Existence result for degenerate cross-diffusion system with application to seawater intrusion

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Existence results for degenerate cross-diffusion systems with application to seawater intrusion

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Abstract: In this paper, we study degenerate parabolic systems, which are strongly coupled. We prove general existence results, but the uniqueness remains an open question. Our proof of existence is based on a crucial entropy estimate which both control the gradient of the solution and the non-negativity of the solution. Our systems are of porous medium type and our theory applies to models in seawater intrusion both in the confined and unconfined cases.

Keywords: Degenerate parabolic system, entropy estimate, porous medium like systems.

1 Introduction

For sake of simplicity, we will work on the torus $\Omega := \mathbb{T}^N = (\mathbb{R}/\mathbb{Z})^N$, with $N \geq 1$.
Let $\Omega_T := (0,T) \times \Omega$ with $T > 0$. Let an integer $m \geq 1$. Our purpose is to study a class of degenerate strongly coupled parabolic systems in two different cases,
Unconfined case:

$$u_t^i = \operatorname{div}\left(u^i \sum_{j=1}^m A_{ij} \nabla u^j\right) \qquad \text{in } \Omega_T, \quad \text{for } i = 1, \dots, m. \tag{1.1}$$

Confined case:

$$\begin{cases} u_t^i = \operatorname{div}\left(u^i \nabla \left(p + \sum\limits_{j=1}^m A_{ij} u^j\right)\right) & \text{in } \Omega_T, \quad \text{for } i = 1, \dots, m, \\ \sum\limits_{i=1}^m u^i(t,x) = 1 & \text{in } \Omega_T, \end{cases} \tag{1.2}$$

with the initial condition

$$u^i(0,x) = u_0^i(x) \geq 0 \qquad \text{a.e. in } \Omega, \quad \text{for } i = 1, \dots, m. \tag{1.3}$$

In the confined case (1.2), p is the pressure defined on Ω_T, which appears as a Lagrange multiplier of the constraint on $u = (u^i)_{1 \leq i \leq m}$, given by the second line of (1.2). In the core of the paper we will assume that $A = (A_{ij})_{1\le i,j\le m}$ is a real $m \times m$ matrix (not necessarily symmetric) that satisfies the following positivity condition: we assume that there exists $\delta_0 > 0$, such that we have

$$\xi^T A \xi \ge \delta_0 |\xi|^2, \quad \text{for all } \xi \in \mathbb{R}^m. \tag{1.4}$$

This condition can be weaken: see Subsection 4.1. Problem (1.1) and (1.2) appear naturally in the modeling of seawater intrusion (see Subsection 1.2).

1.1 Main results

To introduce our main results, we need to define the space $H^1(\Omega)/\mathbb{R}$ and the entropy function Ψ:

The space $\mathbf{H}^1(\Omega)/\mathbb{R}$:

We define $H^1(\Omega)/\mathbb{R}$ as the space of functions of $H^1(\Omega)$, up to addition of constant. A natural norm is

$$\|p\|_{(H^1(\Omega)/\mathbb{R})} = \inf_{c\in\mathbb{R}} \|p - c\|_{H^1(\Omega)} = \left\| p - \frac{1}{|\Omega|}\int_\Omega p \right\|_{H^1(\Omega)}. \tag{1.5}$$

The function Ψ:

We define the nonnegative function Ψ as

$$\Psi(a) - \frac{1}{e} = \begin{cases} a \ln a & \text{for} \quad a > 0, \\ 0 & \text{for} \quad a = 0, \\ +\infty & \text{for} \quad a < 0, \end{cases} \tag{1.6}$$

which is minimal for $a = \dfrac{1}{e}$.

Theorem 1.1 ***(Existence for the unconfined system)***
Assume that A satisfies (1.4). For $i = 1, \dots, m$, let $u_0^i \ge 0$ in Ω satisfying

$$\sum_{i=1}^m \int_\Omega \Psi(u_0^i) < +\infty, \tag{1.7}$$

where Ψ is given in (1.6). Then there exists a function $u = (u^i)_{1\le i\le m} \in (L^2(0,T;H^1(\Omega)) \cap C([0,T);(W^{1,\infty}(\Omega))'))^m$ solution in the sense of distributions of (1.1),(1.3), with $u^i \ge 0$ a.e. in Ω_T, for $i = 1, \dots, m$. Moreover this solution satisfies the following entropy estimate for a.e. $t \in (0,T)$, with $u^i(t) = u^i(t,\cdot)$:

$$\sum_{i=1}^m \int_\Omega \Psi(u^i(t)) + \delta_0 \sum_{i=1}^m \int_0^t \int_\Omega |\nabla u^i|^2 \le \sum_{i=1}^m \int_\Omega \Psi(u_0^i), \tag{1.8}$$

where Ψ is given in (1.6).

Theorem 1.2 ***(Existence for the confined system)***
Assume that A satisfies (1.4). For $i = 1, \dots, m$, let $u_0^i \ge 0$ in Ω satisfying (1.7). Then there exists a function $u = (u^i)_{1\le i\le m} \in (L^2(0,T;H^1(\Omega)) \cap C([0,T);(W^{1,\infty}(\Omega))'))^m$, and a function $p \in L^2(0,T;H^1(\Omega)/\mathbb{R})$ such that (u,p) is a solution in the sense of distributions of

(1.2),(1.3), with $u^i \geq 0$ a.e. in Ω_T, for $i = 1, \ldots, m$. Moreover this solution satisfies the entropy estimate (1.8) and for a.e. $t \in (0,T)$, with $u^i(t) = u^i(t,\cdot)$, $p(t) = p(t,\cdot)$:

$$\int_\Omega |\nabla p(t)|^2 \leq \|A\|^2 \sum_{i=1}^{m} \int_\Omega \left|\nabla u^i(t)\right|^2. \tag{1.9}$$

Here $\|A\|$ is the matrix norm defined as

$$\|A\| = \sup_{|\xi|=1} |A\xi|. \tag{1.10}$$

Notice that the entropy estimate (1.8) guarantees that $\nabla u^i \in L^2(0,T;L^2(\Omega))$, and therefore allows us to define the product $u^i \sum_{i=1}^{m} A_{ij} \nabla u^j$ in (1.1) and (1.2). Similarly, estimate (1.9) guarantees that $\nabla p \in L^2(0,T;L^2(\Omega))$ and allows us to define the product $u^i \nabla p$ in (1.2). When our proofs were obtained, we realized that a similar entropy estimate has been obtained in [7] and [9] for a special system different from ours.

1.2 Application to seawater intrusion

In this subsection, we describe briefly two models of seawater intrusion (confined and unconfined), which are particular cases of our systems (1.1) and (1.2).
An aquifer is an underground layer of a porous and permeable rock through which water can move. On the one hand coastal aquifers contain freshwater and on the other hand saltwater from the sea can enter in the ground and replace the freshwater. We refer to [3] for a general overview on seawater intrusion models.
Now let $\nu = 1 - \varepsilon_0 \in (0,1)$ where

$$\varepsilon_0 = \frac{\gamma_s - \gamma_f}{\gamma_s}$$

with γ_s and γ_f are the specific weight of the saltwater and freshwater respectively. We can distinguish two types of aquifers, confined and unconfined.

Unconfined aquifers:

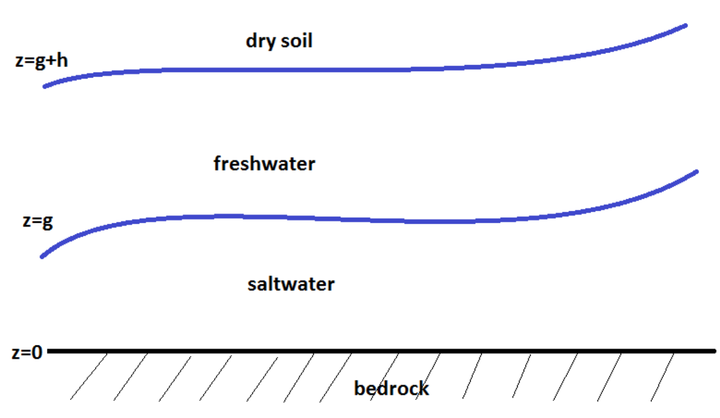

Figure 1: Unconfined aquifer

We assume that in the porous medium, the interface between the saltwater and the bedrock is given as $\{z = 0\}$, the interface between the saltwater and the freshwater, which are assumed to be unmiscible, can be written as $\{z = g(t, x)\}$, and the interface between the freshwater and the dry soil can be written as $\{z = h(t, x) + g(t, x)\}$. Then the evolutions of h and g are given by a coupled nonlinear parabolic system (we refer to see [14]) of the form

$$\begin{cases} h_t = \operatorname{div}\{h\nabla(\nu(h+g))\} & \text{in } \Omega_T, \\ g_t = \operatorname{div}\{g\nabla(\nu h+g)\} & \text{in } \Omega_T, \end{cases} \tag{1.11}$$

This is a particular case of (1.1), where the 2×2 matrix

$$A = \begin{pmatrix} \nu & \nu \\ \nu & 1 \end{pmatrix} \tag{1.12}$$

satisfies (1.4).

Confined aquifers:
A confined aquifer is an aquifer that lies between two impermeable layers. For simplicity we assume that $z = 0$ and $z = 1$ represent these two layers.

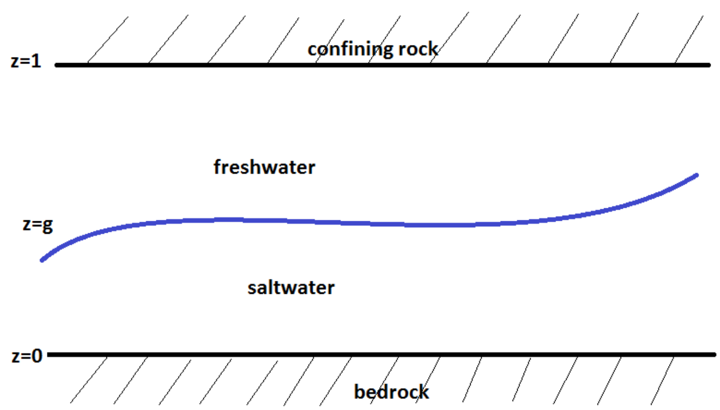

Figure 2: Confined aquifer

Then a natural model is the following system:

$$\begin{cases} h_t = \operatorname{div}\{h\nabla(p+\nu(h+g))\} & \text{in } \Omega_T, \\ g_t = \operatorname{div}\{g\nabla(p+\nu h+g)\} & \text{in } \Omega_T, \\ h+g = 1 & \text{in } \Omega_T, \end{cases}$$

where p is the pressure on the top confined rock. This is a particular case of (1.2) with matrix A given by (1.12).

1.3 Brief review of the litterature

The cross-diffusion systems, in particular the strongly coupled ones (for which the equations are coupled in the highest derivatives terms), are widely presented in different domains such as biology, chemistry, ecology, fluid mechanics and others. They are difficult to treat. Many of the standard results cannot be applied for such problems, such as the maximum principle.

