Existence of traveling waves for Lipschitz discrete dynamics.
Monostable case as a limit of bistable cases

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Abstract: We study discrete monostable dynamics with general Lipschitz non-linearities. This includes also degenerate non-linearities. In the positive monostable case, we show the existence of a branch of traveling waves solutions for velocities $c \geq c^+$, with non existence of solutions for $c < c^+$. We also give certain sufficient conditions to insure that $c^+ \geq 0$ and we give an example where $c^+ < 0$. This model of discrete dynamics can be seen as a generalized Frenkel-Kontorova model for which we can also add a driving force parameter σ. We show that σ can vary in an interval $[\sigma^-, \sigma^+]$. For $\sigma \in (\sigma^-, \sigma^+)$ this corresponds to a bistable case, while for $\sigma = \sigma^+$ this is a positive monostable case, and for $\sigma = \sigma^-$ this is a negative monostable case. We study the velocity function $c = c(\sigma)$ as σ varies in $[\sigma^-, \sigma^+]$. In particular for $\sigma = \sigma^+$ (resp. $\sigma = \sigma^-$), we find vertical branches of traveling waves solutions with $c \geq c^+$ (resp. $c \leq c^-$).

Our method of proof is new and relies on viscosity solutions. Moreover, the monostable case with $c = c^+$ is seen advantageously as a limit situation of the bistable case. For $c >> 1$, the traveling waves are constructed as perturbations of solutions of an associated ODE. Finally to fill the gap between $c = c^+$ and large c, we use certain hull functions that are associated to correctors of a homogenization problem.

Keywords: Traveling waves, degenerate monostable non-linearity, KPP non-linearity, bistable non-linearity, Frenkel-Kontorova model, viscosity solutions, Perron's method.

1   Introduction

1.1   General motivation

Our initial motivation was to study the classical fully overdamped Frenkel-Kontorova model, which is a system of ordinary differential equations

$$(1.1) \qquad \frac{dX_i}{dt} = X_{i+1} - 2X_i + X_{i-1} + f(X_i) + \sigma,$$

where $X_i(t) \in \mathbb{R}$ denotes the position of a particle $i \in \mathbb{Z}$ at time t, $\dfrac{dX_i}{dt}$ is the velocity of this particle, f is the force created by a 1-periodic potential and σ represents the constant driving force. Such external force could be for example $f(x) = 1 - \cos(2\pi x) \geq 0$. This kind of system can be, for instance, used as a model of the motion of a dislocation defect in a crystal (see the book of Braun and Kivshar [8]). This motion is described by particular solutions of the form

$$(1.2) \qquad X_i(t) = \phi(i + ct)$$

with

$$(1.3) \qquad \phi' \geq 0 \quad \text{and} \quad \phi \text{ is bounded.}$$

Such a solution, ϕ, is called a traveling wave solution and c denotes its velocity of propagation. From (1.1) and (1.2), it is equivalent to look for solutions ϕ of

$$(1.4) \qquad c\phi'(z) = \phi(z+1) - 2\phi(z) + \phi(z-1) + f(\phi(z)) + \sigma$$

with $z = i + ct$. For such a model, and under certain conditions on f, we show the existence of traveling waves for each value of σ in an interval $[\sigma^-, \sigma^+]$ (see Theorem 1.6). We also get the whole picture (see Figure 4 for qualitative properties of this picture) of the velocity function $c = c(\sigma)$ with respect to the driving force σ, with vertical branches for $\sigma = \sigma^-$ or $\sigma = \sigma^+$.

When $f > 0 = f(0) = f(1)$ on $(0,1)$ and $\sigma = 0$, we can moreover normalize the limits of the profile ϕ as

$$(1.5) \qquad \phi(-\infty) = 0 \quad \text{and} \quad \phi(+\infty) = 1.$$

This case is called a positive monostable case and is associated here to $\sigma^+ = 0$. Moreover, we can show the existence of a critical velocity c^+ such that the following holds. There exists a branch of traveling waves solutions for all velocity $c \geq c^+$ and there are no solutions for $c < c^+$.

The goal of this paper is to present similar results in a framework more general than (1.4). To this end, given a real function F (whose properties will be specified in Subsections 1.2 and 1.3), we consider the following generalized equation with $\sigma \in \mathbb{R}$

$$(1.6) \qquad c\phi'(z) = F(\phi(z+r_0), \phi(z+r_1), ..., \phi(z+r_N)) + \sigma,$$

where $N \geq 0$ and $r_i \in \mathbb{R}$ for $i = 0, ..., N$ such that

$$(1.7) \qquad r_0 = 0 \quad \text{and} \quad r_i \neq r_j \text{ if } i \neq j,$$

which does not restrict the generality. In (1.6), we are looking for both the profile ϕ and the velocity c.

Equation (1.1) can be seen as a discretization of the following reaction diffusion equation

$$(1.8) \qquad u_t = \Delta u + f(u).$$

In 1937, Fisher [15] and Kolmogorov, Petrovsky and Piskunov [28] studied the traveling waves for equation (1.8) which they proposed as a model describing the spreading of a gene throughout a population. Later, many works have been devoted for such equation that appears in biological models for developments of genes or populations dynamics and in combustion theory (see for instance, Aronson, Weinberger [3, 4] and Hadeler, Rothe [22]). For more developments and applications in biology of reaction diffusion equations, the reader may refer to [31] and to the references cited therein. There is also a considerable work on the existence, uniqueness and stability of traveling waves and their speed of propagation for the homogeneous KPP-Fisher non-linearity (see for example [23, 24, 25, 26, 35]). Such results have been shown also for the inhomogeneous, heterogeneous and random KPP-Fisher non-linearity (see [6, 7, 30]).

Traveling waves were studied also for discrete bistable reaction diffusion equations (see for instance [9, 13]. See also [1] and the references therein. In the monostable case, we distinguish

[27] (for nonlocal non-linearities with integer shifts) and [14, 29, 32, 33] (for problems with linear nonlocal part and with integer shifts also). See also [20] for particular monostable non-linearities with irrational shifts. We also refer to [19, 10, 21, 11, 12, 24, 34] for different positive monostable non-linearities. In the monostable case, we have to underline the work of Hudson and Zinner [27] (see also [34]), where they proved the existence of a branch of solutions $c \geq c^*$ for general Lipschitz non-linearities (with possibly an infinite number of neighbors $N = +\infty$, and possibly p types of different particles, while $p = 1$ in our study) but with integer shifts $r_i \in \mathbb{Z}$. However, they do not state the nonexistence of solutions for $c < c^*$. Their method of proof relies on an approximation of the equation on a bounded domain (applying Brouwer's fixed point theorem) and an homotopy argument starting from a known solution. The full result is then obtained as the size of the domain goes to infinity. Here we underline that our results hold for the fully nonlinear case with real shifts $r_i \in \mathbb{R}$.

Several approaches were used to construct traveling waves for discrete monostable dynamics. We already described the homotopy method of Hudson and Zinner [27]. In a second approach, Chen and Guo [11] proved the existence of a solution starting from an approximated problem. They constructed a fixed point solution of an integral reformulation (approximated on a bounded domain) using the monotone iteration method (with sub and supersolutions). This approach was also used to get the existence of a solution in [18, 12, 20, 21]. A third approach based on recursive method for monotone discrete in time dynamical systems was used by Wienberger et al. [29, 32]. See also [33], where this method is used to solve problems with a linear nonlocal part. In a fourth approach [19], Guo and Hamel used global space-time sub and supersolutions to prove the existence of a solution for periodic monostable equations.

There is also a wide literature about the uniqueness and the asymptotics at infinity of a solution for a monostable non-linearities, see for instance [10, 26] (for a degenerate case), [11, 12] and the references therein. Let us also mention that certain delayed reaction diffusion equations with some KPP-Fisher non-linearities do not admit traveling waves (see for example [18, 34]).

Finally, we mention that our method opens new possibilities to be adapted to more general problems. For example, we can think to adapt our approach to a case with possibly p types of different particles similar to [17]. The case with an infinite number of neighbors $N = +\infty$ could be also studied. We can also think to study fully nonlinear parabolic equations.

1.2 Main results in the monostable case

In this subsection, we consider equation (1.6) with $\sigma = 0$. We study the existence of traveling waves of equation (1.6) (with $\sigma = 0$) for positive degenerate monostable non-linearities and with conditions at infinity given by (1.5).

In order to present our results in this case, we have to introduce some assumptions on $F : [0,1]^{N+1} \to \mathbb{R}$.

Assumption (A_{Lip}):

  i) **Regularity:** $F \in \text{Lip}([0,1]^{N+1})$.

  ii) **Monotonicity:** $F(X_0, X_1, ..., X_N)$ is non-decreasing w.r.t. each X_i for $i \neq 0$.

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Assumption (P_{Lip}):

Positive degenerate monostability:
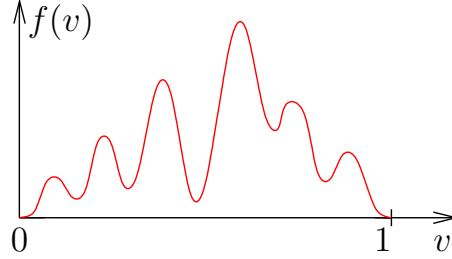Let $f(v) = F(v, ..., v)$ such that $f(0) = f(1) = 0$, $f > 0$ in $(0, 1)$.

Figure 1: Positive degenerate monostable non-linearity f

Our main result is:

Theorem 1.1 (Monostable case: existence of a branch of traveling waves)
Assume (A_{Lip}) and (P_{Lip}). Then there exists a real c^+ such that for all $c \geq c^+$ there exists a traveling wave $\phi : \mathbb{R} \to \mathbb{R}$ solution (in the viscosity sense (see Definition 2.1)) of

(1.9)
$$\begin{cases} c\phi'(z) = F(\phi(z + r_0), \phi(z + r_1), ..., \phi(z + r_N)) & on \quad \mathbb{R} \\ \phi \text{ is non-decreasing over } \mathbb{R} \\ \phi(-\infty) = 0 \quad and \quad \phi(+\infty) = 1. \end{cases}$$

On the contrary for $c < c^+$, there is no solution of (1.9).

Up to our knowledge, Theorem 1.1 is the first result for discrete dynamics with real shifts $r_i \in \mathbb{R}$ in the fully nonlinear case. Even when $r_i \in \mathbb{Z}$, the only result that we know for fully nonlinear dynamics is the one of Hudson and Zinner [27]. However, the nonexistence of solutions for $c < c^+$ is not addressed in [27].

See Figure 2 for an explicit Lipschitz non-linearity example for which our result (Theorem 1.1) is still true, even if $f'(0)$ is not defined. We also prove that the critical velocity c^+ is unstable in the following sense:

Proposition 1.2 (Instability of the minimal velocity c_F^+)
There exists a function F satisfying (A_{Lip}) and (P_{Lip}) with a minimal velocity c_F^+ such that there exists a sequence of functions F_δ (satisfying (A_{Lip}) and (P_{Lip})) with associated critical velocity $c_{F_\delta}^+$ satisfying
$$F_\delta \to F \quad in \quad L^\infty([0, 1]^{N+1})$$
when $\delta \to 0$, but
$$\liminf_{\delta \to 0} c_{F_\delta}^+ > c_F^+.$$

We believe that the critical velocity c^+ contains information about $f'(0)$; similar to classical result in [28] which asserts that the critical velocity of reaction diffusion equation (1.8) is $c^+ = 2\sqrt{f'(0)}$. This shows that when F is only Lipschitz, it becomes very difficult to capture c_F^+ and to show Theorem 1.1 (see its proof, Section 7).
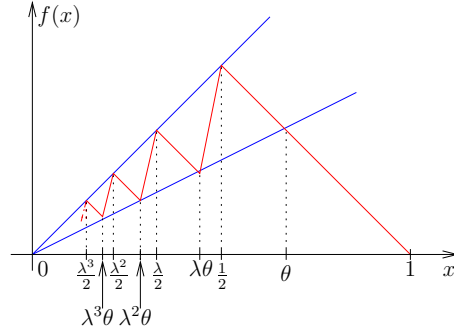
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Figure 2: Lipschitz positive degenerate monostable non-linearity; the rest of the figure over $[0, \frac{\lambda^3}{2}]$ is completed by dilation of center 0 and ratio λ.

Examples of functions F satisfying assumptions (A_{Lip}) and (P_{Lip}) are given for $N = 2$, $r_0 = 0$, $r_1 = -1$, $r_2 = 1$ by

$$(1.10) \qquad F(X_0, X_1, X_2) = X_2 + X_1 - 2X_0 + g(X_0),$$

with for instance non-linearity $g(x) = x(1 - x)$ or $g(x) = x^2(1 - x)^2$.

In the next result, we prove that the critical velocity c^+ (given in Theorem 1.1) is non-negative for particular F, i.e. we need to assume some smoothness and strict monotonicity on F near $\{0\}^{N+1}$; and this is given in assumption (P_{C^1}) (which is stronger than (P_{Lip})):

Assumption (P_{C^1}):

Positive degenerate monostability:
Let $f(v) = F(v, ..., v)$ such that $f(0) = 0 = f(1)$ and $f > 0$ in $(0, 1)$.

Smoothness near $\{0\}^{N+1}$:
F is C^1 over a neighborhood of $\{0\}^{N+1}$ in $[0, 1]^{N+1}$ and $f'(0) > 0$.

Proposition 1.3 (Non-negative c^+ for particular F)
Consider a function F satisfying (A_{Lip}) and (P_{C^1}). Let c^+ given by Theorem 1.1. Then we have $c^+ \geq 0$, if one of the three following conditions i), ii) or iii) holds true:

*i) **Reflection symmetry of F***
Let $X = (X_i)_{i \in \{0,...,N\}} \in [0, 1]^{N+1}$. Assume that for all $i \in \{0, ..., N\}$ there exists $\bar{i} \in \{0, ..., N\}$ such that $r_{\bar{i}} = -r_i$; and

$$F(\overline{X}) = F(X) \quad for \ all \ \ X \in [0, 1]^{N+1},$$

where

$$\overline{X}_i = X_{\bar{i}} \quad for \quad i \in \{0, ..., N\}.$$

*ii) **All the r_i's "shifts" are non-negative***
Assume that $r_i \geq 0$ for all $i \in \{0, ..., N\}$.

*iii) **Strict monotonicity***
Let

$$(1.11) \qquad I = \left\{ i \in \{1, ..., N\} \ such \ that \ there \ exists \ \bar{i} \in \{1, ..., N\} \ \ with \ \ r_{\bar{i}} = -r_i \right\}$$

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and assume that

$$(1.12) \qquad \frac{\partial F}{\partial X_0}(0) + \sum_{i \in I} \min\left(\frac{\partial F}{\partial X_i}(0), \frac{\partial F}{\partial X_{\bar{i}}}(0) \right) > 0.$$

Notice that because of the monotonicity of F in X_j for $j \neq 0$, condition (1.12) is satisfied if

$$\frac{\partial F}{\partial X_0}(0) > 0.$$

Moreover, if

$$(1.13) \qquad I = \{1, ..., N\} \quad \text{and} \quad \frac{\partial F}{\partial X_i}(0) = \frac{\partial F}{\partial X_{\bar{i}}}(0) \quad \text{for all} \quad i \in I,$$

then condition (1.12) is equivalent to $f'(0) > 0$. In particular, under condition i) property (1.13) holds true. This shows that condition iii) is more general than condition i).

Remark that if we replace (P_{C^1}) by (P_{Lip}) assuming i), ii) or iii), we do not know if $c^+ \geq 0$.

Proposition 1.4 (Counter example with $c^+ < 0$)
There exists a function F satisfying (A_{Lip}) and (P_{C^1}) such that the associated critical velocity (given in Theorem 1.1) is negative, i.e. $c^+ < 0$.

Remark 1.5 (Formally: asymptotics of the profile and the minimal velocity)
Given a velocity c, if the associated profile ϕ satisfies (in some strong sense)

$$\phi(x) \sim A|x|^k e^{\lambda x} \quad as \quad x \to -\infty \quad with \quad A > 0, \ \lambda > 0 \ and \ k \geq 0,$$

then formally λ has to solve the equation

$$c\lambda = \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0) e^{\lambda r_i} =: P(\lambda).$$

Note that $P(0) = f'(0)$. If $f'(0) > 0$, we expect that the velocity c satisfies

$$c \geq c^* := \inf_{\lambda > 0} \frac{P(\lambda)}{\lambda}.$$

So it is natural to conjecture that $c^+ \geq c^*$ if F is C^1 in a neighborhood of $\{0\}^{N+1}$. The answer of this conjecture will be addressed in a following version of this work. See [10, 11, 12] for results of this kind for particular non-linearities F, and [27] for fully non-linearities with integer shifts $r_i \in \mathbb{Z}$.

1.3 Main result on the velocity function

In this subsection, we consider equation (1.6) with a constant parameter $\sigma \in \mathbb{R}$ and $F : \mathbb{R}^{N+1} \to \mathbb{R}$. We are interested in the velocities c associated to σ (that we call roughly speaking the "velocity function").

For σ belonging to some interval $[\sigma^-, \sigma^+]$, we prove the existence of a traveling wave and we study the variation of its velocity c with respect to σ.

Let $E = (1, ..., 1)$, $\Theta = (\theta, ..., \theta) \in \mathbb{R}^{N+1}$ and assume that the function F satisfies:

Assumption $(\tilde{A}_{C^1})$:

Regularity: F is globally Lipschitz continuous over $\mathbb{R}^{N+1}$ and C^1 over a neighborhood in $\mathbb{R}^{N+1}$ of the two intervals $]0, \Theta[$ and $]\Theta, E[$.

Monotonicity: $F(X_0, ..., X_N)$ is non-decreasing w.r.t. each X_i for $i \neq 0$.

Periodicity: $F(X_0 + 1, ..., X_N + 1) = F(X_0, ..., X_N)$ for every $X = (X_0, ..., X_N) \in \mathbb{R}^{N+1}$.

Notice that, since F is periodic in E direction, then F is C^1 over a neighborhood of $\mathbb{R}E \backslash (\mathbb{Z}E \cup \mathbb{Z}\Theta)$.

Assumption ($\tilde{B}_{C^1}$):
Define $f(v) = F(v, ..., v)$ such that:

Bistability: $f(0) = f(1)$ and there exists $\theta \in (0, 1)$ such that

$$\begin{cases} f' > 0 \quad \text{on} \quad (0, \theta) \\ f' < 0 \quad \text{on} \quad (\theta, 1). \end{cases}$$
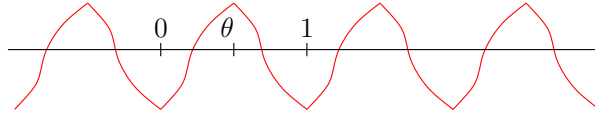

Figure 3: Bistable non-linearity f

See Figure 3 for an example of f satisfying ($\tilde{B}_{C^1}$). Notice that assumptions ($\tilde{A}_{C^1}$) and ($\tilde{B}_{C^1}$) holds true in particular for the Frenkel-Kontorova model for $\beta > 0$:

$$(1.14) \qquad \frac{d}{dt} X_i = X_{i+1} + X_{i-1} - 2X_i - \beta \sin\left(2\pi\left(X_i + \frac{1}{4}\right)\right) + \sigma.$$

Theorem 1.6 (General case: traveling waves and the velocity function)
Under assumptions ($\tilde{A}_{C^1}$) and ($\tilde{B}_{C^1}$), define $\sigma^\pm$ as

$$(1.15) \qquad \begin{cases} \sigma^+ = -\min f \\ \sigma^- = -\max f. \end{cases}$$

Associate for each $\sigma \in [\sigma^-, \sigma^+]$ the solutions $m_\sigma \in [\theta - 1, 0]$ and $b_\sigma \in [0, \theta]$ of $f(s) + \sigma = 0$. Then consider the following equation

$$(1.16) \qquad \begin{cases} c\phi'(z) = F(\phi(z + r_0), \phi(z + r_1), ..., \phi(z + r_N)) + \sigma \quad \text{on} \quad \mathbb{R} \\ \phi \text{ is non-decreasing over } \mathbb{R} \\ \phi(-\infty) = m_\sigma \quad \text{and} \quad \phi(+\infty) = m_\sigma + 1, \end{cases}$$

1- Bistable case: traveling waves for $\sigma \in (\sigma^-, \sigma^+)$
We have

(i) **(Existence of a traveling wave)**
For any $\sigma \in (\sigma^-, \sigma^+)$, there exists a unique real $c := c(\sigma)$, such that there exists a function $\phi_\sigma : \mathbb{R} \to \mathbb{R}$ solution of (1.16) in the viscosity sense.

(ii) **(Continuity and monotonicity of the velocity function)**
The map
$$\sigma \mapsto c(\sigma)$$

is continuous on (σ^-, σ^+) and there exists a constant $K > 0$ such that the function $c(\sigma)$ is non-decreasing and satisfies

$$\frac{dc}{d\sigma} \geq K|c| \quad on \quad (\sigma^-, \sigma^+)$$

in the viscosity sense. In addition, there exists real numbers $c^- \leq c^+$ such that

$$\lim_{\sigma \to \sigma^-} c(\sigma) = c^- \quad and \quad \lim_{\sigma \to \sigma^+} c(\sigma) = c^+.$$

Moreover, either $c^- = 0 = c^+$ or $c^- < c^+$.

2- Monostable cases: vertical branches for $\sigma = \sigma^\pm$
We have

(i) **(Existence of traveling waves for $c \geq c^+$ when $\sigma = \sigma^+$)**
Let $\sigma = \sigma^+$, then for every $c \geq c^+$ there exists a traveling wave ϕ solution of

(1.17)
$$\begin{cases} c\phi'(z) = F(\phi(z+r_0), \phi(z+r_1), ..., \phi(z+r_N)) + \sigma^+ \quad on \quad \mathbb{R} \\ \phi \text{ is non-decreasing over } \mathbb{R} \\ \phi(-\infty) = 0 = m_{\sigma^+} \quad and \quad \phi(+\infty) = 1. \end{cases}$$

Moreover, for any $c < c^+$, there is no solution ϕ of (1.17).

(ii) **(Existence of traveling waves for $c \leq c^-$ when $\sigma = \sigma^-$)**
Let $\sigma = \sigma^-$, then for every $c \leq c^-$, there exists a traveling wave ϕ solution of

(1.18)
$$\begin{cases} c\phi'(z) = F(\phi(z+r_0), \phi(z+r_1), ..., \phi(z+r_N)) + \sigma^- \quad on \quad \mathbb{R} \\ \phi \text{ is non-decreasing over } \mathbb{R} \\ \phi(-\infty) = \theta - 1 = m_{\sigma^-} \quad and \quad \phi(+\infty) = \theta. \end{cases}$$

Moreover, for any $c > c^-$, there is no solution ϕ of (1.18).

Note that for the Frenkel Kontorova model (1.14), we have $\sigma^\pm = \pm 1$ and $c^+ > 0 > c^-$ (cf. Lemma 8.1), and Figure 4 illustrates the graph of the velocity $c(\sigma)$ which has a plateau at the level $c = 0$ in particular if $|\sigma| < \beta - 1$ (see Proposition 2.6).

