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Existence and Uniqueness of Traveling Waves for Fully Overdamped Frenkel–Kontorova Models

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Abstract

In this article, we study the existence and the uniqueness of traveling waves for a discrete reaction–diffusion equation with bistable nonlinearity, namely a generalization of the fully overdamped Frenkel–Kontorova model. This model consists of a system of ODEs which describes the dynamics of crystal defects in lattice solids. Under very weak assumptions, we prove the existence of a traveling wave solution and the uniqueness of the velocity of propagation of this traveling wave. The question of the uniqueness of the profile is also studied by proving Strong Maximum Principle or some weak asymptotics on the profile at infinity.

1. Introduction

In this work, we are interested in the fully overdamped Frenkel–Kontorova (FK) model, which describes the dynamics of crystal defects in a lattice (see, for instance, the book of Braun and Kivshar [8] for an introduction to this model). This model (and its generalization) is a discrete reaction–diffusion equation with "bistable" nonlinearity. For this model, we show the existence and the uniqueness of traveling waves.

1.1. Setting of the Problem

We first give an example of the simplest fully overdamped Frenkel Kontorova model, and then we provide a general framework for which we will establish our results.

(i) The simplest Frenkel–Kontorova model

The simplest fully overdamped FK model is a chain of atoms, where the position $X_i(t) \in \mathbb{R}$ at the time t of the particle $i \in \mathbb{Z}$ solves

$$\frac{\mathrm{d}X_i}{\mathrm{d}t} = X_{i+1} + X_{i-1} - 2X_i - \sin(2\pi(X_i - L)) - \sin(2\pi L), \qquad (1.1)$$

where $\frac{\mathrm{d}X_i}{\mathrm{d}t}$ is the velocity of the ith particle, $-\sin(2\pi L)$ is a constant driving force which will cause the chain of atoms to move and $\sin(2\pi(X_i - L))$ denotes the force created by a periodic potential reflecting the periodicity of the crystal, whose period is assumed to be 1. Set, for simplicity,

$$f_L(x) := -\sin(2\pi(x-L)) - \sin(2\pi L). \tag{1.2}$$

We look for particular *traveling wave* solutions of (1.1), namely solutions of the form

$$X_i(t) = \phi(i + ct), \tag{1.3}$$

with

$$\begin{cases} \phi' \geqq 0 \\ \phi(+\infty) - \phi(-\infty) = 1. \end{cases} \tag{1.4}$$

Here, c is the velocity of propagation of the traveling wave ϕ, and (1.4) reflects the existence of a defect of one lattice space, called dislocation. Expression (1.3) means that the defect moves with velocity c under the driving force $-\sin(2\pi L)$. In addition, ϕ is a phase transition between $\phi(-\infty)$ and $\phi(+\infty)$, which are two "stable" equilibriums of the crystal.

Clearly, if we plug (1.3) into (1.1), the profile ϕ and the velocity c have to satisfy

$$c\phi'(z) = \phi(z+1) + \phi(z-1) - 2\phi(z) + f_L(\phi(z)), \tag{1.5}$$

with $z = i + ct$ and f_L defined in (1.2).

Due to the equivalence (for $c \neq 0$) between the solutions of (1.1) and (1.5), from now on, we will focus on Equation (1.5).

Theorem 1.1. (Existence and uniqueness of traveling waves for the FK model) *There exists a unique real c and a function $\phi : \mathbb{R} \to \mathbb{R}$ solution of*

$$\begin{cases} c\phi'(z) = \phi(z+1) + \phi(z-1) - 2\phi(z) + f_L(\phi(z)) \quad \text{on} \quad \mathbb{R} \\ \phi \text{ is non-decreasing over } \mathbb{R} \\ \phi(-\infty) = 0 \quad \text{and} \quad \phi(+\infty) = 1, \end{cases} \tag{1.6}$$

in classical sense, if $c \neq 0$ and almost everywhere if $c = 0$. Moreover, if $c \neq 0$, then the profile ϕ is unique (up to space translation) and $\phi' > 0$ on $\mathbb{R}$.

This theorem has been proved in several works (see for instance, the pioneering works [25] and [19], and [23] in full generality).

(ii) A simple example not covered by the literature

Define the function G as

$$G(X_{i-1}, X_i, X_{i+1}) := \max\left(\frac{1}{2}, X_{i-1}\right) + \min\left(\frac{1}{2}, X_{i+1}\right) - X_i - \frac{1}{2} + f_L(X_i), \tag{1.7}$$

where f_L is as defined in (1.2), then consider the following system

$$\dot{X}_i = G(X_{i-1}, X_i, X_{i+1}) \quad \text{for } i \in \mathbb{Z}. \tag{1.8}$$

Theorem 1.2. (Existence and uniqueness of traveling waves for example (1.7)) *For any* $L \in (\frac{-1}{4}, \frac{1}{4})\backslash\{0\}$, *the results of Theorem* 1.1 *hold true for system* (1.6) *replaced by the following system*

$$\begin{cases} c\phi'(z) = G(\phi(z-1), \phi(z), \phi(z+1)) \quad \textit{on } \mathbb{R} \\ \phi \textit{ is non-decreasing over } \mathbb{R} \\ \phi(-\infty) = 0 \quad \textit{and} \quad \phi(+\infty) = 1. \end{cases} \tag{1.9}$$

To our knowledge, this result is new. Notice, for instance, that this result is not included in MALLET-PARET's work [23], since G does not satisfy $\frac{\partial G}{\partial X_{i-1}} > 0$ and $\frac{\partial G}{\partial X_{i+1}} > 0$. Such a condition is important in [23] for constructing traveling waves using the deformation (continuation) method.

(iii) General framework

We now consider a generalization of Equation (1.5). To this end, we introduce a real function (with properties to be specified later in Section 1.2):

$$F : [0, 1]^{N+1} \to \mathbb{R}. \tag{1.10}$$

We then consider the equation

$$c\phi'(z) = F(\phi(z + r_0), \phi(z + r_1), \ldots, \phi(z + r_N)), \tag{1.11}$$

where $N \geqq 0$ and $r_i \in \mathbb{R}$ for $i = 0, \ldots, N$. We also normalize the limits of the profile at infinity as follows:

$$\phi(-\infty) = 0, \quad \phi(+\infty) = 1. \tag{1.12}$$

Note that, for $N = 2$ and $F = F_0(X_0, X_1, X_2) = X_2 + X_1 - 2X_0 + f_L(X_0)$, Equation (1.5) is a particular case of (1.11). Moreover, F_0 is compatible with (1.12).

Assume, without loss of generality, for the whole work that:

$$r_0 = 0 \quad \text{and} \quad r_i \neq r_j \quad \text{if } i \neq j.$$

1.2. Main Results

In order to present our results, we have to introduce some assumptions on F defined in (1.10). For later use, we split these assumptions into assumptions (A) and (B).

Assumption (A). **Regularity:** F is globally Lipschitz continuous over $[0, 1]^{N+1}$.
Monotonicity: $F(X_0, \ldots, X_N)$ is non-decreasing with respect to each X_i for $i \neq 0$.
We set $f(v) = F(v, \ldots, v)$.

Assumption (B). **Instability:** $f(0) = 0 = f(1)$ and there exists $b \in (0, 1)$ such that $f(b) = 0$, $f_{|(0,b)} < 0$, $f_{|(b,1)} > 0$ and $f'(b) > 0$.
Smoothness: F is C^1 in a neighborhood of $\{b\}^{N+1}$.

Remark 1.3. 1. The point b is assumed to be unstable; this is the meaning of the condition $f'(b) > 0$.

2. Note that the instability part of Assumption (B) means, in particular, that f is of "Bistable" shape (see [23]).

Theorem 1.4. (Existence of a traveling wave) *Under assumptions* (A) *and* (B), *there exist a real* $c \in \mathbb{R}$ *and a function* $\phi : \mathbb{R} \to \mathbb{R}$ *that solves*

$$\begin{cases} c\phi'(z) = F(\phi(z + r_0), \phi(z + r_1), \dots, \phi(z + r_N)) & \text{on } \mathbb{R} \\ \phi \text{ is non-decreasing over } \mathbb{R} \\ \phi(-\infty) = 0 \quad \text{and} \quad \phi(+\infty) = 1, \end{cases} \tag{1.13}$$

in the classical sense, if $c \neq 0$ *and almost everywhere if* $c = 0$.

Our method for constructing a solution relies on the construction of a hull function for an associated homogenization problem (see the work of FORCADEL, IMBERT, MONNEAU [17]). In order to prove the uniqueness of the traveling wave, we need the following additional assumptions:

Assumption (C). Inverse monotonicity close to $\{0\}^{N+1}$ **and** $E = \{1\}^{N+1}$
There exists $\beta_0 > 0$ such that for $a > 0$, we have

$$\begin{cases} F(X + (a, \dots, a)) < F(X) & \text{for all } X,\ X + (a, \dots, a) \in [0, \beta_0]^{N+1} \\ F(X + (a, \dots, a)) < F(X) & \text{for all } X,\ X + (a, \dots, a) \in [1 - \beta_0, 1]^{N+1}. \end{cases}$$

This condition is important to get the comparison principle (see Theorem 4.1).

Assumption (D+). **(i) All the** r_i **"shifts" have the same sign:** Assume that $r_i \leqq 0$ for all $i \in \{0, \dots, N\}$.
(ii) Strict monotonicity: F is increasing in X_{i^+} with $r_{i^+} > 0$.

Assumption (D−). **(i) All the** r_i **"shifts" have the same sign:** Assume that $r_i \geqq 0$ for all $i \in \{0, \dots, N\}$.
(ii) Strict monotonicity: F is increasing in X_{i^-} with $r_{i^-} < 0$.

Assumption (E+). **(i) Strict monotonicity close to** 0**:** Assume that $\frac{\partial F}{\partial X_{i^+}}(0) > 0$ with $r_{i^+} > 0$.
(ii) Smoothness close to $\{0\}^{N+1}$**:**

There exists $\nabla F(0)$, with $f'(0) < 0$, and there exists $\alpha \in (0, 1)$ and $C_0 > 0$ such that, for all $X \in [0, 1]^{N+1}$,

$$|F(X) - F(0) - X.\nabla F(0)| \leqq C_0 |X|^{1+\alpha}.$$

Assumption (E−). **(i) Strict monotonicity close to** 1**:** Assume, for $E = (1, \dots, 1) \in \mathbb{R}^{N+1}$, that $\frac{\partial F}{\partial X_{i^-}}(E) > 0$ with $r_{i^-} < 0$.

(ii) Smoothness close to $\{1\}^{N+1}$**:**

There exists $\nabla F(E)$ with $f'(1) < 0$ and there exists $\alpha \in (0, 1)$ and $C_0 > 0$ such that, for all $X \in [0, 1]^{N+1}$,

$$|F(X) - F(E) - (X - E).\nabla F(E)| \leqq C_0|X - E|^{1+\alpha},$$

with $E = (1, \dots, 1) \in \mathbb{R}^{N+1}$.

Theorem 1.5. (Uniqueness of the velocity and of the profile) *Assume* (A) *and let* (c, ϕ) *be a solution of*

$$\begin{cases} c\phi'(z) = F(\phi(z + r_0), \phi(z + r_1), \dots, \phi(z + r_N)) \quad \textit{on } \mathbb{R} \\ \phi(-\infty) = 0 \quad \textit{and} \quad \phi(+\infty) = 1. \end{cases} \tag{1.14}$$

(a) Uniqueness of the velocity: *Under the additional Assumption* (C)*, the velocity* c *is unique.*

(b) Uniqueness of the profile ϕ**:** *If* $c \neq 0$, *then under the additional assumptions* (C) *and* (D+)(i) *or* (ii) *or* (E+) *if* $c > 0$ (*resp.* (D−)(i) *or* (ii) *or* (E−) *if* $c < 0$), *the profile* ϕ *is unique* (*up to space translation*) *and* $\phi' > 0$ *on* $\mathbb{R}$.

Remark 1.6. (*Interpreting the assumptions*)

1. **If c** $>$ **0**: Assumptions (D+)(i), (ii) and (E+), respectively, are important to prove a Strong Maximum Principle (see Lemmas 6.2 and 6.4) and the asymptotics of the profile near $-\infty$ (see Lemma 6.6) that we use to prove the uniqueness of the profile of a solution.
2. **If c** $<$ **0**: Under (D−)(i), (ii) and (E−), we get the same results, respectively, as for $c > 0$, however, the asymptotics are proven near $+\infty$ in this case.

We remark, also, that Theorems 1.1 and 1.2 are particular cases of Theorems 1.4 and 1.5. Indeed, existence of the solution in Theorem 1.2 follows from Theorem 1.4 and the fact that $b \neq \frac{1}{2}$ in Assumption (B), when $L \in (-\frac{1}{4}, \frac{1}{4})\backslash\{0\}$. Uniqueness of the profile in Theorem 1.2 follows from Theorem 1.5(b), and the fact that the function G defined in (1.7) verifies assumptions ($E\pm$).

For the whole paper, we define

$$r^* = \max_{i=0,\dots,N} |r_i|, \tag{1.15}$$

and we set the a notation for a general function h :

$$F((h(y + r_i))_{i=0,\dots,N}) := F(h(y + r_0), h(y + r_1), \dots, h(y + r_N)).$$

1.3. Brief Review of the Literature

The study of traveling waves in reaction–diffusion equations was introduced in pioneering works of Fisher [16] and Kolmogorov, Petrovsky and Piskunov [21]. Existence of traveling wave solutions has been obtained, for instance, in [2,7,15,20]. More generally, there is a huge literature about existence, uniqueness and stability of traveling waves with various nonlinearities with applications, in particular, in biology and combustion; refer, for instance, to the references cited in [6,10]. There are also several works on discrete or nonlocal versions of reaction-diffusion equations (see for instance [5,9,11,13,14,18,24,26] and [10,23] and the references cited therein).

As explained above, in the case of bistable nonlinearity f, the existence and uniqueness of traveling waves are well known for the model equation

$$u_t = u_{xx} + f(u). \tag{1.16}$$

Starting from Equation (1.16), and using a continuation method, Bates et al. [4] proved, in particular, the existence of traveling waves for the convolution model

$$u_t = J * u - u + f(u), \tag{1.17}$$

where J is a kernel.

In [23], Mallet-Paret (see also Carpio et al. [9] for the semi-linear case) used also a global continuation method (that is, a homotopy method) to get existence of traveling waves for bistable nonlinearities and to obtain information about the uniqueness and the dependence of solutions on parameters. This continuation argument was applied to connect the discrete dynamical system that he studied and a PDE model for which the existence and uniqueness were known. He proved the continuation between the solutions of the two systems using a general Fredholm alternative method [22] for the linearized traveling wave equations.

Traveling waves were also studied by Chow et al. [12] for lattice dynamical systems (lattice ODEs) and for coupled map lattices (CMLs) that arise as time-discretizations of lattice ODEs. Using a geometric approach, the authors studied the stability of traveling waves for lattice ODEs and proved existence of traveling waves of their time discretized CMLs. More precisely, they constructed a local coordinate system in a tubular neighborhood of the traveling wave solution in the phase space of their system. Such an approach is used to transform lattice ODEs into a nonautonomous time-periodic ODE and traveling waves to periodic solutions of this ODE. In addition, they gain from this transformation the possibility to use the standard tools of dynamical systems and to see traveling waves of a CML as certain orbits for a circle diffeomorphism whose rotational number is equal to the wave speed.

Zinner [25] proved the existence of traveling waves for the discrete Nagumo equation

$$\dot{x}_i = d(x_{i+1} - 2x_i + x_{i-1}) + f(x_i) \quad i \in \mathbb{Z}. \tag{1.18}$$

The construction is done by first introducing a simplified problem (using a projection to 0 or 1 for $|i| \geqq N$) for which the existence is attained by Brouwer's fixed point theorem. Hankerson and Zinner [19] also proved existence of traveling waves [for an equation more general than (1.18)] obtained as the long time limit of the solution with Heaviside initial data, using an interesting lap number argument.

In [10], Chen, Guo and Wu constructed traveling waves for lattice ODEs with bistable nonlinearity. They rephrase the solution ϕ of (1.11) as a fixed point of an integral formulation. First, they considered a simplified problem (using a projection on 0 or 1 for large indices $|i| \geqq N$) and they show, for any $c \neq 0$, the existence of a solution $\phi^{N,c}$ using the monotone iteration method. Finally, they recover the existence of a solution in the limit $N \to +\infty$ for a suitable choice $c = c(N)$ converging to a limit velocity.

In this paper, we introduce a method that is completely new, at least when used to prove the existence of traveling waves. In our approach, the existence of traveling waves relies on the construction of hull functions of slope p (like correctors) for an associated homogenization problem. Passing to the limit $p \to 0$, one important difficulty is to identify a traveling wave joining two stable states. In particular, we have prevented this traveling wave from degenerating to the intermediate unstable state. The uniqueness of the profile is proved using either the strong maximum principle or the weak asymptotics of the profile. Note that using weak asymptotics (in comparison with those of Mallet-Paret [23]) allows us to have weaker assumptions.

We also mention that our method is still effective in higher dimensional problems. Consider, for instance, the model

$$\frac{\mathrm{d}}{\mathrm{d}t} X_I(t) = f(X_I) + \sum_{|J|=1} (X_{I+J} - X_I), \tag{1.19}$$

which describes the interaction of an atom $I \in \mathbb{Z}^n$ with its nearest neighbors ($X_I \in \mathbb{R}$ denotes the position of atom I). We can look for traveling waves $X_I(t) = \phi(ct + \nu \cdot I)$ that propagate in a direction $\nu \in \mathbb{R}^n$ with $|\nu| = 1$. That is, for $z = ct + \nu \cdot I$, we look for a ϕ solution of

$$c\phi'(z) = f(\phi(z)) + \sum_{|J|=1} (\phi(z + \nu \cdot J) - \phi(z)),$$

where f denotes a bistable nonlinearity. Setting $r_j := \nu \cdot J$, we recover an equation of type (1.11) for $N = 2n$. Therefore, the results of higher dimensional problems follow from our one dimensional results (Theorems 1.4 and 1.5) as far as they hold for general r_j shifts.

