Existence of traveling waves for Lipschitz discrete dynamics. Monostable case as a limit of bistable cases

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Abstract: We study discrete monostable dynamics with general Lipschitz non-linearities. This includes also degenerate non-linearities. In the positive monostable case, we show the existence of a branch of traveling waves solutions for velocities $c \geq c^+$, with non existence of solutions for $c < c^+$. We also give certain sufficient conditions to insure that $c^+ \geq 0$ and we give an example when $c^+ < 0$. We as well prove a lower bound of c^+, precisely we show that $c^+ \geq c^*$, where c^* is associated to a linearized problem at infinity. On the other hand, under a KPP condition we show that $c^+ \leq c^*$. We also give an example where $c^+ > c^*$.

This model of discrete dynamics can be seen as a generalized Frenkel-Kontorova model for which we can also add a driving force parameter σ. We show that σ can vary in an interval $[\sigma^-, \sigma^+]$. For $\sigma \in (\sigma^-, \sigma^+)$ this corresponds to a bistable case, while for $\sigma = \sigma^+$ this is a positive monostable case, and for $\sigma = \sigma^-$ this is a negative monostable case. We study the velocity function $c = c(\sigma)$ as σ varies in $[\sigma^-, \sigma^+]$. In particular for $\sigma = \sigma^+$ (resp. $\sigma = \sigma^-$), we find vertical branches of traveling waves solutions with $c \geq c^+$ (resp. $c \leq c^-$).

Our method of proof is new and relies on viscosity solutions. Moreover, the monostable case with $c = c^+$ is seen advantageously as a limit situation of the bistable case. For $c >> 1$, the traveling waves are constructed as perturbations of solutions of an associated ODE. Finally to fill the gap between $c = c^+$ and large c, we use certain hull functions that are associated to correctors of a homogenization problem.

Keywords: Traveling waves, degenerate monostable non-linearity, KPP non-linearity, bistable non-linearity, Frenkel-Kontorova model, viscosity solutions, Perron's method.

Contents

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1 Introduction

1.1 General motivation

Our initial motivation was to study the classical fully overdamped Frenkel-Kontorova model, which is a system of ordinary differential equations

$$\frac{dX_i}{dt} = X_{i+1} - 2X_i + X_{i-1} + f(X_i) + \sigma, \tag{1.1}$$

where $X_i(t) \in \mathbb{R}$ denotes the position of a particle $i \in \mathbb{Z}$ at time t, $\dfrac{dX_i}{dt}$ is the velocity of this particle, f is the force created by a 1-periodic potential and σ represents the constant driving force. Such external force could be for example $f(x) = 1 - \cos(2\pi x) \geq 0$. This kind of system can be, for instance, used as a model of the motion of a dislocation defect in a crystal (see the book of Braun and Kivshar [8]). This motion is described by particular solutions of the form

$$X_i(t) = \phi(i + ct) \tag{1.2}$$

with

$$\phi' \geq 0 \quad \text{and} \quad \phi \text{ is bounded.} \tag{1.3}$$

Such a solution, ϕ, is called a traveling wave solution and c denotes its velocity of propagation. From (1.1) and (1.2), it is equivalent to look for solutions ϕ of

$$c\phi'(z) = \phi(z+1) - 2\phi(z) + \phi(z-1) + f(\phi(z)) + \sigma \tag{1.4}$$

with $z = i + ct$. For such a model, and under certain conditions on f, we show the existence of traveling waves for each value of σ in an interval $[\sigma^-, \sigma^+]$ (see Theorem 1.7). We also get the whole picture (see Figure 4 for qualitative properties of this picture) of the velocity function $c = c(\sigma)$ with respect to the driving force σ, with vertical branches for $\sigma = \sigma^-$ or $\sigma = \sigma^+$.

When $f > 0 = f(0) = f(1)$ on $(0,1)$ and $\sigma = 0$, we can moreover normalize the limits of the profile ϕ as

$$\phi(-\infty) = 0 \quad \text{and} \quad \phi(+\infty) = 1. \tag{1.5}$$

This case is called a positive monostable case and is associated here to $\sigma^+ = 0$. Moreover, we can show the existence of a critical velocity c^+ such that the following holds. There exists a branch of traveling waves solutions for all velocity $c \geq c^+$ and there are no solutions for $c < c^+$.

The goal of this paper is to present similar results in a framework more general than (1.4). To this end, given a real function F (whose properties will be specified in Subsections 1.2 and 1.3), we consider the following generalized equation with $\sigma \in \mathbb{R}$

$$c\phi'(z) = F(\phi(z + r_0), \phi(z + r_1), ..., \phi(z + r_N)) + \sigma, \tag{1.6}$$

where $N \geq 0$ and $r_i \in \mathbb{R}$ for $i = 0, ..., N$ such that

$$r_0 = 0 \quad \text{and} \quad r_i \neq r_j \ \text{ if } \ i \neq j, \tag{1.7}$$

which does not restrict the generality. In (1.6), we are looking for both the profile ϕ and the velocity c.

Equation (1.1) can be seen as a discretization of the following reaction diffusion equation

$$u_t = \Delta u + f(u). \tag{1.8}$$

In 1937, Fisher [15] and Kolmogorov, Petrovsky and Piskunov [28] studied the traveling waves for equation (1.8) which they proposed as a model describing the spreading of a gene throughout a population. Later, many works have been devoted for such equation that appears in biological models for developments of genes or populations dynamics and in combustion theory (see for instance, Aronson, Weinberger [3, 4] and Hadeler, Rothe [22]). For more developments and applications in biology of reaction diffusion equations, the reader may refer to [31] and to the references cited therein. There is also a considerable work on the existence, uniqueness and stability of traveling waves and their speed of propagation for the homogeneous KPP-Fisher non-linearity (see for example [23, 24, 25, 26, 35]). Such results have been shown also for the inhomogeneous, heterogeneous and random KPP-Fisher non-linearity (see [6, 7, 30]).

Traveling waves were studied also for discrete bistable reaction diffusion equations (see for instance [9, 13]). See also [1] and the references therein. In the monostable case, we distinguish [27] (for nonlocal non-linearities with integer shifts) and [14, 29, 32, 33] (for problems with linear nonlocal part and with integer shifts also). See also [20] for particular monostable non-linearities with irrational shifts. We also refer to [19, 10, 21, 11, 12, 24, 34] for different positive monostable non-linearities. In the monostable case, we have to underline the work of Hudson and Zinner [27] (see also [34]), where they proved the existence of a branch of solutions $c \geq c^*$ for general Lipschitz non-linearities (with possibly an infinite number of neighbors $N = +\infty$, and possibly p types of different particles, while $p = 1$ in our study) but with integer shifts $r_i \in \mathbb{Z}$. However, they do not state the nonexistence of solutions for $c < c^*$. Their method of proof relies on an approximation of the equation on a bounded domain (applying Brouwer's fixed point theorem) and an homotopy argument starting from a known solution. The full result is then obtained as the size of the domain goes to infinity. Here we underline that our results hold for the fully nonlinear case with real shifts $r_i \in \mathbb{R}$.

Several approaches were used to construct traveling waves for discrete monostable dynamics. We already described the homotopy method of Hudson and Zinner [27]. In a second approach, Chen and Guo [11] proved the existence of a solution starting from an approximated problem. They constructed a fixed point solution of an integral reformulation (approximated on a bounded domain) using the monotone iteration method (with sub and supersolutions). This approach was also used to get the existence of a solution in [18, 12, 20, 21]. A third approach based on recursive method for monotone discrete in time dynamical systems was used by Wienberger et al. [29, 32]. See also [33], where this method is used to solve problems with a linear nonlocal part. In a fourth approach [19], Guo and Hamel used global space-time sub and supersolutions to prove the existence of a solution for periodic monostable equations.

There is also a wide literature about the uniqueness and the asymptotics at infinity of a solution for a monostable non-linearities, see for instance [10, 26] (for a degenerate case), [11, 12] and the references therein. Let us also mention that certain delayed reaction diffusion equations with some KPP-Fisher non-linearities do not admit traveling waves (see for example [18, 34]).

Finally, we mention that our method opens new possibilities to be adapted to more general problems. For example, we can think to adapt our approach to a case with possibly p types of different particles similar to [17]. The case with an infinite number of neighbors $N = +\infty$ could be also studied. We can also think to study fully nonlinear parabolic equations.

1.2 Main results in the monostable case

In this subsection, we consider equation (1.6) with $\sigma = 0$. We study the existence of traveling waves of equation (1.6) (with $\sigma = 0$) for positive degenerate monostable non-linearities and with conditions at infinity given by (1.5).

In order to present our results in this case, we have to introduce some assumptions on $F : [0,1]^{N+1} \to \mathbb{R}$.

Assumption ($A_{\mathbf{Lip}}$):

i) Regularity: $F \in \mathrm{Lip}([0,1]^{N+1})$.

ii) Monotonicity: $F(X_0, X_1, ..., X_N)$ is non-decreasing w.r.t. each X_i for $i \neq 0$.

Assumption ($P_{\mathbf{Lip}}$):

Positive degenerate monostability:
Let $f(v) = F(v, ..., v)$ such that $f(0) = f(1) = 0$, $f > 0$ in $(0,1)$.

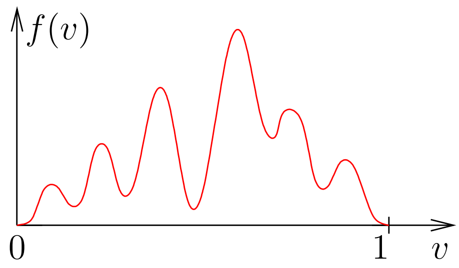

Figure 1: Positive degenerate monostable non-linearity f

Our main result is:

Theorem 1.1 (Monostable case: existence of a branch of traveling waves)
Assume (A_{Lip}) and (P_{Lip}). Then there exists a real c^+ such that for all $c \geq c^+$ there exists a traveling wave $\phi : \mathbb{R} \to \mathbb{R}$ solution (in the viscosity sense (see Definition 2.1)) of

$$(1.9) \qquad \begin{cases} c\phi'(z) = F(\phi(z+r_0), \phi(z+r_1), ..., \phi(z+r_N)) & \text{on} \quad \mathbb{R} \\ \phi \text{ is non-decreasing over } \mathbb{R} \\ \phi(-\infty) = 0 \quad \text{and} \quad \phi(+\infty) = 1. \end{cases}$$

On the contrary for $c < c^+$, there is no solution of (1.9).

Up to our knowledge, Theorem 1.1 is the first result for discrete dynamics with real shifts $r_i \in \mathbb{R}$ in the fully nonlinear case. Even when $r_i \in \mathbb{Z}$, the only result that we know for fully nonlinear dynamics is the one of Hudson and Zinner [27]. However, the nonexistence of solutions for $c < c^+$ is not addressed in [27].

See Figure 2 for an explicit Lipschitz non-linearity example for which our result (Theorem 1.1) is still true, even if $f'(0)$ is not defined. We also prove that the critical velocity c^+ is unstable in the following sense:

Proposition 1.2 (Instability of the minimal velocity c_F^+)
There exists a function F satisfying (A_{Lip}) and (P_{Lip}) with a minimal velocity c_F^+ such that there exists a sequence of functions F_δ (satisfying (A_{Lip}) and (P_{Lip})) with associated critical velocity $c_{F_\delta}^+$ satisfying

$$F_\delta \to F \quad \text{in} \quad L^\infty([0,1]^{N+1})$$

when $\delta \to 0$, but

$$\liminf_{\delta \to 0} c_{F_\delta}^+ > c_F^+.$$

We believe that the critical velocity c^+ contains information about $f'(0)$; similar to classical result in [28] which asserts that the critical velocity of reaction diffusion equation (1.8) is $c^+ = 2\sqrt{f'(0)}$. This shows that when F is only Lipschitz, it becomes very difficult to capture c_F^+ and to show Theorem 1.1 (see its proof, Section 7).

Examples of functions F satisfying assumptions (A_{Lip}) and (P_{Lip}) are given for $N = 2$, $r_0 = 0$, $r_1 = -1$, $r_2 = 1$ by

$$(1.10) \qquad F(X_0, X_1, X_2) = X_2 + X_1 - 2X_0 + g(X_0),$$

with for instance non-linearity $g(x) = x(1-x)$ or $g(x) = x^2(1-x)^2$.

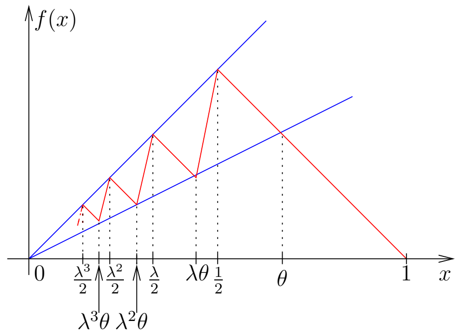

Figure 2: Lipschitz positive degenerate monostable non-linearity; the rest of the figure over $[0, \frac{\lambda^3}{2}]$ is completed by dilation of center 0 and ratio λ.

In the next result, we prove that the critical velocity c^+ (given in Theorem 1.1) is non-negative for particular F, i.e. we need to assume some smoothness and strict monotonicity on F near $\{0\}^{N+1}$; and this is given in assumption (P_{C^1}) (which is stronger than (P_{Lip})):

Assumption (P_{C^1}):

Positive degenerate monostability:
Let $f(v) = F(v, ..., v)$ such that $f(0) = 0 = f(1)$ and $f > 0$ in $(0, 1)$.

Smoothness near $\{0\}^{N+1}$:
F is C^1 over a neighborhood of $\{0\}^{N+1}$ in $[0, 1]^{N+1}$ and $f'(0) > 0$.

Proposition 1.3 (Non-negative c^+ for particular F)
Consider a function F satisfying (A_{Lip}) and (P_{C^1}). Let c^+ given by Theorem 1.1. Then we have $c^+ \geq 0$, if one of the three following conditions i), ii) or iii) holds true:

i) **Reflection symmetry of F**
Let $X = (X_i)_{i\in\{0,...,N\}} \in [0, 1]^{N+1}$. Assume that for all $i \in \{0, ..., N\}$ there exists $\bar{i} \in \{0, ..., N\}$ such that $r_{\bar{i}} = -r_i$; and

$$F(\overline{X}) = F(X) \quad \textit{for all} \quad X \in [0, 1]^{N+1},$$

where

$$\overline{X}_i = X_{\bar{i}} \quad \textit{for} \quad i \in \{0, ..., N\}.$$

ii) **All the r_i's "shifts" are non-negative**
Assume that $r_i \geq 0$ for all $i \in \{0, ..., N\}$.

iii) **Strict monotonicity**
Let

$$I = \left\{ i \in \{1, ..., N\} \textit{ such that there exists } \bar{i} \in \{1, ..., N\} \textit{ with } r_{\bar{i}} = -r_i \right\} \tag{1.11}$$

and assume that

$$\frac{\partial F}{\partial X_0}(0) + \sum_{i\in I} \min\left(\frac{\partial F}{\partial X_i}(0), \frac{\partial F}{\partial X_{\bar{i}}}(0)\right) > 0. \tag{1.12}$$

Notice that because of the monotonicity of F in X_j for $j \neq 0$, condition (1.12) is satisfied if

$$\frac{\partial F}{\partial X_0}(0) > 0.$$

Moreover, if

$$(1.13) \qquad I = \{1, ..., N\} \quad \text{and} \quad \frac{\partial F}{\partial X_i}(0) = \frac{\partial F}{\partial X_{\bar{i}}}(0) \quad \text{for all} \quad i \in I,$$

then condition (1.12) is equivalent to $f'(0) > 0$. In particular, under condition *i)* property (1.13) holds true. This shows that condition *iii)* is more general than condition *i)*.

Remark that if we replace (P_{C^1}) by (P_{Lip}) assuming *i)*, *ii)* or *iii)*, we do not know if $c^+ \geq 0$.

Proposition 1.4 (Counter example with $c^+ < 0$)
There exists a function F satisfying (A_{Lip}) and (P_{C^1}) such that the associated critical velocity (given in Theorem 1.1) is negative, i.e. $c^+ < 0$.

In the following proposition, we give a lower bound of the critical velocity c^+. To this end, assume that

$$(1.14) \qquad \exists\, i_0 \in \{0, ..., N\} \ \text{ such that } \ r_{i_0} > 0 \ \text{ and } \ \frac{\partial F}{\partial X_{i_0}}(0, ..., 0) > 0,$$

Proposition 1.5 (Lower bound for c^+)
Let F be a function satisfying (A_{Lip}) and (P_{C^1}). Let c^+ given by Theorem 1.1. Assume either (1.14) *or $c^+ < 0$, then*

$$c^+ \geq c^*,$$

where

$$(1.15) \qquad c^* := \inf_{\lambda > 0} \frac{P(\lambda)}{\lambda} \quad \textit{with} \quad P(\lambda) := \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0, ..., 0) e^{\lambda r_i}.$$

We can also get the result of Proposition 1.5 under conditions different from (1.14) (see Remark 9.1).
Here, it is natural to ask if we may have $c^+ = c^*$ in general or not. We give for instance in Lemma 9.3, an example of a non-linearity where we have $c^+ > c^*$ which answers the question. On the other hand, we can find a KPP type condition to insure the inequality $c^+ \leq c^*$, as show the following result:

Proposition 1.6 (KPP condition for $c^+ \leq c^*$)
Let F be a function satisfying (A_{Lip}) and (P_{Lip}). Let c^+ given by Theorem 1.1 and assume that F is differentiable at $\{0\}^{N+1}$ in $[0,1]^{N+1}$. If moreover F satisfies the KPP condition:

$$(1.16) \qquad F(X) \leq \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0, ..., 0) X_i \quad \textit{for every} \quad X \in [0,1]^{N+1},$$

then $c^+ \leq c^*$ with c^* defined in (1.15).

1.3 Main result on the velocity function

In this subsection, we consider equation (1.6) with a constant parameter $\sigma \in \mathbb{R}$ and $F : \mathbb{R}^{N+1} \to \mathbb{R}$. We are interested in the velocities c associated to σ (that we call roughly speaking the "velocity function").

For σ belonging to some interval $[\sigma^-, \sigma^+]$, we prove the existence of a traveling wave and we study the variation of its velocity c with respect to σ.
Let $E = (1, ..., 1)$, $\Theta = (\theta, ..., \theta) \in \mathbb{R}^{N+1}$ and assume that the function F satisfies:

Assumption ($\tilde{A}_{C^1}$):

Regularity: F is globally Lipschitz continuous over $\mathbb{R}^{N+1}$ and C^1 over a neighborhood in $\mathbb{R}^{N+1}$ of the two intervals $]0,\Theta[$ and $]\Theta,E[$.

Monotonicity: $F(X_0,...,X_N)$ is non-decreasing w.r.t. each X_i for $i \neq 0$.

Periodicity: $F(X_0+1,...,X_N+1) = F(X_0,...,X_N)$ for every $X = (X_0,...,X_N) \in \mathbb{R}^{N+1}$.

Notice that, since F is periodic in E direction, then F is C^1 over a neighborhood of $\mathbb{R}E \backslash (\mathbb{Z}E \cup \mathbb{Z}\Theta)$.

Assumption ($\tilde{B}_{C^1}$):
Define $f(v) = F(v,...,v)$ such that:

Bistability: $f(0) = f(1)$ and there exists $\theta \in (0,1)$ such that

$$\begin{cases} f' > 0 & \text{on} \quad (0,\theta) \\ f' < 0 & \text{on} \quad (\theta,1). \end{cases}$$

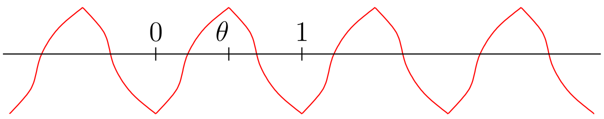

Figure 3: Bistable non-linearity f

See Figure 3 for an example of f satisfying ($\tilde{B}_{C^1}$). Notice that assumptions ($\tilde{A}_{C^1}$) and ($\tilde{B}_{C^1}$) holds true in particular for the Frenkel-Kontorova model for $\beta > 0$:

$$\frac{d}{dt}X_i = X_{i+1} + X_{i-1} - 2X_i - \beta \sin\left(2\pi\left(X_i + \frac{1}{4}\right)\right) + \sigma. \tag{1.17}$$

Theorem 1.7 (General case: traveling waves and the velocity function)
Under assumptions ($\tilde{A}_{C^1}$) and ($\tilde{B}_{C^1}$), define $\sigma^\pm$ as

$$\begin{cases} \sigma^+ = -\min f \\ \sigma^- = -\max f. \end{cases} \tag{1.18}$$

Associate for each $\sigma \in [\sigma^-,\sigma^+]$ the solutions $m_\sigma \in [\theta-1,0]$ and $b_\sigma \in [0,\theta]$ of $f(s)+\sigma = 0$. Then consider the following equation

$$\begin{cases} c\phi'(z) = F(\phi(z+r_0),\phi(z+r_1),...,\phi(z+r_N)) + \sigma \quad \text{on} \quad \mathbb{R} \\ \phi \text{ is non-decreasing over } \mathbb{R} \\ \phi(-\infty) = m_\sigma \quad \text{and} \quad \phi(+\infty) = m_\sigma + 1, \end{cases} \tag{1.19}$$

1- Bistable case: traveling waves for $\sigma \in (\sigma^-,\sigma^+)$
We have

(i) **(Existence of a traveling wave)**
For any $\sigma \in (\sigma^-,\sigma^+)$, there exists a unique real $c := c(\sigma)$, such that there exists a function $\phi_\sigma : \mathbb{R} \to \mathbb{R}$ solution of (1.19) in the viscosity sense.

(*ii*) **(Continuity and monotonicity of the velocity function)**
The map

$$\sigma \mapsto c(\sigma)$$

is continuous on (σ^-, σ^+) *and there exists a constant* $K > 0$ *such that the function* $c(\sigma)$ *is non-decreasing and satisfies*

$$\frac{dc}{d\sigma} \geq K|c| \quad on \ \ (\sigma^-, \sigma^+)$$

in the viscosity sense. In addition, there exists real numbers $c^- \leq c^+$ *such that*

$$\lim_{\sigma \to \sigma^-} c(\sigma) = c^- \quad and \quad \lim_{\sigma \to \sigma^+} c(\sigma) = c^+.$$

Moreover, either $c^- = 0 = c^+$ *or* $c^- < c^+$.

2- Monostable cases: vertical branches for $\sigma = \sigma^\pm$
We have

(*i*) **(Existence of traveling waves for** $c \geq c^+$ **when** $\sigma = \sigma^+$**)**
Let $\sigma = \sigma^+$, *then for every* $c \geq c^+$ *there exists a traveling wave* ϕ *solution of*

$$(1.20) \qquad \begin{cases} c\phi'(z) = F(\phi(z + r_0), \phi(z + r_1), ..., \phi(z + r_N)) + \sigma^+ \quad on \quad \mathbb{R} \\ \phi \ is \ non\text{-}decreasing \ over \ \mathbb{R} \\ \phi(-\infty) = 0 = m_{\sigma^+} \quad and \quad \phi(+\infty) = 1. \end{cases}$$

Moreover, for any $c < c^+$, *there is no solution* ϕ *of (1.20).*

(*ii*) **(Existence of traveling waves for** $c \leq c^-$ **when** $\sigma = \sigma^-$**)**
Let $\sigma = \sigma^-$, *then for every* $c \leq c^-$, *there exists a traveling wave* ϕ *solution of*

$$(1.21) \qquad \begin{cases} c\phi'(z) = F(\phi(z + r_0), \phi(z + r_1), ..., \phi(z + r_N)) + \sigma^- \quad on \quad \mathbb{R} \\ \phi \ is \ non\text{-}decreasing \ over \ \mathbb{R} \\ \phi(-\infty) = \theta - 1 = m_{\sigma^-} \quad and \quad \phi(+\infty) = \theta. \end{cases}$$

Moreover, for any $c > c^-$, *there is no solution* ϕ *of (1.21).*

Note that for the Frenkel Kontorova model (1.17), we have $\sigma^\pm = \pm 1$ and $c^+ > 0 > c^-$ (cf. Lemma 9.4), and Figure 4 illustrates the graph of the velocity $c(\sigma)$ which has a plateau at the level $c = 0$ in particular if $|\sigma| < \beta - 1$ (see Proposition 2.6).

In view of Theorem 1.7, we can ask the following:
Open question 1. For a general F, what is the precise behavior of the function $c(\sigma)$ close to the boundary of the plateau $c = 0$ and close to σ^+ and σ^-?
Open question 2. Can we construct a function F such that $c^+ = 0 = c^-$?
For indications in the direction of open question 1, see for instance [9] (discussion on page 4 after Theorem 1.2).

Remark 1.8 (sign of c^+ **and** c^-**)**
If we can apply Proposition 1.3 for $F + \sigma^+$, *we deduce that* $c^+ \geq 0$. *Similarly, by symmetry (see Lemma 3.7), it is possible to introduce similar assumptions to conclude that* $c^- \leq 0$.

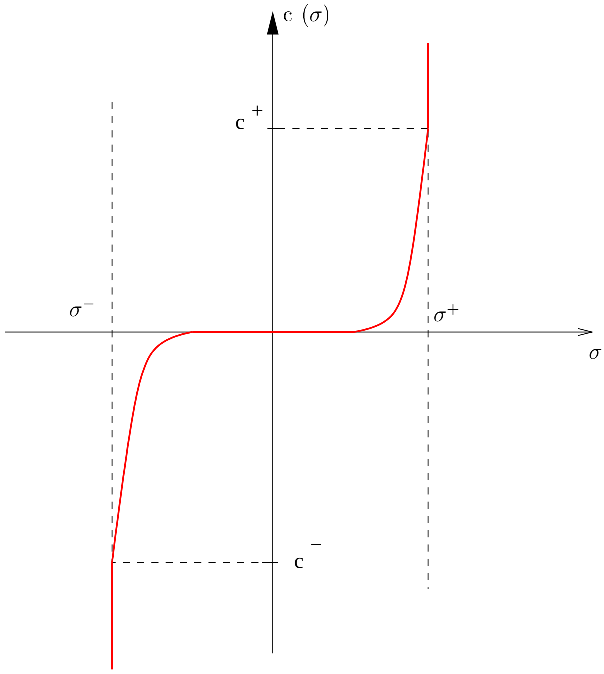

Figure 4: Typical graph of the velocity function $c(\sigma)$ with vertical branches at $\sigma = \sigma^{\pm}$.

Remark 1.9 (Existence of m_σ and b_σ for $\sigma \in [\sigma^-, \sigma^+]$)
Remark that under assumption $(\tilde{B}_{C^1})$ and from the definition of $\sigma^{\pm}$ (see (1.18)), the associated $m_\sigma \in [\theta - 1, 0]$ and $b_\sigma \in [0, \theta]$ exist uniquely for every $\sigma \in [\sigma^-, \sigma^+]$. This implies that the two maps $\sigma \to m_\sigma,\ b_\sigma$ are well defined.

Remark 1.10 (No solution of (1.19) when $\sigma \notin [\sigma^-, \sigma^+]$)
From the definition of $\sigma^{\pm}$ (see (1.18)), we see that the function $f + \sigma = 0$ has no solution if $\sigma \notin [\sigma^-, \sigma^+]$. Moreover, if ϕ is a bounded solution of

$$c\phi'(z) = F(\phi(z + r_0), \phi(z + r_1), ..., \phi(z + r_N)) + \sigma \quad \text{on} \quad \mathbb{R}, \tag{1.22}$$

then $\phi(\pm\infty)$ should solve the equation $f + \sigma = 0$. Thus, we conclude that (1.22) does not admit a bounded solution if $\sigma \notin [\sigma^-, \sigma^+]$.

Notice that Theorem 1.1 is a generalization of Theorem 1.7-2 (i) for $\sigma = \sigma^+$. Also, notice that Theorem 1.7-1 (i) is already proved in [1] (see [1, Proposition 2.3]).

As a notation, we set for a general function h:

$$F((h(z + r_i))_{i=0,...,N}) = F(h(z + r_0), h(z + r_1), ..., h(z + r_N))$$

and we define

$$r^* = \max_{i=0,...,N} |r_i|. \tag{1.23}$$

In the rest of the paper, we will use the notation introduced in Theorem 1.7.

1.4 Organization of the paper

Even if the main results of Subsections 1.2 and 1.3 are very different, the proofs are deeply related (because we use the results in the bistable case to deduce some results in the monostable case).

The paper is composed of three parts.

In a first part, we prove the existence of solutions for (1.9) for large velocities. This part is splitted into two sections (Sections 2 and 3). We recall, in Section 2, the notion of viscosity solutions and some useful results for monotone functions. Section 3 is devoted to the construction of a solution whenever we have positive supersolution with non-zero velocity. We also prove the existence of traveling waves solutions of (1.9) for $c >> 1$, which is applicable in particular for (1.20) and also for (1.21) when $c << -1$ (up to apply a suitable transformation).

We study in a second part the full range of velocities and it is decomposed into three sections (Sections 4, 5 and 6). Precisely, we revisit in Section 4 the results of [1]. In a first subsection, we generalize and precise the result of existence of a traveling wave obtained in [1]. We prove, in a second subsection, results about the passage to the limit in our equation and about the identification of the limits at infinity of the limit profile. In a third subsection, we apply the existence result of traveling waves obtained in Subsection 4.1 and we show the uniqueness of the velocity for solutions of (1.19) as a function of the driving force $\sigma \in (\sigma^-, \sigma^+)$. In Section 5, we prove the continuity and monotonicity of the velocity function over (σ^-, σ^+) and we show that the velocity function attains finite limits $c^\pm$ at $\sigma^\pm$. We also prove, in this section, the existence of solutions of (1.20) (resp. (1.21)) for $c = c^+$ (resp. $c = c^-$). In Section 6, we fill the gap by proving the existence of solutions of (1.20) (resp. (1.21)) for every $c \geq c^+$ (resp. $c \leq c^-$). Moreover, we show that for any $c < c^+$ (resp. $c > c^-$) there is no solution of (1.20) (resp. (1.21)). We prove Theorem 1.7 at the end of Section 6.

The third part is also decomposed into three sections (Sections 7, 8 and 9) and it is dedicated to define and study the critical velocity. For instance, Theorem 1.1 is proved in Section 7, which we split in three subsections. In Subsection 7.1, we recall an extension result to $\mathbb{R}^{N+1}$ of a non-linearity defined on $[0,1]^{N+1}$ and then we prove Theorem 1.1 in the special case where the non-linearity is smooth. Under some additional assumptions, we prove the result of Theorem 1.1 using another approach in Subsection 7.2. In Subsection 7.3, we give the proof of Theorem 1.1 in full generality for Lipschitz non-linearities, where the construction of the critical velocity c^+ follows the lines of the proof of the regular case, but requires a lot of work to adapt it to this very delicate situation. In Section 8, we prove a strong maximum principle (Proposition 8.1), a lower bound (Proposition 8.3) and a Harnack type inequality (Proposition 8.4) for a profile that we use to prove that $c^+ \geq c^*$ in Subsection 9.1. Section 9 is dedicated to properties of the critical velocity c^+. Subsection 9.1 is specified for the proof of Proposition 1.5 where we show that $c^+ \geq c^*$. In this subsection, we also show that $c^+ \leq c^*$ under a KPP type condition (precisely, we prove Proposition 1.6). We as well give an example (see Lemma 9.3) where $c^+ > c^*$. In Subsection 9.2, we prove that c^+ is non-negative under certain assumptions, namely Proposition 1.3. While in Subsection 9.3, we construct a counter-example for which $c^+ < 0$, i.e. Proposition 1.4 and we prove the instability result of Proposition 1.2.

Finally in the Appendix (Section 10), we prove and state two kinds of results (which are used to prove that $c^+ \geq 0$): first, extension by antisymmetry and antisymmetry-reflection (Propositions 10.1 and 10.4) and second, a comparison principle (Propositions 10.6 and 10.7).

1.5 Notations of our assumptions

In our paper, we introduce assumptions (A_{Lip}), (P_{Lip}) and (P_{C^1}) in Section 1.2, assumptions $(\tilde{A}_{C^1})$ and $(\tilde{B}_{C^1})$ in Section 1.3, assumptions $(\tilde{B}'_{C^1})$ and (B_{Lip}) in Section 4.3, assumptions $(\tilde{A}_{\text{Lip}})$, (A_{C^1}) and (P'_{C^1}) in Section 7.1 and assumption $(\tilde{B}_{m,b})$ in Section 4.2.

