The velocity diagram for traveling waves

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Abstract

In this Note, we consider traveling waves in a reaction-diffusion equation in dimension one. Motivated by the motion of dislocations in crystals, we introduce an additive parameter σ in the reaction term, which may be interpreted as an exterior force applied on the crystal. Under certain natural assumptions and for every value of $\sigma \in [\sigma^-, \sigma^+]$, we show the existence of traveling waves ϕ of velocity c . The range $\sigma \in (\sigma^-, \sigma^+)$ corresponds to bistable cases with a unique velocity $c = c(\sigma)$. On the contrary, the case $\sigma = \sigma^+$ is positively monostable with a branch of velocities $c \geq c^+$, while the case $\sigma = \sigma^-$ is negatively monostable with a branch of velocities $c \leq c^-$. This study gives rise to a natural connection between bistable cases and monostable cases in a single velocity diagram. We also give some qualitative properties of the velocity function $\sigma \mapsto c(\sigma)$.

Résumé

Dans cette Note, nous considérons des ondes progressives pour une équation de réaction-diffusion en dimension un. Motivés par le mouvement de dislocations dans les cristaux, nous introduisons un paramètre additif σ dans le terme de réaction, qui peut être interprété comme une force extérieure appliquée au cristal. Sous certaines hypothèses naturelles et pour chaque valeur de $\sigma \in [\sigma^-, \sigma^+]$, nous montrons l'existence d'ondes progressives ϕ se déplaçant à la vitesse c. Le domaine $\sigma \in (\sigma^-, \sigma^+)$ correspond aux cas bistables avec une unique vitesse $c = c(\sigma)$. Au contraire, le cas $\sigma = \sigma^+$ est positivement monostable avec une branche de vitesses $c \geq c^+$, et le cas $\sigma = \sigma^-$ est négativement monostable avec une branche de vitesse $c \leq c^-$. Cette étude met en évidence un lien naturel entre les cas bistables et les cas monostables au sein d'un unique diagramme en vitesse. On donne aussi des propriétés qualitatives de la fonction vitesse $\sigma \mapsto c(\sigma)$.

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1. Introduction

In this Note, we consider particular solutions $u(t,x)$ to the standard reaction-diffusion equation with an additional exterior parameter $\sigma \in \mathbb{R}$

$$u(t,x) = \phi(x+ct) \qquad \text{satisfying} \qquad u_t = u_{xx} + f(u) + \sigma \quad \text{for} \quad (t,x) \in \mathbb{R}\times\mathbb{R} \tag{1}$$

where ϕ is a traveling wave moving with velocity c. Such a model is for instance inspired from the dynamics of a dislocation defect in a crystal where σ is the exterior shear stress applied on the crystal. From this point of view, equation (1) can be seen as an approximation of the fully overdamped Frenkel-Kontorova model (see [8]). We assume that the function f satisfies

$$\begin{cases} \textbf{(Regularity)} & f \quad \text{is Lipschitz-continuous} \\ \textbf{(Periodicity)} & f(v+1) = f(v) \quad \text{for all} \quad v \in \mathbb{R} \\ \textbf{(Decreasing)} & f \text{ decreasing on } (\theta,1) \text{ for } \theta \in (0,1) \\ \textbf{(Stricly increasing)} & f'(v) \geq g(v) > 0 \quad \text{for almost every } v \in (0,\theta) \text{ with } g:(0,\theta)\to\mathbb{R} \text{ continuous} \end{cases} \tag{2}$$

The graph of f is represented on Figure 1.