We get the existence of solutions under very weak assumptions in comparison with similar results in previous works. Our framework is very flexible, and does not require a setting in any particular functional space. We think that our method opens new perspectives and could be used to study many models: for example, fully overdamped FK models with time dependent nonlinearities, accelerated FK models, FK with multi-particles.

1.4. Organization of the Paper

In Section 2, we introduce an extension of F onto $\mathbb{R}^{N+1}$ and we recall, for the extension function, the notion of viscosity solutions, the existence of hull functions for our model and we prove some results about monotone functions. We prove Theorem 1.4 (for the extended function) in Section 3. In Section 4, we prove the uniqueness of the velocity of a profile (Theorem 1.5 part (a) = Proposition 4.5) and a comparison principle result on the half-line. Section 5 is devoted to the asymptotics of a profile near $\pm\infty$ (Proposition 5.1). In Section 6, we prove the uniqueness of the profile (Theorem 1.5 part (b)). Finally, we prove in Appendices A and B the extension result, namely Lemma 2.1 and some results about monotone functions, Lemmas 2.10 and 2.11, respectively.

2. Preliminary Results

This section is divided into four subsections. In the first subsection, we extend the function F onto $\mathbb{R}^{N+1}$. In the second subsection, we recall the definition of a viscosity solution. We apply a result of existence of hull functions associated to the homogenization of our problem with the extended F in the third subsection. We dedicate the fourth subsection to some results about monotone functions that we have used in Section 3.

2.1. Extension of F

The proof of existence of traveling waves is based on the construction of hull functions (like correctors) associated to a homogenization problem (see [17]). To this end, we first need to extend the function F in $\tilde{F}$ defined over $\mathbb{R}^{N+1}$ and satisfying the following assumption:

Assumption ($\tilde{\text{A}}$). **Regularity:** $\tilde{F}$ is globally Lipschitz continuous over $\mathbb{R}^{N+1}$.
Periodicity: $\tilde{F}(X_0+1,\ldots,X_N+1) = \tilde{F}(X_0,\ldots,X_N)$ for every $X=(X_0,\ldots,X_N)\in\mathbb{R}^{N+1}$.
Monotonicity: $\tilde{F}(X_0,\ldots,X_N)$ is non-decreasing with respect to each X_i for $i\neq 0$.

The extension result is the following:

Lemma 2.1. (Extension of F) *Given a function F defined over $Q=[0,1]^{N+1}$ satisfying* (A) *and $F(1,\ldots,1)=F(0,\ldots,0)$, there exists an extension $\tilde{F}$ defined over $\mathbb{R}^{N+1}$, such that*

$$\tilde{F}_{|Q} = F \quad \textit{and} \quad \tilde{F} \textit{ satisfies } (\tilde{A}).$$

The proof of this lemma is postponed to Appendix A.

Remark 2.2. We note that if ϕ is a traveling wave constructed for (1.13) with F replaced by $\tilde{F}$, then ϕ is a traveling wave of (1.13). This is a direct consequence of Lemma 2.1 and the fact that

$$\begin{cases} \phi \text{ is non-decreasing on } \mathbb{R} \\ \phi(-\infty)=0 \text{ and } \phi(+\infty)=1. \end{cases}$$

By convention, we will say that $\tilde{F}$ satisfies (B) (resp. (C), (D) or (E)) if and only if $F = \tilde{F}_{|Q}$ satisfies (B) (resp. (C), (D) or (E)).

We now give a result corresponding to Theorem 1.4 for $\tilde{F}$, whose proof is given in Section 3.

Proposition 2.3. (Result corresponding to Theorem 1.4 for $\tilde{F}$) *Assume that $\tilde{F}$ satisfies* $(\tilde{A})$, (B). *Then there exist a real c and a function ϕ solution of*

$$\begin{cases} c\phi'(z) = \tilde{F}((\phi(z + r_i))_{i=0,\dots,N}) \quad \text{on } \mathbb{R} \\ \phi \text{ is non-decreasing on } \mathbb{R} \\ \phi(-\infty) = 0 \quad \text{and} \quad \phi(+\infty) = 1, \end{cases} \tag{2.1}$$

in the classical sense, if $c \neq 0$, and almost everywhere if $c = 0$.

For simplicity, in the rest of this section and in Section 3, we denote $\tilde{F}$ as F.

Proof of Theorem 1.4. The proof of Theorem 1.4 is a straightforward consequence of Remark 2.2 and Proposition 2.3. □

2.2. Viscosity Solution

Throughout the paper, we will use the notion of viscosity solution that we introduce in this subsection. To this end, we recall that the upper and the lower semi-continuous envelopes, u^* and u_*, of a locally bounded function u are defined as

$$u^*(y) = \limsup_{x \to y} u(x) \quad \text{and} \quad u_*(y) = \liminf_{x \to y} u(x).$$

Definition 2.4. (*Viscosity solution*) Let $u : \mathbb{R} \to \mathbb{R}$ be a locally bounded function, $c \in \mathbb{R}$ and let F be as defined on $\mathbb{R}^{N+1}$.

– The function u is a sub-solution (resp. a super-solution) of

$$cu'(x) = F((u(x + r_i))_{i=0,\dots,N}) \quad \text{on } \mathbb{R}, \tag{2.2}$$

if u is upper semi-continuous (resp. lower semi-continuous) and if for all test functions $\psi \in C^1(\mathbb{R})$ such that $u - \psi$ attains a local maximum (resp. a local minimum) at x^*, we have

$$c\psi'(x^*) \leqq F((u(x^* + r_i))_{i=0,\dots,N}) \text{ (resp. } c\psi'(x^*) \geqq F((u(x^* + r_i))_{i=0,\dots,N})).$$

– A function u is a viscosity solution of (2.2) if u^* is a sub-solution and u_* is a super-solution.

We also recall the stability result for viscosity solutions (see [3, Theorem 4.1]).

Proposition 2.5. (Stability of viscosity solutions) *Consider a function F defined on $\mathbb{R}^{N+1}$ and satisfying $(\tilde{A})$. Assume that $(u_\varepsilon)_\varepsilon$ is a sequence of sub-solutions (resp. super-solutions) of (2.2). Suppose that the functions $(u_\varepsilon)_\varepsilon$ are uniformly locally bounded on $\mathbb{R}$ and let*

$$\overline{u}(x) = \limsup_{\varepsilon\to 0}{}^* u_\varepsilon(x) := \limsup_{(\varepsilon,y)\to(0,x)} u_\varepsilon(y) \quad \text{and} \quad \underline{u}(x) = \liminf_{\varepsilon\to 0}{}_* u_\varepsilon(x) := \liminf_{(\varepsilon,y)\to(0,x)} u_\varepsilon(y),$$

be the relaxed upper and lower semi-limits. If $\overline{u}$ (resp. $\underline{u}$) is finite, then $\overline{u}$ is a sub-solution (resp. $\underline{u}$ is a super-solution) of (2.2).

2.3. On the Hull Function

In this subsection, we first adapt the result of existence of a hull function associated with the homogenization of our problem, then we make the link between the existence of a hull function and the existence of the traveling wave.

Lemma 2.6. (Existence of a hull function [17, Theorem 1.5]) *Let F be a given function satisfying Assumption $(\tilde{\mathrm{A}})$ and $p > 0$. There exists a unique λ_p such that there exists a locally bounded function $h_p : \mathbb{R} \to \mathbb{R}$ satisfying (in the viscosity sense):*

$$\begin{cases} \lambda_p h_p' = F((h_p(y + pr_i))_{i=0,\dots,N}) \quad \text{on } \mathbb{R} \\ h_p(y+1) = h_p(y) + 1 \\ h_p'(y) \geqq 0 \\ |h_p(y+y') - h_p(y) - y'| \leqq 1 \quad \text{for all } y' \in \mathbb{R}. \end{cases} \tag{2.3}$$

Such a function h_p is called a hull function. Moreover, there exists a constant $K > 0$, independent on p, such that

$$|\lambda_p| \leqq K(1+p).$$

Notice that Lemma 2.6 is proved in [17] only for $r_i \in \mathbb{Z}$. However, the proof for the generalization $r_i \in \mathbb{R}$ is still valid (it is exactly the same).

Recalling this, and using the hull function h_p, we define the function ϕ_p as:

$$\phi_p(x) := h_p(px). \tag{2.4}$$

Moreover we set, as a velocity, the ratio

$$c_p := \frac{\lambda_p}{p}. \tag{2.5}$$

Remark 2.7. It is possible that $c_p = 0$ for all $p > 0$. Our proof of existence of traveling waves is done for the general case. However, we state throughout the proof the different situations for the velocity.

Notice that the above ϕ_p satisfies the fol lowing lemma:

Lemma 2.8. (Properties of ϕ_p) *Let $p > 0$ and assume $(\tilde{A})$. Then the function ϕ_p defined in (2.4) satisfies, in the viscosity sense:*

$$\begin{cases} c_p \phi_p' = F((\phi_p(z + r_i))_{i=0,\ldots,N}) & \text{on } \mathbb{R} \\ \phi_p' \geqq 0 \\ \phi_p\left(z + \dfrac{1}{p}\right) = \phi_p(z) + 1. \end{cases} \tag{2.6}$$

Moreover, if $c_p \neq 0$ then there exists $M > 0$ independent on p such that

$$|\phi_p'| \leqq \frac{M}{|c_p|}, \tag{2.7}$$

for $0 < p \leqq \frac{1}{r^*}$, with r^* given in (1.15).

Proof of Lemma 2.8. Let h_p be a viscosity solution given by Lemma 2.6. We then get (2.6) by a change of variables (2.4)–(2.5). We now show (2.7). We choose $p > 0$ such that

$$\frac{1}{p} \geqq r^*.$$

Since ϕ_p is non-decreasing, then we have

$$\begin{cases} |\phi_p(x + r_i) - \phi_p(x)| \leqq \left|\phi_p\left(x + \dfrac{1}{p}\right) - \phi_p(x)\right| = 1 & \text{if } r_i \geqq 0 \\ |\phi_p(x + r_i) - \phi_p(x)| \leqq \left|\phi_p\left(x - \dfrac{1}{p}\right) - \phi_p(x)\right| = 1 & \text{if } r_i \leqq 0 \end{cases}.$$

Moreover, since $F \in \mathrm{Lip}(\mathbb{R}^{N+1})$, then

$$|F((\phi_p(x + r_i)_{i=0,\ldots,N})) - F((\phi_p(x))_{i=0,\ldots,N})| \leqq L \begin{vmatrix} 1 \\ \vdots \\ 1 \end{vmatrix} =: L^1,$$

where L is the Lipschitz constant of F. On the other hand, f is bounded (because f is Lipschitz and periodic) and $F((\phi_p(x))_{i=0,\ldots,N}) = f(\phi_p(x))$, thus

$$|F((\phi_p(x + r_i)_{i=0,\ldots,N}))| \leqq L^1 + |f|_{L^\infty} =: M.$$

This implies that

$$|c_p \phi_p'| \leqq M$$

in the viscosity sense. If in addition $c_p \neq 0$, then we get the Lipschitz bound

$$|\phi_p'| \leqq \frac{M}{|c_p|}.$$

□

2.4. Useful Results about Monotone Functions

In this subsection, we review miscellaneous results about monotone functions that we will use later in Section 3 for the proof of Proposition 2.3. We state Helly's Lemma on the one hand, and the equivalence between viscosity and almost everywhere solutions on the other hand.

Lemma 2.9. (Helly's Lemma, (see [1, Section 3.3, page 70])) *Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of non-decreasing functions on $[a,b]$ verifying $|g_n| \leqq M$ uniformly in n. Then there exists a subsequence $(g_{n_j})_{j\in\mathbb{N}}$ such that*

$$g_{n_j} \to g \quad \text{almost everywhere on } [a,b],$$

with g non-decreasing and $|g| \leqq M$.

Lemma 2.10. (Complement of Helly's Lemma) *Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of non-decreasing functions on a bounded interval I and assume that*

$$g_n \to g \quad \text{almost everywhere on } I.$$

If g is constant on $\mathring{I}$, then for every closed subset interval $I' \subset \mathring{I}$,

$$g_n \to g \quad \text{uniformly on } I'.$$

The proof of Lemma 2.10 is described in Appendix B.

Now we introduce a lemma that shows the equivalence between viscosity and almost everywhere solutions under the monotonicity of the solution.

Lemma 2.11. (Equivalence between viscosity and almost everywhere solutions) *Let F satisfy Assumption $(\tilde{\mathrm{A}})$. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a non-decreasing function. Then ϕ is a viscosity solution of*

$$0 = F((\phi(x+r_i))_{i=0,\ldots,N}) \quad \text{on } \mathbb{R} \tag{2.8}$$

if and only if ϕ is an almost everywhere solution of the same equation.

The proof of Lemma 2.11 is also described Appendix B.

3. Construction of a Traveling Wave: Proof of Proposition 2.3

This section is devoted to the proof of existence of a traveling wave for system (2.1). We control both the velocity of propagation and the finite difference of a solution in the first subsection, then we prove Proposition 2.3 in the second subsection.

3.1. Preliminary Results

We have

Lemma 3.1. (Velocity c_p is bounded) *Under Assumptions* ($\tilde{\mathrm{A}}$) *and* (B), *let* c_p *be the velocity given by* (2.5). *Then there exists* $M_1 > 0$ *such that*

$$|c_p| \leqq M_1$$

for $0 < p \leqq \frac{1}{r^*}$, *with* r^* *given in* (1.15).

Proof of Lemma 3.1. Consider the function ϕ_p given by (2.4) which satisfies (2.6). Let c_p be the associated velocity given by (2.5). Assume by contradiction that when $p \to p_0 \in [0, \frac{1}{r^*}]$

$$\lim_{p \to p_0} c_p = +\infty, \tag{3.1}$$

(the case $c_p \to -\infty$ being similar). Let $\overline{\phi}_p(x) := \phi_p(c_p x)$ solution of

$$\overline{\phi}'_p(x) = F\left(\left(\overline{\phi}_p\left(x + \frac{r_i}{c_p}\right)\right)_{i=0,\dots,N}\right).$$

Since $\overline{\phi}_p$ is invariant with respect to space translations, we may assume that

$$\overline{\phi}_p(0) = b - \varepsilon$$

for some $\varepsilon > 0$ small enough. Moreover, by (2.7) we have

$$|\overline{\phi}'_p| = |c_p \phi'_p| \leqq M$$

for some $M > 0$ independent on p. Thus using Ascoli's Theorem and the diagonal extraction argument, $\overline{\phi}_p$ converges as $p \to p_0$ (up to a subsequence) to some $\overline{\phi}$ locally uniformly on $\mathbb{R}$, and $\overline{\phi}$ classically satisfies

$$\begin{aligned} \overline{\phi}'(x) &= F((\overline{\phi}(x))_{i=0,\dots,N}) \\ &= f(\overline{\phi}(x)) \end{aligned}$$

and $\overline{\phi}(0) = b - \varepsilon$. But $\overline{\phi}'_p \geqq 0$ (because (3.1) implies trivially that $c_p \geqq 0$), thus $\overline{\phi}' \geqq 0$. Hence $f(\overline{\phi}(x)) \geqq 0$ for all x, in particular $f(\overline{\phi}(0)) = f(b - \varepsilon)$, which is a contradiction since $f(b - \varepsilon) < 0$ (see Assumption (B)). $\square$

Next, we introduce an important proposition on the control of the finite difference that will be used in the proof of existence of a traveling wave.

Proposition 3.2. (Control on the finite difference) *Assume that F satisfies $(\tilde{A})$ and let $a > r^*$ with r^* given by (1.15) and $M_0 > 0$. For every $\varepsilon > 0$, there exists $\delta > 0$ such that for any function ϕ which is a (viscosity) solution of*

$$\begin{cases} c\phi'(x) = F((\phi(x+r_i))_{i=0,\dots,N}) \quad \text{on } \mathbb{R} \\ \phi' \geqq 0 \\ \phi(x+1) \leqq \phi(x) + 1 \\ |c| \leqq M_0 \\ |c\phi'| \leqq M_0, \end{cases}$$

and for all $x_0 \in \mathbb{R}$ satisfying

$$\phi_*(x_0 + a) - \phi^*(x_0 - a) \leqq \delta,$$

we have

$$\operatorname{dist}(\alpha, \{0, b\} + \mathbb{Z}) < \varepsilon \quad \text{for all } \alpha \in [\phi_*(x_0), \phi^*(x_0)].$$

Note that $\{0, b\} + \mathbb{Z} \equiv \mathbb{Z} \cup (b + \mathbb{Z})$. Roughly speaking, this proposition says that if ϕ is flat enough around x_0, then $\phi(x_0)$ is close to a zero of f.