Generically, assumptions of type A holds for F, assumptions of type P are positivity assumptions on $f(v) = F(v, ..., v)$, and assumptions of type B are bistable assumptions for f.

Assumptions with *tilde* ($\tilde{}$) means that the functions F and f are considered on $\mathbb{R}^{N+1}$ and $\mathbb{R}$ respectively, and are assumed to be $(1, ..., 1)$-periodic and 1-periodic respectively. On the contrary, assumptions without *tilde* means assumptions for F and f on a finite box $[0, 1]^{N+1}$ and $[0, 1]$ respectively.

The subscript "Lip" means that we only require Lipschitz functions, while the subscript "C^1" means that we require C^1 functions (at least on some part of their domain of definition).

Finally, assumptions with *prime* ($'$) are (locally in the paper) variant of the assumptions without *prime*.

Part I

Vertical branches for large velocities

2 Preliminary results

We recall, in a first subsection, the definition of viscosity solutions, a stability result and Perron's method for constructing a solution. We state, in a second subsection, Helly's Lemma and the equivalence result between viscosity and almost everywhere solutions for non-decreasing functions. In a third subsection, we give an example with a discontinuous viscosity solution.

2.1 Viscosity solution

In the whole paper, we will use the notion of viscosity solutions that we introduce in this subsection. To this end, we recall that the upper and lower semi-continuous envelopes, u^* and u_*, of a locally bounded function u are defined as

$$u^*(y) = \limsup_{x \to y} u(x) \quad \text{and} \quad u_*(y) = \liminf_{x \to y} u(x).$$

Definition 2.1 (Viscosity solution)
Let $I = I' = \mathbb{R}$ (or $I = (-r^*, +\infty)$ and $I' = (0, +\infty)$) and $u : I \to \mathbb{R}$ be a locally bounded function, $c \in \mathbb{R}$ and F defined on $\mathbb{R}^{N+1}$.

- *The function u is a subsolution (resp. a supersolution) on I' of*

  $$cu'(x) = F((u(x + r_i))_{i=0,...,N}) + \sigma, \tag{2.1}$$

  if u is upper semi-continuous (resp. lower semi-continuous) and if for all test function $\psi \in C^1(I)$ such that $u - \psi$ attains a local maximum (resp. a local minimum) at $x^* \in I'$, we have

  $$c\psi'(x^*) \leq F((u(x^* + r_i))_{i=0,...,N}) + \sigma \quad \Big(\text{resp. } c\psi'(x^*) \geq F((u(x^* + r_i))_{i=0,...,N}) + \sigma\Big).$$

- *A function u is a viscosity solution of (2.1) on I' if u^* is a subsolution and u_* is a supersolution on I'.*

We also recall the stability result for viscosity solutions (see [5, Theorem 4.1] and [16, Proposition 2.4] for a similar proof).

Proposition 2.2 (Stability of viscosity solutions)
Consider a function F defined on $\mathbb{R}^{N+1}$ and satisfying $(\tilde{A}_{Lip})$ (introduced in Subsection 7.1). Assume that $(u_\varepsilon)_\varepsilon$ is a sequence of subsolutions (resp. supersolutions) of (2.1).

(*i*) *Let*

$$\overline{u}(x) = \limsup_{\varepsilon\to 0}{}^* u_\varepsilon(x) := \limsup_{(\varepsilon,y)\to(0,x)} u_\varepsilon(y) \quad \text{and} \quad \underline{u}(x) = \liminf_{\varepsilon\to 0}{}_* u_\varepsilon(x) := \liminf_{(\varepsilon,y)\to(0,x)} u_\varepsilon(y),$$

be the relaxed upper and lower semi-limits. If $\overline{u}$ *(resp.* $\underline{u}$*) is finite, then* $\overline{u}$ *is a subsolution (resp.* $\underline{u}$ *is a supersolution) of (2.1).*

(*ii*) *Let* $\mathcal{T}$ *be a nonempty collection of subsolutions of* (2.1) *and set* $U(x) = \sup_{u\in\mathcal{T}} u(x)$*. If* U^* *is finite then* U^* *is a subsolution of* (2.1)*. A similar result holds for supersolutions.*

Next, we state Perron's method that we will use to construct a solution in Section 3.

Proposition 2.3 (Perron's method ([16, Proposition 2.8**]))**
Let $I = (-r^*, +\infty)$ *and* $I' = (0, +\infty)$ *and* F *be a function satisfying* $(\tilde{A}_{Lip})$ *(introduced in Subsection 7.1). Let* u *and* v *defined on* I *satisfying*

$$u \le v \quad \text{on} \quad I,$$

such that u *and* v *are respectively a sub and a supersolution of (2.1) on* I'*. Let* $\mathcal{L}$ *be the set of all functions* $\tilde{v} : I \to \mathbb{R}$*, such that* $u \le \tilde{v}$ *over* I *with* $\tilde{v}$ *supersolution of (2.1) on* I'*. For every* $z \in I$*, let*

$$w(z) = \inf\{\tilde{v}(z) \quad \text{such that} \quad \tilde{v} \in \mathcal{L}\}.$$

Then w *is a solution of (2.1) over* I' *satisfying* $u \le w \le v$ *over* I*.*

2.2 Some results for monotone functions

In this subsection, we state Helly's Lemma for the convergence of a sequence of non-decreasing functions. We also recall the result about the equivalence between the viscosity and almost everywhere solutions. These results will be used later in Sections 4.3, 5, 6 and 7.

Lemma 2.4 (Helly's Lemma, (see [2], Section 3.3**, page** 70**))**
Let $(g_n)_{n\in\mathbb{N}}$ *be a sequence of non-decreasing functions on* $[a, b]$ *verifying* $|g_n| \le M$ *uniformly in* n*. Then there exists a subsequence* $(g_{n_j})_{j\in\mathbb{N}}$ *such that*

$$g_{n_j} \to g \quad \text{a.e. on} \quad [a, b],$$

with g *non-decreasing and* $|g| \le M$*.*

Now, we state the lemma for non-decreasing functions about the equivalence between a viscosity and an almost everywhere solution.

Lemma 2.5 (Equivalence between viscosity and a.e. solutions)
Let F *satisfy assumption* $(\tilde{A}_{Lip})$ *(introduced in Subsection 7.1). Let* $\phi : \mathbb{R} \to \mathbb{R}$ *be a non-decreasing function. Then* ϕ *is a viscosity solution of*

$$0 = F((\phi(x + r_i))_{i=0,\dots,N}) + \sigma \quad \text{on} \quad \mathbb{R}, \tag{2.2}$$

if and only if ϕ *is an almost everywhere solution of the same equation.*

For the proof of Lemma 2.5, we refer the reader to [1, Lemma 2.11].

2.3 Example of discontinuous viscosity solution

We give in this section an example of a discontinuous viscosity solution.

Proposition 2.6 (Discontinuous viscosity solution)
Consider $\beta > 0$, $\sigma \in \mathbb{R}$ and let (c, ϕ) be a solution of

$$\begin{cases} c\phi'(z) = \phi(z+1) - 2\phi(z) + \phi(z-1) + \beta \sin(2\pi\phi(z)) + \sigma \quad \text{on } \mathbb{R} \\ \phi \text{ is non-decreasing} \\ \phi(+\infty) - \phi(-\infty) = 1. \end{cases} \tag{2.3}$$

Then $\sigma^{\pm} = \pm\beta$. Moreover, if $|\sigma| < \beta - 1$, then $\phi \notin C^0$ and $c = 0$.

For the convenience of the reader we give the proof of this result (which is basically contained in Theorem 1.2 in Carpio et al. [9]).

Proof of Proposition 2.6
Clearly, we have $\sigma^{\pm} = \pm\beta$ (see Remark 1.10). Let $|\sigma| < \beta - 1$ and let us show that $\phi \notin C^0(\mathbb{R})$. Assume to the contrary that $\phi \in C^0(\mathbb{R})$.
Notice that because ϕ is non-decreasing and $\phi(+\infty) - \phi(-\infty) = 1$, we deduce that

$$\phi(z+1) - 2\phi(z) + \phi(z-1) \in [-1, 1].$$

Define now

$$\psi(z) = \phi(z+1) - 2\phi(z) + \phi(z-1) + \beta \sin(2\pi\phi(z)) + \sigma.$$

Because $\phi \in C^0$, then looking at the sup and inf of $\sin(2\pi\phi)$, we deduce that

$$\begin{cases} \sup_{\mathbb{R}} \psi \geq \beta + \sigma - 1 > 0 \\ \inf_{\mathbb{R}} \psi \leq -\beta + \sigma + 1 < 0, \end{cases}$$

where the strict inequalities follow from $|\sigma| < \beta - 1$. But $c\phi' = \psi$ which implies that $c\phi'$ changes sign. This is impossible because ϕ is non-decreasing. Therefore, $\phi \notin C^0(\mathbb{R})$, which implies that $c = 0$. □

3 Vertical branches for large velocities

We prove in this section that if (1.9) admits a positive supersolution ϕ ($\phi > 0$), then there exists a solution of (1.9) (cf. Proposition 3.2). Conversely, we also show that if (c, ϕ) is a solution of (1.9), then $(\tilde{c}, \phi)$ is a supersolution of 1.9 for all $\tilde{c} \geq c$ (see Corollary 3.4). As a consequence of Proposition 3.2), we prove that system (1.9) admits a solution for all $c >> 1$ (cf. Proposition 3.5).

The result is applicable in particular for function F defined on $\mathbb{R}^{N+1}$ and satisfying $(\tilde{A}_{C^1})$ and $(\tilde{B}_{C^1})$ with $\sigma = \sigma^+$, which can be always reduced to the case $\sigma^+ = 0$ by adding a constant to F, and hence we may get a solution for (1.20) for $c >> 1$. In this section, we show also the existence of solutions (1.21) for $c << -1$ which follows from the case $\sigma = \sigma^+$ using a transformation result (Lemma 3.7).

Definition 3.1 (Supersolution of (1.9)**)**
We say that (c, ψ) is a supersolution of (1.9) *if (c, ψ) satisfies*

$$\begin{cases} c\psi'(z) \geq F((\psi(z + r_i))_{i=0,\dots,N}) \quad \text{on} \quad \mathbb{R} \\ \psi \text{ is non-decreasing over } \mathbb{R} \\ \psi(-\infty) = 0 \quad \text{and} \quad \psi(+\infty) = 1. \end{cases}$$

Proposition 3.2 (Solution of (1.9) if it admits a positive supersolution)
Consider a function F satisfying (A_{Lip}) and (P_{Lip}). Assume that there exists a continuous supersolution (c,ψ) of (1.9) with $c \neq 0$ and $\psi > 0$. Then there exists a traveling wave ϕ such that (c,ϕ) is a solution of (1.9).

Proof of Proposition 3.2
We have (c,ψ) is a supersolution of (1.9) with $c \neq 0$ and $\psi > 0$. Up to space translation, we may assume that $\psi(0) = \theta \in (0,1)$. We will construct a solution using Perron's method.

Step 1: construction of a subsolution
Consider the constant function $\overline{\psi} = \varepsilon$ with $\varepsilon > 0$ small enough fixed. Then

$$0 = c\overline{\psi}'(x) \le F((\overline{\psi}(x + r_i))_{i=0,\dots,N}) = f(\varepsilon).$$

Hence $(c,\overline{\psi})$ is a subsolution of

$$cw'(x) = F((w(x + r_i))_{i=0,\dots,N}) \quad \text{on} \quad \mathbb{R}. \tag{3.1}$$

Step 2: construction of local solution
Since $\psi(-\infty) = 0$, $\psi(+\infty) = 1$, $\psi > 0$ and ψ is non-decreasing and continuous, then for ε small fixed and up to shift ψ, we can define $k_\varepsilon < 0$ such that

$$\psi(k_\varepsilon) = \varepsilon \quad \text{and} \quad \psi > \varepsilon \quad \text{on} \quad (k_\varepsilon, +\infty). \tag{3.2}$$

Then using Perron's method (Proposition 2.3), there exists a solution ϕ_ε of (3.1) on $(r^* + k_\varepsilon, +\infty)$ such that

$$\varepsilon \le \phi_\varepsilon \le \psi \quad \text{on} \quad (k_\varepsilon, +\infty).$$

Step 3: ϕ_ε is non-decreasing on $(k_\varepsilon, +\infty)$.
Define for $x \in (k_\varepsilon, +\infty)$ the function

$$\overline{\phi}(x) := \inf_{p \ge 0} \phi_\varepsilon(x + p).$$

Clearly, since $\varepsilon \le \phi_\varepsilon(x + p)$ for all $p \ge 0$ and $x \in (k_\varepsilon, +\infty)$, we get $\varepsilon \le \overline{\phi}(x) \le \phi_\varepsilon(x) \le \psi(x)$ for all $x \in (k_\varepsilon, +\infty)$. On the other hand, for all $p \ge 0$, $\phi_\varepsilon(x + p)$ is a solution of (3.1) over $(r^* + k_\varepsilon, +\infty)$, then $(\overline{\phi})_*$ is supersolution of (3.1) over $(k_\varepsilon + r^*, +\infty)$ (using Proposition 2.2 (ii)). Moreover, we have $\varepsilon \le (\overline{\phi})_* \le \psi$. But ϕ_ε is defined as the infimum of supersolutions (recall Proposition 2.3 for Perron's method), thus $\phi_\varepsilon \le (\overline{\phi})_* \le \overline{\phi} \le \phi_\varepsilon$ over $(k_\varepsilon, +\infty)$. Therefore, for every $p \ge 0$,

$$\phi_\varepsilon(x) = \overline{\phi}(x) \le \phi_\varepsilon(x + p) \quad \text{over} \quad (k_\varepsilon, +\infty),$$

and hence ϕ_ε is non-decreasing over $(k_\varepsilon, +\infty)$.

Step 4: passing to the limit $\varepsilon \to 0$

Step 4.1: setting
Since ϕ_ε is a non-decreasing solution of (3.1) on $(r^* + k_\varepsilon, +\infty)$, then $\phi_\varepsilon(+\infty)$ has to solve $f(x) = 0$ (see (3.1)). But ϕ_ε is a non-decreasing and $0 < \varepsilon \le \phi_\varepsilon \le \psi \le 1$ over $(k_\varepsilon, +\infty)$, we conclude that

$$\phi_\varepsilon(+\infty) = 1.$$

Moreover, from equation (3.1) and $c \neq 0$, we deduce in particular that ϕ_ε is Lipschitz on $(r^* + k_\varepsilon, +\infty)$ with

$$|\phi_\varepsilon'| \leq K_0 \quad \text{for a constant } K_0 \text{ independent of } \varepsilon.$$

In addition, since $\phi_\varepsilon(0) \leq \psi(0) = \theta$ and $\phi_\varepsilon(+\infty) = 1$, then there exists $x_\varepsilon \geq 0$ such that $\phi_\varepsilon(x_\varepsilon) = \theta$.

Notice also that for ε small enough, we have $r^* + k_\varepsilon < 0$ and we also have that k_ε is increasing w.r.t. ε and $k_\varepsilon \to -\infty$ as $\varepsilon \to 0$. Indeed, if $k_\varepsilon \to k_0 \in \mathbb{R}$, then $\psi(k_0) = 0$ which is impossible since $\psi > 0$.

Step 4.2**: global non-decreasing solution of** (3.1)
Let $\tilde{\phi}_\varepsilon(x) := \phi_\varepsilon(x + x_\varepsilon)$ which is a solution of (3.1) on $(-d_\varepsilon, +\infty)$, where $d_\varepsilon = x_\varepsilon - (r^* + k_\varepsilon)$. We have $\tilde{\phi}_\varepsilon(0) = \theta$ and $d_\varepsilon \to +\infty$ as $\varepsilon \to 0$ because $k_\varepsilon \to -\infty$ and $x_\varepsilon \geq 0$. We also have

$$|\tilde{\phi}_\varepsilon| \leq K_0 \quad \text{on } (-d_\varepsilon, +\infty).$$

Thus passing to the limit $\varepsilon \to 0$, $\tilde{\phi}_\varepsilon$ converges (using Ascoli's Theorem) to some non-decreasing ϕ solution of

$$\text{(3.3)} \qquad \begin{cases} c\phi'(x) = F((\phi(x + r_i))_{i=0,\ldots,N}) \\ 0 \leq \phi' \leq K_0 \quad \text{on} \quad \mathbb{R} \\ 0 \leq \phi \leq 1 \quad \text{and} \quad \phi(0) = \theta. \end{cases}$$

Let $a = \phi(-\infty)$ or $\phi(+\infty)$. Then it is easy to see that $0 = f(a)$ which implies that

$$\phi(-\infty) = 0 \quad \text{and} \quad \phi(+\infty) = 1.$$

Therefore ϕ is a solution of

$$\text{(3.4)} \qquad \begin{cases} c\phi'(x) = F(\phi(x + r_i))_{i=0,\ldots,N}) \quad \text{on } \mathbb{R} \\ \phi \text{ is non-decreasing} \\ \phi(-\infty) = 0 \quad \text{and} \quad \phi(+\infty) = 1, \end{cases}$$

and this ends the proof. □

Remark 3.3 (Relax of conditions of Proposition 3.2)
Up to adapt the proof of Proposition 3.2, it would be easy to relax the condition with $c = 0$ and/or ψ possibly discontinuous (but still monotone).

Corollary 3.4 (Half line of solutions)
Under the assumptions of Proposition 3.2, assume that (1.9) *admits a solution (c, ϕ) with $\phi > 0$. Then for all $\tilde{c} \geq c$ there exists a solution $\tilde{\phi}$ of* (1.9)*.*

Proof of Corollary 3.4
Let (c, ϕ) be a solution (1.9) and $\tilde{c} \geq c$, we have

Case 1**:** $\tilde{c} \neq 0$

$$\tilde{c}\phi'(z) \geq c\phi'(z) = F((\phi(z + r_i))_{i=0,\ldots,N}).$$

Hence $(\tilde{c}, \phi)$ is a supersolution of (1.9). Since $\phi > 0$, then using Proposition 3.2, we deduce the existence of a solution of (1.9) for every $\tilde{c} \geq c$, if $\tilde{c} \neq 0$.

Case 2**:** $\tilde{c} = 0$
If $\tilde{c} = 0$, then we consider a sequence of solutions (c_n, ϕ_n) with $c_n \neq 0$ and $c_n \to 0 = \tilde{c}$. Since

ϕ_n is monotone and bounded uniformly in n, then using Helly's Lemma (Lemma 2.4) and the diagonal extraction argument, ϕ_n converges up to a subsequence to a non-decreasing function ϕ a.e. Moreover, we can assume (up to translation) that $\phi_n(0) = \frac{1}{2}$, and hence we get that $\phi(0) = \frac{1}{2}$.

In addition, We have

$$c_n \int_{b_1}^{b_2} (\phi_n)'(z)dz = \int_{b_1}^{b_2} \big(F_n((\phi_n(z + r_i))_{i=0,\ldots,N})\big)dz$$

for every $b_1 < b_2$. That is,

$$c_n(\phi_n(b_2) - \phi_n(b_1)) = \int_{b_1}^{b_2} \big(F_n((\phi_n(z + r_i))_{i=0,\ldots,N})\big)dz.$$

But $|F((\phi_n(z + r_i))_{i=0,\ldots,N})| \leq \mathcal{M}_0$ for some $\mathcal{M}_0 > 0$ and

$$F_n((\phi_n(z + r_i))_{i=0,\ldots,N}) \to F((\phi(z + r_i))_{i=0,\ldots,N}) \quad \text{a.e.}$$

Thus, using Lebesgue's dominated convergence theorem, we pass to the limit $n \to +\infty$ and get

$$0 = \int_{b_1}^{b_2} F((\phi(z + r_i))_{i=0,\ldots,N})dz$$

which implies (since b_1 and b_2 are arbitrary) that

$$0 = F((\phi(z + r_i))_{i=0,\ldots,N}) \tag{3.5}$$

almost everywhere. Then by Lemma 2.5, ϕ verifies (3.5) in the viscosity sense. and satisfies (4.6). Moreover, since $\phi(0) = \frac{1}{2}$, we can deduce that ϕ is a solution of (1.9). □

Proposition 3.5 (Existence of traveling waves for $c >> 1$)
Consider a function F satisfying (A_{Lip}) and (P_{Lip}). Then for $c >> 1$, there exists a traveling wave ϕ solution of (1.9).

Proof of Proposition 3.5
The strategy of the proof consists in constructing a positive supersolution for $c >> 1$ of a re-scaled form of the equation

$$c\phi'(y) = F(\phi(y + r_0), \phi(y + r_1), ..., \phi(y + r_N)) \quad \text{on} \quad \mathbb{R}, \tag{3.6}$$

then we conclude by Proposition 3.2.

Step 1: re-scaling equation (3.6)
If ϕ is a supersolution of (3.6) (with $\phi(-\infty) = 0$, $\phi(+\infty) = 1$), then for every $z \in \mathbb{R}$, the function h defined as

$$h(z) := \phi(cz)$$

has to satisfy, knowing that $c >> 1$,

$$h'(z) = F\left(\left(h\left(z + \frac{r_i}{c}\right)\right)_{i=0,\ldots,N}\right) \quad \text{on} \quad \mathbb{R}. \tag{3.7}$$

Step 2: supersolution of (3.7)
In order to construct a supersolution of (3.7), we first mention some useful properties of the solution of the ODE

$$h_0' = F(h_0, ..., h_0) = f(h_0) \geq 0, \tag{3.8}$$

with $h_0(0) = \frac{1}{2}$.

Step 2.1: existence and monotonicity of h_0

Since $f > 0$ on $(0,1)$ and f is Lipschitz over $[0,1]$ (see assumptions (A_{Lip}) and (P_{Lip})), then there exists a C^1 solution h_0 of (3.8) defined on $\mathbb{R}$, with values in $[0,1]$, satisfying

$$h_0' > 0 \quad \text{on} \quad \mathbb{R}. \tag{3.9}$$

Since the constant functions 0 and 1 are respectively a sub and a supersolution of (3.8) (since $f(0) = f(1) = 0$), then

$$0 \le h_0(z) \le 1.$$

We also easily deduce that

$$h_0(-\infty) = 0 \quad \text{and} \quad h_0(+\infty) = 1.$$

Step 2.2: supersolution of (3.7)

The proof is similar to Step 2.2. Let $\varepsilon = \dfrac{1}{c}$ and $0 < \delta = \overline{M}\varepsilon$ with $\overline{M}$ chosen large, and c chosen such that $\overline{a} = 1 + \delta \le 2$. Then consider the function

$$\overline{h}(z) = h_0(\overline{a}z)$$

that we want to show to be a supersolution of (3.7) on $\mathbb{R}$, taking the advantage of the fact that (3.8) is a caricature of (3.7) for large c.

We have

$$\overline{h}(z + \varepsilon r_i) = h_0(\overline{a}z) + \varepsilon \overline{a} r_i \overline{L}_i \quad \text{with} \quad \overline{L}_i = \int_0^1 h_0'(\overline{a}z + \varepsilon \overline{a} r_i t).$$

Because $F \in \text{Lip}([0,1]^{N+1})$ for some Lipschitz constant L, we get

$$\begin{aligned} F((\overline{h}(z+\varepsilon r_i))_{i=0,\dots,N}) - f(h_0(\overline{a}z)) &= F((h_0(\overline{a}z) + \overline{a}\varepsilon r_i \overline{L}_i))_{i=0,\dots,N}) - F((h_0(\overline{a}z))_{i=0,\dots,N}) \\ &\le \varepsilon \overline{a} L \begin{vmatrix} r_0 \overline{L}_0 \\ \vdots \\ r_N \overline{L}_N \end{vmatrix}, \end{aligned}$$

where $r^* = \max_{i=0,\dots,N} |r_i|$ (recall (1.23)).

We now estimate the $\overline{L}_i$'s.

Case 1: $f \in C^1([0,1])$

If $f \in C^1([0,1])$, then for $z \in \mathbb{R}$, we have

$$h_0''(z) = f'(h_0(z))h_0'(z).$$

As $h_0' > 0$ on $\mathbb{R}$ and $f \in C^1([0,1])$, we get for $z \in \mathbb{R}$

$$(\ln(h_0'(z)))' = f'(h_0(z)),$$

where the absolute value of the right hand side is bounded by some constant $\mathcal{K}$. Hence, using the continuity of h_0', for any $b \in \mathbb{R}$ and for all $z \in \mathbb{R}$, we obtain

$$\ln\left(\frac{h_0'(z+b)}{h_0'(z)}\right) \le \mathcal{K}|b|.$$

This implies that

$$h_0'(z+b) \le h_0'(z)e^{\mathcal{K}|b|} \quad \text{for every} \quad z \in \mathbb{R}. \tag{3.10}$$

Case 2: $f \in$ Lip$([0,1])$
We want to show that (3.10) is still true if $f \in \mathrm{Lip}([0,1])$, and the point is to regularize by convolution the function f and then to pass to the limit. Using the extension result (cf. Lemma 7.1), there exists a function $\tilde{F}$ defined over $\mathbb{R}^{N+1}$ and satisfying $(\tilde{A}_{\mathrm{Lip}})$. Moreover, the function $\tilde{f}(v) := \tilde{F}(v, ..., v)$ is nothing but the periodic extension of f with period 1.

Let $\rho_\varepsilon(x) = \frac{1}{\varepsilon}\rho(\frac{x}{\varepsilon})$, where ρ is a mollifier and define the function $\tilde{f}_\varepsilon(x) := \tilde{f} \star \rho_\varepsilon(x)$. Then consider the ODE

$$\begin{cases} h_\varepsilon' = \tilde{f}_\varepsilon(h_\varepsilon) \\ h_\varepsilon(0) = \dfrac{1}{2}. \end{cases} \tag{3.11}$$

Since $\tilde{f}_\varepsilon$ is C^1, then there exists a unique regular solution h_ε defined over $\mathbb{R}$ and satisfies

$$h_\varepsilon'(z+b) \le h_\varepsilon'(z)e^{\mathcal{K}|b|} \quad \text{for every} \quad z \in \mathbb{R}. \tag{3.12}$$

Moreover, since $\tilde{f}_\varepsilon$ is periodic smooth, then there exists some C independent of ε such that

$$|h_\varepsilon'| \le C \quad \text{on} \quad \mathbb{R}.$$

Therefore, using Ascoli's theorem and the extraction diagonal argument, h_ε converges locally uniformly to some h_1 that solves in the classical sense

$$\begin{cases} h_1' = \tilde{f}(h_1) \\ h_1(0) = \dfrac{1}{2}, \end{cases} \tag{3.13}$$

and

$$h_1'(z+b) \le h_1'(z)e^{\mathcal{K}|b|} \quad \text{for every} \quad z \in \mathbb{R}.$$

But the constant functions 0 and 1 are respectively sub and supersolution of (3.13), then

$$0 \le h_1 \le 1,$$

that is, h_1 is a solution of (3.8). Thus by uniqueness, we get that $h_1 = h_0$, and hence h_0 satisfies (3.10).

Consequences in both Case 1 and Case 2
Now, we go back to estimate the $\overline{L}_i$'s. Using (3.10) for $b = \overline{a}\varepsilon r_i t$ and using the fact that $\overline{a} < 1$, we get for every $i \in \{0, ..., N\}$ that

$$0 \le \overline{L}_i = \int_0^1 h_0'(\overline{a}z + \overline{a}\varepsilon r_i t)dt \le h_0'(\overline{a}z)e^{\mathcal{K}\overline{a}\varepsilon|r_i|} \le h_0'(\overline{a}z)e^{\mathcal{K}\varepsilon r^*} =: \overline{K}h_0'(\overline{a}z).$$

This implies that

$$F((\overline{h}(z+\varepsilon r_i))_{i=0,\dots,N}) - f(h_0(\overline{a}z)) \le 2\varepsilon L_1 r^* \overline{K} h_0'(\overline{a}z),$$

where we have used that $\overline{a} \le 2$ and that $L_1 := L \begin{vmatrix} 1 \\ \vdots \\ 1 \end{vmatrix}$.

Therefore, we deduce that with $\delta = \overline{M}\varepsilon$

$$\begin{aligned}
\overline{h}'(z) - F((\overline{h}(z+\varepsilon r_i))_{i=0,...,N}) &= \overline{a}h_0'(\overline{a}z) - F((\overline{h}(z+\varepsilon r_i))_{i=0,...,N}) \\
&= \delta h_0'(\overline{a}z) - \Big(F((\overline{h}(z+\varepsilon r_i))_{i=0,...,N}) - f(h_0(\overline{a}z))\Big) \\
&\geq \varepsilon\Big(\overline{M} - 2L_1 r^*\overline{K}\Big)h_0'(\overline{a}z) \\
&\geq 0,
\end{aligned}$$

if we choose $\overline{M} \geq 2L_1 r^* \overline{K}$. Therefore $\overline{h}$ is a supersolution of (3.7).

Step 3: solution of (3.7) **for** $c >> 1$
We have $\overline{h}(z) = h_0(\overline{a}z)$ is a supersolution of (3.7). Moreover, since $\overline{a} > 0$, $h_0' > 0$ on $\mathbb{R}$ and

$$h_0(-\infty) = 0 \quad \text{and} \quad h_0(+\infty) = 1,$$

we deduce that

$$0 < \overline{h} < 1.$$

Therefore, using Proposition 3.2, we get the existence of solution of (3.7) for $c >> 1$ and hence for (1.9). □

Lemma 3.6 (Vertical branches for $\sigma = \sigma^{\pm}$)
Consider a function F satisfying $(\tilde{A}_{C^1})$ and $(\tilde{B}_{C^1})$. Assume that $\sigma = \sigma^+$ (resp. $\sigma = \sigma^-$), then for $c >> 1$ (resp. $c << -1$), there exists a traveling wave solution of (1.20) *(resp.* (1.21)*).*

Proof of Lemma 3.6
Proving the existence of solution for $c >> 1$ when $\sigma = \sigma^+$ follows exactly from Proposition 3.5 where $\sigma^+ = 0$. However, the proof of the result for $c << -1$ when $\sigma = \sigma^-$ follows from the proof of the case $\sigma = \sigma^+$ and the transformation lemma below (Lemma 3.7). □

Lemma 3.7 (Transformation of solutions)
Let ϕ be a solution of

$$c\phi'(z) = F((\phi(z+r_i))_{i=0,...,N}) + \sigma^- \quad \text{over} \quad \mathbb{R}, \tag{3.14}$$

then

$$\overline{\phi}(z) = \theta - \phi(-z)$$

is a solution of (3.14) with F, c, r_i and σ^- replaced respectively by

$$\begin{cases} \overline{F}(X_0, ..., X_N) = -F((\theta - X_i)_{i=0,...,N}) \\ \overline{c} = -c, \quad \overline{r}_i = -r_i \quad \text{and} \quad \overline{\sigma}^+ = -\sigma^- \end{cases} \tag{3.15}$$

Moreover, if F satisfies $(\tilde{A}_{C^1})$ and $(\tilde{B}_{C^1})$ then $\overline{F}$ satisfies $(\tilde{A}_{C^1})$ and $(\tilde{B})$ (with $\overline{f}(v) = \overline{F}(v, ..., v)$).

Proof of Lemma 3.7
The proof of Lemma 3.7 is straightforward. □

Part II

Study of the full range of velocities

4 Revisiting results of [1]

This section is divided into three subsections. In a first subsection, we generalize the result of existence of traveling waves obtained in [1]. We present, in a second subsection, some techniques to pass to the limit in the equation. In a third subsection, we apply the existence result of first subsection for the bistable case when $\sigma \in (\sigma^-, \sigma^+)$.

4.1 Bistable case

We prove in this subsection the existence of traveling waves for the bistable non-linearity under weaker assumptions. This result is not proved in [1] and it is more general. We will use this general result later in the proof of Theorem 1.1, Section 7.3, Step 1. This result will be used also to prove that $c^+ \geq 0$ (proof of Proposition 1.3), Section 9.2.

In order to present our result, we assume that

Assumption $(B_{\mathbf{Lip}})$**:**
Let $f(v) := F(v, ..., v)$ and assume

Instability: $f(0) = 0 = f(1)$ and there exists $b \in (0,1)$ such that $f(b) = 0$, $f_{|_{(0,b)}} < 0$ and $f_{|_{(b,1)}} > 0$.

