Traveling waves for discrete reaction-diffusion equations in the general monostable case

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Abstract: We consider general fully nonlinear discrete reaction-diffusion equations $u_t = F[u]$, described by some function F. In the positively monostable case, we study monotone traveling waves of velocity c, connecting the unstable state 0 to a stable state 1. Under Lipschitz regularity of F, we show that there is a minimal velocity c_F^+ such that there is a branch of traveling waves with velocities $c \geq c_F^+$, and no traveling waves for $c < c_F^+$. We also show that the map $F \mapsto c_F^+$ is not continuous for the L^∞ norm on F. Assuming more regularity of F close to the unstable state 0, we show that $c_F^+ \geq c_F^*$ where the velocity c_F^* can be computed from the linearization of the equation around the unstable state 0. We show that the inequality can be strict for certain nonlinearities F. On the contrary, under a KPP condition on F, we show the equality $c_F^+ = c_F^*$. Finally, we also give an example where c_F^+ is negative.

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1 Introduction

1.1 General motivation

We were originally motivated by the study of the classical fully overdamped Frenkel-Kontorova model, which is a system of ordinary differential equations

$$\frac{dX_i}{dt} = X_{i+1} - 2X_i + X_{i-1} + f(X_i), \tag{1.1}$$

where $X_i(t) \in \mathbb{R}$ denotes the position of a particle $i \in \mathbb{Z}$ at time t, $\frac{dX_i}{dt}$ is the velocity of this particle, f is the force created by a 1-periodic potential. Such force could be for example $f(x) = 1 - \cos(2\pi x) \geq 0$. This kind of system can be, for instance, used as a model of the motion of a dislocation defect in a crystal (see the book of Braun and Kivshar [10]). This motion is described by particular solutions of the form

$$X_i(t) = \phi(i + ct) \quad \text{with} \quad \phi' \geq 0 \quad \text{and } \phi \text{ bounded}$$

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where ϕ is called a travelling wave moving with velocity $c \in \mathbb{R}$. It satisfies

$$c\phi'(z) = \phi(z+1) - 2\phi(z) + \phi(z-1) + f(\phi(z))$$

In the monostable case, say when the Lipschitz nonlinearity f satisfies $f > 0$ on $(0,1)$ with $f(0) = 0 = f(1)$, we can moreover normalize the limits of the profile as

$$\phi(-\infty) = 0, \quad \phi(+\infty) = 1 \tag{1.2}$$

Then it is possible to show the existence of a branch of solutions (c, ϕ_c) for all velocities $c \geq c^+$ and the non existence of solutions for $c < c^+$ where c^+ is the minimal velocity.

The goal of this paper is to present similar results in a general framework including Frenkel-Kontorova model. To this end, given a real function F (whose properties will be specified later in this Introduction), we consider solutions (c, ϕ) satisfying the limit conditions (1.2) to the following generalized equation

$$c\phi'(z) = F(\phi(z+r_0), \phi(z+r_1), ..., \phi(z+r_N)) \quad \text{with} \quad \phi' \geq 0 \tag{1.3}$$

where $N \geq 0$ and $r_i \in \mathbb{R}$ for $i = 0, ..., N$ such that

$$r_0 = 0 \quad \text{and} \quad r_i \neq r_j \quad \text{if} \quad i \neq j, \tag{1.4}$$

which does not restrict the generality. For simplicity, we will also use the following compact notation

$$F((\phi(z+r_i))_{i=0,...,N}) := F(\phi(z+r_0), \phi(z+r_1), ..., \phi(z+r_N))$$

Notice that in general equations (1.3) do not have a Strong Maximum Principle which creates a further difficulty with respect the standard reaction-diffusion equations.

Equation (1.1) can be seen as a discretization of the following standard reaction-diffusion equation

$$u_t = \Delta u + f(u). \tag{1.5}$$

In 1937, Fisher [17] and Kolmogorov, Petrovsky and Piskunov [29] studied the traveling waves for equation (1.5) which they proposed as a model describing the spreading of a gene throughout a population. Later, many works have been devoted for such equation that appears in biological models for developments of genes or populations dynamics and in combustion theory (see for instance, Aronson, Weinberger [5, 6] and Hadeler, Rothe [23]). For more developments and applications in biology of reaction-diffusion equations, the reader may refer to [32] and to the references cited therein. There is also a considerable work on the existence, uniqueness and stability of traveling waves and their speed of propagation for the homogeneous Fisher-KPP nonlinearity (see for example [24, 25, 26, 27, 36]). Such results have been shown also for the inhomogeneous, heterogeneous and random Fisher-KPP nonlinearities (see [8, 9, 31]).

Traveling waves were studied also for discrete bistable reaction-diffusion equations (see for instance [11, 15]). See also [1] and the references therein. In the monostable case, we distinguish [28] (for nonlocal non-linearities with integer shifts) and [16, 30, 33, 34] (for problems with linear nonlocal part and with integer shifts also). See also [21] for particular monostable nonlinearities with irrational shifts. We also refer to [20, 12, 22, 13, 14, 25, 35] for different positive monostable nonlinearities. In the monostable case, we have to underline the work of Hudson and Zinner [28] (see also [35]), where they proved the existence of a branch of solutions $c \geq c^+$ for general Lipschitz nonlinearities (with possibly an infinite number of neighbors $N = +\infty$, and possibly p types of different particles, while $p = 1$ in our study) but with integer shifts $r_i \in \mathbb{Z}$. However, they do not state the nonexistence of solutions for $c < c^+$. Their method of proof relies on an approximation of the equation on a bounded domain (applying Brouwer's fixed point theorem) and an homotopy argument starting from a known solution. The full result is then obtained as the size of the domain goes to infinity. Here we underline that our results hold for the fully nonlinear case with real shifts $r_i \in \mathbb{R}$.

Several approaches were used to construct traveling waves for discrete monostable dynamics. We already described the homotopy method of Hudson and Zinner [28]. In a second approach, Chen and Guo [13] proved the existence of a solution starting from an approximated problem. They constructed a fixed point solution of an integral reformulation (approximated on a bounded domain) using the monotone iteration method (with sub and supersolutions). This approach was also used to get the existence of a solution in [19, 14, 21, 22]. A third approach based on recursive method for monotone discrete in time dynamical systems was used by Weinberger et al. [30, 33]. See also [34], where this method is used to solve problems with a linear nonlocal part. In a fourth approach [20], Guo and Hamel used global space-time sub and supersolutions to prove the existence of a solution for periodic monostable equations.

There is also a wide literature about the uniqueness and the asymptotics at infinity of a solution for a monostable non-linearities, see for instance [12, 27] (for a degenerate case), [13, 14] and the references therein. Let us also mention that certain delayed reaction diffusion equations with some Fisher-KPP non-linearities do not admit traveling waves (see for example [19, 35]).

The present work has been already announced in a preprint [2] that was accessible since 2014 and also in the PhD thesis in 2014 of the first author. Unfortunately, the life conditions of the two authors did not permit the submission to publication of the manuscript. The present paper corresponds to part III of [2]. The remaining parts of the preprint [2] correspond to [3] (see also [4]).

1.2 Main results

In order to present our results, we consider for $N \geq 1$ a function $F : [0,1]^{N+1} \to \mathbb{R}$, and introduce the following natural assumptions.

Assumption (A_{Lip}):

i) Regularity: $F \in \mathrm{Lip}([0,1]^{N+1})$.

ii) Monotonicity: $F(X_0, X_1, ..., X_N)$ is non-decreasing w.r.t. each X_i for $i \neq 0$.

Assumption (P_{Lip}):

Positive degenerate monostability:
Let $f(v) = F(v, ..., v)$ such that $f(0) = f(1) = 0, \quad f > 0$ in $(0,1)$.

Our main result is:

Theorem 1.1 (Existence of a branch of traveling waves in the monostable case) *Assume (A_{Lip}) and (P_{Lip}). Then there exists a real c^+ such that for all $c \geq c^+$ there exists a traveling wave $\phi : \mathbb{R} \to \mathbb{R}$ solution (in the viscosity sense (see Definition 2.1)) of*

$$(1.6) \qquad \begin{cases} c\phi'(z) = F(\phi(z + r_0), \phi(z + r_1), ..., \phi(z + r_N)) & \text{on } \mathbb{R} \\ \phi \text{ is non-decreasing over } \mathbb{R} \\ \phi(-\infty) = 0 \quad \text{and} \quad \phi(+\infty) = 1. \end{cases}$$

On the contrary for $c < c^+$, there is no solution of (1.6).

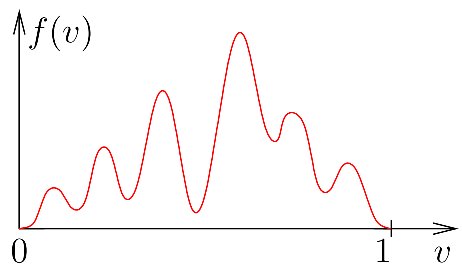

Figure 1: Positive degenerate monostable nonlinearity f

Notice that assumptions of Theorem 1.1 hold true even for equations as degenerate as a simple ODE

$$c\phi' = f(\phi) \geq 0 \quad \text{on} \quad \mathbb{R}$$

for which it is easy to see that $c^+ = 0$. Recall also that under assumptions of Theorem 1.1, the Strong Maximum Principle is not valid for general nonlinearities F (see for instance Remark 4.7).

Up to our knowledge, Theorem 1.1 is the first result for discrete dynamics with real shifts $r_i \in \mathbb{R}$ in the fully nonlinear case. Even when $r_i \in \mathbb{Z}$, the only result that we know for fully nonlinear dynamics is the one of Hudson and Zinner [28]. However, the nonexistence of solutions for $c < c^+$ is not addressed in [28].

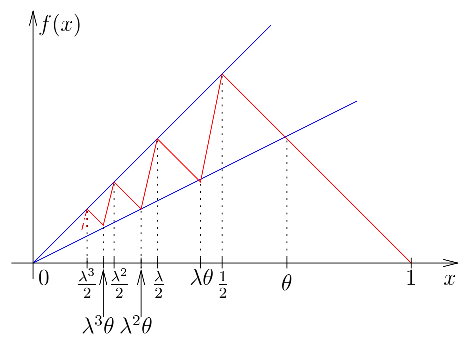

Figure 2: Lipschitz positive degenerate monostable nonlinearity; the rest of the figure over $[0, \frac{\lambda^3}{2}]$ is completed by dilation of center 0 and ratio λ.

See Figure 2 for an explicit Lipschitz non-linearity example for which our result (Theorem 1.1) is still true, even if $f'(0)$ is not defined. We also prove that the minimal velocity c^+ is unstable in the following sense:

Proposition 1.2 (Instability of the minimal velocity c_F^+)
There exists a function F satisfying (A_{Lip}) and (P_{Lip}) with a minimal velocity c_F^+ such that there exists a sequence of functions F_δ (satisfying also (A_{Lip}) and (P_{Lip}) with uniform Lipschitz bound on F_δ as $\delta \to 0$) with associated minimal velocity $c_{F_\delta}^+$ satisfying

$$F_\delta \to F \quad \text{in} \quad L^\infty([0,1]^{N+1})$$

when $\delta \to 0$, but

$$\liminf_{\delta \to 0} c_{F_\delta}^+ > c_F^+.$$

When f is smooth enough, we will see below in Proposition 1.4 that the minimal velocity c^+ contains information about $f'(0)$, similarly to classical result in [29] which asserts that the minimal velocity of reaction-diffusion equation (1.5) is $c^+ = 2\sqrt{f'(0)}$. This shows that when F is only Lipschitz, it becomes much more delicate to capture c_F^+ and to show Theorem 1.1.

Examples of functions F satisfying assumptions (A_{Lip}) and (P_{Lip}) are given for $N = 2$, $r_0 = 0$, $r_1 = -1$, $r_2 = 1$ by

$$F(X_0, X_1, X_2) = X_2 + X_1 - 2X_0 + f(X_0), \tag{1.7}$$

with for instance non-linearity $f(x) = x(1-x)$ or $f(x) = x^2(1-x)^2$.

In the next result, we give some lower bound on the minimal velocity c^+ (given in Theorem 1.1). To this end, we need to assume some smoothness and strict monotonicity on F near $\{0\}^{N+1}$; and this is given in assumption (P_{C^1}) (which is stronger than (P_{Lip})):

Assumption (P_{C^1}):

Positive degenerate monostability:
Let $f(v) = F(v, ..., v)$ such that $f(0) = 0 = f(1)$ and $f > 0$ in $(0, 1)$.

Smoothness near $\{0\}^{N+1}$:
F is C^1 over a neighborhood of $\{0\}^{N+1}$ in $[0, 1]^{N+1}$ and $f'(0) > 0$.

Then we have

Theorem 1.3 (Lower bound for c^+)
Let F be a function satisfying (A_{Lip}) and (P_{C^1}). Let c^+ given by Theorem 1.1. Then we have

$$c^+ \geq c^*,$$

where

$$c^* := \inf_{\lambda > 0} \frac{P(\lambda)}{\lambda} \quad \text{with} \quad P(\lambda) := \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0, ..., 0) e^{\lambda r_i}. \tag{1.8}$$

The proof of Theorem 1.3 is quite involved in comparison to the case of standard reaction-diffusion equations. This is due to the fact Harnack inequality may fail in our context.
More precisely, we have to introduce a discussion assuming or not the following condition

$$\exists\ i_0 \in \{1, ..., N\} \ \text{ such that } \ r_{i_0} > 0 \ \text{ and } \ \frac{\partial F}{\partial X_{i_0}}(0, ..., 0) > 0, \tag{1.9}$$

Notice that under assumption (1.9), we show some sort of discrete Strong Maximum Principle to the right (because $r_{i_0} > 0$) for the associated linear evolution equation (see Proposition 4.1). Under the same assumption, we also show a Harnack inequality for the nonlinear equation satisfied by the traveling wave (see Proposition 4.4), which is of independent interest. Notice that this Harnack inequality also holds for $c = 0$ (somehow because the profile is nondecreasing). On the contrary, if we replace (1.9) by a similar condition where $r_{i_0} < 0$, then Harnack inequality can fail for $c = 0$ (see the counter-example given in Remark 4.7), but still holds true for $c < 0$ (see Proposition 4.6). Using such Harnack inequalities, we can show Theorem 1.3.

Here, it is natural to ask if we may have $c^+ = c^*$ in general or not. Already in the standard case of reaction-diffusion equations, it is known that we may have $c^+ > c^*$ (see for instance [24]). In our case, we give in Lemma 5.2, an example of a nonlinearity where we have $c^+ > c^*$ which shows also that the inequality can be strict also for discrete reaction-diffusion equations. On the other hand, as it may also be expected, we can find a KPP type condition to insure the reverse inequality $c^+ \leq c^*$, as shows the following result.

Proposition 1.4 (KPP condition to get $c^+ \le c^*$)
Let F be a function satisfying (A_{Lip}) and (P_{Lip}). Let c^+ given by Theorem 1.1 and assume that F is differentiable at $\{0\}^{N+1}$ in $[0,1]^{N+1}$. If moreover F satisfies the KPP condition:

$$F(X) \le \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0,...,0)X_i \quad \text{for every} \quad X \in [0,1]^{N+1}, \tag{1.10}$$

then $c^+ \le c^*$ with c^* defined in (1.8).