Among them, the model proposed by Shigesada, Kawasaki and Teramoto in [25] which arises in population dynamics and can be written as

$$u_t^i = \mathrm{div} J^i + f_i(u^1, u^2), \quad J^i = \nabla(\beta_i u^i + \alpha_{i1} u^1 u^i + \alpha_{i2} u^2 u^i) + d_i u^i q, \qquad i = 1, 2, \quad (1.13)$$

where u^i denotes the population density of the i-th species, J^i the corresponding population flows and $q = \nabla U$ where $U(t, x)$ is a prescribed environmental potential, modeling areas where the environmental conditions are more or less favorable. The existence of a solution for this problem was frequently studied with restrictions on the diffusion coefficients. We cite some of the available results for

$$\begin{cases} \alpha_{11} = \alpha_{22} = \alpha_{21} = 0, \alpha_{12} > 0, & \beta_1, \beta_2 \quad > 0, & d_1 = d_2 = 0, & \text{see [23, 24]}, \\ \alpha_{11} = \alpha_{22} = 0, \alpha_{21} = \alpha_{12} = 1, & \beta_1, \beta_2 \quad > 0, & d_1 = d_2 = 0, & \text{see [6]}, \\ \alpha_{11}, \alpha_{22} \geq 0, \alpha_{21} = 0, \alpha_{12} \geq 0 & \beta_1, \beta_2 \quad \geq 0, & d_1 = d_2 = 0, & \text{see [21]}, \\ \alpha_{11} = \alpha_{22} = \alpha_{21} = 0, \alpha_{12} > 0, & \beta_1 = \beta_2 \quad > 0, & d_1 = d_2 = 0, & \text{see [13]}, \\ 0 < \alpha_{21} < 8\alpha_{11}, 0 < \alpha_{12} < 8\alpha_{22}, & \beta_1, \beta_2 \quad > 0, & d_1, d_2 \geq 0, & \text{see [27]}, \\ 0 \leq \alpha_{21} < 8\alpha_{11}, 0 \leq \alpha_{12} < 8\alpha_{22}, & \beta_1, \beta_2 \quad \geq 0, & d_1, d_2 \in \mathbb{R}, & \text{see [11]}, \\ \alpha_{11}, \alpha_{22} \geq 0, 2\alpha_{21} < \alpha_{11}, 2\alpha_{12} < \alpha_{22}, & \beta_1 - \beta_2 \quad > 0, & d_1 = d_2 = 0, & \text{see [16]} \\ \alpha_{11} > \alpha_{21} > 0, \alpha_{22} > \alpha_{12} > 0, & & & \\ \alpha_{12}\alpha_{21} < (\alpha_{11} - \alpha_{21})(\alpha_{22} - \alpha_{12}), & \beta_1 - \beta_2 \quad > 0, & d_1 = d_2 = 0, & \text{see [15]}, \\ \alpha_{11} = \alpha_{22} > 0, \alpha_{21} = \alpha_{12} = 1, & \beta_1, \beta_2 \quad \geq 0, & d_1, d_2 \in \mathbb{R}, & \text{see [7]}. \end{cases}$$

In [28], the existence of a global solution of a problem such:

$$u_t^i - \Delta\left[\left(\beta_i + \sum_{j=1}^m \alpha_{ij} u^j\right) u^i\right] = \left(a_i - \sum_{j=1}^m b_{ij} u^j\right) u^i, \qquad \text{for } i = 1, \ldots, m,$$

is studied where $\beta_i > 0$ and $\alpha_{ij} > 0$. For the stationary problem, we refer to see, for example, [20] where $\beta_i > 0$ and $\alpha_{ij} \geq 0$. In [17], Lepoutre, Pierre and Rolland studied a relaxed model, without a term source (see also [4, 16]) of the form

$$\begin{cases} u_t^i = \Delta[a_i(\tilde{u}) u^i], \quad \tilde{u} = (\tilde{u}^i)_{1 \leq i \leq m}, & \text{for} \quad i = 1, \ldots, m, \\ \tilde{u}^i - \delta_i \Delta \tilde{u}^i = u^i, \quad \text{with } \delta_i > 0, & \text{for} \quad i = 1, \ldots, m, \end{cases} \quad (1.14)$$

in any dimension and for general nonlinearities bounded $a_i \in C([0, \infty)^m; [\eta, \infty))$, for some $\eta > 0$. In [17], they show the existence of a weak solution. Moreover if the functions a_i are locally Lipschitz continuous then it is shown that this solution has more regularity and then is unique.

Another example of such problems is the electochemistry model studied by Choi, Huan and Lui in [8] where they consider the general form

$$u_t^i = \sum_{\ell=1}^n \sum_{j=1}^m \frac{\partial}{\partial x_\ell}\left(a_\ell^{ij}(u) \frac{\partial u^j}{\partial x_\ell}\right), \quad u = (u^i)_{1 \leq i \leq m} \quad \text{for} \quad i = 1, \ldots, m, \qquad (1.15)$$

and prove the existence of a weak solution of (1.15) under assumptions on the matrices $A_l(u) = (a_l^{ij}(u))_{1 \leq i,j \leq m}$: it is continuous in u, its components are uniformly bounded with respect to u and its symmetric part is definite positive. Their strategy of proof seeks to use Galerkin method to prove the existence of solutions to the linearized system and then to apply Schauder fixed-point theorem. Then they apply the results obtained to an electrochemistry

model.
A fourth example of cross-diffusion models is the chemotaxis model introduced in [18]. The global existence for classical solutions of this model is studied by Hillen and Painter in [12]:

$$\left\{\begin{array}{lcl} u_t & = & \nabla(\nabla u - V(u,v)\nabla v), \\ v_t & = & \mu\Delta v + g(u,v), \end{array}\right.$$

where the chemotactic cross-diffusion V is assumed to be bounded, and the function g describes production and degradation of the external stimulus.
A fifth example is a seawater intrusion problem in a confined aquifers, studied in [22]. It consists in a coupled system of an elliptic and a degenerate parabolic equation. The global existence is obtained by using Schauder's fixed point with a parabolic regularization.

1.4 Strategy of the proof

In the unconfined case (1.1) (respectively the confined case (1.2)), the elliptic part of the equation does not have a Lax-Milgram structure. Otherwise, our existence result is mainly based on the entropy estimate (1.8). It is difficult to get this entropy estimate directly (we do not have enough regularity to do it), so we proceed by approximations:
Approximation 1:
We discretize in time system (1.1) (respectively (1.2)), with a time step $\Delta t = \dfrac{T}{K}$, where $K \in \mathbb{N}^*$. Then for a given $u^n = (u^{i,n})_{1\leq i\leq m} \in (H^1(\Omega))^m$, we consider the implicit scheme which is an elliptic system:

$$\frac{u^{i,n+1} - u^{i,n}}{\Delta t} = \mathrm{div}\left\{ u^{i,n+1} \sum_{j=1}^{m} A_{ij} \nabla u^{j,n+1} \right\}. \tag{1.16}$$

Approximation 2:
We regularize the PDE term:

$$\mathrm{div}\left\{ u^{i,n+1} \sum_{i=1}^{m} A_{ij} \nabla u^{j,n+1} \right\}. \tag{1.17}$$

To do that, we take $\eta > 0$ and $0 < \varepsilon < 1 < \ell$, and we choose the following regularization

$$\mathrm{div}\left\{ T^{\varepsilon,\ell}(u^{i,n+1}) \sum_{j=1}^{m} A_{ij} \nabla \rho_\eta \star \rho_\eta \star u^{j,n+1} \right\}, \tag{1.18}$$

where $T^{\varepsilon,\ell}$ is a truncation operator defined as,

$$T^{\varepsilon,\ell}(a) := \left\{\begin{array}{ll} \varepsilon & \text{if } a \leq \varepsilon, \\ a & \text{if } \varepsilon \leq a \leq \ell, \\ \ell & \text{if } a \geq \ell, \end{array}\right. \tag{1.19}$$

and the mollifier $\rho_\eta(x) = \dfrac{1}{\eta^N}\rho(\dfrac{x}{\eta})$ with $\rho \in C_c^\infty(\mathbb{R}^N)$, $\rho \geq 0$, $\displaystyle\int_{\mathbb{R}^N} \rho = 1$ and $\rho(-x) = \rho(x)$.
Now with the convolution by ρ_η in (1.18), the term $\nabla\rho_\eta \star \rho_\eta \star u^{j,n+1}$ behaves like $u^{j,n+1}$.
Note that the convolution $\rho_\eta \star u^{j,n+1}$ is done on $\mathbb{R}^N$, considering $u^{j,n+1}$ as its $\mathbb{Z}^N$- periodic

extension on $\mathbb{R}^N$.
Approximation 3:
Let $\delta > 0$. We then add a term like $\delta\Delta u^i$. This way, the term $\delta\Delta u^i$ remains the main term, and we can apply the Lax-Milgram theorem.
Approximation 4:
We consider div $\left(\delta T^{\varepsilon,\ell}(u^i)\nabla u^i\right)$ instead of $\delta\Delta u^i$, to keep an entropy estimate.
Then by freezing the coefficients to make a linear structure (these coefficients are now called $\delta T^{\varepsilon,\ell}(v^{i,n+1})$), we obtain these following modified systems:
Unconfined case:

$$\frac{u^{i,n+1}-u^{i,n}}{\Delta t} = \operatorname{div}\left\{T^{\epsilon,\ell}(v^{i,n+1})\left(\sum_{j=1}^{m} A_{ij}\nabla\rho_\eta \star \rho_\eta \star u^{j,n+1} + \delta\nabla u^{i,n+1}\right)\right\}, \tag{1.20}$$

Confined case:

$$\begin{cases} \dfrac{u^{i,n+1}-u^{i,n}}{\Delta t} & = \operatorname{div}\left\{T^{\epsilon,\ell}(v^{i,n+1})\left(\nabla p^{n+1} + \sum\limits_{j=1}^{m} A_{ij}\nabla\rho_\eta \star \rho_\eta \star u^{j,n+1} + \delta\nabla u^{i,n+1}\right)\right\}, \\ \sum\limits_{j=1}^{m} u^{i,n+1} & = 1. \end{cases} \tag{1.21}$$

We will look for fixed points solutions $v^{i,n+1} = u^{i,n+1}$ of these modified systems. Finally, we will recover the expected result dropping one after one all the approximations.

1.5 Organization of the paper

In Section 2, we recall some useful tools. In Section 3, we study the unconfined case (1.1). By discretizing our problem on time, in Subsection 3.1, we obtain an elliptic problem. We use the Lax-Milgram theorem to show the existence of a unique solution to a linear problem (when the coefficients $T^{\varepsilon,\ell}(u^{i,n+1})$ are frozen to be $T^{\varepsilon,\ell}(v^{i,n+1})$). We demonstrate, in Subsection 3.2, the existence of a solution of the nonlinear problem, using the Schaefer's fixed point theorem.

Then we pass to the limit when

$$\begin{cases} (\Delta t, \varepsilon) & \to & (0,0) & \text{in Subsection 3.3,} \\ (\ell, \eta, \delta) & \to & (\infty, 0, 0) & \text{in Subsection 3.4.} \end{cases}$$

Generalizations (including more general matrices A or tensors) will be presented in Section 4. Finally, Section 5 contains the Appendix with some technical results.

2 Preliminary tools

Theorem 2.1 (Schaefer's fixed point theorem) *[10, Theorem 4 page 504]*
Let X be a real Banach space. Suppose that

$$\Phi : X \to X$$

is a continuous and compact mapping. Assume further that the set

$$\{u \in X, \quad u = \lambda\Phi(u) \quad \text{for some} \quad \lambda \in [0,1]\}$$

is bounded. Then Φ has a fixed point.

Proposition 2.2 (Aubin's lemma)*[26]*
For any $T > 0$, and $\Omega = \mathbb{T}^N$, let E denote the space

$$E := \left\{ g \in L^2((0,T); H^1(\Omega)) \text{ and } g_t \in L^2((0,T); H^{-1}(\Omega)) \right\},$$

endowed with the Hilbertian norm

$$\|\omega\|_E = \left(\|\omega\|^2_{L^2(0,T;H^1(\Omega))} + \|\omega_t\|^2_{L^2(0,T;H^{-1}(\Omega))} \right)^{\frac{1}{2}}.$$

The embedding

$$E \hookrightarrow L^2((0,T); L^2(\Omega)) \quad \text{is compact.}$$

On the other hand, it follows from [19, Proposition 2.1 and Theorem 3.1, Chapter 1] that the embedding

$$E \hookrightarrow C([0,T]; L^2(\Omega)) \quad \text{is continuous.}$$

Lemma 2.3 ***(Simon's Lemma)****[26]*
Let X, B and Y three Banach spaces, where $X \hookrightarrow B$ with compact embedding and $B \hookrightarrow Y$ with continuous embedding. If $(g^n)_n$ is a sequence such that

$$\|g^n\|_{L^q(0,T;B)} + \|g^n\|_{L^1(0,T;X)} + \|g^n_t\|_{L^1(0,T;Y)} \leq C,$$

where $1 < q \leq \infty$, and C is a constant independent of n, then $(g^n)_n$ is relatively compact in $L^p(0,T;B)$ for all $1 \leq p < q$.

Now we will present the variant of the original result of Simon's lemma [26, Corollary 6, page 87]. First of all, let us define the norm $|.|_{\text{Var}([a,b);Y)}$ where Y is a Banach space.
For a function $g : [a,b) \to Y$, we set

$$|g|_{\text{Var}([a,b);Y)} = \sup \sum_j |g(a_{j+1}) - g(a_j)|_Y \tag{2.1}$$

over all possible finite partitions:

$$a \leq a_0 < \cdots < a_k < b.$$

Theorem 2.4 ***(Variant of Simon's Lemma)***
Let X, B and Y three Banach spaces, where $X \hookrightarrow B$ with compact embedding and $B \hookrightarrow Y$ with continuous embedding. Let $(g^n)_n$ be a sequence such that

$$|g^n|_{L^1(0,T;X)} + |g^n|_{L^q(0,T;B)} + |g^n|_{Var([0,T);Y)} \leq C, \tag{2.2}$$

where $1 < q < \infty$, and C is a constant independent of n. Then $(g^n)_n$ is relatively compact in $L^p(0,T;B)$ for all $1 \leq p < q$.