In view of Theorem 1.6, we can ask the following:
Open question 1. For a general F, what is the precise behavior of the function $c(\sigma)$ close to the boundary of the plateau $c = 0$ and close to σ^+ and σ^-?
Open question 2. Can we construct a function F such that $c^+ = 0 = c^-$?
For indications in the direction of open question 1, see for instance [9] (discussion on page 4 after Theorem 1.2).

Remark 1.7 (sign of c^+ and c^-)
If we can apply Proposition 1.3 for $F + \sigma^+$, we deduce that $c^+ \geq 0$. Similarly, by symmetry (see Lemma 3.3), it is possible to introduce similar assumptions to conclude that $c^- \leq 0$.
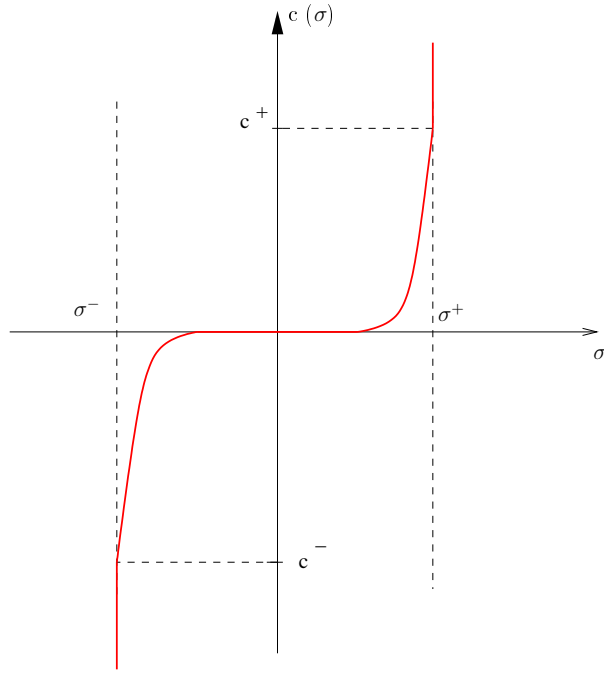
Figure 4: Typical graph of the velocity function $c(\sigma)$ with vertical branches at $\sigma = \sigma^{\pm}$.

Remark 1.8 (Existence of m_σ and b_σ for $\sigma \in [\sigma^-, \sigma^+]$)
Remark that under assumption $(\tilde{B}_{C^1})$ and from the definition of $\sigma^{\pm}$ (see (1.15)), the associated $m_\sigma \in [\theta - 1, 0]$ and $b_\sigma \in [0, \theta]$ exist uniquely for every $\sigma \in [\sigma^-, \sigma^+]$. This implies that the two maps $\sigma \to m_\sigma$, b_σ are well defined.

Remark 1.9 (No solution of (1.16) when $\sigma \notin [\sigma^-, \sigma^+]$)
From the definition of $\sigma^{\pm}$ (see (1.15), we see that the function $f + \sigma = 0$ has no solution if $\sigma \notin [\sigma^-, \sigma^+]$. Moreover, if ϕ is a bounded solution of

$$(1.19) \qquad c\phi'(z) = F(\phi(z + r_0), \phi(z + r_1), ..., \phi(z + r_N)) + \sigma \quad on \quad \mathbb{R},$$

then $\phi(\pm\infty)$ should solve the equation $f + \sigma = 0$. Thus, we conclude that (1.19) does not admit a bounded solution if $\sigma \notin [\sigma^-, \sigma^+]$.

Notice that Theorem 1.1 is a generalization of Theorem 1.6-2 (i) for $\sigma = \sigma^+$. Also, notice that Theorem 1.6-1 (i) is already proved in [1] (see [1, Proposition 2.3]).

As a notation, we set for a general function h:

$$F((h(z + r_i))_{i=0,...,N}) = F(h(z + r_0), h(z + r_1), ..., h(z + r_N))$$

and we define

$$(1.20) \qquad r^* = \max_{i=0,...,N} |r_i|.$$

In the rest of the paper, we will use the notation introduced in Theorem 1.6.

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1.4   Organization of the paper

Even if the main results of Subsections 1.2 and 1.3 are very different, the proofs are deeply related (because we use the results in the bistable case to deduce some results in the monostable case).

We recall, in Section 2, the notion of viscosity solutions and some useful results for monotone functions. Section 3 is devoted to the proof of existence of traveling waves solutions of (1.9) for $c >> 1$, which is applicable in particular for (1.17) and also for (1.18) when $c << -1$ (up to apply a suitable transformation). We use, in Section 4, results in [1] to prove the existence of a traveling wave and the uniqueness of the velocity for solutions of (1.16) as a function of the driving force $\sigma \in (\sigma^-, \sigma^+)$. In Section 5, we prove the continuity and monotonicity of the velocity function over (σ^-, σ^+) and we show that the velocity function attains finite limits $c^\pm$ at $\sigma^\pm$. We also prove, in this section, the existence of solutions of (1.17) (resp. (1.18)) for $c = c^+$ (resp. $c = c^-$). In Section 6, we fill the gap by proving the existence of solutions of (1.17) (resp. (1.18)) for every $c \geq c^+$ (resp. $c \leq c^-$). Moreover, we show that for any $c < c^+$ (resp. $c > c^-$) there is no solution of (1.17) (resp. (1.18)). We prove Theorem 1.6 at the end of Section 6.

Theorem 1.1 is proved in Section 7, which we split in two subsections. In Subsection 7.1, we recall an extension result to $\mathbb{R}^{N+1}$ of a non-linearity defined on $[0,1]^{N+1}$ and then we prove Theorem 1.1 in the special case where the non-linearity is smooth. In Subsection 7.2, we give the proof of Theorem 1.1 in full generality for Lipschitz non-linearities, where the construction of the critical velocity c^+ follows the lines of the proof of the regular case, but requires a lot of work to adapt it to this very delicate situation. Section 8 is dedicated to properties of the critical velocity c^+. In Subsection 8.1, we prove that c^+ is non-negative under certain assumptions, namely Proposition 1.3. While in Subsection 8.2, we construct a counter-example for which $c^+ < 0$, i.e. Proposition 1.4 and we prove the instability result of Proposition 1.2.

Finally the Appendix (Section 9) is divided into two parts. We prove in Subsection 9.1 results about the passage to the limit in our equation and about the identification of the limits at infinity of the limit profile. In Subsection 9.2, we prove and state two kinds of results (which are used to prove that $c^+ \geq 0$): first, extension by antisymmetry and antisymmetry-reflection (Propositions 9.4 and 9.7) and second, a comparison principle (Propositions 9.9 and 9.10).

1.5   Notations of our assumptions

In our paper, we introduce assumptions (A_{Lip}), (P_{Lip}) and (P_{C^1}) in Section 1.2, assumptions $(\tilde{A}_{C^1})$ and $(\tilde{B}_{C^1})$ in Section 1.3, assumptions $(\tilde{B}'_{C^1})$ and (B_{Lip}) in Section 4, assumptions $(\tilde{A}_{\text{Lip}})$, (A_{C^1}) and (P'_{C^1}) in Section 7.1 and assumption $(\tilde{B}_{m,b})$ in Section 9.1.

Generically, assumptions of type A holds for F, assumptions of type P are positivity assumptions on $f(v) = F(v, ..., v)$, and assumptions of type B are bistable assumptions for f.

Assumptions with *tilde* (˜) means that the functions F and f are considered on $\mathbb{R}^{N+1}$ and $\mathbb{R}$ respectively, and are assumed to be $(1, ..., 1)$-periodic and 1-periodic respectively. On the contrary, assumptions without *tilde* means assumptions for F and f on a finite box $[0,1]^{N+1}$

and $[0, 1]$ respectively.

The subscript "Lip" means that we only require Lipschitz functions, while the subscript "C^1" means that we require C^1 functions (at least on some part of their domain of definition).

Finally, assumptions with *prime* ($'$) are (locally in the paper) variant of the assumptions without *prime*.

2 Preliminary results

We recall, in a first subsection, the definition of viscosity solutions, a stability result and Perron's method for constructing a solution. We state, in a second subsection, Helly's Lemma and the equivalence result between viscosity and almost everywhere solutions for non-decreasing functions. In a third subsection, we give an example with a discontinuous viscosity solution.

2.1 Viscosity solution

In the whole paper, we will use the notion of viscosity solutions that we introduce in this subsection. To this end, we recall that the upper and lower semi-continuous envelopes, u^* and u_*, of a locally bounded function u are defined as

$$u^*(y) = \limsup_{x \to y} u(x) \quad \text{and} \quad u_*(y) = \liminf_{x \to y} u(x).$$

Definition 2.1 (Viscosity solution)
Let $I = I' = \mathbb{R}$ (or $I = (-r^*, +\infty)$ and $I' = (0, +\infty)$) and $u : I \to \mathbb{R}$ be a locally bounded function, $c \in \mathbb{R}$ and F defined on $\mathbb{R}^{N+1}$.

   - *The function u is a subsolution (resp. a supersolution) on I' of*

   (2.1) $$cu'(x) = F((u(x + r_i))_{i=0,\dots,N}) + \sigma,$$

   if u is upper semi-continuous (resp. lower semi-continuous) and if for all test function $\psi \in C^1(I)$ such that $u - \psi$ attains a local maximum (resp. a local minimum) at $x^* \in I'$, we have

   $$c\psi'(x^*) \le F((u(x^* + r_i))_{i=0,\dots,N}) + \sigma \quad \left(resp. \ c\psi'(x^*) \ge F((u(x^* + r_i))_{i=0,\dots,N}) + \sigma \right).$$

   - *A function u is a viscosity solution of (2.1) on I' if u^* is a subsolution and u_* is a supersolution on I'.*

We also recall the stability result for viscosity solutions (see [5, Theorem 4.1] and [16, Proposition 2.4] for a similar proof).

Proposition 2.2 (Stability of viscosity solutions)
Consider a function F defined on $\mathbb{R}^{N+1}$ and satisfying $(\tilde{A}_{Lip})$ (introduced in Subsection 7.1). Assume that $(u_\varepsilon)_\varepsilon$ is a sequence of subsolutions (resp. supersolutions) of (2.1).

(*i*) *Let*

$$\overline{u}(x) = \limsup{}^{*} u_{\varepsilon}(x) := \limsup_{(\varepsilon,y)\to(0,x)} u_{\varepsilon}(y) \quad and \quad \underline{u}(x) = \liminf{}_{*} u_{\varepsilon}(x) := \liminf_{(\varepsilon,y)\to(0,x)} u_{\varepsilon}(y),$$

be the relaxed upper and lower semi-limits. If $\overline{u}$ (resp. $\underline{u}$) is finite, then $\overline{u}$ is a subsolution (resp. $\underline{u}$ is a supersolution) of (2.1).

(*ii*) *Let $\mathcal{T}$ be a nonempty collection of subsolutions of (2.1) and set $U(x) = \sup_{u \in \mathcal{T}} u(x)$. If U^{*} is finite then U^{*} is a subsolution of (2.1). A similar result holds for supersolutions.*

Next, we state Perron's method that we will use to construct a solution in Section 3.

Proposition 2.3 (Perron's method ([16, Proposition 2.8]))
Let $I = (-r^{*}, +\infty)$ and $I' = (0, +\infty)$ and F be a function satisfying $(\tilde{A}_{Lip})$ (introduced in Subsection 7.1). Let u and v defined on I satisfying

$$u \leq v \quad on \quad I,$$

such that u and v are respectively a sub and a supersolution of (2.1) on I'. Let $\mathcal{L}$ be the set of all functions $\tilde{v} : I \to \mathbb{R}$, such that $u \leq \tilde{v}$ over I with $\tilde{v}$ supersolution of (2.1) on I'. For every $z \in I$, let

$$w(z) = \inf\{\tilde{v}(z) \quad such \ that \quad \tilde{v} \in \mathcal{L}\}.$$

Then w is a solution of (2.1) over I' satisfying $u \leq w \leq v$ over I.

2.2 Some results for monotone functions

In this subsection, we state Helly's Lemma for the convergence of a sequence of non-decreasing functions. We also recall the result about the equivalence between the viscosity and almost everywhere solutions. These results will be used later in Sections 4, 5, 6 and 7.

Lemma 2.4 (Helly's Lemma, (see [2], Section 3.3, page 70))
Let $(g_{n})_{n \in \mathbb{N}}$ be a sequence of non-decreasing functions on $[a, b]$ verifying $|g_{n}| \leq M$ uniformly in n. Then there exists a subsequence $(g_{n_{j}})_{j \in \mathbb{N}}$ such that

$$g_{n_{j}} \to g \quad a.e. \ on \quad [a, b],$$

with g non-decreasing and $|g| \leq M$.

Now, we state the lemma for non-decreasing functions about the equivalence between a viscosity and an almost everywhere solution.

Lemma 2.5 (Equivalence between viscosity and a.e. solutions)
Let F satisfy assumption $(\tilde{A}_{Lip})$ (introduced in Subsection 7.1). Let $\phi : \mathbb{R} \to \mathbb{R}$ be a non-decreasing function. Then ϕ is a viscosity solution of

$$(2.2) \qquad\qquad 0 = F((\phi(x + r_{i}))_{i=0,...,N}) + \sigma \quad on \quad \mathbb{R},$$

if and only if ϕ is an almost everywhere solution of the same equation.

For the proof of Lemma 2.5, we refer the reader to [1, Lemma 2.11].

2.3 Example of discontinuous viscosity solution

We give in this section an example of a discontinuous viscosity solution.

Proposition 2.6 (Discontinuous viscosity solution)
Consider $\beta > 0$, $\sigma \in \mathbb{R}$ and let (c, ϕ) be a solution of

(2.3)
$$
\begin{cases}
c\phi'(z) = \phi(z+1) - 2\phi(z) + \phi(z-1) + \beta\sin(2\pi\phi(z)) + \sigma & \text{on } \mathbb{R} \\
\phi \text{ is non-decreasing} \\
\phi(+\infty) - \phi(-\infty) = 1.
\end{cases}
$$

Then $\sigma^{\pm} = \pm\beta$. Moreover, if $|\sigma| < \beta - 1$, then $\phi \notin C^0$ and $c = 0$.

For the convenience of the reader we give the proof of this result (which is basically contained in Theorem 1.2 in Carpio et al. [9]).

Proof of Proposition 2.6
Clearly, we have $\sigma^{\pm} = \pm\beta$ (see Remark 1.9). Let $|\sigma| < \beta - 1$ and let us show that $\phi \notin C^0(\mathbb{R})$. Assume to the contrary that $\phi \in C^0(\mathbb{R})$.
Notice that because ϕ is non-decreasing and $\phi(+\infty) - \phi(-\infty) = 1$, we deduce that

$$
\phi(z+1) - 2\phi(z) + \phi(z-1) \in [-1, 1].
$$

Define now
$$
\psi(z) = \phi(z+1) - 2\phi(z) + \phi(z-1) + \beta\sin(2\pi\phi(z)) + \sigma.
$$

Because $\phi \in C^0$, then looking at the sup and inf of $\sin(2\pi\phi)$, we deduce that

$$
\begin{cases}
\sup_{\mathbb{R}} \psi \geq \beta + \sigma - 1 > 0 \\
\inf_{\mathbb{R}} \psi \leq -\beta + \sigma + 1 < 0,
\end{cases}
$$

where the strict inequalities follow from $|\sigma| < \beta - 1$. But $c\phi' = \psi$ which implies that $c\phi'$ changes sign. This is impossible because ϕ is non-decreasing. Therefore, $\phi \notin C^0(\mathbb{R})$, which implies that $c = 0$. $\qquad\square$

3 Vertical branches for large velocities

We devote this section to the proof of existence of traveling waves solutions of (1.9) for $c \gg 1$ and under the assumptions (A_{Lip}) and (P_{Lip}). The result is applicable in particular for function F defined on $\mathbb{R}^{N+1}$ and satisfying $(\tilde{A}_{C^1})$ and $(\tilde{B}_{C^1})$ with $\sigma = \sigma^+$, which can be always reduced to the case $\sigma^+ = 0$ by adding a constant to F, and hence we may get a solution for (1.17) for $c \gg 1$. In this section, we show also the existence of solutions (1.18) for $c \ll -1$ which follows from the case $\sigma = \sigma^+$ using a transformation result (Lemma 3.3).

Proposition 3.1 (Existence of traveling waves for $c \gg 1$)
Consider a function F satisfying (A_{Lip}) and (P_{Lip}). Then for $c \gg 1$, there exists a traveling wave ϕ solution of (1.9).

Proof of Proposition 3.1

The strategy of the proof consists in constructing a sub and a supersolution for $c \gg 1$ of a re-scaled form of the equation

(3.1) $$c\phi'(y) = F(\phi(y + r_0), \phi(y + r_1), ..., \phi(y + r_N)) \quad \text{on} \quad \mathbb{R},$$

and then in proving the existence of a solution using Perron's method.

Step 1: re-scaling equation (3.1)

If ϕ is a solution of (3.1) (with $\phi(-\infty) = 0$, $\phi(+\infty) = 1$), then for every $z \in \mathbb{R}$, the function h defined as

$$h(z) := \phi(cz)$$

has to satisfy

(3.2) $$h'(z) = F\left(\left(h\left(z + \frac{r_i}{c}\right)\right)_{i=0,...,N}\right) \quad \text{on} \quad \mathbb{R}.$$

Step 2: sub and supersolution of (3.2)

In order to construct a sub and a supersolution of (3.2), we first mention some useful properties of the solution of the ODE

(3.3) $$h_0' = F(h_0, ..., h_0) = f(h_0) \geq 0,$$

with $h_0(0) = \frac{1}{2}$.

Step 2.1: existence and monotonicity of h_0

Since $f > 0$ on $(0, 1)$ and f is Lipschitz over $[0, 1]$ (see assumptions (A_{Lip}) and (P_{Lip})), then there exists a C^1 solution h_0 of (3.3) defined on $\mathbb{R}$, with values in $[0, 1]$, satisfying

(3.4) $$h_0' > 0 \quad \text{on} \quad \mathbb{R}.$$

Since the constant functions 0 and 1 are respectively a sub and a supersolution of (3.3) (since $f(0) = f(1) = 0$), then

$$0 \leq h_0(z) \leq 1.$$

We also easily deduce that

$$h_0(-\infty) = 0 \quad \text{and} \quad h_0(+\infty) = 1.$$

Step 2.2: subsolution of (3.2)

Let $\varepsilon = \frac{1}{c}$ and $0 < \delta = \underline{M}\varepsilon$ for $\underline{M}$ chosen large, and c chosen such that $\underline{a} = 1 - \delta > 0$. Then define

$$\underline{h}(z) = h_0(\underline{a}z) \quad \text{over} \quad \mathbb{R}.$$

Our goal is to prove that $\underline{h}$ is a subsolution of (3.2) on $\mathbb{R}$, taking advantage of the fact that (3.3) is a caricature of (3.2) for large c.

First notice that we have

$$\underline{h}(z + \varepsilon r_i) = h_0(\underline{a}z) + \underline{a}\varepsilon r_i \underline{L}_i \quad \text{with} \quad \underline{L}_i = \int_0^1 h_0'(\underline{a}z + \underline{a}\varepsilon r_i t)dt.$$

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Because $F \in \mathrm{Lip}([0,1]^{N+1})$ for some Lipschitz constant L, we get

$$F((\underline{h}(z+\varepsilon r_i))_{i=0,\ldots,N}) - f(h_0(\underline{a}z)) = F((h_0(\underline{a}z) + \underline{a}\varepsilon r_i \underline{L}_i)_{i=0,\ldots,N}) - F((h_0(\underline{a}z))_{i=0,\ldots,N})$$

$$\geq -\varepsilon \underline{a} L \begin{vmatrix} r_0\underline{L}_0 \\ \vdots \\ r_N\underline{L}_N \end{vmatrix} \geq -\varepsilon \underline{a} L r^* \begin{vmatrix} \underline{L}_0 \\ \vdots \\ \underline{L}_N \end{vmatrix},$$

where $r^* = \max\limits_{i=0,\ldots,N} |r_i|$ (recall (1.20)).

We now estimate the $\underline{L}_i$'s.

Case 1: $f \in C^1([0,1])$

If $f \in C^1([0,1])$, then for $z \in \mathbb{R}$, we have

$$h_0''(z) = f'(h_0(z))h_0'(z).$$

As $h_0' > 0$ on $\mathbb{R}$ and $f \in C^1([0,1])$, we get for $z \in \mathbb{R}$

$$(\ln(h_0'(z)))' = f'(h_0(z)),$$

where the absolute value of the right hand side is bounded by some constant $\mathcal{K}$. Hence, using the continuity of h_0', for any $b \in \mathbb{R}$ and for all $z \in \mathbb{R}$, we obtain

$$\ln\left(\frac{h_0'(z+b)}{h_0'(z)}\right) \leq \mathcal{K}|b|.$$

This implies that

$$(3.5) \qquad h_0'(z+b) \leq h_0'(z)e^{\mathcal{K}|b|} \quad \text{for every} \ \ z \in \mathbb{R}.$$

Case 2: $f \in \mathbf{Lip}([0,1])$

We want to show that (3.5) is still true if $f \in \mathrm{Lip}([0,1])$, and the point is to regularize by convolution the function f and then to pass to the limit. Using the extension result (cf. Lemma 7.1), there exists a function $\tilde{F}$ defined over $\mathbb{R}^{N+1}$ and satisfying $(\tilde{A}_{\mathrm{Lip}})$. Moreover, the function $\tilde{f}(v) := \tilde{F}(v,\ldots,v)$ is nothing but the periodic extension of f with period 1.

Let $\rho_\varepsilon(x) = \frac{1}{\varepsilon}\rho(\frac{x}{\varepsilon})$, where ρ is a mollifier and define the function $\tilde{f}_\varepsilon(x) := \tilde{f} \star \rho_\varepsilon(x)$. Then consider the ODE

$$(3.6) \qquad \begin{cases} h_\varepsilon' = \tilde{f}_\varepsilon(h_\varepsilon) \\ h_\varepsilon(0) = \dfrac{1}{2}. \end{cases}$$

Since $\tilde{f}_\varepsilon$ is C^1, then there exists a unique regular solution h_ε defined over $\mathbb{R}$ and satisfies

$$(3.7) \qquad h_\varepsilon'(z+b) \leq h_\varepsilon'(z)e^{\mathcal{K}|b|} \quad \text{for every} \ \ z \in \mathbb{R}.$$

Moreover, since $\tilde{f}_\varepsilon$ is periodic smooth, then there exists some C independent of ε such that

$$|h_\varepsilon'| \leq C \quad \text{on} \quad \mathbb{R}.$$

Therefore, using Ascoli's theorem and the extraction diagonal argument, h_ε converges locally uniformly to some h_1 that solves in the classical sense

(3.8)
$$
\begin{cases}
h_1' = \tilde{f}(h_1) \\
h_1(0) = \dfrac{1}{2},
\end{cases}
$$

and

$$
h_1'(z+b) \leq h_1'(z)e^{\mathcal{K}|b|} \quad \text{for every } z \in \mathbb{R}.
$$

But the constant functions 0 and 1 are respectively sub and supersolution of (3.8), then

$$
0 \leq h_1 \leq 1,
$$

that is, h_1 is a solution of (3.3). Thus by uniqueness, we get that $h_1 = h_0$, and hence h_0 satisfies (3.5).

