Remarks on the pricing of American options

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Content

- The variational inequality for the American option
- Discretization by finite elements
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  - The discrete free boundary
- Solution Procedures
  - A front tracking algorithm
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American options

The American options can be exercised at any time before the maturity T.

Vanilla American Option (put) with Strike K and Maturity T

- spot price: x
- payoff function: $u_\circ(x) = (K - x)_+$.
- interest rate r, (assumed constant for simplicity)
- local volatility: $\sigma(t, x)$. We use sometimes

$$\eta(t, x) = \sigma^2(T - t, x).$$

Pricing

With Black-Scholes assumptions, the value of the American option with payoff $u_\circ$ and maturity T is

$$u(t, x_t) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}^* \left(e^{-r(\tau - t)} u_\circ(x_\tau) | F_t \right)$$

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Using now the time to maturity, we obtain: for $0 < t < T$ and $x > 0$

$$\begin{aligned}
\partial_t u - \frac{\eta(t,x)x^2}{2} \partial^2_{xx} u - rx\partial_x u + ru &\geq 0, \quad u \geq u_\circ \\
\left(\partial_t u - \frac{\eta(t,x)x^2}{2} \partial^2_{xx} u - rx\partial_x u + ru\right)(u - u_\circ) &= 0, \\
u(t = 0) &= u_\circ
\end{aligned}$$

Variational Inequality with Local Volatility

$$V = \left\{ v \in L^2(\mathbb{R}_+), x\frac{\partial v}{\partial x} \in L^2(\mathbb{R}_+) \right\}, \quad \|v\|_V^2 = \int_{\mathbb{R}^2} v^2 + |x\frac{\partial v}{\partial x}|^2.$$

Closed Convex of V : $\mathcal{K} = \{v \in V, v \geq u_\circ \text{ in } \mathbb{R}_+\}$

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Weak problem: find $u \in L^2(0,T;\mathcal{K}) \cap C^0([0,T];L^2(\mathbb{R}_+)),\ \partial_t u \in L^2(0,T;V')$ s.t.

$$\left\langle \frac{\partial u}{\partial t} + A(t)u, v - u \right\rangle \geq 0, \quad \forall v \in \mathcal{K}, \qquad u(t=0) = u_\circ$$

with

$$\langle A(t)v, w\rangle = \int_{\mathbb{R}_+} \left(\frac{\eta}{2} x^2 \frac{\partial v}{\partial x}\frac{\partial w}{\partial x} + (\eta + x\frac{\partial \eta}{\partial x} - r)x\frac{\partial v}{\partial x} w + rvw \right) dx.$$

Assumptions

Due to the particular choice of $u_\circ$ (piecewise affine and convex), mild assumptions suffice to get a lot of information on u:

Assumption I

For two positive constants $\underline{\eta} \leq \overline{\eta}$,

$$0 < \underline{\eta} \leq \eta \leq \overline{\eta} \quad \text{a.e.}$$

and for $M > 0$,

$$|x \frac{\partial \eta}{\partial x}| \leq M \quad \text{a.e.}$$

Results on the V.I.

Under Assumptions I on η, the V.I. has a unique solution u and

- u is continuous and $u(t,0) = K$
- $u \geq u_e$ (u_e price of the European put)
- $\partial_x u(t,.)$ is continuous in x for a.e. $t > 0$ and $-1 \leq \partial_x u \leq 0$.

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- $\partial_x u(t,.)$ is continuous in x for a.e. $t > 0$ and $-1 \leq \partial_x u \leq 0$.
- There exists a function γ , $[0,T] \to [0,K]$, (called price of exercise), s.t.

$$u(t,x) = u_\circ(x) \Leftrightarrow x \leq \gamma(t), \quad \forall t \in (0,T),$$

- The function γ is upper semi-continuous, right continuous, and $\forall t > 0$ $\lim_{\tau \to t_-} \gamma(t)$ exists. Therefore, the set $\partial\{u = u_\circ\}$ is negligeable.

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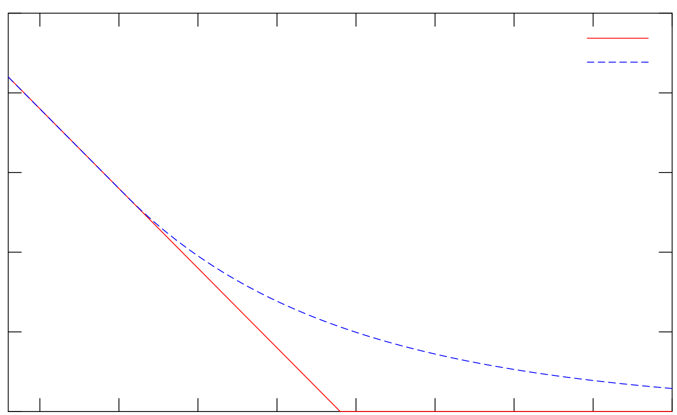
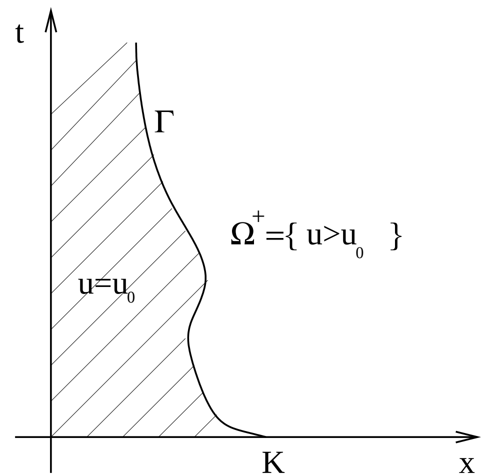
- $$\partial_t u - \frac{\eta(t,x)x^2}{2}\partial^2_{xx}u - rx\partial_x u + ru = \mu, \quad \mu = rK1_{\{u=u_\circ\}}.$$
- Bounds in various norms, independent of η.