Proof of Theorem 2.4
Step 1: Regularization of the sequence
Let $\bar{\rho} \in C^\infty_c(\mathbb{R})$ with $\bar{\rho} \geq 0$, $\int_{\mathbb{R}} \bar{\rho} = 1$ and supp $\bar{\rho} \subset (-1,1)$. For $\varepsilon > 0$, we set

$$\bar{\rho}_\varepsilon(x) = \varepsilon^{-1} \bar{\rho}(\varepsilon^{-1} x).$$

We extend $g^n = g^n(t)$ by zero outside the time interval $[0,T)$. Because $q < +\infty$, we see that for each n, we choose some $0 < \varepsilon_n \to 0$ as $n \to +\infty$ such that

$$|\bar{g}^n - g^n|_{L^q(0,T;B)} \to 0 \quad \text{as} \quad n \to +\infty, \quad \text{with} \quad \bar{g}^n = \bar{\rho}_{\varepsilon_n} \star g^n \tag{2.3}$$

For any $\delta > 0$ small enough, we also have for n large enough (such that $\varepsilon_n < \delta$):

$$|\bar{g}^n|_{L^1(\delta,T-\delta;X)} \le |g^n|_{L^1(0,T;X)} \le C$$

and

$$|\bar{g}^n_t|_{L^1(\delta,T-\delta;Y)} \le |g^n|_{\mathrm{Var}([0,T);Y)} \le C \tag{2.4}$$

Step 2: Checking (2.4)
Note that by definition of $|\cdot|_{\mathrm{Var}([0,T);Y)}$ (see (2.1)), there exists a sequence of step functions f_η which approximates uniformly $\bar{g}^n$ on $[0,T)$ as $\eta \to 0$, with moreover satisfies

$$|f_\eta|_{\mathrm{Var}([0,T);Y)} \to |\bar{g}^n|_{\mathrm{Var}([0,T);Y)}.$$

Therefore we get easily (for $\varepsilon_n < \delta$)

$$|(\bar{\rho}_{\varepsilon_n} \star f_\eta)_t|_{L^1(\delta,T-\delta;Y)} \le |f_\eta|_{\mathrm{Var}([0,T);Y)}$$

which implies (2.4), when we pass to the limit as η goes to zero.
Step 3: Conclusion
We can then apply Corollary 6 in [26] to deduce that $\bar{g}^n$ is relatively compact in $L^p(0,T;B)$ for all $1 \le p < q$. Because of (2.3), we deduce that this is also the case for the sequence $(g^n)_n$, which ends the proof of the Theorem. □

3 Existence for the unconfined case

Our goal is to prove Theorem 1.1 in order to get the existence of a solution for the unconfined system (i.e. system (1.1)).

3.1 Existence for a linear elliptic problem

In this subsection we prove the existence, via Lax-Milgram theorem, of the unique solution for the linear elliptic system given in (1.20).
Let us recall our linear elliptic system. Assume that A is any $m \times m$ real matrix. Let $v^{n+1} = (v^{i,n+1})_{1 \le i \le m} \in (L^2(\Omega))^m$ and $u^n = (u^{i,n})_{1 \le i \le m} \in (H^1(\Omega))^m$. Then for all Δt, ε, ℓ, η, $\delta > 0$, with $\varepsilon < 1 < \ell$ and $\Delta t < \tau$ where τ is given in (3.2), we look for the solution $u^{n+1} = (u^{i,n+1})_{1 \le i \le m}$ of the following system:

$$\begin{cases} \dfrac{u^{i,n+1} - u^{i,n}}{\Delta t} & = \operatorname{div}\left\{ J^i_{\epsilon,\ell,\eta,\delta}(v^{n+1}, u^{n+1}) \right\} \quad \text{in} \quad \mathcal{D}'(\Omega), \\ J^i_{\epsilon,\ell,\eta,\delta}(v^{n+1}, u^{n+1}) & = T^{\epsilon,\ell}(v^{i,n+1}) \left\{ \displaystyle\sum_{j=1}^{m} A_{ij} \nabla \rho_\eta \star \rho_\eta \star u^{j,n+1} + \delta \nabla u^{i,n+1} \right\}, \end{cases} \tag{3.1}$$

where $T^{\varepsilon,\ell}$ is given in (1.19).

Proposition 3.1 ***(Existence for system (3.1))***
Assume that A is any $m \times m$ real matrix. Let Δt, ε, ℓ, η, $\delta > 0$, with $\varepsilon < 1 < \ell$, such that

$$\Delta t < \frac{\delta \varepsilon \eta^2}{{C_0}^2 \ell^2 \|A\|^2} := \tau, \tag{3.2}$$

where

$$C_0 = \|\nabla \rho\|_{L^1(\mathbb{R}^N)}. \tag{3.3}$$

Then for $n \in \mathbb{N}$, for a given $v^{n+1} = (v^{i,n+1})_{1\leq i\leq m} \in (L^2(\Omega))^m$ and $u^n = (u^{i,n})_{1\leq i\leq m} \in (H^1(\Omega))^m$, there exists a unique function $u^{n+1} = (u^{i,n+1})_{1\leq i\leq m} \in (H^1(\Omega))^m$ solution of system (3.1). Moreover, this solution u^{n+1} satisfies the following estimate

$$\left(1 - \frac{\Delta t}{\tau}\right) \left\|u^{n+1}\right\|^2_{(L^2(\Omega))^m} + \Delta t \varepsilon \delta \left\|\nabla u^{n+1}\right\|^2_{(L^2(\Omega))^m} \leq \|u^n\|^2_{(L^2(\Omega))^m}, \tag{3.4}$$

where τ is given in (3.2).

Proof of Proposition 3.1.
The proof is done in two steps using Lax-Milgram theorem.
First of all, let us define for all $u^{n+1} = (u^{i,n+1})_{1\leq i\leq m}$ and $\varphi = (\varphi^i)_{1\leq i\leq m} \in (H^1(\Omega))^m$, the following bilinear form:

$$\begin{aligned} a(u^{n+1}, \varphi) &= \sum_{i=1}^m \int_\Omega u^{i,n+1}\varphi^i + \Delta t \sum_{i,j=1}^m \int_\Omega T^{\varepsilon,\ell}(v^{i,n+1}) A_{ij} \left(\nabla \rho_\eta \star \rho_\eta \star u^{j,n+1}\right) \cdot \nabla \varphi^i \\ &\quad + \Delta t \delta \sum_{i=1}^m \int_\Omega T^{\varepsilon,\ell}(v^{i,n+1}) \nabla u^{i,n+1} \cdot \nabla \varphi^i, \end{aligned}$$

which can be also rewritten as

$$\begin{aligned} a(u^{n+1}, \varphi) &= \left\langle u^{n+1}, \varphi \right\rangle_{(L^2(\Omega))^m} + \Delta t \left\langle T^{\varepsilon,\ell}(v^{n+1}) \nabla \varphi, A \nabla \rho_\eta \star \rho_\eta \star u^{n+1} \right\rangle_{(L^2(\Omega))^m} \\ &\quad + \Delta t \delta \left\langle T^{\varepsilon,\ell}(v^{n+1}) \nabla \varphi, \nabla u^{n+1} \right\rangle_{(L^2(\Omega))^m}, \end{aligned}$$

where $\langle \cdot, \cdot \rangle_{(L^2(\Omega))^m}$ denotes the scalar product on $(L^2(\Omega))^m$, and the following linear form:

$$L(\varphi) = \sum_{i=1}^m \int_\Omega u^{i,n} \varphi^i = \langle u^n, \varphi \rangle_{(L^2(\Omega))^m}.$$

Step 1: Continuity of a
For every $n \in \mathbb{N}$, u^{n+1} and $\varphi \in (H^1(\Omega))^m$, we have

$$\begin{aligned} |a(u^{n+1}, \varphi)| &\leq \|u^{n+1}\|_{(L^2(\Omega)^m} \|\varphi\|_{(L^2(\Omega))^m} + \Delta t \ell \|A\| \|\nabla \rho_\eta \star \rho_\eta \star u^{n+1}\|_{(L^2(\Omega))^m} \|\nabla \varphi\|_{(L^2(\Omega))^m} \\ &\quad + \Delta t \delta \ell \|\nabla u^{n+1}\|_{(L^2(\Omega))^m} \|\nabla \varphi\|_{(L^2(\Omega))^m} \\ &\leq 3 \max(1, \Delta t \ell \|A\|, \Delta t \delta \ell) \|u^{n+1}\|_{(H^1(\Omega))^m} \|\varphi\|_{(H^1(\Omega))^m}. \end{aligned}$$

where $\|A\|$ is given in (1.10) and we have used the fact that

$$\left\|\nabla \rho_\eta \star \rho_\eta \star u^{n+1}\right\|_{(L^2(\Omega))^m} \leq \left\|\nabla u^{n+1}\right\|_{(L^2(\Omega))^m}, \tag{3.5}$$

and

$$\varepsilon \le T^{\varepsilon,\ell}(a) \le \ell, \quad \text{for all } a \in \mathbb{R}. \tag{3.6}$$

Step 2: Coercivity of a
For all $\varphi \in (H^1(\Omega))^m$, we have that $a(\varphi,\varphi) = a_0(\varphi,\varphi) + a_1(\varphi,\varphi)$, where

$$a_0(\varphi,\varphi) = \|\varphi\|^2_{(L^2(\Omega))^m} + \Delta t\delta \left\langle T^{\varepsilon,\ell}(\varphi)\nabla\varphi, \nabla\varphi\right\rangle_{(L^2(\Omega))^m}$$

and

$$a_1(\varphi,\varphi) = \Delta t \left\langle T^{\varepsilon,\ell}(\varphi)\nabla\varphi, A\nabla\rho_\eta \star \rho_\eta \star \varphi\right\rangle_{(L^2(\Omega))^m}.$$

On the one hand, we already have the coercivity of a_0:

$$a_0(\varphi,\varphi) \;\; \ge \;\; \|\varphi\|^2_{(L^2(\Omega))^m} + \Delta t\delta\varepsilon\|\nabla\varphi\|^2_{(L^2(\Omega))^m}.$$

On the other hand, we have

$$\begin{aligned}
|a_1(\varphi,\varphi)| &\le \Delta t\ell \|A\| \|\nabla\rho_\eta \star \rho_\eta \star \varphi\|_{(L^2(\Omega))^m} \|\nabla\varphi\|_{(L^2(\Omega))^m} \\
&\le \Delta t\ell \|A\| \left(\frac{1}{2\alpha} \|\nabla\rho_\eta \star \rho_\eta \star \varphi\|^2_{(L^2(\Omega))^m} + \frac{\alpha}{2} \|\nabla\varphi\|^2_{(L^2(\Omega))^m} \right) \\
&\le \frac{\Delta t\ell^2 \|A\|^2 C_0^2}{2\delta\varepsilon\eta^2} \|\varphi\|^2_{(L^2(\Omega))^m} + \frac{\Delta t\varepsilon\delta}{2} \|\nabla\varphi\|^2_{(L^2(\Omega))^m},
\end{aligned}$$

where in the second line we have used Young's inequality, and chosen $\alpha = \dfrac{\delta\varepsilon}{\|A\|\,\ell}$ in the third line, with C_0 is given in (3.3) and $\|A\|$ is given in (1.10). So we get that

$$a(\varphi,\varphi) \;\; \ge \;\; \left(1 - \frac{\Delta t}{2\tau}\right) \|\varphi\|^2_{(L^2(\Omega))^m} + \frac{\Delta t\varepsilon\delta}{2} \|\nabla\varphi\|^2_{(L^2(\Omega))^m} \tag{3.7}$$

is coercive, since $\Delta t < \tau$ where τ is given in (3.2).
Finally, it is clear that L is linear and continuous on $(H^1(\Omega))^m$, and then by Lax-Milgram theorem there exists a unique solution, u^{n+1}, of system (3.1).

