ for all n. This can be rewritten as $u^0(X_I) - v(X_I, t_n) \le Kt_n + \mu^0$.

The lower bound $u^0(X_I) + Kt_n \ge v(X_I, t_n)$ is proved similarly.

Before continuing the proof, we need a few notations. We put

$$\mu = \sup_{Q_T^\Delta} (u - v),$$

which quantity may be infinite a priori. Moreover, for every $0 < \alpha \le 1$, $0 < \varepsilon \le 1$ and $\sigma > 0$, we set

$$M_\sigma^{\alpha,\varepsilon} = \sup_{\mathbb{R}^2\times[0,T]\times Q_T^\Delta} \psi_\sigma^{\alpha,\varepsilon}(X, t, X_I, t_n)$$

with

$$\psi_\sigma^{\alpha,\varepsilon}(X, t, X_I, t_n) = u(X,t) - v(X_I, t_n) - \frac{|X - X_I|^2}{2\varepsilon} - \frac{(t-t_n)^2}{2\varepsilon} - \sigma t - \alpha|X|^2 - \alpha|X_I|^2.$$

We shall drop the super and subscripts on ψ and M when no ambiguity arises as concerns the value of the parameters.

Since the solution u is Lipschitz continuous, with a Lipschitz constant that depends only on $|\nabla u^0|_{L^\infty}$ and $|c|_{W^{1,\infty}}$, it has a linear growth in the space variable. This growth is also true for v by virtue of step 1. Therefore, there is a constant K so that

$$|u(X,t)| \le K\big(1+|X|\big), \qquad |v(X_I,t_n)| \le K\big(1+|X_I|\big)$$

(recall that $T \le 1$). This implies in particular that the function ψ actually achieves its maximum at some point that we denote (X^*, t^*, X_I^*, t_n^*), i.e.

$$\psi(X^*, t^*, X_I^*, t_n^*) = \max_{\mathbb{R}^2\times[0,T]\times Q_T^\Delta} \psi(X,t,X_I,t_n).$$

Step 2 The maximum point of ψ enjoys the following estimates

$$\alpha|X^*| + \alpha|X_I^*| \le K \tag{18}$$

and

$$|X^* - X_I^*| \le K\varepsilon, \qquad |t^* - t_n^*| \le (K+2\sigma)\varepsilon. \tag{19}$$

To see this, we first use the inequality $\psi(X^*, t^*, X_I^*, t_n^*) \ge \psi(0,0,0,0) \ge 0$ (by (17) and the linear growth of u and v. We obtain that

$$\alpha|X^*|^2 + \alpha|X_I^*|^2 \le K(1+|X^*|+|X_I^*|) \le K + \frac{K^2}{\alpha} + \frac{\alpha}{2}|X^*|^2 + \frac{\alpha}{2}|X_I^*|^2.$$

This clearly implies (18) since $\alpha \le 1$.

The first bound in (19) follows from the Lipschitz regularity of u and from the inequality $\psi(X^*, t^*, X_I^*, t_n^*) \ge \psi(X_I^*, t^*, X_I^*, t_n^*)$ and from (18). This indeed entails that

$$\begin{aligned}\frac{|X^*-X_I^*|^2}{2\varepsilon} &\le u(X^*,t^*) - u(X_I^*,t^*) - \alpha|X^*|^2 + \alpha|X_I^*|^2 \\ &\le K|X^*-X_I^*| + \alpha|X^*-X_I^*|(|X^*|+|X_I^*|) \\ &\le K|X^*-X_I^*|.\end{aligned}$$

This gives the estimate $|X^*-X_I^*| \le K\varepsilon$.

The second bound in (19) is deduced from $\psi(X^*, t^*, X_I^*, t_n^*) \ge \psi(X^*, t_n^*, X_I^*, t_n^*)$ in exactly the same way.

Step 3 We have $\mu \le K$.