Consequences in both Case 1 and Case 2

Now, we go back to estimate the $\underline{L}_i$'s. Using (3.5), we get, for every $i \in \{0, ..., N\}$, that

$$
0 \leq \underline{L}_i = \int_0^1 h_0'(\underline{a}z + \underline{a}\varepsilon r_i t)dt \leq h_0'(\underline{a}z)e^{\mathcal{K}\underline{a}\varepsilon|r_i|} \leq h_0'(\underline{a}z)e^{\mathcal{K}\varepsilon r^*} =: \underline{K}h_0'(\underline{a}z),
$$

using (3.5) for $b = \underline{a}\varepsilon r_i t$ and using the fact that $\underline{a} < 1$.

Therefore, for $L_1 := L \begin{vmatrix} 1 \\ \vdots \\ 1 \end{vmatrix}$, we deduce that with $\delta = \underline{M}\varepsilon$

$$
\begin{aligned}
\underline{h}'(z) - F((\underline{h}(z + \varepsilon r_i))_{i=0,...,N}) &= \underline{a}h_0'(\underline{a}z) - F((\underline{h}(z + \varepsilon r_i))_{i=0,...,N}) \\
&= -\delta h_0'(\underline{a}z) - \left(F((\underline{h}(z + \varepsilon r_i))_{i=0,...,N}) - f(h_0(\underline{a}z))\right) \\
&\leq -\delta h_0'(\underline{a}z) + \varepsilon \underline{a}Lr^* \begin{vmatrix} L_0 \\ \vdots \\ L_N \end{vmatrix} \\
&\leq \varepsilon\left(-\underline{M} + \underline{a}L_1 r^* \underline{K}\right)h_0'(\underline{a}z) \leq 0,
\end{aligned}
$$

where the last inequality follows once $\underline{M} \geq L_1 r^* \underline{K}$ (taking into account that $\underline{a} \leq 1$). Therefore $\underline{h}$ is a subsolution of (3.2).

Step 2.3: supersolution of (3.2)

The proof is similar to Step 2.2. Let $\varepsilon = \dfrac{1}{c}$ and $0 < \delta = \overline{M}\varepsilon$ with $\overline{M}$ chosen large, and c chosen such that $\overline{a} = 1 + \delta \leq 2$. Then consider the function

$$
\overline{h}(z) = h_0(\overline{a}z)
$$

that we want to show to be a supersolution of (3.2). Clearly, we have

$$
\overline{h}(z + \varepsilon r_i) = h_0(\overline{a}z) + \varepsilon\overline{a}r_i\overline{L}_i \quad \text{with} \quad \overline{L}_i = \int_0^1 h_0'(\overline{a}z + \varepsilon\overline{a}r_i t)
$$

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and

$$F((\overline{h}(z + \varepsilon r_i))_{i=0,\dots,N}) - f(h_0(\overline{a}z)) \le \varepsilon \overline{a} L \left| \begin{matrix} r_0 \overline{L}_0 \\ \vdots \\ r_N \overline{L}_N \end{matrix} \right| \le 2\varepsilon L_1 r^* \overline{K} h_0'(\overline{a}z),$$

where for the last inequality we have used that $\overline{a} \le 2$, $\overline{K} = e^{2\mathcal{K}\varepsilon r^*}$ and L_1 given in the above step. Finally, we deduce that

$$
\begin{aligned}
\overline{h}'(z) - F((\overline{h}(z + \varepsilon r_i))_{i=0,\dots,N}) &\ge \varepsilon \left(\overline{M} - 2L_1 r^* \overline{K} \right) h_0'(\overline{a}z) \\
&\ge 0,
\end{aligned}
$$

if we choose $\overline{M} \ge 2L_1 r^* \overline{K}$. Therefore $\overline{h}$ is a supersolution of (3.2).

Step 3: constructing a solution u_s on a half line
Notice that

(3.9) $$\overline{h}(0) = \underline{h}(0) = \theta \quad \text{and} \quad \overline{h}(z) > \underline{h}(z) \quad \text{for all} \quad z > 0,$$

but the supersolution $\overline{h}$ is not above the subsolution $\underline{h}$ on $\mathbb{R}$.

Step 3.1: shifting $\underline{h}$
Define, for $s > 0$, the function

$$\underline{h}_s(z) = \underline{h}(z - s).$$

Using the definitions of $\overline{h}$, $\underline{h}$ and the properties of h_0, we can easily deduce that if $\overline{h}$ is fixed, then there exists a unique $k_s < 0$ such that

(3.10) $$\overline{h}(k_s) = \underline{h}_s(k_s) \quad \text{and} \quad \overline{h} > \underline{h}_s \quad \text{over} \quad (k_s, +\infty).$$

Indeed, solving the equality

$$h_0(\overline{a}z) = h_0(\underline{a}(z - s)),$$

we get the explicit expression

(3.11) $$k_s = \frac{-\mu s}{1 - \mu} \quad \text{with} \quad 0 < \mu = \frac{\underline{a}}{\overline{a}} < 1.$$

Step 3.2: building a monotone solution for (3.2) on $(r^* + k_s, +\infty)$
Since $\overline{h}$ and $\underline{h}_s$ are respectively a super and a subsolution of (3.2) such that

$$\overline{h} > \underline{h}_s \quad \text{over} \quad (k_s, +\infty),$$

then using Perron's method (Proposition 2.3), there exists a solution u_s of (3.2) on $(r^* + k_s, +\infty)$ such that $\underline{h}_s \le u_s \le \overline{h}$ on $(k_s, +\infty)$.

Step 3.3: u_s is non-decreasing on $(k_s, +\infty)$.
Define for $x \in (k_s, +\infty)$ the function

$$\overline{u}(x) := \inf_{p \ge 0} u_s(x + p).$$

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Clearly, since $\underline{h}_s(x) \le \underline{h}_s(x+p) \le u_s(x+p)$ for all $p \ge 0$ and $x \in (k_s, +\infty)$, we get $\underline{h}_s(x) \le \overline{u}(x) \le u_s(x) \le \overline{h}(x)$ for all $x \in (k_s, +\infty)$. On the other hand, for all $p \ge 0$, $u_s(x+p)$ is a solution of (3.2) over $(k_s + r^*, +\infty)$, then $(\overline{u})_*$ is supersolution of (3.2) over $(k_s + r^*, +\infty)$ (using Proposition 2.2 (ii)). But u_s is defined as the infimum of supersolutions (recall Proposition 2.3 for Perron's method), thus $u_s \le (\overline{u})_* \le \overline{u} \le u_s$ over $(k_s, +\infty)$. Therefore, for every $p \ge 0$,

$$u_s(x) = \overline{u}(x) \le u_s(x+p) \quad \text{over} \quad (k_s, +\infty),$$

and hence u_s is non-decreasing over $(k_s, +\infty)$.

Step 4: passing to the limit $s \to +\infty$

Step 4.1: setting
Let u_s be the solution of (3.2) over $(r^* + k_s, +\infty)$ constructed in Step 3. From equation (3.2) we deduce in particular that u_s is Lipschitz on $(r^* + k_s, +\infty)$ with

$$|u'_s| \le K_0 \quad \text{for a constant } K_0 \text{ independent of } s.$$

Now, for s large enough, we have $r^* + k_s < 0$. Hence, since $u_s(0) \le \overline{h}(0) = \theta$ and $u_s \ge \underline{h}_s$ on $(k_s, +\infty)$ with $\underline{h}_s(+\infty) = 1$, then there exists $z_s \ge 0$ such that $u_s(z_s) = \theta$.
  Next, let $d_s = z_s - (k_s + r^*)$. We have from (3.11) that

$$(3.12) \qquad \qquad \lim_{s \to +\infty} d_s = +\infty.$$

Step 4.2: global non-decreasing solution of (3.2)
Let $v_s(x) := u_s(x + z_s)$ which is a solution of (3.2) on $(-d_s, +\infty)$ with $d_s \to +\infty$ as $s \to +\infty$. We have, $v_s(0) = \theta$ and

$$|v'_s| \le K_0 \quad \text{on } (-d_s + \infty).$$

Thus passing to the limit $s \to +\infty$, v_s converges (using Ascoli's Theorem) to some non-decreasing v solution of

$$(3.13) \qquad \begin{cases} v'(z) = F\left(\left(v\left(z + \dfrac{r_i}{c} \right) \right)_{i=0,\ldots,N} \right) \\ 0 \le v' \le K_0 \quad \text{on} \quad \mathbb{R} \\ 0 \le v \le 1 \quad \text{and} \quad v(0) = \theta. \end{cases}$$

Let $a = v(-\infty)$ or $v(+\infty)$. Then it is easy to see that $0 = f(a)$ which implies that

$$v(-\infty) = 0 \quad \text{and} \quad v(+\infty) = 1.$$

Therefore v is a solution of

$$(3.14) \qquad \begin{cases} v'(z) = F\left(\left(v\left(z + \dfrac{r_i}{c} \right) \right)_{i=0,\ldots,N} \right) \quad \text{on} \quad \mathbb{R} \\ v \text{ is non-decreasing} \\ v(-\infty) = 0 \quad \text{and} \quad v(+\infty) = 1, \end{cases}$$

and this ends the proof. $\qquad \qquad \square$

Lemma 3.2 (Vertical branches for $\sigma = \sigma^{\pm}$)
Consider a function F satisfying $(\tilde{A}_{C^1})$ and $(\tilde{B}_{C^1})$. Assume that $\sigma = \sigma^+$ (resp. $\sigma = \sigma^-$), then for $c \gg 1$ (resp. $c \ll -1$), there exists a traveling wave solution of (1.17) (resp. (1.18)).

Proof of Lemma 3.2
Proving the existence of solution for $c \gg 1$ when $\sigma = \sigma^+$ follows exactly from Proposition 3.1 where $\sigma^+ = 0$. However, the proof of the result for $c \ll -1$ when $\sigma = \sigma^-$ follows from the proof of the case $\sigma = \sigma^+$ and the transformation lemma below (Lemma 3.3). $\qquad\square$

Lemma 3.3 (Transformation of solutions)
Let ϕ be a solution of

$$(3.15) \qquad\qquad c\phi'(z) = F((\phi(z + r_i))_{i=0,...,N}) + \sigma^- \quad over \quad \mathbb{R},$$

then

$$\overline{\phi}(z) = \theta - \phi(-z)$$

is a solution of (3.15) with F, c, r_i and σ^- replaced respectively by

$$(3.16) \qquad \begin{cases} \overline{F}(X_0, ..., X_N) = -F((\theta - X_i)_{i=0,...,N}) \\ \overline{c} = -c, \quad \overline{r}_i = -r_i \quad and \quad \overline{\sigma}^+ = -\sigma^- \end{cases}$$

Moreover, if F satisfies $(\tilde{A}_{C^1})$ and $(\tilde{B}_{C^1})$ then $\overline{F}$ satisfies $(\tilde{A}_{C^1})$ and $(\tilde{B})$ (with $\overline{f}(v) = \overline{F}(v, ..., v)$).

Proof of Lemma 3.3
The proof of Lemma 3.3 is straightforward. $\qquad\square$

4   Existence of traveling waves for $\sigma \in (\sigma^-, \sigma^+)$

In this section we prove, for every $\sigma \in (\sigma^-, \sigma^+)$, the existence of a unique velocity $c = c(\sigma)$ and the existence of a traveling wave $\phi = \phi_\sigma$ solution of (1.16).

The main result of this section is:

Proposition 4.1 (Existence and uniqueness of $c = c(\sigma)$ for $\sigma \in (\sigma^-, \sigma^+)$)
Assume that F satisfies $(\tilde{A}_{C^1})$, $(\tilde{B}_{C^1})$ and let $\sigma \in (\sigma^-, \sigma^+)$. Then there exists a unique real $c(\sigma)$ (simply denoted by c_σ) such that there exists a function $\phi_\sigma : \mathbb{R} \to \mathbb{R}$ solution of (1.16) for $c = c_\sigma$ (in the viscosity sense). Moreover, this solution satisfies the following property: there is no $a > r^*$ (r^* is given in (1.20)) and $x \in \mathbb{R}$ such that

$$(4.1) \qquad\qquad \phi_\sigma = b_\sigma \quad on \quad [x - a, x + a],$$

where b_σ, m_σ are defined in Theorem 1.6.

In order to prove Proposition 4.1, we introduce the following lemma:

Lemma 4.2 (Continuity and monotonicity of m_σ, b_σ over $[\sigma^-, \sigma^+]$)
Under the assumptions $(\tilde{A}_{C^1})$ and $(\tilde{B}_{C^1})$, the two maps

$$
\begin{array}{ccc}
[\sigma^-, \sigma^+] & \to & [\theta - 1, 0] \\
\sigma & \mapsto & m_\sigma
\end{array}
\quad and \quad
\begin{array}{ccc}
[\sigma^-, \sigma^+] & \to & [0, \theta] \\
\sigma & \mapsto & b_\sigma,
\end{array}
$$

are continuous. Moreover, the map m_σ is increasing in σ, while b_σ is decreasing.

The proof of Lemma 4.2 is straightforward from the definition of $\sigma^\pm$ and from assumption $(\tilde{B}_{C^1})$.

Proof of Proposition 4.1
Let $\sigma \in (\sigma^-, \sigma^+)$. Let $m_\sigma \in (\theta - 1, 0)$ and $b_\sigma \in (0, \theta)$ (since $\sigma \neq \sigma^\pm$) be the solutions of $f(s) + \sigma = 0$. Because of assumption $(\tilde{B}_{C^1})$, the function $(f + \sigma)_{|[m_\sigma, m_\sigma + 1]}$ is of bistable type, that is $f + \sigma$ satisfies

$$
(\tilde{B}'_{C^1}) \left|
\begin{array}{l}
f(v) + \sigma = 0 \;\; \text{for} \;\; v = m_\sigma, \; b_\sigma \;\; \text{and} \;\; m_\sigma + 1 \\
(f + \sigma)_{|(m_\sigma, b_\sigma)} < 0, \;\; (f + \sigma)_{|(b_\sigma, m_\sigma + 1)} > 0 \;\; \text{and} \;\; f'(b_\sigma) > 0.
\end{array}
\right.
$$

Step 1: existence of a traveling wave
Under assumption $(\tilde{A}_{C^1})$ and property $(\tilde{B}'_{C^1})$, the existence of a traveling wave ϕ_σ and a velocity c_σ solution of (1.16) follows from [1, Proposition 2.3].

Step 2: uniqueness of the velocity c_σ under (M)
Assume that F is decreasing close to $\{m_\sigma\}^{N+1}$ and $\{m_\sigma + 1\}^{N+1}$ in the direction $E = (1, ..., 1)$. That is, there exists $\varepsilon > 0$ small such that F satisfies:

$$
(M) \left|
\begin{array}{l}
F(X + (a, ..., a)) < F(X) \;\; \text{for all} \;\; a > 0 \;\; \text{such that} \;\; X, X + (a, ..., a) \in [m_\sigma, m_\sigma + \varepsilon]^{N+1} \\
F(X + (a, ..., a)) < F(X) \;\; \text{for all} \;\; a > 0 \;\; \text{such that} \;\; X, X + (a, ..., a) \in [m_\sigma + 1 - \varepsilon, m_\sigma + 1]^{N+1}.
\end{array}
\right.
$$

Then under assumptions $(\tilde{A}_{C^1})$ and (M), the velocity c_σ is unique, (as a consequence of [1, Theorem 1.6 (a)])).

Step 3: checking that F satisfies (M)
Since F is C^1 over a neighborhood of $\mathbb{R}E \backslash (\mathbb{Z}E \cup \mathbb{Z}\Theta)$, then for every $\delta > 0$ there exists $\varepsilon = \varepsilon(\delta) > 0$ such that if $X, X + (a, ..., a) \in [m_\sigma, m_\sigma + \varepsilon]^{N+1}$, then

$$
(4.2) \qquad\qquad |\nabla F(X + t(a, ..., a)) - \nabla F(m_\sigma, ..., m_\sigma)| \leq \delta
$$

for all $t \in [0, 1]$. Hence using (4.2), we get

$$
\begin{aligned}
F(X + (a, ..., a)) - F(X) - f'(m_\sigma)a \;\; &= \left(\int_0^1 dt \sum_{i=0}^{N} \left(\frac{\partial F}{\partial X_i}(X + t(a, ..., a)) - \frac{\partial F}{\partial X_i}(m_\sigma, ..., m_\sigma) \right) \right) a \\
&\leq \;\; (N + 1)a\delta.
\end{aligned}
$$

Now, since $f'(m_\sigma) < 0$, we deduce that

$$
F(X + (a, ..., a)) - F(X) \leq (f'(m_\sigma) + (N + 1)\delta)a < 0
$$

for $\delta > 0$ small enough. Similarly, we show that F is decreasing close to $\{m_\sigma + 1\}^{N+1}$.

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Step 4**: verification of** (4.1)

Assume that there exists $a > r^*$ and $x_0 \in \mathbb{R}$ such that

(4.3) $$\phi_\sigma = b_\sigma \quad \text{on} \quad [x_0 - a, x_0 + a].$$

Then proceeding as in [1, Proposition 2.3, Steps 4 and 5] (without any change in Steps 4.1, 4.2, 4.3 and Step 5), we get a contradiction. Indeed in the proof of [1, Proposition 2.3], we were assuming that ϕ_σ is constant on a half line, but condition (4.3) is sufficient to conclude.
□

In fact, we can prove the existence of traveling waves for the bistable non-linearity under weaker assumptions. In the following proposition, we prove a general existence result for traveling waves which we will use later in the proof of Theorem 1.1, Section 7.2, Step 1. This proposition will be used also to prove that $c^+ \geq 0$ (proof of Proposition 1.3), Section 8.1.

Assumption $(B_{\mathbf{Lip}})$**:**

Let $f(v) := F(v, ..., v)$ and assume

Instability: $f(0) = 0 = f(1)$ and there exists $b \in (0, 1)$ such that $f(b) = 0$, $f_{|_{(0,b)}} < 0$ and $f_{|_{(b,1)}} > 0$.

Strict monotonicity: There exists some $\eta > 0$ such that

$$F(X + (\omega, ..., \omega)) - F(X) \geq \eta \, \omega$$

for $\omega > 0$ small enough and for all X close enough to $(b, ..., b)$.

Proposition 4.3 (Existence of c for a Lipschitz bistable non-linearity)

Consider a function F defined over $\mathbb{R}^{N+1}$ and satisfying (A_{Lip}) and (B_{Lip}). Then there exist a real c and a function ϕ solution of (1.9) in the classical sense if $c \neq 0$ and almost everywhere if $c = 0$. Moreover, there is no $a > r^*$ and $x \in \mathbb{R}$ such that

(4.4) $$\phi = b \quad on \quad [x - a, x + a].$$

This result is the analogue of the existence result of [1, Proposition 2.3], assuming that F is less regular near the instability b which is replaced by the strict monotonicity of F near b.

Proof of Proposition 4.3

As it is written above, the proof of Proposition 4.3 is a variant of the proof of [1, Proposition 2.3]. However, in this case, we obtain the contradiction using the strict monotonicity (Step 4.3) while the rest of the proof (Step 0 to Step 4.2 and Step 5) stays the same.

We now prove the contradiction using the strict monotonicity, revisiting Step 4.3 of the proof of [1, Proposition 2.3].

Step 4.3**: getting a contradiction**

We recall that we consider an approximation ϕ_p of the profile ϕ, for some parameter p going to zero, which satisfies

$$c_p \phi_p'(z) = F((\phi_p(x + r_i))_{i=0,...,N}).$$

We construct (see [1, Proposition 2.3]) a local minimum x_p^* of ψ_p satisfying

$$0 < m_p = \psi_p(x_p^*),$$

where $\psi_p(x) = (\phi_p)_*(x+a) - (\phi_p)^*(x-a)$. Then it is possible to see as in [1, Proposition 2.3, Step 4.3], that

$$0 \geq F((a_i)_{i=0,\dots,N}) - F((c_i)_{i=0,\dots,N}),$$

where

$$a_i = \begin{cases} k_i & \text{if} \quad r_i \leq 0 \\ k_i + m_p & \text{if} \quad r_i > 0 \end{cases} \quad \text{and} \quad c_i = \begin{cases} k_i - m_p & \text{if} \quad r_i \leq 0 \\ k_i & \text{if} \quad r_i > 0, \end{cases}$$

and

$$k_i = \begin{cases} (\phi_p)_* (x_p^* + a + r_i) & \text{if} \quad r_i \leq 0 \\ (\phi_p)^* (x_p^* - a + r_i) & \text{if} \quad r_i > 0. \end{cases}$$

Here, the notation c_i is not ambiguous and has nothing to do with the velocity c_p. Since $a_i = c_i + m_p$ for every $i \in \{0, \dots, N\}$, then

$$0 \geq F((c_i + m_p)_{i=0,\dots,N}) - F((c_i)_{i=0,\dots,N}).$$

Now, since $0 < m_p \to 0$ and $k_i \to b$ for all i (see [1, Proposition 2.3, Steps 4.1, 4.2 and 5]), then

$$c_i \to b \quad \text{for all } i \in \{0, \dots, N\}.$$

Therefore, for p small enough, we have c_i close to b and $m_p > 0$ is small enough, thus using the strict monotonicity in (B_{Lip}), we deduce that

$$0 \geq F((c_i + m_p)_{i=0,\dots,N}) - F((c_i)_{i=0,\dots,N}) \geq \eta m_p > 0,$$

which is a contradiction.
Note that, the proof of (4.4) follows exactly as the proof of (4.1). $\qquad \square$

5 Properties of the velocity

We split this section into two subsections. We dedicate a first subsection to the proof of monotonicity and continuity of the velocity function $c(\sigma)$ over (σ^-, σ^+). In a second subsection, we prove that the velocity function attains finite limits $c^\pm$ as σ goes to $\sigma^\pm$ respectively. We also prove the existence of traveling waves solutions of (1.17) (resp. (1.18)) for $c = c^+$ (resp. $c = c^-$).

5.1 Monotonicity and continuity of the velocity

This subsection consists in two results. The monotonicity (Proposition 5.1 and Lemma 5.4) and the continuity (Proposition 5.3) of the velocity function on (σ^-, σ^+).

We start with the following result.