As a corollary of Theorem 1.3, we can show that $c^+ \ge 0$ holds true under certain conditions (see Corollary 6.1).

More generally, contrarily to standard reaction-diffusion equations, we may have $c^+ < 0$, as shows the following counter-example.

Proposition 1.5 (Counter-example with $c^+ < 0$; see Subsection 6.2)
There exists a function F satisfying (A_{Lip}) and (P_{C^1}) such that the associated minimal velocity c^+ is negative.

1.3 Organization of the paper

In Section 2, we recall some useful results about viscosity solutions which are used all over the paper. In Section 3, we give the proof of Theorem 1.1 about the existence of a minimal velocity c^+.

In Section 4, we prove different results about Strong Maximum Principles and Harnack inequalities, which are used in Section 5 to do the proof of Theorem 1.3, proving that $c^+ \ge c^*$.

In Section 6, we present in Corollary 6.1 sufficient conditions to insure the inequality $c^+ \ge 0$, and also prove Proposition 1.5 for an example of negative velocity c^+. Finally in the same section, we show the instability of the minimal velocity (proof of Proposition 1.2).

2 Preliminaries

We recall here some useful results involving viscosity solutions (see for instance [7]). Some of these results are contained in [1].

We first recall the notion of viscosity solutions that we use in this work. To this end, we recall that the upper and lower semi-continuous envelopes, u^* and u_*, of a locally bounded function u are defined as

$$u^*(x) = \limsup_{y\to x} u(y) \quad \text{and} \quad u_*(x) = \liminf_{y\to x} u(y).$$

and that u is upper semi-continuous if and only if $u = u^*$ (and similarly u is lower semi-continuous if and only if $u = u_*$).

For functions $u(x,t)$, we also define similarly (for later use)

$$u^*(x,t) := \limsup_{(y,\tau)\to(x,t)} u(y,\tau), \quad u_*(x,t) := \liminf_{(y,\tau)\to(x,t)} u(y,\tau)$$

Definition 2.1 (Viscosity solution)
Let $I = I' = \mathbb{R}$ (or $I = (-r^*, +\infty)$ and $I' = (0,+\infty)$) and $u : I \to \mathbb{R}$ be a locally bounded function, $c \in \mathbb{R}$ and F continuous defined on $\mathbb{R}^{N+1}$.

- *The function u is a subsolution (resp. a supersolution) on I' of*

$$cu'(x) = F((u(x+r_i))_{i=0,\dots,N}), \tag{2.1}$$

if u is upper semi-continuous (resp. lower semi-continuous) and if for all test function $\psi \in C^1(I)$ such that $u - \psi$ attains a local maximum (resp. a local minimum) at $x^* \in I'$, we have

$$c\psi'(x^*) \le F((u(x^*+r_i))_{i=0,\dots,N}) \quad \Big(\text{resp. } c\psi'(x^*) \ge F((u(x^*+r_i))_{i=0,\dots,N})\Big).$$

- *A function u is a viscosity solution of (2.1) on I' if u^* is a subsolution and u_* is a supersolution on I'.*

Next, we state Perron's method to construct solutions.

Proposition 2.2 (Perron's method ([18, Proposition 2.8]))
Let $I = (-r^*, +\infty)$ and $I' = (0, +\infty)$ and F be a function satisfying (A_{Lip}). Let u and v defined on I with values in $[0,1]$, satisfying

$$u \le v \quad \text{on} \quad I,$$

such that u and v are respectively a sub and a supersolution of (2.1) on I'. Let $\mathcal{L}$ be the set of all functions $\tilde{v} : I \to \mathbb{R}$, such that $u \le \tilde{v}$ over I with $\tilde{v}$ supersolution of (2.1) on I'. For every $z \in I$, let

$$w(z) = \inf\{\tilde{v}(z) \quad \text{such that} \quad \tilde{v} \in \mathcal{L}\}.$$

Then w is a viscosity solution of (2.1) over I' satisfying $u \le w \le v$ over I.

The following result is important and meaningful in our work.

Lemma 2.3 (Equivalence between viscosity and a.e. solutions, [1, Lemma 2.11])
Let F satisfying assumption (A_{Lip}). Let $\phi : \mathbb{R} \to [0,1]$ be a non-decreasing function. Then ϕ is a viscosity solution of

$$c\phi'(x) = F((\phi(x+r_i))_{i=0,\dots,N}) \quad \text{on} \quad \mathbb{R},$$

if and only if ϕ is an almost everywhere solution of the same equation.

Having this result in hands, we have the following useful criterion to pass to the limit.

Proposition 2.4 (Stability by passage to the limit)
Let F satisfying assumption (A_{Lip}). Given $a < b$, let $\phi_n : I := (a - r^*, b + r^*) \to [0,1]$ be a non-decreasing viscosity solution of

$$c_n\phi_n'(x) = F((\phi_n(x+r_i))_{i=0,\dots,N}) \quad \text{on} \quad I' := (a,b)$$

satisfying the bounds

$$|\phi_n|_{L^\infty(I)} \le 1, \quad |c_n| \le C$$

Then up to a subsequence, we have

$$\phi_n \to \phi \quad \text{a.e. on } I, \quad c_n \to c$$

and ϕ is a viscosity solution of

$$c\phi'(x) = F((\phi(x+r_i))_{i=0,\dots,N}) \quad \text{on} \quad I'$$

Proof of Proposition 2.4

The existence of a subsequence converging almost everywhere follows from classical Helly's theorem for monotone functions. The remaining part of the argument follows from the equivalence between viscosity solutions and almost everywhere solutions when $c = 0$. In the case $c \neq 0$, we get bounds on $|\phi_n|_{C^1(I)} \leq C'$, and the result follows for instance from the classical stability of viscosity solutions (or also by a direct argument for ODEs).

Proposition 2.5 (Solution built on a positive nondecreasing supersolution)
Assume that F satisfies (A_{Lip}) and (P_{Lip}). Assume that (c, ψ) is a supersolution in the sense that it satisfies (in the viscosity sense)

$$\begin{cases} c\psi'(z) \geq F((\psi(z + r_i))_{i=0,\dots,N}) \quad on \quad \mathbb{R} \\ \psi \text{ is non-decreasing over } \mathbb{R} \\ \psi(-\infty) = 0 \quad and \quad \psi(+\infty) = 1. \end{cases}$$

and the positivity condition

$$\psi > 0 \quad on \quad \mathbb{R}$$

Then there exists a solution (c, ϕ) of the associated equation, namely of (1.6).

Proof of Proposition 2.5

The proof relies on the method of sub/supersolutions. We refer the reader to the proof of Proposition 3.2 in [3] which can be applied without changes (even if the assumptions are not exactly the same).

3 Minimal velocity c^+ and proof of Theorem 1.1

The goal of this section is the proof of Theorem 1.1, which is done in the fourth and last subsection 3.4. The three first subsections can be seen as preliminaries for the main proof.

In Subsection 3.1, we prove Proposition 3.1, which provides a direct proof of Theorem 1.1 under the additional assumption that F is increasing in some variable X_{i_0} with $r_{i_0} > 0$, and that $c^+ \neq 0$. In Subsection 3.2, we present a lemma in order to extend F from $[0,1]^{N+1}$ to the whole space $\mathbb{R}^{N+1}$. This extension property is then used in Subsection 3.3 for a proof of Theorem 1.1, under additional regularity and nondegeneracy assumptions (A_{C^1}) and (P_{C^1}). The result is presented in Proposition 3.3, and the method of proof is a good preparation (in a simplified setting) for the general proof which is done in the last subsection and which is more technical.

3.1 A direct proof but not general

We now give a natural and simplified proof of Theorem 1.1 under the additional assumption (3.1), which is presented in the following proposition.

Proposition 3.1 (Branch of solutions under additional assumptions)
We work under the assumptions of Theorem 1.1. Let

$$c^+ = \inf \mathcal{E} \quad with \quad \mathcal{E} := \{c \in \mathbb{R} \ \ such\ that \ \ \exists\ (c, \phi) \ \ solution\ of\ (1.6)\}.$$

i) (Existence of c^+)
Then $\mathcal{E} \neq \emptyset$ and $c^+ > -\infty$ with $c^+ \in \mathcal{E}$.

ii) (Branch of velocities under an additional assumption)
Moreover, if the following additional assumption is satisfied

$$c^+ \neq 0 \quad \text{and } F \text{ is increasing in } X_{i_0} \text{ with } r_{i_0} > 0 \tag{3.1}$$

then for every $c \geq c^+$ *there exists a solution of* (1.6)*, and there is no solution for* $c < c^+$*.*

Sketch of the proof of Proposition 3.1

Step 1: $\mathcal{E} \neq \emptyset$
Step 1.1: A supersolution $\overline{\phi}_\varepsilon$
We follow an argument of Proposition 3.4 in [3], that we recall here without too much details (in particular because we will give later a more general method of proof of Theorem 1.1 and then as a corollary, it will give a second proof of Proposition 3.1). With $f(v) := F(v, \dots, v)$, we first solve the ODE

$$h_0' := f(h_0) \geq 0 \quad \text{on} \quad \mathbb{R} \quad \text{with} \quad h_0(0) = \frac{1}{2}$$

Then for $\varepsilon > 0$, we can set

$$\overline{\phi}_\varepsilon(x) = h_0(\varepsilon a_\varepsilon x) \quad \text{with} \quad a_\varepsilon = 1 + M_0 \varepsilon$$

Then for $M_0 > 0$ large enough (depending on $|f'|_{L^\infty(\mathbb{R})}$, on $r^* \geq |r_i|$ and on the Lipschitz constant of F), and $\varepsilon > 0$ small enough, we can insure that

$$\varepsilon^{-1} \overline{\phi}_\varepsilon' \geq F((\overline{\phi}_\varepsilon(x + r_i))_{i=0,\dots,N}) \quad \text{with} \quad \overline{\phi}_\varepsilon > 0 \quad \text{on} \quad \mathbb{R}$$

Step 1.2: construction of a solution ϕ_c
Having a positive increasing supersolution $\overline{\phi}_\varepsilon$ for the velocity $c = \varepsilon^{-1}$, we can then apply Proposition 2.5 which shows the existence of a nondecreasing solution ϕ_c of

$$c\phi_c' = F((\phi_c(x + r_i))_{i=0,\dots,N}) \quad \text{with} \quad \phi_c' \geq 0 \qquad \text{on} \quad \mathbb{R} \quad \text{with} \quad \frac{1}{2} \in [(\phi_c)_*(0), (\phi_c)^*(0)]$$

of velocity $c = \varepsilon^{-1}$ large enough. This forces in particular $\phi_c(-\infty) = 0$ and $\phi_c(+\infty) = 1$. This implies that $\mathcal{E} \neq \emptyset$.

Step 2: $c^+ > -\infty$
Consider a sequence $c_n \in \mathcal{E}$ such that $c_n \to c^+$ and (c_n, ϕ_n) is a solution of (1.6), and assume by contradiction that $c^+ = -\infty$. Setting $\tilde{\phi}_n(x) = \phi_n(|c_n| x)$, and up to translate the profile, we can insure that

$$-\tilde{\phi}_n'(y) = F\left(\left(\tilde{\phi}_n\left(y + \frac{r_i}{|c_n|}\right)\right)_{i=0,\dots,N}\right) \quad \text{with} \quad \tilde{\phi}_n' \geq 0 \quad \text{and} \quad \tilde{\phi}_n(0) = \frac{1}{2} \tag{3.2}$$

This gives a uniform Lipschitz estimate

$$|\tilde{\phi}_n'|_{L^\infty(\mathbb{R})} \leq M$$

Using Ascoli's Theorem, we can pass to the limit $\tilde{\phi}_n \to \tilde{\phi}$ (up to a subsequence) which solves

$$-\tilde{\phi}' = F(\tilde{\phi}, ..., \tilde{\phi}) = f(\tilde{\phi}) \geq 0 \quad \text{with} \quad \tilde{\phi}' \geq 0 \quad \text{and} \quad \tilde{\phi}(0) = \frac{1}{2}$$

This gives a contradiction with the fact that $f > 0$ on $(0, 1)$. Therefore $c^+ > -\infty$.

Step 3: $c^+ \in \mathcal{E}$
Consider again a sequence of solutions (c_n, ϕ_n) such that $c_n \to c^+$, say with

$$\frac{1}{2} \in [(\phi_n)_*(0), (\phi_n)^*(0)].$$

Then we have $\phi_n \to \phi^+$ at least almost everywhere, and by the stability of viscosity solutions (see Proposition 2.4), we see that the limit satisfies (in the viscosity sense)

$$c^+(\phi^+)'(x) = F((\phi^+(x+r_i))_{i=0,\dots,N}) \quad \text{and} \quad (\phi^+)' \geq 0 \quad \text{on} \quad \mathbb{R} \quad \text{with} \quad \frac{1}{2} \in [(\phi^+)_*(0), (\phi^+)^*(0)].$$

This shows that (c^+, ϕ^+) is a solution and then $c^+ \in \mathcal{E}$.

Step 4: branch of solutions $\mathcal{E} = [c^+, +\infty)$ under assumption (3.1)
Step 4.1: $\phi^+ > 0$
If $c^+ < 0$, then we know from [1, Lemma 6.1] that a Strong Maximum Principle holds. Precisely it shows that if $\phi^+(x_0) = 0$, then

$$\phi^+ = 0 \quad \text{on} \quad [x_0, +\infty)$$

which leads to a contradiction with the fact that $\phi(+\infty) = 1$.
If $c^+ > 0$ and assuming moreover that F is increasing in X_{i_0} with $r_{i_0} > 0$, we know from [1, Lemma 6.2] that another Strong Maximum Principle holds. Precisely is shows that if $\phi^+(x_0) = 0$, then

$$\phi^+ = 0 \quad \text{on} \quad \mathbb{R}$$

which leads again to a contradiction with $\phi(+\infty) = 1$.
Because we assumed that $c^+ \neq 0$, this shows that

$$\phi^+ > 0 \quad \text{on} \quad \mathbb{R}$$

Step 4.2: getting solutions ϕ_c
For any $c > c^+$, we see that the nondecreasing function ϕ^+ satisfies

$$c(\phi^+)' \geq F((\phi^+(x+r_i))_{i=0,\dots,N}) \quad \text{with} \quad (\phi^+)' \geq 0 \quad \text{and} \quad \phi^+ > 0 \quad \text{on} \quad \mathbb{R}.$$

Having a positive nondecreasing supersolution ϕ^+, we can proceed as in Step 1.2 and construct a solution ϕ_c of (1.6). This shows that $\mathcal{E} = [c^+, +\infty)$ and ends the proof of the proposition.

3.2 Extension of F

In order to make the proof of Theorem 1.1 (and of its simplified version Proposition 3.3), it will be useful to extend the function F defined on $[0,1]^{N+1}$ to a function $\tilde{F}$ defined on $\mathbb{R}^{N+1}$. This is the following result.

Lemma 3.2 (Extension of F, Lemma 2.1 in [1])
Consider a function F defined over $[0,1]^{N+1}$ and satisfying (A_{Lip}) such that $F(0,...,0) = F(1,...,1) = 0$. There exists an extension $\tilde{F}$ defined over $\mathbb{R}^{N+1}$ such that

$$\tilde{F}_{|_{[0,1]^{N+1}}} = F$$

and $\tilde{F}$ satisfies

Assumption ($\tilde{A}_{\mathbf{Lip}}$):

Regularity: $\tilde{F}$ *is globally Lipschitz continuous over* $\mathbb{R}^{N+1}$.