We first claim that for σ large enough, we must have either $t^* = 0$ or $t_n^* = 0$. Suppose the contrary. Then, the function $(X,t) \mapsto \psi(X,t,X_I^*,t_n^*)$ achieves its maximum at a point in $\mathbb{R}^2 \times (0,T]$. Using the fact that u is a viscosity subsolution of (4), we get the inequality

$$\sigma + p_t^* \le c(X^*,t^*)|p_X^* + 2\alpha X^*| \tag{20}$$

for

$$p_t^* = \frac{t^*-t_n^*}{\varepsilon}, \qquad p_X^* = \frac{X^*-X_I^*}{\varepsilon}.$$

(to handle the case $t^* = T$, we take advantage here of the well-known fact that, for the Cauchy problem, the equation holds also in the viscosity sense at the terminal time).

On the other hand, since $t_n^* > 0$, we know that $\psi(X^*, t^*, X_I^*, t_n^*) \geq \psi(X^*, t^*, \cdot, t_n^* - \Delta t)$. This reads as

$$v(\cdot, t_n^* - \Delta t) \geq \varphi(\cdot, t_n^* - \Delta t) + v(X_I^*, t_n^*) - \varphi(X_I^*, t_n^*)$$

for

$$\varphi(X_I, t_n) = -\frac{|X^* - X_I|^2}{2\varepsilon} - \frac{(t^* - t_n)^2}{2\varepsilon} - \alpha |X_I|^2.$$

Using the fact that the scheme is monotone and commutes with the addition of the constants, we get that

$$v(X_I^*, t_n^*) = \big(Sv(\cdot, t_n^* - \Delta t)\big)(X_I^*) \geq S\varphi(\cdot, t_n^* - \Delta t)(X_I^*) + v(X_I^*, t_n^*) - \varphi(X_I^*, t_n^*),$$

i.e.

$$\varphi(X_I^*, t_n^*) \geq S\varphi(\cdot, t_n^* - \Delta t)(X_I^*).$$

Expliciting the scheme S, we arrive at the supersolution inequality

$$\frac{\varphi(X_I^*, t_n^*) - \varphi(X_I^*, t_n^* - \Delta t)}{\Delta t} \geq H_d\big(X_I^*, t_n^* - \Delta t, D^+\varphi(X_I^*, t_n^* - \Delta t), D^-\varphi(X_I^*, t_n^* - \Delta t)\big).$$

Straightforward computations of the discrete derivatives of φ yield

$$p_t^* + \frac{\Delta t}{2\varepsilon} \geq H_d\big(X_I^*, t_n^* - \Delta t, p_X^* - \frac{\Delta X}{2\varepsilon} - \alpha(2X_I^* + \Delta X), p_X^* + \frac{\Delta X}{2\varepsilon} - \alpha(2X_I^* - \Delta X)\big).$$

Subtracting the above inequality to (20), we obtain

$$\begin{aligned}
\sigma &\leq c(X^*, t^*)|p_X^*| + K\alpha|X^*| + \frac{\Delta t}{2\varepsilon} - H_d(X_I^*, t_n^* - \Delta t, p_X^*, p_X^*) + K\frac{|\Delta X|}{\varepsilon} + K\alpha|X_I^*| + K\alpha|\Delta X| \\
&\leq K\frac{|\Delta X|}{\varepsilon} + \frac{\Delta t}{2\varepsilon} + K(|X^* - X_I^*| + |t^* - t_n^*| + \Delta t)|p_X^*| + K\alpha\big(|X_I^*| + |X^*| + |\Delta X|\big) \\
&< K\frac{|\Delta X|}{\varepsilon} + \frac{\Delta t}{2\varepsilon} + K\varepsilon + K\sigma\varepsilon + K. \qquad (21)
\end{aligned}$$

The first inequality uses the Lipschitz regularity of c; the second one the consistency of H_d and the Lipschitz regularity of c in space and time; the third one the bounds (18) and (19) and the fact that $|\Delta X|, \Delta t \leq 1$. Choosing $\overline{\varepsilon} = 1/(2K)$, we get that for all $\varepsilon \leq \overline{\varepsilon}$

$$\sigma \leq K\frac{|\Delta X| + \Delta t}{\varepsilon} + K.$$

Putting

$$\sigma^*(|\Delta X| + \Delta t, \varepsilon) = K\frac{|\Delta X| + \Delta t}{\varepsilon} + K,$$

we therefore conclude that we must have $t^* = 0$ or $t_n^* = 0$ provided $\sigma \geq \sigma^*$.