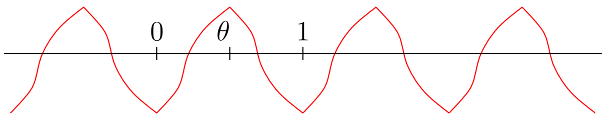

Figure 1. Nonlinearity f

Setting

$$\sigma^+ := -\min f, \quad \sigma^- := -\max f$$

we define for each $\sigma \in [\sigma^-, \sigma^+]$, the unique roots m_σ and b_σ solutions of

$$f(m_\sigma) + \sigma = 0, \quad m_\sigma \in [\theta - 1, 0] \qquad \text{and} \qquad f(b_\sigma) + \sigma = 0, \quad b_\sigma \in [0,\theta]$$

Here the nonlinearity $f^\sigma := f + \sigma$ falls in one the following three cases

$$\begin{cases} \text{bistable case } \sigma \in (\sigma^-, \sigma^+), & f^\sigma < 0 \quad \text{on} \quad (m_\sigma, b_\sigma) \quad \text{and} \quad f^\sigma > 0 \quad \text{on} \quad (b_\sigma, 1+m_\sigma) \\ \text{positive monostable case } \sigma = \sigma^+, & f^{\sigma^+} > 0 \quad \text{on} \quad (m_{\sigma^+}, 1+m_{\sigma^+}) \quad \text{with} \quad b_{\sigma^+} = m_{\sigma^+} = 0 \\ \text{negative monostable case } \sigma = \sigma^-, & f^{\sigma^-} < 0 \quad \text{on} \quad (m_{\sigma^-}, 1+m_{\sigma^-}) \quad \text{with} \quad b_{\sigma^-} = 1 + m_{\sigma^-} = \theta \end{cases}$$

Our goal is to connect bistable and monostable types as the parameter σ varies, as it is very natural from the point of view of the motion of a dislocation defect under the exterior force σ. To this end, we now consider traveling waves ϕ of velocity c associated to the parameter $\sigma \in [\sigma^-, \sigma^+]$, solutions of

$$c\phi' = \phi'' + f(\phi) + \sigma \quad \text{on} \quad \mathbb{R} \qquad \text{with} \qquad \phi(-\infty) = m_\sigma, \quad \phi(+\infty) = 1 + m_\sigma \tag{3}$$

Recall here some known results. When $f \in C^1(\mathbb{R})$ with $f'(m_\sigma) < 0$, the uniqueness of c is known in [7] in the bistable case. In the positive monostable case, when $f \in C^1([0,1])$ with $f'(0) > 0$, the existence of a branch of velocities $c \geq c^+$ is also known (see [3,9,6]).

Then our result is the following velocity diagram.

Theorem 1.1 (Velocity diagram)
Assume that f satisfies (2).
i) (Bistable case $\sigma \in (\sigma^-, \sigma^+)$)
For every $\sigma \in (\sigma^-, \sigma^+)$, there exists a unique velocity $c = c(\sigma)$ and a unique (up to translations) traveling wave ϕ solution of (3). Moreover ϕ is increasing and the map $\sigma \mapsto c(\sigma)$ is continuous increasing for $\sigma \in (\sigma^-, \sigma^+)$ with finite limits

$$c^- := \lim_{\sigma^- < \sigma \to \sigma^-} c(\sigma), \quad c^+ := \lim_{\sigma^+ > \sigma \to \sigma^+} c(\sigma)$$

There exists also $\delta > 0$ such that $0 < \delta \leq \frac{c(\sigma_2) - c(\sigma_1)}{\sigma_2 - \sigma_1}$ for all $\sigma_1, \sigma_2 \in (\sigma^-, \sigma^+)$ with $\sigma_1 \neq \sigma_2$.
If moreover $f \in C^{1,1}(\theta, 1)$ with $f' < 0$ on $(\theta, 1)$, then the map $\sigma \mapsto c(\sigma)$ is locally Lipschitz-continuous inside the interval (σ^-, σ^+).
ii) (Positive monostable case $\sigma = \sigma^+$)
For every $c \geq c^+$, there exists a unique (up to translations) traveling wave ϕ solution of (3). Moreover ϕ is increasing. For all $c < c^+$, there are no traveling waves solutions of (3).
iii) (Negative monostable case $\sigma = \sigma^-$)
For every $c \leq c^-$, there exists a unique (up to translations) traveling wave ϕ solution of (3). Moreover ϕ is increasing. For all $c > c^-$, there are no traveling waves solutions of (3).