Proposition 5.1 (Monotonicity of the velocity)
Assume $(\tilde{A}_{C^1})$, $(\tilde{B}_{C^1})$ and let σ_1, $\sigma_2 \in [\sigma^-, \sigma^+]$ such that $\sigma_1 < \sigma_2$. Let $i = 1, 2$ and associate for each $\sigma = \sigma_i$ a solution (c_i, ϕ_i) of (1.16). Then

$$c_1 \leq c_2.$$

Proof of Proposition 5.1

We argue by contradiction. Let σ_1, $\sigma_2 \in [\sigma^-, \sigma^+]$ such that $\sigma_1 < \sigma_2$. To simplify, we denote m_{σ_i} by m_i. Let

$$\phi_1(-\infty) = m_1 < m_2 = \phi_2(-\infty) < \phi_1(+\infty) = m_1 + 1 < m_2 + 1 = \phi_2(+\infty).$$

Assume to the contrary that $c_2 < c_1$. Let $a \in \mathbb{R}$ and define $\phi_2^a(x) = \phi_2(x + a)$. Hence, for $a \geq 0$ large enough fixed, we get

$$\phi_2^a \geq \phi_1 \quad \text{over} \quad \mathbb{R}.$$

Next, set

$$\left|\begin{array}{l} u_1(t, x) = \phi_1(x + c_1 t) \\ u_2(t, x) = \phi_2^a(x + c_2 t), \end{array}\right.$$

then for $j = 1, 2$, we have

(5.1) $$\partial_t u_j(t, x) = F((u_j(t, x + r_i))_{i=0,\dots,N}) + \sigma_j.$$

Moreover, at time $t = 0$, we have

$$u_2(0, x) = \phi_2^a(x) \geq \phi_1(x) = u_1(0, x) \quad \text{over} \quad \mathbb{R}.$$

Thus applying the comparison principle for equation (5.1) (see [16, Propositions 2.5 and 2.6]), we get

$$u_2(t, x) \geq u_1(t, x) \quad \text{for all} \quad (t, x) \in [0, +\infty) \times \mathbb{R}.$$

Taking $x = y - c_2 t$, we get

$$\phi_2^a(y) \geq \phi_1(y + (c_1 - c_2)t) \quad \text{for all } t \geq 0 \text{ and } y \in \mathbb{R}.$$

Using that $c_1 > c_2$ and passing to the limit $t \to \infty$, we get

$$\phi_2^a(y) \geq \phi_1(+\infty) = m_1 + 1 \quad \text{for all } y \in \mathbb{R}.$$

But $\phi_2^a(-\infty) = m_2 < m_1 + 1$, hence a contradiction. Therefore $c_1 \leq c_2$. $\qquad\square$

Then we have the straightforward consequence of Proposition 5.1.

Corollary 5.2 (Monotonicity and limits of $c(\sigma)$)
Assume $(\tilde{A}_{C^1})$, $(\tilde{B}_{C^1})$. For $\sigma \in (\sigma^-, \sigma^+)$, let $(c(\sigma), \phi_\sigma)$ be a solution of (1.16) given in Proposition 4.1. Then the velocity function is non-decreasing on (σ^-, σ^+). Moreover, the limits

$$\lim_{\sigma \to \sigma^-} c(\sigma) = c^- \quad \text{and} \quad \lim_{\sigma \to \sigma^+} c(\sigma) = c^+$$

exist and satisfy $-\infty \leq c^- \leq c^+ \leq +\infty$.

Proposition 5.3 (Continuity of the velocity function)
Suppose that F satisfies $(\tilde{A}_{C^1})$, $(\tilde{B}_{C^1})$ and let $\sigma \in (\sigma^-, \sigma^+)$. Let $(c(\sigma), \phi_\sigma)$ be a solution of (1.16) given in Proposition 4.1. Then the map $\sigma \mapsto c(\sigma)$ is continuous on (σ^-, σ^+).

Proof of Proposition 5.3

Let $\sigma_0 \in (\sigma^-, \sigma^+)$ and $c_0 := c(\sigma_0)$ be the associated velocity given in Proposition 4.1. Let $\sigma_n \in (\sigma^-, \sigma^+)$ be a sequence such that $\sigma_n \to \sigma_0$ and let $c_n = c(\sigma_n)$. We want to show that $c_n \to c_0$. Assume that ϕ_0 and ϕ_n (for each n) are solutions of (1.16) associated respectively to σ_0 and σ_n (for each n).

Step 1: passing to the limit $n \to +\infty$

As a consequence of the monotonicity of $c(\sigma)$ (Proposition 5.1) and the fact that σ_0, $\sigma_n \in (\sigma^-, \sigma^+)$ for all n, we get that c_n is bounded. Thus, up to a subsequence, we set $\overline{c} = \lim\limits_{n \to +\infty} c_n$.

Recall that (c_n, ϕ_n) solves

$$c_n \phi_n'(z) = F((\phi_n(z + r_i))_{i=0,\dots,N}) + \sigma_n$$

and $\theta - 1 < m_{\sigma_n} \leq \phi_n \leq m_{\sigma_n} + 1 < 1$.

Therefore, passing to the limit $n \to +\infty$ (see Lemma 9.1), ϕ_n converges to a function ϕ almost everywhere, and ϕ solves (in the viscosity sense)

$$(5.2) \qquad \overline{c}\phi'(x) = F((\phi(x + r_i))_{i=0,\dots,N}) + \sigma_0.$$

Moreover, Theorem 9.3 implies that $(\overline{c}, \phi)$ solves (1.16) for $\sigma = \sigma_0$.

Step 2: conclusion

From the uniqueness of the velocity on (σ^-, σ^+) (Proposition 4.1) and the fact that c_0 and $\overline{c}$ are associated to $\sigma_0 \in (\sigma^-, \sigma^+)$, we deduce that $\overline{c} = c_0$. From the uniqueness of the limit $\overline{c}$ (whatever is the subsequence $\sigma_n \to \sigma_0$), we deduce the continuity of the velocity function c. $\square$

Lemma 5.4 (Strict monotonicity)

Under the assumptions of Proposition 5.1, there exists a constant $K > 0$ such that $c(\sigma)$ satisfies

$$(5.3) \qquad \frac{dc}{d\sigma} \geq K|c| \quad on \ (\sigma^-, \sigma^+)$$

in the viscosity sense.

Proof of Lemma 5.4

Clearly, if $c = 0$ then (5.3) holds true.

Let σ_1, $\sigma_2 \in (\sigma^-, \sigma^+)$ with $\sigma_1 < \sigma_2$ and, as in the proof of Proposition 5.1, let us call $c_1 \leq c_2$ the associated velocities and ϕ_1, ϕ_2 the corresponding profiles with $\phi_i(-\infty) = m_{\sigma_i}$ for $i = 1$, 2 and $m_{\sigma_1} < m_{\sigma_2}$. Recall also that $(c, \phi) = (c_i, \phi_i)$ solves for $\sigma = \sigma_i$ and $i = 1$, 2

$$(5.4) \qquad c\phi' = F((\phi(x + r_i))_{i=0,\dots,N}) + \sigma.$$

Suppose that $c_1 > 0$. Since $F \in \mathrm{Lip}(\mathbb{R}^{N+1})$ and ϕ_1 is bounded, then there exists some $C > 0$ such that

$$|F((\phi_1(x + r_i))_{i=0,\dots,N})| \leq C.$$

Therefore

$$0 \leq \phi_1' \leq c_1^{-1}(|\sigma_1| + C).$$

Hence for $\delta = c_1(|\sigma_1| + C)^{-1}$, we have (using (5.4))

$$(c_1 + \delta(\sigma_2 - \sigma_1))\phi'_1 \leq \sigma_2 + F((\phi_1(x + r_i))_{i=0,\dots,N}).$$

But, this means that $(\bar{c}, \phi_1)$, with $\bar{c} = c_1 + \delta(\sigma_2 - \sigma_1)$, is a subsolution of (5.4) with $\sigma = \sigma_2$. Comparing $\phi_1(x + \bar{c}t)$ to $\phi_2(x + c_2 t)$ as in Proposition 5.1, we deduce that $\bar{c} \leq c_2$, that is,

(5.5) $$\frac{c_2 - c_1}{\sigma_2 - \sigma_1} \geq c_1(|\sigma_1| + C)^{-1} =: Kc_1 \quad (\sigma_1 \in (\sigma^-, \sigma^+) \text{ bounded}).$$

Now letting $\sigma_1 \to \sigma_2$, and using the continuity of $c(\sigma)$, inequality (5.3) follows (in the sense of viscosity) in case $c > 0$. Similarly, we prove that $c(\sigma)$ verifies (5.3) for $c < 0$. $\qquad\square$

5.2 Finite threshold velocities $(c^+ < +\infty$ and $c^- > -\infty)$

In this subsection, we show that $c^+ < +\infty$ (resp. $c^- > -\infty$) and we prove the existence of a solution for $c = c^+$ (resp. $c = c^-$) of (1.17) (resp. (1.18)).

In order to prove that $c^+ < +\infty$ and $c^- > -\infty$, we need to start with the following useful lemma.

Lemma 5.5 (Bound on the velocity for $\sigma \in (\sigma^-, \sigma^+)$)
Consider a function F satisfying $(\tilde{A}_{C^1})$ and $(\tilde{B}_{C^1})$. Then there exists $0 < c^+_* < +\infty$ (resp. $-\infty < c^-_* < 0$) such that the followings holds. Let $\sigma_0 \in (\sigma^-, \sigma^+)$ and c_{σ_0} be such that (c_{σ_0}, ϕ_0) is a solution of (1.16) with $\sigma = \sigma_0$. Then

$$-\infty < c^-_* \leq c_{\sigma_0} \leq c^+_* < +\infty.$$

Proof of Lemma 5.5
Notice that from Proposition 3.1, there exists $0 < c^+_* < +\infty$ (resp. $-\infty < c^-_* < 0$) such that for all $c_1 > c^+_*$ (resp. $c_2 < c^-_*$) there exists (c_1, ϕ_1) (resp. (c_2, ϕ_2)) solution of (1.16) for $\sigma = \sigma^+$ (resp. $\sigma = \sigma^-$).

We prove that $c_{\sigma_0} \leq c^+_*$ (the case $c^-_* \leq c_{\sigma_0}$ being similar). Assume to the contrary that $c_{\sigma_0} = c_1 > c^+_*$. Suppose that (c_1, ϕ_1) be a solution of (1.16) for $\sigma = \sigma^+$.

Let $\bar{\sigma}$ be such that

(5.6) $$\sigma^- < \sigma_0 < \bar{\sigma} < \sigma^+$$

and associate a solution $(\bar{c}, \bar{\phi})$ of (1.16) for $\sigma = \bar{\sigma}$. Using Proposition 5.1, we get that

$$\bar{c} \leq c_1 = c_{\sigma_0}.$$

Moreover, using (5.6) and the fact that $c_{\sigma_0} = c_1 > c^+_* > 0$, we deduce from Lemma 5.4 that

$$c_1 = c_{\sigma_0} < \bar{c},$$

which is a contradiction. $\qquad\square$

Then we have the straightforward result:

Corollary 5.6 (Finite limits of c as $\sigma \to \sigma^{\pm}$)
Consider a function F satisfying $(\tilde{A}_{C^1})$ and $(\tilde{B}_{C^1})$. Let c^-, c^+ given by Corollary 5.2 and c_*^+, c_*^- defined in Lemma 5.5. Then

$$-\infty < c_*^- \leq c^- \leq c^+ \leq c_*^+ < +\infty.$$

Lemma 5.7 (Existence of a solution of (1.17) for $c = c^{\pm}$)
Assume $(\tilde{A}_{C^1})$, $(\tilde{B}_{C^1})$ and let $\sigma = \sigma^+$ (resp. $\sigma = \sigma^-$). There exists a profile ϕ^+ (resp. ϕ^-) such that (c^+, ϕ^+) (resp. (c^-, ϕ^-)) solves (1.17) (resp. (1.18)).

Proof of Lemma 5.7

Step 0: preliminary
Assume that $\sigma = \sigma^+$ and let us prove the existence of a solution of (1.17) for c^+ (proving the existence of solution of (1.17) for c^- in the case $\sigma = \sigma^-$ is treated similarly). The goal is to get a solution as a limit of the profiles as $\sigma \to \sigma^+$, recalling that $c^+ = \lim\limits_{\sigma \to \sigma^+} c(\sigma)$.

Consider $\sigma \in (\sigma^-, \sigma^+)$ and let (c_σ, ϕ_σ) be a solution of (1.16), namely

(5.7)
$$\begin{cases} c_\sigma \phi_\sigma'(z) = F(\phi_\sigma(z+r_0), \phi_\sigma(z+r_1), ..., \phi_\sigma(z+r_N)) + \sigma \quad \text{on} \quad \mathbb{R}. \\ \phi_\sigma \text{ is non-decreasing over } \mathbb{R} \\ \phi_\sigma(-\infty) = m_\sigma \quad \text{and} \quad \phi_\sigma(+\infty) = m_\sigma + 1. \end{cases}$$

As in the proof of Proposition 5.3, there exists some constant $M > 0$ independent on σ such that

$$|F(\phi_\sigma(z+r_0), \phi_\sigma(z+r_1), ..., \phi_\sigma(z+r_N)) + \sigma^+| \leq M \quad \text{for all} \quad \sigma \in (\sigma^-, \sigma^+).$$

Moreover, up to translate ϕ_σ, we can assume that (because $m_\sigma \to 0$ as $\sigma \to \sigma^+$)

(5.8)
$$(\phi_\sigma)_*(0) \leq \frac{1}{2} \leq \phi_\sigma^*(0).$$

Step 1: passing to the limit $\sigma \to \sigma^+$
Applying Lemma 9.1, we deduce that there exists some function $\phi = \phi^+$ which satisfies, in viscosity sense

(5.9)
$$\begin{cases} c^+(\phi)'(z) = F(\phi(z+r_0), \phi(z+r_1), ..., \phi(z+r_N)) + \sigma^+ \quad \text{on} \quad \mathbb{R}. \\ \phi \text{ is non-decreasing over } \mathbb{R} \\ 0 = m_{\sigma^+} \leq \phi \leq m_{\sigma^+} + 1 = 1. \end{cases}$$

Step 2: limits of the profile ϕ
Passing to the limit in (5.8), we get

$$0 \leq \phi(-\infty) \leq \phi_*(0) \leq \frac{1}{2} \leq (\phi)^*(0) \leq \phi(+\infty) = 1.$$

Because $\phi(\pm\infty)$ solves

$$f(\phi(\pm\infty)) + \sigma^+ = 0,$$

the solution has to satisfy

$$\phi(-\infty) = m_{\sigma^+} = 0 \quad \text{and} \quad \phi(+\infty) = 1.$$

Therefore $\phi = \phi^+$ solves (1.17). $\qquad \square$

6 Filling the gaps: traveling waves for $c \geq c^+$ and $c \leq c^-$

We prove, in this section, for each $c \geq c^+$ (resp. $c \leq c^-$) the existence of a solution of (1.17) (resp. (1.18)). We also prove that (1.17) (resp. (1.18)) admits no solution for any $c < c^+$ (resp. $c > c^-$).

Proposition 6.1 (Existence of solution for vertical branches of velocities)
Let F be a given function satisfying assumptions $(\tilde{A}_{C^1})$ and $(\tilde{B}_{C^1})$. Let $c^+ < +\infty$ and $c^- > -\infty$ be given by Corollary 5.2. Then for every $c > c^+$ (resp. $c < c^-$), there exists a solution ϕ of (1.17) (resp. (1.18)).

For $c = c^+$ or $c >> 1$ (resp. $c = c^-$ or $c << 1$) we already have the existence of a solution of ϕ of (1.17) (resp. (1.18)). Proposition 6.1 fills the gap for all $c \geq c^+$ (resp. $c \leq c^-$).

In order to prove Proposition 6.1, we will need the following preliminary result that is proved in [16].

Lemma 6.2 (Existence of a hull function ([16, Theorem 1.5 and Theorem 1.6 a1]))
Let F be a given function satisfying assumption $(\tilde{A}_{C^1})$, $p > 0$ and $\sigma \in \mathbb{R}$. There exists a unique $\lambda(\sigma, p) = \lambda_p(\sigma)$ such that there exists a locally bounded function $h_p : \mathbb{R} \to \mathbb{R}$ satisfying (in the viscosity sense):

(6.1)
$$\begin{cases} \lambda_p h'_p(z) = F((h_p(z + pr_i))_{i=0,\dots,N}) + \sigma & on \quad \mathbb{R} \\ h_p(z + 1) = h_p(z) + 1 \\ h'_p(y) \geq 0 \\ |h_p(z + z') - h_p(z) - z'| \leq 1 \quad for \ any \ z, \ z' \in \mathbb{R}. \end{cases}$$

Moreover, there exists a constant $K > 0$, independent on p and σ, such that

(6.2)
$$|\lambda_p - \sigma| \leq K(1 + p)$$

and the function

$$\lambda_p : \mathbb{R} \to \mathbb{R}$$
$$\sigma \to \lambda_p(\sigma)$$

is continuous with $\lambda_p(\pm\infty) = \pm\infty$.

For the proof of Lemma 6.2, we refer the reader to [16, Theorems 1.5 and 1.6]. However, proving that $\lambda_p(\pm\infty) = \pm\infty$ follows from (6.2).

Corollary 6.3 (Existence of ϕ_p)
Let F be a given function satisfying assumption $(\tilde{A}_{C^1})$, $p > 0$ and $c \in (c^+, +\infty)$ fixed. Then there exists $\sigma = \sigma(c, p) \in \mathbb{R}$ such that there exists a function $\phi_p : \mathbb{R} \to \mathbb{R}$ that satisfies in the viscosity sense:

(6.3)
$$\begin{cases} c\phi'_p(z) = F((\phi_p(z + r_i))_{i=0,\dots,N}) + \sigma(c, p) & on \quad \mathbb{R} \\ \phi'_p \ non\text{-}decreasing \\ \phi_p\left(z + \dfrac{1}{p}\right) = \phi_p(z) + 1. \end{cases}$$

Proof of Corollary 6.3
Let $\sigma = \sigma(c,p)$ such that

(6.4)
$$\lambda_p(\sigma) = cp$$

and define the function ϕ_p as:

(6.5)
$$\phi_p(x) = h_p(px),$$

where h_p is given by Lemma 6.2. This gives the result. □

Now, we give the proof of Proposition 6.1.

Proof of Proposition 6.1
Choose $\bar{c} > c^+$ and let $\delta_0 > 0$ such that

$$\bar{c} > c^+ + \delta_0.$$

Step 1: preliminary
Choose $\eta > 0$ small and let $\sigma^+ - \eta \le \sigma_\eta < \sigma^+$. From Proposition 4.1, we know that for σ_η, there exits a solution $(c_{\sigma_\eta}, \phi_{\sigma_\eta})$ of (1.16) such that

$$c_{\sigma_\eta} \le c^+.$$

Moreover, as $c_{\sigma_\eta} = \lim_{p \to 0} c(\sigma_\eta, p)$ with $c(\sigma_\eta, p) = \dfrac{\lambda(\sigma_\eta, p)}{p}$ (see the proof of existence of [1, Proposition 2.3]), then there exists p_η such that for all $0 < p \le p_\eta$, we have

(6.6)
$$|c(\sigma_\eta, p) - c_{\sigma_\eta}| \le \delta_0.$$

Thus, for $0 < p \le p_\eta$, we get

(6.7)
$$c(\sigma_\eta, p) \le c_{\sigma_\eta} + \delta_0 \le c^+ + \delta_0 < \bar{c}.$$

Moreover, since the map $\sigma \mapsto \lambda(\sigma, p) = c(\sigma, p)p$ is continuous with $\lambda(\pm\infty, p) = \pm\infty$ (see Lemma 6.2), then for such $0 < p \le p_\eta$, there exists $\bar{\sigma}_p \in \mathbb{R}$ and a function $\phi_p : \mathbb{R} \to \mathbb{R}$ (see Corollary 6.3) such that

$$c(\bar{\sigma}_p, p) = \bar{c}$$

and $(\bar{c}, \phi_p)$ solves (6.3). Hence from (6.7), we get

$$c(\sigma_\eta, p) < c(\bar{\sigma}_p, p).$$

In addition, since $\lambda(\sigma, p)$ is non-decreasing with respect to σ, then

(6.8)
$$\bar{\sigma}_p > \sigma_\eta \ge \sigma^+ - \eta \quad \text{for } 0 < p \le p_\eta.$$

Step 2: passing to the limit $p \to 0$
Since $\lim_{p \to 0} \lambda(\bar{\sigma}_p, p) = \lim_{p \to 0} \bar{c}p = 0$, we deduce from (6.2) that there exists some $L_0 > 0$ independent of p such that

(6.9)
$$|\bar{\sigma}_p| \le L_0 \quad \text{for } 0 < p \le p_\eta.$$

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Thus
$$\overline{\sigma}_p \to \overline{\sigma}_0 \quad \text{as} \quad p \to 0 \quad \text{(up to a subsequence)}.$$

Recall that ϕ_p is non-decreasing and that
$$\phi_p\left(x + \frac{1}{2p}\right) - \phi_p\left(x + \frac{-1}{2p}\right) = 1.$$

We can also assume that
$$\begin{cases} (\phi_p)^*(0) \geq \dfrac{1}{2} \\ (\phi_p)_*(0) \leq \dfrac{1}{2}. \end{cases}$$

Therefore, since $F \in \mathrm{Lip}(\mathbb{R}^{N+1})$ and due to (6.9), we deduce (as in proof of [1, Lemma 2.8]) that there exists some $M > 0$ independent on n such that
$$|F((\phi_p(x + r_i)_{i=0,\dots,N})) + \overline{\sigma}_n| \leq M.$$

Applying arguments similar to the ones of the proof of Lemma 9.1, we see that ϕ_p converges to some ϕ almost everywhere and ϕ is a viscosity solution of

(6.10)
$$\begin{cases} \overline{c}\phi'(x) = F((\phi(x + r_i))_{i=0,\dots,N}) + \overline{\sigma}_0 \\ \phi \text{ non-decreasing and bounded} \\ \phi(+\infty) - \phi(-\infty) \leq 1, \end{cases}$$

and ϕ satisfies
$$\begin{cases} \phi^*(0) \geq \dfrac{1}{2} \\ \phi_*(0) \leq \dfrac{1}{2}. \end{cases}$$

In addition, we have
$$\overline{\sigma}_0 \geq \sigma^+ - \eta \quad \text{(because of (6.8))}.$$

But $\eta > 0$ is arbitrary, hence
$$\overline{\sigma}_0 \geq \sigma^+.$$

Moreover, since $\overline{\sigma}_0 \leq \sigma^+$ (otherwise, (6.10) admits no solution, see Remark 1.9), thus
$$\overline{\sigma}_0 = \sigma^+.$$

Finally, since $\phi(\pm\infty)$ solves $f + \sigma^+ = 0$, then we conclude that
$$\phi(-\infty) = m_{\sigma^+} = 0 \quad \text{and} \quad \phi(+\infty) = 1,$$

which ends the proof. $\qquad\square$

Lemma 6.4 (Non-existence of solution for $c < c^+$ and $c > c^-$)
Consider a function F and assume $(\tilde{A}_{C^1})$ and $(\tilde{B}_{C^1})$. Let $\sigma = \sigma^+$ (resp. $\sigma = \sigma^-$) and $c^+ < +\infty$ (resp. $c^- > -\infty$) be given by Corollary 5.2. Let (c, ϕ) be a solution of (1.17) (resp. (1.18)), then $c \geq c^+$ (resp. $c \leq c^-$).

Proof of Lemma 6.4

Let $\sigma = \sigma^+$ and (c, ϕ) be a solution of (1.17). We want to prove that $c \geq c^+$ (similarly, we show that there is no solution of (1.18) for $c > c^-$ when $\sigma = \sigma^-$).