Strict monotonicity: There exists some $\eta > 0$ such that

$$F(X + (\omega, ..., \omega)) - F(X) \geq \eta\, \omega$$

for $\omega > 0$ small enough and for all X close enough to $(b, ..., b)$.

Proposition 4.1 (Existence of c for a Lipschitz bistable non-linearity)
Consider a function F defined over $[0,1]^{N+1}$ and satisfying (A_{Lip}) and (B_{Lip}). Then there exist a real c and a function ϕ solution of (1.9) in the classical sense if $c \neq 0$ and almost everywhere if $c = 0$. Moreover, there is no $a > r^*$ and $x \in \mathbb{R}$ such that

$$\phi = b \quad on \quad [x - a, x + a]. \tag{4.1}$$

This result is the analogue of the existence result of [1, Proposition 2.3], assuming that F is less regular near the instability b which is replaced by the strict monotonicity of F near b.

Proof of Proposition 4.1
As it is written above, the proof of Proposition 4.1 is a variant of the proof of [1, Proposition 2.3]. However, in this case, we obtain the contradiction using the strict monotonicity (Step 4.3) while the rest of the proof (Step 0 to Step 4.2 and Step 5) stays the same.
We now prove the contradiction using the strict monotonicity, revisiting Step 4.3 of the proof of [1, Proposition 2.3].

Step 4.3**: getting a contradiction**
We recall that we consider an approximation ϕ_p of the profile ϕ, for some parameter p going to zero, which satisfies

$$c_p \phi_p'(z) = F((\phi_p(x + r_i))_{i=0,...,N}).$$

We construct (see [1, Proposition 2.3]) a local minimum x_p^* of ψ_p satisfying

$$0 < m_p = \psi_p(x_p^*),$$

where $\psi_p(x) = (\phi_p)_*(x+a) - (\phi_p)^*(x-a)$. Then it is possible to see as in [1, Proposition 2.3, Step 4.3], that

$$0 \geq F((a_i)_{i=0,...,N}) - F((c_i)_{i=0,...,N}),$$

where

$$a_i = \begin{cases} k_i & \text{if} \quad r_i \leq 0 \\ k_i + m_p & \text{if} \quad r_i > 0 \end{cases} \quad \text{and} \quad c_i = \begin{cases} k_i - m_p & \text{if} \quad r_i \leq 0 \\ k_i & \text{if} \quad r_i > 0, \end{cases}$$

and

$$k_i = \begin{cases} (\phi_p)_* (x_p^* + a + r_i) & \text{if} \quad r_i \leq 0 \\ (\phi_p)^* (x_p^* - a + r_i) & \text{if} \quad r_i > 0. \end{cases}$$

Here, the notation c_i is not ambiguous and has nothing to do with the velocity c_p.
Since $a_i = c_i + m_p$ for every $i \in \{0, ..., N\}$, then

$$0 \geq F((c_i + m_p)_{i=0,...,N}) - F((c_i)_{i=0,...,N}).$$

Now, since $0 < m_p \to 0$ and $k_i \to b$ for all i (see [1, Proposition 2.3, Steps 4.1, 4.2 and 5]), then

$$c_i \to b \quad \text{for all } i \in \{0, ..., N\}.$$

Therefore, for p small enough, we have c_i close to b and $m_p > 0$ is small enough, thus using the strict monotonicity in (B_{Lip}), we deduce that

$$0 \geq F((c_i + m_p)_{i=0,...,N}) - F((c_i)_{i=0,...,N}) \geq \eta m_p > 0,$$

which is a contradiction.

Verification of (4.1)
Assume that there exists $a > r^*$ and $x_0 \in \mathbb{R}$ such that

$$\phi_\sigma = b_\sigma \quad \text{on} \quad [x_0 - a, x_0 + a]. \tag{4.2}$$

Then proceeding as in [1, Proposition 2.3, Steps 4 and 5] (without any change in Steps 4.1, 4.2, 4.3 and Step 5), we get a contradiction. Indeed in the proof of [1, Proposition 2.3], we were assuming that ϕ_σ is constant on a half line, but condition (4.2) is sufficient to conclude. □

4.2 Results for passing to the limit

The main result of this subsection (Theorem 4.4) identifies the limits of a constructed profile. We also prove some results to pass to the limit in the equation, namely Lemma 4.2. We will use the results of this subsection to prove the continuity of the velocity function later in Proposition 5.4, Subsection 5.1.

We start by introducing the following bistable notation:

Assumption ($\tilde{B}_{m,b}$):
Let $f(x) := F(x, ..., x)$ and $m < b < m + 1$,

Bistability: $f(m) = 0 = f(b) = f(m+1)$, $f < 0$ on (m, b) and $f > 0$ on $(b, m+1)$.

Lemma 4.2 (Passing to the limit)
Consider a sequence of functions F_n satisfying $(\tilde{A}_{Lip})$ and $(\tilde{B}_{m_n,b_n})$ (with $m_n \in [0,1)$) such that

$$Lip(F_n) \leq C \quad \text{independent on } n. \tag{4.3}$$

Let (c_n, ϕ_n) be a solution of

$$\begin{cases} c_n \phi_n'(z) = F_n((\phi_n(z + r_i))_{i=0,\dots,N}) \quad \text{over} \quad \mathbb{R} \\ \phi_n \text{ is non-decreasing on } \mathbb{R} \\ \phi_n(-\infty) = m_n \quad \text{and} \quad \phi_n(+\infty) = m_n + 1. \end{cases} \tag{4.4}$$

Assume that

$$|\phi_n| \leq M \quad \text{for some} \quad M > 0 \text{ independent of } n. \tag{4.5}$$

Assume moreover that there exists a real number c such that $c_n \to c$; and that $F_n \to F$ locally uniformly and $(m_n, b_n) \to (m, b)$ as $n \to +\infty$. Then, up to a subsequence, ϕ_n converges almost everywhere to some ϕ that solves in the viscosity sense

$$\begin{cases} c\phi'(z) = F((\phi(z + r_i))_{i=0,\dots,N}) \quad \text{over} \quad \mathbb{R} \\ \phi \text{ is non-decreasing on } \mathbb{R} \\ m \leq \phi(-\infty) \quad \text{and} \quad \phi(+\infty) \leq m + 1. \end{cases} \tag{4.6}$$

Moreover, either ϕ satisfies

$$m = \phi(-\infty) \quad \text{and} \quad \phi(+\infty) = m + 1$$

or there exists two solutions ϕ^a and ϕ^b such that

$$m = \phi^a(-\infty) \quad \text{and} \quad \phi^a(+\infty) = b$$

and

$$b = \phi^b(-\infty) \quad \text{and} \quad \phi^b(+\infty) = m + 1.$$

Proof of Lemma 4.2

Step 1: passing to the limit
The proof of this result follows from [1, Proposition 2.3, Step 2]. For the convenience of the reader, we give the proof here.

Because of (4.3) and since ϕ_n is bounded, we deduce that there exists a constant $M_0 > 0$ independent of n such that

$$|F_n((\phi_n(z + r_i))_{i=0,\dots,N})| \leq M_0 \quad \text{independent on } n. \tag{4.7}$$

Case 1: $c \neq 0$
Since $|c_n| \geq \frac{|c|}{2}$ for n large, then

$$|\phi_n'| \leq \frac{2M_0}{c} \quad \text{for large} \quad n.$$

Thus ϕ_n is uniformly Lipschitz. Using Ascoli's Theorem and the diagonal extraction argument, we get that ϕ_n converges to ϕ (up to a subsequence) locally uniformly on $\mathbb{R}$. Moreover ϕ is non-decreasing and satisfies (by stability of viscosity solutions)

$$c\phi'(x) = F((\phi(x + r_i))_{i=0,\dots,N}). \tag{4.8}$$

We easily deduce (4.6).

Case 2**:** $c = 0$
Since ϕ_n is monotone and bounded (uniformly in n), then using Helly's Lemma (Lemma 2.4) and the diagonal extraction argument, ϕ_n converges (up to a subsequence) to a non-decreasing ϕ a.e. Our goal is to show that

$$0 = F((\phi(x + r_i))_{i=0,\dots,N}). \tag{4.9}$$

Subcase 2.1**:** $c_n = 0$ **for all** n
We first use the equivalence between viscosity solutions and almost everywhere solutions (Lemma 2.5) and then pass to the limit in (4.4) using Helly's lemma (Lemma 2.4). Hence, we get a solution ϕ of (4.9) almost everywhere. Again, we use Lemma 2.5 to conclude that ϕ is a viscosity solution of (4.9) and satisfies (4.6).

Subcase 2.2**:** $c_n \neq 0$ **for all** n
See Case 2 of the proof of Corollary 3.4 to deduce ϕ verifies (4.9) in the viscosity sense and satisfies (4.6).

Step 2**: limits of the profile**
Since $\phi(\pm\infty)$ solves $f = 0$, then $\phi(\pm\infty) \in \{m, b, m+1\}$. Therefore, either ϕ satisfies

$$m = \phi(-\infty) \quad \text{and} \quad \phi(+\infty) = m + 1$$

or there exists two solutions ϕ^a and ϕ^b such that ϕ^a satisfies

$$m = \phi^a(-\infty) \quad \text{and} \quad \phi^a(+\infty) = b$$

and

$$\begin{cases} (\phi^a)_*(0) \leq \dfrac{m+b}{2} \\ (\phi^a)^*(0) \geq \dfrac{m+b}{2} \end{cases} \tag{4.10}$$

and ϕ^b satisfies

$$b = \phi^b(-\infty) \quad \text{and} \quad \phi^b(+\infty) = m + 1$$

and

$$\begin{cases} (\phi^b)_*(0) \leq \dfrac{m+1+b}{2} \\ (\phi^b)^*(0) \geq \dfrac{m+1+b}{2} \end{cases} \tag{4.11}$$

Solutions ϕ^a and ϕ^b can be obtained as limits of $\phi_n^a(x) = \phi_n(x + a_n)$ and $\phi_n^b(x) = \phi_n(x + b_n)$ for suitable shifts a_n, b_n such that ϕ_n^a and ϕ_n^b satisfies resp. (4.10) and (4.11). □

We recall now the existence result of traveling waves whose a slightly different statement is given in [1, Proposition 2.3]. In order to present the main result of this section, we need to introduce the following technical lemma.

Lemma 4.3 (Controlling the finite difference)
Consider F satisfying $(\tilde{A}_{C^1})$, $\sigma_0 \in (\sigma^-, \sigma^+)$ fixed and $\beta > 0$. Let $a > r^*$ (r^* is given by (1.23)) and

$M_0 > 0$, then for all $\sigma \in [\sigma_0 - \beta, \sigma_0 + \beta] \subset (\sigma^-, \sigma^+)$ and for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all function ϕ (viscosity) solution of

$$\begin{cases} c\phi'(x) = F((\phi(x + r_i))_{i=0,\dots,N}) + \sigma \quad on \ \mathbb{R} \\ \phi' \geq 0 \\ \phi(x+1) \leq \phi(x) + 1 \\ |c| \leq M_0 \\ |c\phi'| \leq M_0, \end{cases}$$

and for all $x_0 \in \mathbb{R}$ satisfying

$$\phi_*(x_0 + a) - \phi^*(x_0 - a) \leq \delta,$$

we have

$$dist\big(\alpha, \{m_\sigma, b_\sigma\} + \mathbb{Z}\big) < \varepsilon \quad for\ all \ \ \alpha \in [\phi_*(x_0), \phi^*(x_0)].$$

Proof of Lemma 4.3
The proof of this lemma follows from a straightforward generalization of [1, Proposition 3.2] for the function F replaced by $F + \sigma$ and $(0, b)$ replaced by (m_σ, b_σ) for $\sigma \in [\sigma_0 - \beta, \sigma_0 + \beta] \subset (\sigma^-, \sigma^+)$ and for some $\beta > 0$. We similarly show that for every $\varepsilon > 0$ there exists $\delta_\sigma(\varepsilon) > 0$ such that the result holds true.

However, we can show that $\delta_\sigma(\varepsilon) = \delta(\varepsilon)$ can be chosen independent of σ and the proof of this generalization follows exactly the same lines. Indeed, we proceed by contradiction assuming that the statement is false for a sequence $\sigma_n \in [\sigma_0 - \beta, \sigma_0 + \beta]$, and consider a sequence of solutions ϕ^n. The presence of σ_n does not create any additional difficulty in the passage to the limit in the equation. □

Theorem 4.4 (Identification of the limits of the profile)
We work under the assumptions of Lemma 4.2 with $F_n = F + \sigma_n$, $m_n = m_{\sigma_n}$, $b_n = b_{\sigma_n}$ and F satisfying $(\tilde{A}_{C^1})$ and $(\tilde{B}_{C^1})$. We assume moreover that the solution (c_n, ϕ_n) of (4.4) is given by Proposition 4.5 for $\sigma_n \in (\sigma^-, \sigma^+)$. Let (c_∞, ϕ_∞) be the solution of (4.6) constructed in Lemma 4.2. If $\sigma_\infty \in (\sigma^-, \sigma^+)$, then we have moreover

$$\phi_\infty(-\infty) = m_{\sigma_\infty} \quad and \quad \phi_\infty(+\infty) = m_{\sigma_\infty} + 1.$$

Proof of Theorem 4.4
Let (c_n, ϕ_n) be a solution of (4.4) given by Proposition 4.5 and (c_∞, ϕ_∞) be a solution of (4.6) for $\sigma_\infty \in (\sigma^-, \sigma^+)$, obtained by passing to the limit $n \to \infty$. Our aim is to show that

$$\phi_\infty(-\infty) = m_{\sigma_\infty} \quad \text{and} \quad \phi_\infty(+\infty) = m_{\sigma_\infty} + 1.$$

For $\varepsilon > 0$ small enough ($\varepsilon < \frac{1}{2}\min(b_{\sigma_n} - m_{\sigma_n}, m_{\sigma_n} + 1 - b_{\sigma_n})$), let $z_n,\ y_n \in \mathbb{R}$ such that

$$\begin{cases} (\phi_n)^*(z_n) \geq b_{\sigma_n} + \varepsilon \\ (\phi_n)_*(z_n) \leq b_{\sigma_n} + \varepsilon \end{cases} \tag{4.12}$$

and

$$\begin{cases} (\phi_n)^*(y_n) \geq b_{\sigma_n} - \varepsilon \\ (\phi_n)_*(y_n) \leq b_{\sigma_n} - \varepsilon. \end{cases} \tag{4.13}$$

Assume moreover that up to translate ϕ_n, we have

$$\begin{cases} (\phi_n)_*(0) \le b_{\sigma_n} \\ (\phi_n)^*(0) \ge b_{\sigma_n}. \end{cases}$$

For every $x \in \mathbb{R}$, set with $a > r^*$

$$\psi_n(x) := (\phi_n)_*(x+a) - (\phi_n)^*(x-a) \ge 0$$

and denote by

$$\overline{m}_n = \min_{[y_n, z_n]} \psi_n(x) = \psi_n(x_n) \ge 0,$$

for some $x_n \in [y_n, z_n]$ since ψ_n is lower semi-continuous.

We claim that $\overline{m}_n > 0$. Indeed, if $\overline{m}_n = 0$, then since $\psi_n(y_n),\ \psi_n(z_n) \ge \delta(\varepsilon) > 0$ (because of (4.12), (4.13) and using Lemma 4.3), we get

$$x_n \in (y_n, z_n).$$

Moreover, we have that

$$0 = \psi_n(x_n) = (\phi_n)_*(x_n + a) - (\phi_n)^*(x_n - a)$$

and ϕ_n is non-decreasing, hence

$$\phi_n = const \quad \text{over} \quad (x_n - a, x_n + a),$$

and ϕ_n solves $f + \sigma_n = 0$.
Now, since

$$b_{\sigma_n} - \varepsilon \le (\phi_n)^*(y_n) \le \phi_n(x_n) \le (\phi_n)_*(z_n) \le b_{\sigma_n} + \varepsilon,$$

we get that

$$\phi_n = b_{\sigma_n} \quad \text{over} \quad (x_n - a, x_n + a).$$

Therefore, for $r^* < \overline{a} < a$, we have

$$\phi_n = b_{\sigma_n} \quad \text{over} \quad [x_n - \overline{a}, x_n + \overline{a}],$$

which is in contradiction with Proposition 4.5. Therefore, $\overline{m}_n > 0$ and the proof of the identification of limits of the profile proceeds similarly as in [1, Proposition 2.3], where now Step 5 is no longer necessary. In particular we avoid the case $\phi(\pm\infty) = b_{\sigma_\infty}$. □

4.3 Application to the existence of traveling waves for $\sigma \in (\sigma^-, \sigma^+)$

In this section we prove, for every $\sigma \in (\sigma^-, \sigma^+)$, the existence of a unique velocity $c = c(\sigma)$ and the existence of a traveling wave $\phi = \phi_\sigma$ solution of (1.19).

The main result of this section is:

Proposition 4.5 (Existence and uniqueness of $c = c(\sigma)$ for $\sigma \in (\sigma^-, \sigma^+)$)
Assume that F satisfies $(\tilde{A}_{C^1})$, $(\tilde{B}_{C^1})$ and let $\sigma \in (\sigma^-, \sigma^+)$. Then there exists a unique real $c(\sigma)$ (simply denoted by c_σ) such that there exists a function $\phi_\sigma : \mathbb{R} \to \mathbb{R}$ solution of (1.19) for $c = c_\sigma$ (in the viscosity sense). Moreover, this solution satisfies the following property: there is no $a > r^*$ (r^* is given in (1.23)) and $x \in \mathbb{R}$ such that

$$\phi_\sigma = b_\sigma \quad \text{on} \quad [x - a, x + a], \tag{4.14}$$

where b_σ, m_σ are defined in Theorem 1.7.

In order to prove Proposition 4.5, we introduce the following lemma:

Lemma 4.6 (Continuity and monotonicity of $m_\sigma,\ b_\sigma$ over $[\sigma^-,\sigma^+]$)
Under the assumptions $(\tilde{A}_{C^1})$ and $(\tilde{B}_{C^1})$, the two maps

$$\begin{array}{ccc} [\sigma^-,\sigma^+] & \to [\theta-1,0] \\ \sigma & \mapsto\ m_\sigma \end{array} \qquad \textit{and} \qquad \begin{array}{ccc} [\sigma^-,\sigma^+] & \to [0,\theta] \\ \sigma & \mapsto\ b_\sigma, \end{array}$$

are continuous. Moreover, the map m_σ is increasing in σ, while b_σ is decreasing.

The proof of Lemma 4.6 is straightforward from the definition of $\sigma^\pm$ and from assumption $(\tilde{B}_{C^1})$.

Proof of Proposition 4.5

Let $\sigma \in (\sigma^-,\sigma^+)$. Let $m_\sigma \in (\theta-1,0)$ and $b_\sigma \in (0,\theta)$ (since $\sigma \neq \sigma^\pm$) be the solutions of $f(s)+\sigma=0$. Because of assumption $(\tilde{B}_{C^1})$, the function $(f+\sigma)_{|[m_\sigma,m_\sigma+1]}$ is of bistable type, that is $f+\sigma$ satisfies

$$(\boldsymbol{\tilde{B}'_{C^1}}) \left| \begin{array}{l} f(v)+\sigma=0 \ \text{ for } \ v=m_\sigma,\ b_\sigma \ \text{ and } \ m_\sigma+1 \\ (f+\sigma)_{|(m_\sigma,b_\sigma)}<0, \quad (f+\sigma)_{|(b_\sigma,m_\sigma+1)}>0 \ \text{ and } \ f'(b_\sigma)>0. \end{array} \right.$$

Step 1: existence of a traveling wave

Since F satisfies $(\tilde{A}_{C^1})$ and $b_\sigma \in (0,\theta)$ (because $\sigma \neq \sigma^\pm$), then F is C^1 near $\{b_\sigma\}^{N+1}$. Therefore, for $\omega>0$ small enough, X close enough to $\{b_\sigma\}^{N+1}$ and for all $\varepsilon>0$, we have

$$\begin{aligned} F(X+(\omega,...,\omega))-F(X) &= \int_0^1 dt \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(X+t(\omega,...,\omega))\omega \\ &\geq (N+1)(f'(b_\sigma)-\varepsilon)\omega \\ &\geq (N+1)\frac{f'(b_\sigma)}{2}\omega \qquad \left(\text{for } \varepsilon \leq \tfrac{f'(b_\sigma)}{2}\right) \\ &= \eta\omega \qquad \left(\text{with } \eta=(N+1)\tfrac{f'(b_\sigma)}{2}\right). \end{aligned}$$

Again, since F satisfies $(\tilde{A}_{C^1})$, which implies in particular that F satisfies (A_{Lip}), then using Proposition 4.1, there exists a traveling wave ϕ_σ and a velocity c_σ solution of (1.19).

Step 2: uniqueness of the velocity c_σ under (M)

Assume that F is decreasing close to $\{m_\sigma\}^{N+1}$ and $\{m_\sigma+1\}^{N+1}$ in the direction $E=(1,...,1)$. That is, there exists $\varepsilon>0$ small such that F satisfies:

$$(\boldsymbol{M}) \left| \begin{array}{l} F(X+(a,...,a))<F(X) \ \text{ for all } a>0 \text{ such that } X,\ X+(a,...,a)\in[m_\sigma,m_\sigma+\varepsilon]^{N+1} \\ F(X+(a,...,a))<F(X) \ \text{ for all } a>0 \text{ such that } X,\ X+(a,...,a)\in[m_\sigma+1-\varepsilon,m_\sigma+1]^{N+1}. \end{array} \right.$$

Then under assumptions $(\tilde{A}_{C^1})$ and (M), the velocity c_σ is unique, (as a consequence of [1, Theorem 1.5 (a)])).

Step 3: checking that F satisfies (M)

Since F is C^1 over a neighborhood of $\mathbb{R}E\backslash(\mathbb{Z}E\cup\mathbb{Z}\Theta)$, then for every $\delta>0$ there exists $\varepsilon=\varepsilon(\delta)>0$ such that if $X,\ X+(a,...,a)\in[m_\sigma,m_\sigma+\varepsilon]^{N+1}$, then

$$|\nabla F(X+t(a,...,a))-\nabla F(m_\sigma,...,m_\sigma)| \leq \delta \tag{4.15}$$

for all $t\in[0,1]$. Hence using (4.15), we get

$$\begin{aligned} F(X+(a,...,a))-F(X)-f'(m_\sigma)a &= \left(\int_0^1 dt \sum_{i=0}^{N}\left(\frac{\partial F}{\partial X_i}(X+t(a,...,a))-\frac{\partial F}{\partial X_i}(m_\sigma,...,m_\sigma)\right)\right)a \\ &\leq (N+1)a\delta. \end{aligned}$$

Now, since $f'(m_\sigma) < 0$, we deduce that

$$F(X + (a, ..., a)) - F(X) \leq (f'(m_\sigma) + (N+1)\delta)a < 0$$

for $\delta > 0$ small enough. Similarly, we show that F is decreasing close to $\{m_\sigma + 1\}^{N+1}$.
Note that, the proof of (4.14) follows exactly as the proof of (4.1). □

5 Properties of the velocity

We split this section into two subsections. We dedicate a first subsection to the proof of monotonicity and continuity of the velocity function $c(\sigma)$ over (σ^-, σ^+). In a second subsection, we prove that the velocity function attains finite limits $c^\pm$ as σ goes to $\sigma^\pm$ respectively. We also prove the existence of traveling waves solutions of (1.20) (resp. (1.21)) for $c = c^+$ (resp. $c = c^-$).

5.1 Monotonicity and continuity of the velocity

This subsection consists in two results. The monotonicity (Corollary 5.2 and Lemma 5.5) and the continuity (Proposition 5.4) of the velocity function on (σ^-, σ^+). We start with the following result.

Proposition 5.1 (Monotonicity of the velocity)
Assume $(\tilde{A}_{C^1})$ *and let* $\sigma \in [\sigma^-, \sigma^+]$. *Let* (c_1, ϕ_1) *and* (c_2, ϕ_2) *be respectively a sub and a supersolution of* (1.19) *such that*

$$\begin{cases} \phi_1(-\infty) < \phi_2(-\infty) \\ \phi_1(+\infty) < \phi_2(+\infty) \\ \phi_1(+\infty) > \phi_2(-\infty). \end{cases} \tag{5.1}$$

Then

$$c_1 \leq c_2.$$

Proof of Proposition 5.1
Assume to the contrary that $c_2 < c_1$. Let $a \in \mathbb{R}$ and define $\phi_2^a(x) = \phi_2(x + a)$. Hence, for $a \geq 0$ large enough fixed, we get

$$\phi_2^a \geq \phi_1 \quad \text{over} \quad \mathbb{R}.$$

Next, set

$$\left| \begin{array}{l} u_1(t, x) = \phi_1(x + c_1 t) \\ u_2(t, x) = \phi_2^a(x + c_2 t), \end{array} \right.$$

then the u_j are respectively a sub and a supersolution for $j = 1, 2$ of the following equation

$$\partial_t u_j(t, x) = F((u_j(t, x + r_i))_{i=0,...,N}) + \sigma_j. \tag{5.2}$$

Moreover, at time $t = 0$, we have

$$u_2(0, x) = \phi_2^a(x) \geq \phi_1(x) = u_1(0, x) \quad \text{over} \quad \mathbb{R}.$$

Thus applying the comparison principle for equation (5.2) (see [16, Propositions 2.5 and 2.6]), we get

$$u_2(t, x) \geq u_1(t, x) \quad \text{for all} \quad (t, x) \in [0, +\infty) \times \mathbb{R}.$$

Taking $x = y - c_2 t$, we get

$$\phi_2^a(y) \geq \phi_1(y + (c_1 - c_2)t) \quad \text{for all } t \geq 0 \text{ and } y \in \mathbb{R}.$$

Using that $c_1 > c_2$ and passing to the limit $t \to \infty$, we get

$$\phi_2^a(y) \geq \phi_1(+\infty) \quad \text{for all } y \in \mathbb{R}.$$

But $\phi_2^a(-\infty) < \phi_1(+\infty)$ (see (5.1)), hence a contradiction. Therefore $c_1 \leq c_2$. □

Corollary 5.2 (Monotonicity of the velocity over $[\sigma^-, \sigma^+]$)
Assume $(\tilde{A}_{C^1})$, $(\tilde{B}_{C^1})$ and let $\sigma_1,\ \sigma_2 \in [\sigma^-, \sigma^+]$ such that $\sigma_1 < \sigma_2$. Let $i = 1, 2$ and associate for each $\sigma = \sigma_i$ a solution (c_i, ϕ_i) of (1.19). Then

$$c_1 \leq c_2.$$

Proof of Corollary 5.2
Let $\sigma_1,\ \sigma_2 \in [\sigma^-, \sigma^+]$ such that $\sigma_1 < \sigma_2$. Since (c_1, ϕ_1) and (c_2, ϕ_2) are two solutions of (1.19), then ϕ_1 and ϕ_2 are respectively a sub and a supersolution of

$$c\phi'(x) = F((\phi(x + r_i))_{i=0,\dots,N}) + \sigma_2.$$

Moreover, for m_{σ_i} denoted by m_i, we have (see Lemma 4.6)

$$\left| \begin{array}{l} \phi_1(-\infty) = m_1 < m_2 = \phi_2(-\infty) \\ \phi_1(+\infty) = m_1 + 1 < m_2 + 1 = \phi_2(+\infty) \\ \phi_1(+\infty) = m_1 + 1 > m_2 = \phi_2(-\infty). \end{array} \right.$$

Therefore, the result of Corollary 5.2 follows from Proposition 5.1. □

Then we have the straightforward consequence of Proposition 5.1.

Corollary 5.3 (Monotonicity and limits of $c(\sigma)$)
Assume $(\tilde{A}_{C^1})$, $(\tilde{B}_{C^1})$. For $\sigma \in (\sigma^-, \sigma^+)$, let $(c(\sigma), \phi_\sigma)$ be a solution of (1.19) given in Proposition 4.5. Then the velocity function is non-decreasing on (σ^-, σ^+). Moreover, the limits

$$\lim_{\sigma \to \sigma^-} c(\sigma) = c^- \quad \text{and} \quad \lim_{\sigma \to \sigma^+} c(\sigma) = c^+$$

exist and satisfy $-\infty \leq c^- \leq c^+ \leq +\infty$.

Proposition 5.4 (Continuity of the velocity function)
Suppose that F satisfies $(\tilde{A}_{C^1})$, $(\tilde{B}_{C^1})$ and let $\sigma \in (\sigma^-, \sigma^+)$. Let $(c(\sigma), \phi_\sigma)$ be a solution of (1.19) given in Proposition 4.5. Then the map $\sigma \mapsto c(\sigma)$ is continuous on (σ^-, σ^+).

Proof of Proposition 5.4
Let $\sigma_0 \in (\sigma^-, \sigma^+)$ and $c_0 := c(\sigma_0)$ be the associated velocity given in Proposition 4.5. Let $\sigma_n \in (\sigma^-, \sigma^+)$ be a sequence such that $\sigma_n \to \sigma_0$ and let $c_n = c(\sigma_n)$. We want to show that $c_n \to c_0$. Assume that ϕ_0 and ϕ_n (for each n) are solutions of (1.19) associated respectively to σ_0 and σ_n (for each n).

Step 1: passing to the limit $n \to +\infty$
As a consequence of the monotonicity of $c(\sigma)$ (Proposition 5.1) and the fact that $\sigma_0,\ \sigma_n \in (\sigma^-, \sigma^+)$ for all n, we get that c_n is bounded. Thus, up to a subsequence, we set $\bar{c} = \lim_{n \to +\infty} c_n$.

Recall that (c_n, ϕ_n) solves

$$c_n \phi_n'(z) = F((\phi_n(z + r_i))_{i=0,\dots,N}) + \sigma_n$$

and $\theta - 1 < m_{\sigma_n} \le \phi_n \le m_{\sigma_n} + 1 < 1$.

Therefore, passing to the limit $n \to +\infty$ (see Lemma 4.2), ϕ_n converges to a function ϕ almost everywhere, and ϕ solves (in the viscosity sense)

$$\bar{c}\phi'(x) = F((\phi(x + r_i))_{i=0,\dots,N}) + \sigma_0. \tag{5.3}$$

Moreover, Theorem 4.4 implies that $(\bar{c}, \phi)$ solves (1.19) for $\sigma = \sigma_0$.

Step 2: conclusion
From the uniqueness of the velocity on (σ^-, σ^+) (Proposition 4.5) and the fact that c_0 and $\bar{c}$ are associated to $\sigma_0 \in (\sigma^-, \sigma^+)$, we deduce that $\bar{c} = c_0$. From the uniqueness of the limit $\bar{c}$ (whatever is the subsequence $\sigma_n \to \sigma_0$), we deduce the continuity of the velocity function c. □

Lemma 5.5 (Strict monotonicity)
Under the assumptions of Proposition 5.1, there exists a constant $K > 0$ such that $c(\sigma)$ satisfies

$$\frac{dc}{d\sigma} \ge K|c| \quad \text{on} \quad (\sigma^-, \sigma^+) \tag{5.4}$$

in the viscosity sense.