Proof of Proposition 3.2. The proof is done by contradiction.
Step 1. construction of a sequence, by contradiction
We assume by contradiction that there exists $\varepsilon_0 > 0$ such that for all $\delta_n \to 0$, there exists ϕ^n solution of

$$\begin{cases} c^n(\phi^n)'(x) = F((\phi^n(x+r_i))_{i=0,\dots,N}) \\ (\phi^n)' \geqq 0 \\ \phi^n(x+1) \leqq \phi^n(x) + 1 \\ |c^n| \leqq M_0 \\ |c^n(\phi^n)'| \leqq M_0, \end{cases} \tag{3.2}$$

such that there exists $x_n \in \mathbb{R}$ satisfying

$$\phi^n_*(x_n + a) - (\phi^n)^*(x_n - a) \leqq \delta_n \to 0, \tag{3.3}$$

and there exists $\alpha_n \in [\phi^n_*(x_n), (\phi^n)^*(x_n)]$ such that

$$\operatorname{dist}(\alpha_n, \{0, b\} + \mathbb{Z}) \geqq \varepsilon_0 > 0. \tag{3.4}$$

Up to replacing $\phi^n(x)$ by $\phi^n(x + e_n) + k_n$ with $e_n \in \mathbb{R}$, $k_n \in \mathbb{Z}$, we can assume that

$$\left| \begin{array}{l} x_n \equiv 0 \\ \phi^n(0) \in [0, 1) \quad \text{for all } n. \end{array} \right. \tag{3.5}$$

Step 2. passing to limit $n \to +\infty$
Because $|c^n| \leqq M_0$ then, up to extracting a subsequence as $n \to +\infty$, we have

$$c^n \to c.$$

Case 1. $c \neq 0$
For n large enough, we have $|c^n| \geqq \frac{|c|}{2} \neq 0$. Hence

$$|(\phi^n)'| \leqq \frac{2M_0}{|c|} \quad \text{for large } n,$$

thus ϕ^n is uniformly Lipschitz continuous, using Ascoli's Theorem and the diagonal extraction argument, $\phi^n \to \phi$ (up to a subsequence) locally uniformly on $\mathbb{R}$. Moreover, ϕ satisfies (in the viscosity sense)

$$\begin{cases} c\phi'(x) = F((\phi(x + r_i))_{i=0,\ldots,N}) \\ \phi' \geqq 0. \end{cases} \tag{3.6}$$

Case 2. $c = 0$
Notice that $\phi^n(x+1) \leqq \phi^n(x) + 1$ implies (using (3.5))

$$\begin{cases} \phi^n(x) \leqq \lceil x \rceil + 1 & \text{for } x \geqq 0 \\ \phi^n(x) \geqq -\lceil |x| \rceil & \text{for } x \leqq 0. \end{cases} \tag{3.7}$$

Therefore, using Helly's Lemma (Lemma 2.9) and the diagonal extraction argument, ϕ^n converges (up to a subsequence) to ϕ locally almost everywhere. Moreover, we have (using Lemma 2.11 if $c^n = 0$)

$$c^n \int_{b_1}^{b_2} (\phi^n)'(z)\, \mathrm{d}z = \int_{b_1}^{b_2} F((\phi^n(z + r_i))_{i=0,\ldots,N})\, \mathrm{d}z$$

for every $b_1 < b_2$. That is,

$$c^n(\phi^n(b_2) - \phi^n(b_1)) = \int_{b_1}^{b_2} F((\phi^n(z + r_i))_{i=0,\ldots,N})\, \mathrm{d}z.$$

But

$$F((\phi^n(z + r_i))_{i=0,\ldots,N}) \to F((\phi(z + r_i))_{i=0,\ldots,N}) \quad \text{almost everywhere}$$

and

$$|F((\phi^n(z + r_i))_{i=0,\ldots,N})| \leqq m_0(1 + |z|)$$

for some constant $m_0 > 0$ (because of (3.7) and the fact that F is globally Lipschitz with f bounded). Thus, using Lebesgue's dominated convergence theorem, we pass to the limit $n \to +\infty$, and we get

$$0 = \int_{b_1}^{b_2} F((\phi(z + r_i))_{i=0,\ldots,N})\, \mathrm{d}z,$$

which implies (since b_1 and b_2 are arbitrary) that

$$0 = F((\phi(z + r_i))_{i=0,\ldots,N}) \quad \text{almost everywhere.}$$

Since $(\phi^n)' \geqq 0$ implies $\phi' \geqq 0$, then by Lemma 2.11, ϕ verifies

$$\begin{cases} 0 = F((\phi(x+r_i))_{i=0,\dots,N}) \\ \phi' \geqq 0 \end{cases} \tag{3.8}$$

in the viscosity sense.
Step 3. getting a contradiction
Passing to the limit in (3.3) with $x_n = 0$ implies that

$$\phi_*(a) \leqq \phi^*(-a).$$

But ϕ is non-decreasing, then $\phi = \text{const} =: k$ over $(-a, a)$. Since $a > r^*$, then from (3.6) and (3.8), we get for $x = 0$

$$\begin{aligned} 0 &= F((\phi(x+r_i))_{i=0,\dots,N}) \\ &= F((k)_{i=0,\dots,N}) = f(k), \end{aligned}$$

hence $k \in \{0, b\} + \mathbb{Z}$. On the other hand, since $\alpha_n \in [\phi^n_*(0), (\phi^n)^*(0)]$, then (up to a subsequence)

$$\alpha_n \to \alpha \in \{k\} = [\phi_*(0), \phi^*(0)].$$

Moreover, if we pass to limit in (3.4), we get

$$\text{dist}(\alpha = k, \{0, b\} + \mathbb{Z}) \geqq \varepsilon_0 > 0,$$

which is a contradiction. □

3.2. Proof of Proposition 2.3

Proof of Proposition 2.3. The proof is done in several steps.
Step 0. introduction
Let $p > 0$ and ϕ_p (given by (2.4)) be a non-decreasing solution of

$$c_p \phi_p'(x) = F((\phi_p(x+r_i))_{i=0,\dots,N})$$

with

$$\phi_p\left(x + \frac{1}{p}\right) = 1 + \phi_p(x)$$

and let c_p be as given by (2.5). Up to translating ϕ_p, let us suppose that

$$\begin{cases} (\phi_p)_*(0) \leqq b \\ (\phi_p)^*(0) \geqq b. \end{cases} \tag{3.9}$$

Our aim is to pass to the limit as p goes to zero.
Step 0.1. introduce z_p and y_p
For any $\varepsilon > 0$ small enough ($\varepsilon < \frac{1}{2}\min(b, 1-b)$), let $z_p,\ y_p \in \mathbb{R}$ such that

$$\begin{cases} (\phi_p)^*(z_p) \geqq b + \varepsilon \\ (\phi_p)_*(z_p) \leqq b + \varepsilon, \end{cases} \tag{3.10}$$

and

$$\begin{cases} (\phi_p)^*(y_p) \geqq b - \varepsilon \\ (\phi_p)_*(y_p) \leqq b - \varepsilon. \end{cases} \tag{3.11}$$

From Proposition 3.2, since $(\phi_p)^*(z_p) > b$ and $(\phi_p)_*(y_p) < b$, we deduce that (for $a > r^*$)

$$(\phi_p)_*(z_p + a) - (\phi_p)^*(z_p - a) \geqq \delta(\varepsilon) > 0 \tag{3.12}$$

and

$$(\phi_p)_*(y_p + a) - (\phi_p)^*(y_p - a) \geqq \delta(\varepsilon) > 0, \tag{3.13}$$

with $\delta(\varepsilon)$ independent of p. Moreover, we notice that

$$y_p \leqq 0. \tag{3.14}$$

(Otherwise, $b - \varepsilon \geqq (\phi_p)_*(y_p) \geqq (\phi_p)^*(0) \geqq b$, a contradiction).
Step 1. viscosity super-solution
Let

$$\psi_p(x) := (\phi_p)_*(x + a) - (\phi_p)^*(x - a).$$

Notice that ψ_p is lower semi-continuous and $\psi_p(x) \geqq 0$ for all $x \in \mathbb{R}$ (because $(\phi_p)_*$ is lower semi-continuous, $(\phi_p)^*$ is upper semi-continuous and ϕ_p is non-decreasing). Since (in the viscosity sense)

$$\begin{cases} c_p((\phi_p)_*)'(x + a) \geqq F(((\phi_p)_*(x + a + r_i))_{i=0,\dots,N}) \\ c_p((\phi_p)^*)'(x - a) \leqq F(((\phi_p)^*(x - a + r_i))_{i=0,\dots,N}), \end{cases}$$

then we can show (using a doubling of variables) the following inequality

$$c_p(\psi_p)'_*(x) \geqq F(((\phi_p)_*(x + a + r_i))_{i=0,\dots,N}) - F(((\phi_p)^*(x - a + r_i))_{i=0,\dots,N}), \tag{3.15}$$

which holds in the viscosity sense.
Step 2. passing to the limit $p \to 0$
Since c_p is bounded (see Lemma 3.1), then

$$c_p \to c,$$

up to a subsequence.
Case 1. $c \neq 0$
For p small enough, we have $|c_p| \geqq \frac{|c|}{2} \neq 0$. From (2.7), we deduce that

$$|\phi_p'| \leqq \frac{2M}{|c|} \quad \text{for } p \text{ small},$$

thus ϕ_p is uniformly Lipschitz continuous, using Ascoli's Theorem and the diagonal extraction argument, $\phi_p \to \phi$ (up to a subsequence) locally uniformly on $\mathbb{R}$.

Moreover, ϕ satisfies, at least in the viscosity sense (using the stability result, Proposition 2.5),

$$\begin{cases} c\phi'(x) = F((\phi(x+r_i))_{i=0,\dots,N}) \\ \phi' \geqq 0, \end{cases} \tag{3.16}$$

and

$$\begin{cases} (\phi)_*(0) \leqq b \\ (\phi)^*(0) \geqq b. \end{cases}$$

Case 2. $c = 0$
Let $R > 0$ and choose p small enough such that $R < \frac{1}{2p}$. Since

$$\phi_p\left(\frac{1}{2p}\right) = 1 + \phi_p\left(\frac{-1}{2p}\right), \tag{3.17}$$

then for all $x \in [-R, R]$, we have

$$|\phi_p(x) - \phi_p(0)| \leqq \left|\phi_p\left(\frac{1}{2p}\right) - \phi_p\left(\frac{-1}{2p}\right)\right| = 1.$$

Notice that (3.9), the monotonicity of ϕ_p and (3.17) implies that

$$b - 1 \leqq \phi_p\left(-\frac{1}{2p}\right) \leqq (\phi_p)_*(0) \leqq b \leqq (\phi_p)^*(0) \leqq \phi_p\left(\frac{1}{2p}\right) \leqq b + 1,$$

thus

$$b - 1 \leqq \phi_p(0) \leqq b + 1.$$

Hence

$$|\phi_p|_{L^\infty[-R,R]} \leqq 3.$$

Using Helly's Lemma (Lemma 2.9) and the diagonal extraction argument, ϕ_p converges locally almost everywhere (up to a subsequence) to non-decreasing function ϕ. Thus, ϕ satisfies

$$\begin{cases} 0 = c\phi'(x) = F((\phi(x+r_i))_{i=0,\dots,N}) \\ \phi' \geqq 0 \end{cases} \tag{3.18}$$

almost everywhere. Moreover, from Lemma 2.11, we deduce that ϕ is a viscosity solution of (3.18) with

$$\begin{cases} \phi_*(0) \leqq b \\ \phi^*(0) \geqq b. \end{cases}$$

Step 3. first properties of the limit ϕ
Step 3.1. the oscillation of ϕ is bounded
Consider any $R > 0$. Choose p_0 such that $R \leqq \frac{1}{2p_0}$ and let $p \in (0, p_0]$. Then

$$\phi_p(R) - \phi_p(-R) \leqq \phi_p\left(\frac{1}{2p_0}\right) - \phi_p\left(\frac{-1}{2p_0}\right) = 1.$$

However, ϕ_p converges (up to a subsequence and at least almost everywhere) to ϕ (see Step 2), thus

$$\phi(R) - \phi(-R) \leqq 1$$

for almost every R. Now let R goes to $+\infty$, we conclude that

$$\phi(+\infty) - \phi(-\infty) \leqq 1.$$

Step 3.2. $\phi(\pm\infty) \in \mathbb{Z} \cup (\{b\} + \mathbb{Z})$
Since (3.16) is invariant by translation, then

$$\phi^n(x) = \phi(x - n)$$

is a viscosity solution of

$$c(\phi^n)'(x) = F((\phi^n(x + r_i))_{i=0,\dots,N}).$$

Moreover, ϕ is non-decreasing and bounded (see Step 3.1), thus $(\phi^n)_n$ is a non-increasing sequence of bounded functions. Therefore, ϕ^n converges pointwise as $n \to +\infty$. Moreover, since

$$\lim_{n\to+\infty} (\phi^n(x) - \phi(-\infty)) = 0,$$

then ϕ^n converges to $\phi(-\infty)$. Now, using the stability for viscosity solutions (see Proposition 2.5), we deduce that $\phi(-\infty)$ is a solution of

$$c(\phi(-\infty))' = F((\phi(-\infty))_{i=0,\dots,N}) = f(\phi(-\infty)).$$

That is,

$$f(\phi(-\infty)) = 0.$$

Similarly, we get $f(\phi(+\infty)) = 0$. Therefore the assertion of the step follows from (B).
Step 4. $\phi(\pm\infty) \notin \{b\} + \mathbb{Z}$
Since $\phi(+\infty) - \phi(-\infty) \leqq 1$ and

$$\begin{cases} \phi_*(0) \leqq b \\ \phi^*(0) \geqq b, \end{cases}$$

we get that $\phi(-\infty) \in \{b-1, 0, b\}$ and $\phi(+\infty) \in \{b, 1, b+1\}$. We want to exclude the cases $\phi(\pm\infty) = b,\ b \pm 1$. Notice that if $\phi(+\infty) = b + 1$, then $\phi(-\infty) = b$. Similarly, if $\phi(-\infty) = b - 1$, then $\phi(+\infty) = b$. Therefore, it is sufficient to

exclude the cases $\phi(\pm\infty) = b$. At the end, this will show that $\phi(+\infty) = 1$ and $\phi(-\infty) = 0$.
Suppose to the contrary that

$$\phi(+\infty) = b,$$

(the case $\phi(-\infty) = b$ being similar). Let $x_0 = 2r^*$, where $r^* = \max_{i=0,\dots,N} |r_i|$. Since

$$b = \phi(+\infty) \geqq \phi^*(0) \geqq b,$$

then $\phi(x) = b$ for all $x > 0$. Hence

$$\phi(x_0) = \phi(x_0 \pm a) = b,$$

for $r^* < a < 2r^*$. Using the uniform convergence of ϕ_p to ϕ (see Lemma 2.10 if $c = 0$), we deduce that

$$\phi_p(x_0) \to b$$

and

$$\psi_p(x_0) = (\phi_p)_*(x_0 + a) - (\phi_p)^*(x_0 - a) \to 0 \quad \text{as } p \to 0.$$

Step 4.1. Equation satisfied by ψ_p at its point of minimum
Since (for z_p and y_p defined in (3.10) and (3.11)) we have

$$\begin{cases} z_p \to +\infty \text{ as } p \to 0 & (\phi \text{ is non-decreasing and } \phi(+\infty) = b) \\ y_p \leqq 0 & (\text{by } (3.14)), \end{cases}$$

then $x_0 \in [y_p, z_p]$ for p small enough. Next, set

$$m_p = \min_{x \in [y_p, z_p]} \psi_p(x) = \psi_p(x_p^*) \geqq 0 \quad \text{with } x_p^* \in [y_p, z_p],$$

thus

$$m_p = \psi_p(x_p^*) \leqq \psi_p(x_0) \to 0 \quad \text{as } p \to 0. \tag{3.19}$$

In addition, since

$$\begin{cases} \psi_p(y_p) \geqq \delta(\varepsilon) > 0 \\ \psi_p(z_p) \geqq \delta(\varepsilon) > 0, \end{cases}$$

then

$$x_p^* \in (y_p, z_p). \tag{3.20}$$

Therefore from (3.15), we get

$$\begin{aligned} 0 = c_p((\psi_p)_*)'(x_p^*) \geqq F(((\phi_p)_*(x_p^* + a + r_i))_{i=0,\dots,N}) \\ - F(((\phi_p)^*(x_p^* - a + r_i))_{i=0,\dots,N}) \end{aligned} \tag{3.21}$$

in the viscosity sense (and pointwisely).

Step 4.2. $\psi_p(x_p^* + r_i) \geqq \psi_p(x_p^*) = m_p$ for all i
Because of (3.20), we have

$$b - \varepsilon \leqq (\phi_p)^*(y_p) \leqq (\phi_p)^*(x_p^*) \leqq (\phi_p)_*(z_p) \leqq b + \varepsilon. \tag{3.22}$$

Therefore, by reasoning similar to that of Step 2, we show that

$$\phi_p(x_p^* + .) \to \phi_0 \quad \text{almost everywhere on } \mathbb{R},$$

and ϕ_0 is a viscosity solution of (3.16).