It is known from Theorem 1.6-1, that for every $\sigma \in (\sigma^-, \sigma^+)$, there exists $(c(\sigma), \phi_\sigma)$ solution of (1.16). Let $\sigma_n \in (\sigma^-, \sigma^+)$ be a sequence such that $\sigma_n \to \sigma^+$, $c(\sigma_n) \to c^+$ and $(c(\sigma_n), \phi_{\sigma_n})$ is a solution of (1.16). Since $\sigma_n < \sigma^+$, Proposition 5.1 implies that

$$c(\sigma_n) \leq c.$$

Therefore, passing to the limit $\sigma_n \to \sigma^+$, we get that

$$c^+ \leq c,$$

which ends the proof. □

Lemma 6.5 (Strict inequality between threshold velocities)
Consider a function F satisfying $(\tilde{A}_{C^1})$, $(\tilde{B}_{C^1})$ and let c^-, c^+ given by Corollary 5.6. If $c^- \neq 0$ or $c^+ \neq 0$, then

$$c^- < c^+.$$

Proof of Lemma 6.5
This is a straightforward consequence of (5.3).

□

Proof of Theorem 1.6
Theorem 1.6 is proved in several propositions and lemmata. In Propositions 4.1, 6.1, 5.3 and Lemma 5.4, we prove, for $\sigma \in (\sigma^-, \sigma^+)$, the existence of traveling waves and the monotonicity and the continuity of the velocity of propagation respectively. Existence of vertical branches of solutions (when $\sigma = \sigma^\pm$) is proved in Lemma 3.2, where we show the existence of traveling waves for $c \gg 1$ and for $c \ll -1$; and in Corollary 5.6, Lemma 5.7, Proposition 6.1, Lemma 6.4 and Lemma 6.5, where we respectively show the existence of finite critical limits $c^\pm$ of the velocity function when σ goes to $\sigma^\pm$, the existence of solutions for the critical limits of velocity, fill the gap and prove the non-existence of solution when $c < c^+$ and $\sigma = \sigma^+$ or when $c > c^-$ and $\sigma = \sigma^-$, and finally prove the inequality between c^+ and c^-. □

7 Proof of Theorem 1.1

We devote this section to the proof of Theorem 1.1 and we split it into two subsections. We recall in a first subsection an extension result over $\mathbb{R}^{N+1}$. For pedagogical reasons, we also prove in this subsection the result of Theorem 1.1 in a simple case where the non-linearity F is assumed to be smooth (cf. Proposition 7.2). In a second subsection, we give the proof of Theorem 1.1 in full generality for Lipschitz non-linearities F.

To prove the result (in any case), we first show the existence of traveling waves for $c \gg 1$ by applying Proposition 3.1. The next step is to define the critical velocity c^+ and then we prove, for all $c \geq c^+$, the existence of traveling wave solutions of system (1.9). Finally, We show the non-existence of solutions of (1.9) for any $c < c^+$.

7.1 Preliminary results

We start this subsection by recalling an extension result of the function F defined on $[0,1]^{N+1}$ into a function $\tilde{F}$ over $\mathbb{R}^{N+1}$. We also prove the result of Theorem 1.1 in a simple case.

Lemma 7.1 (Extension of F)
Consider a function F defined over $[0,1]^{N+1}$ and satisfying (A_{Lip}) such that $F(0,...,0) = F(1,...,1) = 0$. There exists an extension $\tilde{F}$ defined over $\mathbb{R}^{N+1}$ such that

$$\tilde{F}_{|_{[0,1]^{N+1}}} = F$$

and $\tilde{F}$ satisfies

Assumption $(\tilde{A}_{\mathbf{Lip}})$:

  Regularity: $\tilde{F}$ *is globally Lipschitz continuous over $\mathbb{R}^{N+1}$.*

  Monotonicity: $\tilde{F}(X_0,...,X_N)$ *is non-decreasing w.r.t. each X_i for $i \neq 0$.*

  Periodicity: $\tilde{F}(X_0+1,...,X_N+1) = \tilde{F}(X_0,...,X_N)$ *for every $X = (X_0,...,X_N) \in \mathbb{R}^{N+1}$.*

  Lemma 7.1 corresponds to Lemma 2.1 in [1] whose proof is given in the appendix A of [1].

  Notice that the function $\tilde{f}(v) := \tilde{F}(v,...,v)$ is nothing but a periodic extension of f on $\mathbb{R}$ with period 1, that is

$$\tilde{f}_{|_{[0,1]}} = f,$$

hence $\tilde{f}(0) = \tilde{f}(1) = 0$.

  Notice also that ϕ is a solution of

$$\begin{cases} c\phi'(z) = F((\phi(z+r_i))_{i=0,...,N}) & \text{on } \mathbb{R} \\ \phi \text{ is non-decreasing over } \mathbb{R} \\ \phi(-\infty) = 0 \quad \text{and} \quad \phi(+\infty) = 1 \end{cases}$$

if and only if ϕ is a solution of

(7.1)
$$\begin{cases} c\phi'(z) = \tilde{F}((\phi(z+r_i))_{i=0,...,N}) & \text{on } \mathbb{R} \\ \phi \text{ is non-decreasing over } \mathbb{R} \\ \phi(-\infty) = 0 \quad \text{and} \quad \phi(+\infty) = 1, \end{cases}$$

since $\tilde{F}_{|_{[0,1]^{N+1}}} = F$. In particular $\tilde{F}$ satisfies (P_{Lip}) if F satisfies (P_{Lip}).

  In order to prove Theorem 1.1 in a special case when F is smooth (see Proposition 7.2), we need to introduce precise assumptions.

Assumption (A_{C^1}):

  Regularity: $F \in C^1([0,1]^{N+1})$.

  Monotonicity: $F(X_0,...,X_N)$ is non-decreasing w.r.t. each X_i for $i \neq 0$.

Assumption (P'_{C^1}):

Positive degenerate monostability:
  Let $f(v) = F(v, ..., v)$ such that $f(0) = 0 = f(1)$ and $f > 0$ in $(0, 1)$.

Smoothness near $\{0\}^{N+1}$ and $\{1\}^{N+1}$:
  There exists $\delta > 0$ such that

$$\begin{cases} f' > 0 & \text{on} \quad (0, \delta) \\ f' < 0 & \text{on} \quad (1 - \delta, 1) \end{cases}$$

Proposition 7.2 (Vertical branch, simple case)
Consider a function F satisfying (A_{C^1}) and (P'_{C^1}). Then the result of Theorem 1.1 holds true.

Proof of Proposition 7.2
Note that $\sigma^+ = 0$ in this case.

Using Proposition 3.1, we deduce that for $c \gg 1$, there exists a solution of (1.9). Next, from the extension lemma (Lemma 7.1), we see that if F satisfies (A_{C^1}) (which implies (A_{Lip})), then the extended function $\tilde{F}$ satisfies $(\tilde{A}_{C^1})$. Because of assumption (P'_{C^1}) and $\tilde{f}$ is 1-periodic with $\tilde{f} = f$ on $[0, 1]$, there exists $\varepsilon_0 > 0$ small enough, such that for $-\varepsilon_0 < \sigma < 0$, $\tilde{f} + \sigma$ has a bistable shape over $(m_\sigma, m_\sigma + 1)$ where m_σ is defined exactly as in Theorem 1.6. Precisely, by bistable shape we mean that there exists m_σ and b_σ solutions of $\tilde{f} + \sigma = 0$ satisfying $-1 < m_\sigma < 0 < b_\sigma < m_\sigma + 1 < 1$ and

$$\begin{cases} \tilde{f} + \sigma < 0 & \text{on} \quad (m_\sigma, b_\sigma) \\ \tilde{f} + \sigma > 0 & \text{on} \quad (b_\sigma, m_\sigma + 1) \\ \tilde{f}'(b_\sigma) > 0 \quad \text{and} \quad \tilde{f}'(m_\sigma) = \tilde{f}'(m_\sigma + 1) < 0. \end{cases}$$

For $\sigma \in (-\varepsilon_0, 0)$, using Proposition 4.1 (which stays true with $(\tilde{B}_{C^1})$ replaced by (P'_{C^1}) and $\sigma \in (-\varepsilon_0, 0)$ instead of $\sigma \in (\sigma^-, \sigma^+)$), we show the existence of a unique velocity c_σ such that there exists a profile ϕ_σ solution of system (1.16) with F replaced by $\tilde{F}$. From Propositions 5.1 and 5.3 (which stay true similarly for $(\tilde{B}_{C^1})$ replaced by (P'_{C^1}) and $\sigma \in (-\varepsilon_0, 0)$), we get that the map

$$\sigma \mapsto c_\sigma$$

is monotone continuous on $(-\varepsilon_0, 0)$ and we define as in Corollary 5.2 the critical velocity c^+ as

$$\lim_{\sigma \to 0^-} c_\sigma = c^+.$$

Again, up to replace $(\tilde{B}_{C^1})$ by (P'_{C^1}) and $\sigma \in (\sigma^-, \sigma^+)$ by $\sigma \in (-\varepsilon_0, 0)$, we can use Lemma 5.5, Corollary 5.6 and Lemma 5.7, and show that $c^+ < +\infty$ and that (1.9) admits a solution for $c = c^+$. We use Proposition 6.1 (again with $(\tilde{B}_{C^1})$ replaced by (P'_{C^1})) to fill the gap and get the existence of solution (c, ϕ) for each $c \geq c^+$. Finally, the non-existence of solutions for $c < c^+$ follows from Lemma 6.4 (with $(\tilde{B}_{C^1})$ replaced by (P'_{C^1})). $\qquad\square$

7.2 Proof of Theorem 1.1

This subsection is devoted for the proof of Theorem 1.1.

Proof of Theorem 1.1

Let us consider a general function $F : [0,1]^{N+1} \to \mathbb{R}$ and $f(v) = F(v,...,v)$ satisfying (A_{Lip}) and (P_{Lip}). We have to adapt the proof of Proposition 7.2 with a much lower regularity of F (here F is only Lipschitz). To this end, we will introduce an approximation F_δ of F.

Step 0: a δ-approximation

Define for $X = (X_0,...,X_N) \in [0,1]^{N+1}$ and $\delta > 0$ small

$$F_\delta(X) = F(X) - f(X_0) + f_\delta(X_0),$$

where

$$f_\delta(v) = \begin{cases} \max\big(f(\delta) + L_0(v-\delta), 0\big) & \text{on} \quad [0,\delta] \\ \max\big(f(1-\delta) - L_0(v-(1-\delta)), 0\big) & \text{on} \quad [1-\delta, 1] \\ f & \text{on} \quad [\delta, 1-\delta], \end{cases}$$

with a constant $L_0 > 0$ satisfying $L_0 > 2\text{Lip}(F) =: 2L_F^\infty$. Clearly, we have $F_\delta(v,...,v) = f_\delta(v)$.

Set

$$(7.2) \qquad \begin{cases} b_\delta = \delta - \dfrac{f(\delta)}{L_0} > 0 \\ m_\delta = 1 - \delta + \dfrac{f(1-\delta)}{L_0} < 1 \end{cases}$$

which satisfies

$$(7.3) \qquad 0 < b_\delta < \delta < 1 - \delta < m_\delta < 1,$$

and

$$f_\delta(b_\delta) = 0 = f_\delta(m_\delta) \quad \text{and} \quad f_\delta > 0 \quad \text{on} \quad (b_\delta, m_\delta).$$

Let $\tilde{F}$ and $\tilde{F}_\delta$ defined on $\mathbb{R}^{N+1}$ be the extension functions of F and F_δ (which are defined on $[0,1]^{N+1}$) respectively constructed by Lemma 7.1. Define $\tilde{f}_\delta(v) = \tilde{F}_\delta(v,...,v)$ and $\tilde{f}(v) = \tilde{F}(v,...,v)$, then $\tilde{f}_\delta$ and $\tilde{f}$ are 1-periodic with $(\tilde{f}_\delta)_{|[0,1]} = f_\delta$ and $(\tilde{f})_{|[0,1]} = f$. Moreover, since $\tilde{f}_\delta \le \tilde{f}$, we get that

$$\tilde{F}_\delta \le \tilde{F} \quad \text{over} \quad \mathbb{R}^{N+1}.$$

Now, for $\sigma < 0$ small fixed $(0 < -\sigma < \min_{[\delta, 1-\delta]} f)$, define $0 < b_{\delta,\sigma} < m_{\delta,\sigma} < 1$ such that

$$(7.4) \qquad \begin{cases} (\tilde{f}_\delta + \sigma)(b_{\delta,\sigma}) = 0 = (\tilde{f}_\delta + \sigma)(m_{\delta,\sigma}) = (\tilde{f}_\delta + \sigma)(m_{\delta,\sigma} - 1) \\ \tilde{f}_\delta + \sigma < 0 \quad \text{on} \quad (m_{\delta,\sigma} - 1, b_{\delta,\sigma}) \\ \tilde{f}_\delta + \sigma > 0 \quad \text{on} \quad (b_{\delta,\sigma}, m_{\delta,\sigma}). \end{cases}$$

Notice that

$$\begin{cases} m_{\delta,\sigma} \to m_\delta \\ b_{\delta,\sigma} \to b_\delta \end{cases} \quad \text{as} \quad \sigma \to 0^-.$$

For simplicity, we will denote $\tilde{F}$, $\tilde{F}_\delta$, $\tilde{f}$ and $\tilde{f}_\delta$ by F, F_δ, f and f_δ respectively.

Step 1: **existence of a solution of the approximated non-linearity** F_δ

From the definition of f_δ, we see that (for $0 < -\sigma < \min\limits_{[\delta, 1-\delta]} f$)

(7.5) $$b_\delta < b_{\delta,\sigma} < \delta.$$

Now, because of (7.5) and using the definition of F_δ with the fact that F is L_F^∞-Lipschitz, then for X close to $\{b_{\delta,\sigma}\}^{N+1}$ and $\omega > 0$ small enough, we get that

(7.6)
$$F_\delta(X + (\omega, ..., \omega)) - F_\delta(X) = F(X + (\omega, ..., \omega)) - F(X) - f(X_0 + \omega) + f(X_0) + f_\delta(X_0 + \omega) - f_\delta(X_0)$$
$$\geq -2\omega L_F^\infty + \omega L_0$$
$$= \omega(-2L_F^\infty + L_0) = \omega\eta > 0 \quad \text{(because of the condition on } L_0\text{)}.$$

Since $F_\delta + \sigma$ satisfies (A_{Lip}) and (B_{Lip}) with $[0, 1]^{N+1}$ replaced by $[m_{\delta,\sigma} - 1, m_{\delta,\sigma}]^{N+1}$ and b replaced by $b_{\delta,\sigma}$ (see (7.4) and (7.6)), then applying the result of Proposition 4.3 (but now on $[m_{\delta,\sigma} - 1, m_{\delta,\sigma}]^{N+1}$), we deduce that there exists a solution $\phi_{\delta,\sigma}$ that solves in viscosity sense

(7.7)
$$\begin{cases} c_{\delta,\sigma}\phi'_{\delta,\sigma}(x) = F_\delta((\phi_{\delta,\sigma}(x + r_i))_{i=0,...,N}) + \sigma \quad \text{on} \quad \mathbb{R} \\ \phi_{\delta,\sigma} \text{ is non-decreasing over } \mathbb{R} \\ \phi_{\delta,\sigma}(-\infty) = m_{\delta,\sigma} - 1 \quad \text{and} \quad \phi_{\delta,\sigma}(+\infty) = m_{\delta,\sigma}. \end{cases}$$

More precisely, we have used the fact that $F_\delta(\cdot + \{m_{\delta,\sigma} - 1\}^{N+1}) + \sigma$ satisfies (A_{Lip}) and (P_{Lip}) on $[0, 1]^{N+1}$ with b defined by $b_{\delta,\sigma} = b + m_{\delta,\sigma} - 1$, and Proposition 4.3 provides a profile $\phi : \mathbb{R} \to [0, 1]$ such that $\phi + m_{\delta,\sigma} - 1 =: \phi_{\delta,\sigma}$.

Step 2: $c_{\delta,\sigma}$ **is non-decreasing in** σ **for** δ **fixed**

Here, this is a variant of the proof of Proposition 5.1. Let $\delta > 0$ fixed, $-\min\limits_{[\delta, 1-\delta]} f < \sigma_1 < \sigma_2 < 0$ and set $(c_{\delta,\sigma_1}, \phi_{\delta,\sigma_1})$, $(c_{\delta,\sigma_2}, \phi_{\delta,\sigma_2})$ be the associated solutions of (7.7) for σ_1 and σ_2 respectively.
We have

$$m_{\delta,\sigma_1} - 1 < m_{\delta,\sigma_2} - 1 < m_{\delta,\sigma_1} < m_{\delta,\sigma_2};$$

that is $\phi_{\delta,\sigma_1}(\pm\infty) < \phi_{\delta,\sigma_2}(\pm\infty)$, and $\phi_{\delta,\sigma_1}(+\infty) > \phi_{\delta,\sigma_2}(-\infty)$. Thus using the proof of Proposition 5.1, we deduce that $c_{\delta,\sigma_1} \leq c_{\delta,\sigma_2}$.

Step 3: $c_{\delta,\sigma}$ **is non-increasing in** δ **for** σ **fixed**

For $\delta_2 > \delta_1 > 0$, fix σ such that $-\min\limits_{[\delta_1, 1-\delta_1]} f < \sigma < 0$ and associate respectively the two solutions $(c_{\delta_2,\sigma}, \phi_{\delta_2,\sigma})$ and $(c_{\delta_1,\sigma}, \phi_{\delta_1,\sigma})$ of (7.7). From the definition of F_δ, $m_{\delta,\sigma}$ and $b_{\delta,\sigma}$ (see Step 0), we see that

$$F_{\delta_2} \leq F_{\delta_1},$$

hence $(c_{\delta_2,\sigma}, \phi_{\delta_2,\sigma})$ is a subsolution of (7.7) for F_δ replaced by F_{δ_1}. Moreover, we also have that

$$m_{\delta_2,\sigma} - 1 < m_{\delta_1,\sigma} - 1 < m_{\delta_2,\sigma} < m_{\delta_1,\sigma},$$

hence $\phi_{\delta_2,\sigma}(\pm\infty) < \phi_{\delta_1,\sigma}(\pm\infty)$ and $\phi_{\delta_2,\sigma}(+\infty) > \phi_{\delta_1,\sigma}(-\infty)$. Using the proof of Proposition 5.1 (which is still true for sub and supersolutions), we deduce that $c_{\delta_2,\sigma} \leq c_{\delta_1,\sigma}$.

Step 4: **passing to the limit** $\sigma \to 0^- = \sigma^+$

For $\delta > 0$ fixed, let $(c_{\delta,\sigma}, \phi_{\delta,\sigma})$ be a solution of (7.7). Since $F_\delta \leq F$ (see Step 0), we deduce that $(c_{\delta,\sigma}, \phi_{\delta,\sigma})$ is a subsolution for (7.7), with F_δ replaced by F.

On the other hand, let us consider any solution ϕ_{c_0} of

(7.8)
$$\begin{cases} c_0 \phi'_{c_0}(x) = F((\phi_{c_0}(x + r_i))_{i=0,...,N}) & \text{on} \quad \mathbb{R} \\ \phi_{c_0} \text{ is non-decreasing over } \mathbb{R} \\ \phi_{c_0}(-\infty) = 0 \quad \text{and} \quad \phi_{c_0}(+\infty) = 1. \end{cases}$$

From Proposition 3.1, we know that such a solution does exist at least for $c_0 \gg 1$.

Since $\phi_{\delta,\sigma}$ satisfies

$$\phi_{\delta,\sigma}(-\infty) = m_{\delta,\sigma} - 1 \quad \text{and} \quad \phi_{\delta,\sigma}(+\infty) = m_{\delta,\sigma},$$

then $\phi_{\delta,\sigma}(\pm\infty) < \phi_{c_0}(\pm\infty)$ and $\phi_{\delta,\sigma}(+\infty) > \phi_{c_0}(-\infty)$. Thus using the proof of Proposition 5.1 (which is still true for sub and supersolutions), we deduce that

$$c_{\delta,\sigma} \le c_0 \quad \text{for all} \quad \sigma \in (- \min_{[\delta, 1-\delta]} f, 0).$$

Since the map $\sigma \mapsto c_{\delta,\sigma}$ is non-decreasing, then

$$c_{\delta,\sigma} \to c_\delta^+ \quad \text{as} \quad \sigma \to 0^-.$$

Therefore, passing to the limit $\sigma \to 0^-$, using Lemma 9.1, $\phi_{\delta,\sigma}$ converges almost everywhere to some ϕ_δ that solves in the viscosity sense

(7.9)
$$\begin{cases} c_\delta^+ \phi'_\delta(x) = F_\delta((\phi_\delta(x + r_i))_{i=0,...,N}) & \text{on} \quad \mathbb{R} \\ \phi_\delta \text{ is non-decreasing over } \mathbb{R} \\ m_\delta - 1 \le \phi_\delta(-\infty) \quad \text{and} \quad \phi_\delta(+\infty) \le m_\delta. \end{cases}$$

We can insure that ϕ_δ is non constant, assuming that

$$\begin{cases} (\phi_{\delta,\sigma})^*(0) \ge \dfrac{b_\delta + m_\delta}{2} \\ (\phi_{\delta,\sigma})_*(0) \le \dfrac{b_\delta + m_\delta}{2}, \end{cases}$$

and this implies in addition that

$$\phi_\delta(-\infty) \le b_\delta \quad \text{and} \quad \phi_\delta(+\infty) = m_\delta.$$

Step 5: passing to the limit $\delta \to 0^+$

Since $c_{\delta,\sigma} \le c_0$ for any $\delta > 0$ and $\sigma \in (- \min_{[\delta, 1-\delta]} f, 0)$, we get

(7.10)
$$c_\delta^+ \le c_0 \quad \text{for all} \quad \delta \in \left(0, \frac{1}{2}\right).$$

Moreover, since $c_{\delta,\sigma}$ is non-increasing in δ, then c_δ^+ is non-increasing in δ. Hence from (7.10), we get

(7.11)
$$\lim_{\delta \to 0^+} c_\delta^+ = c^+ \le c_0.$$

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We can also assume, up to translation, that the solution ϕ_δ of (7.9) satisfies

$$\begin{cases} (\phi_\delta)^*(0) \geq \dfrac{1}{2} \\ (\phi_\delta)_*(0) \leq \dfrac{1}{2}. \end{cases}$$

Thus passing to the limit $\delta \to 0^+$, using again Lemma 9.1, then ϕ_δ converges, up to a subsequence, almost everywhere to some ϕ which solves in viscosity sense

(7.12)
$$\begin{cases} c^+ \phi'(x) = F((\phi(x + r_i))_{i=0,\dots,N}) \quad \text{on} \quad \mathbb{R} \\ \phi \text{ is non-decreasing over } \mathbb{R} \\ 0 \leq \phi(-\infty) \quad \text{and} \quad \phi(+\infty) \leq 1 \end{cases}$$

and satisfies

(7.13)
$$\begin{cases} (\phi)^*(0) \geq \dfrac{1}{2} \\ (\phi)_*(0) \leq \dfrac{1}{2}. \end{cases}$$

But $\phi(\pm\infty)$ is a solution of $\tilde{f} = 0$, then we get

$$\phi(-\infty) = 0 \quad \text{and} \quad \phi(+\infty) = 1.$$

This implies that if (c_0, ϕ_{c_0}) is a solution of (7.8), then $c_0 \geq c^+$ and moreover there exists such a solution $(c_0, \phi_{c_0}) = (c^+, \phi)$. We also recall that we have solutions of (7.8) for $c_0 \gg 1$. Our goal now is to fill the gap and to show that we have solutions for all $c \geq c^+$.

