Obstacle and optimal stopping problems for American Asian options

Andrea Pascucci

Asian options in the Black&Scholes model

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

Payoff: $\varphi = \varphi(t, S_t, A_t)$

Arithmetic average: $A_t = \frac{1}{t}\int_0^t S_s ds$

Geometric average: $A_t = \exp\left(\frac{1}{t}\int_0^t \log S_s ds\right)$

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- European and American style
- traded in currency and commodity markets to avoid price manipulations
- space augmentation $\longrightarrow$ Markovian formulation for non-uniformly parabolic PDE:

$$Ku = \frac{\sigma^2 S^2}{2}\partial_{SS}u + S\partial_A u + \partial_t u = 0$$

for $(t, S, A) \in [0, T] \times \mathbb{R}_+ \times \mathbb{R}_+$

European Asian options in B&S: reduction to one state variable

Only for European options in the Black & Scholes setting and for particular homogeneous payoffs:

- **Ingersoll** (1987)

  $\varphi(t, S, A) = (S - A)^+ \qquad$ (floating strike)

  $$x = \frac{A}{S} \qquad \Longrightarrow \qquad \frac{\sigma^2 x^2}{2}\partial_{xx} + x\partial_x + \partial_t, \qquad x > 0$$

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- **Rogers & Shi** (1995)

  $\varphi(t,S,A) = (A-K)^+$ (fixed strike)

  $$x = \frac{A-K}{S} \qquad \Longrightarrow \qquad \frac{\sigma^2 x^2}{2}\partial_{xx} + x\partial_x + \partial_t, \qquad \underline{x \in \mathbb{R}}$$

Obstacle problem for American options

X diffusion in $\mathbb{R}^N$ (state variables)

K related Kolmogorov operator

φ payoff function

$$\begin{cases} \max\{Ku, \varphi - u\} = 0, & \text{in }]0,T[\times\mathbb{R}^N \\ u(T,\cdot) = \varphi(T,\cdot), & \text{in } \mathbb{R}^N \end{cases}$$

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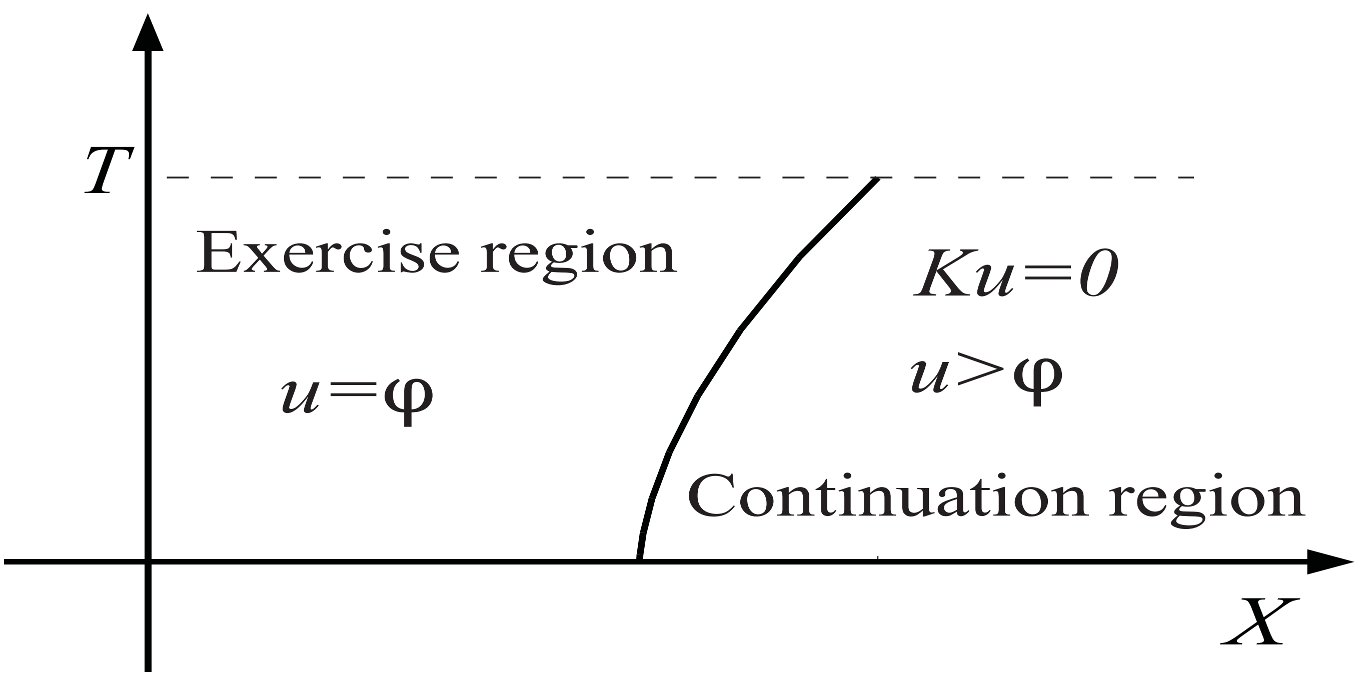