To summarize

$$u_\circ(x) = (K - x)_+.$$





profiles of u at $t = 0$ and $t > 0$.

the free boundary

The option should be exercised when $u(t, x_t) = u_\circ(x_t)$.

Other results

Proposition: the price of exercice is bounded away from 0:
Under Assumption I, there exists $\gamma_0 > 0$ depending only on $\bar{\eta}$ s.t.

$$\gamma(t) \geq \gamma_0, \quad \forall t \in [0, T].$$

More regularity of $\eta \Rightarrow$ continuity of γ.

If η is regular and non increasing, then $\gamma \in \mathcal{C}^1((0, T])$.

Localization

Truncate the domain, i.e. focus on prices in $x \in (0, \bar{x})$, for $\bar{x}$ large enough.
Impose an artificial boundary condition on $x = \bar{x}$: either Dirichlet, Neu-
-mann or transparent condition.
Change V and $A(t)$ accordingly . In particular V becomes

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If $\gamma_0 \in (0, K)$ is known, one can even focus on $[\underline{x}, \bar{x}]$, with $0 \leq \underline{x} < \gamma_0$:

find $u \in L^2(0, T, \mathcal{K}) \cap \mathcal{C}^0([0, T]; L^2(\Omega))$, with $\partial_x u \in L^2(0, T; V')$, s.t.
$u(t = 0) = u_\circ$ and $\langle \partial_t u + A(t)u, v - u\rangle \geq 0, \quad \forall v \in \mathcal{K},$

with the new closed set $\mathcal{K}$:

$$\mathcal{K} = \{v \in V, v \geq u_\circ \text{ in } (0, \bar{x}], u = u_\circ \text{ in } (0, \underline{x}]\}.$$

A Finite Element Method

- Partition $[0,T]$ into subintervals $[t_{n-1}, t_n]$, $1 \leq n \leq N$, with $\Delta t_i = t_i - t_{i-1}$, $\Delta t = \max_i \Delta t_i$.

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- Partition $[0,\bar{x}]$ into subintervals $\omega_i = [x_{i-1},x_i]$, $1 \le i \le N_h + 1$, such that $0 = x_0 < x_1 < \ldots < x_{N_h} < x_{N_h+1} = \bar{x}$.

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- Assume that both K and $\underline{x}$ coincide with nodes of $\mathcal{T}_h$: $\exists \alpha < \kappa$, $0 \leq \alpha < \kappa < N_h + 1$ s.t. $x_\kappa = K$ and $x_{\alpha-1} = \underline{x}$.

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$$V_h = \left\{ v_h \in V,\ \forall \omega \in \mathcal{T}_h, v_{h|\omega} \in \mathcal{P}_1(\omega) \right\}.$$

Since K and $\underline{x}$ are nodes of $\mathcal{T}_h$, $u_\circ \in V_h$ and we can define $\mathcal{K}_h \subset V_h$ by

$$\begin{aligned} \mathcal{K}_h &= \{v \in V_h,\ v \ge u_\circ \text{ in } [0,\bar{x}), \quad v = u_\circ \text{ in } [0,\underline{x}]\} \\ &= \{v \in V_h,\ v(x_i) \ge u_\circ(x_i),\ i = 0,\ldots,N_h+1,\ v(x_i) = u_\circ(x_i),\ i < \alpha\}. \end{aligned}$$

Discrete problem

The discrete problem arising from an implicit Euler scheme is: find $(u^n)_{0 \le n \le N} \in \mathcal{K}_h$ satisfying $u^0 = u_\circ$, and for all n, $1 \le n \le N$,

$$\forall v \in \mathcal{K}_h, \quad \left(u^n - u^{n-1}, v - u^n\right) + \Delta t_n \langle A(t_n) u^n, v - u^n \rangle \ge 0.$$

Let $(w^i)_{i=0,\dots N_h}$ be the nodal basis of V_h, and let $\mathbf{M}$ and $\mathbf{A}^m$ in $^{(N_h+1)\times(N_h+1)}$ be the mass and stiffness matrices defined by

$$\mathbf{M}_{i,j} = (w^i, w^j), \quad \mathbf{A}^m_{i,j} = \langle A(t_m) w^j, w^i \rangle, \quad 0 \le i, j \le N_h.$$

the discrete problem reads, in matrix form

$$\left\{ \begin{array}{rcll} (\mathbf{M}(U^n - U^{n-1}) + \Delta t_n \mathbf{A}^n U^n)_i & \ge & 0, & \text{for } i \ge \alpha, \\ U^n_i & = & U^0_i & \text{for } i < \alpha, \\ U^n & \ge & U^0, & \\ (U^n - U^0)^T (\mathbf{M}(U^n - U^{n-1}) + \Delta t_n \mathbf{A}^n U^n) & = & 0. & \end{array} \right.$$

Results

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- Stability : calling $u_{\Delta t}$ the piecewise affine function in time such that $u_{\Delta t}(t_n) = u^n$, if Δt is small enough,

$$\sup_{0 \le t \le T} \|u_{\Delta t}(t)\|^2 + \int_0^T |u_{\Delta t}(t)|_V^2 dt \le C\|u_\circ\|_V^2.$$

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- Convergence: Assume that the coefficients σ and r are smooth enough so that

$$\lim_{\Delta t\to 0} \sup_{n=1,\dots,N} \sup_{t\in[t_{n-1},t_n]} \sup_{v,w\in V} \frac{|((A(t_n) - A(t))v, w)|}{\|v\|_V \|w\|_V} = 0,$$

then

$$\lim_{h,\Delta t\to 0} \left(\|u - u_{\Delta t}\|_{L^2(0,T;V)} + \|u - u_{\Delta t}\|_{L^\infty(0,T;L^2(0,\bar{x}))}\right) = 0.$$

The discrete exercise boundary

Question : is there is a well defined exercise boundary $t \to \gamma_h(t)$ in the discrete problem too?