Step 6: filling the gap
This step is analogous to the proof of Proposition 6.1. Fix $\bar{c} > c^+$ and let $\beta_0 > 0$ such that

(7.14)
$$\bar{c} > c^+ + \beta_0.$$

Step 6.1 construction of a solution $(\bar{c}, \bar{\phi})$ associated to some $\bar{\sigma}$

Substep 6.1.1: $c^+ = \lim\limits_{\delta \to 0^-} c_\delta^+$
We know from Steps $4, 5$ that there exists a non trivial solution $(c_\delta^+, \phi_\delta)$ of (7.9) and that $c^+ = \lim\limits_{\delta \to 0^-} c_\delta^+$. Thus there exists some $\delta_0 > 0$ such that

(7.15)
$$|c_\delta^+ - c^+| \leq \frac{\beta_0}{3} \quad \text{for all} \quad 0 < \delta \leq \delta_0.$$

Substep 6.1.2: $c_\delta^+ = \lim\limits_{\sigma \to 0^-} c_{\delta,\sigma}$
Similarly, we know from Steps $1, 4$ that, for every $0 < \delta \leq \delta_0$, there exists a solution $(c_{\delta,\sigma}, \phi_{\delta,\sigma})$ of (7.7) and that $c_\delta^+ = \lim\limits_{\sigma \to 0^-} c_{\delta,\sigma}$. Thus there exists some $\sigma_\delta > 0$ such that

(7.16)
$$|c_{\delta,\sigma} - c_\delta^+| \leq \frac{\beta_0}{3} \quad \text{for all} \quad 0 < -\sigma \leq \sigma_\delta.$$

Substep 6.1.3: $c_{\delta,\sigma} = \lim\limits_{p \to 0^+} c_{\delta,\sigma,p}$

Based on the proof of [1, Proposition 2.3], there exists (for every $0 < \delta \leq \delta_0$ and $0 < -\sigma \leq \sigma_\delta$ such that (7.16) holds true) a velocity $c_{\delta,\sigma,p}$, a profile $\phi_{\delta,\sigma,p}$ and some $p_{\delta,\sigma} > 0$ such that $c_{\delta,\sigma,p}$ converges up to a subsequence to $c_{\delta,\sigma}$ as $p \to 0$ and

$$(7.17) \qquad |c_{\delta,\sigma,p} - c_{\delta,\sigma}| \leq \frac{\beta_0}{3} \quad \text{for all } p \text{ of the subsequence such that} \quad 0 < p \leq p_{\delta,\sigma},$$

where $(c_{\delta,\sigma,p}, \phi_{\delta,\sigma,p})$ is a solution of

$$(7.18) \qquad \begin{cases} c_{\delta,\sigma,p}(\phi_{\delta,\sigma,p})'(x) = F_\delta((\phi_{\delta,\sigma,p}(x + r_i))_{i=0,\dots,N}) + \sigma \quad \text{on} \ \ \mathbb{R} \\ (\phi_{\delta,\sigma,p})' \geq 0 \\ \phi_{\delta,\sigma,p}\left(x + \dfrac{1}{p}\right) = 1 + \phi_{\delta,\sigma,p}(x). \end{cases}$$

Substep 6.1.4: **construction of a solution** $(\bar{c}, \bar{\phi})$ **associated to some** $\bar{\sigma}$

Since the map

$$(7.19) \qquad \sigma \mapsto \lambda(\sigma, p) := p c_{\delta,\sigma,p}$$

is continuous with $\lambda(\pm\infty, p) = \pm\infty$ (see Lemma 6.2 applied to F_δ instead of F), then for every $0 < \delta \leq \delta_0$, $0 < -\sigma \leq \sigma_\delta$ and $0 < p \leq p_{\delta,\sigma}$ such that (7.16) and (7.17) hold true, there exists $\bar{\sigma} = \bar{\sigma}_{\delta,p} \in \mathbb{R}$ and a function $\bar{\phi} = \phi_{\delta,\bar{\sigma},p} : \mathbb{R} \to \mathbb{R}$ (see Corollary 6.3) such that

$$c_{\delta,\bar{\sigma},p} = \bar{c}$$

and $(\bar{c}, \bar{\phi})$ solves

$$(7.20) \qquad \begin{cases} \bar{c}\bar{\phi}'(x) = F_\delta((\bar{\phi}(x + r_i))_{i=0,\dots,N}) + \bar{\sigma} \quad \text{on} \ \ \mathbb{R} \\ \bar{\phi}' \geq 0 \\ \bar{\phi}\left(x + \dfrac{1}{p}\right) = 1 + \bar{\phi}(x). \end{cases}$$

Substep 6.1.5: **consequence of Substeps** 6.1.1-6.1.4

For every $0 < \delta \leq \delta_0$, $0 < -\sigma \leq \sigma_\delta$ and $0 < p \leq p_{\delta,\sigma}$, (7.14), (7.15), (7.16) and (7.17) hold true, thus we get

$$c_{\delta,\sigma,p} \leq c^+ + \beta_0 < \bar{c} = c_{\delta,\bar{\sigma},p}.$$

But the map $\sigma \mapsto c_{\delta,\sigma,p}$ is non-decreasing (see Lemma 6.2 and (7.19)), hence we obtain that

$$(7.21) \qquad \sigma < \bar{\sigma} = \bar{\sigma}_{\delta,p}.$$

Step 6.2: **getting a profile for the original problem with velocity** $\bar{c}$

Substep 6.2.0: **a priori estimate on** $\bar{\sigma}$

The couple $(\bar{c}, \phi_{\delta,\bar{\sigma},p})$ is a solution of (7.20), thus for $p < 1$, we get

$$\phi_{\delta,\bar{\sigma},p}(x + 1) - \phi_{\delta,\bar{\sigma},p}(x) \leq 1;$$

and hence we can show that there exists a constant M_0 independent of p and δ such that

$$|F_\delta| \le M_0.$$

Thus integrating (7.20) over $[0,1]$, implies that there exists a constant $K > 0$ such that

$$|\overline{\sigma}| \le K$$

for all $\delta < \frac{1}{2}$ and $p \le 1$.

Substep 6.2.1: passing to the limit $p \to 0$
Since $|f_\delta - f| \le o_\delta(1)$, then we can assume, up to translation, that

$$(7.22) \qquad \begin{cases} (\phi_{\delta,\overline{\sigma},p})^*(0) \ge \gamma_{\delta,\overline{\sigma}} \\ (\phi_{\delta,\overline{\sigma},p})_*(0) \le \gamma_{\delta,\overline{\sigma}} \end{cases} \quad \text{with} \quad |f_\delta(\gamma_{\delta,\overline{\sigma}}) + \overline{\sigma}| \ge \frac{1}{4}osc(f),$$

with for instance $\gamma_{\delta,\overline{\sigma}} \in [m_\delta - 1, m_\delta]$. Hence using the proof of Lemma 9.1 and the last equality of (7.20), we pass to the limit $p \to 0$ and $\phi_{\delta,\overline{\sigma},p}$ converges up to subsequence to a non trivial (because of (7.22)) solution $\phi_{\delta,\overline{\sigma}_{\delta,0}}$ of

$$(7.23) \qquad \begin{cases} \overline{c}\phi'_{\delta,\overline{\sigma}_{\delta,0}}(z) = F_\delta((\phi_{\delta,\overline{\sigma}_{\delta,0}}(z + r_i))_{i=0,\ldots,N}) + \overline{\sigma}_{\delta,0} \quad \text{on} \quad \mathbb{R} \\ \phi_{\delta,\overline{\sigma}_{\delta,0}} \text{ is non-decreasing on } \mathbb{R} \\ \phi_{\delta,\overline{\sigma}_{\delta,0}}(+\infty) - \phi_{\delta,\overline{\sigma}_{\delta,0}}(-\infty) \le 1, \end{cases}$$

where

$$\overline{\sigma}_{\delta,p} \to \overline{\sigma}_{\delta,0}$$

and $|\overline{\sigma}_{\delta,0}| \le K$.

Substep 6.2.2: establishing $\overline{\sigma}_{\delta,0} = 0$
Since $\sigma < \overline{\sigma}_{\delta,p}$ (see (7.21)), then we get $\sigma \le \overline{\sigma}_{\delta,0}$. Thus passing to the limit $\sigma \to 0$, we get

$$\overline{\sigma}_{\delta,0} \ge 0,$$

without any change in equation (7.23). Moreover, since we have

$$0 = f_\delta(\phi_{\delta,\overline{\sigma}_{\delta,0}}(\pm\infty)) + \overline{\sigma}_{\delta,0}$$

and $f_\delta \ge 0$, then we get that

$$\overline{\sigma}_{\delta,0} = 0.$$

Therefore, because of (7.22), $\phi_\delta := \phi_{\delta,\overline{\sigma}_{\delta,0}=0}$ satisfies (7.9) with c_δ^+ replaced by $\overline{c}$.

Substep 6.2.3: passing to the limit $\delta \to 0$
Up to translation, we assume that

$$\begin{cases} (\phi_\delta)^*(0) \ge \dfrac{b_\delta + m_\delta}{2} \\ (\phi_\delta)_*(0) \le \dfrac{b_\delta + m_\delta}{2}, \end{cases}$$

Therefore, passing to the limit using once more Lemma 9.1, ϕ_δ converges up to a subsequence to a solution ϕ of (7.12) and (7.13), with c^+ replaced by $\bar{c}$. This ϕ is non trivial because of (7.13). Moreover, since $\phi(\pm\infty)$ solves $f = 0$, we deduce that ϕ is a solution of (1.9) associated for the velocity $\bar{c}$.

Step 7: no solution for $c < c^+$
This step is analogous to Lemma 6.4. Let (c, ϕ) be a solution of (1.9). Then as a solution of (7.8), we can choose $(c_0, \phi_0) = (c, \phi)$. Therefore, the choice $c_0 = c$ in (7.11), implies that

$$c^+ \leq c,$$

and then there is no a solution of (1.9) for $c < c^+$. $\qquad\square$

8 On the critical velocity

In a first subsection, we prove Proposition 1.3 which asserts that the critical velocity satisfies $c^+ \geq 0$ under additional assumptions. In a second subsection, we give an example (Proposition 1.4) that shows that we can have $c^+ < 0$ when the additional assumptions are not satisfied. We also prove the instability of the critical velocity, namely Proposition 1.2.

8.1 Critical velocity c^+ is non-negative

This subsection is devoted for the proof of Proposition 1.3. Independently, we also show that $c^- < 0 < c^+$ for the Frenkel-Kontorova model (1.14).

Proof of Proposition 1.3
Let (c, ϕ) be a solution of (1.9) given in Theorem 1.1 with c fixed. Our goal is to show that $c \geq 0$; and hence $c^+ \geq 0$. We perform the proof in several steps.

Step 0: preliminary
Define for $X = (X_0, ..., X_N) \in [0, 1]^{N+1}$ and $\delta > 0$ small the function

(8.1) $$F_\delta(X) = F(X) - f(X_0) + f_\delta(X_0),$$

where

$$f_\delta(v) = \begin{cases} f & \text{on} \quad [0, 1 - \delta] \\ \max\left(f(1 - \delta) - L_0(v - (1 - \delta)), 0\right) & \text{on} \quad [1 - \delta, 1], \end{cases}$$

with a constant $L_0 > 2\text{Lip}(F) > 0$ large enough. Let $\delta \in (0, \frac{1}{2})$ and set

$$1_\delta := 1 - \delta + \frac{f(1 - \delta)}{L_0} < 1,$$

(where 1_δ was denoted by m_δ in the proof of Theorem 1.1).

Part I: antisymmetric extension of F_δ and proof for ii)
Using Proposition 9.4, there exists an antisymmetric extension G_δ on $[-1, 1]^{N+1}$ of F_δ such that

$$\begin{cases} (G_\delta)_{|_{[0,1]^{N+1}}} = F_\delta \\ G_\delta(-X) = -G_\delta(X) \quad \text{for all} \quad X \in [-1, 1]^{N+1} \end{cases}$$

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and satisfying (A_{Lip}) over $[-1,1]^{N+1}$ (since F_δ satisfies (A_{Lip}) over $[0,1]^{N+1}$). Moreover, still by Proposition 9.4, since F_δ is C^1 over a neighborhood of $\{0\}^{N+1}$ in $[0,1]^{N+1}$ (because of (P_{C^1}) and (8.1)) and $f'_\delta(0) = f'(0) > 0$, then there exists $\eta > 0$ such that for every X, $X + (a, ..., a) \in [-1,1]^{N+1}$ close to $\{0\}^{N+1}$ with $a > 0$ small, we have

$$(8.2) \qquad\qquad G_\delta(X + (a, ..., a)) - G_\delta(X) \geq \eta a.$$

In addition, the function $g_\delta(v) := G_\delta(v, ..., v)$ satisfies

$$(8.3) \qquad\qquad \begin{cases} g_\delta(-1_\delta) = g_\delta(0) = g_\delta(1_\delta) = 0 \\ (g_\delta)_{|(-1_\delta,0)} < 0 \quad \text{and} \quad (g_\delta)_{|(0,1_\delta)} > 0, \end{cases}$$

(since we have $f_\delta(0) = 0 = f_\delta(1_\delta)$ and $f_\delta > 0$ on $(0,1_\delta)$).

Step I.1: existence of traveling waves for G_δ
Clearly, since G_δ satisfies (8.3) and (8.2), then G_δ satisfies the assumption (B_{Lip}) with $[0,1]^{N+1}$ replaced by over $[-1_\delta, 1_\delta]^{N+1}$ and b replaced by 0. In addition, G_δ satisfies, by construction, the assumption (A_{Lip}) over $[-1_\delta, 1_\delta]^{N+1}$. Thus applying the result of Proposition 4.3 with $[0,1]^{N+1}$ replaced by $[-1_\delta, 1_\delta]^{N+1}$ and b replaced by 0, we deduce that there exists a real c_δ^0 and a function ϕ_δ^0 solution of

$$(8.4) \qquad\qquad \begin{cases} c_\delta^0 (\phi_\delta^0)'(x) = G_\delta((\phi_\delta^0(x + r_i))_{i=0,...,N}) \quad \text{on} \quad \mathbb{R} \\ \phi_\delta^0 \ \text{is non-decreasing over} \ \mathbb{R} \\ \phi_\delta^0(-\infty) = -1_\delta \quad \text{and} \quad \phi_\delta^0(+\infty) = 1_\delta. \end{cases}$$

Step I.2: $c_\delta^0 \geq 0$
We show in this step that c_δ^0 is non-negative under $ii)$, i.e. assuming $r_i \geq 0$ for all $i \in \{0, ..., N\}$. Then $\psi(x) = -\phi_\delta^0(-x)$ satisfies

$$\begin{aligned} -c_\delta^0 \psi'(y) &= -G_\delta((-\psi(y - r_i))_{i=0,...,N}) \\ &= G_\delta((\psi(y - r_i))_{i=0,...,N}) \\ &\leq G_\delta((\psi(y + r_i))_{i=0,...,N}), \end{aligned}$$

hence $(\bar{c} = -c_\delta^0, \psi)$ is a subsolution of (8.4). Using an argument similar to the computation of (7.6) for L_0 large enough (here $L_0 > 2\text{Lip}(F)$), we can show that G_δ is decreasing close to $\{-1_\delta\}^{N+1}$ and $\{1_\delta\}^{N+1}$ inside $[-1_\delta, 1_\delta]^{N+1}$, that is G_δ satisfies (9.18) and (9.20) (for $s = -1_\delta$ and $s' = 1_\delta$). Applying the comparison principle results (Proposition 9.9 and Proposition 9.10) and the ideas of the proof of Proposition 5.1, we deduce that

$$-c_\delta^0 = \bar{c} \leq c_\delta^0,$$

that is

$$(8.5) \qquad\qquad 0 \leq c_\delta^0.$$

Step I.3: comparing c and c_δ^0

Recall that $(c_\delta^0, \phi_\delta^0)$ is a solution of (8.4). Moreover, since $G_\delta = F_\delta \leq F$ over $[0,1]^{N+1}$, then (c, ϕ) is a supersolution for (1.9), with F replaced by G_δ.

Since
$$\phi(-\infty) = 0 \quad \text{and} \quad \phi(+\infty) = 1$$

and $-1_\delta < 0 < 1_\delta < 1$, that is $\phi_\delta^0(\pm\infty) < \phi(\pm\infty)$ and $\phi_\delta^0(+\infty) > \phi(-\infty)$, then using the proof of Proposition 5.1 (which still true for sub and supersolutions), we deduce that

$$0 \leq c_\delta^0 \leq c.$$

Part II: extension of F_δ by antisymmetry-reflection and proof for iii)

In this part, we assume that F (and then F_δ) satisfies the strict monotonicity condition (1.12). Using Remark 9.8, we can assume that the set I defined in (1.11) satisfies

$$I = \{1, ..., N\},$$

i.e. for all $i \in \{1, ..., N\}$, there exists $\bar{i} \in \{1, ..., N\}$ such that $r_i = -r_{\bar{i}}$. Using now Proposition 9.7, there exists an extension $\overline{G}_\delta$ on $[-1,1]^{N+1}$ of F_δ such that

$$\begin{cases} (\overline{G}_\delta)_{|_{[0,1]^{N+1}}} = F_\delta \\ \overline{G}_\delta(-\overline{X}) = -\overline{G}_\delta(X) \quad \text{for all} \quad X \in [-1,1]^{N+1} \end{cases}$$

and satisfying (A_{Lip}) over $[-1,1]^{N+1}$. Since F is C^1 over a neighborhood of $\{0\}^{N+1}$ in $[0,1]^{N+1}$, then (using Proposition 9.7) there exists $\eta > 0$ such that for every $X, X+(a, ..., a) \in [-1,1]^{N+1}$ close to $\{0\}^{N+1}$ with $a > 0$ small, we have

$$(8.6) \qquad \overline{G}_\delta(X + (a, ..., a)) - \overline{G}_\delta(X) \geq \eta a.$$

In addition, the function $\overline{g}_\delta(v) := \overline{G}_\delta(v, ..., v)$ satisfies

$$(8.7) \qquad \begin{cases} \overline{g}_\delta(-1_\delta) = \overline{g}_\delta(0) = \overline{g}_\delta(1_\delta) = 0 \\ (\overline{g}_\delta)_{|_{(-1_\delta, 0)}} < 0 \quad \text{and} \quad (\overline{g}_\delta)_{|_{(0, 1_\delta)}} > 0. \end{cases}$$

Step II.1: existence of traveling waves for $\overline{G}_\delta$

This step is a variant of Step I.1 with G_δ replaced $\overline{G}_\delta$. Thus we deduce that there exists a real $\overline{c}_\delta^0$ and a function $\overline{\phi}_\delta^0$ solution of

$$(8.8) \qquad \begin{cases} \overline{c}_\delta^0(\overline{\phi}_\delta^0)'(x) = \overline{G}_\delta((\overline{\phi}_\delta^0(x + r_i))_{i=0,...,N}) \quad \text{on} \quad \mathbb{R} \\ \overline{\phi}_\delta^0 \text{ is non-decreasing over } \mathbb{R} \\ \overline{\phi}_\delta^0(-\infty) = -1_\delta \quad \text{and} \quad \overline{\phi}_\delta^0(+\infty) = 1_\delta. \end{cases}$$

Step II.2: $\overline{c}_\delta^0 = 0$

Let $\psi(x) = -\overline{\phi}_\delta^0(-x)$, then

$$\begin{aligned} \overline{G}_\delta((\psi(x + r_i))_{i=0,...,N}) &= \overline{G}_\delta((-\overline{\phi}_\delta^0(-x - r_i))_{i=0,...,N}) \\ &= \overline{G}_\delta((-\overline{\phi}_\delta^0(-x + r_{\bar{i}}))_{i=0,...,N}) \\ &= -\overline{G}_\delta((\overline{\phi}_\delta^0(-x + r_i))_{i=0,...,N}) \\ &= -\overline{c}_\delta^0(\overline{\phi}_\delta^0)'(-x) = -\overline{c}_\delta^0\psi'(x). \end{aligned}$$

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Thus $(-\overline{c}_\delta^0, \psi)$ is a solution of (8.8) with $\overline{c}_\delta^0$ replaced by $-\overline{c}_\delta^0$.

Similarly to Step $I.2$ (with $L_0 > 2\text{Lip}(F)$), we can show that $\overline{G}_\delta$ is decreasing close to $\{-1_\delta\}^{N+1}$ and $\{1_\delta\}^{N+1}$ inside $[-1_\delta, 1_\delta]^{N+1}$. By comparison principle, we get

$$\overline{c}_\delta^0 \leq \overline{c} \quad \text{and} \quad \overline{c} \leq \overline{c}_\delta^0,$$

which implies that

(8.9) $$\overline{c}_\delta^0 = 0.$$

Step $II.3$: comparing c and $\overline{c}_\delta^0$

This step is analogous to Step $I.3$ with $(c_\delta^0, \phi_\delta^0)$ replaced by $(\overline{c}_\delta^0, \overline{\phi}_\delta^0)$ and G_δ replaced by $\overline{G}_\delta$. Thus proceeding similarly we show that

$$0 = \overline{c}_\delta^0 \leq c.$$

Part III: proof for i)

Under condition i), we have

$$I = \{1, ..., N\} \quad \text{and} \quad \frac{\partial F}{\partial X_i}(0) = \frac{\partial F}{\partial X_{\overline{i}}}(0) \quad \text{for all} \ \ i \in I,$$

thus condition (1.12) is equivalent to $f'(0) > 0$. Therefore, we can apply iii) which shows that $c^+ \geq 0$. $\qquad\qquad\square$

Lemma 8.1 (Sign of c^+ and c^- for (FK) model (1.14))

Consider the Frenkel-Kontorova model with $\beta > 0$

$$c\phi'(z) = \phi(z+1) + \phi(z-1) - 2\phi(z) - \beta \sin\left(2\pi\left(\phi(z) + \frac{1}{4}\right)\right) + \sigma,$$

with $\sigma \in [-\beta, \beta] = [\sigma^-, \sigma^+]$. Let $c^\pm$ be the critical velocity associated to $\sigma^\pm$. Then

$$c^- < 0 < c^+.$$

Proof of Lemma 8.1

Let $\sigma = \sigma^+ = \beta$ and let us show that $c^+ > 0$. Let ϕ be non-decreasing with $\phi(-\infty) = 0$ and $\phi(+\infty) = 1$. Integrating over the real line the equation

$$c^+ \phi'(z) = \phi(z+1) + \phi(z-1) - 2\phi(z) + f(\phi(z)),$$

where $f(\phi(z)) = -\beta \sin\left(2\pi\left(\phi(z) + \frac{1}{4}\right)\right) + \beta \geq 0$, we get that

$$c^+ = \int_{\mathbb{R}} \left(-\beta \sin\left(2\pi\left(\phi(z) + \frac{1}{4}\right)\right) + \beta\right) dz \geq 0.$$

Since $f > 0$ on $(0,1)$, if $c^+ = 0$, then

$$\phi(z) = 0 \text{ or } 1 \text{ almost everywhere.}$$

This implies that

$$\Delta_1 \phi(z) := \phi(z+1) + \phi(z-1) - 2\phi(z) = 0 \quad \text{almost everywhere.}$$

Consider now the set

$$\mathcal{A} = \{z \in \mathbb{R}, \ \Delta_1 \phi(z) \neq 0\},$$

which has measure zero. Thus the set $\mathcal{A} + \mathbb{Z}$ has also measure zero. Hence for a fixed $a \in \mathbb{R} \backslash (\mathcal{A} + \mathbb{Z}) \neq \emptyset$, we have

$$\Delta_1 \phi(a+k) = 0 \quad \text{for every } k \in \mathbb{Z}.$$

This implies that there exists $\lambda, b \in \mathbb{R}$ (that may depend on a) such that

$$\phi(a+k) = \lambda k + b.$$

But ϕ is bounded, then $\lambda = 0$ and hence $\phi(a+k) = b$, which is a contradiction since $\phi(+\infty) \neq \phi(-\infty)$. Therefore $c^+ > 0$.