Since

$$m_p = \psi_p(x_p^*) = (\phi_p)_*(x_p^* + a) - (\phi_p)^*(x_p^* - a) \to 0 \quad \text{as } p \to 0, \tag{3.23}$$

we deduce that

$$\phi_0 = \text{const:} = k \quad \text{on } (-a, a). \tag{3.24}$$

From Lemma 2.10 and (3.22), we deduce that $k \in [b - \varepsilon, b + \varepsilon]$. Moreover, we have

$$0 = c\phi_0'(0) = F((\phi_0(0 + r_i))_{i=0,\ldots,N}) = f(k),$$

hence $k = b$. Again, using Lemma 2.10

we deduce that

$$\sup_{(x_p^* - a + \delta, x_p^* + a - \delta)} |\phi_p(x) - b| \to 0 \quad \text{for any } \delta > 0.$$

Moreover, because of (3.23), we can even conclude that

$$(\phi_p)_*(x_p^* + a),\ (\phi_p)^*(x_p^* - a) \to b \quad \text{as } p \to 0. \tag{3.25}$$

Now, since

$$\begin{cases} (\phi_p)_*(y_p) \leqq b - \varepsilon \\ (\phi_p)^*(z_p) \geqq b + \varepsilon, \end{cases}$$

then $y_p, z_p \notin (x_p^* - a + \delta, x_p^* + a - \delta)$ for every fixed δ. Since $y_p < x_p^* < z_p$, thus choosing $0 < \delta \leqq a - r^*$ implies that

$$y_p \leqq x_p^* + r_i \leqq z_p \quad \text{for all } i.$$

Therefore,

$$\psi_p(x_p^* + r_i) \geqq \psi_p(x_p^*) = m_p. \tag{3.26}$$

Step 4.3. getting a contradiction
In this step, we assume that $m_p > 0$ (it will be shown in Step 5) and we want to get a contradiction. Set

$$k_i = \begin{cases} (\phi_p)_*(x_p^* + a + r_i) & \text{if } r_i \leqq 0 \\ \left(\phi_p\right)^* (x_p^* - a + r_i) & \text{if } r_i > 0. \end{cases}$$

Hence from (3.26) and using the monotonicity of F together with inequality (3.21), we get

$$0 \geqq F((a_i)_{i=0,\dots,N}) - F((c_i)_{i=0,\dots,N}),$$

where

$$a_i = \begin{cases} k_i & \text{if } r_i \leqq 0 \\ k_i + m_p & \text{if } r_i > 0 \end{cases} \quad \text{and} \quad c_i = \begin{cases} k_i - m_p & \text{if } r_i \leqq 0 \\ k_i & \text{if } r_i > 0. \end{cases}$$

Notice that

$$k_i \in [(\phi_p)^*(x_p^* - a), (\phi_p)_*(x_p^* + a)].$$

Therefore from (3.25) and the fact that $m_p \to 0$, we deduce that

$$a_i \to b \quad \text{and} \quad c_i \to b \quad \text{as } p \to 0.$$

Since F is C^1 near $\{b\}^{N+1}$ and $c_i + t(a_i - c_i) = c_i + tm_p$, then

$$\begin{aligned} 0 &\geqq \int_0^1 \mathrm{d}t \sum_{i=0}^{N} \left((a_i - c_i) \frac{\partial F}{\partial X_i}((c_j + t(a_j - c_j))_{j=0,\dots,N}) \right) \\ &= \int_0^1 \mathrm{d}t \sum_{i=0}^{N} \left(m_p \frac{\partial F}{\partial X_i}((c_j + tm_p)_{j=0,\dots,N}) \right). \end{aligned}$$

Since $m_p > 0$, we get

$$\begin{aligned} 0 &\geqq \int_0^1 \mathrm{d}t \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}((c_j + tm_p)_{j=0,\dots,N}) \\ &= f'(b) + \int_0^1 \mathrm{d}t \left(\sum_{i=0}^{N} \frac{\partial F}{\partial X_i}((c_j + tm_p)_{j=0,\dots,N}) - \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(b, \dots, b) \right). \end{aligned}$$

But F is C^1 near $\{b\}^{N+1}$ and $c_i + tm_p \to b$ for all i, thus

$$\int_0^1 \mathrm{d}t \left(\sum_{i=0}^{N} \frac{\partial F}{\partial X_i}((c_j + tm_p)_{j=0,\dots,N}) - \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(b, \dots, b) \right) \to 0 \quad \text{as } p \to 0.$$

This implies that

$$0 \geqq f'(b) > 0,$$

which is a contradiction because of Assumption (B).

Step 5. $m_p > 0$
We split this step into two cases:
Case 1. F is strongly increasing in some direction
Assume that F verifies, in addition,

$$\frac{\partial F}{\partial X_{i_1}} \geqq \delta_0 > 0, \tag{3.27}$$

for certain i_1 with $r_{i_1} > 0$ (assuming $r_{i_1} < 0$ being similar).
Assume to the contrary that $m_p = 0$. Thus

$$\psi_p(x_p^*) = (\phi_p)_*(x_p^* + a) - (\phi_p)^*(x_p^* - a) = 0.$$

Since ϕ_p is non-decreasing, then

$$\phi_p(x_p^*) = \phi_{p_{|(x_p^*-a, x_p^*+a)}} = k = \text{const},$$

where k is a zero of f, that is,

$$f(k) = 0. \tag{3.28}$$

Let $d \geqq x_p^* + a$ be the first real number such that

$$\phi_p(d+\eta) > k \quad \text{for every } \eta > 0.$$

Choose $0 < \eta < r_{i_1}$ and set

$$x_1 = d + \eta - r_{i_1}.$$

From the definition of d, we deduce that

$$\phi_p = k \quad \text{on a neighborhood of } x_1,$$

hence $\phi_p'(x_1) = 0$. Moreover, we have

$$\begin{cases} \phi_p(x_1 + r_i) \geqq k & \text{for all } i \neq i_1 \\ \phi_p(x_1 + r_{i_1}) = \phi_p(d+\eta) > k & \text{for } i = i_1, \end{cases}$$

therefore

$$\begin{aligned} 0 = c\phi_p'(x_1) &= F((\phi_p(x_1 + r_i))_{i=0,\dots,N}) \\ &\geqq F(k, \dots, \overbrace{\phi_p(x_1 + r_{i_1}))}^{i_1}, \dots, k) \\ &\geqq f(k) + \delta_0(\phi_p(d+\eta) - k) \\ &= \delta_0(\phi_p(d+\eta) - k) > 0, \end{aligned}$$

where we have used (3.28) for the last line. This is a contradiction.
Case 2. create the monotonicity
In fact, we can always assume hypothesis (3.27) for a modification F_p of F, where

$$F_p(X_0, X_1, \dots, X_N) = F(X_0, X_1, \dots, X_N) + p(X_{i_1} - X_0).$$

Then the whole construction works for F replaced by F_p with the additional monotonicity property (3.27) with $\delta_0 = p$. Once we pass to the limit $p \to 0$, we still get the same contradiction as in Step 4.3 and we recover the construction of traveling wave ϕ of (2.1) for the function F. □

4. Uniqueness of the Velocity c

In this section, we prove the uniqueness of the velocity of a traveling wave ϕ solution of (1.14) (part (a) of Theorem 1.5). We show in the first subsection a comparison principle on the half-line, and we prove the uniqueness of the velocity in the second subsection.

4.1. Comparison Principle on the Half-Line

In this subsection, we prove a comparison principle on the half-line that is essentially used to prove the uniqueness of the velocity (in the second subsection of this section) and the uniqueness of the profile ϕ that solves (1.14) (in Section 6).

Theorem 4.1. (Comparison principle on $(-\infty, r^*]$) *Let* $F : [0,1]^{N+1} \to \mathbb{R}$ *satisfying* (A) *and assume that*

$$\left| \begin{array}{l} \text{there exists } \eta_0 > 0 \text{ such that if} \\ Y = (Y_0, \dots, Y_N),\ Y + (a, \dots, a) \in [0, \eta_0]^{N+1} \\ \text{then } F(Y + (a, \dots, a)) < F(Y) \quad \text{if } a > 0. \end{array} \right. \tag{4.1}$$

Let $u, v : (-\infty, r^*] \to [0,1]$ *be, respectively, a sub- and a super-solution of*

$$cu'(x) = F((u(x + r_i))_{i=0,\dots,N}) \quad \text{on } (-\infty, 0) \tag{4.2}$$

in the sense of Definition 2.4. *Moreover, assume that*

$$u \leqq \beta_0 \quad \text{on } (-\infty, r^*]$$

and

$$u \leqq v \quad \text{on } [0, r^*].$$

Then

$$u \leqq v \quad \text{on } (-\infty, r^*].$$

Before giving the proof of this result, we give a corollary which is a comparison principle on $[-r^*, +\infty)$.

Corollary 4.2. (Comparison principle on $[-r^*, +\infty)$) *Let* $F : [0,1]^{N+1} \to \mathbb{R}$ *satisfying* (A) *and assume that:*

$$\left| \begin{array}{l} \text{there exists } \eta_0 > 0 \text{ such that if} \\ X = (X_0, \dots, X_N),\ X + (a, \dots, a) \in [1 - \eta_0, 1]^{N+1} \\ \text{then } F(X + (a, \dots, a)) < F(X) \text{ if } a > 0. \end{array} \right. \tag{4.3}$$

Let $u, v : [-r^*, +\infty) \to [0,1]$ *be, respectively, a sub- and a super-solution of* (4.2) *on* $(0, +\infty)$ *in sense of Definition* 2.4. *Moreover, assume that*

$$v \geqq 1 - \eta_0 \quad \text{on } [-r^*, +\infty),$$

and that

$$u \leqq v \quad on\ [-r^*, 0].$$

Then

$$u \leqq v \quad on\ [-r^*, +\infty).$$

Remark 4.3. (*Inverse monotonicity*) Notice that assumptions (4.1) and (4.3) are satisfied if F is C^1 on a neighborhood of $\{0\}^{N+1}$ and $\{1\}^{N+1}$ in $[0, 1]^{N+1}$ and $f'(0) < 0$, $f'(1) < 0$. This condition means that 0 and 1 are stable equilibria.

Lemma 4.4. (Transformation of a solution of (4.2)) *Let $u, v : (-\infty, r^*] \to [0, 1]$ be, respectively, a sub- and super-solution of* (4.2) *in the sense of Definition* 2.4. *Then*

$$\widehat{u}(x) := 1 - u(-x) \quad and \quad \widehat{v}(x) := 1 - v(-x)$$

are, respectively, a super- and a sub-solution of (4.2) *on* $[-r^*, +\infty)$ *with* F, c *and* r_i (*for all* $i \in \{0, \dots, N\}$) *replaced by* $\widehat{F}$, $\widehat{c}$ *and* $\widehat{r_i}$, *given by*

$$\begin{cases} \widehat{F}(X_0, \dots, X_N) = -F(1 - X_0, \dots, 1 - X_N) \\ \widehat{c} := -c \\ \widehat{r_i} := -r_i. \end{cases} \tag{4.4}$$

Moreover,

$$\widehat{F} : [0, 1]^{N+1} \to \mathbb{R}$$

satisfies (A), (B) *and* (C), *where* b *and* f *are replaced by*

$$\begin{cases} \widehat{b} := 1 - b \\ \widehat{f}(v) := -f(-v) \end{cases}$$

in (B).

Notice that, Lemma 4.4 is still true even though $u,\ v : \mathbb{R} \to [0, 1]$ are a sub and a super-solution of (4.2) on $\mathbb{R}$.

Proof of Lemma 4.4. Let $u : (-\infty, r^*] \to [0, 1]$ be a sub-solution of (4.2) and set $\widehat{u}(x) = 1 - u(-x)$. It is then easy to see that, in the viscosity sense,

$$\begin{aligned} c\widehat{u}'(x) = cu'(-x) &\leqq F((u(-x + r_i))_{i=0,\dots,N}) \\ &= F((1 - \widehat{u}(x - r_i))_{i=0,\dots,N}). \end{aligned}$$

Hence $\widehat{u}$ is a super-solution of (4.2) on $[-r^*, +\infty)$ with F, r_i and c replaced by $\widehat{F}, \widehat{r_i} := -r_i$ and $\widehat{c} := -c$ given in (4.4). Similarly, we show that $\widehat{v}$ is a sub-solution of the same equation on $[-r^*, +\infty)$. □

Proof of Corollary 4.2. Let $u, v : [-r^*, +\infty) \to [0, 1]$ be a sub and super-solution of (4.2) on $(0, +\infty)$ such that $v \geqq 1 - \beta_0$ on $[-r^*, +\infty)$. We set $\widehat{u}(x) = 1 - u(-x)$ and $\widehat{v}(x) = 1 - v(-x)$. It is then easy to see that $\widehat{u}, \widehat{v} \in [0, 1]$, $\widehat{v} \leqq \beta_0$ on $(-\infty, r^*]$.

Using Lemma 4.4, we show that $\widehat{u}$ and $\widehat{v}$ are respectively a super and a sub-solution of (4.2) with (F, c, r_i) replaced by $(\widehat{F}, \widehat{c}, \widehat{r_i})$ defined in (4.4). Moreover, using the fact that F satisfies (4.3), we deduce that $\widehat{F}$ satisfies (4.1).

We then deduce by Theorem 4.1 that

$$\widehat{v} \leqq \widehat{u} \quad \text{on } (-\infty, r^*]$$

that is

$$u \leqq v \quad \text{on } [-r^*, -\infty).$$

□

We now turn to the proof of Theorem 4.1.

Proof of Theorem 4.1. Let $u, v : (-\infty, r^*] \to [0, 1]$ be respectively a sub- and a super-solution of (4.2) such that

$$u \leqq \beta_0 \quad \text{on } (-\infty, r^*],$$

and $u \leqq v$ on $[0, r^*]$.

Step 0. Introduction

Set

$$\overline{v} := \min(v, \beta_0).$$

According to (4.1) we have

$$F(\beta_0, \ldots, \beta_0) < F(0, \ldots, 0) = f(0) = 0,$$

thus the constant β_0 is a super-solution of (4.2). Hence $\overline{v}$ is a super-solution of (4.2) on $(-\infty, 0)$ with $u \leqq \overline{v}$ on $[0, r^*]$. Moreover, since $\overline{v} \leqq v$, it is sufficient to prove the comparison principle (Theorem 4.1) between u and $\overline{v}$ which satisfy in addition $u, \ \overline{v} \in [0, \beta_0]$.

For simplicity, we note $\overline{v}$ as v with $u, v \in [0, \beta_0]$ and $u \leqq v$ on $[0, r^*]$.

Step 1. Doubling the variables

Suppose by contradiction that

$$M = \sup_{x \in (-\infty, r^*]} u(x) - v(x) > 0.$$

Let $\varepsilon, \ \alpha > 0$ and define

$$\begin{aligned} M_{\varepsilon,\alpha} &:= \sup_{x,\, y \in (-\infty, r^*]} \left(u(x) - v(y) - \frac{|x - y|^2}{2\varepsilon} - \alpha |x|^2 \right) \\ &= u(\overline{x}_\varepsilon) - v(\overline{y}_\varepsilon) - \frac{|\overline{x}_\varepsilon - \overline{y}_\varepsilon|^2}{2\varepsilon} - \alpha |\overline{x}_\varepsilon|^2, \end{aligned}$$

for certain $\overline{x}_\varepsilon,\ \overline{y}_\varepsilon \in (-\infty, -r^*]$. Note that the maximum is reached since the function

$$(x, y) \mapsto \psi(x, y) = u(x) - v(y) - \frac{|x-y|^2}{2\varepsilon} - \alpha|x|^2$$

is upper semi-continuous and satisfies $\psi(x, y) \to -\infty$ as $|x|, |y| \to +\infty$. Moreover, for all $\delta > 0$, there exists $x_\delta \in (-\infty, r^*]$ such that

$$M \geqq u(x_\delta) - v(x_\delta) \geqq M - \delta.$$

Hence

$$\begin{aligned} M_{\varepsilon,\alpha} &\geqq u(x_\delta) - v(x_\delta) - \alpha|x_\delta|^2 \\ &\geqq M - \delta - \alpha|x_\delta|^2 \\ &\geqq \frac{M}{2} > 0, \end{aligned}$$

for $\delta = \frac{M}{4}$ and α chosen small enough such that $\alpha < \frac{M}{4|x_\delta|^2}$. Moreover, since $u(\overline{x}_\varepsilon) - v(\overline{y}_\varepsilon) \leqq \beta_0$, we have

$$\frac{|\overline{x}_\varepsilon - \overline{y}_\varepsilon|^2}{2\varepsilon} + \alpha|\overline{x}_\varepsilon|^2 \leqq \beta_0. \tag{4.5}$$

Step 2. There exists α small enough and $\varepsilon \to 0$ such that $\overline{x}_\varepsilon \in [0, r^*]$ or $\overline{y}_\varepsilon \in [0, r^*]$ Assume that $\overline{x}_\varepsilon \in [0, r^*]$ (the case $\overline{y}_\varepsilon \in [0, r^*]$ being similar). Using (4.5), we deduce that $\overline{y}_\varepsilon \in [-\sqrt{2\beta_0\varepsilon}, r^*]$. Then $\overline{x}_\varepsilon$ and $\overline{y}_\varepsilon$ converge (up to a subsequence) to a certain $\overline{x}_0 \in [0, r^*]$ as $\varepsilon \to 0$ (from (4.5), the two limits coincide). We then deduce that

$$\begin{aligned} 0 < \frac{M}{2} &\leqq \limsup_{\varepsilon\to 0}(u(\overline{x}_\varepsilon) - v(\overline{y}_\varepsilon)) \\ &\leqq u(\overline{x}_0) - v(\overline{x}_0) \leqq 0, \end{aligned}$$

which is a contradiction. The last inequality takes place since $u \leqq v$ on $[0, r^*]$.
Step 3. For all α and ε small enough, we have $\overline{x}_\varepsilon,\ \overline{y}_\varepsilon \in (-\infty, 0)$
Step 3.1. Viscosity inequalities
We have

$$u(x) \leqq v(\overline{y}_\varepsilon) + M_{\varepsilon,\alpha} + \frac{|x - \overline{y}_\varepsilon|^2}{2\varepsilon} + \alpha|x|^2 := \phi(x),$$

and $u(\overline{x}_\varepsilon) = \phi(\overline{x}_\varepsilon)$. Thus

$$c\left(\frac{\overline{x}_\varepsilon - \overline{y}_\varepsilon}{\varepsilon} + 2\alpha\overline{x}_\varepsilon\right) = c\phi'(\overline{x}_\varepsilon) \leqq F((u(\overline{x}_\varepsilon + r_i))_{i=0,\dots,N}). \tag{4.6}$$

Similarly, we get

$$c\left(\frac{\overline{x}_\varepsilon - \overline{y}_\varepsilon}{\varepsilon}\right) \geqq F((v(\overline{y}_\varepsilon + r_i))_{i=0,\dots,N}). \tag{4.7}$$

Subtracting (4.7) from (4.6) implies that

$$2c\alpha\overline{x}_\varepsilon \leqq F((u(\overline{x}_\varepsilon + r_i))_{i=0,\dots,N}) - F((v(\overline{y}_\varepsilon + r_i))_{i=0,\dots,N}). \tag{4.8}$$

Note that from (4.5)

$$\alpha|\overline{x}_\varepsilon| \leqq \sqrt{\alpha\beta_0}.$$

This implies that for ε fixed, $\alpha\overline{x}_\varepsilon \to 0$ as $\alpha \to 0$.
Step 3.2. Passing to the limit $\alpha \to 0$
We have

$$u(x) - v(y) - \frac{|x-y|^2}{2\varepsilon} - \alpha|x|^2 \leqq u(\overline{x}_\varepsilon) - v(\overline{y}_\varepsilon) - \frac{|\overline{x}_\varepsilon - \overline{y}_\varepsilon|^2}{2\varepsilon} - \alpha|\overline{x}_\varepsilon|^2.$$

Set

$$\left|\begin{aligned} u_i^\alpha &= u(\overline{x}_\varepsilon + r_i) \\ v_i^\alpha &= v(\overline{y}_\varepsilon + r_i), \end{aligned}\right.$$

then

$$\begin{cases} u_i^\alpha \leqq v_i^\alpha + m_\alpha + \delta_i^\alpha & \text{if } i \neq 0 \\ u_0^\alpha = v_0^\alpha + m_\alpha & \text{if } i = 0, \end{cases}$$

where $m_\alpha = u_0^\alpha - v_0^\alpha$ and $\delta_i^\alpha = 2\alpha\overline{x}_\varepsilon r_i + \alpha|r_i|^2$. For ε fixed, since $u_i^\alpha,\ v_i^\alpha \in [0, \beta_0]$ and $\frac{M}{2} \leqq m_\alpha \leqq \beta_0$, we deduce that as $\alpha \to 0$ and up to a subsequence,

$$\begin{cases} u_i^\alpha \to u_i^0 \\ v_i^\alpha \to v_i^0 \\ m_\alpha \to m_0 \\ \delta_i^\alpha \to 0, \end{cases}$$

with $u_i^0,\ v_i^0 \in [0, \beta_0]$, $0 < \frac{M}{2} \leqq m_0 \leqq \beta_0$ and

$$\begin{cases} u_i^0 \leqq v_i^0 + m_0 & \text{if } i \neq 0 \\ u_0^0 = v_0^0 + m_0 & \text{if } i = 0. \end{cases}$$

Moreover, passing to the limit in (4.8) implies that

$$0 \leqq F((u_i^0)_{i=0,\dots,N}) - F((v_i^0)_{i=0,\dots,N}). \tag{4.9}$$

Step 4. Getting a contradiction
We claim that for all i, there exists $l_i,\ l_i' \geqq 0$ such that

$$u_i^0 + l_i = v_i^0 - l_i' + m_0, \tag{4.10}$$

and

$$\begin{cases} \overline{u}_i^0 := u_i^0 + l_i \leqq \beta_0 \\ \overline{v}_i^0 := v_i^0 - l_i' \geqq 0. \end{cases}$$

Recall that for all $i \in \{0, \dots, N\}$, we have

$$\begin{cases} u_i^0,\ v_i^0 \in [0, \beta_0] \\ u_i^0 \leqq v_i^0 + m_0 \\ u_0^0 - v_0^0 = m_0 \leqq \beta_0. \end{cases}$$

If for some i, $u_i^0 = v_i^0 + m_0$, then it suffices to take $l_i = l_i' = 0$. Assume then that $u_i^0 < v_i^0 + m_0$.