- Jaillet, Lamberton Lapeyre : Yes if the volatility is constant.
  Main reason: the solution to the discrete problem is nondecreasing w.r.t. t.

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- In the general case, this may not hold.

Theorem: Assume that $\mathcal{T}_h$ is uniform and that $\underline{x} > 0$. For h and $\frac{h^2}{\Delta t}$ small enough *s.t.* $\mathbf{A}_\alpha^n$ and $\mathbf{M}_\alpha + \Delta t_n \mathbf{A}_\alpha^n$ are M-matrices $\forall n$, there exist N real numbers γ_h^n, s.t.

$$\underline{x} \le \gamma_h^n < K,$$

γ_h^n is a node of $\mathcal{T}_h$,

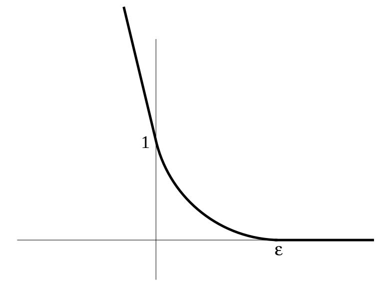
$$\forall i, 0 \le i \le N_h, \quad u^n(x_i) = u_\circ(x_i) \Leftrightarrow x_i \le \gamma_h^n.$$

proof: penalized monotonous problem

- Choose $\mathcal{V}_\epsilon(v) = \mathcal{V}(\frac{v}{\epsilon})$, $\mathcal{V}$ a smooth nonincreasing convex function

with

$$\begin{aligned} \mathcal{V}(0) &= 1, \quad \text{and} \quad \mathcal{V}(u) = 0, \quad u \geq 1, \\ 0 &\geq \mathcal{V}'(u) \geq -2 \quad u \in \mathbb{R}. \end{aligned}$$



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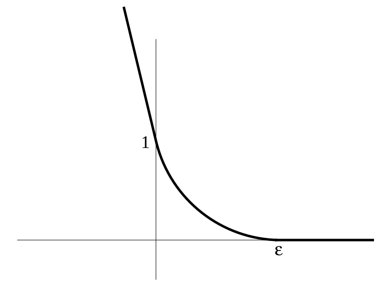

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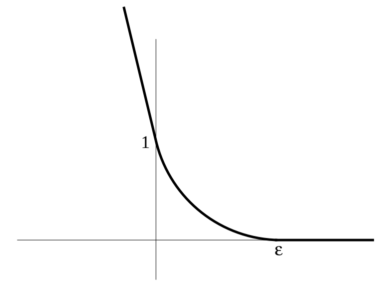

- Introduce $\tilde{V}_h = \{v_h \in V_h,\ v_h(x_i) = 0,\ \forall i < \alpha\}$.
- $n = 0 \dots N$: find $u_\epsilon^n \in V_h$ s.t. $u_\epsilon^n - u_\circ \in \tilde{V}_h$, and $\forall v \in \tilde{V}_h$,

$$\frac{(u_\epsilon^n - u_\epsilon^{n-1}, v)}{\Delta t_n} + \langle A(t_n) u_\epsilon^n, v\rangle - rK \sum_{\alpha \leq i \leq \kappa} \frac{|\tilde{\Omega}_i|}{2} \mathcal{V}_\epsilon(u_\epsilon^n(x_i) - u_\circ(x_i)) v(x_i) = 0,$$

where $\tilde{\Omega}_i = \text{supp}(w_i) \cap (0, K)$, so $\frac{|\tilde{\Omega}_i|}{2} = \int_0^K w_i$.

- discrete maximum principle for proving that $\frac{u_\epsilon^n(x_i)-u_\epsilon^n(x_{i-1})}{h} \geq -1$. (needs a strong assumption on the mesh).
- pass to the limit when $\epsilon \to 0$.

Solution Procedures

A free boundary tracking algorithm

Assume that there is a free boundary in the discrete problem as in the theorem above. Then at time step n, one just needs to look for $\gamma_h(t_n)$.

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- solve the discrete problem corresponding to

$$\frac{u^n - u^{n-1}}{\Delta t_n} - \frac{\eta(x,t_n)x^2}{2}\frac{\partial^2 u^n}{\partial x^2} - rx\frac{\partial u^n}{\partial x} + ru^n = 0 \text{ for } \gamma_h^n < x < \bar{x},$$

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  and $u^n(\bar{x}) = 0$,
- if u^n is solution to the V.I., stop else shift the point γ_h^n to the next node on the mesh left/right according to which constraint is violated by u^n.

Algorithm

choose k such that $\gamma_h^{n-1} = x_k$; set found=false;

while(not found)

.. solve $$\left\{ \begin{array}{rcll} (\mathbf{M}(U^n - U^{n-1}) + \Delta t_n \mathbf{A}^n U^n)_i & = & 0, & \text{for } i \geq k, \\ U_i^n & = & U_i^0 & \text{for } i < k. \end{array} \right.$$

.. **if** $((U^n - U^0)_{k+1} < 0$)

.. found=false; $k = k + 1$;

.. **else** $\{$

.. compute $a = (\mathbf{M}(U^n - U^{n-1}) + \Delta t_n \mathbf{A}^n U^n)_{k-1}$;

.. **if** ($a < 0$)

.. found=false; $k = k - 1$;

.. **else** found=true

.. $\}$

In our tests, the average number of iterations is ~ 2.