Similarly, for $\sigma = \sigma^- = -\beta$, we show that $c^- < 0$, since $f - 2\beta < 0$ on $\left(-\frac{1}{2}, \frac{1}{2}\right)$. $\qquad \square$

8.2 Instability of critical velocity

In this section, we show that the critical velocity c^+ given in Theorem 1.1 is unstable in the sense of Proposition 1.2, which we prove in this section.

Before proving Proposition 1.2, we give an example of a non-linearity F for which the associated critical velocity is negative. This example will be the proof of Proposition 1.4.

Proof of Proposition 1.4

The aim is to construct a function F satisfying (A_{Lip}) and (P_{C^1}) such that the associated critical velocity satisfies $c^+ < 0$. To this end, we will construct a function $f \in \mathrm{Lip}([0,1])$, which is linear in a neighborhood of zero with $f'(0) > 0$, such that there exists a couple (c, ϕ) with $c < 0$ solution of

$$(8.10) \qquad \begin{cases} c\phi'(x) = \phi(x-1) - \phi(x) + f(\phi(x)) \quad \text{on} \quad \mathbb{R} \\ \phi' \geq 0 \\ \phi(-\infty) = 0 \quad \text{and} \quad \phi(+\infty) = 1. \end{cases}$$

Let $c = -\mu$ with $0 < \mu < 1$ and

$$\phi(x) = \begin{cases} \dfrac{1}{2} e^{\gamma x} & \text{on} \quad (-\infty, 0] \\[2mm] 1 - \dfrac{1}{2} e^{-\gamma x} & \text{on} \quad [0, +\infty) \end{cases}$$

with $\gamma > 0$. We claim that $\phi \in C^1(\mathbb{R})$ and $(-\mu, \phi)$ solves

$$(8.11) \qquad \begin{cases} 0 < \phi(x) - \phi(x-1) - \mu\phi'(x) \quad \text{on} \quad \mathbb{R} \\ \phi' > 0 \\ \phi(-\infty) = 0 \quad \text{and} \quad \phi(+\infty) = 1, \end{cases}$$

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which is possible to check for $0 < \gamma << 1$.

Therefore, it is sufficient to define the function f as

(8.12) $$f(\phi(x)) := \phi(x) - \phi(x-1) - \mu\phi'(x) > 0 \quad \text{for all } x \in \mathbb{R}.$$

Notice that, when $x \to +\infty$, $\phi(+\infty) = 1$ and $\phi'(x) \to 0$, thus $f(1) = 0$. Similarly, we have $f(0) = 0$. Moreover, since $\phi \in C^{1,1}(\mathbb{R})$, we have that $f \in \text{Lip}((0,1))$. In fact, by a direct tedious calculation, one can deduce that

$$f(x) = \begin{cases} (1 - e^{-\gamma} - \mu\gamma)x & \text{for } x \in \left[0, \dfrac{1}{2}\right] \\[2mm] 1 + (1 + \mu\gamma)(x-1) + \dfrac{e^{-\gamma}}{4(x-1)} & \text{for } x \in \left[\dfrac{1}{2}, 1 - \dfrac{1}{2}e^{-\gamma}\right] \\[2mm] (1 - e^{\gamma} + \mu\gamma)(x-1) & \text{for } x \in \left[1 - \dfrac{1}{2}e^{-\gamma}, 1\right], \end{cases}$$

and this implies that $f \in \text{Lip}([0,1])$ and $1 > f'(0) > 0$. We can even check that f is concave and C^1 except at the point $x = \frac{1}{2}$, where it is neither concave nor C^1.

Remark that to get more regular non-linearities, one can consider

(8.13) $$f_\varepsilon(x) := \Big((\phi(\cdot) - \phi(\cdot - 1) - \mu\phi'(\cdot)) \star \rho_\varepsilon\Big)(x),$$

where ρ_ε satisfies $\rho_\varepsilon \geq 0$, $\rho_\varepsilon(x) = \frac{1}{\varepsilon}\rho(\frac{x}{\varepsilon})$ (ρ is a mollifier) and $supp\,\rho_\varepsilon \subset B_\varepsilon(0)$. However, in this case, $\rho_\varepsilon \star \phi$ is a solution of (8.10), with f replaced by f_ε, and then $f_\varepsilon \in C^\infty([0,1])$ with $f'_\varepsilon(0) > 0$. $\qquad\square$

Now, we give the proof of the instability result, namely Proposition 1.2.

Proof of Proposition 1.2

We have seen, in Proposition 1.4, that there exists a function F satisfying (A_{Lip}) and (P_{C^1}) such that the associated critical velocity $c_F^+ := c^+$ satisfies

(8.14) $$c_F^+ < 0.$$

Our goal is to build a sequence of functions F_δ with a critical velocity $c_{F_\delta}^+$ such that

$$F_\delta \to F \quad \text{in} \quad L^\infty([0,1]^{N+1})$$

as $\delta \to 0$, and prove that

(8.15) $$\liminf_{\delta \to 0} c_{F_\delta}^+ > c_F^+.$$

Step 1: construction of F_δ

Define for $X = (X_0, ..., X_N) \in [0,1]$ and $\delta > 0$ small the function

(8.16) $$F_\delta(X) = F(X) - f(X_0) - f_\delta(X_0),$$

where

(8.17) $$f_\delta(v) = \begin{cases} \max\big(f(\delta) + L_0(v - \delta), 0\big) & \text{on} \quad [0, \delta] \\ f & \text{on} \quad [\delta, 1], \end{cases}$$

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with a constant $L_0 > 0$ satisfying $L_0 > 2\text{Lip}(F) =: 2L_F^\infty$.

By construction of f_δ, we clearly have

$$\|F_\delta - F\|_{L^\infty} \;=\; \|f - f_\delta\|_{L^\infty} \to 0 \quad \text{as} \quad \delta \to 0.$$

Step 2: existence of $c_{F_\delta}^+$
Set

$$0_\delta = \delta - \frac{f(\delta)}{L_0} > 0,$$

(where 0_δ was denoted by b_δ in the proof of Theorem 1.1).

Since F_δ satisfies (A_{Lip}) and (P_{Lip}) with $[0,1]^{N+1}$ replaced by $[0_\delta, 1]^{N+1}$, then applying the result of Theorem 1.1, we deduce that there exists a minimal velocity $c_{F_\delta}^+$ and a profile ϕ solution of

(8.18)
$$\begin{cases} c_{F_\delta}^+ \phi'(z) = F_\delta(\phi(z+r_0), \phi(z+r_1), ..., \phi(z+r_N)) & \text{on} \quad \mathbb{R} \\ \phi \text{ is non-decreasing over } \mathbb{R} \\ \phi(-\infty) = 0_\delta \quad \text{and} \quad \phi(+\infty) = 1. \end{cases}$$

Step 3: establishing (8.15)
Our aim is to show that $c_{F_\delta}^+ \geq 0$. Since F_δ is non-decreasing w.r.t X_i for all $i \neq 0$, then for $X = (X_0, X') \in [0_\delta, 1]^{N+1}$, we have

$$F_\delta(X_0, X') \geq F_\delta(X_0, 0_\delta, ..., 0_\delta) := A(X_0).$$

Moreover, for $X_0 \; X_0 + h \in [0_\delta, \delta]$ with $h > 0$, we have

$$\begin{aligned} A(X_0 + h) - A(X_0) &= F(X_0 + h, 0_\delta + h, ..., 0_\delta + h) - F(X_0, 0_\delta, ..., 0_\delta) - f(X_0 + h) \\ &\quad + f(X_0) + f_\delta(X_0 + h) - f_\delta(X_0) \\ &\geq -2hL_F^\infty + hL_0 \\ &= h(L_0 - 2L_F^\infty) > 0, \end{aligned}$$

where we have used that F is L_F^∞-Lipschitz (in the second line) and that $L_0 > 2L_F^\infty$ in the last inequality. This implies that A is increasing over $[0_\delta, \delta]$, but $A(0_\delta) = F_\delta(0_\delta, 0_\delta, ..., 0_\delta) = 0$. Hence, we get $A \geq 0$ over $[0_\delta, \delta]$.

Therefore, we deduce that

$$F_\delta \geq 0 \quad \text{over} \quad [0_\delta, \delta] \times [0_\delta, 1]^N.$$

Now since $\phi(-\infty) = 0_\delta$, then for $z << -1$ very negative, we get that $\phi(z+r_0) = \phi(z) \in [0_\delta, \delta]$. Hence, for all $\phi(z) \in [0_\delta, \delta]$, we obtain from (8.18) that

$$c_{F_\delta}^+ \phi'(z) = F_\delta(\phi(z+r_0), \phi(z+r_1), ..., \phi(z+r_N)) \geq 0,$$

but $\phi' \geq 0$, thus we deduce that

$$c_{F_\delta}^+ \geq 0.$$

This implies that (because of (8.14))

$$\liminf_{\delta \to 0} c_{F_\delta}^+ \geq 0 > c_F^+.$$

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Step 4**: conclusion**
Let

$$(8.19) \qquad \widehat{\phi}(x) = \frac{\phi(x) - 0_\delta}{1 - 0_\delta}, \quad c^+_{\widehat{F}_\delta} = (1 - 0_\delta)c^+_{F_\delta}$$

and

$$\widehat{F}_\delta((X_i)_{i=0,\dots,N}) = F_\delta(((1 - 0_\delta)X_i + 0_\delta)_{i=0,\dots,N}).$$

Then we have

$$(8.20) \qquad \begin{cases} c^+_{\widehat{F}_\delta} \widehat{\phi}'(z) = \widehat{F}_\delta((\widehat{\phi}(z + r_i))_{i=0,\dots,N}) & \text{on} \quad \mathbb{R} \\ \widehat{\phi} \text{ is non-decreasing over } \mathbb{R} \\ \widehat{\phi}(-\infty) = 0 \quad \text{and} \quad \widehat{\phi}(+\infty) = 1 \end{cases}$$

and $c^+_{\widehat{F}_\delta}$ is the critical velocity associated to $\widehat{F}_\delta$ which is defined on $[0,1]^{N+1}$. Moreover, we still have $|\widehat{F}_\delta - F| \to 0$ as $\delta \to 0$ and $\widehat{F}_\delta$ satisfies (A_{Lip}) and (P_{Lip}) on $[0,1]^{N+1}$. In addition, since $0_\delta \to 0$ as $\delta \to 0$, then from (8.19) we still have

$$\liminf_{\delta \to 0} c^+_{\widehat{F}_\delta} = \liminf_{\delta \to 0} c^+_{F_\delta} \geq 0 > c^+_F.$$

Therefore, up to rename $\widehat{F}_\delta$ as F_δ, this ends the proof of Proposition 1.2. $\qquad \square$

9 Appendix: useful results

In this appendix, we present some techniques to pass to the limit in the equation, in a first subsection. In a second subsection, we prove some results that we use to prove that the critical velocity c^+ is non-negative.

9.1 Results for passing to the limit

The main result of this subsection (Theorem 9.3) identifies the limits of a constructed profile. We also prove some results to pass to the limit in the equation, namely Lemma 9.1.

We start by introducing the following bistable notation:

Assumption $(\tilde{B}_{m,b})$:
Let $f(x) := F(x, \dots, x)$ and $m < b < m + 1$,

Bistability: $f(m) = 0 = f(b) = f(m+1)$, $f < 0$ on (m, b) and $f > 0$ on $(b, m+1)$.

Lemma 9.1 (Passing to the limit)
Consider a sequence of functions F_n satisfying $(\tilde{A}_{Lip})$ and $(\tilde{B}_{m_n,b_n})$ (with $m_n \in [0,1)$) such that

$$(9.1) \qquad Lip(F_n) \leq C \quad \text{independent on } n.$$

Let (c_n, ϕ_n) be a solution of

(9.2)
$$\begin{cases} c_n \phi'_n(z) = F_n((\phi_n(z + r_i))_{i=0,...,N}) \quad over \quad \mathbb{R} \\ \phi_n \text{ is non-decreasing on } \mathbb{R} \\ \phi_n(-\infty) = m_n \quad and \quad \phi_n(+\infty) = m_n + 1. \end{cases}$$

Assume that

(9.3)
$$|\phi_n| \leq M \quad for \; some \; M > 0 \; independent \; of \; n.$$

Assume moreover that there exists a real number c such that $c_n \to c$; and that $F_n \to F$ locally uniformly and $(m_n, b_n) \to (m, b)$ as $n \to +\infty$. Then, up to a subsequence, ϕ_n converges almost everywhere to some ϕ that solves in the viscosity sense

(9.4)
$$\begin{cases} c\phi'(z) = F((\phi(z + r_i))_{i=0,...,N}) \quad over \quad \mathbb{R} \\ \phi \text{ is non-decreasing on } \mathbb{R} \\ m \leq \phi(-\infty) \quad and \quad \phi(+\infty) \leq m + 1. \end{cases}$$

Moreover, either ϕ satisfies

$$m = \phi(-\infty) \quad and \quad \phi(+\infty) = m + 1$$

or there exists two solutions ϕ^a and ϕ^b such that

$$m = \phi^a(-\infty) \quad and \quad \phi^a(+\infty) = b$$

and

$$b = \phi^b(-\infty) \quad and \quad \phi^b(+\infty) = m + 1.$$

Proof of Lemma 9.1

Step 1: passing to the limit
The proof of this result follows from [1, Proposition 2.3, Step 2]. For the convenience of the reader, we give the proof here.

Because of (9.1) and since ϕ_n is bounded, we deduce that there exists a constant $M_0 > 0$ independent of n such that

(9.5)
$$|F_n((\phi_n(z + r_i))_{i=0,...,N})| \leq M_0 \quad \text{independent on } n.$$

Case 1: $c \neq 0$
Since $|c_n| \geq \frac{|c|}{2}$ for n large, then

$$|\phi'_n| \leq \frac{2M_0}{c} \quad \text{for large } n.$$

Thus ϕ_n is uniformly Lipschitz. Using Ascoli's Theorem and the diagonal extraction argument, we get that ϕ_n converges to ϕ (up to a subsequence) locally uniformly on $\mathbb{R}$. Moreover ϕ is non-decreasing and satisfies (by stability of viscosity solutions)

(9.6)
$$c\phi'(x) = F((\phi(x + r_i))_{i=0,...,N}).$$

We easily deduce (9.4).

Case 2: $c = 0$

Since ϕ_n is monotone and bounded (uniformly in n), then using Helly's Lemma (Lemma 2.4) and the diagonal extraction argument, ϕ_n converges (up to a subsequence) to a non-decreasing ϕ a.e. Our goal is to show that

$$(9.7) \qquad 0 = F((\phi(x + r_i))_{i=0,\ldots,N}).$$

Subcase 2.1: $c_n = 0$ **for all** n

We first use the equivalence between viscosity solutions and almost everywhere solutions (Lemma 2.5) and then pass to the limit in (9.2) using Helly's lemma (Lemma 2.4). Hence, we get a solution ϕ of (9.7) almost everywhere. Again, we use Lemma 2.5 to conclude that ϕ is a viscosity solution of (9.7) and satisfies (9.4).

Subcase 2.2: $c_n \neq 0$ **for all** n

We have

$$c_n \int_{b_1}^{b_2} (\phi_n)'(z)dz = \int_{b_1}^{b_2} \left(F_n((\phi_n(z + r_i))_{i=0,\ldots,N})\right)dz$$

for every $b_1 < b_2$. That is,

$$c_n(\phi_n(b_2) - \phi_n(b_1)) = \int_{b_1}^{b_2} \left(F_n((\phi_n(z + r_i))_{i=0,\ldots,N})\right)dz.$$

But, we have (9.5) and

$$F_n((\phi_n(z + r_i))_{i=0,\ldots,N}) \to F((\phi(z + r_i))_{i=0,\ldots,N}) \quad \text{a.e.}$$

Thus, using Lebesgue's dominated convergence theorem, we pass to the limit $n \to +\infty$ and get

$$0 = \int_{b_1}^{b_2} F((\phi(z + r_i))_{i=0,\ldots,N})dz$$

which implies (since b_1 and b_2 are arbitrary) that

$$0 = F((\phi(z + r_i))_{i=0,\ldots,N}) \quad \text{a.e.}$$

Then by Lemma 2.5, ϕ verifies (9.7) in the viscosity sense and satisfies (9.4).

Step 2: limits of the profile

Since $\phi(\pm\infty)$ solves $f = 0$, then $\phi(\pm\infty) \in \{m, b, m + 1\}$. Therefore, either ϕ satisfies

$$m = \phi(-\infty) \quad \text{and} \quad \phi(+\infty) = m + 1$$

or there exists two solutions ϕ^a and ϕ^b such that ϕ^a satisfies

$$m = \phi^a(-\infty) \quad \text{and} \quad \phi^a(+\infty) = b$$

and

$$(9.8) \qquad \begin{cases} (\phi^a)_*(0) \leq \dfrac{m + b}{2} \\ (\phi^a)^*(0) \geq \dfrac{m + b}{2} \end{cases}$$

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and ϕ^b satisfies

$$b = \phi^b(-\infty) \quad \text{and} \quad \phi^b(+\infty) = m + 1$$

and

(9.9)
$$\begin{cases} (\phi^b)_*(0) \le \dfrac{m+1+b}{2} \\ (\phi^b)^*(0) \ge \dfrac{m+1+b}{2} \end{cases}$$

Solutions ϕ^a and ϕ^b can be obtained as limits of $\phi_n^a(x) = \phi_n(x + a_n)$ and $\phi_n^b(x) = \phi_n(x + b_n)$ for suitable shifts a_n, b_n such that ϕ_n^a and ϕ_n^b satisfies resp. (9.8) and (9.9). $\qquad\square$

We recall now the existence result of traveling waves whose a slightly different statement is given in [1, Proposition 2.3]. In order to present the main result of this section, we need to introduce the following technical lemma.

Lemma 9.2 (Controlling the finite difference)
Consider F satisfying $(\tilde{A}_{C^1})$, $\sigma_0 \in (\sigma^-, \sigma^+)$ fixed and $\beta > 0$. Let $a > r^*$ (r^* is given by (1.20)) and $M_0 > 0$, then for all $\sigma \in [\sigma_0 - \beta, \sigma_0 + \beta] \subset (\sigma^-, \sigma^+)$ and for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all function ϕ (viscosity) solution of

$$\begin{cases} c\phi'(x) = F((\phi(x + r_i))_{i=0,\dots,N}) + \sigma \quad \text{on} \quad \mathbb{R} \\ \phi' \ge 0 \\ \phi(x+1) \le \phi(x) + 1 \\ |c| \le M_0 \\ |c\phi'| \le M_0, \end{cases}$$

and for all $x_0 \in \mathbb{R}$ satisfying

$$\phi_*(x_0 + a) - \phi^*(x_0 - a) \le \delta,$$

we have

$$\text{dist}\big(\alpha, \{m_\sigma, b_\sigma\} + \mathbb{Z}\big) < \varepsilon \quad \text{for all} \quad \alpha \in [\phi_*(x_0), \phi^*(x_0)].$$

Proof of Lemma 9.2
The proof of this lemma follows from a straightforward generalization of [1, Proposition 3.2] for the function F replaced by $F + \sigma$ and $(0, b)$ replaced by (m_σ, b_σ) for $\sigma \in [\sigma_0 - \beta, \sigma_0 + \beta] \subset (\sigma^-, \sigma^+)$ and for some $\beta > 0$. We similarly show that for every $\varepsilon > 0$ there exists $\delta_\sigma(\varepsilon) > 0$ such that the result holds true.

However, we can show that $\delta_\sigma(\varepsilon) = \delta(\varepsilon)$ can be chosen independent of σ and the proof of this generalization follows exactly the same lines. Indeed, we proceed by contradiction assuming that the statement is false for a sequence $\sigma_n \in [\sigma_0 - \beta, \sigma_0 + \beta]$, and consider a sequence of solutions ϕ^n. The presence of σ_n does not create any additional difficulty in the passage to the limit in the equation. $\qquad\square$

Theorem 9.3 (Identification of the limits of the profile)
We work under the assumptions of Lemma 9.1 with $F_n = F + \sigma_n$, $m_n = m_{\sigma_n}$, $b_n = b_{\sigma_n}$ and F satisfying $(\tilde{A}_{C^1})$ and $(\tilde{B}_{C^1})$. We assume moreover that the solution (c_n, ϕ_n) of (9.2) is

given by Proposition 4.1 for $\sigma_n \in (\sigma^-, \sigma^+)$. Let (c_∞, ϕ_∞) be the solution of (9.4) constructed in Lemma 9.1. If $\sigma_\infty \in (\sigma^-, \sigma^+)$, then we have moreover

$$\phi_\infty(-\infty) = m_{\sigma_\infty} \quad and \quad \phi_\infty(+\infty) = m_{\sigma_\infty} + 1.$$

Proof of Theorem 9.3

Let (c_n, ϕ_n) be a solution of (9.2) given by Proposition 4.1 and (c_∞, ϕ_∞) be a solution of (9.4) for $\sigma_\infty \in (\sigma^-, \sigma^+)$, obtained by passing to the limit $n \to \infty$. Our aim is to show that

$$\phi_\infty(-\infty) = m_{\sigma_\infty} \quad \text{and} \quad \phi_\infty(+\infty) = m_{\sigma_\infty} + 1.$$

For $\varepsilon > 0$ small enough ($\varepsilon < \frac{1}{2} \min(b_{\sigma_n} - m_{\sigma_n}, m_{\sigma_n} + 1 - b_{\sigma_n})$), let $z_n, \ y_n \in \mathbb{R}$ such that

(9.10)
$$\begin{cases} (\phi_n)^*(z_n) \geq b_{\sigma_n} + \varepsilon \\ (\phi_n)_*(z_n) \leq b_{\sigma_n} + \varepsilon \end{cases}$$

and

(9.11)
$$\begin{cases} (\phi_n)^*(y_n) \geq b_{\sigma_n} - \varepsilon \\ (\phi_n)_*(y_n) \leq b_{\sigma_n} - \varepsilon. \end{cases}$$

Assume moreover that up to translate ϕ_n, we have

$$\begin{cases} (\phi_n)_*(0) \leq b_{\sigma_n} \\ (\phi_n)^*(0) \geq b_{\sigma_n}. \end{cases}$$

For every $x \in \mathbb{R}$, set with $a > r^*$

$$\psi_n(x) := (\phi_n)_*(x + a) - (\phi_n)^*(x - a) \geq 0$$

and denote by

$$\overline{m}_n = \min_{[y_n, z_n]} \psi_n(x) = \psi_n(x_n) \geq 0,$$

for some $x_n \in [y_n, z_n]$ since ψ_n is lower semi-continuous.