Case 1. $u_i^0,\ v_i^0 \in (v_0^0, u_0^0)$

Set $l_i = u_0^0 - u_i^0$ and $l_i' = v_i^0 - v_0^0$. Then

$$\begin{cases} \overline{u}_i^0 = u_i^0 + l_i = u_0^0 \leqq \beta_0 \\ \overline{v}_i^0 = v_i^0 - l_i' = v_0^0 \geqq 0, \end{cases}$$

and $\overline{u}_i^0 = \overline{v}_i^0 + m_0$.

Case 2. $u_i^0 > u_0^0$ and $v_i^0 > v_0^0$

Since $u_i^0 - v_0^0 > m_0$, then there exists $l_i' < v_i^0 - v_0^0$ such that

$$u_i^0 = v_i^0 - l_i' + m_0$$

and $\overline{v}_i^0 = v_i^0 - l_i' > v_0^0 \geqq 0$. Thus, it is sufficient to take $l_i = 0$.

Case 3. $u_i^0 < u_0^0$ and $v_i^0 < v_0^0$

This case can be treated as Case 2 by taking $l_i' = 0$ and $l_i < u_0^0 - u_i^0$.

Finally, going back to (4.9), since F is non-decreasing, we deduce that

$$\begin{aligned} 0 &\leqq F((u_i^0)_{i=0,\dots,N}) - F((v_i^0)_{i=0,\dots,N}) \\ &\leqq F((\overline{u}_i^0)_{i=0,\dots,N}) - F((\overline{v}_i^0)_{i=0,\dots,N}) \\ &= F((\overline{u}_i^0)_{i=0,\dots,N}) - F((\overline{u}_i^0 - m_0)_{i=0,\dots,N}) \\ &< 0. \end{aligned}$$

The last inequality takes place since F satisfies (4.1) for $\overline{u}_i^0,\ \overline{u}_i^0 - m_0 \in [0, \beta_0]$ and $m_0 > 0$. Therefore, we get a contradiction. □

4.2. Uniqueness of the Velocity

This subsection is devoted to proving the uniqueness of the velocity c of a traveling wave that solves (1.14).

Proposition 4.5. (Uniqueness of c)
Under assumption (A), *consider the function F defined on* $[0, 1]^{N+1}$. *Let* (c_j, ϕ_j) *be a solution of* (1.14) *for* $j = 1, 2$. *If, in addition, F satisfies* (C), *then* $c_1 = c_2$.

Proof of Proposition 4.5. Suppose that for $j = 1, 2$, (c_j, ϕ_j) is a solution of (1.14) and assume by contradiction that $c_1 < c_2$. We have,

$$\phi_j(-\infty) = 0 \quad \text{and} \quad \phi_j(+\infty) = 1.$$

We set $\delta = \min(\beta_0, \frac{1}{4})$ where β_0 is given in Assumption (C). Up to translation of ϕ_1 and ϕ_2, we can assume that

$$\phi_1(x) \geqq 1 - \delta \quad \forall x \geqq -r^*$$

and

$$\phi_2(x) \leqq \delta \quad \forall x \leqq r^*.$$

This implies that

$$\phi_2 \leqq \phi_1 \text{ over } [-r^*, r^*].$$

Moreover, since $c_1 < c_2$, we have

$$c_1\phi_2'(x) \leqq c_2\phi_2'(x) = F((\phi_2(x + r_i))_{i=0,\dots,N}).$$

Hence (c_1, ϕ_2) is a sub-solution of (1.14). Since

$$\phi_1 \geqq 1 - \delta \quad \text{on } [-r^*, +\infty),$$

we deduce using Corollary 4.2 that

$$\phi_2 \leqq \phi_1 \text{ over } [-r^*, +\infty).$$

Similarly, since

$$\phi_2 \leqq \delta \quad \text{on } (-\infty, r^*],$$

we deduce using Theorem 4.1 that

$$\phi_2 \leqq \phi_1 \text{ over } (-\infty, r^*].$$

Therefore,

$$\phi_2 \leqq \phi_1 \quad \text{over } \mathbb{R}.$$

Next, set

$$\left|\begin{aligned} &u_1(t, x) = \phi_1(x + c_1 t)\\ &u_2(t, x) = \phi_2(x + c_2 t), \end{aligned}\right.$$

then for $j = 1, 2$, we have

$$\partial_t u_j(t, x) = F((u_j(t, x + r_i))_{i=0,\dots,N}). \tag{4.11}$$

Moreover, at time $t = 0$, we have

$$u_1(0, x) = \phi_1(x) \geqq \phi_2(x) = u_2(0, x) \quad \text{over } \mathbb{R},$$

thus applying the comparison principle for Equation (4.11) (see [17]), we get

$$u_1 \geqq u_2 \quad \forall t \geqq 0 \quad \forall\, x \in \mathbb{R}.$$

Taking $x = y - c_1 t$, we get

$$\phi_1(y) \geqq \phi_2(y + (c_2 - c_1)t), \quad \forall\, t \geqq 0, \ \forall\, y \in \mathbb{R}.$$

Using that $c_1 < c_2$, and passing to the limit $t \to +\infty$, we get

$$\phi_1(y) \geqq \phi_2(+\infty) = 1, \quad \forall\, y \in \mathbb{R}.$$

But $\phi_1(-\infty) = 0$, hence there is a contradiction. Therefore $c_1 \geqq c_2$. Similarly, we show that $c_2 \geqq c_1$, hence $c_1 = c_2$. $\square$

5. Asymptotics for the Profile

In this section, our main result is the asymptotics near $\pm\infty$ for solutions $\phi : \mathbb{R} \to [0,1]$ of

$$c\phi'(x) = F((\phi(x + r_i))_{i=0,\dots,N}) \quad \text{on } \mathbb{R}, \tag{5.1}$$

namely Proposition 5.1.

Proposition 5.1. (Asymptotics near $\pm\infty$) *Consider a function F defined on $[0,1]^{N+1}$ satisfying* (A) *and* (C)*, and assume that $c \neq 0$. Then*

(i) **asymptotics near $-\infty$**
Let $\phi : \mathbb{R} \to [0,1]$ be a solution of (5.1)*, satisfying*

$$\phi(-\infty) = 0 \quad \textit{and} \quad \phi \geqq \delta > 0 \quad \textit{on } [0, r^*]$$

for some $\delta > 0$ and assume (E+)(ii)*. If there exists a unique $\lambda^+ > 0$ solution of*

$$c\lambda = \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0, \dots, 0) e^{\lambda r_i}, \tag{5.2}$$

then for any sequence $(x_n)_n$, $x_n \to -\infty$, there exists a subsequence $(x_{n'})_{n'}$ and $A > 0$ such that

$$\frac{\phi(x + x_{n'})}{e^{\lambda^+ x_{n'}}} \longrightarrow A e^{\lambda^+ x} \quad \textit{locally uniformly on } \mathbb{R} \textit{ as } n' \to +\infty.$$

(ii) **asymptotics near $+\infty$**
Let $\phi : \mathbb{R} \to [0,1]$ be a solution of (5.1)*, satisfying*

$$\phi(+\infty) = 1 \quad \textit{and} \quad \phi \leqq 1 - \delta < 1 \quad \textit{on } [0, r^*]$$

for some $\delta > 0$ and assume (E−)(ii)*. If there exists a unique $\lambda^- < 0$ solution of*

$$c\lambda = \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(1, \dots, 1) e^{\lambda r_i}, \tag{5.3}$$

then for any sequence $(x_n)_n$, $x_n \to +\infty$, there exists a subsequence $(x_{n'})_{n'}$ and $A > 0$ such that

$$\frac{1 - \phi(x + x_{n'})}{e^{\lambda^- x_{n'}}} \longrightarrow A e^{\lambda^- x} \quad \textit{locally uniformly on } \mathbb{R} \textit{ as } n' \to +\infty.$$

5.1. Uniqueness and Existence of $\lambda^{\pm}$

In this subsection, we address the question of the existence and uniqueness of $\lambda^{\pm}$.

Lemma 5.2. (Uniqueness and existence of λ^{+}) *Assume* (A) *and suppose that* $\nabla F(0)$ *exists with* $f'(0) < 0$. *Then there is at most one solution* $\lambda^{+} > 0$ *of* (5.2). *Moreover, if* $c < 0$ *or, if we assume* (E+)(i), *then there exists a (unique) solution* $\lambda^{+} > 0$ *of* (5.2).

Proof of Lemma 5.2. *Step 1*. Uniqueness
Let

$$g(\lambda) := \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0, \dots, 0)e^{\lambda r_i} - c\lambda. \tag{5.4}$$

Because of Assumption (A), the function g is convex and

$$g(0) = f'(0) < 0.$$

Thus, there exists at most one solution $\lambda^{+} > 0$ of (5.2) and if λ^{+} exists, then we have

$$g < 0 \quad \text{on } (0, \lambda^{+}) \quad \text{and} \quad g > 0 \quad \text{on } (\lambda^{+}, +\infty). \tag{5.5}$$

Step 2. Existence
Assume $c < 0$. We have

$$g(\lambda) \geqq \frac{\partial F}{\partial X_0}(0, \dots, 0) - c\lambda,$$

which implies that $\lim_{\lambda \to +\infty} g(\lambda) = +\infty$. On the other hand, if we assume (E+)(i), then

$$g(\lambda) \geqq \frac{\partial F}{\partial X_0}(0, \dots, 0) + \frac{\partial F}{\partial X_{i_+}}(0, \dots, 0)e^{\lambda r_{i_+}} - c\lambda,$$

which implies that $\lim_{\lambda \to +\infty} g(\lambda) = +\infty$.

Therefore, there exists a unique $\lambda^{+} > 0$ such that $g(\lambda^{+}) = 0$. $\square$

In the same way (or using Lemma 4.4), we can prove the following lemma concerning λ^{-}

Lemma 5.3. (Uniqueness and existence of λ^{-})
Assume (A) *and suppose that* $\nabla F(1, \dots, 1)$ *exists with* $f'(1) < 0$. *Then there is at most one solution* $\lambda^{-} < 0$ *of* (5.3). *Moreover, if* $c > 0$ *or, if we assume* (E−)(i), *then there exists a* (*unique*) *solution* $\lambda^{-} < 0$ *of* (5.3).

5.2. Proof of Proposition 5.1

In this subsection, we prove that any solution of (5.1) is exponentially bounded (from above and below) near $-\infty$. Finally, we prove Proposition 5.1(i).

Lemma 5.4. (Exponential bounds for a solution of (5.1) near $-\infty$) *Assume* (A), (C) *and* (E+)(ii). *Let* $\phi : (-\infty, 0] \to [0, 1]$ *be a solution of* (5.1) *on* $(-\infty, -r^*)$ *satisfying* $\phi(-\infty) = 0$ *and assume that there exists* $\lambda^+ > 0$, *a solution of* (5.2). *Then there exists* k_2 *such that*

$$\phi(x) \leqq k_2 e^{\lambda^+ x} \quad \textit{for all } x \leqq 0.$$

Moreover, if

$$\phi \geqq \delta > 0 \quad \textit{on} \quad [-r^*, 0] \quad \textit{for some } \delta > 0, \tag{5.6}$$

then there exists $k_1 > 0$ *such that*

$$k_1 e^{\lambda^+ x} \leqq \phi(x) \quad \textit{for all } x \leqq 0.$$

Remark 5.5. Notice that the exponential bounds of Lemma 5.4 do not hold if we do not assume (E+)(ii). To show this, it suffices to define $f(u) = -u'$ with $u(x) = -xe^x$. A simple computation then gives that

$$\begin{cases} f(0) = 0 \\ f'(0) = -1 \\ f'(u) - f'(0) \sim_{u \to 0} \frac{-1}{\ln u}, \end{cases}$$

and so f does not satisfy (E+)(ii) and u is not exponentially bounded.

Proof of Lemma 5.4. The idea of the proof is to construct a sub- and super-solution of

$$c\phi'(x) = F((\phi(x + r_i))_{i=0,\ldots,N}) \quad \text{on } (-\infty, -r^*), \tag{5.7}$$

then, using the comparison principle (Theorem 4.1), we deduce the existence of k_1 and k_2. Let $\lambda^+ > 0$ be the solution of (5.2) and consider the perturbation $\lambda^+ < \lambda' < (1 + \alpha)\lambda^+$ with α given in Assumption (E+)(ii).
Step 1. existence of k_1
Step 1.1. construction of a sub-solution of (5.7)
Set

$$\underline{\phi}(x) = A\left(e^{\lambda^+ x} + e^{\lambda' x}\right)$$

defined on $(-\infty, 0]$, where $A > 0$ will be chosen such that $\underline{\phi}$ is a sub-solution of (5.7). Since λ^+ is a solution of (5.2), then for $x \in (-\infty, -r^*)$ we have

$$\begin{aligned} c\underline{\phi}'(x) &= c\lambda^+ A e^{\lambda^+ x} + cA\lambda' e^{\lambda' x} \\ &= \nabla F(0, \ldots, 0).((Ae^{\lambda^+(x+r_i)})_{i=0,\ldots,N}) + cA\lambda' e^{\lambda' x} \\ &= \nabla F(0, \ldots, 0).((\underline{\phi}(x + r_i))_{i=0,\ldots,N}) - Ae^{\lambda' x}\big(\nabla F(0, \ldots, 0) \\ &\quad \times ((e^{\lambda' r_i})_{i=0,\ldots,N}) - c\lambda'\big) \\ &\leqq F((\underline{\phi}(x + r_i))_{i=0,\ldots,N}) + C_0|\Phi(x)|^{1+\alpha} - Ae^{\lambda' x} g(\lambda'), \end{aligned}$$

where for the last line we have used (E+)(ii), $\Phi(x) = ((\underline{\phi}(x + r_i))_{i=0,\dots,N})$ and g as defined in (5.4). Using the fact that for $x \in (-\infty, -r^*)$, we have $\underline{\phi}(x + r_i) \leqq 2Ae^{\lambda^+(x+r^*)}$. Therefore, we get

$$\begin{aligned} c\underline{\phi}'(x) &- F((\underline{\phi}(x + r_i))_{i=0,\dots,N}) \\ &\leqq A\left(2^{1+\alpha} C_0 A^\alpha e^{(1+\alpha)\lambda^+(x+r^*)} |E|^{1+\alpha} - e^{\lambda' x} g(\lambda')\right) \\ &\leqq A\left(2^{1+\alpha} C_0 A^\alpha e^{(1+\alpha)\lambda^+ r^*} |E|^{1+\alpha} - g(\lambda')\right) e^{\lambda' x}, \end{aligned}$$

with $E = (1, \dots, 1) \in \mathbb{R}^{N+1}$. Since $g(\lambda') > 0$ (see (5.5)),

$$c\underline{\phi}'(x) - F((\underline{\phi}(x + r_i))_{i=0,\dots,N}) \leqq 0 \quad \text{for } A \text{ small enough.}$$

This shows that $\underline{\phi}$ is a sub-solution of (5.7) on $(-\infty, -r^*)$.
Step 1.2. applying the comparison principle
Up to a decreasing $A > 0$, let us assume that $2A \leqq \min(\delta, \beta_0)$ with δ given in (5.6) and β_0 given in Assumption (C) (this is possible since A can be chosen as small as we want). Thus

$$\phi \geqq \delta \geqq 2A \geqq \underline{\phi} \quad \text{on } [-r^*, 0]$$

and

$$\underline{\phi} \leqq 2A \leqq \beta_0 \quad \text{on } (-\infty, 0].$$

Hence using the comparison principle (Theorem 4.1 and a shift of the functions), we deduce that

$$\underline{\phi}(x) \leqq \phi(x) \quad \text{for all } x \leqq 0.$$

This implies that ϕ satisfies

$$k_1 := A \leqq \frac{\phi(x)}{e^{\lambda^+ x}} \quad \text{for all } x \leqq 0.$$

Step 2. existence of k_2
Step 2.1. construction of a super-solution of (5.7)
Define for $x \in (-\infty, 0]$ the function

$$\overline{\phi}(x) = A\left(2e^{\lambda^+ x} - e^{\lambda' x}\right).$$

Repeating the same proof as in Step 1, we get

$$\begin{aligned} c\overline{\phi}'(x) &- F((\overline{\phi}(x + r_i))_{i=0,\dots,N}) \\ &\geqq A\left(-2^{1+\alpha} C_0 A^\alpha e^{(1+\alpha)\lambda^+(x+r^*)} |E|^{1+\alpha} + e^{\lambda' x} g(\lambda')\right) \\ &\geqq A\left(-2^{1+\alpha} C_0 A^\alpha e^{(1+\alpha)\lambda^+ r^*} |E|^{1+\alpha} + g(\lambda')\right) e^{\lambda' x}, \end{aligned}$$

with $E = (1, \dots, 1) \in \mathbb{R}^{N+1}$. Again, since $g(\lambda') > 0$, then $\overline{\phi}$ is a super-solution of (5.7) for $A > 0$ small enough.