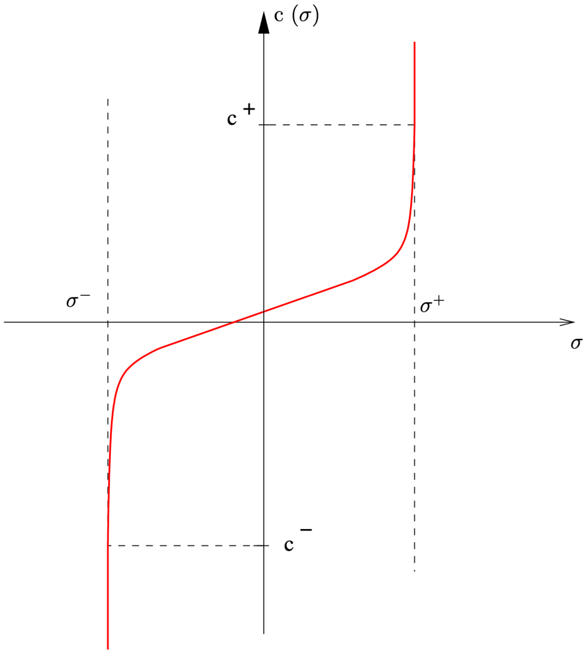

Figure 2. Schematic velocity diagram: the velocity function $c(\sigma)$ with vertical branches at $\sigma = \sigma^\pm$.

The result of Theorem 1.1 is presented in the velocity diagram in Figure 2, and there are cases where $\frac{dc}{d\sigma}$ blows up at $\sigma = \sigma^\pm$ (see Remark 1). This result allows to understand the branch of solutions $c \geq c^+$ in the positive monostable case as a sort of continuation of the increasing velocity $\sigma \mapsto c(\sigma)$ for $\sigma \in (\sigma^-, \sigma^+)$. This phenomenon is also related to the fact that for $\sigma \in \mathbb{R} \setminus [\sigma^-, \sigma^+]$, there are no roots of $f + \sigma = 0$ and then no possible traveling waves. We see that Theorem 1.1 gives the whole picture of the velocity diagram which connects bistable to monostable cases.

Let us also mention that part of the results stated in Theorem 1.1 stays true (loosing possibly the strict monotonicity of the velocity function $\sigma \mapsto c(\sigma)$ and loosing possibly the uniqueness of the profile (up to translations)) for traveling waves of discrete reaction-diffusion equations of the type

$$u(t,x) = \phi(x+ct) \quad \text{satisfying} \quad u_t = F(u(t,x+r_0),\dots,u(t,x+r_N))$$

where $r_0 = 0$ and the $r_i \in \mathbb{R}$ are discrete shifts for $i = 1,\dots,N$ with $N \geq 1$, under certain periodicity and monotonicity assumptions on F (in order to insure a comparison principle, but possibly loosing the strong comparison principle) joint to the previous assumption (2) on $f(v) := F(v,\dots,v)$. The results are given in [1].

2. Sketch of the proof of the theorem

The originality of this Note is probably more in the statement of the theorem than in the proof itself. Part of the arguments of the proof are classical. Some of those arguments are for instance developed in [1] for discrete reaction-diffusion equations, and are simply adapted here. For those reasons, we only indicate here the sketch of the proof of the theorem.

Step 1: existence and uniqueness for $\sigma \in (\sigma^-,\sigma^+)$
A standard way to build a solution (c,ϕ) consists to use the method of sub/supersolutions to build a solution on a large finite interval $[-R,R]$ and to adjust the velocity c in such a way that the transition arises in the middle of the interval for instance, and finally to pass to the limit as $R \to +\infty$ (like in [4] for ignition type nonlinearities). A different method that is used in [2] and is more generally useful to get results on the velocity in the case of degenerate equations without strong maximum principle. This method consists to solve the cell problem for homogenization (similarly to what is done in [8]) finding solutions (λ_p, h_p) with $\lambda_p \in \mathbb{R}$, associated to a fixed slope $p > 0$