A Regularized Active Set Strategy

Aim: for pricing, use an algorithm of Ito and Kunish based on active sets.
Semi-discrete problem: find $u^n \in \mathcal{K}$ such that

$$\forall v \in \mathcal{K}, \quad \left(u^n - u^{n-1}, v - u^n\right) + \Delta t_n a_{t_n}(u^n, v - u^n) \geq 0.$$

Primal-dual formulation for $c > 0$,

$$\begin{array}{rcl} \forall v \in V, \quad \left(\frac{u^n - u^{n-1}}{\Delta t_n}, v\right) + a_{t_n}(u^n, v) - \langle \mu, v \rangle & = & 0, \\ \mu & = & \max(0, \mu - c(u^n - u^0)). \end{array}$$

In iterative algorithms, μ^m may not be a function, whereas μ is generally a function.
Remedy: one parameter family of regularized problems based on modifying the equation for μ:

$$\mu = \alpha \max(0, \mu - c(u^n - u^0)), \qquad 0 < \alpha < 1$$

Equation for μ equivalent to

$$\mu = \max(0, -\chi(u^n - u^0)), \qquad \chi = c\alpha/(1-\alpha) \in (0, +\infty),$$

which can be relaxed:

$$\mu = \max(0, \bar{\mu} - \chi(u^n - u^0)),$$

where $\bar{\mu}$ is fixed.

Finally

$$\forall v \in V, \quad \left(\frac{u^n - u^{n-1}}{\Delta t_n}, v\right) + a_{t_n}(u^n, v) - \langle \mu, v \rangle = 0,$$
$$\mu = \max(0, \bar{\mu} - \chi(u^n - u^0)),$$

it has a unique solution, with μ a square integrable function.

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$$\text{Set} \quad \begin{cases} \mathcal{A}^{-,k+1} = \{x : \bar{\mu}^k(x) - \chi(u^{n,k}(x) - u^0(x)) > 0\} \\ \mathcal{A}^{+,k+1} = (0,\bar{x}) \backslash \mathcal{A}^{-,k+1}. \end{cases}$$

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  - Solve for $u^{n,k+1} \in V$ s.t. $\forall v \in V$,

$$\left(\frac{u^{n,k+1} - u^{n-1}}{\Delta t_n}, v \right) + a_{t_n}(u^{n,k+1}, v) - (\bar{\mu} - \chi(u^{n,k+1} - u^0), 1_{\mathcal{A}^{-,k+1}} v) = 0.$$

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- Loop
  -

$$\text{Set} \quad \begin{cases} \mathcal{A}^{-,k+1} = \{x : \bar{\mu}^k(x) - \chi(u^{n,k}(x) - u^0(x)) > 0\} \\ \mathcal{A}^{+,k+1} = (0, \bar{x}) \backslash \mathcal{A}^{-,k+1}. \end{cases}$$

  - Solve for $u^{n,k+1} \in V$ s.t. $\forall v \in V$,

$$\left(\frac{u^{n,k+1} - u^{n-1}}{\Delta t_n}, v \right) + a_{t_n}(u^{n,k+1}, v) - (\bar{\mu} - \chi(u^{n,k+1} - u^0), 1_{\mathcal{A}^{-,k+1}} v) = 0.$$

  -

$$\text{set} \quad \mu^{k+1} = \begin{cases} 0 & \text{on} \quad \mathcal{A}^{+,k+1}, \\ \bar{\mu} - \chi(u^{n,k+1} - u^0) & \text{on} \quad \mathcal{A}^{-,k+1} \end{cases} \quad \text{and} \quad k = k + 1$$

Calling A_n the operator: $\langle A_n v, w\rangle = \left(\frac{v}{\Delta t_n}, w\right) + a_{t_n}(v, w)$ and

$$F(v,\mu) = \begin{pmatrix} A_n v + \mu - \frac{u^{n-1}}{\Delta t_n} \\ \mu - \max(0, \bar{\mu} - \chi(v - u^0)) \end{pmatrix},$$

it is proved that $G(v,\mu)$ defined by

$$G(v,\mu)h = \begin{pmatrix} A_n h_1 + h_2 \\ h_2 - \chi 1_{\{\bar{\mu}-\chi(v-u^0)>0\}} h_1 \end{pmatrix}$$

is a generalized derivative in $V \times L^2$ of F in the sense that

$$\lim_{\|h\|\to 0} \frac{\|F(v+h_1, \mu+h_2) - F(v,\mu) - G(v+h_1, \mu+h_2)h\|}{\|h\|} = 0.$$

Semi-smooth Newton

Note that in the algorithm above,

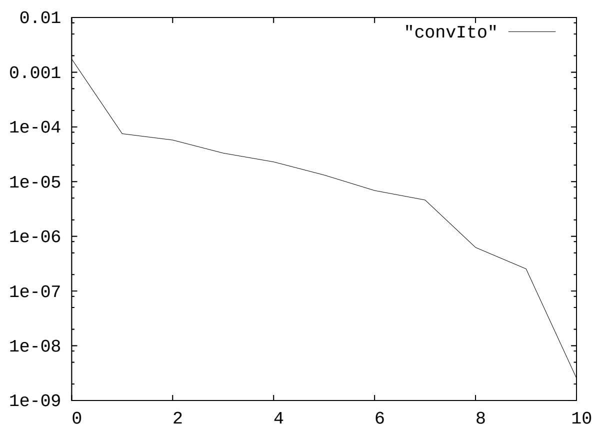
$$G(u^{n,k}, \mu^k)h = \begin{pmatrix} A_n h_1 + h_2 \\ h_2 - \chi 1_{\mathcal{A}^{-,k+1}} h_1 \end{pmatrix}.$$

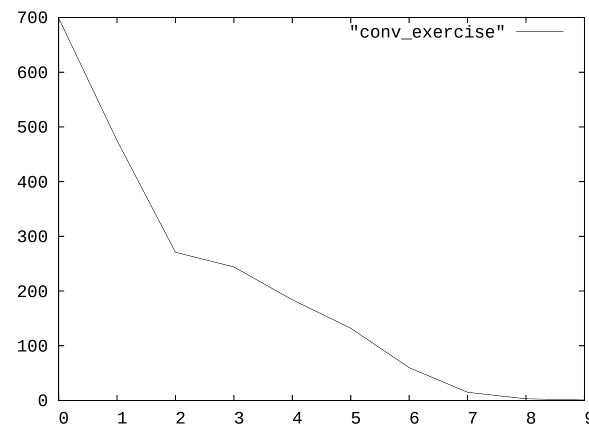
Thus the primal-dual active set algorithm above can be seen as a semi-smooth Newton method applied to F, i.e.

$$(u^{n,k+1}, \mu^{k+1}) = (u^{n,k}, \mu^k) + G^{-1}(u^{n,k}, \mu^k)F(u^{n,k}, \mu^k).$$

Convergence

Ito and Kunish have proved that the convergence is superlinear, if the initial guess is not too far from the solution.