Step 2.2. applying the comparison principle
Define, for $a > 0$ large enough, the function $\tilde{\phi}(x) = \phi(x - a)$ such that

$$\tilde{\phi} \leqq \min\left(\beta_0, Ae^{-\lambda^+ r^*}\right) \quad \text{on } (-\infty, 0],$$

with β_0 given in Assumption (C). This is possible because we assume that $\phi(-\infty) = 0$. Thus

$$\tilde{\phi} \leqq Ae^{-\lambda^+ r^*} \leqq \overline{\phi} \quad \text{on } [-r^*, 0].$$

Hence, applying the comparison principle result (Theorem 4.1, up to a shift of the functions), we deduce that

$$\tilde{\phi} \leqq \overline{\phi} \quad \text{on } (-\infty, 0].$$

This implies that

$$\frac{\phi(x)}{e^{\lambda^+ x}} \leqq 2Ae^{\lambda^+ a} \quad \text{for all } x \leqq -a.$$

Using the fact that $\phi \leqq 1$, we get

$$\frac{\phi(x)}{e^{\lambda^+ x}} \leqq k_2 \quad \text{for all } x \leqq 0,$$

where $k_2 := \max\left(2Ae^{\lambda^+ a}, \max_{x\in[-a,0]} \frac{\phi(x)}{e^{\lambda^+ x}}\right)$. □

We only prove Proposition 5.1(i) (the proof of Proposition 5.1(ii) being similar).

Proof of Proposition 5.1(i). Let $\phi : [0, 1] \to \mathbb{R}$ be a solution of (5.1) such that

$$\phi(-\infty) = 0 \quad \text{and} \quad \phi \geqq \delta \quad \text{for some } \delta > 0.$$

We recall, from Lemma 5.4, that

$$0 < k_1 \leqq \frac{\phi(x)}{e^{\lambda^+ x}} \leqq k_2 < +\infty \quad \text{for all } x \leqq 0, \tag{5.8}$$

where λ^+ is the solution of (5.2).
Step 1. Shifting and rescaling ϕ
For a sequence $x_n \to -\infty$ and for all $x \leqq 0$, define the function v_n as

$$v_n(x - x_n) := \frac{\phi(x)}{e^{\lambda^+ x}}.$$

For $i = 0, \ldots, N$, we have

$$\begin{aligned} c\phi'(x) &= ce^{\lambda^+ x}(v_n'(x - x_n) + \lambda^+ v_n(x - x_n)) \\ &= F((v_n(x + r_i - x_n)e^{\lambda^+(x+r_i)})_i). \end{aligned} \tag{5.9}$$

That is, for $y = x - x_n$,

$$\begin{aligned} c(v_n'(y) + \lambda^+ v_n(y)) &= e^{-\lambda^+(y+x_n)} F((v_n(y+r_i)e^{\lambda^+(y+x_n+r_i)})_i) \\ &= e^{-\lambda^+(y+x_n)}[F((v_n(y+r_i)e^{\lambda^+(y+x_n+r_i)})_i) \\ &\quad - \nabla F(0).((v_n(y+r_i)e^{\lambda^+(y+x_n+r_i)})_i)] \\ &\quad + \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0) v_n(y+r_i) e^{\lambda^+ r_i}. \end{aligned}$$

From Assumption (E+)(ii), we then have

$$\begin{aligned} c(v_n'(y) + \lambda^+ v_n(y)) = &\sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0) v_n(y+r_i) e^{\lambda^+ r_i} \\ &+ O\left(e^{-\lambda^+(y+x_n)} |(v_n(y+r_i)e^{\lambda^+(y+x_n+r_i)})_i|^{1+\alpha}\right), \end{aligned}$$

that is,

$$\begin{aligned} c(v_n'(y) + \lambda^+ v_n(y)) \ = &\sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0) v_n(y+r_i) e^{\lambda^+ r_i} \\ &+ O\left(e^{\lambda^+ \alpha (y+x_n)} |(v_n(y+r_i)e^{\lambda^+ r_i})_i|^{1+\alpha}\right). \end{aligned} \tag{5.10}$$

Step 2. Passing to the limit $n \to +\infty$
Because of (5.8), we have

$$0 < k_1 \leqq v_n(y) \leqq k_2 < +\infty \quad \text{for } y \leqq -x_n, \tag{5.11}$$

and for any compact set $K \subset \mathbb{R}$

$$e^{\lambda^+ \alpha (y+x_n)} |(v_n(y+r_i)e^{\lambda^+ r_i})_i|^{1+\alpha} \to 0 \quad \text{as } n \to +\infty \quad (\text{because } x_n \to -\infty)$$

uniformly in $y \in K$. As $c \neq 0$, we get from (5.10) that there exists some $C_K > 0$ (independent of n) such that

$$|v_n'| \leqq C_K \quad \text{on } K.$$

Applying Ascoli's theorem, there exists a subsequence $v_{n'}$ such that

$$v_{n'} \longrightarrow v_\infty \quad \text{locally uniformly on } \mathbb{R}.$$

Moreover, v_∞ satisfies

$$c(v_\infty'(y) + \lambda^+ v_\infty(y)) = \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0) v_\infty(y+r_i) e^{\lambda^+ r_i}, \tag{5.12}$$

and (using (5.11))

$$k_1 \leqq v_\infty \leqq k_2 \quad \text{on } \mathbb{R}. \tag{5.13}$$

Step 3. Applying Fourier transform
Applying a Fourier transform to (5.12) implies that

$$\hat{v}_\infty(\xi)G(\xi) = 0,$$

where $G(\xi) = c(i\xi + \lambda^+) - \sum_{j=0}^{N} \frac{\partial F}{\partial X_j}(0, \ldots, 0)e^{\lambda^+ r_j} e^{i\xi r_j}$.
Step 3.1. $G(\xi) = 0 \iff \xi = 0$
Clearly, if $\xi = 0$ then $G(\xi) = 0$ (because λ^+ solves (5.2)).
Assume that $G(\xi) = 0$ with $\xi \in \mathbb{R}$. Hence

$$c\lambda^+ = \sum_{j=0}^{N} \frac{\partial F}{\partial X_j}(0, \ldots, 0)e^{\lambda^+ r_j} \cos(\xi r_j) \tag{5.14}$$

and

$$c\xi = \sum_{j=0}^{N} \frac{\partial F}{\partial X_j}(0, \ldots, 0)e^{\lambda^+ r_j} \sin(\xi r_j). \tag{5.15}$$

Using the fact that $\frac{\partial F}{\partial X_j}(0) \geqq 0$ for $j \neq 0$, we deduce from (5.2) and (5.14) that for all $j \in \{1, \ldots, N\}$, we have

$$\begin{cases} \dfrac{\partial F}{\partial X_j}(0, \ldots, 0) = 0 \\ \text{or} \\ \xi r_j = 0 \mod(2\pi) \quad \text{and} \quad \dfrac{\partial F}{\partial X_j}(0) > 0. \end{cases} \tag{5.16}$$

Substituting (5.16) in (5.15), taking into consideration that $r_0 = 0$, implies that $c\xi = 0$ and thus $\xi = 0$, because $c \neq 0$.
Step 3.2. $v_\infty = \text{const}$
From step 3.1, we deduce that $\operatorname{supp}\{\hat{v}\} \subset \{0\}$. Therefore,

$$\hat{v}(0) = \sum_{\text{finite}} c_k \delta_0^{(k)}.$$

The inverse Fourier transform implies that v_∞ is a polynomial. But v_∞ is bounded (see (5.13)), hence

$$v_\infty = \text{const} := A.$$

Consequently,

$$\frac{\phi(x + x_{n'})}{e^{\lambda^+(x+x_{n'})}} = v_{n'}(x) \to A.$$

□

6. Uniqueness of the Profile and Proof of Theorem 1.5

In this section, we prove the uniqueness of the profile (under Assumption (D) or (E)). Under Assumption (D) we will use a Strong Maximum Principle, while under Assumption (E) we will need the asymptotics joined with a Half Strong Maximum Principle (just on the half-line, see Lemma 6.1). We show, in the first subsection, three different kinds of Strong Maximum Principle satisfied by (1.14) when $c \neq 0$. In the second subsection, we prove the uniqueness of the profile and Theorem 1.5.

6.1. Different Kinds of Strong Maximum Principle

Here, we prove three different kinds of Strong Maximum Principle for (1.14) when $c \neq 0$. We also add a technical lemma (Lemma 6.5) that allows us to compare two different solutions on $\mathbb{R}$ with at least one contact point.

We prove the Strong Maximum Principle (Lemmas 6.1, 6.3 and 6.4) for $c > 0$. However, when $c < 0$, the corresponding results can be deduced from the case $c > 0$ using the transformation of Lemma 4.4.

Lemma 6.1. (Half Strong Maximum Principle) *Let $F : [0, 1]^{N+1} \to \mathbb{R}$ satisfying Assumption* (A) *and let $\phi_1, \phi_2 : \mathbb{R} \to [0, 1]$ be a viscosity sub- and a super-solution of* (2.2)*, respectively. Assume that*

$$\begin{cases} \phi_2 \geqq \phi_1 & \text{on } \mathbb{R} \\ \phi_2(0) = \phi_1(0). \end{cases}$$

If $c > 0$ (resp. $c < 0$), then

$$\phi_1 = \phi_2 \quad \text{for all } x \leqq 0 \quad (\text{resp. } x \geqq 0).$$

Proof of Lemma 6.1. Assume that $c > 0$ and let $w(x) := \phi_2(x) - \phi_1(x)$. Since ϕ_2 is a super-solution and ϕ_1 is a sub-solution of (2.2), then using the doubling of variable method we show that w is a viscosity super-solution of

$$cw'(x) \geqq F((\phi_2(x + r_i))_{i=0,\dots,N}) - F((\phi_1(x + r_i))_{i=0,\dots,N}) \quad \text{on } \mathbb{R}.$$

But F is non-decreasing with respect to X_i for all $i \neq 0$, thus we get

$$cw'(x) \geqq F(\phi_1(x) + w(x), (\phi_1(x + r_i))_{i=1,\dots,N}) - F((\phi_1(x + r_i))_{i=0,\dots,N}).$$

Now, since F is globally Lipschitz, then

$$w'(x) \geqq \frac{-L}{c} w(x), \tag{6.1}$$

where L is the Lipschitz constant of F.

Notice that $y(x) = w(x_0)e^{\frac{-L}{c}(x-x_0)}$ satisfies the equality in inequality (6.1) for any $x_0 \in \mathbb{R}$. As $y(x_0) = w(x_0)$, then using the comparison principle for the ODE (6.1), we deduce that

$$w(x) \geqq w(x_0)e^{\frac{-L}{c}(x-x_0)} \quad \text{for all } x \geqq x_0. \tag{6.2}$$

If $w(x_0) > 0$, then $w(x) > 0$ for all $x \geqq x_0$. This implies that

$$\phi_2 > \phi_1 \quad \text{for all } x \geqq x_0.$$

Finally, since $\phi_2(0) = \phi_1(0)$, then we deduce that

$$\phi_2 = \phi_1 \quad \text{for all } x \leqq 0,$$

(otherwise, if there is $x_1 < 0$ such that $\phi_2(x_1) > \phi_1(x_1)$, then from the above argument, we deduce that $\phi_2(0) > \phi_1(0)$, a contradiction). □

Lemma 6.2. (Strong Maximum Principle under (D±)(ii)) *Let* $F : [0,1]^{N+1} \to \mathbb{R}$ *satisfying* (A). *Let* $\phi_1, \phi_2 : \mathbb{R} \to [0,1]$, *respectively, be a viscosity sub- and super-solution of* (2.2) *such that*

$$\phi_2 \geqq \phi_1 \quad \text{on } \mathbb{R} \quad \text{and} \quad \phi_2(0) = \phi_1(0).$$

(*a*) *If* F *is increasing with respect to* X_{i_0} *for certain* $i_0 \neq 0$ *then*

$$\phi_2(kr_{i_0}) = \phi_1(kr_{i_0}) \quad \text{for all } k \in \mathbb{N}.$$

(*b*) *If we assume, moreover, that* F *satisfies* (D+)(ii) *if* $c > 0$, *or* (D−)(ii) *if* $c < 0$, *then*

$$\phi_1(x) = \phi_2(x) \quad \text{for all } x \in \mathbb{R}.$$

Proof of Lemma 6.2. (*a*) Assume for simplicity that $i_0 = 1$. Let ϕ_1, ϕ_2, respectively, be a viscosity sub-solution and a viscosity super-solution of (2.2). Then using the doubling of variable method, we can show that the function $w = \phi_2 - \phi_1$ satisfies

$$cw'(x) \geqq F((\phi_2(x+r_i))_{i=0,\dots,N}) - F((\phi_1(x+r_i))_{i=0,\dots,N}) \quad \text{on } \mathbb{R} \tag{6.3}$$

in the viscosity sense. As w is a viscosity super-solution of (6.3), $w(0) = 0$ and $w \geqq 0$ on $\mathbb{R}$, we deduce that

$$0 \geqq F((\phi_2(r_i))_{i=0,\dots,N}) - F((\phi_1(r_i))_{i=0,\dots,N}) \quad \text{at } x = 0.$$

Thus, using the fact that $\phi_2(0) = \phi_1(0)$ and that F is monotone with respect to X_i for all $i \neq 0$, we get

$$F((\phi_2(r_i))_{i=0,\dots,N}) = F((\phi_1(r_i))_{i=0,\dots,N}).$$

Next, since F is increasing with respect to X_1, we deduce that

$$\phi_2 = \phi_1 \quad \text{at } x = r_1,$$

(otherwise, $F((\phi_2(r_i))_{i=0,\dots,N}) > F((\phi_1(r_i))_{i=0,\dots,N})$, because F is non-decreasing with respect to X_i for $i \neq 0, 1$ and increasing with respect to X_1). Therefore, upon repeating the above argument for $x = r_1$, we show that

$$\phi_2(kr_1) = \phi_1(kr_1) \quad \text{for all } k \in \mathbb{N}.$$

(*b*) Assume that $c > 0$ and that F satisfies (D+)(ii) (the other case being similar). By contradiction, suppose that there exists $x \in \mathbb{R}$ such that $\phi_1(x) < \phi_2(x)$. Let $k \in \mathbb{N}$ big enough such that $kr_{i_+} > x$. Then, using Lemma 6.1 (up to shifting the functions), and the fact that $\phi_1(kr_{i_+}) = \phi_2(kr_{i_+})$, we get that $\phi_1(x) = \phi_2(x)$, which is a contradiction. □

Lemma 6.3. (Comparison principle, under (D±)(i)) *Assume that* $c > 0$ (*resp.* $c < 0$) *and let* F *satisfy* (A) *and* (D+)(i) (*resp.* (D−)(i)). *Let* ϕ_1, ϕ_2 *be, respectively, a viscosity sub- and a viscosity super-solution of* (2.2). *Assume that* $\phi_1(0) = \phi_2(0)$ *and*

$$\phi_1 \leqq \phi_2 \quad on\ [-r^*, 0] \quad (resp.\ on\ [0, r^*]),$$

then

$$\phi_1(x) \leqq \phi_2(x) \quad for\ all\ x \geqq -r^* \quad (resp.\ x \leqq r^*).$$

Proof of Lemma 6.3. Assume that $c > 0$ (the case $c < 0$ being similar). If $r^* = 0$, then the result follows from the comparison principle for ODEs.