Step 3: Proof of estimate (3.4)
Using (3.7) and the fact that $a(u^{n+1}, u^{n+1}) = L(u^{n+1})$ we get

$$\begin{aligned}
\left(1 - \frac{\Delta t}{2\tau}\right) \left\|u^{n+1}\right\|^2_{(L^2(\Omega))^m} + \frac{\Delta t\varepsilon\delta}{2} \left\|\nabla u^{n+1}\right\|^2_{(L^2(\Omega))^m} &\le \left\langle u^{i,n}, u^{i,n+1}\right\rangle_{(L^2(\Omega))^m} \\
&\le \frac{1}{2} \|u^n\|^2_{(L^2(\Omega))^m} + \frac{1}{2} \left\|u^{n+1}\right\|^2_{(L^2(\Omega))^m},
\end{aligned}$$

which gives us the estimate (3.4). □

3.2 Existence for the nonlinear time discrete problem

In this subsection we prove the existence, using Schaefer's fixed point theorem, of a solution for the nonlinear time discrete system (3.10) given below. Moreover, we also show that this solution satisfies a suitable entropy estimate.

First, to present our result we need to choose a function $\Psi_{\varepsilon,\ell}$ which is continuous, convex and satisfies that $\Psi''_{\varepsilon,\ell}(x) = \dfrac{1}{T^{\varepsilon,\ell}(x)}$, where $T^{\varepsilon,\ell}$ is given in (1.19). So let

$$\Psi_{\varepsilon,\ell}(a) - \frac{1}{e} = \begin{cases} \frac{a^2}{2\varepsilon} + a\ln\epsilon - \frac{\varepsilon}{2} & \text{if } a \le \varepsilon, \\ a\ln a & \text{if } \varepsilon < a \le \ell, \\ \frac{a^2}{2\ell} + a\ln\ell - \frac{\ell}{2} & \text{if } a > f\ell. \end{cases} \tag{3.8}$$

Let us introduce our nonlinear time discrete system: Assume that A satisfies (1.4). Let $u^0 = (u^{i,0})_{1\le i\le m} := u_0 = (u_0^i)_{1\le i\le m}$ that satisfies

$$\sum_{i=1}^{m} \int_\Omega \Psi_{\varepsilon,\ell}(u_0^i) < +\infty, \tag{3.9}$$

such that $u_0^i \ge 0$ in Ω for $i = 1,\dots,m$. Then for all Δt, ε, ℓ, η, $\delta > 0$, with $\varepsilon < 1 < \ell$ and $\Delta t < \tau$ where τ is given in (3.2), for $n \in \mathbb{N}$, we look for a solution $u^{n+1} = (u^{i,n+1})_{1\le i\le m}$ of the following system:

$$\begin{cases} \dfrac{u^{i,n+1} - u^{i,n}}{\Delta t} = \operatorname{div}\left\{ J^i_{\varepsilon,\ell,\eta,\delta}(u^{n+1}, u^{n+1}) \right\} & \text{in } \mathcal{D}'(\Omega), \text{ for } n \ge 0 \\ u^{i,0}(x) = u_0^i(x) & \text{in } \Omega, \end{cases} \tag{3.10}$$

where $J^i_{\varepsilon,\ell,\eta,\delta}$ is given in system (3.1), and $T^{\varepsilon,\ell}$ is given in (1.19).

Proposition 3.2 ***(Existence for system (3.10))***
Assume that A satisfies (1.4). Let $u_0 = (u_0^i)_{1\le i\le m}$ that satisfies (3.9), such that $u_0^i \ge 0$ a.e. in Ω for $i = 1,\dots,m$. Then for all Δt, ε, ℓ, η, $\delta > 0$, with $\varepsilon < 1 < \ell$ and $\Delta t < \tau$ where τ is given in (3.2), there exists a sequence of functions $u^{n+1} = (u^{i,n+1})_{1\le i\le m} \in (H^1(\Omega))^m$ for $n \in \mathbb{N}$, solution of system (3.10), that satisfies the following entropy estimate:

$$\sum_{i=1}^{m} \int_\Omega \Psi_{\varepsilon,\ell}(u^{i,n+1}) + \delta\Delta t \sum_{i=1}^{m}\sum_{k=0}^{n} \int_\Omega |\nabla u^{i,k+1}|^2 + \delta_0 \Delta t \sum_{i=1}^{m}\sum_{k=0}^{n} \int_\Omega |\nabla\rho_\eta \star u^{i,k+1}|^2 \le \sum_{i=1}^{m} \int_\Omega \Psi_{\varepsilon,\ell}(u_0^i), \tag{3.11}$$

where $\Psi_{\varepsilon,\ell}$ is given in (3.8).

Proof of Proposition 3.2.
Our proof is based on the Schaefer's fixed point theorem. So we need to define, for a given $w := u^n = (u^{i,n})_{1\le i\le m} \in (L^2(\Omega))^m$ and $v := v^{n+1} = (v^{i,n+1})_{1\le i\le m} \in (L^2(\Omega))^m$, the map Φ as:

$$\begin{array}{cccc} \Phi & : & (L^2(\Omega))^m & \to & (L^2(\Omega))^m \\ & & v & \mapsto & u \end{array}$$

where $u := u^{n+1} = (u^{i,n+1})_{1\le i\le m} = \Phi(v^{n+1}) \in (H^1(\Omega))^m$ is the unique solution of system (3.1), given by Proposition 3.1.

Step 1: Continuity of Φ

Let us consider the sequence v_k such that

$$\begin{cases} v_k \in (L^2(\Omega))^m, \\ \\ v_k \longrightarrow v \quad \text{in} \quad (L^2(\Omega))^m. \end{cases}$$

We want to prove that the sequence $u_k = \Phi(v_k) \longrightarrow u = \Phi(v)$ to get the continuity of Φ. From the estimate (3.4), we deduce that u_k is bounded in $(H^1(\Omega))^m$. Therefore, up to a subsequence, we have

$$\begin{cases} u_k \rightharpoonup u \quad \text{weakly in} \;\; (H^1(\Omega))^m, \\ \text{and} \\ u_k \to u \quad \text{strongly in} \;\; (L^2(\Omega))^m, \end{cases}$$

where the strong convergence arises because Ω is compact. Thus, by the definition of the truncation operator $T^{\varepsilon,\ell}$, we can see that $T^{\varepsilon,\ell}$ is continuous and bounded, then by dominated convergence theorem, we have that

$$T^{\varepsilon,\ell}(v_k^i) \longrightarrow T^{\varepsilon,\ell}(v^i) \quad \text{in } L^2(\Omega), \quad \text{for } i = 1, \dots, m.$$

Now we have

$$\frac{u_k^i - w^i}{\Delta t} = \operatorname{div}\left\{J^i_{\varepsilon,\ell,\eta,\delta}(v_k, u_k)\right\} \qquad \text{in } \mathcal{D}'(\Omega). \tag{3.12}$$

This system also holds in $H^{-1}(\Omega)$, because $J^i_{\varepsilon,\ell,\eta,\delta}(v_k, u_k) \in L^2(\Omega)$. Hence by multiplying this system by a test function in $(H^1(\Omega))^m$ and integrating over Ω for the bracket $\langle \cdot, \cdot \rangle_{H^{-1}(\Omega)\times H^1(\Omega)}$, we can pass directly to the limit in (3.12) when k tends to $+\infty$, and we get

$$\frac{u^i - w^i}{\Delta t} = \operatorname{div}\{J^i_{\varepsilon,\ell,\eta,\delta}(v, u)\} \qquad \text{in } \mathcal{D}'(\Omega). \tag{3.13}$$

where we used in particular the weak L^2 - strong L^2 convergence in the product $T^{\varepsilon,\ell}(v_k)\nabla u_k$. Then $u = (u^i)_{1\le i\le m} = \Phi(v)$ is a solution of system (3.1). Finally, by uniqueness of the solutions of (3.1), we deduce that the limit u does not depend on the choice of the subsequence, and then that the full sequence converges:

$$u_k \to u \quad \text{strongly in} \quad (L^2(\Omega))^m, \quad \text{with} \quad u = \Phi(v).$$

Step 2: Compactness of Φ

By the definition of Φ we can see that for a bounded sequence $(v_k)_k$ in $(L^2(\Omega))^m$, $\Phi(v_k) = u_k$ converges strongly in $(L^2(\Omega))^m$ up to a subsequence, which implies the compactness of Φ.

Step 3: A priori bounds on the solutions of $v = \lambda\Phi(v)$

Let us consider a solution v of

$$v = \lambda\Phi(v) \quad \text{for some} \quad \lambda \in [0, 1].$$

By (3.4) we see that there exists a constant $C_1 = C_1(\Delta t, \varepsilon, ...)$ such that for any given $w \in (L^2(\Omega))^m$, we have $\|\Phi(v)\|_{(H^1(\Omega))^m} \le C_1 \|w\|_{(L^2(\Omega))^m}$. Hence $v = \lambda\Phi(v)$ is bounded.

Step 4: Existence of a solution

Now, we can apply Schaefer's fixed point Theorem (Theorem 2.1), to deduce that Φ has a fixed point u^{n+1} on $(L^2(\Omega))^m$. This implies the existence of a solution u^{n+1} of system (3.10).

Step 5: Proof of estimate (3.11)

We have,

$$\begin{aligned}
&\sum_{i=1}^{m}\int_\Omega \frac{\Psi_{\varepsilon,\ell}(u^{i,n+1})-\Psi_{\varepsilon,\ell}(u^{i,n})}{\Delta t}\\
\le\ & \sum_{i=1}^{m}\int_\Omega \left(\frac{u^{i,n+1}-u^{i,n}}{\Delta t}\right)\Psi'_{\varepsilon,\ell}(u^{i,n+1})\\
=\ & \sum_{i=1}^{m}\left\langle \frac{u^{i,n+1}-u^{i,n}}{\Delta t}, \Psi'_{\varepsilon,\ell}(u^{i,n+1})\right\rangle_{H^{-1}(\Omega)\times H^1(\Omega)}\\
=\ & -\sum_{i=1}^{m}\left\langle \delta T^{\varepsilon,\ell}(u^{i,n+1})\nabla u^{i,n+1}+T^{\varepsilon,\ell}(u^{i,n+1})\sum_{j=1}^{m}A_{ij}\nabla\rho_\eta\star\rho_\eta\star u^{j,n+1}, \Psi''_{\varepsilon,\ell}(u^{i,n+1})\nabla u^{i,n+1}\right\rangle_{L^2(\Omega)}\\
=\ & -\sum_{i=1}^{m}\left\{\delta\int_\Omega|\nabla u^{i,n+1}|^2+\int_\Omega\sum_{j=1}^{m}\nabla\rho_\eta\star u^{i,n+1}A_{ij}\nabla\rho_\eta\star u^{j,n+1}\right\}\\
\le\ & -\sum_{i=1}^{m}\delta\int_\Omega|\nabla u^{i,n+1}|^2-\delta_0\sum_{i=1}^{m}\int_\Omega|\nabla\rho_\eta\star u^{i,n+1}|^2,
\end{aligned}$$

where we have used, in the second line, the convexity inequality on $\Psi_{\varepsilon,\ell}$. In the third line, we used the fact that $\dfrac{u^{i,n+1}-u^{i,n}}{\Delta t}\in H^{-1}(\Omega)$ and that $\nabla\Psi'_{\varepsilon,\ell}(u^{i,n+1})=\Psi''_{\varepsilon,\ell}(u^{i,n+1})\nabla u^{i,n+1}\in L^2(\Omega)$ because $\Psi'_{\varepsilon,\ell}(u^{i,n+1})\in C^1(\mathbb{R})$, see [5, Proposition IX.5, page 155]. Thus, in the fourth line we use that $u^{i,n+1}$ is a solution for system (3.10) where we have applied an integration by parts. In the fifth line, we used the transposition of the convolution (see for instance [5, Proposition IV.16, page 67]), and the fact that $\check{\rho}_\eta(x)=\rho_\eta(-x)=\rho_\eta(x)$. Finally, in the last line we use that A satisfies (1.4).

Then by a straightforward recurrence we get estimate (3.11). This ends the proof of Proposition 3.2. □

3.3 Passage to the limit when $(\Delta t,\varepsilon)\to(0,0)$

In this subsection we pass to the limit when $(\Delta t,\varepsilon)\to(0,0)$ in system (3.10) to get the existence of a solution for the continuous approximate system (3.16) given below.