Optimal stopping and fair price of American options

u "weak" solution to the obstacle problem for

the diffusion X

$$u(t,x) = \sup_{\tau \in [t,T]} E\left[\varphi(\tau, X_{\tau}^{t,x})\right]$$

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$u(t, X_t)$ fair price, i.e. value of a self-financing strategy such that

- $u(t, X_t) \geq \varphi(t, X_t)$ for any $t \in [0,T]$
- $u(\tau, X_\tau) = \varphi(\tau, X_\tau)$ for some $\tau \in [0,T]$

Classical results for uniformly parabolic PDEs

- Variational solutions:
  Bensoussan&Lions (1978)
  Kinderlehrer&Stampacchia (1980)

- Obstacle and optimal stopping:
  van Moerbeke (1975)
  Bensoussan&Lions (1978)

- American options:
  Bensoussan (1984)
  Karatzas (1988)
  Jaillet&Lamberton&Lapeyre (1990)

- Strong solutions:
  Friedman (1975)

- Viscosity solutions:
  Barles (1997)
  Oksendal&Reikvam (1998)
  Gatarec&Swiech (1999)

American Asian options: numerical results

Barraquand and Pudet 1996

Barles 1997

Wu, You and Kwok 1999

Hansen and Jorgensen 2000

Meyer 2000

Ben-Ameur, Breton and L'Ecuyer 2002

Marcozzi 2003

Fu and Wu 2003

Jiang and Dai 2004

d'Halluin, Forsyth and Labahn 2005

Huang and Thulasiram 2005

Dai and Kwok 2006

Main results

1) Di Francesco, P. and Polidoro (2007)

Existence of a strong solution to the obstacle problem for non-uniformly parabolic pricing PDEs for Asian options

Ingredients:

- a priori estimates (Sobolev and Schauder type)
- penalization technique
- existence results for quasi-linear Cauchy-Dirichlet problem

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2) P. (2007)

Solution to the optimal stopping and pricing problems for Asian options

Ingredients:

- upper Gaussian estimates for the transition density of the state process
- generalized Itô formula

Linear SDEs and Kolmogorov PDEs with constant coefficients

$$dX_t = BX_t dt + \sigma dW_t$$

W d-dimensional Brownian motion

B constant $N \times N$ matrix

σ constant $N \times d$ matrix

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Solution:

$$X_t = e^{tB} \left(x + \int_0^t e^{-sB} \sigma dW_s \right), \qquad x \in \mathbb{R}^N$$

X_t is a Gaussian process:

- Mean
$$E(X_t) = e^{tB} x$$
- Covariance matrix
$$\mathcal{C}(t) = \int_0^t e^{sB} \sigma \left(e^{sB} \sigma \right)^T \sigma ds$$

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$\rightarrow X$ has a transition density/fundamental solution of the Kolmogorov op.

$$K = \mathrm{div}(A\nabla) + \underbrace{\langle Bx, \nabla \rangle + \partial_t}_{Y}$$

where $(x,t) \in \mathbb{R}^{N+1}$ and

$$A = \frac{1}{2}\sigma\sigma^* = \begin{pmatrix} I_d & 0 \\ 0 & 0 \end{pmatrix}$$

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Example: $N = 2$, $d = 1$

$$\begin{cases} dX_t^1 = dW_t \\ dX_t^2 = X_t^1 dt \end{cases}$$

that is,

$$X_t^2 = \int_0^t X_s^1 ds \qquad \text{(average for Asian option)}$$

Non Euclidean structure

$$K = \mathrm{div}(A\nabla) + \underbrace{\langle Bx, \nabla \rangle + \partial_t}_{Y}, \qquad (t,x) \in \mathbb{R}^{N+1}$$