Let us assume that $r^* > 0$. Since $\phi_1 \leqq \phi_2$ on $[-r^*, 0]$ and $r_i < 0$ for all $i \neq 0$ (see Assumption (D+)(i)), then for all $x \in [0, \min_{i\neq 0}(-r_i)]$, the function $w(x) := \phi_1(x) - \phi_2(x)$ satisfies (in the viscosity sense)

$$\begin{aligned} cw'(x) &\leqq F((\phi_1(x + r_i))_{i=0,\dots,N}) - F((\phi_2(x + r_i))_{i=0,\dots,N}) \\ &\leqq F(w(x) + \phi_2(x), (\phi_2(x + r_i))_{i\neq 0}) - F((\phi_2(x + r_i))_{i=0,\dots,N}) \\ &\leqq L|w(x)| \quad \text{(because } F \text{ is } L-\text{Lipschitz)}, \end{aligned}$$

where we have used in the second line the fact that $\phi_1(x + r_i) \leqq \phi_2(x + r_i)$ for $i \neq 0$, because $-r^* \leqq x + r_i \leqq 0$ for all $i \neq 0$. But $w(0) = 0$ and $y \equiv 0$ is a solution of $cw' = L|w|$, then using the comparison principle of the ODE, we deduce that

$$w \leqq 0 \quad \text{for all } x \in [0, \min_{i\neq 0}(-r_i)].$$

This implies that

$$\phi_1 \leqq \phi_2 \quad \text{for all } x \in [0, \min_{i\neq 0}(-r_i)].$$

Finally, the result of this lemma ($\phi_1 \leqq \phi_2$ for all $x \geqq -r^*$) follows by repeating the above argument several times, each on the new extended interval. □

Lemma 6.4. (Strong Maximum principle under (D±)(i)) *Assume* $c > 0$ (*resp.* $c < 0$) *and let* F *satisfy* (A) *and* (D+)(i) (*resp.* (D−)(i)). *Let* ϕ_1, ϕ_2 *be two solutions of* (2.2) *such that*

$$\phi_1(0) = \phi_2(0) \quad and \quad \phi_1 \leqq \phi_2 \quad on\ \mathbb{R}.$$

Then

$$\phi_1(x) = \phi_2(x) \quad for\ all\ x \in \mathbb{R}.$$

Proof of Lemma 6.4. Let $c > 0$ (the case $c < 0$ is deduced from the case $c > 0$ using Lemma 4.4). Using Lemma 6.1, we deduce that

$$\phi_1 = \phi_2 \quad \text{for all } x \leqq 0.$$

Thus, it is sufficient to prove that $\phi_1 \geqq \phi_2$ for all $x \geqq 0$ (because $\phi_1 \leqq \phi_2$ for $x \geqq 0$). We have,

$$\phi_1(0) = \phi_2(0) \quad \text{and} \quad \phi_1 \geqq \phi_2 \quad \text{on } [-r^*, 0] \quad (\text{since } \phi_1 = \phi_2\ \forall x \leqq 0),$$

and ϕ_2, ϕ_1, respectively, are a viscosity sub- and super-solution of (2.2). Using the comparison principle (Lemma 6.3), we deduce that

$$\phi_1 \geqq \phi_2 \quad \text{for all } x \geqq -r^*.$$

Therefore, $\phi_1(x) = \phi_2(x)$ for all $x \in \mathbb{R}$. □

Lemma 6.5. (Ordering two solutions of (1.14) up to translation) *Assume that* $c \neq 0$ *and let* $F : [0, 1]^{N+1} \to \mathbb{R}$ *satisfying* (A) *and* (C). *Let* ϕ_1 *and* ϕ_2 *be two solutions of* (1.14). *There exists a shift* $a^* \in \mathbb{R}$ *and some* $x_0 \in [-r^*, r^*]$ *such that* $\phi_2^{a^*}(x) := \phi_2(x + a^*)$ *and* ϕ_1 *satisfy*

$$\begin{cases} \phi_2^{a^*} \geqq \phi_1 & \text{on } \mathbb{R} \\ \phi_2^{a^*}(x_0) = \phi_1(x_0). \end{cases}$$

Proof of Lemma 6.5. The idea of the proof is to translate ϕ_2 and then to compare the translation with ϕ_1.
Step 1. Family of solutions above ϕ_1
For $a \in \mathbb{R}$, let us define

$$\phi_2^a(x) := \phi_2(x + a).$$

For some $a > 0$ large enough, (because of the conditions at $\pm\infty$ in (1.14)), we have

$$\phi_2^{\overline{a}} \geqq \phi_1 \quad \text{on } [-r^*, r^*] \quad \text{for all } \overline{a} \geqq a,$$

and then using the comparison principle (Theorem 4.1 and Corollary 4.2), we deduce that for all $\overline{a} \geqq a$, we have

$$\phi_2^{\overline{a}} \geqq \phi_1 \quad \text{on } \mathbb{R}.$$

Step 2. There exists a^* such that $\phi_2^{a^*}$ and ϕ_1 touch at $x_0 \in [-r^*, r^*]$
Let

$$a^* = \inf\{a \in \mathbb{R}, \quad \phi_2^{\overline{a}} \geqq \phi_1 \quad \text{on } \mathbb{R} \quad \text{for all } \overline{a} \geqq a\}.$$

Recall that $c \neq 0$ and then $\phi_i \in C^1(\mathbb{R})$ for $i = 1, 2$.

Assume by contradiction that

$$\inf_{[-r^*, r^*]} \left(\phi_2^{a^*} - \phi_1\right) \geqq \delta > 0.$$

Then for all $0 \leqq \varepsilon \leqq \varepsilon_0$ with ε_0 small enough, we have

$$\phi_2^{a^*-\varepsilon} - \phi_1 \geqq 0 \quad \text{on } [-r^*, r^*].$$

Applying the comparison principle (Theorem 4.1 and Corollary 4.2), we get

$$\phi_2^{a^*-\varepsilon} - \phi_1 \geqq 0 \quad \text{on } \mathbb{R},$$

which is a contradiction with the definition of a^*. Thus

$$\inf_{[-r^*,r^*]} \phi_2^{a^*} - \phi_1 = 0.$$

Hence, there exists $x_0 \in [-r^*, r^*]$ such that

$$\phi_2^{a^*} = \phi_1 \quad \text{at } x_0,$$

knowing that $\phi_2^{a^*}(x) \geqq \phi_1(x)$ for all $x \in \mathbb{R}$. □

6.2. Proof of Theorem 1.5(b)

We devote this subsection to the proof of the uniqueness of the profile by means of several lemmas. The proof of Theorem 1.5 is given at the end of this subsection. All the proofs are made in the case $c > 0$, since the case $c < 0$ is similar (or is deduced using Lemma 4.4).

Lemma 6.6. (Uniqueness of the profile, under (E+)) *Assume that $c > 0$ and let F satisfy* (A), (C) *and* (E+). *Let $\phi : \mathbb{R} \to [0, 1]$ be a solution of* (1.14), *then ϕ is unique up to a space translation. Moreover, ϕ is non-decreasing.*

Proof of Lemma 6.6. Assume that $c > 0$ and let $\phi_1, \phi_2 : \mathbb{R} \to [0, 1]$ be two solutions of (1.14). The goal of the proof is to show that there exists a translation $\phi_2^{a^*}$ of ϕ_2 such that $\phi_2^{a^*} = \phi_1$. To simplify the notation, we denote r_{i_+} (introduced in (E+)) by r_1.
Step 1. Constructing a translation and applying Lemma 6.1
Using Lemma 6.5, there exists $a^* \in \mathbb{R}$ such that the translation $\phi_2^{a^*}$ of ϕ_2 satisfies:

$$\begin{cases} \phi_2^{a^*} \geqq \phi_1 & \text{on } \mathbb{R} \\ \phi_2^{a^*}(x_0) = \phi_1(x_0). \end{cases} \tag{6.4}$$

Since $c > 0$, then applying Lemma 6.1, we deduce that

$$\phi_2^{a^*} = \phi_1 \quad \text{for all } x \leqq x_0. \tag{6.5}$$

Step 2. Asymptotics of ϕ_1 and $\phi_2^{a^*}$
Using Lemma 5.2 and Proposition 5.1, we get that there exists a subsequence (n') of $(n)_{n\in\mathbb{N}}$ (because $x_0 - nr_1 \to -\infty$ as $n \to +\infty$) and two constants $A_1, A_2 > 0$ such that

$$\frac{\phi_2^{a^*}(x_0 - n'r_1 + x)}{e^{\lambda^+(x_0-n'r_1+x)}} \to A_1 \quad \text{locally uniformly on } \mathbb{R}. \tag{6.6}$$

$$\frac{\phi_1(x_0 - n'r_1 + x)}{e^{\lambda^+(x_0-n'r_1+x)}} \to A_2 \quad \text{locally uniformly on } \mathbb{R}.$$

Using Equation (6.5), we deduce that $A_1 = A_2 := A$.
Step 3. Exchange ϕ_1 and ϕ_2
Applying Lemma 6.5, upon exchanging ϕ_1 and ϕ_2, we deduce that there exists $b^* \geqq 0$ and y_0 such that

$$\begin{cases} \phi_1^{b^*}(x) := \phi_1(x + b^*) \geqq \phi_2 & \text{on } \mathbb{R} \\ \phi_1^{b^*}(y_0) = \phi_2(y_0). \end{cases}$$

Moreover, from Lemma 6.1, we get

$$\phi_1^{b^*}(x) = \phi_2(x) \quad \text{for all } x \leqq y_0 \quad (\text{since } c > 0).$$

Now, using and Lemma 5.2 and Proposition 5.1 and since $y_0 - n'r_1 \to -\infty$ as $n' \to +\infty$, we get the existence of a subsequence of (n') (still denoted by (n')) such that

$$\frac{\phi_1^{b^*}(y_0 - n'r_1 + x)}{e^{\lambda^+(y_0-n'r_1+x)}}, \quad \frac{\phi_2(y_0 - n'r_1 + x)}{e^{\lambda^+(y_0-n'r_1+x)}} \to B \quad \text{locally uniformly on } \mathbb{R}. \tag{6.7}$$

Step 4. Conclusion, $\phi_1 = \phi_2^{a^*}$
For any fixed $x \in \mathbb{R}$, we have

$$\frac{\phi_2(x_0 + a^* - n'r_1 + x)}{e^{\lambda^+(x_0-n'r_1+x)}} \to A, \tag{6.8}$$

$$\frac{\phi_1(x_0 - n'r_1 + x)}{e^{\lambda^+(x_0-n'r_1+x)}} \to A, \tag{6.9}$$

$$\frac{\phi_1(y_0 + b^* - n'r_1 + x)}{e^{\lambda^+(y_0-n'r_1+x)}} \to B \tag{6.10}$$

and

$$\frac{\phi_2(y_0 - n'r_1 + x)}{e^{\lambda^+(y_0-n'r_1+x)}} \to B. \tag{6.11}$$

For $x = y_0 + b^*$, Equation (6.9) implies that

$$\frac{\phi_1(x_0 - n'r_1 + y_0 + b^*)}{e^{\lambda^+(x_0-n'r_1+y_0)}} \to Ae^{\lambda^+ b^*}.$$

Also, Equation (6.10) with $x = x_0$ implies that

$$\frac{\phi_1(x_0 - n'r_1 + y_0 + b^*)}{e^{\lambda^+(x_0 - n'r_1 + y_0)}} \to B,$$

thus

$$Ae^{\lambda^+ b^*} = B.$$

Similarly, if we substitute $x = y_0$ in (6.8) and $x = x_0 + a^*$ in (6.11), we show that

$$A = Be^{\lambda^+ a^*}.$$

Therefore,

$$a^* = -b^*.$$

But

$$\phi_2^{a^*}(x) = \phi_2(x + a^*) \geqq \phi_1(x)$$

and

$$\phi_1^{b^*}(x) = \phi_1(x + b^*) = \phi_1(x - a^*) \geqq \phi_2(x),$$

hence we get

$$\phi_2(x + a^*) = \phi_1(x).$$

Moreover, $\phi_2(x + a) \geqq \phi_1(x)$ for all $a \geqq a^*$, which shows that the profile is nondecreasing. □

Lemma 6.7. (Uniqueness of the profile, under (D+)(i) or (ii)) *Assume that* $c > 0$ *and let* F *satisfy* (A) *and* (C). *Let* $\phi : \mathbb{R} \to [0, 1]$ *be a solution of* (1.14). *If, in addition,* F *satisfies* (D+)(i) *or* (ii), *then* ϕ *is unique up to a space translation.*

Proof of Lemma 6.7. The proof follows from Lemma 6.5 and the Strong Maximum Principle (Lemma 6.4 or Lemma 6.2). □

Lemma 6.8. (Monotonicity of the profile) *Assume that* $c > 0$ (*resp.* $c < 0$) *and let* $F : [0, 1]^{N+1} \to \mathbb{R}$ *satisfying* (A) *and* (C). *Let* $\phi : \mathbb{R} \to [0, 1]$ *be a solution of* (1.14). *If* F *satisfies* (D+)(i) *or* (ii) *or* (E+) (*resp.* (D−)(i) *or* (ii) *or* (E−)), *then* $\phi' > 0$ *on* $\mathbb{R}$.

Proof of Lemma 6.8. Assume that $c > 0$ (the proof when $c < 0$ being similar) and let ϕ be a solution of (1.14).
Step 1. ϕ is non-decreasing
The goal is to show that $\phi(x + a) \geqq \phi(x)$ for all $a \geqq 0$. As in the proof of Lemma 6.5, we deduce that for $a \geqq 0$ large enough and for all $\overline{a} \geqq a$, we have

$$\phi^{\overline{a}}(x) := \phi(x + \overline{a}) \geqq \phi(x) \quad \text{on } [-r^*, r^*].$$

Thus, using the comparison principle (Theorem 4.1 and Corollary 4.2), we deduce that for all $\overline{a} \geqq a$, we have

$$\phi^{\overline{a}}(x) \geqq \phi(x) \quad \text{on } \mathbb{R}.$$

Set

$$a^* = \inf\{a \geqq 0,\ \phi^{\overline{a}}(x) \geqq \phi(x) \text{ on } \mathbb{R} \quad \text{for all } \overline{a} \geqq a\}.$$

We want to prove that $a^* = 0$. By definition of a^*, there exists some x_0 such that

$$\begin{cases} \phi^{a^*} \geqq \phi & \text{on } \mathbb{R} \\ \phi^{a^*}(x_0) = \phi(x_0). \end{cases} \tag{6.12}$$

Case 1. F satisfies (E+)
From Lemma 6.6, ϕ is nondecreasing and then $a^* = 0$.
Case 2. F satisfies (D+)(i) or (ii)
Using (6.12) and the Strong Maximum Principle (Lemma 6.2 or Lemma 6.4), we get that $\phi^{a^*} = \phi$, that is, ϕ is periodic of period a^*. But $\phi(-\infty) = 0$ and $\phi(+\infty) = 1$, thus $a^* = 0$.
Step 2. ϕ is increasing
Let $a > 0$, we want to show that $\phi(x + a) > \phi(x)$. From Step 1, we have $\phi(x + a) \geqq \phi(x)$. Assume that there exists x_0 such that

$$\phi(x_0 + a) = \phi(x_0).$$

Repeating the same argument, as in Step 1, under (D+)(i) or (ii) or (E+), we prove that $a = 0$, which is a contradiction. Thus

$$\phi(x + a) > \phi(x) \quad \text{on } \mathbb{R} \quad \text{for any } a > 0.$$

Step 3. $\phi' > 0$
For $a > 0$, we define

$$w_a(x) = \frac{\phi(x + a) - \phi(x)}{a}.$$

Using the same arguments as in the proof of Lemma 6.1 (see (6.2)), we get that for all $x_0 \in \mathbb{R}$

$$w_a(x) \geqq w_a(x_0)e^{\frac{-L}{c}(x-x_0)} \quad \text{for all } x \geqq x_0.$$

Passing to the limit $a \to 0$, we get that

$$\phi'(x) \geqq \phi'(x_0)e^{\frac{-L}{c}(x-x_0)} \geqq 0 \quad \text{for all } x \geqq x_0. \tag{6.13}$$

By contradiction, assume that there exists x_1 such that $\phi'(x_1) = 0$. This implies that

$$\phi'(x) = 0 \quad \text{for all } x \leqq x_1. \tag{6.14}$$

Indeed, if there exists $x_0 < x_1$ such that $\phi'(x_0) > 0$, then (6.13) implies that

$$\phi'(x_1) \geqq \phi'(x_0) e^{\frac{-L}{c}(x_1 - x_0)} > 0,$$

which is a contradiction.

But ϕ is increasing so (6.14) is a contradiction and so $\phi' > 0$. □

Proof of Theorem 1.5. (a) Uniqueness of the velocity

The proof of the uniqueness of the velocity follows from Proposition 4.5 in Section 4.

(b) Uniqueness of the profile and strict monotonicity

The uniqueness and the strict monotonicity of the solution when $c > 0$ is shown in Lemmas 6.6, 6.7 and 6.8. However the case $c < 0$ is a consequence of Lemma 4.4 and the previous results. □

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Appendix A: Construction of a Monotone Lipschitz Continuous Periodic Extension of F

We devote this Appendix to the proof of Lemma 2.1. To this end, we need to start with two useful results about the orthogonal projection. For any convex set K in $\mathbb{R}^d$ and for any $y \in \mathbb{R}^d$, we call

$$Proj_{|K}(y)$$

the orthogonal projection of y on K.

Lemma A.1. (Characterization of the orthogonal projection) *Let* $N \geqq 1$ *and* $y = (y_1, \ldots, y_N) \in \mathbb{R}^N$. *Then*

$$\mathrm{Proj}_{|_{[0,1]^N}}(y) = ((\mathrm{Proj}_{|_{[0,1]}}(y_i))_{i=1,\ldots,N}).$$

Proof of Lemma A.1. Let $y = (y_1, \ldots, y_N) \in \mathbb{R}^N$ and set $y_i^0 = \mathrm{Proj}_{|_{[0,1]}}(y_i)$. By definition of the orthogonal projection, we have

$$(y_i - y_i^0, \overline{y}_i - y_i^0) \leqq 0 \quad \forall \, \overline{y}_i \in [0, 1].$$

This implies that

$$(y - y^0, \overline{y} - y^0) \leqq 0 \quad \forall \, \overline{y} = (\overline{y}_1, \ldots, \overline{y}_N) \in [0, 1]^N, \tag{A.1}$$

with $y^0 = (y_1^0, \dots, y_N^0)$. But (A.1) is a characterization of the orthogonal projection of y on $[0,1]^N$, thus

$$y^0 = \mathrm{Proj}_{|_{[0,1]^N}}(y).$$

$\square$

Using the above lemma, one can easily check the following consequences:

Corollary A.2. (Ordering and a kind of linearity) *Let* $y = (y_1, \dots, y_N), z = (z_1, \dots, z_N) \in \mathbb{R}^N$ *and set* $e = (1, \dots, 1) \in \mathbb{R}^N$. *Assume that*

$$y \geqq z$$

in the sense that $y_i \geqq z_i$ *for all* $i \in \{1, \dots, N\}$. *Let* $Q_0 = [0,1]^N$, *then*

(i) Order preservation

We have

$$Proj_{|_{Q_0}}(y) \geqq Proj_{|_{Q_0}}(z).$$

(ii) "Linearity"

We have

$$Proj_{|_{Q_0}}(y+e) = Proj_{|_{Q_0-e}}(y) + e,$$

where $Q_0 - e = [-1,0]^N$.

After these preliminary results, we now go back to the proof of Lemma 2.1.