First, let us define the function $\Psi_{0,\ell}$ as

$$\Psi_{0,\ell}(a)-\frac{1}{e}:=\begin{cases} +\infty & \text{if } a<0,\\ 0 & \text{if } a=0,\\ a\ln a & \text{if } 0<a\le\ell,\\ \frac{a^2}{2\ell}+a\ln\ell-\frac{\ell}{2} & \text{if } a>\ell.\end{cases}\tag{3.14}$$

Now let us introduce our continuous approximate system. Assume that A satisfies (1.4). Let $u_0 = (u_0^i)_{1\le i\le m}$ satisfying

$$C_3 := \sum_{i=1}^m \int_\Omega \Psi_{0,\ell}(u_0^i) < +\infty, \tag{3.15}$$

which implies that $u_0^i \ge 0$ a.e. in Ω for $i = 1,\dots,m$. Then for all ℓ, η, $\delta > 0$, with $1 < \ell < +\infty$, we look for a solution $u = (u^i)_{1\le i\le m}$ of the following system:

$$\begin{cases} u_t^i &= div\left\{J^i_{0,\ell,\eta,\delta}(u)\right\} & \text{in } \mathcal{D}'(\Omega_T),\\ J^i_{0,\ell,\eta,\delta}(u) &= T^{0,\ell}(u^i)\left\{\sum_{j=1}^m A_{ij}\nabla\rho_\eta \star \rho_\eta \star u^j + \delta\nabla u^i\right\}, & \\ u^i(0,x) &= u_0^i(x) & \text{in } \Omega. \end{cases} \tag{3.16}$$

where $T^{0,\ell}$ is given in (1.19) with $\varepsilon = 0$, and we recall here $\Omega_T := (0,T)\times\Omega$.

Proposition 3.3 *(**Existence for system (3.16)**)*
Assume that A satisfies (1.4). Let $u_0 = (u_0^i)_{1\le i\le m}$ satisfying (3.15). Then for all ℓ, η, $\delta > 0$ with $1 < \ell < +\infty$ there exists a function $u = (u^i)_{1\le i\le m} \in (L^2(0,T;H^1(\Omega)) \cap C([0,T);L^2(\Omega)))^m$, with $u^i \ge 0$ a.e. in Ω_T, solution of system (3.16) that satisfies the following entropy estimate for a.e. $t \in (0,T)$ with $u^i(t) = u^i(t,\cdot)$

$$\int_\Omega \sum_{i=1}^m \Psi_{0,\ell}(u^i(t)) + \delta\int_0^t\int_\Omega\sum_{i=1}^m \left|\nabla u^i\right|^2 + \delta_0\int_0^t\int_\Omega\sum_{i=1}^m\left|\nabla\rho_\eta\star u^i\right|^2 \le \int_\Omega\sum_{i=1}^m \Psi_{0,\ell}(u_0^i). \tag{3.17}$$

Proof of Proposition 3.3.

Our proof is based on the variant of Simon's Lemma (Theorem 2.4). Recall that $\Delta t = \dfrac{T}{K}$ where $K \in \mathbb{N}^*$ and $T > 0$ is given. We denote by C a generic constant independent of Δt and ε. For all $n \in \{0,\dots,K-1\}$ and $i = 1,\dots,m$, set $t_n = n\Delta t$ and let the piecewise continuous function in time:

$$U^{i,\Delta t}(t,x) := u^{i,n+1}(x), \quad \text{for } t \in (t_n, t_{n+1}], \tag{3.18}$$

with $U^{i,\Delta t}(0,x) := u_0^i(x)$ satisfying (3.9).

Step 1: Upper bound on $\left\|\nabla \mathbf{U}^{\Delta t}\right\|_{(L^2(0,T;L^2(\Omega)))^m}$ and $\left\|\mathbf{U}^{\Delta t}\right\|_{(L^2(0,T;L^2(\Omega)))^m}$
We will prove that $U^{\Delta t} = (U^{i,\Delta t})_{1\le i\le m}$ satisfies

$$\int_0^T \|\nabla U^{\Delta t}(t)\|^2_{(L^2(\Omega))^m} \;\le\; C.$$

For all $n \in \{0,\dots,K-1\}$ and $i = 1,\dots,m$, for $t \in (t_n, t_{n+1}]$ we have

$$\nabla U^{i,\Delta t}(t,x) = \nabla u^{i,n+1}(x).$$

Then

$$\int_{t_n}^{t_{n+1}} \|\nabla U^{i,\Delta t}(t)\|^2_{L^2(\Omega)} \;=\; \Delta t\|\nabla u^{i,n+1}\|^2_{L^2(\Omega)}.$$

Hence

$$\begin{aligned}
\int_0^T \|\nabla U^{\Delta t}(t)\|^2_{(L^2(\Omega))^m} &= \Delta t \sum_{k=0}^{K-1} \|\nabla u^{k+1}\|^2_{(L^2(\Omega)^m} \\
&\leq \frac{C_2}{\delta},
\end{aligned}$$

where we have used the entropy estimate (3.11) with

$$C_2 = \sum_{i=1}^{m} \int_\Omega \Psi_{\varepsilon,\ell}(u_0^i). \tag{3.19}$$

Hence, using Poincaré-Wirtinger's inequality we can get similarly an upper bound on $\int_0^T \left\|U^{i,\Delta t}\right\|^2_{(L^2(\Omega))^m}$ independently of Δt (using the fact that $\int_\Omega u^{i,n+1} = \int_\Omega u^{i,n} = \int_\Omega u^{i,0}$ by equation (3.10)) .

Step 2: Upper bound on $\left\|\mathbf{U}^{\Delta t}\right\|_{(\mathrm{Var}([0,T);H^{-1}(\Omega)))^m}$
We will prove that

$$\left\|U^{\Delta t}\right\|_{(\mathrm{Var}([0,T);H^{-1}(\Omega)))^m} \leq C.$$

Thus we have for $i = 1, \ldots, m$

$$\begin{aligned}
\left\|U^{i,\Delta t}\right\|_{\mathrm{Var}([0,T);H^{-1}(\Omega))} &= \sup \sum_{n=0}^{K-1} \left\|U^{i,\Delta t}(t_{n+1}) - U^{i,\Delta t}(t_n)\right\|_{H^{-1}(\Omega)} \\
&= \sup \sum_{n=0}^{K-1} \left\|u^{i,n+1} - u^{i,n}\right\|_{H^{-1}(\Omega)} \\
&= \Delta t \sup \sum_{n=0}^{K-1} \left\|\frac{u^{i,n+1} - u^{i,n}}{\Delta t}\right\|_{H^{-1}(\Omega)} \\
&\leq \Delta t \sup \sum_{n=0}^{K-1} \left\|T^{\varepsilon,\ell}(u^{i,n+1}) \left(\sum_{j=1}^{m} A_{ij} \nabla\rho_\eta \star \rho_\eta \star u^{j,n+1} + \delta \nabla u^{i,n+1}\right)\right\|_{L^2(\Omega)} \\
&\leq \ell \Delta t \sup \sum_{n=0}^{K-1} \left\{\|A\|_\infty \sum_{j=1}^{m} \left\|\nabla\rho_\eta \star u^{i,n+1}\right\|_{L^2(\Omega)} + \delta \left\|\nabla u^{i,n+1}\right\|_{L^2(\Omega)}\right\} \\
&\leq C,
\end{aligned}$$

where

$$\|A\|_\infty = \max_{1\leq i\leq m} \sum_{j=1}^{m} |A_{ij}|\,, \tag{3.20}$$

and we have used in the last inequality the entropy estimate (3.11), and the fact that

$$\Delta t \sum_{n=0}^{K-1} \left\|\nabla u^{i,n+1}\right\|_{L^2(\Omega)} \leq \sqrt{T} \left(\Delta t \sum_{n=0}^{K-1} \left\|\nabla u^{i,n+1}\right\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}.$$

Step 3: $\mathbf{U^{i,\Delta t}} \in \mathbf{L^p(0,T,L^2}(\Omega))$ with $\mathbf{p} > 2$

The estimate (3.11) gives us that $U^{i,\Delta t} \in L^\infty(0,T;L^1(\Omega)) \cap L^2(0,T;H^1(\Omega))$ for $i = 1,\dots,m$. Using Sobolev injections we get $H^1(\Omega) \hookrightarrow L^{2+\alpha(N)}(\Omega)$, with $\alpha(N) > 0$, and then $U^{i,\Delta t} \in L^2(0,T;L^{2+\alpha(N)}(\Omega))$. Hence by interpolation, we find that $U^{i,\Delta t} \in L^p(0,T;L^2(\Omega))$ with $\left(\frac{1}{p},\frac{1}{2}\right) = (1-\theta)\left(\frac{1}{\infty},\frac{1}{2}\right) + \theta\left(\frac{1}{2},\frac{1}{2+\alpha(N)}\right)$ and $\theta \in (0,1)$, which gives us that $p = \dfrac{4+4\alpha(N)}{2+\alpha(N)} > 2$.

Step 4: Passage to the limit when $(\Delta\mathbf{t},\varepsilon) \to (0,0)$

We have now that

$$\left\|U^{i,\Delta t}\right\|_{L^p(0,T;L^2(\Omega))} + \left\|U^{i,\Delta t}\right\|_{L^1(0,T;H^1(\Omega))} + \left\|U^{i,\Delta t}\right\|_{\mathrm{Var}([0,T);H^{-1}(\Omega))} \leq C.$$

Then by noticing that $H^1(\Omega) \overset{compact}{\hookrightarrow} L^2(\Omega) \overset{continous}{\hookrightarrow} H^{-1}(\Omega)$, and applying the variant of Simon's Lemma (Theorem 2.4), we deduce that $(U^{i,\Delta t})_{\Delta t}$ is relatively compact in $L^2(0,T;L^2(\Omega))$, and that there exists a function $U = (U^i)_{1\leq i\leq m} \in (L^2(0,T;H^1(\Omega)))^m$ such that, when $(\Delta t,\varepsilon) \to (0,0)$, we have (up to a subsequence)

$$\begin{cases} \nabla U^{i,\Delta t} \rightharpoonup \nabla U^i & \text{weakly in } L^2(0,T;L^2(\Omega)), \\ U^{i,\Delta t} \to U^i & \text{strongly in } L^2(0,T;L^2(\Omega)). \end{cases}$$

In addition we have that

$$\frac{U^{i,\Delta t}(t+\Delta t) - U^{i,\Delta t}(t)}{\Delta t} \to U^i_t \quad \text{in } \mathcal{D}'(\Omega_T).$$

Now since we have for $t \in (t_{n-1},t_n)$

$$\frac{u^{i,n+1} - u^{i,n}}{\Delta t} = \frac{U^{i,\Delta t}(t+\Delta t) - U^{i,\Delta t}(t)}{\Delta t},$$

system (3.10) can be written as

$$\frac{U^{i,\Delta t}(t+\Delta t) - U^{i,\Delta t}(t)}{\Delta t} = \mathrm{div}\left\{J^i_{\varepsilon,\ell,\eta,\delta}(U^{i,\Delta t},U^{i,\Delta t})\right\} \quad \text{in } \mathcal{D}'(\Omega), \tag{3.21}$$

where $J^i_{\varepsilon,\ell,\eta,\delta}$ is given in (3.1). Hence by multiplying this system by a test function in $\mathcal{D}(\Omega)$ and integrating over Ω, we can pass directly to the limit when $(\Delta t,\varepsilon) \to (0,0)$ in (3.21) to get

$$U^i_t = \mathrm{div}\left(T^{0,\ell}(U^i)\left(\sum_{j=1}^m A_{ij}\nabla\rho_\eta \star \rho_\eta \star U^j + \delta\nabla U^i\right)\right) \quad \text{in } \mathcal{D}'(\Omega_T),$$

where we used the weak L^2 - strong L^2 convergence in the products such $T^{\varepsilon,\ell}(U^{i,\Delta t})\nabla U^{i,\Delta t}$ to get the existence of a solution of system (3.16).