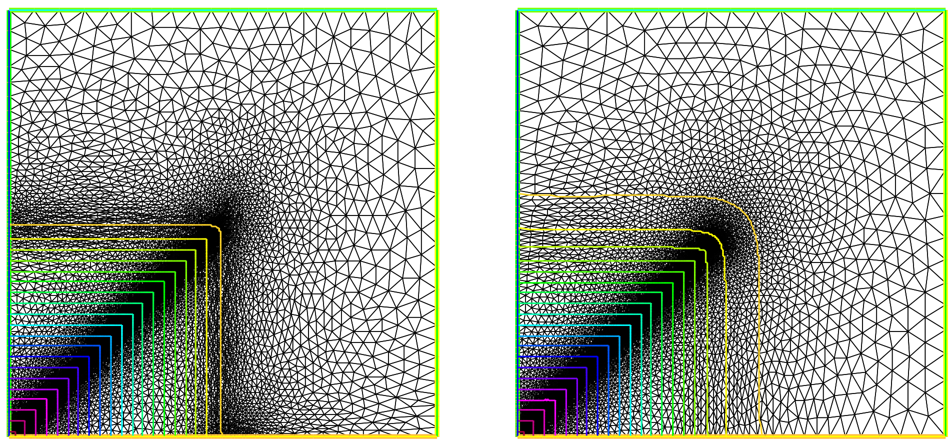
Convergence for one time step in the pricing of an American put on a basket of two assets. $\chi = 10^7$.

Left: norm of the increment of u. Right $\delta(\#(\mathcal{A}^x-))$.

A two-dimensional example

Mesh adaption and active set strategy

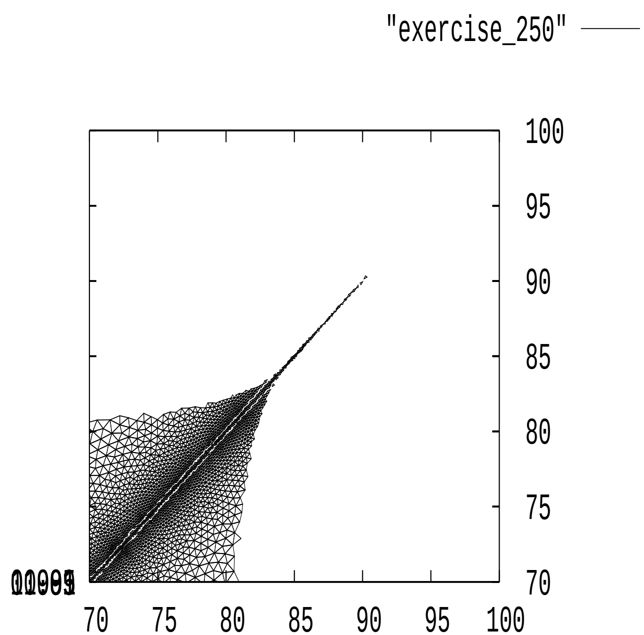
A put on a basket of two assets: $u_\circ(x_1, x_2) = \min((K - x_1)_+, (K - x_2)_+)$



mesh and price at $t = 1$ month and $t = 1$ year to maturity

the active set

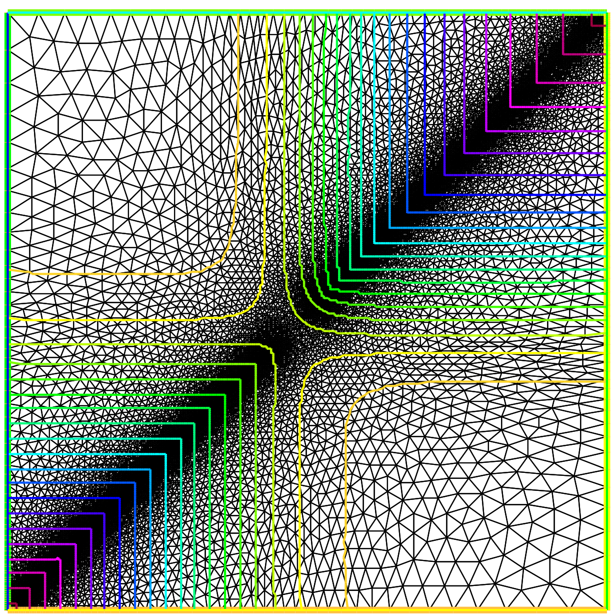
The exercise region one year to maturity (zoom)