Whenever $t_n^* = 0$, we deduce from the Lipschitz regularity of u and the bound (19) that

$$M = \psi(X^*, t^*, X_I^*, 0) \leq u(X^*, t^*) - \tilde{u}^0(X_I^*) \leq K(|X^* - X_I^*| + t^*) + \mu^0 \leq K(1 + \sigma)\varepsilon + \mu^0.$$

Similarly, whenever $t^* = 0$, we deduce from step 1 and from (19) that

$$
\begin{aligned}
M = \psi(X^*, 0, X_I^*, t_n^*) &\le u^0(X^*) - v(X_I^*, t_n^*) \\
&\le u^0(X^*) - u^0(X_I^*) + K t_n^* + \mu^0 \le K(|X^* - X_I^*| + t_n^*) + \mu^0 \le K(1+\sigma)\varepsilon + \mu^0.
\end{aligned}
$$

To sum up, we have shown that

$$
M_\sigma^{\alpha,\varepsilon} \le K(1+\sigma)\varepsilon + \mu^0 \qquad \text{provided } \sigma \ge \sigma^*(|\Delta X|, \varepsilon) \text{ and } \varepsilon \le \overline{\varepsilon}.
$$

Specializing to the case $\varepsilon = \overline{\varepsilon}$, we deduce that, for every (X_I, t_n), we have

$$
u(X_I, t_n) - v(X_I, t_n) - \sigma^*(|\Delta X| + \Delta t, \overline{\varepsilon}) T - 2\alpha |X_I|^2 \le M^{\alpha,\overline{\varepsilon}}_{\sigma^*(|\Delta X|+\Delta t, \overline{\varepsilon})} \le K\big(1 + \sigma^*(|\Delta X| + \Delta t, \overline{\varepsilon})\big)\overline{\varepsilon} + \mu^0.
$$

Sending $\alpha \to 0$, taking the supremum over (X_I, t_n) and recalling that $|\Delta X| + \Delta t \le 1$, $T \le 1$, we conclude that

$$
\mu \le K\big(1 + \sigma^*(|\Delta X| + \Delta t, \overline{\varepsilon})\big) \le K\big(1 + \sigma^*(1, \overline{\varepsilon})\big) + \mu^0.
$$

This completes the proof of step 3.

We again allow $0 < \varepsilon \le 1$ to be arbitrary and we now assume that $\sigma \le 1$.

Step 4 Inequality (18) can be strengthened to

$$
\alpha |X^*|^2 + \alpha |X_I^*|^2 \le K \tag{22}
$$

This simply follows from the inequality $\psi(X^*, t^*, X_I^*, t_n^*) \ge \psi(0,0,0,0) \ge 0$ together with step 3 and (19). We get indeed

$$
\alpha |X^*|^2 + \alpha |X_I^*|^2 \le u(X^*, t^*) - v(X_I^*, t_n^*) \le K(|X^* - X_I^*| + |t^* - t_n^*|) + \mu \le \mu + K(1+\sigma)\varepsilon \le K.
$$

Step 5 We have the bound $\mu \le K\big(T(|\Delta X| + \Delta t)\big)^{1/2}$ if $|\Delta X| + \Delta t \le \frac{T}{K}$.

We argue as in step 3, using (22) instead of (18). The only difference is that the inequality (21) becomes

$$
\sigma < K \frac{|\Delta X| + \Delta t}{\varepsilon} + K\varepsilon + K\Delta t + K\alpha^{1/2}.
$$

Following the reasoning of step 3, we conclude that

$$
M_\sigma^{\alpha,\varepsilon} \le K\varepsilon + \mu^0 \qquad \text{provided} \quad K \frac{|\Delta X| + \Delta t}{\varepsilon} + K\varepsilon + K\Delta t + K\alpha^{1/2} \le \sigma \le 1.
$$