Monotonicity: $\tilde{F}(X_0, ..., X_N)$ *is non-decreasing w.r.t. each* X_i *for* $i \neq 0$.

Periodicity: $\tilde{F}(X_0 + 1, ..., X_N + 1) = \tilde{F}(X_0, ..., X_N)$ *for every* $X = (X_0, ..., X_N) \in \mathbb{R}^{N+1}$.

Notice that the function $\tilde{f}(v) := \tilde{F}(v, ..., v)$ is nothing but a periodic extension of f on $\mathbb{R}$ with period 1, that is

$$\tilde{f}_{|[0,1]} = f,$$

hence $\tilde{f}(0) = \tilde{f}(1) = 0$.

Notice also that ϕ is a solution of (1.6) if and only if ϕ solves

$$\begin{cases} c\phi'(z) = \tilde{F}((\phi(z + r_i))_{i=0,...,N}) \quad \text{on} \;\; \mathbb{R} \\ \phi \;\; \text{is non-decreasing over} \;\; \mathbb{R} \\ \phi(-\infty) = 0 \quad \text{and} \quad \phi(+\infty) = 1, \end{cases}$$

3.3 A simplified proof assuming more regularity on F

In this subsection, and for some pedagogical reasons, we prove a simplified version of Theorem 1.1 in a special case when F is smooth (see Proposition 3.3 below). The arguments of this simplified proof will be also used in the proof of the general Theorem 1.1, but under more technicalities. To state our result, we need to introduce the following assumptions including additional smoothness.

Assumption (A_{C^1}):

Regularity: $F \in C^1([0, 1]^{N+1})$.

Monotonicity: $F(X_0, ..., X_N)$ is non-decreasing w.r.t. each X_i for $i \neq 0$.

Assumption (P'_{C^1}):

Positive monostability:
Let $f(v) = F(v, ..., v)$ such that $f(0) = 0 = f(1)$ and $f > 0$ in $(0, 1)$.

Nondegeneracy near $\{0\}^{N+1}$ and $\{1\}^{N+1}$:
There exists $\delta > 0$ such that

$$\begin{cases} f' > 0 \quad \text{on} \quad (0, \delta) \\ f' < 0 \quad \text{on} \quad (1 - \delta, 1) \end{cases}$$

Proposition 3.3 (Branch of solutions under smoothness assumptions)
Consider a function F *satisfying* (A_{C^1}) *and* (P'_{C^1}). *Then the result of Theorem 1.1 holds true.*

In order to give a proof of Proposition 3.3, we will use the following result.

Lemma 3.4 (Existence of a hull function ([18, Theorem 1.5 and Theorem 1.6, a1,a2]))
Assume that $\tilde{F}$ *satisfies* ($\tilde{A}_{Lip}$) *and let* $p > 0$ *and* $\sigma \in \mathbb{R}$. *There exists a unique real* $\lambda(\sigma, p) = \lambda_p(\sigma)$ *such that there exists a locally bounded function* $h_p : \mathbb{R} \to \mathbb{R}$ *satisfying (in the viscosity sense):*

$$\begin{cases} \lambda_p h_p'(z) = F((h_p(z + pr_i))_{i=0,...,N}) + \sigma \quad \textit{on} \quad \mathbb{R} \\ h_p(z + 1) = h_p(z) + 1 \\ h_p'(z) \geq 0 \\ |h_p(z + z') - h_p(z) - z'| \leq 1 \quad \textit{for any } z, \; z' \in \mathbb{R}. \end{cases}$$

Moreover, there exists a constant $K > 0$, independent on p and σ, such that

$$|\lambda_p - \sigma| \le K(1+p)$$

and the function

$$\lambda_p : \mathbb{R} \to \mathbb{R}$$
$$\sigma \mapsto \lambda_p(\sigma)$$

is continuous nondecreasing with $\lambda_p(\pm\infty) = \pm\infty$.

Proof of Proposition 3.3

Step 1: extension of F

We first extend F in $\tilde{F}$ on $\mathbb{R}^{N+1}$ using Lemma 3.2. We then consider a perturbation of the equation using an additional parameter σ. We consider solutions (c, ϕ) to

$$(3.3) \qquad \begin{cases} c\phi'(z) = \tilde{F}(\phi(z+r_0), \phi(z+r_1), ..., \phi(z+r_N)) + \sigma \quad \text{on} \quad \mathbb{R} \\ \phi \text{ is non-decreasing over } \mathbb{R} \\ \phi(-\infty) = m_\sigma \quad \text{and} \quad \phi(+\infty) = 1 + m_\sigma. \end{cases}$$

Here for

$$\sigma_\delta := -\min\{f(\delta), f(1-\delta)\} < 0$$

and for $\sigma \in (\sigma_\delta, 0]$ and $\tilde{f}(v) = \tilde{F}(v, \dots, v)$, we consider the unique roots m_σ, b_σ of

$$\tilde{f}(m_\sigma) + \sigma = 0, \quad m_\sigma \in (-\delta, 0], \quad \text{and} \quad \tilde{f}(b_\sigma) + \sigma = 0, \quad b_\sigma \in [0, \delta)$$

Now up to decrease $\delta > 0$ (and then $|\sigma_\delta|$), we can assume that we have for all $\sigma \in (\sigma_\delta, 0)$

$$\begin{cases} \tilde{f} + \sigma < 0 \quad \text{on} \quad (m_\sigma, b_\sigma) \\ \tilde{f} + \sigma > 0 \quad \text{on} \quad (b_\sigma, m_\sigma + 1) \\ \tilde{f}'(b_\sigma) > 0 \quad \text{and} \quad \tilde{f}'(m_\sigma) = \tilde{f}'(m_\sigma + 1) < 0. \end{cases}$$

which means that $\tilde{f} + \sigma$ is of bistable type on $[m_\sigma, 1 + m_\sigma]$.

Step 2: existence of solutions in the bistable case $\sigma \in (\sigma_\delta, 0)$

Then from Theorems 1.2 and 1.6 a) in [1], and assuming (A_{C^1}) and (P'_{C^1}), we know that for each $\sigma \in (\sigma_\delta, 0)$ as above, there exists a unique velocity $c = c(\sigma)$ such that there exists a solution (c, ϕ_c) of (3.3).

Step 3: definition of c^+

Step 3.1: bound from above on the velocity

Moreover from Step 1.2 of the proof of Proposition 3.1, we know the existence of solutions $(c_\varepsilon, \phi_{c_\varepsilon})$ for $\sigma = 0$ with $c_\varepsilon = \varepsilon^{-1}$ large enough. Then up to translate the profiles, we get $\phi_{c_\varepsilon}(x) \ge \phi_c(x)$, and the comparison for the evolution equation (see for instance [1])

$$u_t = \tilde{F}((u(x+r_i))_{i=0,\dots,N})$$

implies that $\phi_{c_\varepsilon}(x + c_\varepsilon t) \ge \phi_c(x + ct)$ which implies (using $\phi_{c_\varepsilon}(-\infty) < \phi_c(+\infty)$)

$$c_\varepsilon \ge c = c(\sigma)$$

A similar arguments allows to see that the map $\sigma \mapsto c(\sigma)$ is nondecreasing for $\sigma \in (\sigma_\delta, 0)$. Hence we can define

$$c^+ := \lim_{0 > \sigma \to 0^-} c(\sigma) \le c_\varepsilon < +\infty$$

Step 3.2: existence of a solution (c^+, ϕ^+) for $\sigma = 0$
We consider a sequence $\sigma_n \to 0^-$ and the associated sequence of solutions (c_n, ϕ_n) of (3.3) with $\sigma = \sigma_n$ and $c_n = c(\sigma_n)$. Up to translate ϕ_n, we can assume that

$$\frac{1}{2} \in [(\phi_n)_*(0), (\phi_n)^*(0)]$$

Then up to extract a subsequence, we have $\phi_n \to \phi^+$ at least almost everywhere. Moreover from the stability of viscosity solutions (see Proposition 2.4), we see that (c^+, ϕ^+) is still a solution of (3.3) for $\sigma = 0$, is a solution of (1.6).
Step 3.3: no solutions for $c < c^+$ and $\sigma = 0$
Assume that $(\tilde{c}, \tilde{\phi})$ is a solution for $\sigma = 0$. Then the comparison argument used in Step 3.1 shows that

$$c(\sigma) \leq \tilde{c} \quad \text{for all} \quad \sigma \in (\sigma_\delta, 0)$$

Taking the limit $\sigma \to 0^-$, we get

$$c^+ \leq \tilde{c}$$

This shows that for all $c < c^+$, there are no solutions (c, ϕ) for $\sigma = 0$.

Step 4: filling the gap: existence of solutions for each $c > c^+$
We follow the proof of Proposition 5.2 in [3].
Step 4.1: change of variables
We consider $c > c^+$. We want to show that there exists a solution ϕ_c of (1.6). To this end, we want to use the structure with perturbation σ, even in the absence of strong maximum principle. This is done in Lemma 3.4. Given c, we can choose $\sigma = \sigma(c, p)$ such that

$$\lambda_p = cp$$

This shows that the change of variables

$$\phi_p(x) := h_p(px)$$

satisfies

$$\begin{cases} c\phi_p'(x) = \tilde{F}((\phi_p(x + r_i))_{i=0,\dots,N}) + \sigma(c,p) \quad \text{on} \quad \mathbb{R} \\ \phi_p'(x) \geq 0 \\ \phi_p(x + \dfrac{1}{p}) = \phi_p(x) + 1 \end{cases}$$

Up to translate the profile, we can also assume that for some parameter $\theta \in [0, 1]$ we have

$$\theta \in [(\phi_p)_*(0), (\phi_p)^*(0)]$$

Step 4.2: passing to the limit $p \to 0^+$
Then we have sufficient compactness such that, up to extract a subsequence, we can to pass to the limit $\phi_p \to \phi$ almost everywhere and $\sigma(c, p) \to \sigma_0$ as $p \to 0^+$, and get

$$\begin{cases} c\phi'(x) = \tilde{F}((\phi(x + r_i))_{i=0,\dots,N}) + \sigma_0 \quad \text{on} \quad \mathbb{R} \\ \phi'(x) \geq 0 \\ \phi(+\infty) - \phi(-\infty) \leq 1 \\ \theta \in [(\phi_*(0), (\phi^*(0)] \end{cases}$$

At infinity, we get

$$\tilde{f}(\phi(\pm\infty)) + \sigma_0 = 0 \quad \text{with} \quad \phi(-\infty) \leq \theta \leq \phi(+\infty)$$

Because $\tilde{f} \geq 0$, we deduce that $\sigma_0 \leq 0$. Assume by contradiction that

$$\sigma_0 < 0$$

and let $\sigma \in (\sigma_\delta, 0)$ be such that

$$\sigma_0 < \sigma < 0$$

and choose

$$\theta = 0$$

This implies that

$$\phi(-\infty) < m_\sigma, \quad 0 < \phi(+\infty) < 1 + m_\sigma$$

Hence, up to translation, we can compare the profiles and get by comparison for all time $t \geq 0$ that

$$\phi^+(x + c^+ t) \geq \phi(x + ct)$$

which implies $c^+ \geq c$. Contradiction. We deduce that

$$\sigma_0 = 0$$

Now choosing

$$\theta = \frac{1}{2}$$

we deduce that

$$\tilde{f}(\phi(\pm\infty)) = 0, \quad \phi(-\infty) \leq \frac{1}{2} \leq \phi(+\infty), \quad \phi(+\infty) - \phi(-\infty) \leq 1$$

The fact that $\tilde{f} > 0$ on $(0,1)$ implies that

$$\phi(-\infty) = 0, \quad \phi(+\infty) = 1$$

and this shows that (c, ϕ) is a solution of the equation for $\sigma = 0$, i.e. of (1.6). Because this is true for each $c > c^+$, this ends the proof of the proposition.

3.4 Proof of Theorem 1.1

We are now ready to give a general proof of Theorem 1.1.

Proof of Theorem 1.1
The main idea consists to come back to the proof of Proposition 3.3, by approximation and comparison.
Step 1: definition of the approximation $\tilde{F}_\delta$
Given F defined on $[0,1]^{N+1}$ satisfying (A_{Lip}) and (P_{Lip}), we set for $X = (X_0, ..., X_N) \in [0,1]^{N+1}$ and $\delta > 0$ small

$$F_\delta(X) = F(X) - f(X_0) + f_\delta(X_0)$$

where

$$f_\delta(v) = \begin{cases} \max\big(f(\delta) + L_0(v - \delta), 0\big) & \text{on} \quad [0, \delta] \\ \max\big(f(1-\delta) - L_0(v - (1-\delta)), 0\big) & \text{on} \quad [1-\delta, 1] \\ f & \text{on} \quad [\delta, 1-\delta], \end{cases}$$

with a constant $L_0 > 0$ satisfying $L_0 > \text{Lip}(f)$. Notice that the choice of the constant L_0 allows to see that the map $\delta \mapsto f_\delta$ in nonincreasing for $\delta > 0$ small. Clearly, we also have $F_\delta(v, ..., v) = f_\delta(v)$.