Proof of Lemma 5.5
Clearly, if $c = 0$ then (5.4) holds true.
Let $\sigma_1,\ \sigma_2 \in (\sigma^-, \sigma^+)$ with $\sigma_1 < \sigma_2$ and, as in the proof of Proposition 5.1, let us call $c_1 \le c_2$ the associated velocities and $\phi_1,\ \phi_2$ the corresponding profiles with $\phi_i(-\infty) = m_{\sigma_i}$ for $i = 1,\ 2$ and $m_{\sigma_1} < m_{\sigma_2}$. Recall also that $(c, \phi) = (c_i, \phi_i)$ solves for $\sigma = \sigma_i$ and $i = 1,\ 2$

$$c\phi' = F((\phi(x + r_i))_{i=0,\dots,N}) + \sigma. \tag{5.5}$$

Suppose that $c_1 > 0$. Since $F \in \mathrm{Lip}(\mathbb{R}^{N+1})$ and ϕ_1 is bounded, then there exists some $C > 0$ such that

$$|F((\phi_1(x + r_i))_{i=0,\dots,N})| \le C.$$

Therefore

$$0 \le \phi_1' \le c_1^{-1}(|\sigma_1| + C).$$

Hence for $\delta = c_1(|\sigma_1| + C)^{-1}$, we have (using (5.5))

$$(c_1 + \delta(\sigma_2 - \sigma_1))\phi_1' \le \sigma_2 + F((\phi_1(x + r_i))_{i=0,\dots,N}).$$

But, this means that $(\bar{c}, \phi_1)$, with $\bar{c} = c_1 + \delta(\sigma_2 - \sigma_1)$, is a subsolution of (5.5) with $\sigma = \sigma_2$. Comparing $\phi_1(x + \bar{c}t)$ to $\phi_2(x + c_2 t)$ as in Proposition 5.1, we deduce that $\bar{c} \le c_2$, that is,

$$\frac{c_2 - c_1}{\sigma_2 - \sigma_1} \ge c_1(|\sigma_1| + C)^{-1} =: Kc_1 \quad (\sigma_1 \in (\sigma^-, \sigma^+) \text{ bounded}). \tag{5.6}$$

Now letting $\sigma_1 \to \sigma_2$, and using the continuity of $c(\sigma)$, inequality (5.4) follows (in the sense of viscosity) in case $c > 0$. Similarly, we prove that $c(\sigma)$ verifies (5.4) for $c < 0$. □

5.2 Finite threshold velocities ($c^+ < +\infty$ and $c^- > -\infty$)

In this subsection, we show that $c^+ < +\infty$ (resp. $c^- > -\infty$) and we prove the existence of a solution for $c = c^+$ (resp. $c = c^-$) of (1.20) (resp. (1.21)).

In order to prove that $c^+ < +\infty$ and $c^- > -\infty$, we need to start with the following useful lemma.

Lemma 5.6 (Bound on the velocity for $\sigma \in (\sigma^-, \sigma^+)$)
Consider a function F satisfying $(\tilde{A}_{C^1})$ and $(\tilde{B}_{C^1})$. Then there exists $0 < c_*^+ < +\infty$ (resp. $-\infty < c_*^- < 0$) such that the followings holds. Let $\sigma_0 \in (\sigma^-, \sigma^+)$ and c_{σ_0} be such that (c_{σ_0}, ϕ_0) is a solution of (1.19) with $\sigma = \sigma_0$. Then

$$-\infty < c_*^- \leq c_{\sigma_0} \leq c_*^+ < +\infty.$$

Proof of Lemma 5.6
Notice that from Proposition 3.5, there exists $0 < c_*^+ < +\infty$ (resp. $-\infty < c_*^- < 0$) such that for all $c_1 > c_*^+$ (resp. $c_2 < c_*^-$) there exists (c_1, ϕ_1) (resp. (c_2, ϕ_2)) solution of (1.19) for $\sigma = \sigma^+$ (resp. $\sigma = \sigma^-$).

We prove that $c_{\sigma_0} \leq c_*^+$ (the case $c_*^- \leq c_{\sigma_0}$ being similar). Assume to the contrary that $c_{\sigma_0} = c_1 > c_*^+$. Suppose that (c_1, ϕ_1) be a solution of (1.19) for $\sigma = \sigma^+$.

Let $\overline{\sigma}$ be such that

$$\sigma^- < \sigma_0 < \overline{\sigma} < \sigma^+ \tag{5.7}$$

and associate a solution $(\overline{c}, \overline{\phi})$ of (1.19) for $\sigma = \overline{\sigma}$. Using Proposition 5.1, we get that

$$\overline{c} \leq c_1 = c_{\sigma_0}.$$

Moreover, using (5.7) and the fact that $c_{\sigma_0} = c_1 > c_*^+ > 0$, we deduce from Lemma 5.5 that

$$c_1 = c_{\sigma_0} < \overline{c},$$

which is a contradiction. $\square$

Then we have the straightforward result:

Corollary 5.7 (Finite limits of c as $\sigma \to \sigma^{\pm}$)
Consider a function F satisfying $(\tilde{A}_{C^1})$ and $(\tilde{B}_{C^1})$. Let c^-, c^+ given by Corollary 5.3 and c_*^+, c_*^- defined in Lemma 5.6. Then

$$-\infty < c_*^- \leq c^- \leq c^+ \leq c_*^+ < +\infty.$$

Lemma 5.8 (Existence of a solution of (1.20) for $c = c^{\pm}$)
Assume $(\tilde{A}_{C^1})$, $(\tilde{B}_{C^1})$ and let $\sigma = \sigma^+$ (resp. $\sigma = \sigma^-$). There exists a profile ϕ^+ (resp. ϕ^-) such that (c^+, ϕ^+) (resp. (c^-, ϕ^-)) solves (1.20) (resp. (1.21)).

Proof of Lemma 5.8

Step 0: preliminary
Assume that $\sigma = \sigma^+$ and let us prove the existence of a solution of (1.20) for c^+ (proving the existence of solution of (1.20) for c^- in the case $\sigma = \sigma^-$ is treated similarly). The goal is to get a solution as a limit of the profiles as $\sigma \to \sigma^+$, recalling that $c^+ = \lim_{\sigma \to \sigma^+} c(\sigma)$.

Consider $\sigma \in (\sigma^-, \sigma^+)$ and let (c_σ, ϕ_σ) be a solution of (1.19), namely

$$(5.8) \qquad \begin{cases} c_\sigma \phi'_\sigma(z) = F(\phi_\sigma(z+r_0), \phi_\sigma(z+r_1), ..., \phi_\sigma(z+r_N)) + \sigma \quad \text{on} \quad \mathbb{R}. \\ \phi_\sigma \text{ is non-decreasing over } \mathbb{R} \\ \phi_\sigma(-\infty) = m_\sigma \quad \text{and} \quad \phi_\sigma(+\infty) = m_\sigma + 1. \end{cases}$$

As in the proof of Proposition 5.4, there exists some constant $M > 0$ independent on σ such that

$$|F(\phi_\sigma(z+r_0), \phi_\sigma(z+r_1), ..., \phi_\sigma(z+r_N)) + \sigma^+| \leq M \quad \text{for all} \quad \sigma \in (\sigma^-, \sigma^+).$$

Moreover, up to translate ϕ_σ, we can assume that (because $m_\sigma \to 0$ as $\sigma \to \sigma^+$)

$$(5.9) \qquad (\phi_\sigma)_*(0) \leq \frac{1}{2} \leq \phi^*_\sigma(0).$$

Step 1: passing to the limit $\sigma \to \sigma^+$
Applying Lemma 4.2, we deduce that there exists some function $\phi = \phi^+$ which satisfies, in viscosity sense

$$(5.10) \qquad \begin{cases} c^+(\phi)'(z) = F(\phi(z+r_0), \phi(z+r_1), ..., \phi(z+r_N)) + \sigma^+ \quad \text{on} \quad \mathbb{R}. \\ \phi \text{ is non-decreasing over } \mathbb{R} \\ 0 = m_{\sigma^+} \leq \phi \leq m_{\sigma^+} + 1 = 1. \end{cases}$$

Step 2: limits of the profile ϕ
Passing to the limit in (5.9), we get

$$0 \leq \phi(-\infty) \leq \phi_*(0) \leq \frac{1}{2} \leq (\phi)^*(0) \leq \phi(+\infty) = 1.$$

Because $\phi(\pm\infty)$ solves

$$f(\phi(\pm\infty)) + \sigma^+ = 0,$$

the solution has to satisfy

$$\phi(-\infty) = m_{\sigma^+} = 0 \quad \text{and} \quad \phi(+\infty) = 1.$$

Therefore $\phi = \phi^+$ solves (1.20). □

6 Filling the gaps: traveling waves for $c \geq c^+$ and $c \leq c^-$

We prove, in this section, for each $c \geq c^+$ (resp. $c \leq c^-$) the existence of a solution of (1.20) (resp. (1.21)). We also prove that (1.20) (resp. (1.21)) admits no solution for any $c < c^+$ (resp. $c > c^-$).

Proposition 6.1 (Existence of solution for vertical branches of velocities)
Let F be a given function satisfying assumptions $(\tilde{A}_{C^1})$ and $(\tilde{B}_{C^1})$. Let $c^+ < +\infty$ and $c^- > -\infty$ be given by Corollary 5.3. Then for every $c > c^+$ (resp. $c < c^-$), there exists a solution ϕ of (1.20) (resp. (1.21)).

For $c = c^+$ or $c >> 1$ (resp. $c = c^-$ or $c << 1$) we already have the existence of a solution of ϕ of (1.20) (resp. (1.21)). Proposition 6.1 fills the gap for all $c \geq c^+$ (resp. $c \leq c^-$).

In order to prove Proposition 6.1, we will need the following preliminary result that is proved in [16].

Lemma 6.2 (Existence of a hull function ([16, Theorem 1.5 and Theorem 1.6 a1]))
Let F be a given function satisfying assumption $(\tilde{A}_{C^1})$, $p > 0$ and $\sigma \in \mathbb{R}$. There exists a unique $\lambda(\sigma, p) = \lambda_p(\sigma)$ such that there exists a locally bounded function $h_p : \mathbb{R} \to \mathbb{R}$ satisfying (in the viscosity sense):

$$\begin{cases} \lambda_p h_p'(z) = F((h_p(z + pr_i))_{i=0,\dots,N}) + \sigma \quad \text{on} \quad \mathbb{R} \\ h_p(z+1) = h_p(z) + 1 \\ h_p'(y) \geq 0 \\ |h_p(z + z') - h_p(z) - z'| \leq 1 \quad \text{for any } z,\ z' \in \mathbb{R}. \end{cases} \tag{6.1}$$

Moreover, there exists a constant $K > 0$, independent on p and σ, such that

$$|\lambda_p - \sigma| \leq K(1 + p) \tag{6.2}$$

and the function

$$\begin{aligned} \lambda_p : \mathbb{R} &\to \mathbb{R} \\ \sigma &\to \lambda_p(\sigma) \end{aligned}$$

is continuous with $\lambda_p(\pm\infty) = \pm\infty$.

For the proof of Lemma 6.2, we refer the reader to [16, Theorems 1.5 and 1.6]. However, proving that $\lambda_p(\pm\infty) = \pm\infty$ follows from (6.2).

Corollary 6.3 (Existence of ϕ_p)
Let F be a given function satisfying assumption $(\tilde{A}_{C^1})$, $p > 0$ and $c \in (c^+, +\infty)$ fixed. Then there exists $\sigma = \sigma(c, p) \in \mathbb{R}$ such that there exists a function $\phi_p : \mathbb{R} \to \mathbb{R}$ that satisfies in the viscosity sense:

$$\begin{cases} c\phi_p'(z) = F((\phi_p(z + r_i))_{i=0,\dots,N}) + \sigma(c, p) \quad \text{on} \quad \mathbb{R} \\ \phi_p' \text{ non-decreasing} \\ \phi_p\left(z + \frac{1}{p}\right) = \phi_p(z) + 1. \end{cases} \tag{6.3}$$

Proof of Corollary 6.3
Let $\sigma = \sigma(c, p)$ such that

$$\lambda_p(\sigma) = cp \tag{6.4}$$

and define the function ϕ_p as:

$$\phi_p(x) = h_p(px), \tag{6.5}$$

where h_p is given by Lemma 6.2. This gives the result. □

Now, we give the proof of Proposition 6.1.
Proof of Proposition 6.1
Choose $\overline{c} > c^+$ and let $\delta_0 > 0$ such that

$$\overline{c} > c^+ + \delta_0.$$

Step 1: preliminary
Choose $\eta > 0$ small and let $\sigma^+ - \eta \leq \sigma_\eta < \sigma^+$. From Proposition 4.5, we know that for σ_η, there exits a solution $(c_{\sigma_\eta}, \phi_{\sigma_\eta})$ of (1.19) such that

$$c_{\sigma_\eta} \leq c^+.$$

Moreover, as $c_{\sigma_\eta} = \lim_{p\to 0} c(\sigma_\eta, p)$ with $c(\sigma_\eta, p) = \dfrac{\lambda(\sigma_\eta, p)}{p}$ (see the proof of existence of [1, Proposition 2.3]), then there exists p_η such that for all $0 < p \le p_\eta$, we have

$$|c(\sigma_\eta, p) - c_{\sigma_\eta}| \le \delta_0. \tag{6.6}$$

Thus, for $0 < p \le p_\eta$, we get

$$c(\sigma_\eta, p) \le c_{\sigma_\eta} + \delta_0 \le c^+ + \delta_0 < \overline{c}. \tag{6.7}$$

Moreover, since the map $\sigma \mapsto \lambda(\sigma, p) = c(\sigma, p)p$ is continuous with $\lambda(\pm\infty, p) = \pm\infty$ (see Lemma 6.2), then for such $0 < p \le p_\eta$, there exists $\overline{\sigma}_p \in \mathbb{R}$ and a function $\phi_p : \mathbb{R} \to \mathbb{R}$ (see Corollary 6.3) such that

$$c(\overline{\sigma}_p, p) = \overline{c}$$

and $(\overline{c}, \phi_p)$ solves (6.3). Hence from (6.7), we get

$$c(\sigma_\eta, p) < c(\overline{\sigma}_p, p).$$

In addition, since $\lambda(\sigma, p)$ is non-decreasing with respect to σ, then

$$\overline{\sigma}_p > \sigma_\eta \ge \sigma^+ - \eta \quad \text{for } 0 < p \le p_\eta. \tag{6.8}$$

Step 2: passing to the limit $p \to 0$
Since $\lim_{p\to 0} \lambda(\overline{\sigma}_p, p) = \lim_{p\to 0} \overline{c}p = 0$, we deduce from (6.2) that there exists some $L_0 > 0$ independent of p such that

$$|\overline{\sigma}_p| \le L_0 \quad \text{for } 0 < p \le p_\eta. \tag{6.9}$$

Thus

$$\overline{\sigma}_p \to \overline{\sigma}_0 \quad \text{as} \quad p \to 0 \quad \text{(up to a subsequence)}.$$

Recall that ϕ_p is non-decreasing and that

$$\phi_p\left(x + \frac{1}{2p}\right) - \phi_p\left(x + \frac{-1}{2p}\right) = 1.$$

We can also assume that

$$\begin{cases} (\phi_p)^*(0) \ge \dfrac{1}{2} \\ (\phi_p)_*(0) \le \dfrac{1}{2}. \end{cases}$$

Therefore, since $F \in \text{Lip}(\mathbb{R}^{N+1})$ and due to (6.9), we deduce (as in the proof of [1, Lemma 2.8]) that there exists some $M > 0$ independent on n such that

$$|F((\phi_p(x + r_i)_{i=0,\dots,N})) + \overline{\sigma}_n| \le M.$$

Applying arguments similar to the ones of the proof of Lemma 4.2, we see that ϕ_p converges to some ϕ almost everywhere and ϕ is a viscosity solution of

$$\begin{cases} \overline{c}\phi'(x) = F((\phi(x + r_i))_{i=0,\dots,N}) + \overline{\sigma}_0 \\ \phi \text{ non-decreasing and bounded} \\ \phi(+\infty) - \phi(-\infty) \le 1, \end{cases} \tag{6.10}$$

and ϕ satisfies

$$\begin{cases} \phi^*(0) \geq \frac{1}{2} \\ \phi_*(0) \leq \frac{1}{2}. \end{cases}$$

In addition, we have

$$\overline{\sigma}_0 \geq \sigma^+ - \eta \quad \text{(because of (6.8))}.$$

But $\eta > 0$ is arbitrary, hence

$$\overline{\sigma}_0 \geq \sigma^+.$$

Moreover, since $\overline{\sigma}_0 \leq \sigma^+$ (otherwise, (6.10) admits no solution, see Remark 1.10), thus

$$\overline{\sigma}_0 = \sigma^+.$$

Finally, since $\phi(\pm\infty)$ solves $f + \sigma^+ = 0$, then we conclude that

$$\phi(-\infty) = m_{\sigma^+} = 0 \quad \text{and} \quad \phi(+\infty) = 1,$$

which ends the proof. □

Lemma 6.4 (Non-existence of solution for $c < c^+$ and $c > c^-$)
Consider a function F and assume $(\tilde{A}_{C^1})$ and $(\tilde{B}_{C^1})$. Let $\sigma = \sigma^+$ (resp. $\sigma = \sigma^-$) and $c^+ < +\infty$ (resp. $c^- > -\infty$) be given by Corollary 5.3. Let (c, ϕ) be a solution of (1.20) (resp. (1.21)), then $c \geq c^+$ (resp. $c \leq c^-$).

Proof of Lemma 6.4
Let $\sigma = \sigma^+$ and (c, ϕ) be a solution of (1.20). We want to prove that $c \geq c^+$ (similarly, we show that there is no solution of (1.21) for $c > c^-$ when $\sigma = \sigma^-$).

It is known from Theorem 1.7-1, that for every $\sigma \in (\sigma^-, \sigma^+)$, there exists $(c(\sigma), \phi_\sigma)$ solution of (1.19). Let $\sigma_n \in (\sigma^-, \sigma^+)$ be a sequence such that $\sigma_n \to \sigma^+$, $c(\sigma_n) \to c^+$ and $(c(\sigma_n), \phi_{\sigma_n})$ is a solution of (1.19). Since $\sigma_n < \sigma^+$, Proposition 5.1 implies that

$$c(\sigma_n) \leq c.$$

Therefore, passing to the limit $\sigma_n \to \sigma^+$, we get that

$$c^+ \leq c,$$

which ends the proof. □

Lemma 6.5 (Strict inequality between threshold velocities)
Consider a function F satisfying $(\tilde{A}_{C^1})$, $(\tilde{B}_{C^1})$ and let c^-, c^+ given by Corollary 5.7. If $c^- \neq 0$ or $c^+ \neq 0$, then

$$c^- < c^+.$$

Proof of Lemma 6.5
This is a straightforward consequence of (5.4). □

Proof of Theorem 1.7
Theorem 1.7 is proved in several propositions and lemmata. In Propositions 4.5, 6.1, 5.4 and Lemma 5.5, we prove, for $\sigma \in (\sigma^-, \sigma^+)$, the existence of traveling waves and the monotonicity and

the continuity of the velocity of propagation respectively. Existence of vertical branches of solutions (when $\sigma = \sigma^{\pm}$) is proved in Lemma 3.6, where we show the existence of traveling waves for $c >> 1$ and for $c << -1$; and in Corollary 5.7, Lemma 5.8, Proposition 6.1, Lemma 6.4 and Lemma 6.5, where we respectively show the existence of finite critical limits $c^{\pm}$ of the velocity function when σ goes to $\sigma^{\pm}$, the existence of solutions for the critical limits of velocity, fill the gap and prove the non-existence of solution when $c < c^+$ and $\sigma = \sigma^+$ or when $c > c^-$ and $\sigma = \sigma^-$, and finally prove the inequality between c^+ and c^-. □

Part III
Definition and study of the critical velocity

7 Definition of the critical velocity: proof of Theorem 1.1

We devote this section to the proof of Theorem 1.1 and we split it into two subsections. We recall in a first subsection an extension result over $\mathbb{R}^{N+1}$. For pedagogical reasons, we also prove in this subsection the result of Theorem 1.1 in a simple case where the non-linearity F is assumed to be smooth (cf. Proposition 7.2). we prove, in a second subsection, the existence of branch of solution using an approach different from that we use to prove Theorem 1.1 but under some addition assumptions (cf. Proposition 7.3). This result is less general then Theorem 1.1. In a third subsection, we give the proof of Theorem 1.1 in full generality for Lipschitz non-linearities F.

To prove the result (in any case), we first show the existence of traveling waves for $c >> 1$ by applying Proposition 3.5. The next step is to define the critical velocity c^+ and then we prove, for all $c \geq c^+$, the existence of traveling wave solutions of system (1.9). Finally, We show the non-existence of solutions of (1.9) for any $c < c^+$.

7.1 Preliminary results

We start this subsection by recalling an extension result of the function F defined on $[0,1]^{N+1}$ into a function $\tilde{F}$ over $\mathbb{R}^{N+1}$. We also prove the result of Theorem 1.1 in a simple case.

Lemma 7.1 (Extension of F)
Consider a function F defined over $[0,1]^{N+1}$ and satisfying (A_{Lip}) such that $F(0,...,0) = F(1,...,1) = 0$. There exists an extension $\tilde{F}$ defined over $\mathbb{R}^{N+1}$ such that

$$\tilde{F}_{|_{[0,1]^{N+1}}} = F$$

and $\tilde{F}$ satisfies

Assumption $(\tilde{A}_{\mathbf{Lip}})$:

Regularity: *$\tilde{F}$ is globally Lipschitz continuous over $\mathbb{R}^{N+1}$.*

Monotonicity: *$\tilde{F}(X_0, ..., X_N)$ is non-decreasing w.r.t. each X_i for $i \neq 0$.*

Periodicity: *$\tilde{F}(X_0 + 1, ..., X_N + 1) = \tilde{F}(X_0, ..., X_N)$ for every $X = (X_0, ..., X_N) \in \mathbb{R}^{N+1}$.*

Lemma 7.1 corresponds to Lemma 2.1 in [1] whose proof is given in the appendix A of [1].

Notice that the function $\tilde{f}(v) := \tilde{F}(v, ..., v)$ is nothing but a periodic extension of f on $\mathbb{R}$ with period 1, that is

$$\tilde{f}_{|_{[0,1]}} = f,$$

hence $\tilde{f}(0) = \tilde{f}(1) = 0$.

Notice also that ϕ is a solution of

$$\begin{cases} c\phi'(z) = F((\phi(z+r_i))_{i=0,...,N}) & \text{on } \mathbb{R} \\ \phi \ \text{ is non-decreasing over } \ \mathbb{R} \\ \phi(-\infty) = 0 \quad \text{and} \quad \phi(+\infty) = 1 \end{cases}$$

if and only if ϕ is a solution of

$$(7.1) \qquad \begin{cases} c\phi'(z) = \tilde{F}((\phi(z+r_i))_{i=0,...,N}) & \text{on } \mathbb{R} \\ \phi \ \text{ is non-decreasing over } \ \mathbb{R} \\ \phi(-\infty) = 0 \quad \text{and} \quad \phi(+\infty) = 1, \end{cases}$$

since $\tilde{F}_{|_{[0,1]^{N+1}}} = F$. In particular $\tilde{F}$ satisfies (P_{Lip}) if F satisfies (P_{Lip}).

In order to prove Theorem 1.1 in a special case when F is smooth (see Proposition 7.2), we need to introduce precise assumptions.

Assumption (A_{C^1}):

Regularity: $F \in C^1([0,1]^{N+1})$.

Monotonicity: $F(X_0, ..., X_N)$ is non-decreasing w.r.t. each X_i for $i \neq 0$.

Assumption (P'_{C^1}):

Positive degenerate monostability:
Let $f(v) = F(v, ..., v)$ such that $f(0) = 0 = f(1)$ and $f > 0$ in $(0,1)$.

Smoothness near $\{0\}^{N+1}$ and $\{1\}^{N+1}$:
There exists $\delta > 0$ such that

$$\begin{cases} f' > 0 & \text{on} \quad (0,\delta) \\ f' < 0 & \text{on} \quad (1-\delta, 1) \end{cases}$$

Proposition 7.2 (Vertical branch, simple case)
Consider a function F satisfying (A_{C^1}) and (P'_{C^1}). Then the result of Theorem 1.1 holds true.

Proof of Proposition 7.2
Note that $\sigma^+ = 0$ in this case.

Using Proposition 3.5, we deduce that for $c >> 1$, there exists a solution of (1.9). Next, from the extension lemma (Lemma 7.1), we see that if F satisfies (A_{C^1}) (which implies (A_{Lip})), then the extended function $\tilde{F}$ satisfies $(\tilde{A}_{C^1})$. Because of assumption (P'_{C^1}) and $\tilde{f}$ is 1-periodic with $\tilde{f} = f$ on $[0,1]$, there exists $\varepsilon_0 > 0$ small enough, such that for $-\varepsilon_0 < \sigma < 0$, $\tilde{f} + \sigma$ has a bistable shape over $(m_\sigma, m_\sigma + 1)$ where m_σ is defined exactly as in Theorem 1.7. Precisely, by bistable shape we mean that there exists m_σ and b_σ solutions of $\tilde{f} + \sigma = 0$ satisfying $-1 < m_\sigma < 0 < b_\sigma < m_\sigma + 1 < 1$ and

$$\begin{cases} \tilde{f} + \sigma < 0 & \text{on} \quad (m_\sigma, b_\sigma) \\ \tilde{f} + \sigma > 0 & \text{on} \quad (b_\sigma, m_\sigma + 1) \\ \tilde{f}'(b_\sigma) > 0 \ \text{ and } \ \tilde{f}'(m_\sigma) = \tilde{f}'(m_\sigma + 1) < 0. \end{cases}$$

For $\sigma \in (-\varepsilon_0, 0)$, using Proposition 4.5 (which stays true with $(\tilde{B}_{C^1})$ replaced by (P'_{C^1}) and $\sigma \in (-\varepsilon_0, 0)$ instead of $\sigma \in (\sigma^-, \sigma^+)$), we show the existence of a unique velocity c_σ such that there exists a profile ϕ_σ solution of system (1.19) with F replaced by $\tilde{F}$. From Propositions 5.1 and 5.4 (which stay true similarly for $(\tilde{B}_{C^1})$ replaced by (P'_{C^1}) and $\sigma \in (-\varepsilon_0, 0)$), we get that the map

$$\sigma \mapsto c_\sigma$$

is monotone continuous on $(-\varepsilon_0, 0)$ and we define as in Corollary 5.3 the critical velocity c^+ as

$$\lim_{\sigma \to 0^-} c_\sigma = c^+.$$

Again, up to replace $(\tilde{B}_{C^1})$ by (P'_{C^1}) and $\sigma \in (\sigma^-, \sigma^+)$ by $\sigma \in (-\varepsilon_0, 0)$, we can use Lemma 5.6, Corollary 5.7 and Lemma 5.8, and show that $c^+ < +\infty$ and that (1.9) admits a solution for $c = c^+$. We use Proposition 6.1 (again with $(\tilde{B}_{C^1})$ replaced by (P'_{C^1})) to fill the gap and get the existence of solution (c, ϕ) for each $c \geq c^+$. Finally, the non-existence of solutions for $c < c^+$ follows from Lemma 6.4 (with $(\tilde{B}_{C^1})$ replaced by (P'_{C^1})). □

7.2 Another approach of the proof of Theorem 1.1 under additional assumptions

We introduce in this subsection another proof for the existence of branch of solutions of (1.9) under some additional assumptions. This result is less general then the result of Theorem 1.1, but more general then Proposition 7.2.

Proposition 7.3 (Existence of branch of solutions under additional assumptions)
We work under the assumptions of Theorem 1.1. Let

$$c^+ = \inf \mathcal{E} \quad \textit{with} \quad \mathcal{E} := \{c \in \mathbb{R} \ \ \textit{such that} \ \ \exists\ (c, \phi) \ \ \textit{solution of}\ (1.9)\}. \tag{7.2}$$

Then $c^+ > -\infty$ and $c^+ \in \mathcal{E}$. Moreover, if F is increasing in X_{i^+} with $r_{i^+} > 0$ and $c^+ \neq 0$, then for every $c \geq c^+$ there exists a solution of (1.9).

Remark 7.4 (The set $\mathcal{E}$ is nonempty)
Proposition 3.5 implies directly that $\mathcal{E} \neq \emptyset$. Moreover, from the definition of c^+, we see that for all $c < c^+$ there is no solution of (1.9).

Proof of Proposition 7.3
Let c^+ be defined in (7.2). We first want to shoe that $c^+ \in \mathcal{E}$.

Step 1: $c^+ \in \mathcal{E}$
Assume by contradiction that $c^+ \notin \mathcal{E}$. From the definition of c^+ (see (7.2)), there exists a sequence $c_n \in \mathcal{E}$ such that $c_n \to c^+$ and (c_n, ϕ_n) is a solution of (1.9).

Case 1: $c^+ = -\infty$
Set $\overline{\phi}_n(x) = \phi_n(|c_n|x)$, then we have

$$-\overline{\phi}'_n(y) = F\left(\left(\overline{\phi}_n\left(y + \frac{r_i}{|c_n|}\right)\right)_{i=0,\dots,N}\right). \tag{7.3}$$

Since $\overline{\phi}_n$ is invariant with respect to space translation and F is Lipschitz, we may assume that

$$\overline{\phi}_n(0) = \frac{1}{2}.$$

Moreover, since F Lipschitz over $[0,1]^{N+1}$, then we can show that there exists a constant $M > 0$ independent of c_n such that $|\overline{\phi}_n'| \leq M$. Using Ascoli's Theorem, we pass to the limit $c_n \to -\infty$ in (7.3) and we get that $\overline{\phi}_n$ converges (up to a subsequence) locally uniformly to $\overline{\phi}$ which solves

$$-\overline{\phi}' = F(\overline{\phi}(y), ..., \overline{\phi}(y)) = f(\overline{\phi}(y))$$

and satisfies

$$\overline{\phi}(0) = \frac{1}{2}.$$

But $\overline{\phi}' \geq 0$ (since $\overline{\phi}_{c^+}' \geq 0$), hence

$$0 \geq -\overline{\phi}'(0) = f(\overline{\phi}(0)) = f\left(\frac{1}{2}\right) > 0.$$

Contradiction. Thus $c^+ > -\infty$.

Case 2: $c^+ > -\infty$

Case 2.1: $c^+ \neq 0$
If $c^+ \neq 0$, then passing to the limit using Ascoli's theorem as in Case 1, we can deduce that there exists a solution (c^+, ϕ^+) of (1.9) and hence $c^+ \in \mathcal{E}$.

Case 2.2: $c^+ = 0$
See Case 2 of the proof of Corollary 3.4.

Step 2: filling the gap
Let $c > c^+$, we want to prove the existence of a solution of (1.9) for c.

Step 2.1: $c^+ \neq 0$
Let ϕ^+ be a solution of 1.9 associated for c^+. We first show that $\phi^+ > 0$. We distinguish the following two cases:

Case 1: $c^+ < 0$
Assume that $\phi^+(x_0) = 0$ for some $x_0 \in \mathbb{R}$. Since $c^+ < 0$, then using the strong maximum principle ([1, Lemma 6.1]) we get that

$$\phi^+ = 0 \quad \text{on} \quad [x_0, +\infty),$$

which is a contradiction since $\phi^+(+\infty) = 1$.

Case 2: $c^+ > 0$
Assume that $\phi^+(x_0) = 0$ for some $x_0 \in \mathbb{R}$. Using again the strong maximum principle for $c^+ > 0$ ([1, Lemma 6.2]), which is based on the fact that F is increasing in X_{i^+} with $r_{i^+} > 0$, we get a similar contradiction.

Step 2.2: conclusion: $\mathcal{E} = [c^+, +\infty)$
Since (c^+, ϕ^+) is a solution of (1.9) with $\phi^+ > 0$, then we deduce from Corollary 3.4 that there exists a solution of (1.9) for every $c > c^+$. This implies that $\mathcal{E} = [c^+, +\infty)$.

□

7.3 Proof of Theorem 1.1

This subsection is devoted for the proof of Theorem 1.1.

Proof of Theorem 1.1
Let us consider a general function $F : [0,1]^{N+1} \to \mathbb{R}$ and $f(v) = F(v, ..., v)$ satisfying (A_{Lip}) and (P_{Lip}). We have to adapt the proof of Proposition 7.2 with a much lower regularity of F (here F is only Lipschitz). To this end, we will introduce an approximation F_δ of F.

Step 0: a δ-approximation
Define for $X = (X_0, ..., X_N) \in [0,1]^{N+1}$ and $\delta > 0$ small

$$F_\delta(X) = F(X) - f(X_0) + f_\delta(X_0),$$

where

$$f_\delta(v) = \begin{cases} \max\big(f(\delta) + L_0(v-\delta), 0\big) & \text{on} \quad [0,\delta] \\ \max\big(f(1-\delta) - L_0(v-(1-\delta)), 0\big) & \text{on} \quad [1-\delta, 1] \\ f & \text{on} \quad [\delta, 1-\delta], \end{cases}$$

with a constant $L_0 > 0$ satisfying $L_0 > 2\text{Lip}(F) =: 2L_F^\infty$. Clearly, we have $F_\delta(v, ..., v) = f_\delta(v)$.