This implies as before that

$$
u(X_I, t_n) - v(X_I, t_n) \le K\varepsilon + \mu^0 + \big(K \frac{|\Delta X| + \Delta t}{\varepsilon} + K\varepsilon + K\Delta t + K\alpha^{1/2}\big) T + 2\alpha |X_I|^2
$$

provided the quantity within parentheses is ≤ 1. Sending $\alpha \to 0$, taking the supremum over (X_I, t_n) and choosing $\varepsilon = \big(T(|\Delta X| + \Delta t)\big)^{1/2}$, we conclude that

$$
\sup_{Q_T^\Delta}(u - v) = \mu \le K\big(T(|\Delta X| + \Delta t)\big)^{1/2} + \sup_{Q_0^\Delta}(u^0 - v^0) \tag{23}
$$

provided $\frac{|\Delta X|+\Delta t}{T}$, $|\Delta X|$, Δt are small enough and (17) is assumed.

In the general case, we first remark that $\overline{u} = u + \mu^1$ with $\mu^1 = \sup_{Q_0^\Delta}(v^0 - u^0)$ satisfies $\overline{u}^0(X_I) \geq v_I^0$ and $\overline{u}_t = c(x, y, t)|\nabla \overline{u}|$. Then (23) is true with $\overline{u}$ in place of u, i.e.

$$\sup_{Q_T^\Delta}(u + \mu^1 - v) \leq K\big(T(|\Delta X| + \Delta t)\big)^{1/2} + \sup_{Q_0^\Delta}(u^0 + \mu^1 - v^0),$$

which still implies (23) in the general case.

The lower bound for the error is obtained by exchanging u and v. As the proof is similar to the above, we omit it. □

4 Approximation of the non-local eikonal equation

To solve numerically the non-local Hamilton-Jacobi equation (9), we need to approximate the velocity c. We mimic the continuous setting.

For every mesh Δ and every $T \leq \overline{T}$, we consider two subsets U^Δ and V^Δ of E^Δ and put $U_T^\Delta = U^\Delta \cap E_T^\Delta$ and $V_T^\Delta = V^\Delta \cap E_T^\Delta$. For all $T \leq \overline{T}$, we assume that $G^\Delta(V_T^\Delta) \subset U_T^\Delta$ and that the set U_T^Δ is consistent with U_T in the sense that

$$\{(u)^\Delta \mid u \in U_T\} \subset U_T^\Delta$$

where $(u)^\Delta$ is the restriction to Q_T^Δ of the continuous function u. Similarly, we assume that V_T^Δ is consistent with V_T, i.e.

$$\{(c)^\Delta \mid c \in V_T\} \subset V_T^\Delta$$

We also assume that V^Δ is equibounded, i.e. that there is a constant K_4 so that, for every mesh Δ,

$$\sup_{Q_T^\Delta}|c^\Delta| \leq K_4, \qquad \text{for all } c^\Delta \in V^\Delta. \tag{24}$$

We shall say that a mesh Δ satisfies the CFL condition uniformly if

$$\Delta t \leq \frac{L_0}{K_4}\,\Delta x, \qquad \Delta t \leq \frac{L_0}{K_4}\,\Delta y. \tag{25}$$

Note that every mesh satisfying the uniform CFL condition satisfies the classical CFL condition (15) for every $c^\Delta \in V_T^\Delta$.

We approximate the nonlocal velocity mapping $c : U \to V$ by a map $c^\Delta : U^\Delta \to V^\Delta$ so that $c^\Delta(U_T^\Delta) \subset V_T^\Delta$ for all $T \leq \overline{T}$.

The main assumption is the following.

Consistency for the discrete velocity c^Δ There is a constant K_5 such that, for every mesh Δ, for every $T \leq \overline{T}$ and for every $u \in U$, we have

$$\sup_{Q_T^\Delta}|c[u] - c^\Delta[(u)^\Delta]| \leq K_5|\Delta X| \tag{26}$$

We also assume that the discrete operators G^Δ and c^Δ satisfy stability assumptions that are similar to those satisfied by their continuous analogues.