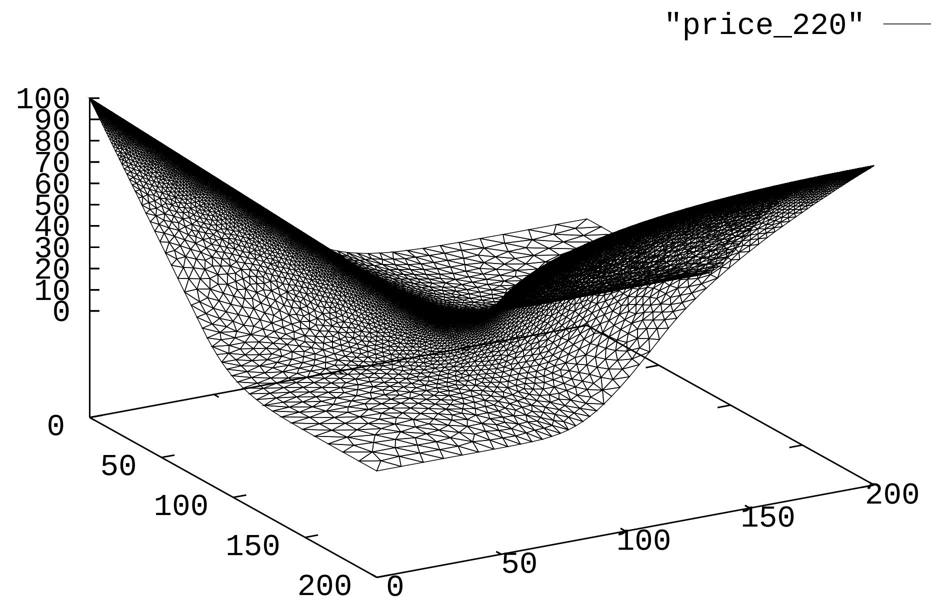


Another example

Mesh adaption and active set strategy

$$u_\circ(x_1,x_2) = \min((K-x_1)_+,(K-x_2)_+) + (K\min((x_1-K)_+,(x_2-K)_+))^{\frac{1}{2}}$$





mesh and price at $t = 11$ months to maturity

A posteriori error indicators

- **Aim** Use a posteriori error indicators to adaptively refine the time-price mesh. The error indicators are computed from the discrete approximation to the solution.

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Let us first consider the semi-discrete problem for European options with a non uniform time grid s.t. $\Delta t_n \leq \rho_{\Delta t} \Delta t_{n\pm 1}$:
find $(u^n)_{0\leq n\leq N} \in L^2(\Omega) \times V_0^N$ satisfying

$$
\begin{aligned}
&u^0 = u_0, \\
&\forall n, 1 \leq n \leq N, \quad \forall v \in V_0, \quad \left(u^n - u^{n-1}, v\right) + \Delta t_n a_{t_n}(u^n, v) = 0.
\end{aligned}
$$

Stability

Under assumptions I, Gårding's inequality

$$\forall t \in [0,T],\ \forall v \in V_0, \qquad a_t(v,v) \geq \frac{1}{4}\sigma_{\min}^2 |v|_V^2 - \lambda \|v\|^2.$$

Assume $2\lambda\Delta t < 1$. Introduce the norm for the sequence $(v^m)_{1\leq m\leq n}$:

$$[[(v^m)]]_n = \left(\left(\prod_{i=1}^{n} (1 - 2\lambda\Delta t_i) \right) \|v^n\|^2 + \frac{1}{2}\sigma_{\min}^2 \sum_{m=1}^{n} \Delta t_m \left(\prod_{i=1}^{m-1} (1 - 2\lambda\Delta t_i) \right) |v^m|_V^2 \right)^{\frac{1}{2}},$$

we have

$$[[(u^m)]]_n \leq \|u^0\|.$$

The Fully Discrete Problem

- $\forall 0 \leq n \leq N$, let $(\mathcal{T}_{nh})$ be a family of grids of $\Omega = (0, \bar{x})$. The grids $\mathcal{T}_{nh}$ for different values of n are not independent: indeed, each $\mathcal{T}_{nh}$ is obained from $\mathcal{T}_{n-1,h}$ by cutting some elements of $\mathcal{T}_{n-1,h}$ or by gluing together elements of $\mathcal{T}_{n-1,h}$.

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- Define for $0 \le n \le N$,

$$V_{nh} = \left\{ v_h \in V, \ \forall \omega \in \mathcal{T}_{nh}, v_{h|\omega} \in \mathcal{P}_1 \right\}, \qquad V_{nh}^0 = V_{nh} \cap V_0.$$

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- Assuming that $u_0 \in V_{0h}$, the fully discrete problem reads:

find $(u_h^n)_{0 \leq n \leq N}, u_h^n \in V_{nh}^0$ s.t. $u_h^0 = u_0$ and

$$\forall 1 \leq n \leq N, \quad \forall v_h \in V_{nh}^0, \quad \left(u_h^n - u_h^{n-1}, v_h\right) + \Delta t_n a_{t_n}(u_h^n, v_h) = 0.$$

Error indicators for adapting the time grid

Assume that the function u_0 belong to V_{1h}. Then, there exists a constant $\alpha \leq \frac{1}{2}$ such that if $\Delta t \leq \frac{\alpha}{\lambda}$:

$$
\begin{aligned}
&[[u - u_{\Delta t}]](t_n) \\
\leq \quad & c \left(\frac{L}{\sigma_{\min}^2} c(u_0) \Delta t + \frac{\mu}{\sigma_{\min}^2} (1 + \rho_{\Delta t}) [[u_{\Delta t} - u_{h,\Delta t}]](t_n) + \frac{\mu}{\sigma_{\min}^2} \Big(\sum_{m=1}^{n} \eta_m^2 \Big)^{\frac{1}{2}} \right),
\end{aligned}
$$

where

$$
\eta_m^2 = \Delta t_m e^{-2\lambda t_{m-1}} \frac{\sigma_{\min}^2}{2} |u_h^m - u_h^{m-1}|_V^2,
$$

and c, $c(u_0)$ and L are positive constants which can be computed.

Error indicators for adapting the price grid

Assume that $u_0 \in V_{1h}$. Then the following a posteriori error estimate holds between $(u^n)_{0\le n\le N}$ and $(u_h^n)_{0\le n\le N}$: for all t_n, $1 \le n \le N$,

$$
\begin{aligned}
&[[(u_{\Delta t} - u_{h,\Delta t})]]^2(t_n) \\
\le \quad & \frac{c}{\sigma_{\min}^2} \max(2, 1+\rho_{\Delta t}) \sum_{m=1}^{n} \Delta t_m \prod_{i=1}^{m-1} (1 - 2\lambda \Delta t_i) \sum_{\omega \in \mathcal{T}_{mh}} \eta_{m,\omega}^2,
\end{aligned}
$$

where

$$
\eta_{m,\omega} = \frac{h_\omega}{x_{\max}(\omega)} \| \frac{u_h^m - u_h^{m-1}}{\Delta t_m} - rx \frac{\partial u_h^m}{\partial x} + r u_h^m \|_{L^2(\omega)} \quad .
$$

Remark no jump terms, because dimension $= 1$.