Step 5: Recovering the initial condition
Let $\bar\rho \in C_c^\infty(\mathbb{R})$ with $\bar\rho \geq 0$, $\int_{\mathbb{R}} \bar\rho = 1$ and supp $\bar\rho \subset (-\frac{1}{2}, \frac{1}{2})$. We set

$$\bar\rho_{\Delta t}(t) = \Delta t^{-1} \bar\rho(\Delta t^{-1} t).$$

Then we have

$$\begin{aligned}
\left\| \bar\rho_{\Delta t} \star U_t^{\Delta t} \right\|^2_{(L^2(0,T;H^{-1}(\Omega)))^m} &= \sum_{i=1}^m \int_0^T \left\| \sum_{n=0}^{K-1} (u^{i,n+1} - u^{i,n}) \delta_{t_{n+1}} \star \bar\rho_{\Delta t} \right\|^2_{H^{-1}(\Omega)} \\
&= \sum_{i=1}^m \sum_{n=0}^{K-1} \int_0^T (\Delta t \bar\rho_{\Delta t}(t - t_{n+1}))^2 \left\| \frac{u^{i,n+1} - u^{i,n}}{\Delta t} \right\|^2_{H^{-1}(\Omega)} \\
&= C_4 \Delta t \sum_{i=1}^m \sum_{n=0}^{K-1} \left\| \frac{u^{i,n+1} - u^{i,n}}{\Delta t} \right\|^2_{H^{-1}(\Omega)} \\
&\leq C_4 \Delta t \sum_{n=0}^{K-1} \sum_{i=1}^m \int_\Omega \left| T^{\varepsilon,\ell}(u^{i,n+1}) \left(\sum_{j=1}^m A_{ij} \nabla \rho_\eta \star \rho_\eta \star u^{j,n+1} + \delta \nabla u^{i,n+1} \right) \right|^2 \\
&\leq C_4 2\ \ell^2 \Delta t \sum_{n=0}^{K-1} \sum_{i=1}^m \int_\Omega \left\{ \left(\sum_{j=1}^m A_{ij} \nabla \rho_\eta \star \rho_\eta \star u^{j,n+1} \right)^2 + \delta^2 \left| \nabla u^{i,n+1} \right|^2 \right\} \\
&\leq C_4\ 2\ \ell^2 \Delta t \sum_{n=0}^{K-1} \int_\Omega \left\{ \|A\|^2 \left\| \nabla \rho_\eta \star \rho_\eta \star u^{n+1} \right\|^2_{(L^2(\Omega))^m} + \delta^2 \left\| \nabla u^{n+1} \right\|^2_{(L^2(\Omega))^m} \right\} \\
&\leq C_4\ 2\ \ell^2 \Delta t \sum_{n=0}^{K-1} \left\{ \|A\|^2 \|\nabla \rho_\eta \star u^{n+1}\|^2_{(L^2(\Omega))^m} + \delta^2 \left\| \nabla u^{n+1} \right\|^2_{(L^2(\Omega))^m} \right\} \\
&\leq C_4\ 2\ \ell^2 C_2 \left(\frac{\|A\|^2}{\delta_0} + \delta \right),
\end{aligned}$$

where $C_4 := \displaystyle\int_0^T \bar\rho(t)\, dt$. Moreover, for all $\varphi \in \mathcal{D}(\Omega_T)$ we have

$$\begin{aligned}
\int_0^T \int_\Omega \bar\rho_{\Delta t} \star U_t^{i,\Delta t} \cdot \varphi &= \int_0^T \int_\Omega (\bar\rho_{\Delta t} \star U^{i,\Delta t})_t \cdot \varphi = -\int_0^T \int_\Omega \bar\rho_{\Delta t} \star U^{i,\Delta t} \cdot \varphi_t \\
&= -\int_0^T \int_\Omega U^{i,\Delta t} \cdot \varphi_t \star \bar\rho_{\Delta t} \to -\int_0^T \int_\Omega U^i \cdot \varphi_t = \int_0^T \int_\Omega U_t^i \cdot \varphi,
\end{aligned}$$

which implies that $\bar\rho_{\Delta t} \star U_t^{i,\Delta t} \rightharpoonup U_t^i$ weakly in $L^2(0,T;H^{-1}(\Omega))$ when $\Delta t \to 0$. Similarly we can prove that $\bar\rho_{\Delta t} \star U^{i,\Delta t} \to U^i$ strongly in $L^2(0,T;L^2(\Omega))$. Then we deduce that $U^i \in \{g \in L^2(0,T;H^1(\Omega)); g_t \in L^2(0,T;H^{-1}(\Omega))\}$. And now $U^i(0,x)$ has sense, by Proposition 2.2, and we have that $U^i(0,x) = u_0^i(x)$ by Proposition 5.1.

Step 6: Proof of estimate (3.17)
By Step 4 we have that there exists a function $U^i \in L^2(0,T;H^1(\Omega))$ such that we have the following convergences when $(\Delta t, \varepsilon) \to (0,0)$

$$\left\{ \begin{array}{lcl} U^{i,\Delta t} & \to & U^i \\ \nabla U^{i,\Delta t} & \rightharpoonup & \nabla U^i \\ \nabla \rho_\eta \star U^{i,\Delta t} & \to & \nabla \rho_\eta \star U^i \end{array} \right| \quad \text{in} \quad L^2(0,T;L^2(\Omega)).$$

Now using the fact that the norm L^2 is weakly lower semicontinuous, with a sequence of integers n_t (depending on Δt) such that $t_{n_t+1} \to t \in (0,T)$ and

$$U^{i,\Delta t}(t) = U^{i,\Delta t}(t_{n_t+1}) = u^{n_t+1},$$

we get

$$\int_0^t \int_\Omega \left|\nabla U^i\right|^2 \le \liminf_{(\Delta t,\varepsilon)\to(0,0)} \int_0^{t_{n_t+1}} \int_\Omega \left|\nabla U^{i,\Delta t}\right|^2 = \liminf_{(\Delta t,\varepsilon)\to(0,0)} \Delta t \sum_{k=0}^{n_t} \int_\Omega |\nabla u^{i,k+1}|^2, \tag{3.22}$$

and

$$\int_0^t \int_\Omega \left|\nabla \rho_\eta \star U^i\right|^2 \le \liminf_{(\Delta t,\varepsilon)\to(0,0)} \int_0^{t_{n_t+1}} \int_\Omega \left|\nabla \rho_\eta \star U^{i,\Delta t}\right|^2 = \liminf_{(\Delta t,\varepsilon)\to(0,0)} \Delta t \sum_{k=0}^{n_t} \int_\Omega |\nabla \rho_\eta \star u^{i,k+1}|^2. \tag{3.23}$$

Moreover, since we have $U^{i,\Delta t} \to U^i$ in $L^2(0,T;L^2(\Omega))$, we get that for a.e. $t \in (0,T)$ (up to a subsequence) $U^{i,\Delta t}(t,\cdot) \to U^i(t,\cdot)$ in $L^2(\Omega)$. For such t we have (up to a subsequence) $U^{i,\Delta t}(t,\cdot) \to U^i(t,\cdot)$ for a.e. in Ω. Noticing that $\Psi_{\varepsilon,\ell}(a) \to \Psi_{0,\ell}(a)$ for a.e. $a \in \mathbb{R}$ and applying Lemma 5.2 we get that for a.e. $t \in (0,T)$

$$\sum_{i=1}^m \int_\Omega \Psi_{\varepsilon,\ell}(U^i(t)) \le \liminf_{(\Delta t,\varepsilon)\to(0,0)} \sum_{i=1}^m \int_\Omega \Psi_{\varepsilon,\ell}(U^{i,\Delta t}(t)) = \liminf_{(\Delta t,\varepsilon)\to(0,0)} \sum_{i=1}^m \int_\Omega \Psi_{\varepsilon,\ell}(u^{i,n_t+1}). \tag{3.24}$$

Therefore (3.22),(3.23) and (3.24) with the entropy estimate (3.11) give us tht for a.e. $t \in (0,T)$

$$\begin{aligned}
&\sum_{i=1}^m \int_\Omega \Psi_{\varepsilon,\ell}(U^i) + \delta \sum_{i=1}^m \int_0^t \int_\Omega \left|\nabla U^i\right|^2 + \delta_0 \sum_{i=1}^m \int_0^t \int_\Omega \left|\nabla \rho_\eta \star U^i\right|^2 \\
\le\ & \liminf_{\Delta t\to 0} \sum_{i=1}^m \int_\Omega \Psi_{\varepsilon,\ell}(u^{i,n_t+1}) + \liminf_{(\Delta t,\varepsilon)\to(0,0)} \delta \Delta t \sum_{i=1}^m \sum_{k=0}^{n_t} \int_\Omega |\nabla u^{i,k+1}|^2 \\
&+ \liminf_{\Delta t\to 0} \delta_0 \Delta t \sum_{i=1}^m \sum_{k=0}^{n_t} \int_\Omega |\nabla \rho_\eta \star u^{i,k+1}|^2 \\
\le\ & \sum_{i=1}^m \int_\Omega \Psi_{\varepsilon,\ell}(u_0^i) \le \sum_{i=1}^m \int_\Omega \Psi_{0,\ell}(u_0^i),
\end{aligned}$$

which implies our result.

Step 7: Non-negativity of $\mathbf{U^i}$
Let $\Omega^\varepsilon := \left\{U^{i,\Delta t} \le \varepsilon\right\}$. By estimate (3.11), we get that there exists a positive constant C

independent of ε and Δt such that for all $i = 1, \dots, m$ we have

$$\begin{aligned}
C &\geq \int_{\Omega} \Psi_{\varepsilon,\ell}(U^{i,\Delta t}) \\
&\geq \int_{\Omega^\varepsilon} \Psi_{\varepsilon,\ell}(U^{i,\Delta t}) \\
&= \int_{\Omega^\varepsilon} \frac{1}{e} + \frac{(U^{i,\Delta t})^2}{2\varepsilon} + U^{i,\Delta t} \ln \varepsilon - \frac{\varepsilon}{2} \\
&\geq \int_{\Omega^\varepsilon} \frac{1}{e} + \frac{(U^{i,\Delta t})^2}{2\varepsilon} + \varepsilon \ln \varepsilon - \frac{1}{2} \\
&\geq \int_{\Omega^\varepsilon} \frac{(U^{i,\Delta t})^2}{2\varepsilon} - \frac{1}{2}.
\end{aligned}$$

So we get that

$$\int_{\Omega^\varepsilon} \frac{(U^{i,\Delta t})^2}{2\varepsilon} \leq C + \frac{1}{2}. \tag{3.25}$$

Now by passing to the limit when $\varepsilon \to 0$ in (3.25) we deduce that we have certainly $\int_{\Omega^-} |U^i|^2 = 0$, where $\Omega^- := \{U^i \leq 0\}$, which gives us that $(U^i)^- = 0$ in $L^2(\Omega)$, where $(U^i)^- = \min(0, U^i)$. □

Remark 3.4 ***(Another method by Lions-Magenes)***
Note that it would be also possible to use a theorem in Lions-Magenes [19, Chap. 3, Theorem 4.1, page 257]. This would prove in particular the existence of a unique solution for the following system:

$$\left\{ \begin{array}{lll}
u_t^i & = \operatorname{div}\left\{ J^i_{\epsilon,\ell,\eta,\delta}(v,u) \right\} & in \quad \mathcal{D}'(\Omega_T), \\
J^i_{\epsilon,\ell,\eta,\delta}(v,u) & = T^{\epsilon,\ell}(v^i) \left\{ \sum_{j=1}^m A_{ij} \nabla \rho_\eta \star \rho_\eta \star u^j + \delta \nabla u^i \right\}, & \\
u^i(0,x) & = u_0^i(x) & in \quad \Omega,
\end{array} \right. \tag{3.26}$$

where $T^{\varepsilon,\ell}$ is given in (1.19).
It would then be possible to find a fixed point solution $v = u$ of (3.26) to recover a solution of (3.16). We would have to justify again the entropy inequality (3.17). Note that our method with time discretisation will be also applicable easily to the confined case (see the proof of Theorem 1.2), which we are not aware of a reference for an analogue result to Theorem 4.1 in [19] applicable directly to the confined case.

3.4 Passage to the limit when $(\ell, \eta, \delta) \to (\infty, 0, 0)$

In this subsection we pass to the limit when $(\ell, \eta, \delta) \to (\infty, 0, 0)$ in system (3.16) to get the existence result, announced in Theorem 1.1, of a solution for system (1.1),(1.3).
Let us recall system (1.1),(1.3). Asume that A satisfies (1.4). Let $u_0 = (u_0^i)_{1 \leq i \leq m}$ satisfying (1.7). Then we look for a solution $u = (u^i)_{1 \leq i \leq m}$ of the following system:

$$\left\{ \begin{array}{lll}
u_t^i & = div \left\{ u^i \sum_{j=1}^m A_{ij} \nabla u^j \right\} & \text{in } \mathcal{D}'(\Omega_T), \\
u^i(0,x) & = u_0^i(x) & \text{a.e. in } \Omega.
\end{array} \right. \tag{3.27}$$

Proof of Theorem 1.1

Let C be a constant independent of ℓ, η and δ that can be different from line to line, and $u^\ell := (u^{i,\ell})_{1\le i\le m}$ a solution of system (3.16), where we drop the indices η and δ to keep light notations. The proof is accomplished by passing to the limit when $(\ell,\eta,\delta)\to(\infty,0,0)$ in (3.16) and using Simon's lemma (Lemma 2.3), in order to get the existence result.