Stability property of the operator G^Δ There is a constant K_6 so that, for every mesh Δ satisfying the uniform CFL condition (25), for all $0 \le T \le \overline{T}$ and all $c_1^\Delta, c_2^\Delta \in V_T^\Delta$,

$$\sup_{Q_T^\Delta} |G^\Delta(c_2^\Delta) - G^\Delta(c_1^\Delta)| \le K_6 T \sup_{Q_T^\Delta} |c_2^\Delta - c_1^\Delta|. \tag{27}$$

As for the continuous case, this assumption is satisfied if the functions in U_T^Δ are equi-lipschitz, i.e. if there is a constant K_6' such that every $u^\Delta \in U_T^\Delta$ satisfies $|D^+ u^\Delta| \le K_6'$.

Stability property of the velocity c^Δ There is a constant K_7 so that, for all meshes Δ, for all $0 \le T \le \overline{T}$ and all $u_1^\Delta, u_2^\Delta \in U_T^\Delta$,

$$\sup_{Q_T^\Delta} |c^\Delta[u_2^\Delta] - c^\Delta[u_1^\Delta]| \le K_7 \big(\sup_{Q_T^\Delta} |u_2^\Delta - u_1^\Delta| + |\Delta X| \big). \tag{28}$$

To resume, we have the following (non-commutative) diagram

$$\begin{array}{ccc} V_T & \rightleftarrows^{G}_{c} & U_T \\ \downarrow^{(\cdot)^\Delta} & & \downarrow^{(\cdot)^\Delta} \\ V_T^\Delta & \rightleftarrows^{G^\Delta}_{c^\Delta} & U_T^\Delta \end{array}$$

To simplify the presentation, we suppose that c^Δ is stationary, i.e. that there is a mapping $\overline{c}^\Delta$ to that $c^\Delta(u^\Delta)(\cdot, t_n) = \overline{c}^\Delta\big(u^\Delta(\cdot, t_n)\big)$. By the definition of the explicit marching scheme (10), this implies that the operator $G^\Delta : U^\Delta \to U^\Delta$ admits a unique fixed point v, even without making the stability assumptions. The function $v \in U^\Delta$ is the solution of the discrete eikonal equation

$$v_I^0 = \tilde{u}^0(X_I), \qquad v_I^{n+1} = v_I^n + \Delta t\, c^\Delta[v](X_I, t_n)\, E_d^{\operatorname{sign}(c^\Delta(X_I, t_n))}(D^+ v^n, D^- v^n), \tag{29}$$

where $\tilde{u}^0(X_I)$ is an approximation of $u^0(X_I)$ (which can be chosen equal to $u^0(X_I)$).

5 Rate of convergence for the non local eikonal equation

The main result of this section is the proof that the rate of convergence for the non-local eikonal equation (9) is the same as the one for the local eikonal equation (4).

By Theorem 1, the non-local equation (9) has a unique solution $u \in U_{T^*}$. It is the fixed point of the operator $G \circ c$. Moreover, the approximated equation (29) has a unique solution $v \in U_{T^*}^\Delta$. It is the fixed point of $G^\Delta \circ c^\Delta$.

Theorem 3 (Discrete-continuous error estimate)
Assume that $T \le T^* \wedge 1$ and $\sup_{Q_0^\Delta} |u^0 - v^0| \le 1$.

Suppose that the data of the continuous problem U, V, G and c satisfies the assumptions of section 1, in particular assumptions (6), (7) *and* (8). *Suppose that the lattice Δ satisfies the uniform CFL condition* (25). *Suppose that the data of the discrete problem U^Δ, V^Δ, G^Δ and c^Δ satisfies the assumptions of sections 2 and 4, in particular assumptions* (24), (26), (27) *and* (28)*. Let* $u \in U_{T^*}$ *be the unique solution of the non-local equation* (9) *and* $v \in U^{\Delta}_{T^*}$ *be the unique solution of the corresponding approximated equation* (29)*.*

Then there exists a positive constant K *such that, for all* $0 \le T \le T^* \wedge 1$*,*

$$\sup_{Q^{\Delta}_T} |u - v| \le \frac{K}{(1 - KT)^+} \left(\sqrt{T|\Delta X|} + \sup_{Q^{\Delta}_0} |u^0 - v^0| \right) \qquad \textit{provided } |\Delta X| \le T/K.$$