We set

$$\begin{cases} b_\delta = \delta - \dfrac{f(\delta)}{L_0} > 0 \\ 1+m_\delta = 1-\delta + \dfrac{f(1-\delta)}{L_0} < 1 \end{cases}$$

which satisfy

$$0 < b_\delta < \delta < 1-\delta < 1+m_\delta < 1,$$

and

$$f_\delta(b_\delta) = 0 = f_\delta(1+m_\delta) \quad \text{and} \quad f_\delta > 0 \ \text{ on } \ (b_\delta, 1+m_\delta)$$

and moreover the comparison

$$0 \le f_\delta \le f \quad \text{on} \quad [0,1]$$

Let $\tilde f$ and $\tilde f_\delta$ be the 1-periodic extensions to $\mathbb{R}$ of the functions f, f_δ. Now let $\tilde F$ defined on $\mathbb{R}^{N+1}$ as the extension of the functions F to $\mathbb{R}^{N+1}$ given by Lemma 3.2, which satisfies $\tilde f(X_0) = \tilde F(X_0, \dots, X_0)$. We also define for $X = (X_0, ..., X_N) \in [0,1]^{N+1}$

$$\tilde F_\delta(X) = \tilde F(X) - \tilde f(X_0) + \tilde f_\delta(X_0)$$

Because $\tilde f_\delta \le \tilde f$, we see that we have the comparison

$$\tilde F_\delta \le \tilde F \quad \text{over} \quad \mathbb{R}^{N+1}.$$

Now given $\delta > 0$, for $\sigma < 0$ small fixed ($0 < -\sigma < \min_{[\delta,1-\delta]} f$), we define uniquely $0 < b_{\delta,\sigma} < 1+m_{\delta,\sigma} < 1$ such that

$$\begin{cases} \big(\tilde f_\delta + \sigma\big)(b_{\delta,\sigma}) = 0 = \big(\tilde f_\delta + \sigma\big)(1+m_{\delta,\sigma}) = \big(\tilde f_\delta + \sigma\big)(m_{\delta,\sigma}) \\ \tilde f_\delta + \sigma < 0 \quad \text{on} \quad (m_{\delta,\sigma}, b_{\delta,\sigma}) \\ \tilde f_\delta + \sigma > 0 \quad \text{on} \quad (b_{\delta,\sigma}, 1+m_{\delta,\sigma}) \\ \tilde f_\delta'(b_{\delta,\sigma}) = L_0 > 0, \quad \tilde f_\delta'(m_{\delta,\sigma}) = -L_0 < 0 \end{cases}$$

which shows that $\tilde f_\delta + \sigma$ is of bistable type on $[m_{\delta,\sigma}, 1+m_{\delta,\sigma}]$. Notice also that

$$\begin{cases} (-\delta, m_\delta] \ni m_{\delta,\sigma} & \to m_\delta \\ [b_\delta, \delta) \ni b_{\delta,\sigma} & \to b_\delta \end{cases} \quad \text{as} \quad \sigma \to 0^-.$$

Step 2: existence of a solution $(c_{\delta,\sigma}, \phi_{\delta,\sigma})$ for the nonlinearity F_δ for $\sigma < 0$ small
We are in the bistable case. Hence as in Step 2 of the proof of Proposition 3.3, still from Theorems 1.2 and 1.6 a) in [1], we see that there exists a unique velocity $c_{\delta,\sigma}$ such that there exists a solution $\phi_{\delta,\sigma}$ of

$$(3.4) \qquad \begin{cases} c_{\delta,\sigma}\phi_{\delta,\sigma}'(x) = \tilde F_\delta((\phi_{\delta,\sigma}(x+r_i))_{i=0,\dots,N}) + \sigma \quad \text{on} \quad \mathbb{R} \\ \phi_{\delta,\sigma} \quad \text{is non-decreasing over } \mathbb{R} \\ \phi_{\delta,\sigma}(-\infty) = m_{\delta,\sigma} \quad \text{and} \quad \phi_{\delta,\sigma}(+\infty) = 1+m_{\delta,\sigma}. \end{cases}$$

Step 2.1: $c_{\delta,\sigma}$ is nondecreasing in σ for δ fixed
A variant of Step 3.1 of the proof of Proposition 3.3 shows that the map

$$(3.5) \qquad \sigma \mapsto c_{\delta,\sigma} \quad \text{is nondecreasing}$$

which follows from the fact that the map $\sigma \mapsto m_{\delta,\sigma}$ is nondecreasing.
Step 2.2: $c_{\delta,\sigma}$ is nonincreasing in δ for σ fixed
Similarly to Step 2.1 above, we deduce that the map

$$\delta \mapsto c_{\delta,\sigma} \quad \text{is nonincreasing}$$

which follows from the fact that the map $\delta \mapsto m_{\delta,\sigma}$ is nonincreasing.

Step 3: definition of c^+
Step 3.1: first, passing to the limit $\sigma \to 0^-$
As in Step 3.1 of the proof of Proposition 3.3, we know that there exists a solution $(c_\varepsilon, \phi_{c_\varepsilon})$ of

$$\begin{cases} c_\varepsilon \phi'(z) = \tilde{F}((\phi_{c_\varepsilon}(x+r_i))_{i=0,\dots,N}) \quad \text{on} \quad \mathbb{R} \\ \phi_{c_\varepsilon} \text{ is non-decreasing over } \mathbb{R} \\ \phi_{c_\varepsilon}(-\infty) = 0 \quad \text{and} \quad \phi_{c_\varepsilon}(+\infty) = 1. \end{cases}$$

with $c_\varepsilon = \varepsilon^{-1}$ large enough. In particular it satisfies for $\sigma < 0$

$$\begin{cases} c_\varepsilon \phi'_{c_\varepsilon}(z) \geq \tilde{F}_\delta((\phi_{c_\varepsilon}(x+r_i))_{i=0,\dots,N}) + \sigma \quad \text{on} \quad \mathbb{R} \\ \phi_{c_\varepsilon}(-\infty) > m_\delta > m_{\delta,\sigma}, \quad \phi_{c_\varepsilon}(+\infty) > 1 + m_{\delta,\sigma} > \phi_{c_\varepsilon}(-\infty) \end{cases}$$

Again the comparison $\phi_{c_\varepsilon}(x + c_\varepsilon t) \geq \phi_{\delta,\sigma}(x + c_{\delta,\sigma} t)$ implies

$$c_{\delta,\sigma} \leq c_\varepsilon < +\infty$$

We can then define (using the monotonicity in σ)

$$c_\delta^+ := \lim_{0>\sigma\to 0^-} c_{\delta,\sigma} \leq c_\varepsilon < +\infty$$

and from (3.5), we deduce the following monotonicity

$$\delta \mapsto c_\delta^+ \quad \text{is nonincreasing.}$$

Now up to extract a subsequence, we have $\phi_{\delta,\sigma} \to \phi_\delta^+$ almost everywhere as $\sigma \to 0^-$ and up to translate the profile $\phi_{\delta,\sigma}$ correctly, we can get (passing to the limit in (3.4))

$$\begin{cases} c_\delta^+ (\phi_\delta^+)'(x) = \tilde{F}_\delta((\phi_\delta^+(x+r_i))_{i=0,\dots,N}) \quad \text{on} \quad \mathbb{R} \\ \phi_\delta^+ \text{ is nondecreasing over } \mathbb{R} \\ m_\delta \leq \phi_\delta^+(-\infty) \quad \text{and} \quad \phi_\delta^+(+\infty) \leq 1 + m_\delta \\ \dfrac{b_\delta + m_\delta}{2} \in [(\phi_\delta^+)_*(0), (\phi_\delta^+)^*(0)] \end{cases} \tag{3.6}$$

This shows in particular that

$$m_\delta \leq \phi_\delta^+(-\infty) \leq b_\delta, \quad \phi_\delta^+(+\infty) = 1 + m_\delta \tag{3.7}$$

Step 3.2: second, passing to the limit $\delta \to 0^+$
Using the monotinicity of the map $\delta \mapsto c_\delta^+$ and the fact that $c_\delta^+ \leq c_\varepsilon < +\infty$, we can define the finite limit

$$c^+ := \lim_{\delta\to 0^+} c_\delta^+$$

Again up to extract a subsequence, we have $\phi_\delta \to \phi^+$ almost everywhere and up to translate the profile ϕ_δ^+ correctly, we can get (passing to the limit in (3.6) with $b_\delta, m_\delta \to 0$)

$$\begin{cases} c^+ (\phi^+)'(x) = \tilde{F}((\phi^+(x+r_i))_{i=0,\dots,N}) \quad \text{on} \quad \mathbb{R} \\ \phi^+ \text{ is non-decreasing over } \mathbb{R} \\ 0 \leq \phi^+(-\infty) \quad \text{and} \quad \phi^+(+\infty) \leq 1 \\ \dfrac{1}{2} \in [(\phi^+)_*(0), (\phi^+)^*(0)] \end{cases} \tag{3.8}$$

which implies

$$\phi^+(-\infty) = 0 \quad \text{and} \quad \phi^+(+\infty) = 1.$$

This shows that (c^+, ϕ^+) is a solution of (1.6).

Step 3.3: no solutions for $c < c^+$
Assume that (c, ϕ_c) is a solution of (1.6). Then we can apply the reasoning of Step 3.1 with $(c_\varepsilon, \phi_{c_\varepsilon})$ replaces by (c, ϕ_c). We finally get

$$c^+ \le c$$

and we deduce that there is no solution (c, ϕ_c) of (1.6) with $c < c^+$.

Step 4: filling the gap: existence of solutions for each $c > c^+$
Recall that we have reached both the existence of a solution (c^+, ϕ^+) for $\tilde{F}$ and $\sigma = 0$, and also for $\delta > 0$ small enough the existence of $(c_\delta^+, \phi_\delta)$ solution of (3.6)-(3.7) with $\tilde{F}_\delta$ and $\sigma = 0$, where $c_\delta^+ \le c^+$.
Step 4.1: changes of variables
We choose any $c > c^+ \ge c_\delta^+$ and proceed exactly as in Step 4.1 of the proof of Proposition 3.3, but with $\tilde{F}$ replaced by $\tilde{F}_\delta$. We get here $\lambda_p = cp$ for $\sigma = \sigma_\delta(c, p)$.
Step 4.2: passing to the limit $p \to 0^+$
Again we get $\sigma_\delta(c, p) \to \sigma_0 \in \mathbb{R}$, and for any given $\theta \in [0, 1]$, we get the existence of some $\phi = \phi_\delta$ solution of

$$\begin{cases} c\phi_\delta'(x) = \tilde{F}_\delta((\phi_\delta(x + r_i))_{i=0,\dots,N}) + \sigma_0 \quad \text{on} \quad \mathbb{R} \\ \phi_\delta'(x) \ge 0 \\ \phi_\delta(+\infty) - \phi_\delta(-\infty) \le 1 \\ \theta \in [(\phi_\delta)_*(0), (\phi_\delta)^*(0)] \end{cases}$$

At infinity, we get

$$\tilde{f}_\delta(\phi_\delta(\pm\infty)) + \sigma_0 = 0 \quad \text{with} \quad \phi_\delta(-\infty) \le \theta \le \phi_\delta(+\infty)$$

Because $\tilde{f}_\delta \ge 0$, we deduce that $\sigma_0 \le 0$. Assume by contradiction that

$$\sigma_0 < 0$$

and let $\sigma \in (\sigma_\delta, 0)$ be such that

$$\sigma_0 < \sigma < 0$$

and choose

$$\theta = 0$$

This implies that

$$\phi_\delta(-\infty) < m_{\delta,\sigma} \le m_\delta, \quad 0 < \phi_\delta(+\infty) < 1 + m_{\delta,\sigma} \le 1 + m_\delta$$

Recall also that (3.7) means

$$m_\delta \le \phi_\delta^+(-\infty) \le b_\delta, \quad \phi_\delta^+(+\infty) = 1 + m_\delta$$

Again, up to translation, we can compare the profiles and get by comparison for all time $t \ge 0$ that

$$\phi_\delta^+(x + c_\delta^+ t) \ge \phi_\delta(x + ct)$$

which implies $c_\delta^+ \ge c$. Contradiction. We deduce that

$$\sigma_0 = 0$$

Step 4.3: passing to the limit $\delta \to 0^+$
Now for the choice $\theta = \frac{1}{2}$ and up to extract a subsequence, we get $\phi_\delta \to \phi$ a.e. as $\delta \to 0^+$ and the limit solves

$$\begin{cases} c\phi'(x) = \tilde{F}((\phi(x+r_i))_{i=0,\dots,N}) \quad \text{on} \quad \mathbb{R} \\ \phi'(x) \geq 0 \\ \phi(+\infty) - \phi(-\infty) \leq 1 \\ \dfrac{1}{2} \in [\phi_*(0), \phi^*(0)] \end{cases}$$

with

$$\tilde{f}(\phi(\pm\infty)) = 0$$

Again because $\tilde{f} > 0$ on $(0,1)$, it forces

$$\phi(-\infty) = 0, \quad \phi(-\infty) = 1$$

and this shows that (c, ϕ) is a solution of the equation for $\sigma = 0$, i.e. of (1.6). Because this is true for each $c > c^+$, this ends the proof of the theorem.

4 Preliminaries on Harnack inequalities

The goal of this section is to prove Harnack inequalities (Propositions 4.4 and 4.6), that we will use in the next section to show that $c^+ \geq c^*$ under certain assumptions. Recall that such Harnack inequalities may fail in general (see Remark 4.7).

We start with the following strong maximum principle for a linear evolution problem.

Proposition 4.1 (A strong maximum principle for a linear evolution problem)
Let F be a function satisfying $(\tilde{A}_{Lip})$ and differentiable at $\{0\}^{N+1}$. Assume that

$$\exists \ i_0 \in \{0, ..., N\} \ \text{ such that } \ r_{i_0} \in \mathbb{R} \ \text{ and } \ \frac{\partial F}{\partial X_{i_0}}(0, ..., 0) > 0.$$

Let $T > 0$ and $u \ : \mathbb{R} \times [0, T) \to [0, +\infty)$ be a lower semi-continuous function which is a (viscosity) supersolution of the linear equation

$$u_t(x,t) = \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0, ..., 0) u(x + r_i, t) \quad \text{for} \quad (x,t) \in \mathbb{R} \times (0,T). \tag{4.1}$$

If $u(x_0, t_0) = 0$ for some $(x_0, t_0) \in \mathbb{R} \times (0, T)$, then

$$u(x_0 + kr_{i_0}, t) = 0 \quad \text{for all} \ \ k \in \mathbb{N} \ \ \text{and} \ \ 0 \leq t \leq t_0.$$

Proof of Proposition 4.1
Let u be a lower semi-continuous supersolution of (4.1) such that $u \geq 0$ and assume that there exists some $(x_0, t_0) \in \mathbb{R} \times (0, T)$ such that $u(x_0, t_0) = u_*(x_0, t_0) = 0$.
Step 1: $u(x_0, t) = 0$ for all $t \in [0, t_0]$
Step 1.1: $u(x_0, \cdot)$ is a viscosity supersolution of (4.2) on $(0, T)$
Because $u \geq 0$ and $\frac{\partial F}{\partial X_i}(0, \dots, 0) \geq 0$ for $i \neq 0$, we deduce that u satisfies in the viscosity sense

$$u_t(x,t) \geq -Lu \quad \text{with} \quad L := \left| \frac{\partial F}{\partial X_0}(0, \dots, 0) \right|$$

Recall also that to check the inequality in the viscosity sense, we have to replace as usual the derivatives of u (where we need them) by the derivative of the test function, i.e. here we have only to do it for u_t.

Now setting $v(t) = u_*(x_0, t)$, we claim that v satisfies in the viscosity sense

$$v_t \geq -Lv \quad \text{on} \quad (0, T). \tag{4.2}$$

This is indeed quite classical but we still explain it. Consider a smooth test function ϕ touching $v_* = v$ from below at some time $t_0 \in (0, T)$. We can moreover assume that the contact is strict, i.e. that

$$\phi \leq v_* \quad \text{with equality only at } t_0$$

Then, classically, we penalize ϕ around the space position x_0 as a new function

$$\phi_\varepsilon(x, t) := \phi(t) - \varepsilon^{-1}|x - x_0|^2$$

Now for any $r, \rho > 0$ small enough we can define the cylinder

$$Q_{\rho,r} := I_x \times I_t \subset \mathbb{R} \times (0, T) \quad \text{with} \quad I_x := [x_0 - \rho, x_0 + \rho], \quad I_t := [t_0 - r, t_0 + r]$$

Moreover, for $\rho > 0$ small enough (depending on r), we have (from the strict contact)

$$\phi_\varepsilon < u_* \quad \text{on} \quad I_x \times (\partial I_t)$$

Now for $\varepsilon > 0$ small enough (depending on r, ρ), we also get that

$$\phi_\varepsilon < u_* \quad \text{on} \quad (\partial I_x) \times I_t$$

We can moreover choose $r = r_\rho$ and $\rho = \rho_\varepsilon$ as sequences as $\varepsilon \to 0$ such that

$$r_\varepsilon, \rho_\varepsilon \to 0^+$$

and because

$$\phi_\varepsilon = u_* \quad \text{at} \quad (x_0, t_0) =: P_0$$

we deduce that

$$\phi_\varepsilon \leq u_* - c_\varepsilon \quad \text{with equality at} \quad P_\varepsilon \in \text{Int}(Q_{\rho_\varepsilon, r_\varepsilon})$$

with

$$(x_\varepsilon, t_\varepsilon) = P_\varepsilon \to P_0 \quad \text{and} \quad 0 \geq c_\varepsilon$$

In particular w ehave

$$\varepsilon^{-1}|x_\varepsilon - x_0|^2 + u_*(x_\varepsilon, t_\varepsilon) - \phi(t_\varepsilon) = c_\varepsilon$$

and taking the lim inf as $\varepsilon \to 0$, we deduce that (up to extract a subsequence)

$$c_\varepsilon \to 0, \quad \varepsilon^{-1}|x_\varepsilon - x_0|^2 \to 0, \quad u_*(x_\varepsilon, t_\varepsilon) \to u_*(x_0, t_0)$$

On the other hand the viscosity inequality for u_* gives with $\tilde{\phi}_\varepsilon := \phi_\varepsilon + c_\varepsilon$ that

$$\partial_t \tilde{\phi}_\varepsilon(P_\varepsilon) \geq -L\tilde{\phi}_\varepsilon(P_\varepsilon)$$

and then at the limit

$$\partial_t \phi(t_0) \geq -L\phi(t_0)$$

which means precisely that v is a viscosity supersolution, i.e. satisfies (4.2).