Set

$$\begin{cases} b_\delta = \delta - \dfrac{f(\delta)}{L_0} > 0 \\ m_\delta = 1 - \delta + \dfrac{f(1-\delta)}{L_0} < 1 \end{cases} \tag{7.4}$$

which satisfies

$$0 < b_\delta < \delta < 1 - \delta < m_\delta < 1, \tag{7.5}$$

and

$$f_\delta(b_\delta) = 0 = f_\delta(m_\delta) \quad \text{and} \quad f_\delta > 0 \ \text{ on } \ (b_\delta, m_\delta).$$

Let $\tilde{F}$ and $\tilde{F}_\delta$ defined on $\mathbb{R}^{N+1}$ be the extension functions of F and F_δ (which are defined on $[0,1]^{N+1}$) respectively constructed by Lemma 7.1. Define $\tilde{f}_\delta(v) = \tilde{F}_\delta(v, ..., v)$ and $\tilde{f}(v) = \tilde{F}(v, ..., v)$, then $\tilde{f}_\delta$ and $\tilde{f}$ are 1-periodic with $(\tilde{f}_\delta)_{|[0,1]} = f_\delta$ and $(\tilde{f})_{|[0,1]} = f$. Moreover, since $\tilde{f}_\delta \le \tilde{f}$, we get that

$$\tilde{F}_\delta \le \tilde{F} \quad \text{over} \quad \mathbb{R}^{N+1}. \tag{7.6}$$

Now, for $\sigma < 0$ small fixed ($0 < -\sigma < \min\limits_{[\delta, 1-\delta]} f$), define $0 < b_{\delta,\sigma} < m_{\delta,\sigma} < 1$ such that

$$\begin{cases} \big(\tilde{f}_\delta + \sigma\big)(b_{\delta,\sigma}) = 0 = \big(\tilde{f}_\delta + \sigma\big)(m_{\delta,\sigma}) = \big(\tilde{f}_\delta + \sigma\big)(m_{\delta,\sigma} - 1) \\ \tilde{f}_\delta + \sigma < 0 \quad \text{on} \quad (m_{\delta,\sigma} - 1, b_{\delta,\sigma}) \\ \tilde{f}_\delta + \sigma > 0 \quad \text{on} \quad (b_{\delta,\sigma}, m_{\delta,\sigma}). \end{cases} \tag{7.7}$$

Notice that

$$\begin{cases} m_{\delta,\sigma} & \to m_\delta \\ b_{\delta,\sigma} & \to b_\delta \end{cases} \quad \text{as} \ \ \sigma \to 0^-.$$

For simplicity, we will denote $\tilde{F}$, $\tilde{F}_\delta$, $\tilde{f}$ and $\tilde{f}_\delta$ by F, F_δ, f and f_δ respectively.

Step 1: existence of a solution of the approximated non-linearity F_δ
From the definition of f_δ, we see that (for $0 < -\sigma < \min_{[\delta,1-\delta]} f$)

$$b_\delta < b_{\delta,\sigma} < \delta. \tag{7.8}$$

Now, because of (7.8) and using the definition of F_δ with the fact that F is L_F^∞-Lipschitz, then for X close to $\{b_{\delta,\sigma}\}^{N+1}$ and $\omega > 0$ small enough, we get that
(7.9)

$$\begin{aligned} F_\delta(X+(\omega,...,\omega))-F_\delta(X) &= F(X+(\omega,...,\omega))-F(X)-f(X_0+\omega)+f(X_0)+f_\delta(X_0+\omega)-f_\delta(X_0) \\ &\geq -2\omega L_F^\infty + \omega L_0 \\ &= \omega(-2L_F^\infty + L_0) = \omega\eta > 0 \quad \text{(because of the condition on } L_0). \end{aligned}$$

Since $F_\delta + \sigma$ satisfies (A_{Lip}) and (B_{Lip}) with $[0,1]^{N+1}$ replaced by $[m_{\delta,\sigma}-1, m_{\delta,\sigma}]^{N+1}$ and b replaced by $b_{\delta,\sigma}$ (see (7.7) and (7.9)), then applying the result of Proposition 4.1 (but now on $[m_{\delta,\sigma} - 1, m_{\delta,\sigma}]^{N+1}$), we deduce that there exists a solution $\phi_{\delta,\sigma}$ that solves in viscosity sense

$$\begin{cases} c_{\delta,\sigma}\phi'_{\delta,\sigma}(x) = F_\delta((\phi_{\delta,\sigma}(x+r_i))_{i=0,\dots,N}) + \sigma \quad \text{on} \quad \mathbb{R} \\ \phi_{\delta,\sigma} \quad \text{is non-decreasing over } \mathbb{R} \\ \phi_{\delta,\sigma}(-\infty) = m_{\delta,\sigma} - 1 \quad \text{and} \quad \phi_{\delta,\sigma}(+\infty) = m_{\delta,\sigma}. \end{cases} \tag{7.10}$$

More precisely, we have used the fact that $F_\delta(\cdot + \{m_{\delta,\sigma} - 1\}^{N+1}) + \sigma$ satisfies (A_{Lip}) and (P_{Lip}) on $[0,1]^{N+1}$ with b defined by $b_{\delta,\sigma} = b + m_{\delta,\sigma} - 1$, and Proposition 4.1 provides a profile $\phi : \mathbb{R} \to [0,1]$ such that $\phi + m_{\delta,\sigma} - 1 =: \phi_{\delta,\sigma}$.

Step 2: $c_{\delta,\sigma}$ is non-decreasing in σ for δ fixed
Here, this is a variant of the proof of Proposition 5.1. Let $\delta > 0$ fixed, $-\min_{[\delta,1-\delta]} f < \sigma_1 < \sigma_2 < 0$ and set $(c_{\delta,\sigma_1}, \phi_{\delta,\sigma_1})$, $(c_{\delta,\sigma_2}, \phi_{\delta,\sigma_2})$ be the associated solutions of (7.10) for σ_1 and σ_2 respectively.

We have

$$m_{\delta,\sigma_1} - 1 < m_{\delta,\sigma_2} - 1 < m_{\delta,\sigma_1} < m_{\delta,\sigma_2};$$

that is $\phi_{\delta,\sigma_1}(\pm\infty) < \phi_{\delta,\sigma_2}(\pm\infty)$, and $\phi_{\delta,\sigma_1}(+\infty) > \phi_{\delta,\sigma_2}(-\infty)$. Thus using the proof of Proposition 5.1, we deduce that $c_{\delta,\sigma_1} \leq c_{\delta,\sigma_2}$.

Step 3: $c_{\delta,\sigma}$ is non-increasing in δ for σ fixed
For $\delta_2 > \delta_1 > 0$, fix σ such that $-\min_{[\delta_1,1-\delta_1]} f < \sigma < 0$ and associate respectively the two solutions $(c_{\delta_2,\sigma}, \phi_{\delta_2,\sigma})$ and $(c_{\delta_1,\sigma}, \phi_{\delta_1,\sigma})$ of (7.10). From the definition of F_δ, $m_{\delta,\sigma}$ and $b_{\delta,\sigma}$ (see Step 0), we see that

$$F_{\delta_2} \leq F_{\delta_1},$$

hence $(c_{\delta_2,\sigma}, \phi_{\delta_2,\sigma})$ is a subsolution of (7.10) for F_δ replaced by F_{δ_1}. Moreover, we also have that

$$m_{\delta_2,\sigma} - 1 < m_{\delta_1,\sigma} - 1 < m_{\delta_2,\sigma} < m_{\delta_1,\sigma},$$

hence $\phi_{\delta_2,\sigma}(\pm\infty) < \phi_{\delta_1,\sigma}(\pm\infty)$ and $\phi_{\delta_2,\sigma}(+\infty) > \phi_{\delta_1,\sigma}(-\infty)$. Using the proof of Proposition 5.1 (which is still true for sub and supersolutions), we deduce that $c_{\delta_2,\sigma} \leq c_{\delta_1,\sigma}$.

Step 4: passing to the limit $\sigma \to 0^- = \sigma^+$
For $\delta > 0$ fixed, let $(c_{\delta,\sigma}, \phi_{\delta,\sigma})$ be a solution of (7.10). Since $F_\delta \leq F$ (see Step 0), we deduce that $(c_{\delta,\sigma}, \phi_{\delta,\sigma})$ is a subsolution for (7.10), with F_δ replaced by F.

On the other hand, let us consider any solution ϕ_{c_0} of

$$(7.11)\qquad \begin{cases} c_0\phi'_{c_0}(x) = F((\phi_{c_0}(x+r_i))_{i=0,\dots,N}) \quad \text{on} \quad \mathbb{R} \\ \phi_{c_0} \ \text{ is non-decreasing over } \mathbb{R} \\ \phi_{c_0}(-\infty) = 0 \quad \text{and} \quad \phi_{c_0}(+\infty) = 1. \end{cases}$$

From Proposition 3.5, we know that such a solution does exist at least for $c_0 >> 1$.

Since $\phi_{\delta,\sigma}$ satisfies

$$\phi_{\delta,\sigma}(-\infty) = m_{\delta,\sigma} - 1 \quad \text{and} \quad \phi_{\delta,\sigma}(+\infty) = m_{\delta,\sigma},$$

then $\phi_{\delta,\sigma}(\pm\infty) < \phi_{c_0}(\pm\infty)$ and $\phi_{\delta,\sigma}(+\infty) > \phi_{c_0}(-\infty)$. Thus using the proof of Proposition 5.1 (which is still true for sub and supersolutions), we deduce that

$$c_{\delta,\sigma} \le c_0 \quad \text{for all} \quad \sigma \in (-\min_{[\delta,1-\delta]} f, 0).$$

Since the map $\sigma \mapsto c_{\delta,\sigma}$ is non-decreasing, then

$$c_{\delta,\sigma} \to c_\delta^+ \quad \text{as} \quad \sigma \to 0^-.$$

Therefore, passing to the limit $\sigma \to 0^-$, using Lemma 4.2, $\phi_{\delta,\sigma}$ converges almost everywhere to some ϕ_δ that solves in the viscosity sense

$$(7.12)\qquad \begin{cases} c_\delta^+\phi'_\delta(x) = F_\delta((\phi_\delta(x+r_i))_{i=0,\dots,N}) \quad \text{on} \quad \mathbb{R} \\ \phi_\delta \ \text{ is non-decreasing over } \mathbb{R} \\ m_\delta - 1 \le \phi_\delta(-\infty) \quad \text{and} \quad \phi_\delta(+\infty) \le m_\delta. \end{cases}$$

We can insure that ϕ_δ is non constant, assuming that

$$\begin{cases} (\phi_{\delta,\sigma})^*(0) \ge \dfrac{b_\delta + m_\delta}{2} \\ (\phi_{\delta,\sigma})_*(0) \le \dfrac{b_\delta + m_\delta}{2}, \end{cases}$$

and this implies in addition that

$$\phi_\delta(-\infty) \le b_\delta \quad \text{and} \quad \phi_\delta(+\infty) = m_\delta.$$

Step 5: passing to the limit $\delta \to 0^+$

Since $c_{\delta,\sigma} \le c_0$ for any $\delta > 0$ and $\sigma \in (-\min_{[\delta,1-\delta]} f, 0)$, we get

$$(7.13)\qquad c_\delta^+ \le c_0 \quad \text{for all} \quad \delta \in \left(0, \frac{1}{2}\right).$$

Moreover, since $c_{\delta,\sigma}$ is non-increasing in δ, then c_δ^+ is non-increasing in δ. Hence from (7.13), we get

$$(7.14)\qquad \lim_{\delta \to 0^+} c_\delta^+ = c^+ \le c_0.$$

We can also assume, up to translation, that the solution ϕ_δ of (7.12) satisfies

$$\begin{cases} (\phi_\delta)^*(0) \ge \dfrac{1}{2} \\ (\phi_\delta)_*(0) \le \dfrac{1}{2}. \end{cases}$$

Thus passing to the limit $\delta \to 0^+$, using again Lemma 4.2, then ϕ_δ converges, up to a subsequence, almost everywhere to some ϕ which solves in viscosity sense

$$(7.15)\qquad \begin{cases} c^+\phi'(x) = F((\phi(x+r_i))_{i=0,\dots,N}) \quad \text{on} \quad \mathbb{R} \\ \phi \ \text{ is non-decreasing over } \mathbb{R} \\ 0 \le \phi(-\infty) \quad \text{and} \quad \phi(+\infty) \le 1 \end{cases}$$

and satisfies

$$(7.16)\qquad \begin{cases} (\phi)^*(0) \ge \dfrac{1}{2} \\ (\phi)_*(0) \le \dfrac{1}{2}. \end{cases}$$

But $\phi(\pm\infty)$ is a solution of $\tilde{f} = 0$, then we get

$$\phi(-\infty) = 0 \quad \text{and} \quad \phi(+\infty) = 1.$$

This implies that if (c_0, ϕ_{c_0}) is a solution of (7.11), then $c_0 \ge c^+$ and moreover there exists such a solution $(c_0, \phi_{c_0}) = (c^+, \phi)$. We also recall that we have solutions of (7.11) for $c_0 >> 1$. Our goal now is to fill the gap and to show that we have solutions for all $c \ge c^+$.

Step 6: filling the gap
This step is analogous to the proof of Proposition 6.1. Fix $\overline{c} > c^+$ and let $\beta_0 > 0$ such that

$$(7.17)\qquad \overline{c} > c^+ + \beta_0.$$

Step 6.1 construction of a solution $(\overline{c}, \overline{\phi})$ associated to some $\overline{\sigma}$

Substep 6.1.1: $c^+ = \lim_{\delta\to 0^-} c^+_\delta$
We know from Steps 4, 5 that there exists a non trivial solution $(c^+_\delta, \phi_\delta)$ of (7.12) and that $c^+ = \lim_{\delta\to 0^-} c^+_\delta$. Thus there exists some $\delta_0 > 0$ such that

$$(7.18)\qquad |c^+_\delta - c^+| \le \frac{\beta_0}{3} \quad \text{for all} \quad 0 < \delta \le \delta_0.$$

Substep 6.1.2: $c^+_\delta = \lim_{\sigma\to 0^-} c_{\delta,\sigma}$
Similarly, we know from Steps 1, 4 that, for every $0 < \delta \le \delta_0$, there exists a solution $(c_{\delta,\sigma}, \phi_{\delta,\sigma})$ of (7.10) and that $c^+_\delta = \lim_{\sigma\to 0^-} c_{\delta,\sigma}$. Thus there exists some $\sigma_\delta > 0$ such that

$$(7.19)\qquad |c_{\delta,\sigma} - c^+_\delta| \le \frac{\beta_0}{3} \quad \text{for all} \quad 0 < -\sigma \le \sigma_\delta.$$

Substep 6.1.3: $c_{\delta,\sigma} = \lim_{p\to 0^+} c_{\delta,\sigma,p}$
Based on the proof of [1, Proposition 2.3], there exists (for every $0 < \delta \le \delta_0$ and $0 < -\sigma \le \sigma_\delta$ such that (7.19) holds true) a velocity $c_{\delta,\sigma,p}$, a profile $\phi_{\delta,\sigma,p}$ and some $p_{\delta,\sigma} > 0$ such that $c_{\delta,\sigma,p}$ converges up to a subsequence to $c_{\delta,\sigma}$ as $p \to 0$ and

$$(7.20)\qquad |c_{\delta,\sigma,p} - c_{\delta,\sigma}| \le \frac{\beta_0}{3} \quad \text{for all } p \text{ of the subsequence such that} \quad 0 < p \le p_{\delta,\sigma},$$

where $(c_{\delta,\sigma,p}, \phi_{\delta,\sigma,p})$ is a solution of

$$(7.21)\qquad \begin{cases} c_{\delta,\sigma,p}(\phi_{\delta,\sigma,p})'(x) = F_\delta((\phi_{\delta,\sigma,p}(x+r_i))_{i=0,\dots,N}) + \sigma \quad \text{on } \mathbb{R} \\ (\phi_{\delta,\sigma,p})' \geq 0 \\ \phi_{\delta,\sigma,p}\left(x + \dfrac{1}{p}\right) = 1 + \phi_{\delta,\sigma,p}(x). \end{cases}$$

Substep 6.1.4: construction of a solution $(\overline{c}, \overline{\phi})$ associated to some $\overline{\sigma}$
Since the map

$$(7.22)\qquad \sigma \mapsto \lambda(\sigma, p) := pc_{\delta,\sigma,p}$$

is continuous with $\lambda(\pm\infty, p) = \pm\infty$ (see Lemma 6.2 applied to F_δ instead of F), then for every $0 < \delta \leq \delta_0$, $0 < -\sigma \leq \sigma_\delta$ and $0 < p \leq p_{\delta,\sigma}$ such that (7.19) and (7.20) hold true, there exists $\overline{\sigma} = \overline{\sigma}_{\delta,p} \in \mathbb{R}$ and a function $\overline{\phi} = \phi_{\delta,\overline{\sigma},p} : \mathbb{R} \to \mathbb{R}$ (see Corollary 6.3) such that

$$c_{\delta,\overline{\sigma},p} = \overline{c}$$

and $(\overline{c}, \overline{\phi})$ solves

$$(7.23)\qquad \begin{cases} \overline{c}\overline{\phi}'(x) = F_\delta((\overline{\phi}(x+r_i))_{i=0,\dots,N}) + \overline{\sigma} \quad \text{on } \mathbb{R} \\ \overline{\phi}' \geq 0 \\ \overline{\phi}\left(x + \dfrac{1}{p}\right) = 1 + \overline{\phi}(x). \end{cases}$$

Substep 6.1.5: consequence of Substeps 6.1.1-6.1.4
For every $0 < \delta \leq \delta_0$, $0 < -\sigma \leq \sigma_\delta$ and $0 < p \leq p_{\delta,\sigma}$, (7.17), (7.18), (7.19) and (7.20) hold true, thus we get

$$c_{\delta,\sigma,p} \leq c^+ + \beta_0 < \overline{c} = c_{\delta,\overline{\sigma},p}.$$

But the map $\sigma \mapsto c_{\delta,\sigma,p}$ is non-decreasing (see Lemma 6.2 and (7.22)), hence we obtain that

$$(7.24)\qquad \sigma < \overline{\sigma} = \overline{\sigma}_{\delta,p}.$$

Step 6.2: getting a profile for the original problem with velocity $\overline{c}$

Substep 6.2.0: a priori estimate on $\overline{\sigma}$
The couple $(\overline{c}, \phi_{\delta,\overline{\sigma},p})$ is a solution of (7.23), thus for $p < 1$, we get

$$\phi_{\delta,\overline{\sigma},p}(x+1) - \phi_{\delta,\overline{\sigma},p}(x) \leq 1;$$

and hence we can show that there exists a constant M_0 independent of p and δ such that

$$|F_\delta| \leq M_0.$$

Thus integrating (7.23) over $[0, 1]$, implies that there exists a constant $K > 0$ such that

$$|\overline{\sigma}| \leq K$$

for all $\delta < \frac{1}{2}$ and $p \leq 1$.

Substep 6.2.1**: passing to the limit** $p \to 0$
Since $|f_\delta - f| \le o_\delta(1)$, then we can assume, up to translation, that

$$(7.25) \qquad \begin{cases} (\phi_{\delta,\overline{\sigma},p})^*(0) \ge \gamma_{\delta,\overline{\sigma}} \\ (\phi_{\delta,\overline{\sigma},p})_*(0) \le \gamma_{\delta,\overline{\sigma}} \end{cases} \quad \text{with} \quad |f_\delta(\gamma_{\delta,\overline{\sigma}}) + \overline{\sigma}| \ge \frac{1}{4} osc(f),$$

with for instance $\gamma_{\delta,\overline{\sigma}} \in [m_\delta - 1, m_\delta]$. Hence using the proof of Lemma 4.2 and the last equality of (7.23), we pass to the limit $p \to 0$ and $\phi_{\delta,\overline{\sigma},p}$ converges up to subsequence to a non trivial (because of (7.25)) solution $\phi_{\delta,\overline{\sigma}_{\delta,0}}$ of

$$(7.26) \qquad \begin{cases} \overline{c}\phi'_{\delta,\overline{\sigma}_{\delta,0}}(z) = F_\delta((\phi_{\delta,\overline{\sigma}_{\delta,0}}(z + r_i))_{i=0,\dots,N}) + \overline{\sigma}_{\delta,0} \quad \text{on} \quad \mathbb{R} \\ \phi_{\delta,\overline{\sigma}_{\delta,0}} \text{ is non-decreasing on } \mathbb{R} \\ \phi_{\delta,\overline{\sigma}_{\delta,0}}(+\infty) - \phi_{\delta,\overline{\sigma}_{\delta,0}}(-\infty) \le 1, \end{cases}$$

where

$$\overline{\sigma}_{\delta,p} \to \overline{\sigma}_{\delta,0}$$

and $|\overline{\sigma}_{\delta,0}| \le K$.

Substep 6.2.2**: establishing** $\overline{\sigma}_{\delta,0} = 0$
Since $\sigma < \overline{\sigma}_{\delta,p}$ (see (7.24)), then we get $\sigma \le \overline{\sigma}_{\delta,0}$. Thus passing to the limit $\sigma \to 0$, we get

$$\overline{\sigma}_{\delta,0} \ge 0,$$

without any change in equation (7.26). Moreover, since we have

$$0 = f_\delta(\phi_{\delta,\overline{\sigma}_{\delta,0}}(\pm\infty)) + \overline{\sigma}_{\delta,0}$$

and $f_\delta \ge 0$, then we get that

$$\overline{\sigma}_{\delta,0} = 0.$$

Therefore, because of (7.25), $\phi_\delta := \phi_{\delta,\overline{\sigma}_{\delta,0}=0}$ satisfies (7.12) with c^+_δ replaced by $\overline{c}$.

Substep 6.2.3**: passing to the limit** $\delta \to 0$
Up to translation, we assume that

$$\begin{cases} (\phi_\delta)^*(0) \ge \dfrac{b_\delta + m_\delta}{2} \\ (\phi_\delta)_*(0) \le \dfrac{b_\delta + m_\delta}{2}, \end{cases}$$

Therefore, passing to the limit using once more Lemma 4.2, ϕ_δ converges up to a subsequence to a solution ϕ of (7.15) and (7.16), with c^+ replaced by $\overline{c}$. This ϕ is non trivial because of (7.16). Moreover, since $\phi(\pm\infty)$ solves $f = 0$, we deduce that ϕ is a solution of (1.9) associated for the velocity $\overline{c}$.

Step 7: no solution for $c < c^+$
This step is analogous to Lemma 6.4. Let (c, ϕ) be a solution of (1.9). Then as a solution of (7.11), we can choose $(c_0, \phi_0) = (c, \phi)$. Therefore, the choice $c_0 = c$ in (7.14), implies that

$$c^+ \le c,$$

and then there is no a solution of (1.9) for $c < c^+$. □

8 Preliminary for the critical velocity: Harnack inequality

We prove in this subsection a Harnack inequality (Proposition 8.4) for the profile that we use in Subsection 9.1 to show that $c^+ \geq c^*$. Our approach is inspired by Hamel [23]. The proof will use a strong maximum principle for a linear evolution problem that we also prove in this subsection.

Proposition 8.1 (Strong maximum principle for a linear evolution problem)
Let F be a function satisfying (A_{Lip}) and differentiable at $\{0\}^{N+1}$. Assume that

$$\exists\ i_0 \in \{0, ..., N\} \quad \text{such that} \quad r_{i_0} > 0 \quad \text{and} \quad \frac{\partial F}{\partial X_{i_0}}(0, ..., 0) > 0. \tag{8.1}$$

Let $T > 0$ and $u : \mathbb{R}\times[0, T) \to [0, +\infty)$ be a lower semi-continuous function which is a supersolution of

$$u_t(x, t) = \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0, ..., 0)u(x + r_i, t) \quad \text{for} \quad (x, t) \in \mathbb{R} \times (0, T). \tag{8.2}$$

If $u(x_0, t_0) = 0$ for some $(x_0, t_0) \in \mathbb{R} \times (0, T)$, then

$$u(x_0 + kr_{i_0}, t) = 0 \quad \text{for all} \quad k \in \mathbb{N} \quad \text{and} \quad 0 \leq t \leq t_0.$$

Proof of Proposition 8.1
Let u be a supersolution of (8.2) such that $u \geq 0$ and assume that there exists some $(x_0, t_0) \in \mathbb{R} \times (0, T)$ such that $u(x_0, t_0) = 0$.

Step 1: $u(x_0, s) = 0$ for all $0 \leq s \leq t_0$

Step 1.1: preliminary
Since u is a supersolution of (8.2) on $\mathbb{R} \times (0, T)$, then u satisfies in the viscosity sense

$$u_t(x, t) \geq \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0, ..., 0)u(x + r_i, t) \quad \text{for all} \quad (x, t) \in \mathbb{R} \times (0, T).$$

Because

$$\frac{\partial F}{\partial X_i}(0, ..., 0) \geq 0 \quad \text{for all} \quad i \neq 0 \tag{8.3}$$

and $\left|\frac{\partial F}{\partial X_0}(0, ..., 0)\right| \leq L$, where L is the Lipschitz constant of F, we get in the viscosity sense (using $u \geq 0$):

$$u_t(x, t) \geq -Lu(x, t) \quad \text{for all} \quad (x, t) \in \mathbb{R} \times (0, T). \tag{8.4}$$

Step 1.2: $u(x_0, \cdot)$ is a viscosity supersolution of (8.4) on $(0, T)$
We now set $v(t) = u(x_0, t)$. We claim that v satisfies in the viscosity sense

$$v_t \geq -Lv \quad \text{on} \quad (0, T). \tag{8.5}$$

In order to prove our claim, let ϕ be a test function such that

$$\begin{cases} v_* \geq \phi & \text{on} \quad (0, T) \\ v_*(t_0) = \phi(t_0) & \text{for some } t_0 \in (0, T) \\ v_* > \phi & \text{for all} \quad t \neq t_0. \end{cases} \tag{8.6}$$

For every $\varepsilon > 0$, define the function

$$\psi_\varepsilon(x,t) := \phi(t) - \frac{1}{\varepsilon}(x - x_0)^2.$$

Then

$$\psi_\varepsilon(x_0,t_0) = \phi(t_0) = v_*(t_0) = u_*(x_0,t_0).$$

Using the definition of ψ_ε and (8.6), we deduce that for any $r_\varepsilon > 0$ small enough such that $[t_0 - r_\varepsilon, t_0 + r_\varepsilon] \subset (0,T)$, we have

$$\begin{cases} \psi_\varepsilon(x_0 \pm r_\varepsilon, t) = \phi(t) - \dfrac{r_\varepsilon^2}{\varepsilon} \le v_*(t) - \dfrac{r_\varepsilon^2}{\varepsilon} = u_*(x_0,t) - \dfrac{r_\varepsilon^2}{\varepsilon} < u_*(x_0,t) \\ \psi_\varepsilon(x, t_0 \pm r_\varepsilon) = \phi(t_0 \pm r_\varepsilon) - \dfrac{1}{\varepsilon}(x-x_0)^2 < v_*(t_0 \pm r_\varepsilon) = u_*(x_0, t_0 \pm r_\varepsilon) \end{cases}$$

Therefore, since u_* is lower semi-continuous, then for every $\varepsilon > 0$ there exists $c_\varepsilon \ge 0$ such that

$$\begin{aligned} \psi_\varepsilon - c_\varepsilon \le u_* \quad &\text{on} \quad (x_0 - r_\varepsilon, x_0 + r_\varepsilon) \times (t_0 - r_\varepsilon, t_0 + r_\varepsilon) \\ = \ &\text{at } P_\varepsilon = (x_\varepsilon, t_\varepsilon) \in (x_0 - r_\varepsilon, x_0 + r_\varepsilon) \times (t_0 - r_\varepsilon, t_0 + r_\varepsilon), \end{aligned}$$

with $P_\varepsilon = (x_\varepsilon, t_\varepsilon) \to (x_0, t_0)$ when $\varepsilon \to 0$ and $r_\varepsilon \to 0$.

Now, since u satisfies (8.4) in the viscosity sense and $\psi_\varepsilon - c_\varepsilon$ is a test function, then we deduce that

$$\phi_t(t_\varepsilon) = (\psi_\varepsilon)_t(P_\varepsilon) \ge -Lu_*(P_\varepsilon). \tag{8.7}$$

This implies that

$$\phi_t(t_0) \ge -L \liminf_{\varepsilon \to 0} u_*(P_\varepsilon) = -Lu_*(x_0,t_0) = -Lv_*(t_0).$$

Thus v satisfies (8.5) in the viscosity sense and hence $u(x_0, \cdot)$ satisfies (8.4) on $(0,T)$ in the viscosity sense.

Step 1.3: conclusion
Let $0 \le s_0 < t_0$ and set $w(t) = e^{-L(t-s_0)} v_*(s_0)$ which is a solution of $w_t = -Lw$. Because $v_*(s_0) \ge w^*(s_0)$, we deduce from the comparison principle that

$$v(t) \ge w(t) \quad \text{on} \quad [s_0, T). \tag{8.8}$$

In particular, evaluating (8.8) at $t = t_0$, we get

$$0 = v(t_0) \ge e^{-L(t_0 - s_0)} v_*(s_0),$$

which implies that

$$0 \ge v_*(s_0) = v(s_0) = u(x_0, s_0),$$

and this is true for any $s_0 \in [0, t_0]$. Because $u \ge 0$, we deduce that

$$u(x_0, s) = 0 \quad \text{for all} \quad 0 \le s \le t_0.$$

Step 2: $u(x_0 + r_{i_0}, t_0) = 0$
Note that for the test function $\phi \equiv 0$, we have

$$\begin{cases} u(x,t) \ge \phi(x,t) & \text{for all} \quad (x,t) \in \mathbb{R} \times (0,T) \\ u(x_0,t_0) = \phi(x_0,t_0) & \text{for} \quad (x_0,t_0) \in \mathbb{R} \times (0,T). \end{cases}$$

Therefore, the supersolution viscosity inequality implies that

$$\begin{aligned}0=\phi_t(x_0,t_0) &\geq \sum_{i=0}^{N}\frac{\partial F}{\partial X_i}(0,...,0)u(x_0+r_i,t_0)\\ &\geq \frac{\partial F}{\partial X_0}(0,...,0)u(x_0,t_0)+\frac{\partial F}{\partial X_{i_0}}(0,...,0)u(x_0+r_{i_0},t_0),\end{aligned}$$

where we have used (8.3) and the fact that $u\geq 0$. Because $u(x_0,t_0)=0$, we conclude that

$$0\geq \frac{\partial F}{\partial X_{i_0}}(0,...,0)u(x_0+r_{i_0},t_0).$$

By assumption (8.1), we recall that $\frac{\partial F}{\partial X_{i_0}}(0,...,0)>0$. Therefore, since $u\geq 0$, we deduce that

$$u(x_0+r_{i_0},t_0)=0.$$

Step 3: $u(x_0+kr_{i_0},s)=0$ for $k\in\mathbb{N}$ and $0\leq s\leq t_0$
Since $u(x_0+r_{i_0},t_0)=0$, then by Step 2, we deduce that $u(x_0+kr_{i_0},t_0)=0$ for $k\in\mathbb{N}$. Using Step 1, we get that $u(x_0+kr_{i_0},s)=0$ for all $0\leq s\leq t_0$ and $k\in\mathbb{N}$. □

Now, we give a lower bound for a solution of the nonlinear problem.

Lemma 8.2 (Existence of a solution for the nonlinear problem)
Consider a function F satisfying $(\tilde{A}_{Lip})$, (P_{Lip}) and let $\varepsilon\in(0,1]$. Then there exists $\psi:\mathbb{R}\times(0,+\infty)\to\mathbb{R}$ a viscosity solution of

$$\psi_t(x,t)=F((\psi(x+r_i,t))_{i=0,...,N})\quad on\quad \mathbb{R}\times(0,+\infty) \tag{8.9}$$

with initial condition satisfying

$$\psi^*(\cdot,0)=\varepsilon H^*\quad and\quad \psi_*(\cdot,0)=\varepsilon H_*, \tag{8.10}$$

where $H=1_{[0,+\infty)}$ is the Heaviside function.

Proof of Lemma 8.2
The proof is done in steps.