Step 1: $\mathbf{u^{i,\ell}\in L^p(0,T,L^2(\Omega))}$ with $\mathbf{p}>2$

The estimate (3.17) gives us that $u^{i,\ell}\in L^\infty(0,T;L^1(\Omega))\cap L^2(0,T;H^1(\Omega))$ for $i=1,\dots,m$. Using Sobolev injections we get $H^1(\Omega)\hookrightarrow L^{2+\alpha(N)}(\Omega)$, with $\alpha(N)>0$, and then $u^{i,\ell}\in L^2(0,T,L^{2+\alpha(N)}(\Omega))$. Hence by interpolation, we find that $u^{i,\ell}\in L^p(0,T,L^2(\Omega))$ with $\left(\frac{1}{p},\frac{1}{2}\right)=(1-\theta)\left(\frac{1}{\infty},\frac{1}{2}\right)+\theta\left(\frac{1}{2},\frac{1}{2+\alpha(N)}\right)$ and $\theta\in(0,1)$, which gives us that $p=\dfrac{4+4\alpha(N)}{2+\alpha(N)}>2$.

Step 2: Upper bound on $\mathbf{u_t^{i,\ell}}$

$$
\begin{aligned}
\sum_{i=1}^m \|u_t^{i,\ell}\|_{L^1(0,T;(W^{1,\infty}(\Omega))')} &= \sum_{i=1}^m \int_0^T \left\|u_t^{i,\ell}\right\|_{(W^{1,\infty}(\Omega))'}\\
&= \sum_{i=1}^m \int_0^T \left\| div\left\{T^{0,\ell}(u^{i,\ell})\left(\sum_{j=1}^m A_{ij}\nabla\rho_\eta\star\rho_\eta\star u^{j,\ell}+\delta\nabla u^{i,\ell}\right)\right\}\right\|_{(W^{1,\infty}(\Omega))'}\\
&\le \sum_{i=1}^m \int_0^T\int_\Omega \left|T^{0,\ell}(u^{i,\ell})\left\{\sum_{j=1}^m A_{ij}\nabla\rho_\eta\star\rho_\eta\star u^{j,\ell}+\delta\nabla u^{i,\ell}\right\}\right|\\
&\le \sum_{i=1}^m \int_0^T\int_\Omega \left\{u^{i,\ell}\rho_\eta\star\left|\sum_{j=1}^m A_{ij}\nabla\rho_\eta\star u^{j,\ell}\right|+\delta u^{i,\ell}\left|\nabla u^{i,\ell}\right|\right\}\\
&\le \sum_{i=1}^m \int_0^T\int_\Omega \left\{\left(\rho_\eta\star u^{i,\ell}\right)\left|\sum_{j=1}^m A_{ij}\nabla\rho_\eta\star u^{j,\ell}\right|+\delta u^{i,\ell}\left|\nabla u^{i,\ell}\right|\right\}\\
&\le \frac{1}{2}\left\|\rho_\eta\star u^\ell\right\|^2_{(L^2(0,T;L^2(\Omega)))^m}+\frac{\|A\|^2}{2}\left\|\nabla\rho_\eta\star u^\ell\right\|^2_{(L^2(0,T;L^2(\Omega)))^m}\\
&\quad+\frac{\delta}{2}\left\|\nabla u^\ell\right\|^2_{(L^2(0,T;L^2(\Omega)))^m}+\frac{\delta}{2}\left\|u^\ell\right\|^2_{(L^2(0,T;L^2(\Omega)))^m}\\
&\le C,
\end{aligned}
$$

where we have used in the last inequality the fact that $u^{i,\ell}$ satisfies the entropy estimate (3.17) (and again Poincaré-Wirtinger's inequality).

Step 3: Passage to the limit when $(\ell,\eta,\delta)\to(\infty,0,0)$

We have now that

$$
\left\|u^{i,\ell}\right\|_{L^p(0,T;L^2(\Omega))}+\left\|u^{i,\ell}\right\|_{L^1(0,T;H^1(\Omega))}+\left\|u_t^{i,\ell}\right\|_{L^1(0,T;(W^{1,\infty}(\Omega))')}\le C.
$$

Then by noticing that $H^1(\Omega) \overset{compact}{\hookrightarrow} L^2(\Omega) \overset{continous}{\hookrightarrow} (W^{1,\infty}(\Omega))'$, and applying Simon's Lemma (Lemma 2.3), we deduce that $(u^{i,\ell})_\ell$ is relatively compact in $L^2(0,T;L^2(\Omega))$, and that there exists a function $u^i \in \{g \in L^2(0,T;H^1(\Omega)), g_t \in L^1(0,T;(W^{1,\infty}(\Omega))')\}$ such that, when $(\ell,\eta,\delta) \to (\infty,0,0)$, we have (up to a subsequence)

$$\begin{cases} u^{i,\ell} \rightharpoonup u^i & \text{weakly in } L^2(0,T;H^1(\Omega)), \\ \nabla u^{j,\ell} \rightharpoonup \nabla u^j & \text{weakly in } L^2(0,T;L^2(\Omega)), \\ u_t^{i,\ell} \rightharpoonup u_t^i & \text{weakly in } D'(\Omega_T), \\ u^{i,\ell} \to u^i & \text{strongly in } L^2(0,T;L^2(\Omega)). \end{cases}$$

This implies in particular

$$\begin{cases} \nabla \rho_\eta \star \rho_\eta \star u^{j,\ell} \rightharpoonup \nabla u^j & \text{weakly in } L^2(0,T;L^2(\Omega)), \\ T^{0,\ell}(u^{i,\ell}) \to u^i & \text{strongly in } L^2(0,T;L^2(\Omega)), \end{cases}$$

where we have used in the last convergence the nonnegativity of u^i a.e. (because $u^{i,\ell} \to u^i$ a.e.). We recall problem (3.16)

$$u_t^{i,\ell} = \operatorname{div}\left\{ T^{0,\ell}(u^{i,\ell}) \left(\sum_{j=1}^m A_{ij} \nabla \rho_\eta \star \rho_\eta \star u^{j,\ell} + \delta \nabla u^{i,\ell} \right) \right\} \quad \text{in } \mathcal{D}'(\Omega_T). \tag{3.28}$$

Hence by multiplying this system by a test function in $\mathcal{D}(\Omega_T)$ and integrating over Ω_T we can pass directly to the limit in (3.28) as $(\ell,\eta,\delta) \to (\infty,0,0)$, and we get

$$u_t^i = \operatorname{div}\left\{ u^i \sum_{j=1}^m A_{ij} \nabla u^j \right\} \quad \text{in } \mathcal{D}'(\Omega_T).$$

where we used in particular the weak L^2 - strong L^2 convergence in the products such $T^{0,\ell}(u^{i,\ell}) \nabla u^{i,\ell}$. Then $u = (u^i)_{1 \le i \le m}$ is a solution of system (3.27).

Step 4: Recovering the initial condition

First of all, let $q = \dfrac{2p}{p+2} > 1$, where $p > 2$ is given in Step 1 of this proof. It remains to prove that for $i = 1, \ldots, m$

$$\left\| u_t^{i,\ell} \right\|_{L^q(0,T;(W^{1,\infty})'(\Omega))} < C.$$

Thus we have that

$$
\begin{aligned}
\|u_t^{i,\ell}\|_{L^q(0,T;(W^{1,\infty}(\Omega))')} = & \left(\int_0^T \left\|u_t^{i,\ell}\right\|^q_{(W^{1,\infty}(\Omega))'}\right)^{\frac{1}{q}} \\
= & \left(\int_0^T \left\| div\left\{T^{0,\ell}(u^{i,\ell})\left(\sum_{j=1}^m A_{ij}\nabla\rho_\eta \star \rho_\eta \star u^{j,\ell} + \delta\nabla u^{i,\ell}\right)\right\}\right\|^q_{(W^{1,\infty}(\Omega))'}\right)^{\frac{1}{q}} \\
\leq & \left(\int_0^T \left(\int_\Omega \left|T^{0,\ell}(u^{i,\ell})\left\{\sum_{j=1}^m A_{ij}\nabla\rho_\eta \star \rho_\eta \star u^{j,\ell} + \delta\nabla u^{i,\ell}\right\}\right|\right)^q\right)^{\frac{1}{q}} \\
\leq & \left(\int_0^T \left(\int_\Omega (\rho_\eta \star u^{i,\ell})\left|\sum_{j=1}^m A_{ij}\nabla\rho_\eta \star u^{j,\ell}\right| + \delta u^{i,\ell}\left|\nabla u^{i,\ell}\right|\right)^q\right)^{\frac{1}{q}} \\
\leq & \left\|u^{i,\ell}\right\|_{L^p(0,T;L^2(\Omega))}\left\|\sum_{j=1}^m A_{ij}\nabla\rho_\eta \star u^{j,\ell}\right\|_{L^2(0,T;L^2(\Omega))} + \delta\left\|u^{i,\ell}\right\|_{L^p(0,T;L^2(\Omega))}\left\|\nabla u^{i,\ell}\right\|_{L^2(0,T;L^2(\Omega))} \\
\leq & \left\|u^{i,\ell}\right\|_{L^p(0,T;L^2(\Omega))}\left(\|A\|_\infty \sum_{j=1}^m \left\|\nabla\rho_\eta \star u^{j,\ell}\right\|_{L^2(0,T;L^2(\Omega))} + \delta\left\|\nabla u^{i,\ell}\right\|_{L^2(0,T;L^2(\Omega))}\right) \\
\leq & C,
\end{aligned}
$$

where we have used in the fifth line Holder's inequality and in the last line the entropy estimate (3.17) and Step 1 of this proof, with $\|A\|_\infty$ is given in (3.20). Moreover since we have that $W^{1,1}(0,T;(W^{1,\infty}(\Omega))') \hookrightarrow C([0,T);(W^{1,\infty}(\Omega))')$ then $u^i(0,x)$ has sense and we have that $u^i(0,x) = u_0^i(x)$ for all $i = 1,\dots,m$ by Proposition 5.1.

Step 5: Proof of the estimate (1.8)
The proof is similar to Step 6 of the proof of Proposition 3.3. Hence by using Fatou's Lemma on the sequence $(\Psi_{0,\ell}(u^{i,\ell}))_\ell$, and the fact that the norm L^2 is lower semicontinuous, then by passing to the limit inferior in estimate (3.17), we get the estimate (1.8). □

4 Generalizations

4.1 Generalization on the matrix A

Assumption (1.4) can be weaken. Indead, we can assume that $A = (A_{ij})_{1\leq i,j\leq m}$ is a real $m \times m$ matrix that satisfies a positivity condition, in the sense that there exist two positive definite diagonal $m \times m$ matrices L and R and $\delta_0 > 0$, such that we have

$$\zeta^T LAR\zeta \geq \delta_0|\zeta|^2, \quad \text{for all} \quad \zeta \in \mathbb{R}^m. \tag{4.1}$$

Remark 4.1 ***(Comments on the positivity condition (4.1))***
The assumption of positivity condition (4.1), generalize our problem for A not necessarily having a symmetric part positive definite. Here is an example of such a matrix, whose symmetric part is not definite positive, but the symmetric part of LAR is definite positive

for some suitable positive diagonal matrices L and R.
We consider

$$A = \begin{pmatrix} 1 & -a \\ 2a & 1 \end{pmatrix} \text{ with } |a| > 2.$$

Indeed,

$$A^{sym} = \frac{A^T + A}{2} = \begin{pmatrix} 1 & \frac{a}{2} \\ \frac{a}{2} & 1 \end{pmatrix},$$

satisfying $\det(A^{sym}) = 1 - \dfrac{a^2}{4} < 0$. *And let*

$$L = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

On the other hand,

$$B = L.A.I_2 = \begin{pmatrix} 2 & -2a \\ 2a & 1 \end{pmatrix},$$

satisfies that

$$B^{sym} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix},$$

is definite positive.

Proposition 4.2 *(**The case where** $L = I_2$)*
Let A be a matrix that satisfies the positivity condition (4.1) with $L = I_2$. Then $\bar{u}$ is a solution for system (1.1) with the matrix $\bar{A} = A\,R$ (instead of A) if and only if $u^i = R_{ii}\,\bar{u}^i$ is a solution for system (1.1) with the matrix A.