Step 1.2: conclusion

Setting for any $s_0 \in (0, t_0)$

$$w(t) := e^{-L(t - s_0)} v(s_0)$$

we see that w is a solution of the ODE

$$\partial_t w = -Lw$$

while v is a supersolution on (s_0, t_0) which coincides with w at time $t = s_0$. Then the comparison principle for ODEs (in the viscosity sense) implies that

$$w \leq v \quad \text{on} \quad [s_0, t_0]$$

Now the fact that $u(x_0, t_0) = 0 = v(t_0)$ implies that

$$0 = v(s_0) = u_*(s_0, x_0)$$

This shows that

$$u_*(t, x_0) = 0 \quad \text{for all} \quad t \in [0, t_0]$$

Because u is lower semi-continuous, this gives the expected result for $u = u_*$.
Step 2: $u(x_0 + r_{i_0}, t_0) = 0$
Using the test function $\phi \equiv 0$ at (x_0, t_0) we get

$$\begin{aligned} 0 = \phi_t(x_0, t_0) \quad &\geq \quad \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0, ..., 0) u(x_0 + r_i, t_0) \\ &\geq \quad \frac{\partial F}{\partial X_0}(0, ..., 0) u(x_0, t_0) + \frac{\partial F}{\partial X_{i_0}}(0, ..., 0) u(x_0 + r_{i_0}, t_0), \end{aligned}$$

Because $u(x_0, t_0) = 0$ and $\frac{\partial F}{\partial X_{i_0}}(0, ..., 0) > 0$, we deduce that

$$u(x_0 + r_{i_0}, t_0) = 0.$$

Step 3: $u(x_0 + kr_{i_0}, t) = 0$ **for** $k \in \mathbb{N}$ **and** $t \in [0, t_0]$
We just apply Steps 1 and 2 iteratively. This ends the proof of the Proposition.

Now, we introduce a nonlinear problem whose linearization around 0 is the linear problem studied in Proposition 4.1 for which we have a strong maximum principle under certain assumptions. In the next result we first show the existence of a solution to the nonlinear problem, and later we will give a bound from below on this solution under certain assumptions.

Lemma 4.2 (Existence of a solution to the nonlinear problem)
Consider a function F satisfying $(\tilde{A}_{Lip})$ such that $F_{|[0,1]^{N+1}}$ satisfies (P_{Lip}) and let $\varepsilon \in (0, 1]$. Then there exists $\psi : \mathbb{R} \times (0, +\infty) \to \mathbb{R}$ viscosity solution of

$$\psi_t(x, t) = F((\psi(x + r_i, t))_{i=0,...,N}) \quad \text{on} \quad \mathbb{R} \times (0, +\infty) \tag{4.3}$$

with initial condition satisfying

$$\psi^*(\cdot, 0) = \varepsilon H^* \quad \text{and} \quad \psi_*(\cdot, 0) = \varepsilon H_*, \tag{4.4}$$

where $H = 1_{[0,+\infty)}$ is the Heaviside function.

Proof of Lemma 4.2
The construction of ψ is naturally done by approximation.

Step 1: construction of ψ_δ solution of (4.3)
Let $\delta > 0$ and for

$$H(x) \le H_\delta(x) := \begin{cases} 0 & \text{if} \quad x \le -\delta \\ \dfrac{x}{\delta} + 1 & \text{if} \quad x \in [-\delta, 0] \\ 1 & \text{if} \quad x \ge 0 \end{cases}$$

Because H_δ is bounded and uniformly continuous, we know ([18, Corollary 2.9]) that there is a unique continuous solution ψ_δ of (4.3) with the prescribed initial data for $\varepsilon \in (0, 1]$

$$\psi_\delta(x, 0) := \varepsilon H_\delta(x) \quad \text{for all} \quad x \in \mathbb{R}$$

Step 2: properties of ψ_δ
Recall that equation (4.3) admits a comparison principle (see [18, Proposition 2.5]). We then deduce that

$$0 \le \psi_\delta \le 1$$

Moreover, since the map $\delta \mapsto H_\delta$ is nondecreasing for $\delta > 0$, we deduce the same property for ψ_δ. Moreover since $H_\delta(x+h) \ge H_\delta(x)$ for all $h \ge 0$, we deduce the same property for ψ_δ, which shows that the map $x \mapsto \psi_\delta(x,t)$ is nondecreasing.
Similarly, using the bound

$$\sup_{[0,1]^{N+1}} |F| \le C_0$$

we can deduce that

$$|\psi_\delta(x,t) - \psi_\delta(x)| \le C_0 t$$

and similarly that

$$|\psi_\delta(x,t) - \psi_\delta(x,s)| \le C_0 |t-s|$$

Step 3: the limit $\delta \to 0$
Since the map $\delta \mapsto \psi_\delta$ is nondecreasing for $\delta > 0$, we can define the pointwise limit

$$\psi := \lim_{\delta \to 0^+} \psi_\delta$$

Using the stablity of viscosity solutions, we deduce that ψ_* and ψ^* are respectively supersolution and subsolution of (4.3) on $\mathbb{R} \times (0, +\infty)$. Moreover we deduce also that ψ satisfies

$$\begin{cases} \psi \quad \text{is nondecreasing w.r.t.} \quad x \\ |\psi(x,t) - \psi(x,s)| \le C_0|t-s| \quad \text{for all} \quad x \in \mathbb{R},\ t,s \in [0,+\infty). \end{cases}$$

Moreover, the fact that those properties are also satisfied by ψ_δ uniformly in $\delta > 0$, joint to the fact that

$$\psi_\delta(x, 0) = \varepsilon H_\delta(x)$$

and the good convergence $H_\delta \to H$ outside the origin, implies easily that at the limit we have

$$(\psi^*)(x,0) = \varepsilon H^*(x), \quad (\psi_*)(x,0) = \varepsilon H_*(x)$$

This ends the proof of the lemma.

Proposition 4.3 (Lower bound on a solution to the nonlinear problem)
Consider a function F satisfying $(\tilde{A}_{Lip})$ such that $F_{|[0,1]^{N+1}}$ satisfies (P_{Lip}). Assume moreover that F is C^1 over a neighborhood of $\{0\}^{N+1}$ in $[0,1]^{N+1}$ and

$$\exists\ i_0 \in \{1, ..., N\} \quad \textit{such that} \quad r_{i_0} > 0 \quad \textit{and} \quad \frac{\partial F}{\partial X_{i_0}}(0, ..., 0) > 0.$$

Then there exists $\varepsilon_0 \in (0,1]$ *and* $T_0 > 0$ *such that for all* $\delta \in (0, T_0)$ *and* $R > 0$*, there exists* $\kappa = \kappa(\delta, R) > 0$ *such that for every* $0 < \varepsilon \leq \varepsilon_0$*, the function* $\psi = \psi_\varepsilon$ *given by Lemma 4.2 with initial conditions* (4.4) *satisfies*

$$\psi_\varepsilon(x,t) \geq \kappa\varepsilon \quad \textit{for all} \quad (x,t) \in [-R,R] \times [\delta, T_0]. \tag{4.5}$$

Proof of Proposition 4.3

We first give an upper bound proportional to ε on the solution $\psi = \psi_\varepsilon$ of (4.3) and then prove the lower bound by contradiction.

Step 1: refined upper bound on ψ on $\mathbb{R} \times [0, 2T_0]$

Let

$$M(t) := \sup_{x \in \mathbb{R}} \psi^*(x,t)$$

It is easy to see that $M(0) = \varepsilon$ and that M satisfies in the viscosity sense the ODE inequality

$$\partial_t M \leq F(M, \dots, M) = f(M) \quad \text{on} \quad (0, +\infty)$$

Then it is natural to introduce the solution M_0 of the ODE

$$\begin{cases} M_0'(t) = f(M_0(t)) \geq 0 \quad \text{for} \quad (0,+\infty) \\ M_0(0) = \varepsilon. \end{cases}$$

Using

$$L_1 := \mathrm{Lip}(f)$$

we get

$$0 \leq f(v) \leq 2L_1\varepsilon \quad \text{for} \quad v \in [0, 2\varepsilon]$$

and then

$$M_0(t) \leq \varepsilon + 2L_1\varepsilon t \leq 2\varepsilon \quad \text{for all} \quad t \in [0, 2T_0] \quad \text{with} \quad T_0 := \frac{1}{4L_1}$$

The comparison of the subsolution M with the solution M_0 shows that $M \leq M_0$ and then

$$0 \leq \psi_\varepsilon(x,t) \leq 2\varepsilon \quad \text{for all} \quad t \in [0, 2T_0]$$

Step 2: establishing (4.5)

Given T_0 as in Step 1, assume by contradiction that (4.5) is false. Then there exist $\delta \in (0, T_0)$, $R > 0$ and sequences $\varepsilon_n \to 0$, $\kappa_n \to 0$ and points such that

$$\psi_{\varepsilon_n}(P_n) \leq \kappa_n \varepsilon_n \quad \text{with} \quad P_n = (x_n, t_n) \in [-R, R] \times [\delta, T_0]$$

Then we can define

$$\overline{\psi}_n(x,t) := \frac{1}{\varepsilon_n}\psi_{\varepsilon_n}(x,t) \quad \text{for all} \quad (x,t) \in \mathbb{R} \times [0, 2T_0]$$

which satisfies as $n \to +\infty$

$$\begin{cases} 0 \leq \overline{\psi}_n \leq 2 \quad \text{over} \quad \mathbb{R} \times [0, 2T_0] \\ \overline{\psi}_n(P_n) \leq \kappa_n \to 0 \\ (\overline{\psi}_n)_*(x, t=0) = H_*(x) \end{cases}$$

and

$$(\overline{\psi}_n)_t(x,t) = \frac{1}{\varepsilon_n} F(\varepsilon_n(\overline{\psi}_n(x + r_i, t))_{i=0,\dots,N}).$$

Step 2.1: uniform lower bound on $\overline{\psi}_n$
Denote by $Z = (\overline{\psi}_n(x + r_i, t))_{i=0,...,N}$. Since F is C^1 over a neighborhood of $\{0\}^{N+1}$, then for ε_n small enough, we get with $L := \sup_{[0,1]^{N+1}} \left|\frac{\partial F}{\partial X_0}\right|$

$$\begin{aligned}(\overline{\psi}_n)_t(x,t) &= \frac{1}{\varepsilon_n}F(\varepsilon_n(\overline{\psi}_n(x+r_i,t))_{i=0,...,N}) \\ &= \sum_{i=0}^{N}\int_0^1 \frac{\partial F}{\partial X_i}(s\varepsilon_n Z)\overline{\psi}_n(x+r_i,t)ds \\ &\geq -L\overline{\psi}_n(x,t),\end{aligned}$$

where we have used the fact that $\overline{\psi}_n \geq 0$ and $\frac{\partial F}{\partial X_i} \geq 0$ for all $i \neq 0$. Hence $\overline{\psi}_n$ is a supersolution of the linear equation

$$w_t(x,t) = -Lw(x,t). \tag{4.6}$$

Setting for $\eta > 0$ small

$$\tilde{H}_\eta(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ \dfrac{x}{\eta} & \text{if} \quad 0 \leq x \leq \eta \\ 1 & \text{if} \quad x \geq \eta \end{cases}$$

we see that

$$\phi(x,t) := e^{-Lt}H_\eta(x)$$

is a subsolution of (4.6), which satisfies moreover

$$\phi(x,t=0) = H_\eta(x) \leq H_*(x) \leq (\overline{\psi}_n)_*(x,t=0)$$

Therefore, using a comparison principle for (4.6), we deduce the following lower bound

$$e^{-Lt}H_\eta(x) \leq \overline{\psi}_n(x,t) \quad \text{for all} \quad (x,t) \in \mathbb{R}\times[0,2T_0).$$

Step 2.2: passing to the limit and getting a contradiction
Using our bounds, we can define the semi-relaxed limit

$$\overline{\psi}_\infty = \liminf_{n\to+\infty}{}_*\overline{\psi}_n$$

which satisfies (up to extract subsequences) with $P_n \to P_\infty = (x_\infty, t_\infty) \in [-R,R]\times[\delta,T_0]$

$$\begin{cases} 0 \leq \overline{\psi}_\infty \leq 2 \quad \text{on} \quad \mathbb{R}\times[0,2T_0) \\ \overline{\psi}_\infty(P_\infty) = 0 \\ e^{-Lt}H_\eta(x) \leq \overline{\psi}_\infty(x,t) \quad \text{for all} \quad (x,t)\in\mathbb{R}\times[0,2T_0). \end{cases}$$

and passing also to the limit in the equation, we deduce that

$$\partial_t\overline{\psi}_\infty(x,t) \geq \sum_{i=0}^{N}\frac{\partial F}{\partial X_i}(0,...,0)\overline{\psi}_\infty(x+r_i,t) \quad \text{on} \quad \mathbb{R}\times[0,2T_0)$$

Then the strong maximum principle (Proposition 4.1) shows for all $k\in\mathbb{N}$ that

$$\overline{\psi}_\infty(x_\infty + kr_{i_0}, t) = 0 \quad \text{for all} \quad 0 \leq t \leq t_\infty.$$

For $k >> 1$, this leads to a contradiction

$$1 = H_\eta(x_\infty + kr_{i_0}) \leq \overline{\psi}_\infty(x_\infty + kr_{i_0}, 0) = 0.$$

We conclude that (4.5) holds true and this ends the proof of the proposition.

We are now ready to give the main result of this section.