$$\lambda_p h_p' = h_p'' + f(h_p) + \sigma, \quad h_p(x + \frac{1}{p}) = h_p(x) + 1, \quad h_p' \geq 0$$

and to define (c,ϕ) as a limit of (λ_p, h_p) as $p \to 0$ (similarly to what is done in [2,1]). The strict increasing property of assumption (2) on f insures that the root b_σ is unstable, and then that the limit profile ϕ does not split in two profiles, one in the range (m_σ, b_σ) and one in the range $(b_\sigma, 1 + m_\sigma)$. This insures that the profile ϕ is a true transition in the range $(m_\sigma, 1 + m_\sigma)$. The decreasing property of assumption (2) insures the comparison principle at infinity. This allows to prove uniqueness of the velocity denoted $c = c(\sigma)$. Using moreover the standard sliding method [5] (based on the strong maximum principle), this allows to get uniqueness (up to translations) and the monotonicity of the profile $\phi =: \phi_\sigma$.

Step 2: continuity and monotonicity of $c(\sigma)$ for $\sigma \in (\sigma^-,\sigma^+)$
The continuity of the map $\sigma \mapsto c(\sigma)$ follows from the uniqueness of the velocity. The fact that the maps $\sigma \mapsto m_\sigma$ and $\sigma \mapsto b_\sigma$ are increasing, insures the comparison at infinity of the associated profiles for two values $\sigma_1 < \sigma_2$, and this shows that the map $\sigma \mapsto c(\sigma)$ is nondecreasing. Moreover the strong comparison principle implies that the map $\sigma \mapsto c(\sigma)$ is increasing.

Step 3: vertical branch of solutions for large velocities $c >> 1$ for $\sigma = \sigma^+$
Using the solution g to the ODE $g' = f(g) + \sigma^+ \geq 0$, say with $g(0) = \delta > 0$, and using the Lipschitz regularity of f, we deduce that $\bar{g}_\varepsilon(x) := g(\varepsilon x)$ satisfies the supersolution inequality for all $\varepsilon > 0$

$$c_\varepsilon \bar{g}'_\varepsilon \ge \bar{g}''_\varepsilon + f(\bar{g}_\varepsilon) + \sigma^+ \quad \text{with} \quad c_\varepsilon = \frac{1}{\varepsilon} + \varepsilon |f'|_{L^\infty(\mathbb{R})}$$

Using the fact that $\underline{g}_\varepsilon := \delta$ is a subsolution on $(0,+\infty)$, we can then build a solution $g_{\varepsilon,\delta}$ on $(0,+\infty)$. Up to translate $g_{\varepsilon,\delta}$, we can pass to the limit $\delta \to 0$ and get a solution $\phi := g_\varepsilon$ associated to the velocity $c := c_\varepsilon$, which works for all velocities $c \ge 2\sqrt{|f'|_{L^\infty(\mathbb{R})}}$.

Step 4: definition of $c^\pm$

If (c,ϕ) is a solution for the parameter σ^+, then we can compare it to a solution of velocity $c(\sigma)$ for $\sigma \in (\sigma^-,\sigma^+)$, and get $c(\sigma) \le c$. This implies that $\lim_{\sigma^+ > \sigma \to \sigma^+} c(\sigma) =: c^+ \le c$. In particular from Step 3, we deduce that $c^+ < +\infty$, and that there are no solutions (c,ϕ) with $c < c^+$ for the parameter σ^+. We get similar results for c^-.