Proof of Lemma 2.1. The proof is split into two main steps. In the first step (the main part of the proof), we construct the extension $\tilde{F}$ of F on $[0,1] \times \mathbb{R}^N$. In the second step, we extend $\tilde{F}$ on the whole $\mathbb{R} \times \mathbb{R}^N$. The function $\tilde{F}$ that we want to construct must satisfy

$$\begin{cases} \tilde{F}_{|_Q} = F & \text{for } Q := [0,1]^{N+1} \\ \tilde{F}(X+E) = \tilde{F}(X) & \text{with } E = (1,\dots,1) \in \mathbb{R}^{N+1}. \end{cases}$$

This implies that for any $y \in Q_0 = [0,1]^N$ and $e = (1, \dots, 1) \in \mathbb{R}^N$, we have (see Fig. 1)

$$\begin{cases} \tilde{F}(1, y+e) = \tilde{F}(0, y) = F(0, y) \\ \tilde{F}(0, y-e) = \tilde{F}(1, y) = F(1, y). \end{cases}$$

Step 1. extension on $[0,1] \times \mathbb{R}^N$
Recall that $Q_0 = [0,1]^N$, $e = (1, \dots, 1) \in \mathbb{R}^N$, then set

$$Q_{-1} := Q_0 - e \quad \text{and} \quad Q_1 := Q_0 + e.$$

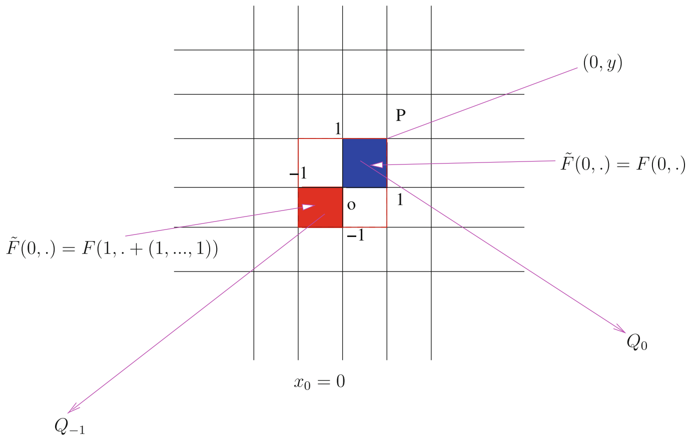

Fig. 1. A cut of $\{x_0 = 0\} \times \mathbb{R}^N$

We first define the auxiliary functions G_α on $[0, 1] \times Q_\alpha$ for $\alpha = -1, 0, 1$. For $y = (y_1, \ldots, y_N) \in \mathbb{R}^N$, we set

$$\left|\begin{array}{ll} G_0(x, y) = F(x, y) & \text{for all} \quad (x, y) \in [0, 1] \times Q_0 \\ G_{-1}(x, y) = F(1, y + e) - F(1, e) & \text{for all} \quad (x, y) \in [0, 1] \times Q_{-1} \\ G_1(x, y) = F(0, y - e) - F(0, 0) & \text{for all} \quad (x, y) \in [0, 1] \times Q_1. \end{array}\right. \tag{A.2}$$

By construction and by using (Assumption (A)), we note that G_α is Lipschitz continuous and non-decreasing with respect to y_i for all $i \in \{1, \ldots, N\}$ on $[0, 1] \times Q_\alpha$, for $\alpha = -1, 0, 1$. Moreover, we have

$$\left|\begin{array}{ll} G_{-1}(x, 0) = 0 & \text{for all } x \in [0, 1] \\ G_1(x, e) = 0 & \text{for all } x \in [0, 1]. \end{array}\right. \tag{A.3}$$

Now, for every $y \in \mathbb{R}^N$, we set for each $\alpha = -1, 0, 1$,

$$Y_\alpha(y) = \mathrm{Proj}_{|Q_\alpha}(y).$$

Then we define the extension G of F on $[0, 1] \times \mathbb{R}^N$ by:

$$G(x, y) = G_0(x, Y_0(y)) + (1 - x)G_{-1}(x, Y_{-1}(y)) + xG_1(x, Y_1(y)).$$

Clearly, because of (A.3), we have

$$G(x, y) = F(x, y) \quad \text{for any } (x, y) \in [0, 1] \times Q_0.$$

Step 1.1. $G(0, z) = G(1, z + e)$ for any $z \in \mathbb{R}^N$.
From the definition of G, we have for any $z \in \mathbb{R}^N$

$$G(1, z) = G_0(1, Y_0(z)) + G_1(1, Y_1(z))$$
$$G(0, z) = G_0(0, Y_0(z)) + G_{-1}(0, Y_{-1}(z)).$$

Therefore,

$$
\begin{aligned}
G(1, z+e) &= G_0(1, Y_0(z+e)) + G_1(1, Y_1(z+e)) \\
&= G_0(1, Y_{-1}(z)+e) + G_1(1, Y_0(z)+e) \\
&= F(1, Y_{-1}(z)+e) + F(0, Y_0(z)) - F(0,0) \\
&= F(1, Y_{-1}(z)+e) + G_0(0, Y_0(z)) - F(1,e) \\
&= G_0(0, Y_0(z)) + G_{-1}(0, Y_{-1}(z)) \\
&= G(0, z),
\end{aligned}
$$

where the second equality follows from Corollary A.2(ii), while the third follows from (A.2) and the fourth follows from the fact that $F(1, e) = F(0, 0)$.
Step 1.2. $G(x, y)$ is monotone in y_i
The result of this step follows from the fact that the orthogonal projection preserves the order (Corollary A.2(i)) and that for any $\alpha = -1, 0, 1$, G_α is non-decreasing on $[0, 1] \times Q_\alpha$ with respect to y_i for all $i \in \{1, \ldots, N\}$.
Step 1.3. G is globally Lipschitz
Let $(x, y),\ (\overline{x}, \overline{y}) \in [0, 1] \times \mathbb{R}^N$, then

$$
\begin{aligned}
|G(x,y) - G(\overline{x},\overline{y})| \leqq\ & |G_0(x, Y_0(y)) - G_0(\overline{x}, Y_0(\overline{y}))| \\
& + |\overline{x} - x|.|G_{-1}(x, Y_{-1}(y))| \\
& + |1 - \overline{x}|.|G_{-1}(x, Y_{-1}(y)) - G_{-1}(\overline{x}, Y_{-1}(\overline{y}))| + |x \\
& - \overline{x}|.|G_1(x, Y_1(y))| \\
& + |\overline{x}|.|G_1(x, Y_1(y)) - G_1(\overline{x}, Y_1(\overline{y}))|.
\end{aligned}
$$

Since for $\alpha = -1, 0, 1$, the functions G_α are Lipschitz continuous and bounded on $[0, 1] \times Q_\alpha$ and using the fact that the orthogonal projection is 1-Lipschitz, we conclude that

$$
|G(x,y) - G(\overline{x},\overline{y})| \leqq M|(x - \overline{x}, y - \overline{y})|,
$$

where $M = L_0 + L_{-1} + L_1 + M_{-1} + M_1$, with L_α is the Lipschitz constant of G_α, M_α the L^∞ norm of G_α for $\alpha = -1, 0, 1$.
Step 2. extension on $\mathbb{R} \times \mathbb{R}^N$
Let $k \in \mathbb{Z}$ and set

$$
\tilde{F}(x+k, y+ke) = G(x,y) \quad \text{for all } (x,y) \in [0,1] \times \mathbb{R}^N.
$$

First of all, $\tilde{F}$ is well defined because of Step 1.1. Moreover by construction, we have the periodicity property

$$
\tilde{F}(x+1, y+e) = \tilde{F}(x,y) \quad \text{for any } (x,y) \in \mathbb{R} \times \mathbb{R}^N.
$$

In addition, $\tilde{F}$ is Lipschitz continuous, non-decreasing in each y_i for $i \in \{1, \ldots, N\}$. $\square$

Appendix B: Proof of Miscellaneous Properties of Monotone Functions

Appendix B is dedicated to the proof of some results about monotone functions, namely Lemmas 2.10 and 2.11
We first prove Lemma 2.10.

Proof of Lemma 2.10. Assume for simplicity that $g = 0$ on $\mathring{I}$. Suppose to the contrary that there exists a closed interval $I_0 \subset \mathring{I}$, $\delta > 0$ and a subsequence $x_{n_j} \in I_0$ with $x_{n_j} \to x_0 \in I_0$ such that

$$|g_{n_j}(x_{n_j})| \geqq \delta.$$

Assume that $g_{n_j}(x_{n_j}) \geqq \delta$ (the case $g_{n_j}(x_{n_j}) \leqq -\delta$ being similar). Set $\varepsilon > 0$ and consider a closed interval I_ε such that $I_0 \subset\subset I_\varepsilon \subset \mathring{I}$. Since $g_{n_j}(x)$ is non-decreasing in x, then

$$g_{n_j}(x) \geqq \delta \quad \text{for all } x \in (I_\varepsilon \backslash I_0) \cap (\{x \geqq x_{n_j}\}) := I_+.$$

Choose $x_1 \in I_+$ such that $g_{n_j}(x_1) \to g(x_1)$ (g_n converges almost everywhere on I_+). Thus

$$0 = g(x_1) \geqq \delta > 0,$$

a contradiction. □

Now, we give the proof of Lemma 2.11. To this end, we recall and prove the following result:

Lemma B.1. (Properties of monotone functions) *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a non-decreasing function.*

(i) Countable set of jumps:

The set

$$\mathcal{S} = \{x \ \textit{such that } \phi \textit{ is discontinuous at } x\} \tag{B.1}$$

is at most countable.

(ii) Density of test points:

Let $x_0 \in \mathbb{R}$, there exists a sequence of functions $\psi_n^+ \in C^\infty(\mathbb{R})$ (resp. $\psi_n^- \in C^\infty(\mathbb{R})$) and a real sequence $(x_n^+)_n$ (resp. $(x_n^-)_n$) such that

$$x_n^+ \to x_0 \quad (\textit{resp. } x_n^- \to x_0)$$

and $\phi^* - \psi_n^+$ (resp. $\phi_* - \psi_n^-$) attains a local maximum (resp. a local minimum) at x_n^+ (resp. at x_n^-) for all n.

The meaning of point (ii) is that the set of points where ϕ^* is tested (in the sense of Definition 2.4) from above (resp. ϕ_* is tested from below) is dense in $\mathbb{R}$.

Proof of Lemma B.1. (a) Proof of (i):
This is classical.

(b) Proof of (ii) for ϕ^*:

Let $x_0 \in \mathbb{R}$. We want to prove that there exists $\psi_n \in C^\infty(\mathbb{R})$ and $x_n \to x_0$ such that $\phi^* - \psi_n$ reaches a local maximum at x_n. For every $\varepsilon > 0$ and for any $b \in \mathbb{R}$, we define the test function

$$\psi_n^b = \frac{1}{\varepsilon}\left(x - \left(x_0 + \frac{1}{n}\right)\right)^2 + b,$$

then we set

$$\beta = \inf \mathcal{E} \quad \text{for } \mathcal{E} = \left\{ b \in \mathbb{R}, \quad \psi_n^b(x) \geqq \phi^*(x) \quad \forall\, x \in \left[x_0, x_0 + \frac{2}{n}\right]\right\}.$$

Indeed, since ϕ^* is locally bounded (because ϕ is a real non-decreasing function) and $\mathcal{E}$ is bounded from below (by definition of $\mathcal{E}$), then $\mathcal{E} \neq \emptyset$. From the definition of β, there always exists $x_n \in [x_0, x_0 + \frac{2}{n}]$ such that

$$\psi_n^\beta(x_n) = \phi^*(x_n) \quad \text{and} \quad \psi_n^\beta(x) \geqq \phi^*(x) \quad \text{on } I = \left[x_0, x_0 + \frac{2}{n}\right]. \tag{B.2}$$

We want to show that x_n belongs to the interior of I (at least for ε large enough). We have

$$\psi_n^\beta(x_0) = \frac{1}{\varepsilon n^2} + \beta > \beta = \psi_n^\beta\left(x_0 + \frac{2}{n}\right) \geqq \phi^*\left(x_0 + \frac{2}{n}\right) \geqq \phi^*(x_0), \tag{B.3}$$

the last two inequalities are true because of (B.2) and the fact that ϕ^* is non-decreasing respectively. Assuming

$$\frac{1}{\varepsilon} > n^2\left(\phi^*\left(x_0 + \frac{2}{n}\right) - \phi^*(x_0)\right),$$

we get

$$\begin{aligned}\psi_n^\beta\left(x_0 + \frac{2}{n}\right) &> \phi^*\left(x_0 + \frac{2}{n}\right) - \phi^*(x_0) + \beta \\ &\geqq \phi^*\left(x_0 + \frac{2}{n}\right),\end{aligned}$$

where the last inequality follows from (B.3). This implies that $\phi^* - \psi_n^\beta$ reaches a local maximum at $x_n \in (x_0, x_0 + \frac{2}{n})$ and $x_n \to x_0$ as $n \to +\infty$.

(c) Proof of (ii) for ϕ_*:

Applying argument (b) for $\phi(x)$ replaced by $-\phi(-x)$, we get the result. □

Proof of Lemma 2.11. We set

$$\mathcal{T} = \bigcup_{i=0}^{N} (\mathcal{S} - \{r_i\})$$

with $\mathcal{S}$ defined in (B.1). Using Lemma B.1(i), we get that $\mathcal{T}$ is countable.
Step 1. viscosity sense implies almost everywhere sense
Assume that ϕ is a viscosity solution of (2.8) (see Definition 2.4) and let $x_0 \in \mathbb{R}\backslash\mathcal{T}$.

By definition, ϕ is continuous at $x_0 + r_i$ for all $i = 0, \ldots, N$. There exist two sequences of real numbers $(x_n^+)_n$ and $(x_n^-)_n$ such that ϕ^* is tested from above at x_n^+ and ϕ_* is tested from below at x_n^- by smooth functions (the sets of such points is dense in $\mathbb{R}$ (by Lemma B.1(ii)), and such that

$$\lim_{n\to+\infty} x_n^{\pm} = x_0.$$

Moreover, from Definition 2.4, we have

$$0 \leqq F((\phi^*(x_n^+ + r_i))_{i=0,\ldots,N}) \tag{B.4}$$

and

$$0 \geqq F((\phi_*(x_n^- + r_i))_{i=0,\ldots,N}). \tag{B.5}$$

Now, using the fact that

$$\lim_{n\to+\infty} \phi^*(x_n^+ + r_i) = \phi(x_0 + r_i) \quad \text{for } i = 0, \ldots, N,$$

and that F is Lipschitz continuous (see $(\tilde{A})$), we pass to the limit $n \to +\infty$ in (B.4), and we get

$$\begin{aligned} 0 &\leqq \limsup_{n\to+\infty} F((\phi^*(x_n^+ + r_i))_{i=0,\ldots,N}) \\ &\leqq F((\phi(x_0 + r_i))_{i=0,\ldots,N}). \end{aligned}$$

Similarly, we show that

$$\begin{aligned} 0 &\geqq \liminf_{n\to+\infty} F((\phi_*(x_n^- + r_i))_{i=0,\ldots,N}) \\ &\geqq F((\phi(x_0 + r_i))_{i=0,\ldots,N}). \end{aligned}$$

Thus

$$0 = F((\phi(x_0 + r_i))_{i=0,\ldots,N}),$$

hence ϕ solves Equation (2.8) at x_0. But $x_0 \in \mathbb{R}\backslash\mathcal{T}$ is arbitrary, thus ϕ solves (2.8) pointwisely on $\mathbb{R}\backslash\mathcal{T}$. Since $\mathcal{T}$ is countable, we get that ϕ satisfies (2.8) almost everywhere.

Step 2. almost everywhere sense implies viscosity sense
Let $x_0 \in \mathbb{R}$. We want to show that ϕ is a viscosity sub-solution at x_0. Let $\psi \in C^1$ such that $\phi \leqq \psi$ with equality at x_0, and we want to prove that

$$0 \leqq F((\phi^*(x_0 + r_i))_{i=0,\dots,N}).$$

Case 1. $x_0 \notin \mathcal{T}$
If $x_0 \notin \mathcal{T}$, then ϕ is continuous at $x_0 + r_i$ for all i. Because ϕ solves (2.8) almost everywhere on $\mathbb{R}$, then there exists a sequence $x_n \to x_0$ such that ϕ solves (2.8) at x_n. Hence we have

$$0 = F((\phi(x_n + r_i))_{i=0,\dots,N}).$$

Passing to the limit $n \to +\infty$, we get

$$0 \leqq F((\phi^*(x_0 + r_i))_{i=0,\dots,N}) = F((\phi(x_0 + r_i))_{i=0,\dots,N}).$$

Case 2. $x_0 \in \mathcal{T}$
Now, assume that $x_0 \in \mathcal{T}$. Since $\mathcal{T}$ is countable, then choose $a_k > a_{k+1} > 0$ such that $a_k \to 0$ and $x_0 + a_k \notin \mathcal{T}$ for all k. Since $x_0 + a_k \notin \mathcal{T}$, then we deduce from Case 1 that

$$0 \leqq F((\phi(x_0 + a_k + r_i))_{i=0,\dots,N}).$$

Now letting $a_k \to 0$, we get

$$\begin{aligned} 0 &\leqq \limsup_{a_k \to 0} F((\phi(x_0 + a_k + r_i))_{i=0,\dots,N}) \\ &= F((\lim_{a_k \to 0} \phi(x_0 + a_k + r_i))_{i=0,\dots,N}) \\ &\leqq F((\phi^*(x_0 + r_i))_{i=0,\dots,N}). \end{aligned}$$

Here, we use the fact that $\phi^*(x) = \lim_{k \to +\infty} \phi(x + a_k)$ for any $x \in \mathbb{R}$ (because ϕ is non-decreasing and $a_k > 0$ with $a_k \to 0$). Hence ϕ is a viscosity sub-solution of (2.8) at x_0.
Similarly, we show that ϕ is a viscosity super-solution at any point, and then ϕ is a viscosity solution. □

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