Proposition 4.4 (Harnack inequality)
Let F be a function satisfying (A_{Lip}), (P_{Lip}) and assume that F is C^1 over a neighborhood of $\{0\}^{N+1}$ in $[0,1]^{N+1}$. Assume moreover that

$$\exists\ i_0 \in \{1, ..., N\} \quad \text{such that} \quad r_{i_0} > 0 \quad \text{and} \quad \frac{\partial F}{\partial X_{i_0}}(0, ..., 0) > 0.$$

Let $c \in \mathbb{R}$. Then for every $\rho > 0$ there exists constants $\overline{\kappa}_1 = \overline{\kappa}_1(\rho, c) > 1$ and $\overline{\kappa}_0 = \overline{\kappa}_0(\rho, c) > 1$ such that for any solution u of

$$\begin{cases} cu'(x) = F((u(x + r_i))_{i=0,...,N}) & \text{on} \quad \mathbb{R} \\ u' \geq 0 \\ u(-\infty) = 0 \quad \text{and} \quad u(+\infty) = 1. \end{cases}$$

we have

$$\sup_{B_\rho(x)} u \leq \overline{\kappa}_1 \inf_{B_\rho(x)} u \quad \text{for all} \quad x \in \mathbb{R}.$$

and

$$u(x + r^*) \leq \overline{\kappa}_0 u(x),$$

where $r^* = \max_{i=0,...,N} |r_i|$.

Proof of Proposition 4.4
Let $\tilde{F}$ be the extension of F on $\mathbb{R}^{N+1}$ given by Lemma 3.2. Then it is easy to check that the function

$$\overline{u}(x, t) := u(x + ct)$$

satisfies in the viscosity sense the equation

$$\overline{u}_t(x, t) = \tilde{F}((\overline{u}(x + r_i, t))_{i=0,...,N}) \quad \text{for all} \quad (x, t) \in \mathbb{R} \times (0, +\infty)$$

and

$$\overline{u}(x, 0) = u(x).$$

Let $x_0 \in \mathbb{R}$ such that $1 \geq u(x_0) > 0$. Since u is nondecreasing, we have

$$\overline{u}(x, 0) \geq u(x_0)H(x - x_0),$$

where $H = 1_{[0,+\infty)}$ is the Heaviside function. For $\varepsilon \in (0, 1]$ that will be fixed later, let $\psi_\varepsilon = \psi$ be the solution given by Lemma 4.2 with initial condition $\psi_\varepsilon(x, 0) = \varepsilon H(x)$ in the sense of (4.4) and let

$$\overline{v}(x, t) := \psi_\varepsilon(x - x_0, t).$$

Now, using Proposition 4.3, we deduce that there exists some $\varepsilon_0 \in (0, 1]$ and T_0 such that for all $\delta \in (0, T_0)$ and $R > 0$ there exists a constant $\kappa = \kappa(\delta, R) > 0$ such that if $\varepsilon \leq \varepsilon_0$, then

$$\overline{v}(x, t) \geq \varepsilon\kappa \quad \text{for all} \quad (x, t) \in [x_0 - R, x_0 + R] \times [\delta, T_0]. \tag{4.7}$$

Case 1: $u(x_0) \le \varepsilon_0$
We now choose
$$\varepsilon = u(x_0) > 0$$
In particular, we have
$$\overline{u}(x,0) \ge \overline{v}^*(x,0) \quad \text{for all} \quad x \in \mathbb{R}.$$
Using the comparison principle (see [18, Proposition 2.5]), we deduce that
$$\overline{u} \ge \overline{v} \quad \text{for all} \quad (x,t) \in \mathbb{R} \times (0,+\infty).$$
From (4.7), we deduce that
$$\overline{u} \ge \kappa u(x_0) \quad \text{on} \quad [x_0 - R, x_0 + R] \times [\delta, T_0].$$
Because $\overline{u}(x,t) = u(x+ct)$, we conclude that
$$\inf_{(x,t)\in[x_0-R,x_0+R]\times[\delta,T_0]} u(x+ct) \ge \kappa u(x_0).$$
Now, for any $\rho > 0$, we can find $R_\rho > 0$ large enough such that
$$\overline{B}_{2\rho}(x_0) \subset \overline{B}_{R_\rho}(x_0) + ct \quad \text{for all} \quad t \in [\delta, T_0].$$
Therefore, since u is nondecreasing, we deduce that
$$u(x_0 - 2\rho) = \inf_{x\in\overline{B}_{2\rho}(x_0)} u(x) \ge \inf_{(x,t)\in[x_0-R_\rho,x_0+R_\rho]\times[\delta,T_0]} u(x+ct) \ge \kappa u(x_0)$$
with $\kappa = \kappa(R_\rho)$.
Case 2: $u(x_0) > \varepsilon_0$
Then choosing
$$\varepsilon = \varepsilon_0$$
we deduce again that
$$u(x_0 - 2\rho) \ge \kappa\varepsilon_0 \ge \kappa\varepsilon_0 u(x_0)$$

Conclusion
Hence setting
$$\kappa_1 := \frac{1}{\kappa\varepsilon_0}$$
we see that in both cases 1 and 2, we get
$$u(x_0) \le \overline{\kappa}_1 u(x_0 - 2\rho) \tag{4.8}$$
Hence for $y := x_0 - \rho$, we get in particular
$$\sup_{B_\rho(y)} u \le \overline{\kappa}_1 \inf_{B_\rho(y)} u$$
Moreover the choice $2\rho := r^*$ in (4.8) gives
$$u(x + r^*) \le \overline{\kappa}_0 u(x) \quad \text{with} \quad \overline{\kappa}_0 := \overline{\kappa}_{1|\rho := r^*/2}$$
This ends the proof of the proposition.

Because the proofs are similar to the original ones, we now give without proofs two results (Propositions 4.5 and 4.6) which are direct adaptations of the proofs of Propositions 4.3 and 4.4.

Proposition 4.5 (Lower bound on a positive segment $[\delta, R]$ when $r_{i_0} < 0$; variant of Proposition 4.3)
Consider a function F satisfying $(\tilde{A}_{Lip})$ such that $F_{|[0,1]^{N+1}}$ satisfies (P_{Lip}). Assume moreover that F is C^1 over a neighborhood of $\{0\}^{N+1}$ in $[0,1]^{N+1}$ and

$$\exists\ i_0 \in \{1, ..., N\} \quad \text{such that} \quad r_{i_0} < 0 \quad \text{and} \quad \frac{\partial F}{\partial X_{i_0}}(0, ..., 0) > 0.$$

Then there exists $\varepsilon_0 \in (0,1]$ and $T_0 > 0$ such that for all $\delta \in (0, T_0)$ and $R > 0$, there exists $\kappa = \kappa(\delta, R) > 0$ such that for every $0 < \varepsilon \leq \varepsilon_0$, the function $\psi = \psi_\varepsilon$ given by Lemma 4.2 with initial conditions (4.4) satisfies

$$\psi_\varepsilon(x,t) \geq \kappa\varepsilon \quad \text{for all} \quad (x,t) \in [\delta, R] \times [\delta, T_0].$$

Based on Proposition 4.5, we can then show the following result.

Proposition 4.6 (Harnack inequality for $c < 0$ when $r_{i_0} < 0$; variant of Proposition 4.4)
Let F be a function satisfying (A_{Lip}), (P_{Lip}) and assume that F is C^1 over a neighborhood of $\{0\}^{N+1}$ in $[0,1]^{N+1}$. Assume moreover that

$$\exists\ i_0 \in \{1, ..., N\} \quad \text{such that} \quad r_{i_0} < 0 \quad \text{and} \quad \frac{\partial F}{\partial X_{i_0}}(0, ..., 0) > 0.$$

Let $c \in (-\infty, 0)$. Then for every $\rho > 0$ there exists constants $\overline{\kappa}_1 = \overline{\kappa}_1(\rho, c) > 1$ and $\overline{\kappa}_0 = \overline{\kappa}_0(\rho, c) > 1$ such that for any solution u of

$$\begin{cases} cu'(x) = F((u(x + r_i))_{i=0,\dots,N}) \quad \text{on} \quad \mathbb{R} \\ u' \geq 0 \\ u(-\infty) = 0 \quad \text{and} \quad u(+\infty) = 1. \end{cases}$$

we have

$$\sup_{B_\rho(x)} u \leq \overline{\kappa}_1 \inf_{B_\rho(x)} u \quad \text{for all} \quad x \in \mathbb{R}.$$

and

$$u(x + r^*) \leq \overline{\kappa}_0 u(x),$$

where $r^* = \max\limits_{i=0,\dots,N} |r_i|$.

Notice that Harnack inequality may fail for $c = 0$ as shows following remark.

Remark 4.7 (When Harnack inequality fails for $c = 0$)
i) (A traveling wave equation)
We consider the equation

$$cu' = u(x - \frac{1}{2}) - u(x) + f(u(x))$$

For $c = 0$, we can plug

$$u(x) := \begin{cases} 0 & \text{if} \quad x \leq 0 \\ x & \text{if} \quad 0 \leq x \leq \frac{1}{2} \\ 1 - \frac{1}{2}e^{-2(x-\frac{1}{2})} & \text{if} \quad x \geq \frac{1}{2} \end{cases}$$

and check that

$$f(u(x)) = u(x) - u(x-\frac{1}{2}) = \begin{cases} 0 & \textit{if} \quad x \le 0 \\ x & \textit{if} \quad 0 \le x \le \frac{1}{2} \\ 1 - \frac{1}{2}e^{-2(x-\frac{1}{2})} - (x-\frac{1}{2}) & \textit{if} \quad \frac{1}{2} \le x \le 1 \\ 1 - \frac{1}{2}e^{-2(x-\frac{1}{2})} - \left\{1 - \frac{1}{2}e^{-2(x-1)}\right\} & \textit{if} \quad x \ge 1 \end{cases}$$

which shows that we can take

$$f(v) := \begin{cases} v & \textit{for} \quad 0 \le v \le \frac{1}{2} \\ v + \frac{1}{2}\ln\{2(1-v)\} & \textit{for} \quad \frac{1}{2} \le v \le 1 - \frac{1}{2}e^{-1} \\ (e-1)(1-v) & \textit{for} \quad 1 - \frac{1}{2}e^{-1} \le v \le 1 \end{cases}$$

which is Lipschitz and satisfies $f > 0 = f(0) = f(1)$ on $(0,1)$. Moreover $f'(0) = 1$ and we can check that $c^* = 0$. This example shows that there is no standard Strong Maximum Principle, and then no Harnack inequality here for $c = 0 = c^*$ and $r_{i_0} = -\frac{1}{2} < 0$.

ii) (Lack of diffusion in a discrete equation)

Consider the related equation

$$u_t(x,t) = u(x-1,t) - u(x,t) \quad \textit{for} \quad (x,t) \in \mathbb{R} \times (0,+\infty)$$

For the initial data

$$u(x,0) = \begin{cases} 0 & \textit{if} \quad x < 0 \\ 1 & \textit{if} \quad x \in [0,1) \\ 0 & \textit{if} \quad x > 1 \end{cases}$$

we get the exact solution

$$u(x,t) = u(x,0) = \begin{cases} 0 & \textit{if} \quad x < 0 \\ \dfrac{t^n}{n!}e^{-t} & \textit{if} \quad x \in n + [0,1) \quad \textit{with} \quad n \in \mathbb{N} \end{cases}$$

This example shows clearly that this discrete equation creates no diffusion at all to the left (i.e. no infinite velocity to the left).

5 Comparison of the minimal velocity c^+ with c^*

The main result of this section is the proof of Theorem 1.3 which states that $c^+ \ge c^*$. Part of our arguments are inspired by Hamel [24], where some comparisons $c^+ \ge c^*$ are also obtained under certain conditions for various (standard) reaction-diffusion equations. We finally show in Lemma 5.2 an example where the inequality is strict: $c^+ > c^*$.

In order to prove Theorem 1.3, we will need the following result.

Lemma 5.1 (Lower bound on the velocity for the linear problem)
Let F be a function satisfying (A_{Lip}) and differentiable at $\{0\}^{N+1}$. Assume moreover that $f(v) = F(v,...,v)$ satisfies

$$(5.1) \qquad f'(0) = \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0,...,0) > 0.$$

Let $c \neq 0$ and assume that there exists $a_0 > 0$ and $C_0 > 0$ such that ϕ is a solution of

$$(5.2) \qquad \begin{cases} c\phi'(x) = \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0,...,0)\phi(x+r_i) \quad \text{on} \quad \mathbb{R} \\ \phi' \geq 0 \\ \phi > 0 \\ 1 \leq \frac{\phi(x+a_0)}{\phi(x)} \leq C_0 \quad \text{for all} \quad x \in \mathbb{R}. \end{cases}$$

Then

$$c \geq c^*,$$

where c^* is given in (1.8).

Proof of Lemma 5.1
Step 0: preliminary
Let $a \in (0, a_0)$ and let

$$K^* = \inf E \quad \text{with} \quad E = \{k \geq 1 \ \text{ such that } \ \phi(x+a) \leq k\phi(x) \quad \text{for all } x \in \mathbb{R}\}.$$

We have $E \neq \emptyset$ because $C_0 \in E$. By definition of $K^* \geq 1$, we have

$$\phi(x+a) \leq K^*\phi(x) \quad \text{for every } \ x \in \mathbb{R}.$$

If $K^* = 1$, then ϕ is constant and the first equation of (5.2) gives

$$0 = \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0,...,0) = f'(0)$$

which is a contradiction with (5.1). Therefore $K^* > 1$, and there exists $\lambda > 0$ such that

$$K^* = e^{\lambda a}.$$

Again by definition of K^*, for every $\varepsilon > 0$, there exists $x_\varepsilon \in \mathbb{R}$ such that

$$\phi(x_\varepsilon + a) > (K^* - \varepsilon)\phi(x_\varepsilon).$$

Setting

$$\phi_\varepsilon(x) := \frac{\phi(x+x_\varepsilon)}{\phi(x_\varepsilon)} \quad \text{with} \quad \phi_\varepsilon(0) = 1$$

we get

$$\phi_\varepsilon(x+a) \leq K^*\phi_\varepsilon(x) \quad \text{and} \quad \phi_\varepsilon(a) > (K^* - \varepsilon)\phi_\varepsilon(0).$$

Step 1: passing to limit $\varepsilon \to 0$
Since $c \neq 0$, we can bound both ϕ_ε and ϕ'_ε on any bounded interval uniformly w.r.t. ε. Therefore, using Ascoli's Theorem, we deduce that ϕ_ε converges to some ϕ_0 locally uniformly and ϕ_0 satisfies (in the viscosity sense)

$$(5.3) \qquad \begin{cases} c\phi_0'(x) = \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0,...,0)\phi_0(x+r_i) \quad \text{on} \quad \mathbb{R} \\ \phi_0' \geq 0 \\ \phi_0(x+a) \leq K^*\phi_0(x) \\ \phi_0(0) = 1 \\ \phi_0(a) \geq K^*\phi_0(0). \end{cases}$$

Now, let $w(x) = K^*\phi_0(x) - \phi_0(x+a)$. Then from (5.3), we deduce that w satisfies

$$\begin{cases} cw'(x) = \displaystyle\sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0,...,0) w(x+r_i) \quad \text{on} \quad \mathbb{R} \\ w \geq 0 \quad \text{on} \quad \mathbb{R} \\ w(0) = 0. \end{cases}$$

Then using the half strong maximum principle [1, Lemma 6.1], we get that $w(x) = 0$ for all $cx \leq 0$, i.e.