Step 5: full vertical branch of solutions for $c \ge c^+$

Using $(c(\sigma),\phi_\sigma)$ the solution given in Step 1 for $\sigma \in (\sigma^-,\sigma^+)$, we can pass to the limit and get $\phi_\sigma \to \phi^+$ as $\sigma \to \sigma^+$, which is a traveling wave of velocity c^+ for $\sigma = \sigma^+$. Then for every $c \ge c^+$, we see that (c,ϕ^+) is a supersolution of the equation for $\sigma = \sigma^+$. Then using a construction as in Step 3, we get the existence of a solution (c,ϕ) for $\sigma = \sigma^+$, for each $c \ge c^+$. Finally the decreasing property of f in (2) implies the comparison principle at $x = +\infty$. Let us consider intervals $(a_\delta,+\infty)$ where $\phi > \phi(a_\delta) = \delta > 0$. Using the sliding method for all $\delta > 0$ small enough with $m_{\sigma^+} = 0$, we can easily deduce the monotonicity and uniqueness of the profile ϕ (up to translations).

Step 6: full vertical branch of solutions for $c \le c^-$

Proceeding as in Step 5, we get a full branch of solutions (c,ϕ) for all $c \le c^-$, and no solutions for $c > c^-$.

Step 7: bounds on $\dfrac{dc}{d\sigma}$

Recall that for any $\sigma \in (\sigma^-,\sigma^+)$, we have $c^- \le c(\sigma) \le c^+$, and the interior regularity theory for the elliptic equation satisfied by the profile $\phi = \phi_\sigma$ gives a bound $|\phi'_\sigma|_{L^\infty(\mathbb{R})} \le \delta^{-1}$ uniformly in $\sigma \in (\sigma^-,\sigma^+)$, for some $\delta > 0$. Consider $\sigma_1,\sigma_2 \in (\sigma^-,\sigma^+)$ with $\sigma_1 < \sigma_2$, and the associated solutions (c_i,ϕ_i) for $i = 1,2$ with $c_i = c(\sigma_i)$, $\phi_i = \phi_{\sigma_i}$ and $m_i = m_{\sigma_i}$.

Step 7.1: bound from below

We deduce that

$$\underline{c}_2 \phi'_1 \le \phi''_1 + f(\phi_1) + \sigma_2 \quad \text{with} \quad \underline{c}_2 := c_1 + \delta(\sigma_2 - \sigma_1)$$

The comparison implies (up to an initial shift of the profiles) that $\phi_2(x + c_2 t) \ge \phi_1(x + \underline{c}_2 t)$ and then the fact that $\phi_2(-\infty) < \phi_1(+\infty)$ implies that $c_2 \ge c_1 + \delta(\sigma_2 - \sigma_1)$ which implies the bound from below on $\frac{dc}{d\sigma}$.

Step 7.2: bound from above

For $c = c(\sigma)$, let us consider the equation $c\zeta' = \zeta'' + f'(\phi_\sigma)\zeta$ satisfied by $\zeta := \phi'_\sigma$. Using the fact that $f' < -\mu < 0$ in a neighborhood of m_σ, we can introduce the roots of $\lambda^2 - c\lambda - \mu = 0$ which are $\lambda_\pm := \frac{c \pm \sqrt{c^2+4\mu}}{2}$ and show by comparison that $\zeta(y) \le \zeta(x) e^{\lambda_+(y-x)}$ for $y \le x$ and $\zeta(x)$ small enough. This gives by integration for $\phi_\sigma(x) - m_\sigma = \int_{-\infty}^x dy\ \zeta(y)$ that $\phi'_\sigma \ge \lambda_+(\phi_\sigma - m_\sigma)$ where $\phi_\sigma - m_\sigma$ is small enough (using for instance interior estimates for elliptic equations). Now for $\sigma_1 < \sigma_2$, we set $\overline{\phi}_2 := \psi + m_2$ with $\psi := \phi_1 - m_1$. Using the fact that $f(m_2+\psi) - f(m_2) - \{f(m_1+\psi) - f(m_1)\} = \int_0^1 dt\ \int_0^1 ds\, \{f''(t\psi + m_1 + s(m_2 - m_1)) - f''(t\psi + m_1)\} \cdot \psi(m_2 - m_1)$ and $f'(m_\sigma)\frac{dm_\sigma}{d\sigma} = -1$ with $f'(m_\sigma) \le -\underline{\mu} < 0$ and $\lambda_+ \ge \underline{\lambda}_+ > 0$ for $\sigma \in [\sigma_1,\sigma_2]$, we get on an interval $(-\infty,-R]$ where ψ is small enough