Step 1: construction of ψ_δ solution of (8.9)
Let $\delta>0$ and define

$$H_\delta=\begin{cases}0 & \text{if}\quad x\leq-\delta\\ \dfrac{x}{\delta}+1 & \text{if}\quad x\in[-\delta,0]\\ 1 & \text{if}\quad x\geq 0\end{cases}$$

Then for every $x\in\mathbb{R}$, we have $H_\delta(x)$ is non-increasing as δ decreases to zero and we also have

$$H_\delta(x)\geq H(x).$$

Since for any given $\delta>0$, the function H_δ is bounded uniformly continuous, then using [16, Corollary 2.9], we deduce that for every $\delta>0$, there exists a unique continuous solution ψ_δ of (8.9) satisfying

$$\psi_\delta(x,0)=\varepsilon H_\delta(x). \tag{8.11}$$

Step 2: properties of ψ_δ

Since $H_\delta(x)$ is non-increasing when δ decreases to zero and $H_\delta(x) \geq 0$, then using the comparison principle (see [16, Proposition 2.5]), we deduce that ψ_δ is non-increasing as δ decreases to zero and $\psi_\delta(x,t) \geq 0$ for all $(x,t) \in \mathbb{R} \times (0,+\infty)$.

Moreover, since $H_\delta(x+h) \geq H_\delta(x)$ for every $h \geq 0$ and $\delta > 0$ fixed, then by comparison principle ([16, Proposition 2.5]), we deduce that

$$\psi_\delta(x+h,t) \geq \psi_\delta(x,t),$$

i.e. ψ_δ is non-decreasing w.r.t. x. Also, since 0 and 1 are two solutions of (8.9) and $0 \leq \varepsilon H_\delta \leq 1$, then from the comparison principle we get that

$$0 \leq \psi_\delta \leq 1.$$

Now, let $C_0 = \sup\limits_{[0,1]^{N+1}} |F|$ and for $h \geq 0$, we set $\psi_\delta^\pm(x,t) := \psi_\delta(x,h) \pm C_0 t$ for $t \geq 0$. Then ψ_δ^+ is a supersolution and ψ_δ^- is a subsolution of (8.9) with

$$\psi_\delta^-(x,0) \leq \psi_\delta(x,h) \leq \psi_\delta^+(x,0).$$

Hence, using the comparison principle, we get for all $t \geq 0$

$$\psi_\delta^-(x,t) \leq \psi_\delta(x,h+t) \leq \psi_\delta^+(x,t), \tag{8.12}$$

i.e.

$$\psi_\delta(x,h) - C_0 t \leq \psi_\delta(x,h+t) \leq \psi_\delta(x,h) + C_0 t.$$

Because this true for any $t,\ h \geq 0$, we deduce that

$$|\psi_\delta(x,t) - \psi_\delta(x,s)| \leq C_0|t-s| \quad \text{for all} \quad x \in \mathbb{R},\ t,s \in [0,+\infty). \tag{8.13}$$

Step 3: the limit $\delta \to 0$

Since ψ_δ is non-increasing as δ decreases to zero and $\psi_\delta(x,t) \geq 0$ for all $(x,t) \in \mathbb{R} \times (0,+\infty)$. Then ψ_δ^+ converges pointwisely to some function $\psi \geq 0$, as $\delta \to 0$.

Using the stability of viscosity solutions (Proposition 2.2 (ii), applied for $\sup -\psi_\delta$), we deduce that ψ_* is a supersolution of (8.9). Moreover, since ψ_δ is non-decreasing w.r.t. x and satisfies (8.13), then

$$\begin{cases} \psi \ \text{ is non-decreasing w.r.t. } \ x \\ |\psi(x,t) - \psi(x,s)| \leq C_0|t-s| \quad \text{for all} \quad x \in \mathbb{R},\ t,s \in [0,+\infty). \end{cases}$$

This implies that

$$\psi^* = \limsup_{\delta \to 0}{}^* \psi_\delta.$$

Hence, using Proposition 2.2 (i), we deduce that ψ^* is a subsolution of (8.9). Therefore, ψ solves (8.9) in the viscosity sense.

In addition, since $H_\eta(x) \geq H(x) \geq H_\delta(x-\delta)$, for every $\eta,\ \delta > 0$, then

$$\psi_\eta(x,t) \geq \psi_\delta(x-\delta,t) \quad \text{for every} \quad \eta,\ \delta > 0.$$

Passing to the limit $\eta \to 0$, we obtain

$$\psi(x,t) \geq \psi_\delta(x-\delta,t) \quad \text{for every} \quad \delta > 0,$$

this implies that for every $\delta > 0$, we have

$$\psi_\delta(x,t) \geq \psi(x,t) \geq \psi_\delta(x-\delta,t) \quad \text{for every } (x,t) \in \mathbb{R} \times [0,+\infty). \tag{8.14}$$

Moreover, we have $\psi_\delta \in C^0$ and

$$\begin{cases} \psi_\delta(x,0) = 0 = \psi_\delta(x-\delta,0) & \text{for} \quad x \le -\delta \\ \psi_\delta(x,0) = \varepsilon = \psi_\delta(x-\delta,0) & \text{for} \quad x \ge \delta. \end{cases}$$

Hence, for every $\delta > 0$, we get

$$\psi^*(x,0) = \psi_*(x,0) = \begin{cases} 0 & \text{for} \quad x \le -\delta \\ \varepsilon & \text{for} \quad x \ge \delta. \end{cases}$$

Therefore, we obtain that

$$\psi^*(x,0) = \psi_*(x,0) = \begin{cases} 0 & \text{for} \quad x < 0 \\ \varepsilon & \text{for} \quad x > 0. \end{cases}$$

Using again (8.14), we get for $(x,t) = (0,0)$ that

$$\varepsilon \ge \psi^*(0,0) \ge \psi_*(0,0) \ge 0.$$

Finally, since ψ^* is upper semi-continuous and ψ_* is lower semi-continuous, we deduce that

$$\psi^*(x,0) = \varepsilon H^*(x) \quad \text{and} \quad \psi_*(x,0) = \varepsilon H_*(x).$$

□

Proposition 8.3 (Lower bound on a solution of the evolution nonlinear problem)
Consider a function F satisfying $(\tilde{A}_{Lip})$ and (P_{Lip}). Assume moreover that F is C^1 over a neighborhood of $\{0\}^{N+1}$ in $[0,1]^{N+1}$ and

$$(8.15) \qquad \exists\ i_0 \in \{0,...,N\} \quad \text{such that} \quad r_{i_0} > 0 \quad \text{and} \quad \frac{\partial F}{\partial X_{i_0}}(0,...,0) > 0.$$

Then there exists $\varepsilon_0 \in (0,1]$ and $T_0 > 0$ such that for all $\delta \in (0,T_0)$ and $R > 0$, there exists $\kappa = \kappa(\delta,R) > 0$ such that for every $0 < \varepsilon \le \varepsilon_0$, the function $\psi = \psi_\varepsilon$ given by Lemma 8.2 with initial conditions (8.10) *satisfies*

$$(8.16) \qquad \psi_\varepsilon(x,t) \ge \kappa\varepsilon \quad \text{for all} \quad (x,t) \in [-R,R] \times [\delta,T_0].$$

Proof of Proposition 8.3
We first give an upper bound on the solution ψ of (8.9) and then we prove Proposition 8.3 by contradiction.

Step 0: upper bound on ψ on $(0,2T_0)$
Let

$$M(t) := \sup_{x\in\mathbb{R}} \psi(x,t).$$

Then $M(0) = \varepsilon$ (since $\psi^*(x,0) = \varepsilon H^*(x)$). Since ψ is a solution of (8.9) then, using the viscosity techniques, we can show that M^* is a subsolution, i.e. satisfies in the viscosity sense

$$v_t(t) \le F(M^*(t),...,M^*(t)) = f(M^*(t)).$$

Using the comparison principle for the ODE $x' = f(x)$, we deduce that

$$(8.17) \qquad M^*(t) \le M_0(t) \quad \text{over} \quad [0,\infty),$$

where M_0 is a solution of

$$\begin{cases} M_0'(t) = f(M_0(t)) \geq 0 \quad \text{for} \quad (0, +\infty) \\ M_0(0) = \varepsilon. \end{cases}$$

Now, because M_0 is non-decreasing, if $M_0(t) \leq 2\varepsilon$ then

$$M_0'(t) \leq \sup_{[0,2\varepsilon]} f \leq 2L_1\varepsilon,$$

where L_1 is the Lipschitz constant of f (because $f(0) = 0$). Thus we get

$$M_0(t) \leq \varepsilon + 2tL_1\varepsilon < 2\varepsilon \quad \text{if} \quad t < \frac{1}{2L_1}.$$

Therefore for

$$T_0 = \frac{1}{4L_1}, \tag{8.18}$$

we get $M^*(t) \leq M_0(t) \leq 2\varepsilon$ on $[0, 2T_0]$, which implies that $\psi_\varepsilon = \psi$ satisfies

$$\psi_\varepsilon(x,t) \leq 2\varepsilon \quad \text{for} \quad (x,t) \in \mathbb{R} \times [0, 2T_0]. \tag{8.19}$$

Step 1: establishing (8.16)
Assume to that contrary that (8.16) is false. Then there exist $\delta \in (0, T_0)$ (with T_0 given in (8.18)), $R > 0$ and two sequences $\varepsilon_n \to 0$, $\kappa_n \to 0$ as $n \to +\infty$ and points

$$P_n = (x_n, t_n) \in [-R, R] \times [\delta, T_0] \tag{8.20}$$

such that

$$\psi_{\varepsilon_n}(P_n) \leq \kappa_n \varepsilon_n.$$

Define

$$\overline{\psi}_n(x,t) := \frac{1}{\varepsilon_n}\psi_{\varepsilon_n}(x,t) \quad \text{for all} \quad (x,t) \in \mathbb{R} \times (0, 2T_0).$$

Then we have (using (8.19)),

$$\begin{cases} 0 \leq \overline{\psi}_n \leq 2 \quad \text{over} \quad \mathbb{R} \times [0, 2T_0) \\ \overline{\psi}_n(P_n) \leq \kappa_n \to 0 \\ (\overline{\psi}_n)_*(x, t=0) = H_*(x) \end{cases}$$

and

$$(\overline{\psi}_n)_t(x,t) = \frac{1}{\varepsilon_n} F(\varepsilon_n(\overline{\psi}_n(x + r_i, t))_{i=0,\dots,N}). \tag{8.21}$$

Step 1.1: uniform lower bound of $\overline{\psi}_n$
Denote by $Z = (\overline{\psi}_n(x + r_i, t))_{i=0,\dots,N}$. Since F is C^1 over a neighborhood of $\{0\}^{N+1}$, then for ε_n small enough, we can show that

$$\begin{aligned} (\overline{\psi}_n)_t(x,t) &= \frac{1}{\varepsilon_n} F(\varepsilon_n(\overline{\psi}_n(x + r_i, t))_{i=0,\dots,N}) \\ &= \int_0^1 \frac{\partial F}{\partial X_0}(s\varepsilon_n Z)\overline{\psi}_n(x,t)ds + \sum_{i=1}^N \int_0^1 \frac{\partial F}{\partial X_i}(s\varepsilon_n Z)\overline{\psi}_n(x + r_i, t)ds \\ &\geq -L\overline{\psi}_n(x,t) + \frac{1}{2}\frac{\partial F}{\partial X_{i_0}}(0, \dots, 0)\overline{\psi}_n(x + r_{i_0}, t), \end{aligned}$$

where we have used the fact that $\overline{\psi}_n \geq 0$ and $\frac{\partial F}{\partial X_i} \geq 0$ for all $i \neq 0$. Hence $\overline{\psi}_n$ is a supersolution of the equation

$$w_t(x,t) = -Lw(x,t) + \frac{1}{2}\frac{\partial F}{\partial X_{i_0}}(0,...,0)w(x+r_{i_0},t). \tag{8.22}$$

Now, let

$$H_\eta(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ \frac{1}{\eta}x & \text{if} \quad 0 \leq x \leq \eta \\ 1 & \text{if} \quad x \geq 1 \end{cases}$$

for $\eta > 0$ small. Since $\frac{\partial F}{\partial X_{i_0}}(0,...,0) \geq 0$, then by a simple calculation, we can show that the function

$$\phi(x,t) = e^{-Lt}H_\eta(x)$$

(with L the Lipschitz constant of F) is a subsolution of (8.22). Moreover, we have

$$(\overline{\psi}_n)_*(x,t=0) = H_*(x) \geq H_\eta(x) = \phi(x,t=0).$$

Therefore, using a comparison principle for (8.22), we deduce that

$$e^{-Lt}H_\eta(x) \leq \overline{\psi}_n(x,t) \quad \text{for all} \quad (x,t) \in \mathbb{R} \times [0,2T_0). \tag{8.23}$$

Step 1.2: passing to the limit and getting a contradiction
Since $\overline{\psi}_n(x,t)$ is uniformly bounded on $\mathbb{R} \times [0,2T_0)$ and

$$(\overline{\psi}_n)_t(x,t) \geq \sum_{i=0}^{N} \int_0^1 \frac{\partial F}{\partial X_i}(s\varepsilon_n Z)\overline{\psi}_n(x+r_i,t)ds,$$

then using the fact that F is C^1 over a neighborhood of $\{0\}^{N+1}$ and $\varepsilon_n \to 0$, we deduce that $\overline{\psi}_\infty = \liminf_{n\to+\infty} {}_*\overline{\psi}_n$ satisfies in the viscosity sense on $\mathbb{R} \times [0,2T_0)$

$$\begin{cases} (\overline{\psi}_\infty)_t(x,t) \geq \displaystyle\sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0,...,0)\overline{\psi}_\infty(x+r_i,t) \\ 0 \leq \overline{\psi}_\infty \leq 2 \end{cases}$$

and

$$e^{-Lt}H_\eta(x) \leq \overline{\psi}_\infty(x,t) \quad \text{for all} \quad (x,t) \in \mathbb{R} \times [0,2T_0). \tag{8.24}$$

In addition, we also have $P_n \to P_\infty = (x_\infty, t_\infty)$ in $[-R,R] \times [\delta, T_0]$, hence using the fact that $\overline{\psi}_n(P_n) \to 0$, we get

$$\overline{\psi}_\infty(P_\infty) = 0.$$

Using the strong maximum principle (Proposition 8.1) that holds for supersolutions, we deduce for $k \in \mathbb{N}$ that

$$\overline{\psi}_\infty(x_\infty + kr_{i_0}, t) = 0 \quad \text{for all} \quad 0 \leq t \leq t_\infty.$$

But $r_{i_0} > 0$, hence for $t = 0$, $k >> 1$ and using (8.24), we get

$$1 = H_\eta(x_\infty + kr_{i_0}) \leq \overline{\psi}_\infty(x_\infty + kr_{i_0}, 0) = 0,$$

which is a contradiction. □

In the following proposition, we give a Harnack type inequality.

Proposition 8.4 (Harnack inequality)
Let F be a function satisfying (A_{Lip}), (P_{Lip}) and assume that F is C^1 over a neighborhood of $\{0\}^{N+1}$ in $[0,1]^{N+1}$. Assume moreover that

$$\exists\ i_0 \in \{0,...,N\} \quad \textit{such that} \quad r_{i_0} > 0 \quad \textit{and} \quad \frac{\partial F}{\partial X_{i_0}}(0,...,0) > 0. \tag{8.25}$$

Let (c,u) with $c \neq 0$ be a solution of

$$\begin{cases} cu'(x) = F((u(x+r_i))_{i=0,\dots,N}) \quad \textit{on} \quad \mathbb{R} \\ u' \geq 0 \\ u(-\infty) = 0 \quad \textit{and} \quad u(+\infty) = 1. \end{cases} \tag{8.26}$$

Then for every $\rho > 0$ there exists a constant $\overline{\kappa}_1 = \overline{\kappa}_1(\rho) > 1$ such that for every $x \in \mathbb{R}$, we have

$$\sup_{B_\rho(x)} u \leq \overline{\kappa}_1 \inf_{B_\rho(x)} u. \tag{8.27}$$

Moreover, there exists $\overline{\kappa}_0 > 1$ such that

$$u(x+r^*) \leq \overline{\kappa}_0 u(x), \tag{8.28}$$

where $r^* = \max_{i=0,\dots,N} |r_i|$.

We refer the reader to Remark 9.1 for comments on assumption (8.25).

Proof of Proposition 8.4
Let $\tilde{F}$ be the extension of F on $\mathbb{R}^{N+1}$ given by Lemma 7.1. Then define the function

$$\overline{u}(x,t) = u(x+ct),$$

where $u \in C^1$, because $c \neq 0$. Thus $\overline{u}$ satisfies

$$\overline{u}_t(x,t) = \tilde{F}((\overline{u}(x+r_i,t))_{i=0,\dots,N}) \quad \text{for all} \quad (x,t) \in \mathbb{R} \times (0,+\infty) \tag{8.29}$$

and

$$\overline{u}(x,0) = u(x). \tag{8.30}$$

Let $x_0 \in \mathbb{R}$ such that $1 \geq u(x_0) > 0$. Since u is non-decreasing, then for all $x \in \mathbb{R}$ we have

$$\overline{u}(x,0) \geq u(x_0)H(x-x_0), \tag{8.31}$$

where $H = 1_{[0,+\infty)}$ is the Heaviside function.

For $\varepsilon \in (0,1]$ that will be fixed later, let $\psi_\varepsilon = \psi$ be the solution given by Lemma 8.2 with initial conditions (8.10) and let

$$\overline{v}(x,t) = \psi_\varepsilon(x-x_0,t).$$

Now, using Proposition 8.3, we deduce that there exists some $\varepsilon_0 \in (0,1]$ and T_0 such that for all $\delta \in (0,T_0)$ and $R > 0$ there exists a constant $\kappa = \kappa(\delta,R) > 0$ such that if $\varepsilon \leq \varepsilon_0$, then

$$\overline{v}(x,t) \geq \varepsilon\kappa \quad \text{for all} \quad (x,t) \in [x_0-R, x_0+R] \times [\delta, T_0]. \tag{8.32}$$

We now choose

$$\varepsilon = \min(\varepsilon_0, u(x_0)).$$

In particular, we have

$$\overline{u}(x,0) \geq \overline{v}^*(x,0) \quad \text{for all} \quad x \in \mathbb{R}.$$

Using the comparison principle (see [16, Proposition 2.5]), we deduce that

$$\overline{u} \geq \overline{v} \quad \text{for all} \quad (x,t) \in \mathbb{R} \times (0,+\infty). \tag{8.33}$$

From (8.32), we deduce that

$$\overline{u} \geq \kappa_1 u(x_0) \quad \text{on} \quad [x_0 - R, x_0 + R] \times [\delta, T_0], \tag{8.34}$$

with $\kappa_1 = \varepsilon_0 \kappa$ (using $\varepsilon \in (0,1]$, $u(x_0) \in (0,1]$ and the definition of ε). Because $\overline{u}(x,t) = u(x+ct)$, we conclude that

$$\inf_{(x,t)\in[x_0-R,x_0+R]\times[\delta,T_0]} u(x+ct) \geq \kappa_1 u(x_0).$$

Now, for any $r > 0$, we can find $R_r > 0$ large enough such that $B_r(x_0) \subset \overline{B}_{R_r}(x_0) + c[\delta, T_0]$. Therefore, since u is continuous and non-decreasing, then

$$u(x_0 - r) = \inf_{x \in B_r(x_0)} u(x) \geq \inf_{(x,t)\in[x_0-R,x_0+R]\times[\delta,T_0]} u(x+ct) \geq \kappa_1 u(x_0) \tag{8.35}$$

with $\kappa_1 = \kappa_1(r)$.

Let $\rho = \frac{r}{2}$ and choose y_0 such that $B_\rho(y_0) = (x_0 - r, x_0)$, i.e. $y_0 - \rho = x_0 - r$ and $y_0 + \rho = x_0$. Thus, using again the fact that u is non-decreasing, we get

$$\sup_{B_\rho(y_0)} u = u(y_0 + \rho) = u(x_0)$$

and

$$u(x_0 - r) = u(y_0 - \rho) = \inf_{B_\rho(y_0)} u.$$

Therefore, we deduce from (8.35) that

$$\sup_{B_\rho(y_0)} u \leq \overline{\kappa}_1 \inf_{B_\rho(y_0)} u \quad \text{with} \quad \overline{\kappa}_1 = \frac{1}{\kappa_1}. \tag{8.36}$$

Using (8.36) for $2\rho \geq r^*$ and $\overline{\kappa}_0 = \overline{\kappa}_1(r^*) = \big(\varepsilon_0 \kappa(\delta, R_{r^*})\big)^{-1}$, setting $z_0 = y_0 - \rho$ and using the monotonicity of u, we get

$$u(z_0 + r^*) \leq u(z_0 + 2\rho) = u(y_0 + \rho) = \sup_{B_\rho(y_0)} u \leq \overline{\kappa}_0 \Big(\inf_{B_\rho(y_0)} u \Big) = \overline{\kappa}_0 u(y_0 - \rho) = \overline{\kappa}_0 u(z_0). \tag{8.37}$$

Finally, since x_0 is chosen arbitrary at the beginning of the reasoning, we deduce that (8.36) and (8.37) do hold for any y_0, z_0. This shows (8.27) and (8.28), and ends the proof. □

9 Properties of the critical velocity

In a first subsection, we prove that $c^+ \geq c^*$, precisely Proposition 1.5. We also show, if F satisfies the KPP condition (1.16), that $c^* \geq c^+$ (see Proposition 1.6). In this subsection, we also give an example where $c^+ > c^*$ (Lemma 9.3). We prove in a second subsection Proposition 1.3 which asserts that the critical velocity satisfies $c^+ \geq 0$ under additional assumptions. In a third subsection, we give an example (Proposition 1.4) that shows that we can have $c^+ < 0$ when the additional assumptions are not satisfied. We also prove the instability of the critical velocity, namely Proposition 1.2.

9.1 Lower bound for c^+

In this subsection, we prove a lower bound for the critical velocity c^+ given in Theorem 1.1. Precisely, we show in Proposition 1.5 that $c^+ \geq c^*$. In Lemma 9.3, we give an example where $c^+ > c^*$. In this subsection, we also prove that and $c^* \geq c^+$ under a KPP condition (see Proposition 1.6).

We start with the proof of Proposition 1.5

Proof of Proposition 1.5
Under assumptions (A_{Lip}) and (P_{Lip}), let c^+ given by Theorem 1.1. We want to show that $c^+ \geq c^*$ with c^* given in (1.15).

Part I: proving that $c^+ \geq c^*$ under the assumption (1.14)
Let $c \geq c^+$ such that $c \neq 0$ and let us prove that $c \geq c^*$. Associate for c a profile ϕ such that (c, ϕ) is a solution of (1.9) (this is always possible since $c \geq c^+$, see Theorem 1.1).

Step $I.1$: $\frac{\phi'(x)}{\phi(x)}$ is globally bounded
From Harnack inequality (8.27), we deduce that if $\phi(x_0) = 0$ at some point $x_0 \in \mathbb{R}$, then $\phi \equiv 0$ which is impossible for a solution of (1.9). Therefore $\phi > 0$.

We have
$$c\frac{\phi'(x)}{\phi(x)} = \frac{1}{\phi(x)} F((\phi(x + r_i))_{i=0,...,N}).$$
We also know, using the monotonicity of F w.r.t. X_i for $i \neq 0$ and $F(0, ..., 0) = 0$, that
$$\begin{aligned} F(\phi(x), \phi(x + r_1), ..., \phi(x + r_N)) &= F(\phi(x), \phi(x + r_1), ..., \phi(x + r_N)) - F(0, ..., 0) \\ &\leq F(\phi(x), \phi(x + r^*), ..., \phi(x + r^*)) - F(0, ..., 0), \end{aligned}$$
where $r^* = \max_{i=0,...,N} |r_i|$. Since F is Lipschitz (with constant Lipschitz L), then
$$F(\phi(x), \phi(x + r_1), ..., \phi(x + r_N)) \leq L \begin{vmatrix} \phi(x) \\ \phi(x + r^*) \\ \vdots \\ \phi(x + r^*) \end{vmatrix} \leq L_1 \phi(x + r^*) \quad \text{with} \quad L_1 = L \begin{vmatrix} 1 \\ \vdots \\ 1 \end{vmatrix}$$
and hence ($c \neq 0$)
$$0 \leq \frac{\phi'(x)}{\phi(x)} \leq \frac{1}{|c|} L_1 \frac{\phi(x + r^*)}{\phi(x)}.$$
From Proposition 8.4, we know that there exists a constant $\overline{\kappa}_0 > 1$ such that

$$\phi(x + r^*) \leq \overline{\kappa}_0 \phi(x), \tag{9.1}$$

therefore, we deduce that

$$0 \leq \frac{\phi'(x)}{\phi(x)} \leq \mathcal{M} := \frac{\overline{\kappa}_0 L_1}{|c|}. \tag{9.2}$$

Step $I.2$: proving that $c \geq c^*$
Since ϕ satisfies (9.2), then $\limsup_{x \to -\infty} \dfrac{\phi'(x)}{\phi(x)} = \lambda$ exists and $\lambda = \lim_{n \to +\infty} \dfrac{\phi'(x_n)}{\phi(x_n)}$ for some $x_n \to -\infty$ as $n \to +\infty$. Let
$$\phi_n(x) := \frac{\phi(x + x_n)}{\phi(x_n)} \geq 0,$$

then $\phi_n(0) = 1$ and ϕ_n satisfies

$$c\phi_n'(x) = \frac{1}{\phi(x_n)} F((\phi(x + x_n + r_i))_{i=0,\dots,N}) \quad \text{on} \quad \mathbb{R}. \tag{9.3}$$

Now, since for all i, $\phi(x + x_n + r_i) \to 0$ as $n \to +\infty$, $F(0, ..., 0) = 0$ and F is C^1 over a neighborhood of $\{0\}^{N+1}$ in $[0, 1]^{N+1}$, then we see that we can write for n large enough

$$c\phi_n'(x) = \sum_{i=0}^{N} \int_0^1 \frac{\partial F}{\partial X_i}(s\phi(x + x_n + r_i))\phi_n(x + r_i)ds \quad \text{on} \quad \mathbb{R}. \tag{9.4}$$

From (8.28), we deduce that for $k \in \mathbb{N}\backslash\{0\}$, we have

$$\phi(x_n + kr^*) \le (\overline{\kappa}_0)^k \phi(x_n) \quad \text{and} \quad \phi(x + r^*) \le \overline{\kappa}_0 \phi(x),$$

with $\overline{\kappa}_0 > 1$. Hence for $x \in [(k-1)r^*, kr^*]$, we get

$$0 \le \phi_n(x) = \frac{\phi(x + x_n)}{\phi(x_n)} \le (\overline{\kappa}_0)^k \le (\overline{\kappa}_0)^{\frac{x}{r^*}+1} \le \overline{\kappa}_0 e^{\mu x} \quad \text{with} \quad \mu = \frac{\ln \overline{\kappa}_0}{r^*}. \tag{9.5}$$

This implies that

$$0 \le \phi_n(x) \le \overline{\kappa}(x) := \overline{\kappa}_0 e^{\mu x^+}.$$

From (9.2), we have

$$0 \le \frac{\phi_n'}{\phi_n} \le \mathcal{M},$$

which implies that

$$0 \le \phi_n'(x) \le \mathcal{M}\overline{\kappa}(x). \tag{9.6}$$

Therefore, using Ascoli's Theorem and the extraction diagonal argument, we deduce that ϕ_n converges locally uniformly to some ϕ_∞ which satisfies (in the viscosity sense)

$$\begin{cases} c\phi_\infty'(x) = \displaystyle\sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0, ..., 0)\phi_\infty(x + r_i) \quad \text{on} \quad \mathbb{R} \\ \phi_\infty' \ge 0 \\ \phi_\infty(0) = 1 \\ \phi_\infty(x + r^*) \le \overline{\kappa}_0 \phi_\infty(x). \end{cases} \tag{9.7}$$

Therefore, using Lemma 9.2 below (with $a_0 = r^* > 0$), we deduce that

$$c \ge c^*. \tag{9.8}$$

Step $I.3$: conclusion ($c^+ \ge c^*$)
Since (9.8) holds true for any $c \ge c^*$ with $c \ne 0$, we deduce that $c^+ \ge c^*$.

Part II: proving $c^+ \ge c^*$ if $c^+ < 0$
Since $c^+ < 0$, we deduce from Proposition 1.3 *ii)* that there exists some $r_{i_1} < 0$. Let $\varepsilon > 0$ and define the function

$$F_\varepsilon(X_0, ..., X_N) := F(X_0, ..., X_N) + \varepsilon(X_{i_1} - X_0).$$

Using Theorem 1.1, there exists a critical velocity c_ε^+ such that for every $c \ge c_\varepsilon^+$ there exists a solution of (1.9) with F replaced by F_ε.

Let (c^+, ϕ^+) be a solution of (1.9) given by Theorem 1.1. Since ϕ^+ is non-decreasing and $r_{i_1} < 0$ $(\phi^+(x + r_{i_1}) - \phi^+(x) \leq 0)$, then

$$c^+(\phi^+)'(x) = F((\phi^+(x + r_i))_{i=0,...,N}) \geq F_\varepsilon((\phi^+(x + r_i))_{i=0,...,N}).$$

In addition, if $\phi^+(x_0) = 0$ for some $x_0 \in \mathbb{R}$, then using the strong maximum principle ([1, Lemma 6.1]), we deduce that

$$\phi^+ = 0 \quad \text{on} \quad [x_0, +\infty)$$

because $c^+ < 0$, which is a contradiction since $\phi^+(+\infty) = 1$. Therefore, (c^+, ϕ^+) is a supersolution of (1.9) for F replaced by F_ε with $\phi^+ > 0$.

Using now Proposition 3.2, we deduce that there exists a solution $(c^+, \overline{\phi})$ of (1.9) with F replaced by F_ε. But c^+_ε is the minimal velocity associated to F_ε, thus we deduce that

$$c^+ \geq c^+_\varepsilon.$$

This implies in particular that $c^+_\varepsilon < 0$, and since

$$\frac{\partial F_\varepsilon}{\partial X_{i_1}}(0, ..., 0) = \frac{\partial F}{\partial X_{i_1}}(0, ..., 0) + \varepsilon > 0,$$

then by Remark 9.1 below, we get that

$$c^+ \geq c^+_\varepsilon \geq c^*_\varepsilon. \tag{9.9}$$

However from (1.15), we have

$$c^*_\varepsilon = \inf_{\lambda > 0} \frac{P_\varepsilon(\lambda)}{\lambda}$$

with

$$\begin{aligned} P_\varepsilon(\lambda) &= \sum_{i=0}^{N} \frac{\partial F_\varepsilon}{\partial X_i}(0, ..., 0)e^{\lambda r_i} \\ &= \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0, ..., 0)e^{\lambda r_i} + \varepsilon(e^{\lambda r_{i_1}} - 1) \\ &= P(\lambda) + \varepsilon(e^{\lambda r_{i_1}} - 1). \end{aligned}$$

Hence

$$c^*_\varepsilon = \inf_{\lambda > 0} \left(\frac{P(\lambda)}{\lambda} + \frac{\varepsilon}{\lambda}(e^{\lambda r_{i_1}} - 1) \right).$$

Thus passing to the limit $\varepsilon \to 0$, we get that

$$\lim_{\varepsilon \to 0} c^*_\varepsilon = \inf_{\lambda > 0} \frac{P(\lambda)}{\lambda} = c^*.$$

Therefore, we deduce from (9.9) that

$$c^+ \geq c^*,$$

which ends the proof. □

Remark 9.1 (About the assumption (1.14))
It is possible to show that Proposition 1.5 still holds true if we replace (1.14) by

$$\exists\, i_1 \in \{0, ..., N\} \quad \textit{such that} \quad r_{i_1} < 0 \quad \textit{and} \quad \frac{\partial F}{\partial X_{i_1}}(0, ..., 0) > 0$$

if

$$c^+ < 0.$$

In order to see it, we can prove a lower bound (analogue to Proposition 8.3) with

$$\psi_\varepsilon \geq \kappa\varepsilon \quad \textit{on} \quad [\delta, R] \times [\delta, T_0]$$

for $\delta > 0$ (this lower bound is obtained with a variant of the strong maximum principle, Proposition 8.1).