$$K^*\phi_0(x) = \phi_0(x+a) \quad \text{for all} \quad cx \leq 0.$$

Step 2: establishing $c \geq c^*$
Because of estimate (5.3), we see that $\phi_0 > 0$. Hence we can define

$$\phi_{0,n}(x) := \frac{\phi_0(x-cn)}{\phi_0(-cn)}.$$

Then $\phi_{0,n}(0) = 1$ and

$$K^*\phi_{0,n}(x) = \phi_{0,n}(x+a) \quad \text{for all} \quad c(x-cn) \leq 0.$$

Step 2.1: passing to the limit $n \to +\infty$
As before, we can pass to the limit $\phi_{0,n} \to \phi_{0,\infty}$ satisfying

$$\begin{cases} c\phi_{0,\infty}'(x) = \displaystyle\sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0,...,0) \phi_{0,\infty}(x+r_i) \quad \text{on} \quad \mathbb{R} \\ \phi_{0,\infty}' \geq 0 \\ \phi_{0,\infty}(0) = 1. \end{cases}$$

with moreover

$$K^*\phi_{0,\infty}(x) = \phi_{0,\infty}(x+a) \quad \text{for all} \quad x \in \mathbb{R}.$$

Step 2.2: conclusion
Let

$$z(x) = \frac{\phi_{0,\infty}(x)}{e^{\lambda x}} \geq 0$$

which satisfies $z \in C^1$ and

$$cz'(x) + c\lambda z(x) = \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0,...,0) e^{\lambda r_i} z(x+r_i) \quad \text{on} \quad \mathbb{R} \tag{5.4}$$

and

$$z(x+a) = z(x)$$

Let x_0 be a minimum of the a-periodic function $z \geq 0$. Assume by contradiction that $z(x_0) = 0$. Then this implies

$$\sum_{i=1}^{N} \frac{\partial F}{\partial X_i}(0,...,0) e^{\lambda r_i} z(x_0+r_i) = 0.$$

Case 1: there exists some index $i_0 \in \{1,\ldots,N\}$ such that $r_{i_0} \neq 0$ and $\frac{\partial F}{\partial X_{i_0}}(0,\ldots,0) > 0$
Since $\frac{\partial F}{\partial X_i}(0,...,0) \geq 0$ for all $i = 1,...,N$, we deduce that

$$z(x_0 + r_{i_0}) = 0.$$

Repeating the same process, we get that $z = 0$ on $x_0 + r_{i_0}\mathbb{N}$. Since z is a-periodic, then $z = 0$ on $x_0 + r_{i_0}\mathbb{N} + a\mathbb{Z} \equiv x_0 + a(\frac{r_{i_0}}{a}\mathbb{N} + \mathbb{Z})$.

Since $a \in (0, a_0)$ is arbitrary, then we can choose $a \in (0, a_0)$ such that $\frac{r_{i_0}}{a} \in \mathbb{R}\backslash\mathbb{Q}$. Therefore, $x_0 + a(\frac{r_{i_0}}{a}\mathbb{N} + \mathbb{Z})$ is dense in $\mathbb{R}$. By continuity of z, this implies

$$z = 0 \quad \text{on} \quad \mathbb{R},$$

which is a contradiction with $z(0) = 1$.

Therefore $z \geq z(x_0) > 0$ and we get

$$\begin{aligned} c\lambda z(x_0) &= \frac{\partial F}{\partial X_0}(0, ..., 0)e^{\lambda r_0} z(x_0) + \sum_{i=1}^{N} \frac{\partial F}{\partial X_i}(0, ..., 0)e^{\lambda r_i} z(x_0 + r_i) \\ &\geq \frac{\partial F}{\partial X_0}(0, ..., 0)e^{\lambda r_0} z(x_0) + \sum_{i=1}^{N} \frac{\partial F}{\partial X_i}(0, ..., 0)e^{\lambda r_i} z(x_0) \\ &= z(x_0)P(\lambda) \quad \text{with} \quad P(\lambda) := \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0, ..., 0)e^{\lambda r_i}. \end{aligned}$$

Hence

$$c \geq \frac{P(\lambda)}{\lambda} \geq \inf_{\lambda'>0} \frac{P(\lambda')}{\lambda'} =: c^*.$$

Case 2: we have $\frac{\partial F}{\partial X_i}(0, \dots, 0) = 0$ for all $i \in \{1, \dots, N\}$

Then we deduce from (5.4) that z satisfies

$$cz' = kz \quad \text{with} \quad k := f'(0) - c\lambda \quad \text{with} \quad \frac{\partial F}{\partial X_0}(0, \dots, 0) = f'(0) > 0$$

Because $c \neq 0$ and z is a-periodic with $z(0) = 1$, we deduce that z is constant and that $k = 0$, i.e.

$$c = \frac{f'(0)}{\lambda} > c^* := \inf_{\lambda'>0} \frac{P(\lambda')}{\lambda'} = 0$$

Hence in Cases 1 and 2, we get $c \geq c^*$ and this ends the proof of the proposition.

Proof of Theorem 1.3

Under assumptions (A_{Lip}) and (P_{C^1}), let c^+ given by Theorem 1.1. We want to show that $c^+ \geq c^*$ with c^* given in (1.8).

We now introduce the following condition

$$\text{(5.5)} \qquad \exists\, i_0 \in \{1, ..., N\} \ \text{ such that } \ r_{i_0} > 0 \ \text{ and } \ \frac{\partial F}{\partial X_{i_0}}(0, ..., 0) > 0$$

and we will distinguish the cases where this assumption is satisfied or not.

Step 1: proving that $c^+ \geq c^*$ under the assumption (5.5)

Let $c \geq c^+$, and let (c, ϕ) be a solution of (1.6). Because of assumption (5.5), we know that Harnack inequality holds true (see Proposition 4.4). Hence we deduce that $\phi > 0$.

Step 1.1: $\frac{\phi'(x)}{\phi(x)}$ is globally bounded when $c \neq 0$

We have

$$c\frac{\phi'(x)}{\phi(x)} = \frac{1}{\phi(x)} F((\phi(x + r_i))_{i=0,\dots,N}).$$

Using $F(0,\dots,0)=0$, the fact that F is Lipschitz with

$$|F(X)| \le L \max_{i=0,\dots,N} |X_i|$$

and the monotonicity of ϕ with $|r_i| \le r^*$, we deduce that

$$\frac{\phi'(x)}{\phi(x)} \le \frac{L}{|c|}\frac{\phi(x+r^*)}{\phi(x)} \le \frac{L\overline{\kappa}_0}{|c|} =: \mathcal{M}$$

where the constant $\overline{\kappa}_0 > 1$ comes from the following Harnack inequality (see Proposition 4.4)

$$\phi(x+r^*) \le \overline{\kappa}_0 \phi(x)$$

Hence we get the bound

$$0 \le \frac{\phi'(x)}{\phi(x)} \le \mathcal{M}.$$

Step 1.2: proving that $c \ge c^*$ and conclusion
Gven a sequence $x_n \to -\infty$ we set

$$\phi_n(x) := \frac{\phi(x+x_n)}{\phi(x_n)} \ge 0$$

which satisfies

$$c\phi_n'(x) = \frac{1}{\varepsilon_n} F(\varepsilon_n(\phi_n(x+r_i))_{i=0,\dots,N}) \quad \text{on} \quad \mathbb{R}, \quad \text{with} \quad \varepsilon_n := \phi(x_n) \to 0$$

and

$$\phi_n(x+r^*) \le \overline{\kappa}_0 \phi_n(x)$$

Moreover, because we have $\phi_n(0)=1$ with $0 \le \frac{\phi_n'(x)}{\phi_n(x)} \le \mathcal{M}$ this implies the bounds

$$0 \le \phi_n(x) \le \max(1, e^{\mathcal{M}x}), \quad 0 \le \phi_n'(x) \le \mathcal{M}\phi_n(x)$$

Now, using Ascoli's Theorem (and some classical diagonal argument), we deduce that ϕ_n converges locally uniformly to some ϕ_∞ which satisfies (at least in the viscosity sense)

$$\begin{cases} c\phi_\infty'(x) = \displaystyle\sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0,\dots,0)\phi_\infty(x+r_i) \quad \text{on} \quad \mathbb{R} \\ \phi_\infty' \ge 0 \\ \phi_\infty(0) = 1 \\ \phi_\infty(x+r^*) \le \overline{\kappa}_0 \phi_\infty(x) \end{cases}$$

where the third three lines imply in particular that

$$\phi_\infty > 0 \quad \text{on} \quad \mathbb{R}$$

Then using Lemma 5.1, we deduce that

$$c \ge c^*.$$

Because this is true for every $c \ge c^+$ with $c \ne 0$, we deduce that $c^+ \ge c^*$.

Step 2: proving $c^+ \geq c^*$ when assumption (5.5) is not satisfied
Then we have

$$\frac{\partial F}{\partial X_i}(0,\dots,0) = 0 \quad \text{for all} \quad r_i > 0$$

Because by assumption we have

$$0 < f'(0) = \sum_{i=0,\dots,N} \frac{\partial F}{\partial X_i}(0,\dots,0) \quad \text{with} \quad \frac{\partial F}{\partial X_i}(0,\dots,0) \geq 0 \quad \text{for all} \quad i \neq 0$$

we deduce that

$$c^* = \inf_{\lambda>0} \frac{P(\lambda)}{\lambda} \leq 0$$

Assume by contradiction that

$$c^+ < c^* \leq 0 \tag{5.6}$$

Up to increase the integer $N \geq 1$, we can always assume that there exists some index $i_1 \in \{1,\dots,N\}$ such that

$$r_{i_1} < 0$$

Let (c^+, ϕ^+) be a solution of (1.6) given by Theorem 1.1. Using the half strong maximum principle [1, Lemma 6.1] with $c^+ < 0$, we get that

$$\phi^+(x_0) = 0 \quad \text{implies} \quad \phi^+(x) = 0 \quad \text{for all} \quad x \geq x_0$$

Hence we deduce that $\phi^+ > 0$. Now let $\varepsilon > 0$ and let us define the function

$$F_\varepsilon(X_0, \dots, X_N) := F(X_0, \dots, X_N) + \varepsilon(X_{i_1} - X_0).$$

Because ϕ^+ is nondecreasing, we see that $\phi^+ > 0$ satisfies

$$c^+(\phi^+)'(x) = F((\phi^+(x+r_i))_{i=0,\dots,N}) \geq F_\varepsilon((\phi^+(x+r_i))_{i=0,\dots,N}).$$

Then we can apply Proposition 2.5 which shows the existence of a nondecreasing solution (c^+, ϕ_ε) of

$$\begin{cases} c^+\phi_\varepsilon' = F_\varepsilon((\phi_\varepsilon(x+r_i))_{i=0,\dots,N}) \quad \text{on} \quad \mathbb{R} \\ \phi_\varepsilon' \geq 0 \\ \phi_\varepsilon(-\infty) = 0, \quad \phi_\varepsilon(+\infty) = 1 \end{cases}$$

Because

$$c^+ < 0 \quad \text{and} \quad \frac{\partial F_\varepsilon}{\partial X_{i_1}}(0,\dots,0) \geq \varepsilon > 0 \quad \text{with} \quad r_{i_1} < 0$$

we can apply Harnack inequality (Proposition 4.6). Proceeding exactly as in Step 1, we get a function ϕ_∞ solution of

$$\begin{cases} c^+\phi_\infty'(x) = \displaystyle\sum_{i=0}^N \frac{\partial F_\varepsilon}{\partial X_i}(0,\dots,0)\phi_\infty(x+r_i) \quad \text{on} \quad \mathbb{R} \\ \phi_\infty' \geq 0 \\ \phi_\infty(0) = 1 \\ \phi_\infty(x+r^*) \leq \overline{\kappa}_0 \phi_\infty(x) \end{cases}$$

which implies again from Lemma 5.1 that

$$0 > c^+ \geq c_\varepsilon := \inf_{\lambda>0} \frac{P_\varepsilon(\lambda)}{\lambda} \quad \text{with} \quad P_\varepsilon(\lambda) = P(\lambda) + \varepsilon(e^{r_{i_1}\lambda} - 1)$$

In the limit $\varepsilon \to 0^+$, we recover

$$c^+ \geq \lim_{\varepsilon \to 0^+} c^*_\varepsilon = c^*$$

which is in contradiction with our assumption (5.6). Hence (5.6) is false, and this shows that

$$c^+ \geq c^*$$

Finally we have shown this result assuming or not assumption (5.5). Hence the result holds in all cases and this ends the proof of the theorem.

Now, we give the proof of Proposition 1.4, where we show that $c^+ \leq c^*$ under a KPP type condition.

Proof of Proposition 1.4

The proof is quite simple. Consider any $c > c^* = \inf_{\lambda>0} \frac{P(\lambda)}{\lambda}$ and choose some $\lambda_0 > 0$ such that we still have

$$c > c(\lambda_0) := \frac{P(\lambda_0)}{\lambda_0} \geq c^*$$

We also set

$$\phi_0(x) := e^{\lambda_0 x} \quad \text{and} \quad G(X) := \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0, ..., 0) X_i.$$

This shows that

$$c\phi_0' \geq c(\lambda_0)\phi_0' = G((\phi_0(x + r_i))_{i=0,...,N}) \geq F((\phi_0(x + r_i))_{i=0,...,N})$$

Hence ϕ_0 is a supersolution of the equation with velocity c. This is also the case of

$$\bar{\phi}_0 := \min(\phi_0, 1)$$

which is then a positive nondecreasing supersolution. Then we can apply Proposition 2.5 which shows the existence of a nondecreasing solution ϕ of

$$\begin{cases} c\phi' = F((\phi(x + r_i))_{i=0,...,N}) \quad \text{on} \quad \mathbb{R} \\ \phi' \geq 0 \\ \phi(-\infty) = 0, \quad \phi(+\infty) = 1 \end{cases}$$

This implies by definition of c^+ that

$$c^+ \leq c$$

Because this is true for any $c > c^*$, we deduce that

$$c^+ \leq c^*$$

and this ends the proof of the proposition.

Now, we give an example of nonlinearity where we have $c^+ > c^*$.

Lemma 5.2 (Example with $c^+ > c^*$)

Consider the function $F : [0, 1]^3 \to \mathbb{R}$ defined as

$$F(X_0, X_{-1}, X_1) := g(X_1) + g(X_{-1}) - 2g(X_0) + f(X_0),$$

with $r_0 = 0$, $r_{\pm 1} = \pm 1$ and f, $g : [0, 1] \to \mathbb{R}$ are C^1 over a neighborhood of 0, Lipschitz on $[0, 1]$ and satisfying

$$\begin{cases} f(0) = f(1) = 0 \\ f > 0 \quad on \quad (0, 1) \\ f'(0) > 0 \end{cases} \quad and \quad \begin{cases} g'(0) = 0 \\ g(1) = 1 + g(0) \\ g' \geq 0. \end{cases}$$

Let c^+ given by Theorem 1.1, then

$$c^+ > c^* = 0,$$

where c^* is defined in (1.8).

An example of such g is $g(x) = x - \frac{1}{2\pi}\sin(2\pi x)$.

Proof of Lemma 5.2
Since $g'(0) = 0$ and $f'(0) > 0$, then $P(\lambda) = f'(0) > 0$. Thus we get that $c^* = \inf_{\lambda>0} \frac{P(\lambda)}{\lambda} = 0$. By Theorem 1.3, we have that $c^+ \geq c^* = 0$. We want to show that $c^+ > c^*$.