The constant K *only depends on the constant* K_0*,* $\dots K_7$*,* L_0*,* L_1 *and* $\left|\nabla u^0\right|_{L^\infty(\mathbb{R}^2)}$*.*

In particular, there are $T^{**} > 0$ *and* K' *depending only on* K *such that*

$$\sup_{Q^{\Delta}_{T^{**}}} |u - v| \le K' \left(\sqrt{|\Delta X|} + \sup_{Q^{\Delta}_0} |u^0 - v^0| \right).$$

Remark 5.1 *The assumptions on the continuous problem were only needed to guarantee the existence of a solution to* (9)*. If instead of making these assumptions we suppose that* (9) *has a solution so that* $c[u] \in W^{1,\infty}(\mathbb{R}^2 \times [0, T^*))$*, then the conclusion of the result prevails with a constant* K *that depends only on* K_0*,* K_5*,* K_6*,* K_7*,* L_0*,* L_1*,* $\left|\nabla u^0\right|_{L^\infty(\mathbb{R}^2)}$*,* T^* *and* $|c[u]|_{W^{1,\infty}(\mathbb{R}^2 \times [0,T^*))}$*. Moreover the consistency of the discrete velocity (26) only needs to be satisfied by the solution* u*, as shows the the proof of the Theorem below.*

Remark 5.2 *Iterating Theorem 3 on several intervals of time, it is possible to get an estimate on* $Q^{\Delta}_{\overline{T}}$ *with* $\overline{T} > 1$*.*

Proof Since u is the fixed point of $G \circ c$ and v is the fixed point of $G^{\Delta} \circ c^{\Delta}$, we have the inequality

$$\begin{aligned} \sup_{Q^{\Delta}_T} |u - v| &= \sup_{Q^{\Delta}_T} |G(c[u]) - G^{\Delta}(c^{\Delta}[v])| \\ &\le \sup_{Q^{\Delta}_T} |G(c[u]) - G^{\Delta}((c[u])^{\Delta})| + \sup_{Q^{\Delta}_T} |G^{\Delta}((c[u])^{\Delta}) - G^{\Delta}(c^{\Delta}[v])| \end{aligned}$$

The function $G^{\Delta}((c[u])^{\Delta})$ is simply the discrete solution associated to the eikonal equation (4) with velocity $c[u]$. Since $c[u]$ is bounded in $W^{1,\infty}$ uniformly in u by assumption (6), we deduce from Theorem 2 that

$$\sup_{Q^{\Delta}_T} |G(c[u]) - G^{\Delta}((c[u])^{\Delta})| \le K \sqrt{T|\Delta X|} + \sup_{Q^{\Delta}_0} |u^0 - v^0|,$$

provided $|\Delta X| \le T/K$, for a constant K that depends only on K_0, $\left|\nabla u^0\right|_{L^\infty}$, L_0 and L_1.

To estimate the second term in the inequality, we apply the stability properties (27) and (28) and the consistency (26):

$$\begin{aligned} \sup_{Q^{\Delta}_T} |G^{\Delta}((c[u])^{\Delta}) - G^{\Delta}(c^{\Delta}[v])| &\le K_6 T \sup_{Q^{\Delta}_T} |c[u] - c^{\Delta}[v]| \\ &\le K_6 T \sup_{Q^{\Delta}_T} \left(|c[u] - c^{\Delta}[(u)^{\Delta}]| + |c^{\Delta}[(u)^{\Delta}] - c^{\Delta}[v]| \right) \\ &\le K_6 (K_5 + K_7) T \left(|\Delta X| + \sup_{Q^{\Delta}_T} |u - v| \right) \end{aligned}$$

We conclude that

$$\sup_{Q_T^\Delta}|u-v| \le K\sqrt{T|\Delta X|} + KT|\Delta X| + KT\sup_{Q_T^\Delta}|u-v| + \sup_{Q_0^\Delta}|u^0-v^0|.$$

This gives the required estimate since $T|\Delta X| \le 1$. □

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