Proof of Proposition 4.1
Let u be a solution for system (1.1) with the matrix $\bar{A} = A\,R$ (instead of A), i.e.

$$\bar{u}^i_t = \operatorname{div}\left(\bar{u}^i \sum_{j=1}^{m} A_{ij} R_{jj} \nabla \bar{u}^j\right).$$

This implies

$$R_{ii}\bar{u}^i_t = \operatorname{div}\left(R_{ii}\bar{u}^i \sum_{j=1}^{m} A_{ij} R_{jj} \nabla \bar{u}^j\right),$$

which means

$$u^i_t = \operatorname{div}\left(u^i \sum_{j=1}^{m} A_{ij} \nabla u^j\right).$$

□

Proposition 4.3 *(**The case where** $R = I_2$)*
Let $u^{n+1} = (u^{i,n+1})_{1\leq i\leq m}$ be a solution of system (3.10) with a matrix A satisfying the

positivity condition (4.1) with $R = I_2$ and L a positive diagonal matrix. Then u^{n+1} satisfies the following entropy estimate

$$\sum_{i=1}^{m}\int_\Omega L_{ii}\Psi_{\varepsilon,\ell}(u^{i,n+1}) \quad + \quad \delta\Delta t \min_{1\le i\le m}\{L_{ii}\}\sum_{i=1}^{m}\sum_{k=0}^{n}\int_\Omega |\nabla u^{i,k+1}|^2$$
$$+ \quad \delta_0\Delta t\sum_{i=1}^{m}\sum_{k=0}^{n}\int_\Omega |\nabla\rho_\eta\star u^{i,k+1}|^2 \le \sum_{i=1}^{m}\int_\Omega L_{ii}\Psi_{\varepsilon,\ell}(u_0^i)$$

Proof of Proposition 4.2
We have that (from fifth line of the computation in Step 5 of the proof of Proposition 3.2)

$$\begin{aligned}
\sum_{i=1}^{m}\int_\Omega L_{ii}\left(\frac{\Psi_{\varepsilon,\ell}(u^{i,n+1})-\Psi_{\varepsilon,\ell}(u^{i,n})}{\Delta t}\right) \quad &\le \quad -\int_\Omega\sum_{i=1}^{m}\sum_{j=1}^{m}L_{ii}A_{ij}(\nabla\rho_\eta\star\rho_\eta\star u^{j,n+1})\cdot\nabla u^{i,n+1}\\
&\quad -\delta\sum_{i=1}^{m}\int_\Omega L_{ii}|\nabla u^{i,n+1}|^2\\
&\le \quad \int_\Omega\sum_{i=1}^{m}\sum_{j=1}^{m}(\nabla\rho_\eta\star u^{j,n+1})L_{ii}A_{ij}(\nabla\rho_\eta\star u^{i,n+1})\\
&\quad -\delta\sum_{i=1}^{m}\int_\Omega L_{ii}|\nabla u^{i,n+1}|^2\\
&\le \quad -\delta_0\int_\Omega\sum_{i=1}^{m}|\nabla\rho_\eta\star u^{i,n+1}|^2-\delta\min_{1\le i\le m}\{L_{ii}\}\sum_{i=1}^{m}\int_\Omega|\nabla u^{i,n+1}|^2,
\end{aligned}$$

where we have used, in the last line, the fact that the matrix A satisfies (4.1) with $R = I_2$. Then by a straightforward recurrence we get (4.2). □

Corollary 4.4 *Theorem 1.1 and Theorem 1.2 still hold true if we replace condition (1.4) by condition (4.1).*

4.2 Generalisation on the problem

4.2.1 The tensor case

Our study can be applied on a generalized systems of the form

$$u_t^i = \sum_{j=1}^{m}\sum_{k=1}^{N}\sum_{l=1}^{N}\frac{\partial}{\partial x_k}\left(f_i(u^i)A_{ijkl}\frac{\partial u^j}{\partial x_l}\right) \qquad \text{for } i = 1,\dots,m, \tag{4.2}$$

where f_i satisfies

$$\begin{cases}
f_i\in C(\mathbb{R}), & \\
0\le f_i(a)\le C(1+|a|) & \text{for } a\in\mathbb{R} \text{ and } C>0,\\
c\,|a|\le f_i(a) & \text{for } a\in[0,a_0] \quad \text{with } a_0, c>0.\\
\displaystyle\int_{a_0}^{A}\frac{1}{f_i(a)}da<+\infty & \text{for all} \quad A\ge a_0.
\end{cases}$$

An example for such f_i is

$$f_i(a) = \max\left(0, \min\left(a, \sqrt{|a-1|}\right)\right).$$

Moreover, $A = (A_{ijkl})_{i,j,k,l}$ is a tensor of order 4 that satisfies the following positivity condition: there exists $\delta_0 > 0$ such that

$$\sum_{i,j,k,l} A_{ijkl}\, \eta^i\, \eta^j\, \zeta_k\, \zeta_l \;\geq \delta_0 |\eta|^2 |\zeta|^2 \quad \text{for all } \eta \in \mathbb{R}^m, \zeta \in \mathbb{R}^N. \tag{4.3}$$

The entropy function Ψ_i is chosen such that Ψ_i is nonnegative, lower semi-continuous, convex and satisfies that $\Psi_i''(a) = \dfrac{1}{f_i(a)}$ for $i = 1, \ldots, m$. Our solution satisfies the following entropy estimate for a.e. $t > 0$

$$\sum_{i=1}^m \int_\Omega \Psi_i(u^i(t)) + \delta_0 \sum_{i=1}^m \int_0^t \int_\Omega |\nabla u^i|^2 \leq \sum_{i=1}^m \int_\Omega \Psi_i(u_0^i). \tag{4.4}$$

To get this entropy we can apply the same strategy announced in Subsection 1.4 where $f_i(u^i)$ will be replaced by $T^{\varepsilon,\ell}(f_i(v^i))$ where $T^{\varepsilon,\ell}$ is the truncation operator given in (1.19) and we use the fact that

$$\begin{aligned} \int_\Omega \sum_{i,j,k,l} \frac{\partial u^i}{\partial x_k} A_{ijkl} \frac{\partial u^j}{\partial x_l} &= c \int_\Omega \sum_{i,j,k,l} \overline{\widehat{\left(\frac{\partial u^i}{\partial x_k}\right)}} A_{ijkl} \widehat{\left(\frac{\partial u^j}{\partial x_l}\right)} \\ &= c \int_\Omega \sum_{i,j,k,l} \zeta_k\, \widehat{u^i}\, A_{ijkl}\, \zeta_l\, \widehat{u^j} \\ &\geq c\, \delta_0 \int_\Omega |\zeta \widehat{u}|^2 = \delta_0 \int_\Omega |\nabla u|^2, \end{aligned}$$

with $c = \dfrac{1}{(2\pi)^N}$.

4.2.2 The variables coefficients case

Here the coefficients $A_{ij}(x,u)$ may depend continuously of (x,u). Then we have to take $\rho_\eta \star (A_{ij}(x,u)(\nabla \rho_\eta \star u^j))$ instead of $A_{ij} \nabla(\rho_\eta \star \rho_\eta \star u^j)$ in the approximate problem. We can consider a problem

$$u_t^i = \operatorname{div}\left(u^i \sum_{j=1}^m A_{ij}(x,u) \nabla u\right) + g^i(x,u),$$

where the source terms are not too large as u goes to infinity.

4.2.3 Laplace-type equations

Moreover, our theory applies to models of the form

$$u_t^i = \Delta(a_i(u) u^i) \quad \text{with } u = (u^i)_{1 \leq i \leq m}, \tag{4.5}$$

under these assumptions:

$$\left\{\begin{array}{l} a^i(u) \geq 0 \quad \text{if} \quad u^j \geq 0 \quad \text{for } j=1,\dots,m, \\ a_i \quad \text{is sublinear}, \\ a_i \in C^1, \\ Sym\left(\left(\dfrac{\partial a_i}{\partial u_j}\right)_{i,j}\right) \geq \delta_0 I \quad \text{with} \quad \delta_0 > 0, \end{array}\right. \tag{4.6}$$

where Sym denotes the symmetric part of a matrix. We can consider a particular case of (4.5) where $a^i(u) = \displaystyle\sum_{j=1}^m A_{ij} u^j$. Then problem (4.5) can be written as

$$u_t^i = \operatorname{div}\left\{u^i \sum_{j=1}^m A_{ij} \nabla u^j + \left(\sum_{j=1}^m A_{ij} u_j\right) \nabla u^i\right\}, \tag{4.7}$$

which can be also solved under these assumptions:

$$\left\{\begin{array}{l} A_{ij} \geq 0 \quad \text{for} \quad i,j=1,\dots,m, \\ Sym(A) \geq \delta_0 I. \end{array}\right.$$

5 Appendix: Technical results

In this section we will present some technical results that are used in our proofs.

Proposition 5.1 ***(Recovering the initial condition)***
Let Y be a Banach space with the norm $\|.\|_Y$ such that $L^2(\Omega) \hookrightarrow Y$ with a continuous embedding. Consider a sequence $(g_m)_m \in L^2(0,T;L^2(\Omega))$ such that $(g_m)_t$ is uniformly bounded in $L^q(0,T;Y)$ with $1 < q \leq 2$, and that there exists a function $g \in L^2(0,T;L^2(\Omega))$ such that $g_m \to g$ strongly in $L^2(0,T;L^2(\Omega))$. Let g_0 in Y. We assume that $(g_m)_{|t=0} \to g_0$ in Y. Then we have

$$g_{|t=0} = g_0 \qquad \text{in } Y.$$

Proof of Proposition 5.1
We have that for all $t \in (0,T)$

$$\begin{aligned} \|g_m(t) - g_m(0)\|_Y &= \left\|\int_0^t (g_m)_\tau(\tau)\right\|_Y \\ &\leq \int_0^t \|(g_m)_\tau(\tau)\|_Y \, ds \\ &\leq t^{\frac{q-1}{q}} \, \|(g_m)_\tau(\tau)\|_{L^q(0,T;Y)}, \end{aligned}$$

where we have used in the last line Holder's inequality. Moreover, Using the fact that $(g_m)_\tau$ is uniformly bounded in $L^q(0,T;Y)$, then there exists a constant C, such that

$$\|g_m(t) - g_m(0)\|_Y \leq Ct^{\frac{q-1}{q}}.$$

Now let $\varphi \in C_c^\infty((0,+\infty),\mathbb{R})$, be such that $\varphi \geq 0$. Using the above inequality, we get that

$$\int_0^t \|g_m(s) - g_m(0)\|_Y^q \varphi(s) \, ds \leq C^q \int_0^t s^{q-1} \varphi(s) \, ds. \tag{5.1}$$

Now since we have $L^2(0,T;L^2(\Omega)) \hookrightarrow L^q(0,T;Y)$ with a continuous embedding we get that

$$g_m \to g \quad \text{strongly in } L^q(0,T;Y).$$

By passing to the limit in m in (5.1), we deduce that

$$\int_0^t \left(\|g(s)-g_0\|_Y^q - C^q s^{q-1}\right)\varphi(s) \le 0.$$

Since $\varphi \ge 0$ is arbitrary, we deduce that for almost every t, we have

$$\|g(t)-g_0\|_Y \le C t^{\frac{q-1}{q}}.$$

Particularly, for $t=0$, we have

$$\|g(0)-g_0\|_Y = 0.$$

This implies the result. □

Lemma 5.2 ***(Convergence result)***
Let $(a_\varepsilon)_\varepsilon$ a real sequence such that $a_\varepsilon \to a_0$ when $\varepsilon \to 0$. Then we have

$$\Psi_{0,\ell}(a_0) \le \liminf_{\varepsilon\to 0} \Psi_{\varepsilon,\ell}(a_\varepsilon),$$

where $\Psi_{\varepsilon,\ell}$ and $\Psi_{0,\ell}$ are given in (3.8) and (3.14) respectively.

Proof of Lemma 5.2
Case 1: suppose that $a_0 > 0$ the result is easily obtained.
Case 2: suppose that $a_0 < 0$. Then we have $\Psi_{0,\ell}(a_0) = +\infty \le \liminf_{\varepsilon\to 0} \Psi_{\varepsilon,\ell}(a_\varepsilon) = +\infty$.
Case 3: suppose that $a_0 = 0$. Let $(b_\varepsilon)_\varepsilon$ a sequence that decreases to 0 when $\varepsilon \to 0$ with $b_\varepsilon > a_\varepsilon$. We can prove that $\Psi_{\varepsilon,\ell}(b_\varepsilon) \to 0 = \Psi_{0,\ell}(0)$. Now since $\Psi_{\varepsilon,\ell}$ is decreasing with respect to the variable a we get $\Psi_{\varepsilon,\ell}(a_\varepsilon) \ge \Psi_{\varepsilon,\ell}(b_\varepsilon)$. This ends the proof. □

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