Upper Bounds for the Error Indicators

For that, we introduce the notation $|||v^n|||$, for $(v^n)_{1\le n\le N}$, $v^n \in V_0$:

$$|||v^n|||^2 = \frac{\sigma_{\min}^2}{2}\Delta t_n \prod_{i=1}^{n-1}(1-2\lambda\Delta t_i)|v^n|_V^2.$$

$$\eta_n \le c \left(\begin{array}{l} |||u^n - u_h^n||| + \sqrt{\rho_{\Delta t}}|||u^{n-1} - u_h^{n-1}||| \\ +\dfrac{e^{-\lambda t_{n-1}}}{\sigma_{\min}}(\|\dfrac{\partial}{\partial t}(u - u_{\Delta t})\|_{L^2(t_{n-1},t_n;V_0')} + \|u - u_{\Delta t}\|_{L^2(t_{n-1},t_n;V_0)}) \\ +(\frac{L}{\sigma_{\min}^2}(\max(1,\rho_{\Delta t}))^{\frac{1}{2}} + \frac{\lambda\mu}{\sigma_{\min}^2})\Delta t_n||u^0|| \end{array} \right)$$

and

$$\eta_{n,\omega} \le C\left(\|\frac{u^{n-1} - u_h^{n-1} - u^n + u_h^n}{\Delta t_n}\|_{V_0'(K_\omega)} + \mu\|S\frac{\partial(u^n - u_h^n)}{\partial S}\|_{L^2(K_\omega)}\right).$$

For European options, the indicators are reliable and efficient.

What about the American options?

The same bounds hold, i.e. error $\lesssim$ indicators hold for American options. Therefore the indicators can be used.

On the contrary, the opposite bounds are not proved. Efficiency is not guaranteed.