$$\overline{c}_2^+ \overline{\phi}'_2 \ge \overline{\phi}''_2 + f(\overline{\phi}_2) + \sigma_2 \quad \text{on} \quad (-\infty,-R], \quad \text{with} \quad \overline{c}_2^+ := c_1 + \left\{ \frac{|f''|_{L^\infty(\mathbb{R})}}{\underline{\mu}\underline{\lambda}_+} \right\} (\sigma_2 - \sigma_1)$$

We get a similar inequality on $[R,+\infty)$ where $1-\psi$ is small enough, with some velocity $\overline{c}_2^-$. Using Harnack inequality, we can show that there exists some $K_R>0$ such that $\phi_1' \geq (K_R)^{-1}$ on $[-R,R]$, which implies

$$\overline{c}_2^0\overline{\phi}_2' \geq \overline{\phi}_2'' + f(\overline{\phi}_2) + \sigma_2 \quad \text{on} \quad [R,R], \quad \text{with} \quad \overline{c}_2^0 := c_1 + K_R\left\{1 + \frac{|f'|_{L^\infty(\mathbb{R})}}{\underline{\mu}}\right\}(\sigma_2-\sigma_1)$$

Using the fact that all the constants can be taken uniformly for $\sigma \in [\sigma_1,\sigma_2] \subset\subset [\sigma^-,\sigma^+]$, we deduce that $c_2 \leq \max(\overline{c}_2^-, \overline{c}_2^0, \overline{c}_2^+)$ which implies the bound from above on $\frac{dc}{d\sigma}$. This ends the sketch of the proof of the theorem.

Remark 1 *Choosing a normalization like for instance $\phi_\sigma(0)=\theta$, we can show that $\bar{\psi} := \frac{d}{d\sigma}(\phi_\sigma - m_\sigma)$ formally satisfies an equation that we can multiply by $e^{-cx}\phi_\sigma'(x)$ and integrate by parts to get with $I_1 := \int_{\mathbb{R}} e^{-cx}\phi_\sigma'^2 dx = \int_{\mathbb{R}} e^{-cx}(\phi_\sigma - m_\sigma)(f(\phi_\sigma)-f(m_\sigma))dx$ and $I_2 = \int_{\mathbb{R}} e^{-cx}\left\{1-\frac{f'(\phi_\sigma)}{f'(m_\sigma)}\right\}\phi_\sigma' dx$, the relation*

$$\left(\frac{dc}{d\sigma}\right)\cdot I_1 = I_2 \quad \textit{with} \quad I_2 := \begin{cases} \dfrac{c}{-f'(m_\sigma)}\displaystyle\int_{\mathbb{R}} e^{-cx}\left\{f(\phi_\sigma)-f(m_\sigma)-(\phi_\sigma-\phi_\sigma(\mp\infty))f'(m_\sigma)\right\}dx & \textit{if} \quad \pm c>0 \\ 1 & \textit{if} \quad c=0 \end{cases}$$

When $f \in C^{1,1}(\theta,1)$ and $f'<0$ on $(\theta,1)$, then we can justify the above computation at least at every point of differentiability of c. This is the case for instance when we consider the 1-periodic function f defined by $f(v)=\frac{1}{2}-|v-\frac{1}{2}|$ for $v\in[0,1]$. Then $\theta=\frac{1}{2}$ and $c^+ = 2\sqrt{f'(0^+)}$. Moreover we can then show that the integral I_2 blows up much faster than I_1 as $\sigma\to\sigma^+$, which implies that $\frac{dc}{d\sigma}$ is not bounded as $\sigma\to\sigma^+$ (and by symmetry also as $\sigma\to\sigma^-$). For general nonlinearities f, it would be interesting to refine the analysis on the behaviour of c at $\sigma=\sigma^\pm$ which is out of the scope of this Note.

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