Assume to the contrary that $c^+ = 0$ and let ϕ be a solution of (1.6) with F replaced by F^0. Using the equivalence between the viscosity solution and almost everywhere solutions (see Lemma 2.3), we deduce that ϕ is an almost everywhere solution of

$$0 = F((\phi(z + r_i))_{i=0,...,N}). \tag{5.7}$$

That is there exists a set $\mathcal{N}$ of measure zero such that for every $z \notin \mathcal{N}$, equation (5.7) holds true.

Let $\mathcal{N}_0 = \cup_{k\in\mathbb{Z}}(\mathcal{N} + k)$ and choose $z_0 \in \mathbb{R}\backslash\mathcal{N}_0$. Then equation (5.7) holds true for every $z_0 + k$ with $k \in \mathbb{Z}$. Hence

$$g(\phi(z_0 + k + 1)) + g(\phi(z_0 + k - 1)) - 2g(\phi(z_0 + k)) = -f(\phi(z_0 + k)) \leq 0 \quad \text{for every} \quad k \in \mathbb{Z}. \tag{5.8}$$

Let h be the piecewise affine function which is affine on each interval $[k, k + 1]$ and satisfying $h(z_0 + k) = g(\phi(z_0 + k))$ with $k \in \mathbb{Z}$. Thus, it is easy to conclude using (5.8) that h is concave. Moreover, h is bounded because g is bounded on $[0, 1]$ and $0 \leq \phi \leq 1$. Therefore, h is constant. This implies that

$$g(\phi(z_0)) = g(\phi(z_0 + k)) = const \quad \text{for all} \quad k \in \mathbb{Z}.$$

Moreover, since $g' \geq 0$, $\phi(-\infty) = 0$ and $\phi(+\infty) = 1$, we conclude that $g = const$ on $[0, 1]$, which is a contradiction with $g(1) = 1+g(0)$. Hence, we get $c^+ > 0 = c^*$. This ends the proof of the lemma.

6 Properties of the minimal velocity

This section is decomposed in two subsections. In the first subsection, we show Corollary 6.1 which gives sufficient conditions to insure that $c^+ \geq 0$. In the second subsection, we give the proof of Proposition 1.5 which shows an example where $c^+ < 0$. Finally, using this example we show the instability of the minimal velocity by L^∞ approximation of the nonlinearity F. This is the proof of Proposition 1.2.

6.1 Nonnegativity of the minimal velocity c^+

Let us now give a corollary of Theorem 1.3.

Corollary 6.1 (Non-negative c^+ for particular F)
Consider a function F satisfying (A_{Lip}) and (P_{C^1}). Let c^+ given by Theorem 1.1. Then we have $c^+ \geq c^* \geq 0$, if one of the three following conditions i), ii) or iii) holds true:

i) **(Reflection symmetry of F)**
Let $X = (X_i)_{i\in\{0,...,N\}} \in [0, 1]^{N+1}$. Assume that for all $i \in \{0, ..., N\}$ there exists $\bar{i} \in \{0, ..., N\}$ such that $r_{\bar{i}} = -r_i$; and

$$F(\overline{X}) = F(X) \quad \textit{for all} \;\; X \in [0, 1]^{N+1},$$

where

$$\overline{X}_i = X_{\bar{i}} \quad for \quad i \in \{0, ..., N\}.$$

$ii)$ **(All the r_i's "shifts" are non-negative)**
Assume that $r_i \geq 0$ for all $i \in \{0, ..., N\}$.

$iii)$ **(Strict monotonicity)**
Let

$$I = \left\{ i \in \{1, ..., N\} \text{ such that there exists } \bar{i} \in \{1, ..., N\} \text{ with } r_{\bar{i}} = -r_i \right\}$$

and assume that

$$\frac{\partial F}{\partial X_0}(0) + \sum_{i \in I} \min \left(\frac{\partial F}{\partial X_i}(0), \frac{\partial F}{\partial X_{\bar{i}}}(0) \right) > 0. \tag{6.1}$$

Notice that in the first version of the manuscript [2], we gave a direct proof of Corollary 6.1, without using Theorem 1.3 that was not available at that time. The proof there was done using extension lemmata, joint to approximation procedures (only close to the root $\phi(+\infty) = 1$) as in our construction of c^+ in the proof of Theorem 1.1.

Notice that because of the monotonicity of F in X_j for $j \neq 0$, condition (6.1) is satisfied if

$$\frac{\partial F}{\partial X_0}(0) > 0.$$

Moreover, if

$$I = \{1, ..., N\} \quad \text{and} \quad \frac{\partial F}{\partial X_i}(0) = \frac{\partial F}{\partial X_{\bar{i}}}(0) \quad \text{for all} \quad i \in I, \tag{6.2}$$

then condition (6.1) is equivalent to $f'(0) > 0$. In particular, under condition $i)$ property (6.2) holds true. This shows that condition $iii)$ is more general than condition $i)$.

Remark that if we replace (P_{C^1}) by (P_{Lip}) assuming for instance $i)$ or $ii)$, we do not know if $c^+ \geq 0$.

Proof of Corollary 6.1
Step 1: Study of c^*
We first show that $c^* \geq 0$ in each case.
Step 1.1: case i)
The reflection symmetry shows that

$$\frac{\partial F}{\partial X_{\bar{i}}}(0) = \frac{\partial F}{\partial X_i}(0) \quad \text{for all} \quad i = 0, \ldots, N$$

and then

$$P(\lambda) = \sum_{i=0,\ldots,N} \frac{\partial F}{\partial X_i}(0) \cdot e^{r_i \lambda} = \frac{\partial F}{\partial X_0}(0) + \sum_{i=1,\ldots,N} \frac{\partial F}{\partial X_i}(0) \cdot \cosh(r_i \lambda) \geq \sum_{i=0,\ldots,N} \frac{\partial F}{\partial X_i}(0) = f'(0) > 0$$

Hence

$$c^* := \inf_{\lambda > 0} \frac{P(\lambda)}{\lambda} \geq 0$$

Step 1.2: case ii)
For the computation of $P(\lambda)$ we can assume that $r_i \geq 0$ for all indices $i = 0, \dots, N$. Then we have

$$P(\lambda) = \sum_{i=0,\dots,N} \frac{\partial F}{\partial X_i}(0) \cdot e^{r_i \lambda} \geq \sum_{i=0,\dots,N} \frac{\partial F}{\partial X_i}(0) = f'(0) > 0$$

and again

$$c^* \geq 0$$

Step 1.3: case iii)
We have

$$P(\lambda) = \sum_{i=0,\dots,N} \frac{\partial F}{\partial X_i}(0) \cdot e^{r_i \lambda} \geq \frac{\partial F}{\partial X_0}(0) + \sum_{i \in I} \min\left(\frac{\partial F}{\partial X_i}(0), \frac{\partial F}{\partial X_{\bar{i}}}(0)\right) \cosh(r_i \lambda) =: Q(\lambda)$$

which implies

$$P(\lambda) \geq Q(\lambda) \geq Q(0) > 0$$

and again

$$c^* \geq 0$$

Step 2: conclusion
Using Theorem 1.3, we deduce that $c^+ \geq c^* \geq 0$, which ends the proof of the corollary.

6.2 Instability of the minimal velocity c^+

In this subsection, we show that the minimal velocity c^+ given in Theorem 1.1 is unstable in the sense of Proposition 1.2. Before proving proving it, we give an example of a nonlinearity F for which the associated minimal velocity is negative (Proposition 1.5).

Proof of Proposition 1.5
The aim is to construct a function F satisfying (A_{Lip}) and (P_{C^1}) such that the associated minimal velocity satisfies $c^+ < 0$. To this end, we will construct a function $f \in \text{Lip}([0,1])$, which is linear in a neighborhood of zero with $f'(0) > 0$, such that there exists a couple (c, ϕ) with $c < 0$ solution of

$$\begin{cases} c\phi'(x) = \phi(x-1) - \phi(x) + f(\phi(x)) & \text{on} \quad \mathbb{R} \\ \phi' \geq 0 \\ \phi(-\infty) = 0 \quad \text{and} \quad \phi(+\infty) = 1. \end{cases} \tag{6.3}$$

Let $c = -\mu$ with $0 < \mu < 1$ and

$$\phi(x) = \begin{cases} \frac{1}{2} e^{\gamma x} & \text{on} \quad (-\infty, 0] \\ 1 - \frac{1}{2} e^{-\gamma x} & \text{on} \quad [0, +\infty) \end{cases}$$

with $\gamma > 0$. We claim that $\phi \in C^1(\mathbb{R})$ and $(-\mu, \phi)$ solves

$$\begin{cases} 0 < \phi(x) - \phi(x-1) - \mu\phi'(x) & \text{on} \quad \mathbb{R} \\ \phi' > 0 \\ \phi(-\infty) = 0 \quad \text{and} \quad \phi(+\infty) = 1, \end{cases}$$

which is possible to check for $0 < \gamma << 1$.

Therefore, it is sufficient to define the function f as

$$f(\phi(x)) := \phi(x) - \phi(x-1) - \mu\phi'(x) > 0 \quad \text{for all } x \in \mathbb{R}.$$

Notice that, when $x \to +\infty$, $\phi(+\infty) = 1$ and $\phi'(x) \to 0$, thus $f(1) = 0$. Similarly, we have $f(0) = 0$. Moreover, since $\phi \in C^{1,1}(\mathbb{R})$, we have that $f \in \mathrm{Lip}((0,1))$. In fact, by a direct tedious calculation, one can deduce that

$$f(v) = \begin{cases} (1 - e^{-\gamma} - \mu\gamma)v & \text{for } v \in \left[0, \dfrac{1}{2}\right] \\ 1 + (1+\mu\gamma)(v-1) + \dfrac{e^{-\gamma}}{4(v-1)} & \text{for } v \in \left[\dfrac{1}{2}, 1 - \dfrac{1}{2}e^{-\gamma}\right] \\ (1 - e^{\gamma} + \mu\gamma)(v-1) & \text{for } v \in \left[1 - \dfrac{1}{2}e^{-\gamma}, 1\right], \end{cases}$$

and this implies that $f \in \mathrm{Lip}([0,1])$ and $1 > f'(0) > 0$. We can even check that f is concave and C^1 except at the point $v = \frac{1}{2}$, where it is neither concave nor C^1. This ends the proof of the proposition.

Remark 6.2 *Notice that to get more regular nonlinearities, one can consider*

$$(6.4) \qquad f_\varepsilon(x) := \Big(\big(\phi(\cdot) - \phi(\cdot - 1) - \mu\phi'(\cdot)\big) \star \rho_\varepsilon \Big)(x),$$

where ρ_ε satisfies $\rho_\varepsilon \geq 0$, $\rho_\varepsilon(x) = \frac{1}{\varepsilon}\rho(\frac{x}{\varepsilon})$ (ρ is a mollifier) and $\operatorname{supp} \rho_\varepsilon \subset B_\varepsilon(0)$. However, in this case, $\rho_\varepsilon \star \phi$ is a solution of (6.3), with f replaced by f_ε, and then $f_\varepsilon \in C^\infty([0,1])$ with $f'_\varepsilon(0) > 0$.

Now, we give the proof of the instability result, namely Proposition 1.2.

Proof of Proposition 1.2

Let us consider the function F given in the proof of Proposition 1.5, namely for $X = (X_0, \ldots, X_N)$

$$(6.5) \qquad \begin{cases} F(X) = F(X_0, X_1) := X_1 - X_0 + f(X_0) \quad \text{with} \quad r_0 = 0, \quad r_1 = -1, \quad N = 1 \\ f(v) = f'(0) \cdot v \quad \text{for} \quad v \in [0, \frac{1}{2}], \quad \text{with} \quad f'(0) > 0 \end{cases}$$

satisfying (A_{Lip}) and (P_{C^1}) with associated minimal velocity $c_F^+ := c^+$ satisfying

$$c_F^+ < 0$$

Our goal is to build a sequence of functions $\hat{F}_\delta$ satisfying (A_{Lip}) and (P_{C^1}) with a minimal velocity $c^+_{\hat{F}_\delta} \geq 0$ such that

$$\hat{F}_\delta \to F \quad \text{in} \quad L^\infty([0,1]^{N+1}) \quad \text{as} \quad \delta \to 0^+$$

which will prove that

$$\liminf_{\delta \to 0} c^+_{\hat{F}_\delta} > c^+_F.$$

We first construct F_δ, and then $\hat{F}_\delta$.

Step 1: construction of F_δ

For $X = (X_0, ..., X_N) \in [0,1]^{N+1}$ and $\delta > 0$ small, define the function

$$F_\delta(X) = F(X) - f(X_0) - f_\delta(X_0),$$

where

$$f_\delta(v) = \begin{cases} \max\big(f(\delta) + L_0(v-\delta), 0\big) & \text{on} \quad [0,\delta] \\ f & \text{on} \quad [\delta,1], \end{cases}$$

with a constant $L_0 > 0$ satisfying $L_0 > \max(\mathrm{Lip}(f), 1)$.
By construction of f_δ, we clearly have

$$\|F_\delta - F\|_{L^\infty} \quad = \quad \|f - f_\delta\|_{L^\infty} \to 0 \quad \text{as} \quad \delta \to 0.$$

Step 2: rescaling and existence of $c^+_{F_\delta}$
We now introduce the root 0_δ of f_δ

$$0_\delta := \delta - \frac{f(\delta)}{L_0} > 0,$$

which satisfies

$$f_\delta > 0 = f_\delta(0_\delta) = f_\delta(1) \quad \text{on} \quad (0_\delta, 1)$$

Since F_δ satisfies (A_{Lip}) and (P_{C^1}) with $[0,1]^{N+1}$ replaced by $[0_\delta,1]^{N+1}$, it is natural to rescale F_δ in

$$\hat{F}_\delta((X_i)_{i=0,\dots,N}) := F_\delta((0_\delta + (1-0_\delta)X_i)_{i=0,\dots,N})$$

which now satisfies (A_{Lip}) and (P_{C^1}) on $[0,1]^{N+1}$. Hence we can apply Theorem 1.1, and deduce that there exists a minimal velocity $c^+_{\hat{F}_\delta}$.
Now using (6.5), notice that

$$\frac{\partial \hat{F}_\delta}{\partial X_1}(0) = (1-0_\delta) > 0, \qquad \frac{\partial \hat{F}_\delta}{\partial X_1}(0) = (1-0_\delta)(-1+L_0) > 0$$

Moreover, we can apply Theorem 1.3 which gives

$$c^+_{\hat{F}_\delta} \geq c^*_{\hat{F}_\delta} := \inf_{\lambda>0} \frac{P(\lambda)}{\lambda} \quad \text{with} \quad P(\lambda) := \sum_{i=0,1} \frac{\partial \hat{F}_\delta}{\partial X_i}(0) e^{r_i\lambda} \geq 0$$

ie

$$c^+_{\hat{F}_\delta} \geq 0 > c^+_F$$

Finally, we deduce that (using for instance the uniform continuity of F)

$$\|\hat{F}_\delta - F\|_{L^\infty} \to 0 \quad \text{as} \quad \delta \to 0$$

which ends the proof of the proposition.

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