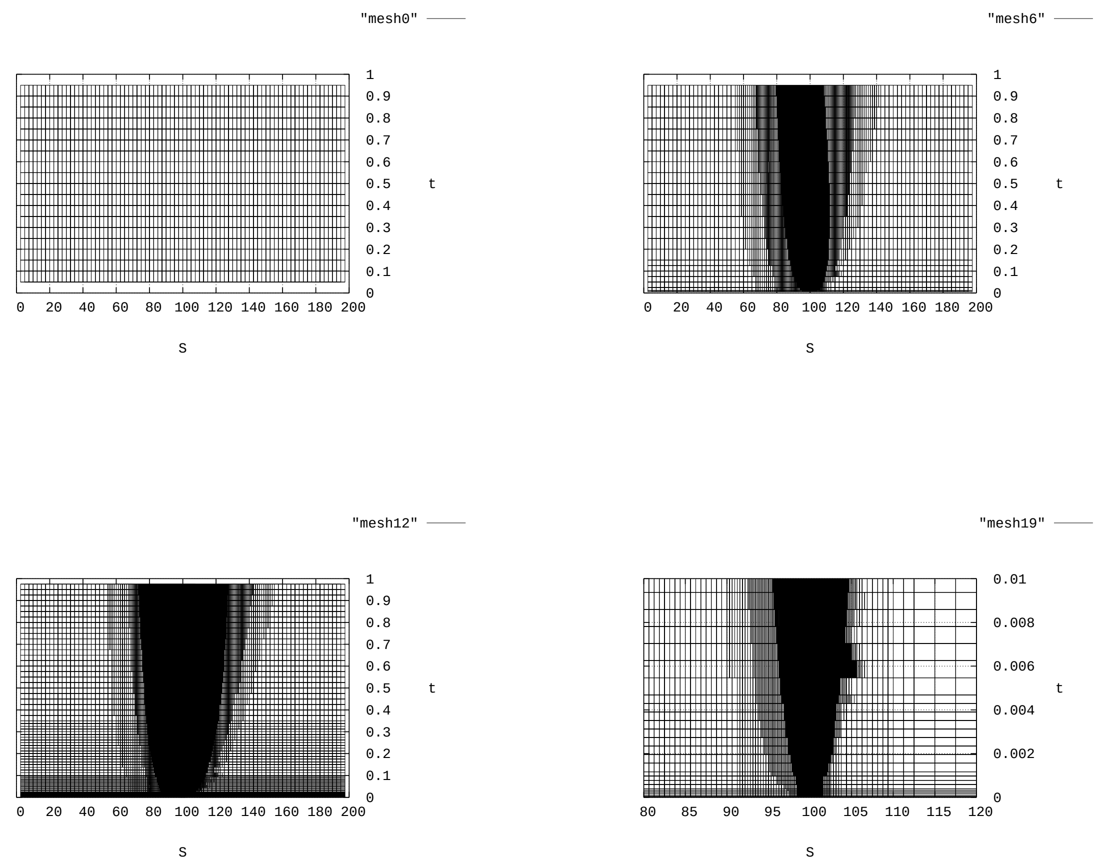
Error vs. Indicators

European option with Constant volatility

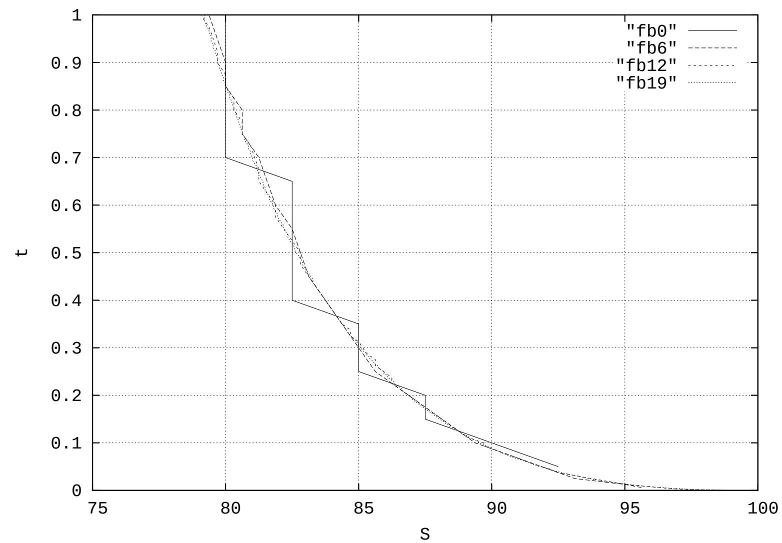
error	5.67	5.66	5.67	4.66	3.73	3.25	3.26	2.53	2.53	1.95	1.45	1.06
estim. err.	12.27	8.56	6.62	5.38	4.58	4.19	3.39	2.95	2.56	2.21	1.85	1.59
error	1.06	0.77	0.77	0.57	0.57	0.41	0.41	0.30				
estim. err.	1.48	1.29	1.03	0.90	0.77	0.67	0.52	0.44				

$\sigma\|u-u_{h,\Delta t}\|_{L^2((0,T);V)}$ and $\left(\sum_m(\eta_m^2+\frac{\Delta t_m}{\sigma^2}\sum_\omega \eta_{m,\omega}^2)\right)^{\frac{1}{2}}$ for the different meshes

Adaptive mesh refinement

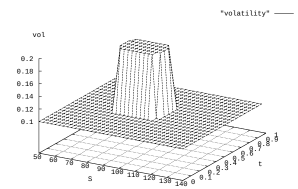


The exercise boundary

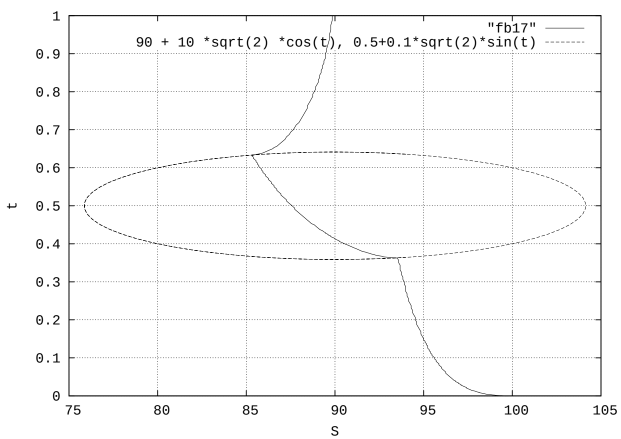


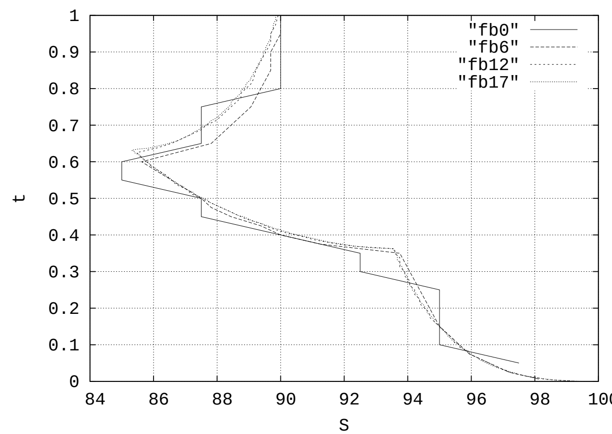
The exercise boundaries for different mesh refinements

Non uniform volatility



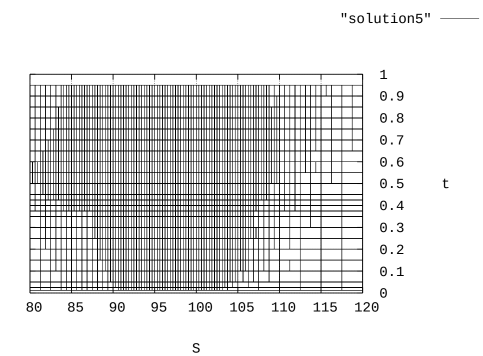
The local volatility surface

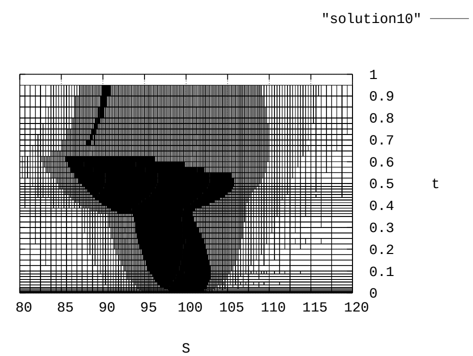




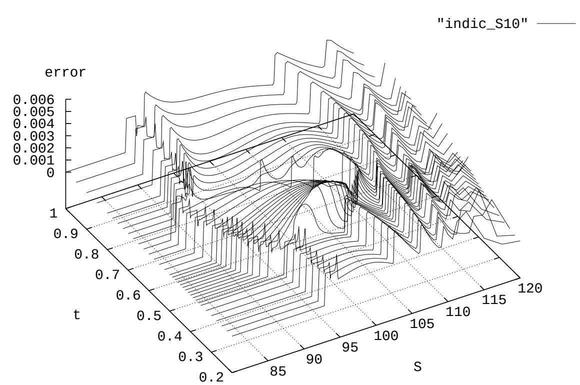
Left: the exercise boundary. Right: the exercise boundaries for different mesh refinements

Non uniform volatility





Mesh refinements: the mesh is refined near the exercise boundary



Error indicators.