From this, we can deduce a Harnack inequality for solution of (8.26) with $c < 0$ (analogue to Proposition 8.4). Again using this Harnack inequality, we can conclude that $c^+ \geq c^*$ as in the proof of Proposition 1.5.

Lemma 9.2 (Lower bound for c^+ for linear problem)
Let F be a function satisfying (A_{Lip}) and differentiable at $\{0\}^{N+1}$ in $[0,1]^{N+1}$. Assume moreover that F satisfies (1.14) and

$$f'(0) = \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0, ..., 0) > 0, \tag{9.10}$$

where we recall that $f(v) = F(v, ..., v)$. Let $c \neq 0$ and assume that there exists $a_0 > 0$ and $C_0 > 0$ such that ϕ is a solution of

$$\begin{cases} c\phi'(x) = \displaystyle\sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0, ..., 0)\phi(x + r_i) \quad \textit{on} \quad \mathbb{R} \\ \phi' \geq 0 \\ \phi > 0 \\ 1 \leq \dfrac{\phi(x + a_0)}{\phi(x)} \leq C_0 \quad \textit{for all} \quad x \in \mathbb{R}. \end{cases} \tag{9.11}$$

Then

$$c \geq c^*,$$

where c^* is given in (1.15).

Proof of Lemma 9.2
Step 0: preliminary
Let $a \in (0, a_0)$ and let

$$K^* = \inf E \quad \text{with} \quad E = \{k \geq 1 \;\; \text{such that} \;\; k\phi(x) \geq \phi(x + a) \quad \text{for all } x \in \mathbb{R}\}.$$

We deduce from (9.11) that $E \neq \emptyset$ because $C_0 \in E$. By definition of K^*, we have

$$K^*\phi(x) \geq \phi(x + a) \quad \text{for every} \;\; x \in \mathbb{R}. \tag{9.12}$$

We have $K^* \geq 1$. If $K^* = 1$, then ϕ is constant and the first equation of (9.11) gives

$$0 = \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0, ..., 0) = f'(0)$$

which is a contradiction with (9.10). Therefore $K^* > 1$, and there exists $\lambda > 0$ such that

$$K^* = e^{\lambda a}. \tag{9.13}$$

Again by definition of K^*, for every $\varepsilon > 0$, there exists $x_\varepsilon \in \mathbb{R}$ such that

$$(K^* - \varepsilon)\phi(x_\varepsilon) < \phi(x_\varepsilon + a). \tag{9.14}$$

Let

$$\phi_\varepsilon(x) := \frac{\phi(x + x_\varepsilon)}{\phi(x_\varepsilon)}.$$

Then $\phi_\varepsilon(0) = 1$,

$$K^*\phi_\varepsilon(x) \geq \phi_\varepsilon(x + a) \tag{9.15}$$

and (9.14) can be rewritten as

$$(K^* - \varepsilon)\phi_\varepsilon(0) < \phi_\varepsilon(a). \tag{9.16}$$

Step 1: passing to limit $\varepsilon \to 0$
Since $c \neq 0$, we can bound both ϕ_ε and ϕ'_ε on any bounded interval uniformly w.r.t. ε (as in Step 2 of the proof of Proposition 1.5). Therefore, using Ascoli Theorem and the extraction diagonal argument, we deduce that ϕ_ε converges to some ϕ_0 locally uniformly and ϕ_0 satisfies (in the viscosity sense)

$$\begin{cases} c\phi'_0(x) = \displaystyle\sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0, ..., 0)\phi_0(x + r_i) \quad \text{on} \quad \mathbb{R} \\ \phi'_0 \geq 0 \\ \phi_0(0) = 1 \\ K^*\phi_0(0) \leq \phi_0(a) \quad \text{(using (9.16))} \\ K^*\phi_0(x) \geq \phi_0(x + a) \quad \text{(using (9.15))}. \end{cases} \tag{9.17}$$

Now, let $w(x) = K^*\phi_0(x) - \phi_0(x + a)$. Then from (9.17), we deduce that w satisfies

$$\begin{cases} cw'(x) = \displaystyle\sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0, ..., 0)w(x + r_i) \quad \text{on} \quad \mathbb{R} \\ w \geq 0 \quad \text{on} \quad \mathbb{R} \\ w(0) = 0. \end{cases} \tag{9.18}$$

Then using the half strong maximum principle [1, Lemma 6.1], we get that $w(x) = 0$ for all $cx \leq 0$, i.e.

$$k^*\phi_0(x) = \phi_0(x + a) \quad \text{for all} \quad cx \leq 0. \tag{9.19}$$

Step 2: establishing $c \geq c^*$
Let

$$\phi_{0,n}(x) := \frac{\phi_0(x - cn)}{\phi_0(-cn)}.$$

Then $\phi_{0,n}(0) = 1$. Moreover, using (9.19), we have

$$K^* \frac{\phi_0(x - cn)}{\phi(-cn)} = \frac{\phi_0(x - cn + a)}{\phi_0(-cn)} \quad \text{for all} \quad c(x - cn) \leq 0.$$

Hence

$$K^*\phi_{0,n}(x) = \phi_{0,n}(x+a) \quad \text{for all} \quad cx \le c^2 n. \tag{9.20}$$

Step 2.1: passing to the limit $n \to +\infty$
As before, we can pass to the limit and show that $\phi_{0,n} \to \phi_{0,\infty}$ with

$$\begin{cases} c\phi_{0,\infty}'(x) = \displaystyle\sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0,...,0)\phi_{0,\infty}(x+r_i) \quad \text{on} \quad \mathbb{R} \\ \phi_{0,\infty}' \ge 0 \\ \phi_{0,\infty}(0) = 1. \end{cases} \tag{9.21}$$

Moreover, passing to the limit in (9.20), we deduce that

$$K^*\phi_{0,\infty}(x) = \phi_{0,\infty}(x+a) \quad \text{for all} \quad x \in \mathbb{R}. \tag{9.22}$$

Step 2.2: conclusion
Let

$$z(x) = \frac{\phi_{0,\infty}(x)}{e^{\lambda x}}.$$

Recall that $\phi_{0,\infty} \in C^1$ (because $c \neq 0$). Then $z \in C^1$ and satisfies

$$cz'(x) + c\lambda z(x) = \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0,...,0)e^{\lambda r_i} z(x+r_i) \quad \text{on} \quad \mathbb{R}. \tag{9.23}$$

We also have

$$z(x+a) = \frac{\phi_{0,\infty}(x+a)}{e^{\lambda(x+a)}} = \frac{K^*\phi_{0,\infty}(x)}{e^{\lambda a}e^{\lambda x}} = z(x),$$

where we have used (9.22) and (9.13).

Because z is a-periodic (and continuous), there exists $x_0 \in \mathbb{R}$ such that z attain it's minimum at x_0. We claim that $z(x_0) \neq 0$. Indeed, if $z(x_0) = 0$, then we deduce from (9.23) that

$$\sum_{i=1}^{N} \frac{\partial F}{\partial X_i}(0,...,0)e^{\lambda r_i} z(x_0+r_i) = 0.$$

Since $\frac{\partial F}{\partial X_i}(0,...,0) \ge 0$ for all $i = 1,...,N$ and F satisfies (1.14), we deduce that

$$z(x_0 + r_{i_0}) = 0.$$

Repeating the same process, we get that $z = 0$ on $x_0 + r_{i_0}\mathbb{N}$. Since z is a-periodic, then $z = 0$ on $x_0 + r_{i_0}\mathbb{N} + a\mathbb{Z} \equiv x_0 + a(\frac{r_{i_0}}{a}\mathbb{N} + \mathbb{Z})$.

Since $a \in (0, a_0)$ is arbitrary, then we can choose $a \in (0, a_0)$ such that $\frac{r_{i_0}}{a} \in \mathbb{R}\backslash\mathbb{Q}$. Therefore, $x_0 + a(\frac{r_{i_0}}{a}\mathbb{N} + \mathbb{Z})$ is dense in $\mathbb{R}$. This implies, since z is continuous, that

$$z = 0 \quad \text{on} \quad \mathbb{R},$$

which is a contradiction with $z(0) = 1$.

Therefore, $z(x_0) \neq 0$. Again, since $z(x_0) = \min z \geq 0$, then using (9.23), we get that

$$\begin{aligned} c\lambda z(x_0) &= \frac{\partial F}{\partial X_0}(0,...,0)e^{\lambda r_0} z(x_0) + \sum_{i=1}^{N} \frac{\partial F}{\partial X_i}(0,...,0)e^{\lambda r_i} z(x_0 + r_i) \\ &\geq \frac{\partial F}{\partial X_0}(0,...,0)e^{\lambda r_0} z(x_0) + \sum_{i=1}^{N} \frac{\partial F}{\partial X_i}(0,...,0)e^{\lambda r_i} z(x_0) \\ &= z(x_0) \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0,...,0)e^{\lambda r_i}. \end{aligned}$$

Using the fact that $z(x_0) \neq 0$, we deduce that

$$c\lambda \geq \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0,...,0)e^{\lambda r_i}.$$

Recall that $\lambda > 0$. Therefore, we get

$$c \geq \frac{P(\lambda)}{\lambda} \geq \inf_{\lambda>0} \frac{P(\lambda)}{\lambda} = c^*,$$

where $P(\lambda) = \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0,...,0)e^{\lambda r_i}$. This ends the proof. □

Now, we give the proof of Proposition 1.6, where we show that $c^+ \leq c^*$ under a KPP type condition.

Proof of Proposition 1.6

The goal is to prove that for any real $c > c^*$ ($c^* < +\infty$), we have $c^+ \leq c$.

For such c, we have $c > c^* = \inf_{\lambda>0} \frac{P(\lambda)}{\lambda}$, hence there exists some $\lambda_0 > 0$ such that

$$c > \frac{P(\lambda_0)}{\lambda_0}.$$

This implies that $\phi(x) = e^{\lambda_0 x}$ satisfies

$$c\phi'(x) > G((\phi(x + r_i))_{i=0,...,N}), \tag{9.24}$$

where $G(X) = \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0,...,0)X_i$. Let $\tilde{F}$ be the extension over $\mathbb{R}^{N+1}$ of F (given by Lemma 7.1). The goal is now to construct a supersolution of

$$cw'(x) = \tilde{F}((w(x + r_i))_{i=0,...,N}) \quad \text{on} \quad \mathbb{R}. \tag{9.25}$$

Step 1: $\overline{\phi}(x) := \min(1, \phi(x))$ **is a supersolution of** (9.25)

We recall that $\phi(0) = 1$. Let $x < 0$, we have

$$\begin{cases} \overline{\phi}(x + r_i) = \phi(x + r_i) & \text{for} \quad r_i \leq 0 \\ \overline{\phi}(x + r_i) \leq \phi(x + r_i) & \text{for} \quad r_i > 0. \end{cases}$$

Since F is non-decreasing w.r.t. X_i for $i \neq 0$, then G satisfies the same property, hence

$$\begin{aligned} G((\phi(x + r_i))_{i=0,...,N}) &\geq G((\overline{\phi}(x + r_i))_{i=0,...,N}) \\ &\geq F((\overline{\phi}(x + r_i))_{i=0,...,N}), \end{aligned}$$

where we have used (1.16) and the fact that $0 \leq \overline{\phi}(x) \leq 1$. But $\overline{\phi}(x) = \phi(x)$ is a test function for $x < 0$ and ϕ satisfies (9.24), thus we get for $x < 0$:

$$c\overline{\phi}'(x) = c\phi'(x) > G((\phi(x+r_i))_{i=0,...,N}) \geq F((\overline{\phi}(x+r_i))_{i=0,...,N}).$$

Similarly for $x > 0$, we have

$$\begin{cases} \overline{\phi}(x+r_i) \leq 1 & \text{for} \quad r_i < 0 \\ \overline{\phi}(x+r_i) = 1 & \text{for} \quad r_i \geq 0. \end{cases}$$

Moreover, since $\overline{\phi}(x) = 1$ is a test function for $x > 0$, we get

$$c\overline{\phi}'(x) = 0 = F(1, ..., 1) \geq F((\overline{\phi}(x+r_i))_{i=0,...,N}).$$

Now for $x = 0$, we have $\overline{\phi}(0) = 1 = \phi(0)$ is a supersolution of (9.25) because there is no test function touching $\overline{\phi}$ from below at $x = 0$ (see Definition 2.1). Finally, since $0 \leq \overline{\phi}(x) \leq 1$, then $\tilde{F}((\overline{\phi}(x+r_i))_{i=0,...,N}) = F((\overline{\phi}(x+r_i))_{i=0,...,N})$ and hence $\overline{\phi}$ is a supersolution of (9.25).

Step 2: subsolution of (9.25)
Let (c^+, ϕ^+) be a solution of (9.25) given by Theorem 1.1. We know, from the proof of Theorem 1.1 (see (7.10)), that

$$c^+ = \lim_{\delta \to 0}(\lim_{\sigma \to 0^-} c_{\delta,\sigma}) \quad \text{and} \quad \phi^+ = \lim_{\delta \to 0}(\lim_{\sigma \to 0^-} \phi_{\delta,\sigma})$$

where $\delta > 0$, $\sigma < 0$ are small enough and $(c_{\delta,\sigma}, \phi_{\delta,\sigma})$ is a solution of (with $\tilde{F}_\delta = F_\delta$)

$$c_{\delta,\sigma}\phi'_{\delta,\sigma}(x) = F_\delta((\phi_{\delta,\sigma}(x+r_i))_{i=0,...,N}) + \sigma$$

and $\phi_{\delta,\sigma}(-\infty) = m_{\delta,\sigma} - 1$, $\phi_{\delta,\sigma}(+\infty) = m_{\delta,\sigma}$ with $m_{\delta,\sigma} - 1 < 0 < m_{\delta,\sigma} < 1$.

Since $F_\delta = \tilde{F}_\delta \leq \tilde{F}$ (see (7.6)) and $\sigma < 0$, then we deduce that $(c_{\delta,\sigma}, \phi_{\delta,\sigma})$ is a subsolution of (9.25) with (c, w) is replaced by $(c_{\delta,\sigma}, \phi_{\delta,\sigma})$.

Step 3: establishing $c^+ \leq c^*$
Using the proof of Proposition 5.1, we deduce that

$$c_{\delta,\sigma} \leq c.$$

Passing to the limit $\sigma \to 0^-$ and then $\delta \to 0$ (as in the proof of Theorem 1.1), we deduce that

$$c^+ \leq c \quad \text{for all} \quad c > c^*. \tag{9.26}$$

This implies that

$$c^+ \leq c^*.$$

□

Now, we give an example of non-linearity where we have $c^+ > c^*$.

Lemma 9.3 (Example with $c^+ > c^*$)
Consider the function $F^0 : [0,1]^3 \to \mathbb{R}$ defined as

$$F^0(X_0, X_{-1}, X_1) := g(X_1) + g(X_{-1}) - 2g(X_0) + f(X_0),$$

with $r_0 = 0$, $r_{\pm 1} = \pm 1$ *and* $f,\ g : [0,1] \to \mathbb{R}$ *are* C^1 *over a neighborhood of* 0, *Lipschitz on* $[0,1]$ *and satisfying*

$$\begin{cases} f(0) = f(1) = 0 \\ f > 0 \quad on \quad (0,1) \\ f'(0) > 0 \end{cases} \quad and \quad \begin{cases} g'(0) = 0 \\ g(1) = 1 + g(0) \\ g' \geq 0. \end{cases}$$

Let c^+ *given by Theorem 1.1 (with* F *replaced by* F^0*), then*

$$c^+ > c^*,$$

where c^* *is defined in* (1.15)*.*

An example of such g is $g(x) = x - \frac{1}{2\pi}\sin(2\pi x)$.

Proof of Lemma 9.3

Since $g'(0) = 0$ and $f'(0) > 0$, then $P(\lambda) = f'(0) > 0$. Thus we get that $c^* = \inf_{\lambda>0} \frac{P(\lambda)}{\lambda} = 0$. By Proposition 1.3 *i*), we have that $c^+ \geq 0 = c^*$. We want to show that $c^+ \neq 0$.

Assume to the contrary that $c^+ = 0$ and let ϕ be a solution of (1.9) with F replaced by F^0. Using the equivalence between the viscosity solution and almost everywhere solutions (see Lemma 2.5), we deduce that ϕ is an almost everywhere solution of

$$0 = F((\phi(z + r_i))_{i=0,\dots,N}). \tag{9.27}$$

That is there exists a set $\mathcal{N}$ of measure zero such that for every $z \notin \mathcal{N}$, equation (9.27) holds true.

Let $\mathcal{N}_0 = \cup_{k\in\mathbb{Z}}(\mathcal{N} + k)$ and choose $z_0 \in \mathbb{R}\backslash\mathcal{N}_0$ (set $\mathcal{N}_0$ has also a zero measure), then equation (9.27) holds true for every $z_0 + k$ with $k \in \mathbb{Z}$. Hence

$$g(\phi(z_0+k+1)) + g(\phi(z_0+k-1)) - 2g(\phi(z_0+k)) = -f(\phi(z_0+k)) \leq 0 \quad \text{for every} \quad k \in \mathbb{Z}. \tag{9.28}$$

Let h be the piecewise affine function which is affine on each interval $[k, k+1]$ and satisfying $h(z_0 + k) = g(\phi(z_0 + k))$ with $k \in \mathbb{Z}$. Thus, it is easy to conclude using (9.28) that h is concave. Moreover, h is bounded because g is bounded on $[0,1]$ and $0 \leq \phi \leq 1$. Therefore, h is constant. This implies that

$$g(\phi(z_0)) = g(\phi(z_0 + k)) = const \quad \text{for all} \quad k \in \mathbb{Z}.$$

Moreover, since $g' \geq 0$, $\phi(-\infty) = 0$ and $\phi(+\infty) = 1$, we conclude that $g = const$ on $[0,1]$, which is a contradiction with $g(1) = 1 + g(0)$. Hence, we get $c^+ > 0 = c^*$. □

9.2 Critical velocity c^+ is non-negative

This subsection is devoted for the proof of Proposition 1.3. Independently, we also show that $c^- < 0 < c^+$ for the Frenkel-Kontorova model (1.17).

Proof of Proposition 1.3

Let (c, ϕ) be a solution of (1.9) given in Theorem 1.1 with c fixed. Our goal is to show that $c \geq 0$; and hence $c^+ \geq 0$. We perform the proof in several steps.

Step 0: preliminary

Define for $X = (X_0, ..., X_N) \in [0,1]^{N+1}$ and $\delta > 0$ small the function

$$F_\delta(X) = F(X) - f(X_0) + f_\delta(X_0), \tag{9.29}$$

where

$$f_\delta(v) = \begin{cases} f & \text{on} \quad [0, 1-\delta] \\ \max\left(f(1-\delta) - L_0(v - (1-\delta)), 0\right) & \text{on} \quad [1-\delta, 1], \end{cases}$$

with a constant $L_0 > 2\mathrm{Lip}(F) > 0$ large enough. Let $\delta \in (0, \frac{1}{2})$ and set

$$1_\delta := 1 - \delta + \frac{f(1-\delta)}{L_0} < 1,$$

(where 1_δ was denoted by m_δ in the proof of Theorem 1.1).

Part I: antisymmetric extension of F_δ and proof for ii)
Using Proposition 10.1, there exists an antisymmetric extension G_δ on $[-1,1]^{N+1}$ of F_δ such that

$$\begin{cases} (G_\delta)_{|_{[0,1]^{N+1}}} = F_\delta \\ G_\delta(-X) = -G_\delta(X) \quad \text{for all} \quad X \in [-1,1]^{N+1} \end{cases}$$

and satisfying (A_{Lip}) over $[-1,1]^{N+1}$ (since F_δ satisfies (A_{Lip}) over $[0,1]^{N+1}$). Moreover, still by Proposition 10.1, since F_δ is C^1 over a neighborhood of $\{0\}^{N+1}$ in $[0,1]^{N+1}$ (because of (P_{C^1}) and (9.29)) and $f_\delta'(0) = f'(0) > 0$, then there exists $\eta > 0$ such that for every X, $X + (a,...,a) \in [-1,1]^{N+1}$ close to $\{0\}^{N+1}$ with $a > 0$ small, we have

$$G_\delta(X + (a,...,a)) - G_\delta(X) \geq \eta a. \tag{9.30}$$

In addition, the function $g_\delta(v) := G_\delta(v,...,v)$ satisfies

$$\begin{cases} g_\delta(-1_\delta) = g_\delta(0) = g_\delta(1_\delta) = 0 \\ (g_\delta)_{|_{(-1_\delta,0)}} < 0 \quad \text{and} \quad (g_\delta)_{|_{(0,1_\delta)}} > 0, \end{cases} \tag{9.31}$$

(since we have $f_\delta(0) = 0 = f_\delta(1_\delta)$ and $f_\delta > 0$ on $(0, 1_\delta)$).

Step $I.1$: existence of traveling waves for G_δ
Clearly, since G_δ satisfies (9.31) and (9.30), then G_δ satisfies the assumption (B_{Lip}) with $[0,1]^{N+1}$ replaced by over $[-1_\delta, 1_\delta]^{N+1}$ and b replaced by 0. In addition, G_δ satisfies, by construction, the assumption (A_{Lip}) over $[-1_\delta, 1_\delta]^{N+1}$. Thus applying the result of Proposition 4.1 with $[0,1]^{N+1}$ replaced by $[-1_\delta, 1_\delta]^{N+1}$ and b replaced by 0, we deduce that there exists a real c_δ^0 and a function ϕ_δ^0 solution of

$$\begin{cases} c_\delta^0 (\phi_\delta^0)'(x) = G_\delta((\phi_\delta^0(x + r_i))_{i=0,...,N}) \quad \text{on} \quad \mathbb{R} \\ \phi_\delta^0 \ \text{ is non-decreasing over } \mathbb{R} \\ \phi_\delta^0(-\infty) = -1_\delta \quad \text{and} \quad \phi_\delta^0(+\infty) = 1_\delta. \end{cases} \tag{9.32}$$

Step $I.2$: $c_\delta^0 \geq 0$
We show in this step that c_δ^0 is non-negative under ii), i.e. assuming $r_i \geq 0$ for all $i \in \{0,...,N\}$. Then $\psi(x) = -\phi_\delta^0(-x)$ satisfies

$$\begin{aligned} -c_\delta^0 \psi'(y) &= -G_\delta((-\psi(y - r_i))_{i=0,...,N}) \\ &= G_\delta((\psi(y - r_i))_{i=0,...,N}) \\ &\leq G_\delta((\psi(y + r_i))_{i=0,...,N}), \end{aligned}$$

hence $(\overline{c} = -c_\delta^0, \psi)$ is a subsolution of (9.32). Using an argument similar to the computation of (7.9) for L_0 large enough (here $L_0 > 2\mathrm{Lip}(F)$), we can show that G_δ is decreasing close to $\{-1_\delta\}^{N+1}$ and $\{1_\delta\}^{N+1}$ inside $[-1_\delta, 1_\delta]^{N+1}$, that is G_δ satisfies (10.7) and (10.9) (for $s = -1_\delta$ and $s' = 1_\delta$).

Applying the comparison principle results (Proposition 10.6 and Proposition 10.7) and the ideas of the proof of Proposition 5.1, we deduce that

$$-c^0_\delta = \overline{c} \leq c^0_\delta,$$

that is

$$0 \leq c^0_\delta. \tag{9.33}$$

Step $I.3$: comparing c and c^0_δ
Recall that $(c^0_\delta, \phi^0_\delta)$ is a solution of (9.32). Moreover, since $G_\delta = F_\delta \leq F$ over $[0,1]^{N+1}$, then (c,ϕ) is a supersolution for (1.9), with F replaced by G_δ.

Since

$$\phi(-\infty) = 0 \quad \text{and} \quad \phi(+\infty) = 1$$

and $-1_\delta < 0 < 1_\delta < 1$, that is $\phi^0_\delta(\pm\infty) < \phi(\pm\infty)$ and $\phi^0_\delta(+\infty) > \phi(-\infty)$, then using the proof of Proposition 5.1 (which still true for sub and supersolutions), we deduce that

$$0 \leq c^0_\delta \leq c.$$

Part II: extension of F_δ by antisymmetry-reflection and proof for *iii)*
In this part, we assume that F (and then F_δ) satisfies the strict monotonicity condition (1.12). Using Remark 10.5, we can assume that the set I defined in (1.11) satisfies

$$I = \{1, ..., N\},$$

i.e. for all $i \in \{1,...,N\}$, there exists $\bar{i} \in \{1,...,N\}$ such that $r_i = -r_{\bar{i}}$. Using now Proposition 10.4, there exists an extension $\overline{G}_\delta$ on $[-1,1]^{N+1}$ of F_δ such that

$$\begin{cases} (\overline{G}_\delta)_{|_{[0,1]^{N+1}}} = F_\delta \\ \overline{G}_\delta(-X) = -\overline{G}_\delta(X) \quad \text{for all} \quad X \in [-1,1]^{N+1} \end{cases}$$

and satisfying (A_{Lip}) over $[-1,1]^{N+1}$. Since F is C^1 over a neighborhood of $\{0\}^{N+1}$ in $[0,1]^{N+1}$, then (using Proposition 10.4) there exists $\eta > 0$ such that for every X, $X + (a,...,a) \in [-1,1]^{N+1}$ close to $\{0\}^{N+1}$ with $a > 0$ small, we have

$$\overline{G}_\delta(X + (a,...,a)) - \overline{G}_\delta(X) \geq \eta a. \tag{9.34}$$

In addition, the function $\overline{g}_\delta(v) := \overline{G}_\delta(v,...,v)$ satisfies

$$\begin{cases} \overline{g}_\delta(-1_\delta) = \overline{g}_\delta(0) = \overline{g}_\delta(1_\delta) = 0 \\ (\overline{g}_\delta)_{|_{(-1_\delta,0)}} < 0 \quad \text{and} \quad (\overline{g}_\delta)_{|_{(0,1_\delta)}} > 0. \end{cases} \tag{9.35}$$

Step $II.1$: existence of traveling waves for $\overline{G}_\delta$
This step is a variant of Step $I.1$ with G_δ replaced $\overline{G}_\delta$. Thus we deduce that there exists a real $\overline{c}^0_\delta$ and a function $\overline{\phi}^0_\delta$ solution of

$$\begin{cases} \overline{c}^0_\delta(\overline{\phi}^0_\delta)'(x) = \overline{G}_\delta((\overline{\phi}^0_\delta(x + r_i))_{i=0,...,N}) \quad \text{on} \quad \mathbb{R} \\ \overline{\phi}^0_\delta \ \text{ is non-decreasing over } \mathbb{R} \\ \overline{\phi}^0_\delta(-\infty) = -1_\delta \quad \text{and} \quad \overline{\phi}^0_\delta(+\infty) = 1_\delta. \end{cases} \tag{9.36}$$

Step $II.2$: $\overline{c}^0_\delta = 0$
Let $\psi(x) = -\overline{\phi}^0_\delta(-x)$, then

$$\begin{aligned}\overline{G}_\delta((\psi(x+r_i))_{i=0,...,N}) &= \overline{G}_\delta((-\overline{\phi}^0_\delta(-x-r_i))_{i=0,...,N})\\ &= \overline{G}_\delta((-\overline{\overline{\phi}^0_\delta(-x+r_i)})_{i=0,...,N})\\ &= -\overline{G}_\delta((\overline{\phi}^0_\delta(-x+r_i))_{i=0,...,N})\\ &= -\overline{c}^0_\delta(\overline{\phi}^0_\delta)'(-x) = -\overline{c}^0_\delta\psi'(x).\end{aligned}$$

Thus $(-\overline{c}^0_\delta, \psi)$ is a solution of (9.36) with $\overline{c}^0_\delta$ replaced by $-\overline{c}^0_\delta$.

Similarly to Step $I.2$ (with $L_0 > 2\text{Lip}(F)$), we can show that $\overline{G}_\delta$ is decreasing close to $\{-1_\delta\}^{N+1}$ and $\{1_\delta\}^{N+1}$ inside $[-1_\delta, 1_\delta]^{N+1}$. By comparison principle, we get

$$\overline{c}^0_\delta \le \overline{c} \quad \text{and} \quad \overline{c} \le \overline{c}^0_\delta,$$

which implies that

$$\overline{c}^0_\delta = 0. \tag{9.37}$$

Step $II.3$: comparing c and $\overline{c}^0_\delta$
This step is analogous to Step $I.3$ with $(c^0_\delta, \phi^0_\delta)$ replaced by $(\overline{c}^0_\delta, \overline{\phi}^0_\delta)$ and G_δ replaced by $\overline{G}_\delta$. Thus proceeding similarly we show that

$$0 = \overline{c}^0_\delta \le c.$$

Part III: proof for $i)$
Under condition $i)$, we have

$$I = \{1, ..., N\} \quad \text{and} \quad \frac{\partial F}{\partial X_i}(0) = \frac{\partial F}{\partial X_{\bar{i}}}(0) \ \text{ for all } \ i \in I,$$

thus condition (1.12) is equivalent to $f'(0) > 0$. Therefore, we can apply $iii)$ which shows that $c^+ \ge 0$.

Complement: another proof for $i)$ and $ii)$
We show in this complement the result of Proposition 1.3 $i)$ and $ii)$ using a different approach.

The proof is done by contradiction. Assume to the contrary that

$$c^+ < 0.$$

Using Proposition 1.5, we deduce that

$$c^+ \ge c^*.$$

Since F satisfies (P_{C^1}), then

$$P(0) = f'(0) > 0 \quad \text{(see (1.15))}.$$

Moreover, we have that P is convex and that $P'(0) = \sum_{i=1}^{N} r_i \frac{\partial F}{\partial X_i}(0, ..., 0)$.

Getting a contradiction
Clearly, if F satisfies the reflection symmetry condition $i)$, then we get that $P'(0) = 0$. Similarly, if $r_i \ge 0$ for all $i \in \{1, ..., N\}$, then $P'(0) \ge 0$. But P is convex, hence we deduce that

$$P(\lambda) > 0 \quad \text{for every} \quad \lambda > 0.$$

Therefore, we get that $c^* \ge 0$, which is a contradiction with $0 > c^+ \ge c^*$. This implies that

$$c^+ > 0$$

under the conditions $i)$ and $ii)$. □

Lemma 9.4 (Sign of c^+ and c^- for (FK) model (1.17))
Consider the Frenkel-Kontorova model with $\beta > 0$

$$c\phi'(z) = \phi(z+1) + \phi(z-1) - 2\phi(z) - \beta \sin\left(2\pi\left(\phi(z) + \frac{1}{4}\right)\right) + \sigma,$$

with $\sigma \in [-\beta, \beta] = [\sigma^-, \sigma^+]$. Let $c^\pm$ be the critical velocity associated to $\sigma^\pm$. Then

$$c^- < 0 < c^+.$$

Proof of Lemma 9.4
Let $\sigma = \sigma^+ = \beta$ and let us show that $c^+ > 0$. Let ϕ be non-decreasing with $\phi(-\infty) = 0$ and $\phi(+\infty) = 1$. Integrating over the real line the equation

$$c^+\phi'(z) = \phi(z+1) + \phi(z-1) - 2\phi(z) + f(\phi(z)),$$

where $f(\phi(z)) = -\beta \sin\left(2\pi\left(\phi(z) + \frac{1}{4}\right)\right) + \beta \geq 0$, we get that

$$c^+ = \int_{\mathbb{R}} \left(-\beta \sin\left(2\pi\left(\phi(z) + \frac{1}{4}\right)\right) + \beta\right) dz \geq 0.$$

Since $f > 0$ on $(0, 1)$, if $c^+ = 0$, then

$$\phi(z) = 0 \text{ or } 1 \quad \text{almost everywhere.}$$

This implies that

$$\Delta_1\phi(z) := \phi(z+1) + \phi(z-1) - 2\phi(z) = 0 \quad \text{almost everywhere.}$$

Consider now the set

$$\mathcal{A} = \{z \in \mathbb{R},\ \Delta_1\phi(z) \neq 0\},$$

which has measure zero. Thus the set $\mathcal{A} + \mathbb{Z}$ has also measure zero. Hence for a fixed $a \in \mathbb{R}\backslash(\mathcal{A} + \mathbb{Z}) \neq \emptyset$, we have

$$\Delta_1\phi(a+k) = 0 \quad \text{for every } k \in \mathbb{Z}.$$

This implies that there exists $\lambda, b \in \mathbb{R}$ (that may depend on a) such that

$$\phi(a+k) = \lambda k + b.$$

But ϕ is bounded, then $\lambda = 0$ and hence $\phi(a+k) = b$, which is a contradiction since $\phi(+\infty) \neq \phi(-\infty)$. Therefore $c^+ > 0$.