We claim that $\overline{m}_n > 0$. Indeed, if $\overline{m}_n = 0$, then since $\psi_n(y_n), \ \psi_n(z_n) \geq \delta(\varepsilon) > 0$ (because of (9.10), (9.11) and using Lemma 9.2), we get

$$x_n \in (y_n, z_n).$$

Moreover, we have that

$$0 = \psi_n(x_n) = (\phi_n)_*(x_n + a) - (\phi_n)^*(x_n - a)$$

and ϕ_n is non-decreasing, hence

$$\phi_n = const \quad \text{over} \quad (x_n - a, x_n + a),$$

and ϕ_n solves $f + \sigma_n = 0$.
Now, since

$$b_{\sigma_n} - \varepsilon \leq (\phi_n)^*(y_n) \leq \phi_n(x_n) \leq (\phi_n)_*(z_n) \leq b_{\sigma_n} + \varepsilon,$$

we get that
$$\phi_n = b_{\sigma_n} \quad \text{over} \quad (x_n - a, x_n + a).$$

Therefore, for $r^* < \overline{a} < a$, we have

$$\phi_n = b_{\sigma_n} \quad \text{over} \quad [x_n - \overline{a}, x_n + \overline{a}],$$

which is in contradiction with Proposition 4.1. Therefore, $\overline{m}_n > 0$ and the proof of the identification of limits of the profile proceeds similarly as in [1, Proposition 2.3], where now Step 5 is no longer necessary. In particular we avoid the case $\phi(\pm\infty) = b_{\sigma_\infty}$. $\qquad\square$

9.2   Useful results used for the proof of $c^+ \geq 0$

This subsection is dedicated for the useful tools that we use to prove that the critical velocity is non-negative, i.e $c^+ \geq 0$.

Proposition 9.4 (Extension by antisymmetry)
Let F be a function defined over $Q = [0,1]^{N+1}$ satisfying (A_{Lip}) and such that $F(0,...,0) = 0$. Then there exists an antisymmetric extension G defined over $[-1,1]^{N+1}$ such that

$$\begin{cases} G_{|Q} = F \\ G(-X) = -G(X) \end{cases}$$

and G satisfies (A_{Lip}) over $[-1,1]^{N+1}$.
Moreover, if F is C^1 over a neigborhood of $\{0\}^{N+1}$ in $[0,1]^{N+1}$ and $f'(0) > 0$ ($f(v) := F(v,...,v)$), then there exists $\eta > 0$ such that for every $a > 0$ small and $X = (X_0,...,X_N) \in [-1,1]^{N+1}$ such that X, $X + (a,...,a)$ are close enough to $\{0\}^{N+1}$, we have

$$(9.12) \qquad\qquad G(X + (a,...,a)) - G(X) \geq \eta a.$$

Remark 9.5 (Reflection)
Note that if F is invariant by reflection symmetry, then it is possible to show that G also; precisely, we mean that if $F(\overline{X}) = F(X)$ for $\overline{X}_i = X_{\overline{i}}$ with $r_{\overline{i}} = -r_i$, then

$$G(\overline{X}) = G(X).$$

We recall before proving Proposition 9.4 the following properties of the orthogonal projection which can be easily shown:

Lemma 9.6 (Some properties of orthogonal projection)
Let $X = (X_i)_{i=0,...,N} \in [-1,1]^{N+1}$ and call $Proj_{|Q}(X)$ the orthogonal projection of X on $Q = [0,1]^{N+1}$. Then
$$Proj_{|Q}(X) = (Proj_{|[0,1]}(X_i))_{i=0,...,N}.$$

Moreover, we have
i) Order preservation
Let $Y = (Y_i)_{i=0,...,N} \in [-1,1]^{N+1}$ and assume that $X \geq Y$ in sense that $X_i \geq Y_i$ for all $i \in \{0,...,N\}$, then
$$Proj_{|Q}(X) \geq Proj_{|Q}(Y).$$

ii) "Antisymmetry"
Let $Q' = [-1, 0]^{N+1} = -Q$, then

$$Proj_{|Q'}(-X) = -Proj_{|Q}(X).$$

Proof of Proposition 9.4
Let $X = (X_i)_{i=0,...,N} \in [-1, 1]^{N+1}$, then define the extension function G by:

$$(9.13) \qquad G(X) = F(Proj_{|Q}(X)) - F(-Proj_{|Q'}(X)),$$

where we recall that $Q' = [-1, 0]^{N+1}$. For $X \in Q$, we clearly have $G(X) = F(X)$.

Step 1: $G(-X) = -G(X)$
We have

$$
\begin{aligned}
G(-X) &= F(Proj_{|Q}(-X)) - F(-Proj_{|Q'}(-X)) \\
&= F(-Proj_{|Q'}(X)) - F(Proj_{|Q}(X)) \\
&= -G(X),
\end{aligned}
$$

where we have used in the second line the antisymmetry in Lemma 9.6.

Step 2: G **satisfies** $(A_{\mathbf{Lip}})$
Since F is globally Lipschitz and the orthogonal projection is 1-Lipschitz, then G is globally Lipschitz on $[-1, 1]^{N+1}$.
We now prove that G is non-decreasing w.r.t X_i for all $i \neq 0$. Let $X = (X_i)_{i=0,...,N}$, $Y = (Y_i)_{i=0,...,N} \in [-1, 1]^{N+1}$ such that

$$
\begin{cases}
X_i \geq Y_i & \text{for all } i \in \{1, ..., N\} \\
X_0 = Y_0,
\end{cases}
$$

and let us show that $G(X) \geq G(Y)$. In fact, since the orthogonal projection preserve the ordering (see Lemma 9.6) and since F is non-decreasing w.r.t X_i for all $i \in \{1, ..., N\}$, we conclude that G is non-decreasing w.r.t X_i for all $i \in \{1, ..., N\}$ over $[-1, 1]^{N+1}$.

Step 3: checking (9.12)
We first give some notations for the projection function. Consider $X = (X_0, ..., X_N) \in [-1, 1]^{N+1}$, then from Lemma 9.6, we have

$$Proj_{|Q}(X) = (Proj_{|[0,1]}(X_i))_{i=0,...,N} = \left(\begin{cases} X_i & \text{if } X_i \geq 0 \\ 0 & \text{if } X_i \leq 0 \end{cases} \right)_{i=0,...N} =: X^+.$$

Similarly, we have (with $Q' = -Q$)

$$Proj_{|Q'}(X) = \left(\begin{cases} 0 & \text{if } X_i \geq 0 \\ X_i & \text{if } X_i \leq 0 \end{cases} \right)_{i=0,...N} =: X^-$$

We also define

$$Q_\Sigma = \left\{ X = (X_0, ..., X_N) \in [-1, 1]^{N+1}, \quad \sigma_i X_i \in [0, 1] \quad \text{for} \quad i = 0, ..., N \right\},$$

where $\Sigma = (\sigma_0, ..., \sigma_N)$ and $\sigma_i = \pm 1$.

Now, we go back to the proof of (9.12) which is splitted in two cases. Let X, $X + (a, ..., a)$ close to $\{0\}^{N+1}$ with $a > 0$ small:

Case 1: X, $X + (a, ..., a) \in Q_\Sigma$

From the definition of G (see (9.13)) and the notations introduced at the beginning of this step, we have

$$G(X + (a, ..., a)) - G(X) = F((X + aE)^+) - F(X^+) - \Big(F(-(X + aE)^-) - F(-X^-) \Big),$$

where $E = (1, ..., 1)$. Thus, we get

$$G(X + (a, ..., a)) - G(X) = a\Theta.\nabla F(X^+) + o(|a\Theta|) + a\Gamma.\nabla F(-X^-) + o(|a\Gamma|),$$

where $a\Theta = (X + aE)^+ - X^+$ with $\Theta = (\theta_i)_{i=0,...,N}$, where

$$\theta_i = \begin{cases} 1 & \text{if} \quad \sigma_i = 1 \\ 0 & \text{if} \quad \sigma_i = -1 \end{cases}$$

and $a\Gamma = (X + aE)^- - X^-$ with $\Gamma = (\gamma_i)_{i=0,...,N}$, where

$$\gamma_i = \begin{cases} 0 & \text{if} \quad \sigma_i = 1 \\ 1 & \text{if} \quad \sigma_i = -1. \end{cases}$$

Hence, we obtain

$$\begin{aligned} G(X + (a, ..., a)) - G(X) \quad &= \quad a(\Theta + \Gamma).\nabla F(0) + a\Theta.(\nabla F(X^+) - \nabla F(0)) + o(a) \\ &+ \quad a\Gamma.(\nabla F(-X^-) - \nabla F(0)), \end{aligned}$$

but $a(\Theta + \Gamma) = (a, ..., a)$, therefore,

$$\begin{aligned} G(X + (a, ..., a)) - G(X) \quad &= \quad a\Big\{ f'(0) + \Theta.(\nabla F(X^+) - \nabla F(0)) + o(1) \\ &+ \quad \Gamma.(\nabla F(-X^-) - \nabla F(0)) \Big\}. \end{aligned}$$

Now, since F is C^1 over a neighborhood of X (X close to $\{0\}^{N+1}$), then we get

$$G(X + (a, ..., a)) - G(X) = a \left\{ f'(0) + o(X^+) + o(X^-) + o(1) \right\} \geq a\frac{f'(0)}{2} > 0$$

for X close enough to $\{0\}^{N+1}$.

Case 2: $X \in Q_\Sigma$ and $X + aE \in Q_{\widehat{\Sigma}}$

There exists an integer $p \geq 1$ such that

$$G(X + aE) - G(X) = \sum_{k=0}^{p} \Big(G(X + t_k E) - G(X + t_{k-1} E) \Big),$$

where $0 = t_0 < t_1 < ... < t_p = a$ such that for $k = 1, ..., p$, we have $X + [t_{k-1}, t_k]E \in Q_{\Sigma_k}$, with $\Sigma = \Sigma_0$ and $\widehat{\Sigma} = \Sigma_p$. Therefore, using Case 1 for each segment, we deduce that

$$G(X + aE) - G(X) \geq \eta a,$$

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with $\eta = \frac{f'(0)}{2} > 0$.

$\square$

We now introduce an extension by antisymmetry-reflection of F :

Proposition 9.7 (Extension by antisymmetry-reflection)
Let F be a function defined on $Q = [0,1]^{N+1}$ satisfying (A_{Lip}) and such that $F(0,...,0) = 0$. Let $X = (X_i)_{i=0,...,N} \in [0,1]^{N+1}$ and assume that

(9.14) $\qquad$ *for all $i \in \{1,...,N\}$ there exists $\bar{i} \in \{1,...,N\}$ such that $r_{\bar{i}} = -r_i$.*

Then there exists a function $\overline{G}$ defined on $[-1,1]^{N+1}$ which satisfies (A_{Lip}) on $[-1,1]^{N+1}$ such that

$$\begin{cases} \overline{G}_{|Q} = F \\ \overline{G}(-X) = -\overline{G}(X) \quad \text{(antisymmetric-reflection)}, \end{cases}$$

where we recall that $\overline{X}_i = X_{\bar{i}}$ with $r_{\bar{i}} = -r_i$.
$\qquad$ *Moreover, if F is C^1 over a neighborhood of $\{0\}^{N+1}$ and*

(9.15) $$\frac{\partial F}{\partial X_0}(0) + \sum_{i=1}^{N} \min\left(\frac{\partial F}{\partial X_i}(0), \frac{\partial F}{\partial X_{\bar{i}}}(0) \right) > 0,$$

then there exists $\eta > 0$ such that for every $a > 0$ small and $X = (X_0,...,X_N) \in [-1,1]^{N+1}$ such that X, $X + (a,...,a)$ are close enough to $\{0\}^{N+1}$, we have

(9.16) $$\overline{G}(X + (a,...,a)) - \overline{G}(X) \geq \eta a.$$

Remark 9.8 (On the reflection condition (9.14))
Notice that we can always assume that the reflection condition (9.14) *is satisfied up to modify the function F. Indeed, if F does not satisfy the reflection condition* (9.14)*, i.e. we have*

$$\{i_1,...,i_M\} = \{i \in \{1,...,N\}, \text{ such that } -r_i \notin \{r_j\}_{j=1,...,N}\}$$

with $M \geq 1$, then let us define

$$r_{N+j} = -r_{i_j} \quad \text{for} \quad j = 1,...,M.$$

Therefore, for each $i \in \{1,...,N+M\}$ there exists $\bar{i} \in \{1,...,N+M\}$ such that $r_{\bar{i}} = -r_i$. Now, for $\tilde{X} = (X, X')$ with $X' = (X_{N+1},...,X_{N+M})$, set

$$\tilde{F}(\tilde{X}) = F(X).$$

Thus $\tilde{F}$ satisfies (9.14) *with N replaced by $\tilde{N} = N + M$ and if moreover ϕ solves*

$$c\phi'(z) = F((\phi(z + r_i))_{i=0,...,N}),$$

then it solves $c\phi'(z) = \tilde{F}((\phi(z + r_i))_{i=0,...,\tilde{N}})$.
$\qquad$ *In addition, if F is C^1 in a neighborhood of $\{0\}^{N+1}$, then $\tilde{F}$ is C^1 in a neighborhood of $\{0\}^{\tilde{N}+1}$, and*

$$\frac{\partial F}{\partial X_0}(0) + \sum_{i \in I} \min\left(\frac{\partial F}{\partial X_i}(0), \frac{\partial F}{\partial X_{\bar{i}}}(0) \right) = \frac{\partial \tilde{F}}{\partial X_0}(0) + \sum_{i=1}^{\tilde{N}} \min\left(\frac{\partial \tilde{F}}{\partial X_i}(0), \frac{\partial \tilde{F}}{\partial X_{\bar{i}}}(0) \right),$$

with $I = \{i \in \{1,...,N\} \text{ such that there exists } \bar{i} \in \{1,...,N\} \text{ with } r_i = -r_{\bar{i}}\}$.

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Proof of Proposition 9.7
The proof is very similar to the proof of Proposition 9.4, so we give only a few details. Let $X \in [-1, 1]^{N+1}$, then define the extension function $\overline{G}$ by

$$(9.17) \qquad \overline{G}(X) = F(Proj_{|Q}(X)) - F(-Proj_{|Q'}(\overline{X})),$$

where we recall that $\overline{X}_i = X_{\overline{i}}$ with $r_{\overline{i}} = -r_i$.

Step 1: $\overline{G}(-\overline{X}) = -\overline{G}(X)$
We have

$$
\begin{aligned}
\overline{G}(-\overline{X}) &= F(Proj_{|Q}(-\overline{X})) - F(-Proj_{|Q'}(\overline{-\overline{X}})) \\
&= F(-Proj_{|Q'}(\overline{X})) - F(-Proj_{|Q'}(-X)) \quad \text{(using Lemma 9.6 } ii) \text{ and } \overline{-\overline{X}} = -\overline{X}) \\
&= F(-Proj_{|Q'}(\overline{X})) - F(Proj_{|Q}(X)) \quad \text{(using again Lemma 9.6 } ii)) \\
&= -\overline{G}(X).
\end{aligned}
$$

Step 2: $\overline{G}$ **satisfies** $(A_{\mathbf{Lip}})$ **on** $[-1, 1]^{N+1}$
This step is an analogous of Step 2 in the proof of Proposition 9.4.

Step 3: checking (9.16)
We have
$$\overline{G}(X) = F(X^+) - F(-(\overline{X})^-)$$

Let $\Sigma = (\sigma_0, \sigma_1, ..., \sigma_N)$ and define $\overline{\Sigma} = (\overline{\sigma}_0, \overline{\sigma}_1, ..., \overline{\sigma}_N)$ such that $\overline{\sigma}_i = \sigma_{\overline{i}}$ for all $i = 0, ..., N$; and then recall

$$Q_\Sigma = \left\{ X = (X_0, ..., X_N) \in [-1, 1]^{N+1} \quad \text{such that} \quad \sigma_i X_i \in [0, 1] \quad \text{for} \quad i = 0, ..., N \right\}.$$

We have
$$X \in Q_\Sigma \Longleftrightarrow \overline{X} \in Q_{\overline{\Sigma}}.$$

Let X, $X + aE$ close enough to $\{0\}^{N+1}$ with $a > 0$ small and $E = (1, ..., 1)$.

Case 1: X, $X + aE \in Q_\Sigma$
Since F is C^1 over a neighborhood of $\{0\}^{N+1}$, then (as in the proof of Proposition 9.4, Step 3) we have

$$
\begin{aligned}
\overline{G}(X + aE) - \overline{G}(X) &= F((X + aE)^+) - F(X^+) - \left(F(-(\overline{X + aE})^-) - F(-(\overline{X})^-) \right) \\
&= F((X + aE)^+) - F(X^+) - \left(F(-(\overline{X} + aE)^-) - F(-(\overline{X})^-) \right) \\
&= a\Theta . \nabla F(X^+) + o(|a\Theta|) + a\overline{\Gamma} . \nabla F(-(\overline{X})^-) + o(|a\overline{\Gamma}|),
\end{aligned}
$$

where $a\Theta = (X + aE)^+ - X^+$ with $\Theta = (\theta_i)_{i=0,...,N}$, where

$$\theta_i = \begin{cases} 1 & \text{if} \quad \sigma_i = 1 \\ 0 & \text{if} \quad \sigma_i = -1 \end{cases}$$

and $a\overline{\Gamma} = (\overline{X} + aE)^- - (\overline{X})^-$ with $\overline{\Gamma} = (\overline{\gamma}_i)_{i=0,...,N}$, where

$$\overline{\gamma}_i = \begin{cases} 0 & \text{if} \quad \overline{\sigma}_i := \sigma_{\overline{i}} = 1 \\ 1 & \text{if} \quad \overline{\sigma}_i := \sigma_{\overline{i}} = -1. \end{cases}$$

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Hence, using the fact that F is C^1, we get

$$\overline{G}(X + aE) - \overline{G}(X) \quad = \quad a\Big\{(\Theta + \overline{\Gamma}).\nabla F(0) + \Theta.(\nabla F(X^+) - \nabla F(0)) + o(1)$$
$$+ \ \overline{\Gamma}.(\nabla F(-(\overline{X})^-) - \nabla F(0))\Big\}$$
$$\geq \quad \frac{a}{2}(\Theta + \overline{\Gamma}).\nabla F(0) > 0,$$

if $(\Theta + \overline{\Gamma}).\nabla F(0) > 0$. This is true because

$$(\Theta + \overline{\Gamma}).\nabla F(0) \ = \ \sum_{i=0}^{N} \theta_i \frac{\partial F}{\partial X_i}(0) + \sum_{j=0}^{N} \overline{\gamma}_j \frac{\partial F}{\partial X_j}(0)$$
$$= \ \sum_{i=0}^{N} \left(\theta_i \frac{\partial F}{\partial X_i}(0) + \overline{\gamma}_{\overline{i}} \frac{\partial F}{\partial X_{\overline{i}}}(0)\right)$$
$$\geq \ \sum_{i=0}^{N} \left((\theta_i + \overline{\gamma}_{\overline{i}}) \min\left(\frac{\partial F}{\partial X_i}(0), \frac{\partial F}{\partial X_{\overline{i}}}(0)\right)\right)$$
$$= \ \frac{\partial F}{\partial X_0}(0) + \sum_{i=1}^{N} \min\left(\frac{\partial F}{\partial X_i}(0), \frac{\partial F}{\partial X_{\overline{i}}}(0)\right) > 0 \quad \text{(using (9.15))},$$

where we have used in the fourth line the fact that $\theta_i + \overline{\gamma}_{\overline{i}} = 1$ for all $i = 0, ..., N$, which follows from the definition of θ_i and $\overline{\gamma}_{\overline{i}}$ and the fact that $\sigma_i = \sigma_{\overline{i}}$.

Case 2: $X \in Q_\Sigma$ and $X + aE \in Q_{\widehat{\Sigma}}$
This case is exactly the same as Case 2 of Step 3 in the proof of Proposition 9.4. However, in this case, we can choose

$$\eta = \frac{a}{2}(\Theta + \overline{\Gamma}).\nabla F(0) > 0.$$

$\square$

Here, we recall two comparison principle results on half lines that we will also use to prove that $c^+ \geq 0$.

Proposition 9.9 (Comparison principle on $[-r^*, +\infty)$)
Let $F : [s, s']^{N+1} \to \mathbb{R}$ satisfying (A_{Lip}) over $[s, s']^{N+1}$ and assume that:

(9.18)
$$\left| \begin{array}{l} \text{there exists } \eta_0 > 0 \text{ such that if} \\ X = (X_0, ..., X_N), \ X + (\alpha, ..., \alpha) \in [s' - \eta_0, s']^{N+1} \\ \text{then } F(X + (\alpha, ..., \alpha)) < F(X) \text{ if } \alpha > 0. \end{array} \right.$$

Let $u, v : [-r^*, +\infty) \to [s, s']$ be respectively a sub and a supersolution of

(9.19)
$$cu'(x) = F((u(x + r_i))_{i=0,...,N}) \quad on \quad (0, +\infty)$$

in sense of Definition 2.1. Moreover, assume that

$$v \geq s' - \eta_0 \quad on \quad [-r^*, +\infty),$$

and that

$$u \le v \quad on \quad [-r^*, 0].$$

Then

$$u \le v \ on \ [-r^*, +\infty).$$

Similarly, we have the following proposition on the half line $(-\infty, -r^*]$:

Proposition 9.10 (Comparison principle on $(-\infty, -r^*]$)
Let $F : [s, s']^{N+1} \to \mathbb{R}$ satisfying (A_{Lip}) over $[s, s']^{N+1}$ and assume that:

(9.20) $\quad \left| \begin{array}{l} \text{there exists } \eta_1 > 0 \text{ such that if} \\ X = (X_0, ..., X_N), \ X + (\alpha, ..., \alpha) \in [s, s + \eta_1]^{N+1} \\ \text{then } F(X + (\alpha, ..., \alpha)) < F(X) \text{ if } \alpha > 0. \end{array} \right.$

Let $u, v : (-\infty, r^*] \to [s, s']$ be respectively a sub and a supersolution of

(9.21) $\qquad cu'(x) = F((u(x + r_i))_{i=0,...,N}) \quad on \quad (-\infty, 0)$

in sense of Definition 2.1. Moreover, assume that

$$u \le s + \eta_1 \quad on \quad (-\infty, r^*],$$

and that

$$u \le v \quad on \quad [0, r^*].$$

Then

$$u \le v \ on \ (-\infty, r^*].$$

For the proof of Proposition 9.9 and Proposition 9.10, we refer the reader for [1, Theorem 4.1 and Corollary 4.2] which is done for F defined on $[0, 1]^{N+1}$ instead of $[s, s']^{N+1}$.

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