Similarly, for $\sigma = \sigma^- = -\beta$, we show that $c^- < 0$, since $f - 2\beta < 0$ on $\left(-\frac{1}{2}, \frac{1}{2}\right)$. □

9.3 Instability of critical velocity

In this section, we show that the critical velocity c^+ given in Theorem 1.1 is unstable in the sense of Proposition 1.2, which we prove in this section.

Before proving Proposition 1.2, we give an example of a non-linearity F for which the associated critical velocity is negative. This example will be the proof of Proposition 1.4.

Proof of Proposition 1.4
The aim is to construct a function F satisfying (A_{Lip}) and (P_{C^1}) such that the associated critical

velocity satisfies $c^+ < 0$. To this end, we will construct a function $f \in \text{Lip}([0,1])$, which is linear in a neighborhood of zero with $f'(0) > 0$, such that there exists a couple (c, ϕ) with $c < 0$ solution of

$$\begin{cases} c\phi'(x) = \phi(x-1) - \phi(x) + f(\phi(x)) \quad \text{on} \quad \mathbb{R} \\ \phi' \geq 0 \\ \phi(-\infty) = 0 \quad \text{and} \quad \phi(+\infty) = 1. \end{cases} \tag{9.38}$$

Let $c = -\mu$ with $0 < \mu < 1$ and

$$\phi(x) = \begin{cases} \frac{1}{2} e^{\gamma x} & \text{on} \quad (-\infty, 0] \\ 1 - \frac{1}{2} e^{-\gamma x} & \text{on} \quad [0, +\infty) \end{cases}$$

with $\gamma > 0$. We claim that $\phi \in C^1(\mathbb{R})$ and $(-\mu, \phi)$ solves

$$\begin{cases} 0 < \phi(x) - \phi(x-1) - \mu\phi'(x) \quad \text{on} \quad \mathbb{R} \\ \phi' > 0 \\ \phi(-\infty) = 0 \quad \text{and} \quad \phi(+\infty) = 1, \end{cases} \tag{9.39}$$

which is possible to check for $0 < \gamma << 1$.

Therefore, it is sufficient to define the function f as

$$f(\phi(x)) := \phi(x) - \phi(x-1) - \mu\phi'(x) > 0 \quad \text{for all } x \in \mathbb{R}. \tag{9.40}$$

Notice that, when $x \to +\infty$, $\phi(+\infty) = 1$ and $\phi'(x) \to 0$, thus $f(1) = 0$. Similarly, we have $f(0) = 0$. Moreover, since $\phi \in C^{1,1}(\mathbb{R})$, we have that $f \in \text{Lip}((0,1))$. In fact, by a direct tedious calculation, one can deduce that

$$f(x) = \begin{cases} (1 - e^{-\gamma} - \mu\gamma)x & \text{for} \quad x \in \left[0, \frac{1}{2}\right] \\ 1 + (1 + \mu\gamma)(x-1) + \dfrac{e^{-\gamma}}{4(x-1)} & \text{for} \quad x \in \left[\frac{1}{2}, 1 - \frac{1}{2}e^{-\gamma}\right] \\ (1 - e^{\gamma} + \mu\gamma)(x-1) & \text{for} \quad x \in \left[1 - \frac{1}{2}e^{-\gamma}, 1\right], \end{cases}$$

and this implies that $f \in \text{Lip}([0,1])$ and $1 > f'(0) > 0$. We can even check that f is concave and C^1 except at the point $x = \frac{1}{2}$, where it is neither concave nor C^1.

Remark that to get more regular non-linearities, one can consider

$$f_\varepsilon(x) := \Big(\big(\phi(\cdot) - \phi(\cdot - 1) - \mu\phi'(\cdot) \big) \star \rho_\varepsilon \Big)(x), \tag{9.41}$$

where ρ_ε satisfies $\rho_\varepsilon \geq 0$, $\rho_\varepsilon(x) = \frac{1}{\varepsilon}\rho(\frac{x}{\varepsilon})$ (ρ is a mollifier) and $supp\, \rho_\varepsilon \subset B_\varepsilon(0)$. However, in this case, $\rho_\varepsilon \star \phi$ is a solution of (9.38), with f replaced by f_ε, and then $f_\varepsilon \in C^\infty([0,1])$ with $f'_\varepsilon(0) > 0$. $\square$

Now, we give the proof of the instability result, namely Proposition 1.2.

Proof of Proposition 1.2

We have seen, in Proposition 1.4, that there exists a function F satisfying (A_{Lip}) and (P_{C^1}) such that the associated critical velocity $c_F^+ := c^+$ satisfies

$$c_F^+ < 0. \tag{9.42}$$

Our goal is to build a sequence of functions F_δ with a critical velocity $c^+_{F_\delta}$ such that

$$F_\delta \to F \quad \text{in} \quad L^\infty([0,1]^{N+1})$$

as $\delta \to 0$, and prove that

$$\liminf_{\delta\to 0} c^+_{F_\delta} > c^+_F. \tag{9.43}$$

Step 1: construction of F_δ
Define for $X = (X_0, ..., X_N) \in [0,1]$ and $\delta > 0$ small the function

$$F_\delta(X) = F(X) - f(X_0) - f_\delta(X_0), \tag{9.44}$$

where

$$f_\delta(v) = \begin{cases} \max\big(f(\delta) + L_0(v-\delta), 0\big) & \text{on} \quad [0,\delta] \\ f & \text{on} \quad [\delta, 1], \end{cases} \tag{9.45}$$

with a constant $L_0 > 0$ satisfying $L_0 > 2\mathrm{Lip}(F) =: 2L^\infty_F$.

By construction of f_δ, we clearly have

$$\|F_\delta - F\|_{L^\infty} \;=\; \|f - f_\delta\|_{L^\infty} \to 0 \quad \text{as} \quad \delta \to 0.$$

Step 2: existence of $c^+_{F_\delta}$
Set

$$0_\delta = \delta - \frac{f(\delta)}{L_0} > 0,$$

(where 0_δ was denoted by b_δ in the proof of Theorem 1.1).

Since F_δ satisfies (A_{Lip}) and (P_{Lip}) with $[0,1]^{N+1}$ replaced by $[0_\delta,1]^{N+1}$, then applying the result of Theorem 1.1, we deduce that there exists a minimal velocity $c^+_{F_\delta}$ and a profile ϕ solution of

$$\begin{cases} c^+_{F_\delta}\phi'(z) = F_\delta(\phi(z+r_0), \phi(z+r_1), ..., \phi(z+r_N)) \quad \text{on} \quad \mathbb{R} \\ \phi \text{ is non-decreasing over } \mathbb{R} \\ \phi(-\infty) = 0_\delta \quad \text{and} \quad \phi(+\infty) = 1. \end{cases} \tag{9.46}$$

Step 3: establishing (9.43)
Our aim is to show that $c^+_{F_\delta} \geq 0$. Since F_δ is non-decreasing w.r.t. X_i for all $i \neq 0$, then for $X = (X_0, X') \in [0_\delta, 1]^{N+1}$, we have

$$F_\delta(X_0, X') \geq F_\delta(X_0, 0_\delta, ..., 0_\delta) := A(X_0).$$

Moreover, for X_0 $X_0 + h \in [0_\delta, \delta]$ with $h > 0$, we have

$$\begin{aligned} A(X_0+h) - A(X_0) &= F(X_0+h, 0_\delta+h, ..., 0_\delta+h) - F(X_0, 0_\delta, ..., 0_\delta) - f(X_0+h) \\ &\qquad + f(X_0) + f_\delta(X_0+h) - f_\delta(X_0) \\ &\geq -2hL^\infty_F + hL_0 \\ &= h(L_0 - 2L^\infty_F) > 0, \end{aligned}$$

where we have used that F is L^∞_F-Lipschitz (in the second line) and that $L_0 > 2L^\infty_F$ in the last inequality. This implies that A is increasing over $[0_\delta, \delta]$, but $A(0_\delta) = F_\delta(0_\delta, 0_\delta, ..., 0_\delta) = 0$. Hence, we get $A \geq 0$ over $[0_\delta, \delta]$.

Therefore, we deduce that

$$F_\delta \geq 0 \quad \text{over} \quad [0_\delta, \delta] \times [0_\delta, 1]^N.$$

Now since $\phi(-\infty) = 0_\delta$, then for $z << -1$ very negative, we get that $\phi(z + r_0) = \phi(z) \in [0_\delta, \delta]$. Hence, for all $\phi(z) \in [0_\delta, \delta]$, we obtain from (9.46) that

$$c^+_{F_\delta} \phi'(z) = F_\delta(\phi(z + r_0), \phi(z + r_1), ..., \phi(z + r_N)) \geq 0,$$

but $\phi' \geq 0$, thus we deduce that

$$c^+_{F_\delta} \geq 0.$$

This implies that (because of (9.42))

$$\liminf_{\delta \to 0} c^+_{F_\delta} \geq 0 > c^+_F.$$

Step 4: conclusion
Let

$$\widehat{\phi}(x) = \frac{\phi(x) - 0_\delta}{1 - 0_\delta}, \quad c^+_{\widehat{F}_\delta} = (1 - 0_\delta) c^+_{F_\delta} \tag{9.47}$$

and

$$\widehat{F}_\delta((X_i)_{i=0,...,N}) = F_\delta(((1 - 0_\delta)X_i + 0_\delta)_{i=0,...,N}).$$

Then we have

$$\begin{cases} c^+_{\widehat{F}_\delta} \widehat{\phi}'(z) = \widehat{F}_\delta((\widehat{\phi}(z + r_i))_{i=0,...,N}) \quad \text{on} \quad \mathbb{R} \\ \widehat{\phi} \text{ is non-decreasing over } \mathbb{R} \\ \widehat{\phi}(-\infty) = 0 \quad \text{and} \quad \widehat{\phi}(+\infty) = 1 \end{cases} \tag{9.48}$$

and $c^+_{\widehat{F}_\delta}$ is the critical velocity associated to $\widehat{F}_\delta$ which is defined on $[0, 1]^{N+1}$. Moreover, we still have $|\widehat{F}_\delta - F| \to 0$ as $\delta \to 0$ and $\widehat{F}_\delta$ satisfies (A_{Lip}) and (P_{Lip}) on $[0, 1]^{N+1}$. In addition, since $0_\delta \to 0$ as $\delta \to 0$, then from (9.47) we still have

$$\liminf_{\delta \to 0} c^+_{\widehat{F}_\delta} = \liminf_{\delta \to 0} c^+_{F_\delta} \geq 0 > c^+_F.$$

Therefore, up to rename $\widehat{F}_\delta$ as F_δ, this ends the proof of Proposition 1.2. □

10 Appendix: Useful results used for the proof of $c^+ \geq 0$

This subsection is dedicated for the useful tools that we use to prove that the critical velocity is non-negative, i.e $c^+ \geq 0$.

Proposition 10.1 (Extension by antisymmetry)
Let F be a function defined over $Q = [0, 1]^{N+1}$ satisfying (A_{Lip}) and such that $F(0, ..., 0) = 0$. Then there exists an antisymmetric extension G defined over $[-1, 1]^{N+1}$ such that

$$\begin{cases} G_{|_Q} = F \\ G(-X) = -G(X) \end{cases}$$

and G satisfies (A_{Lip}) over $[-1, 1]^{N+1}$.

Moreover, if F is C^1 over a neigborhood of $\{0\}^{N+1}$ in $[0, 1]^{N+1}$ and $f'(0) > 0$ ($f(v) := F(v, ..., v)$), then there exists $\eta > 0$ such that for every $a > 0$ small and $X = (X_0, ..., X_N) \in [-1, 1]^{N+1}$ such that X, $X + (a, ..., a)$ are close enough to $\{0\}^{N+1}$, we have

$$G(X + (a, ..., a)) - G(X) \geq \eta a. \tag{10.1}$$

Remark 10.2 (Reflection)
Note that if F is invariant by reflection symmetry, then it is possible to show that G also; precisely, we mean that if $F(\overline{X}) = F(X)$ for $\overline{X}_i = X_{\overline{i}}$ with $r_{\overline{i}} = -r_i$, then

$$G(\overline{X}) = G(X).$$

We recall before proving Proposition 10.1 the following properties of the orthogonal projection which can be easily shown:

Lemma 10.3 (Some properties of orthogonal projection)
Let $X = (X_i)_{i=0,...,N} \in [-1,1]^{N+1}$ and call $Proj_{|_Q}(X)$ the orthogonal projection of X on $Q = [0,1]^{N+1}$. Then

$$Proj_{|_Q}(X) = (Proj_{|_{[0,1]}}(X_i))_{i=0,...,N}.$$

Moreover, we have
i) Order preservation
Let $Y = (Y_i)_{i=0,...,N} \in [-1,1]^{N+1}$ and assume that $X \geq Y$ in sense that $X_i \geq Y_i$ for all $i \in \{0,...,N\}$, then

$$Proj_{|_Q}(X) \geq Proj_{|_Q}(Y).$$

ii) "Antisymmetry"
Let $Q' = [-1,0]^{N+1} = -Q$, then

$$Proj_{|_{Q'}}(-X) = -Proj_{|_Q}(X).$$

Proof of Proposition 10.1
Let $X = (X_i)_{i=0,...,N} \in [-1,1]^{N+1}$, then define the extension function G by:

$$G(X) = F(Proj_{|_Q}(X)) - F(-Proj_{|_{Q'}}(X)), \tag{10.2}$$

where we recall that $Q' = [-1,0]^{N+1}$. For $X \in Q$, we clearly have $G(X) = F(X)$.

Step 1: $G(-X) = -G(X)$
We have

$$\begin{aligned} G(-X) &= F(Proj_{|_Q}(-X)) - F(-Proj_{|_{Q'}}(-X)) \\ &= F(-Proj_{|_{Q'}}(X)) - F(Proj_{|_Q}(X)) \\ &= -G(X), \end{aligned}$$

where we have used in the second line the antisymmetry in Lemma 10.3.

Step 2: G satisfies $(A_{\mathbf{Lip}})$
Since F is globally Lipschitz and the orthogonal projection is 1-Lipschitz, then G is globally Lipschitz on $[-1,1]^{N+1}$.
We now prove that G is non-decreasing w.r.t. X_i for all $i \neq 0$. Let $X = (X_i)_{i=0,...,N}$, $Y = (Y_i)_{i=0,...,N} \in [-1,1]^{N+1}$ such that

$$\begin{cases} X_i \geq Y_i & \text{for all } i \in \{1,...,N\} \\ X_0 = Y_0, \end{cases}$$

and let us show that $G(X) \geq G(Y)$. In fact, since the orthogonal projection preserve the ordering (see Lemma 10.3) and since F is non-decreasing w.r.t. X_i for all $i \in \{1,...,N\}$, we conclude that G is non-decreasing w.r.t. X_i for all $i \in \{1,...,N\}$ over $[-1,1]^{N+1}$.

Step 3: checking (10.1)
We first give some notations for the projection function. Consider $X = (X_0,...,X_N) \in [-1,1]^{N+1}$, then from Lemma 10.3, we have

$$Proj_{|_Q}(X) = (Proj_{|_{[0,1]}}(X_i))_{i=0,...,N} = \left(\begin{Bmatrix} X_i & \text{if} & X_i \geq 0 \\ 0 & \text{if} & X_i \leq 0 \end{Bmatrix} \right)_{i=0,...N} =: X^+.$$

Similarly, we have (with $Q' = -Q$)

$$Proj_{|_{Q'}}(X) = \left(\begin{Bmatrix} 0 & \text{if} & X_i \geq 0 \\ X_i & \text{if} & X_i \leq 0 \end{Bmatrix} \right)_{i=0,...N} =: X^-$$

We also define

$$Q_\Sigma = \left\{ X = (X_0,...,X_N) \in [-1,1]^{N+1}, \quad \sigma_i X_i \in [0,1] \quad \text{for} \quad i = 0,...,N \right\},$$

where $\Sigma = (\sigma_0,...,\sigma_N)$ and $\sigma_i = \pm 1$.

Now, we go back to the proof of (10.1) which is splitted in two cases. Let X, $X + (a,...,a)$ close to $\{0\}^{N+1}$ with $a > 0$ small:

Case 1: $X,\ X + (a,...,a) \in Q_\Sigma$
From the definition of G (see (10.2)) and the notations introduced at the beginning of this step, we have

$$G(X + (a,...,a)) - G(X) = F((X + aE)^+) - F(X^+) - \Big(F(-(X + aE)^-) - F(-X^-) \Big),$$

where $E = (1,...,1)$. Thus, we get

$$G(X + (a,...,a)) - G(X) = a\Theta.\nabla F(X^+) + o(|a\Theta|) + a\Gamma.\nabla F(-X^-) + o(|a\Gamma|),$$

where $a\Theta = (X + aE)^+ - X^+$ with $\Theta = (\theta_i)_{i=0,...,N}$, where

$$\theta_i = \begin{cases} 1 & \text{if} \quad \sigma_i = 1 \\ 0 & \text{if} \quad \sigma_i = -1 \end{cases}$$

and $a\Gamma = (X + aE)^- - X^-$ with $\Gamma = (\gamma_i)_{i=0,...,N}$, where

$$\gamma_i = \begin{cases} 0 & \text{if} \quad \sigma_i = 1 \\ 1 & \text{if} \quad \sigma_i = -1. \end{cases}$$

Hence, we obtain

$$\begin{aligned} G(X + (a,...,a)) - G(X) \quad &= \quad a(\Theta + \Gamma).\nabla F(0) + a\Theta.(\nabla F(X^+) - \nabla F(0)) + o(a) \\ &+ \quad a\Gamma.(\nabla F(-X^-) - \nabla F(0)), \end{aligned}$$

but $a(\Theta + \Gamma) = (a,...,a)$, therefore,

$$\begin{aligned} G(X + (a,...,a)) - G(X) \quad &= \quad a\Big\{ f'(0) + \Theta.(\nabla F(X^+) - \nabla F(0)) + o(1) \\ &+ \quad \Gamma.(\nabla F(-X^-) - \nabla F(0)) \Big\}. \end{aligned}$$

Now, since F is C^1 over a neighborhood of X (X close to $\{0\}^{N+1}$), then we get

$$G(X+(a,...,a)) - G(X) = a\left\{f'(0) + o(X^+) + o(X^-) + o(1)\right\} \geq a\frac{f'(0)}{2} > 0$$

for X close enough to $\{0\}^{N+1}$.

Case 2: $X \in Q_\Sigma$ **and** $X + aE \in Q_{\widehat{\Sigma}}$
There exists an integer $p \geq 1$ such that

$$G(X+aE) - G(X) = \sum_{k=0}^{p} \Big(G(X+t_kE) - G(X+t_{k-1}E)\Big),$$

where $0 = t_0 < t_1 < ... < t_p = a$ such that for $k = 1,...,p$, we have $X + [t_{k-1}, t_k]E \in Q_{\Sigma_k}$, with $\Sigma = \Sigma_0$ and $\widehat{\Sigma} = \Sigma_p$. Therefore, using Case 1 for each segment, we deduce that

$$G(X+aE) - G(X) \geq \eta a,$$

with $\eta = \frac{f'(0)}{2} > 0$. □

We now introduce an extension by antisymmetry-reflection of F :

Proposition 10.4 (Extension by antisymmetry-reflection)
Let F be a function defined on $Q = [0,1]^{N+1}$ satisfying (A_{Lip}) and such that $F(0,...,0) = 0$. Let $X = (X_i)_{i=0,...,N} \in [0,1]^{N+1}$ and assume that

$$\text{(10.3)} \qquad \textit{for all } \; i \in \{1,...,N\} \; \textit{ there exists } \; \bar{i} \in \{1,...,N\} \; \textit{ such that } \; r_{\bar{i}} = -r_i.$$

Then there exists a function $\overline{G}$ defined on $[-1,1]^{N+1}$ which satisfies (A_{Lip}) on $[-1,1]^{N+1}$ such that

$$\begin{cases} \overline{G}_{|Q} = F \\ \overline{G}(-\overline{X}) = -\overline{G}(X) \quad \textit{(antisymmetric-reflection)}, \end{cases}$$

where we recall that $\overline{X}_i = X_{\bar{i}}$ with $r_{\bar{i}} = -r_i$.

Moreover, if F is C^1 over a neighborhood of $\{0\}^{N+1}$ and

$$\text{(10.4)} \qquad \frac{\partial F}{\partial X_0}(0) + \sum_{i=1}^{N} \min\left(\frac{\partial F}{\partial X_i}(0), \frac{\partial F}{\partial X_{\bar{i}}}(0)\right) > 0,$$

then there exists $\eta > 0$ such that for every $a > 0$ small and $X = (X_0,...,X_N) \in [-1,1]^{N+1}$ such that X, $X + (a,...,a)$ are close enough to $\{0\}^{N+1}$, we have

$$\text{(10.5)} \qquad \overline{G}(X+(a,...,a)) - \overline{G}(X) \geq \eta a.$$

Remark 10.5 (On the reflection condition (10.3)**)**
Notice that we can always assume that the reflection condition (10.3) *is satisfied up to modify the function F. Indeed, if F does not satisfy the reflection condition* (10.3)*, i.e. we have*

$$\{i_1,...,i_M\} = \{i \in \{1,...,N\}, \; \textit{such that } -r_i \notin \{r_j\}_{j=1,...,N}\}$$

with $M \geq 1$, then let us define

$$r_{N+j} = -r_{i_j} \quad \textit{for} \quad j = 1,...,M.$$

Therefore, for each $i \in \{1, ..., N+M\}$ *there exists* $\bar{i} \in \{1, ..., N+M\}$ *such that* $r_{\bar{i}} = -r_i$*. Now, for* $\tilde{X} = (X, X')$ *with* $X' = (X_{N+1}, ..., X_{N+M})$*, set*

$$\tilde{F}(\tilde{X}) = F(X).$$

Thus $\tilde{F}$ *satisfies* (10.3) *with* N *replaced by* $\tilde{N} = N + M$ *and if moreover* ϕ *solves*

$$c\phi'(z) = F((\phi(z + r_i))_{i=0,...,N}),$$

then it solves $c\phi'(z) = \tilde{F}((\phi(z + r_i))_{i=0,...,\tilde{N}})$*.*

In addition, if F *is* C^1 *in a neighborhood of* $\{0\}^{N+1}$*, then* $\tilde{F}$ *is* C^1 *in a neighborhood of* $\{0\}^{\tilde{N}+1}$*, and*

$$\frac{\partial F}{\partial X_0}(0) + \sum_{i \in I} \min\left(\frac{\partial F}{\partial X_i}(0), \frac{\partial F}{\partial X_{\bar{i}}}(0)\right) = \frac{\partial \tilde{F}}{\partial X_0}(0) + \sum_{i=1}^{\tilde{N}} \min\left(\frac{\partial \tilde{F}}{\partial X_i}(0), \frac{\partial \tilde{F}}{\partial X_{\bar{i}}}(0)\right),$$

with $I = \left\{i \in \{1, ..., N\} \quad \text{such that there exists } \ \bar{i} \in \{1, ..., N\} \quad \text{with} \quad r_i = -r_{\bar{i}}\right\}$*.*

Proof of Proposition 10.4

The proof is very similar to the proof of Proposition 10.1, so we give only a few details. Let $X \in [-1, 1]^{N+1}$, then define the extension function $\overline{G}$ by

$$\overline{G}(X) = F(Proj_{|_Q}(X)) - F(-Proj_{|_{Q'}}(\overline{X})), \tag{10.6}$$

where we recall that $\overline{X}_i = X_{\bar{i}}$ with $r_{\bar{i}} = -r_i$.

Step 1: $\overline{G}(-\overline{X}) = -\overline{G}(X)$

We have

$$\begin{aligned}
\overline{G}(-\overline{X}) &= F(Proj_{|_Q}(-\overline{X})) - F(-Proj_{|_{Q'}}(\overline{-\overline{X}})) \\
&= F(-Proj_{|_{Q'}}(\overline{X})) - F(-Proj_{|_{Q'}}(-X)) \quad \text{(using Lemma 10.3 } ii\text{) and } \overline{-X} = -\overline{X}) \\
&= F(-Proj_{|_{Q'}}(\overline{X})) - F(Proj_{|_Q}(X)) \quad \text{(using again Lemma 10.3 } ii\text{))} \\
&= -\overline{G}(X).
\end{aligned}$$

Step 2: $\overline{G}$ **satisfies** $(A_{\mathbf{Lip}})$ **on** $[-1, 1]^{N+1}$

This step is an analogous of Step 2 in the proof of Proposition 10.1.

Step 3: checking (10.5)

We have

$$\overline{G}(X) = F(X^+) - F(-(\overline{X})^-)$$

Let $\Sigma = (\sigma_0, \sigma_1, ..., \sigma_N)$ and define $\overline{\Sigma} = (\overline{\sigma}_0, \overline{\sigma}_1, ..., \overline{\sigma}_N)$ such that $\overline{\sigma}_i = \sigma_{\bar{i}}$ for all $i = 0, ..., N$; and then recall

$$Q_\Sigma = \left\{X = (X_0, ..., X_N) \in [-1, 1]^{N+1} \quad \text{such that} \quad \sigma_i X_i \in [0, 1] \quad \text{for} \quad i = 0, ..., N\right\}.$$

We have

$$X \in Q_\Sigma \Longleftrightarrow \overline{X} \in Q_{\overline{\Sigma}}.$$

Let $X, \ X + aE$ close enough to $\{0\}^{N+1}$ with $a > 0$ small and $E = (1, ..., 1)$.

Case 1: $X,\ X + aE \in Q_\Sigma$
Since F is C^1 over a neighborhood of $\{0\}^{N+1}$, then (as in the proof of Proposition 10.1, Step 3) we have

$$\begin{aligned}\overline{G}(X+aE)-\overline{G}(X) &= F((X+aE)^+)-F(X^+)-\Big(F(-(\overline{X+aE})^-)-F(-(\overline{X})^-)\Big)\\ &= F((X+aE)^+)-F(X^+)-\Big(F(-(\overline{X}+aE)^-)-F(-(\overline{X})^-)\Big)\\ &= a\Theta.\nabla F(X^+)+o(|a\Theta|)+a\overline{\Gamma}.\nabla F(-(\overline{X})^-)+o(|a\overline{\Gamma}|),\end{aligned}$$

where $a\Theta = (X+aE)^+ - X^+$ with $\Theta = (\theta_i)_{i=0,...,N}$, where

$$\theta_i = \begin{cases} 1 & \text{if} \quad \sigma_i = 1 \\ 0 & \text{if} \quad \sigma_i = -1 \end{cases}$$

and $a\overline{\Gamma} = (\overline{X}+aE)^- - (\overline{X})^-$ with $\overline{\Gamma} = (\overline{\gamma}_i)_{i=0,...,N}$, where

$$\overline{\gamma}_i = \begin{cases} 0 & \text{if} \quad \overline{\sigma}_i := \sigma_{\overline{i}} = 1 \\ 1 & \text{if} \quad \overline{\sigma}_i := \sigma_{\overline{i}} = -1. \end{cases}$$

Hence, using the fact that F is C^1, we get

$$\begin{aligned}\overline{G}(X+aE)-\overline{G}(X) &= a\Big\{(\Theta+\overline{\Gamma}).\nabla F(0)+\Theta.(\nabla F(X^+)-\nabla F(0))+o(1)\\ &\quad + \overline{\Gamma}.(\nabla F(-(\overline{X})^-)-\nabla F(0))\Big\}\\ &\geq \frac{a}{2}(\Theta+\overline{\Gamma}).\nabla F(0) > 0,\end{aligned}$$

if $(\Theta+\overline{\Gamma}).\nabla F(0) > 0$. This is true because

$$\begin{aligned}(\Theta+\overline{\Gamma}).\nabla F(0) &= \sum_{i=0}^{N}\theta_i\frac{\partial F}{\partial X_i}(0)+\sum_{j=0}^{N}\overline{\gamma}_j\frac{\partial F}{\partial X_j}(0)\\ &= \sum_{i=0}^{N}\left(\theta_i\frac{\partial F}{\partial X_i}(0)+\overline{\gamma}_{\overline{i}}\frac{\partial F}{\partial X_{\overline{i}}}(0)\right)\\ &\geq \sum_{i=0}^{N}\left((\theta_i+\overline{\gamma}_{\overline{i}})\min\left(\frac{\partial F}{\partial X_i}(0),\frac{\partial F}{\partial X_{\overline{i}}}(0)\right)\right)\\ &= \frac{\partial F}{\partial X_0}(0)+\sum_{i=1}^{N}\min\left(\frac{\partial F}{\partial X_i}(0),\frac{\partial F}{\partial X_{\overline{i}}}(0)\right) > 0 \quad \text{(using (10.4))},\end{aligned}$$

where we have used in the fourth line the fact that $\theta_i + \overline{\gamma}_{\overline{i}} = 1$ for all $i = 0, ..., N$, which follows from the definition of θ_i and $\overline{\gamma}_{\overline{i}}$ and the fact that $\sigma_i = \sigma_{\overline{\overline{i}}}$.

Case 2: $X \in Q_\Sigma$ **and** $X + aE \in Q_{\widehat{\Sigma}}$
This case is exactly the same as Case 2 of Step 3 in the proof of Proposition 10.1. However, in this case, we can choose

$$\eta = \frac{a}{2}(\Theta+\overline{\Gamma}).\nabla F(0) > 0.$$

□

Here, we recall two comparison principle results on half lines that we will also use to prove that $c^+ \geq 0$.

Proposition 10.6 (Comparison principle on $[-r^*, +\infty)$)
Let $F : [s, s']^{N+1} \to \mathbb{R}$ satisfying (A_{Lip}) over $[s, s']^{N+1}$ and assume that:

$$(10.7) \qquad \left| \begin{array}{l} \text{there exists } \eta_0 > 0 \text{ such that if} \\ X = (X_0, ..., X_N),\ X + (\alpha, ..., \alpha) \in [s' - \eta_0, s']^{N+1} \\ \text{then } F(X + (\alpha, ..., \alpha)) < F(X) \text{ if } \alpha > 0. \end{array} \right.$$

Let u, $v : [-r^*, +\infty) \to [s, s']$ be respectively a sub and a supersolution of

$$(10.8) \qquad cu'(x) = F((u(x + r_i))_{i=0,...,N}) \quad on \quad (0, +\infty)$$

in sense of Definition 2.1. Moreover, assume that

$$v \geq s' - \eta_0 \quad on \quad [-r^*, +\infty),$$

and that

$$u \leq v \quad on \quad [-r^*, 0].$$

Then

$$u \leq v \ on \ [-r^*, +\infty).$$

Similarly, we have the following proposition on the half line $(-\infty, -r^*]$:

Proposition 10.7 (Comparison principle on $(-\infty, -r^*]$)
Let $F : [s, s']^{N+1} \to \mathbb{R}$ satisfying (A_{Lip}) over $[s, s']^{N+1}$ and assume that:

$$(10.9) \qquad \left| \begin{array}{l} \text{there exists } \eta_1 > 0 \text{ such that if} \\ X = (X_0, ..., X_N),\ X + (\alpha, ..., \alpha) \in [s, s + \eta_1]^{N+1} \\ \text{then } F(X + (\alpha, ..., \alpha)) < F(X) \text{ if } \alpha > 0. \end{array} \right.$$

Let u, $v : (-\infty, r^*] \to [s, s']$ be respectively a sub and a supersolution of

$$(10.10) \qquad cu'(x) = F((u(x + r_i))_{i=0,...,N}) \quad on \quad (-\infty, 0)$$

in sense of Definition 2.1. Moreover, assume that

$$u \leq s + \eta_1 \quad on \quad (-\infty, r^*],$$

and that

$$u \leq v \quad on \quad [0, r^*].$$

Then

$$u \leq v \ on \ (-\infty, r^*].$$

For the proof of Proposition 10.6 and Proposition 10.7, we refer the reader for [1, Theorem 4.1 and Corollary 4.2] which is done for F defined on $[0, 1]^{N+1}$ instead of $[s, s']^{N+1}$.

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