B**-Translations:** $z = (t,x),\ \zeta = (\tau, \xi)$

$$\ell_\zeta(z) = \zeta \circ z := (t+\tau, x + e^{tB}\xi)$$

B**-Dilations:**

$$L = \partial_{xx} + x\partial_y + \partial_t$$

is homogeneous w.r.t.

$$(t,x,y) \xrightarrow{\delta_\lambda} (\lambda^2 t, \lambda x, \lambda^3 y)$$

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B**-Homogeneous norm:**

$$\|(t,x,y)\|_B = |t|^{\frac{1}{2}} + |x| + |y|^{\frac{1}{3}}$$

B**-Hölder continuity:**

$$|u(z) - u(\zeta)| \le c\|\zeta^{-1} \circ z\|_B^\alpha$$

All depends **only on the matrix** B!

Kolmogorov equations with Hölder coeff.

$$K = \sum_{i,j=1}^{d} a_{ij}(t,x)\partial_{x_i x_j} + \sum_{i=1}^{d} a_i(t,x)\partial_{x_i} + a(t,x) + \underbrace{\langle Bx, \nabla \rangle + \partial_t}_{Y}$$

- $(t,x) \in \mathbb{R} \times \mathbb{R}^N$ but $d \leq N$

- $(a_{ij}) \sim I_{\mathbb{R}^d}$

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- $(t,x) \in \mathbb{R} \times \mathbb{R}^N$ but $d \le N$

- $(a_{ij}) \sim I_{\mathbb{R}^d}$

- a_{ij}, a_i, a are B-Hölder continuous

- B is constant and

$$K_0 = \triangle_{\mathbb{R}^d} + Y$$

verifies the Hörmander condition

Example 1: geometric Asian options

Log-price:

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t$$

Geometric average

$$dA_t = X_t dt$$

Pricing PDE

$$\frac{\sigma^2(t,x)}{2}\partial_{xx}u + x\partial_y u + \partial_t u = 0 \qquad (t,x,y) \in \mathbb{R}^3$$

Arithmetic Asian options can be studied as well

Pricing PDE

$$\frac{\sigma^2(t,x)}{2}\partial_{xx}u + e^x\partial_y u + \partial_t u = 0 \qquad (t,x,y) \in \mathbb{R}^3$$

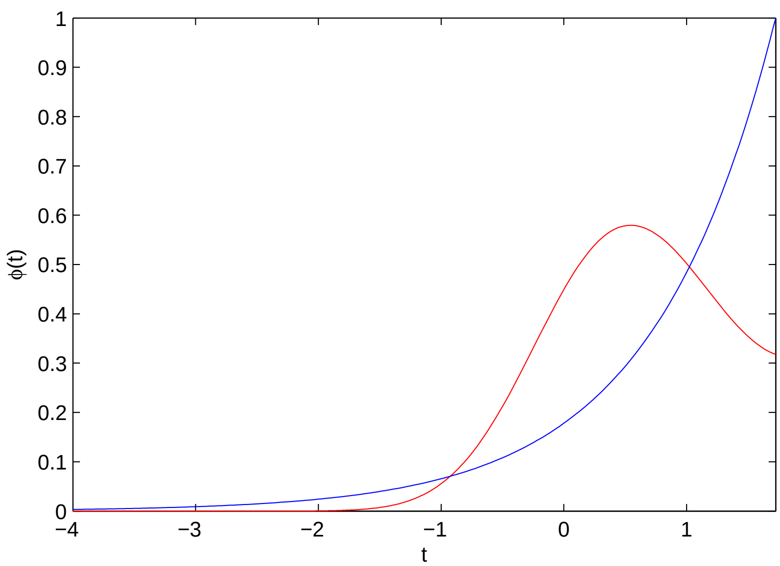
Example 2: path dependent volatility

Weight: $\psi \in L^1(]-\infty, T]),\ \psi > 0$ on $[0, T]$

$$\Psi(t) = \int_{-\infty}^{t} \psi(s) ds$$

Average: $$A_t = \frac{1}{\Psi(t)} \int_{-\infty}^{t} \psi(s) Z_s ds$$

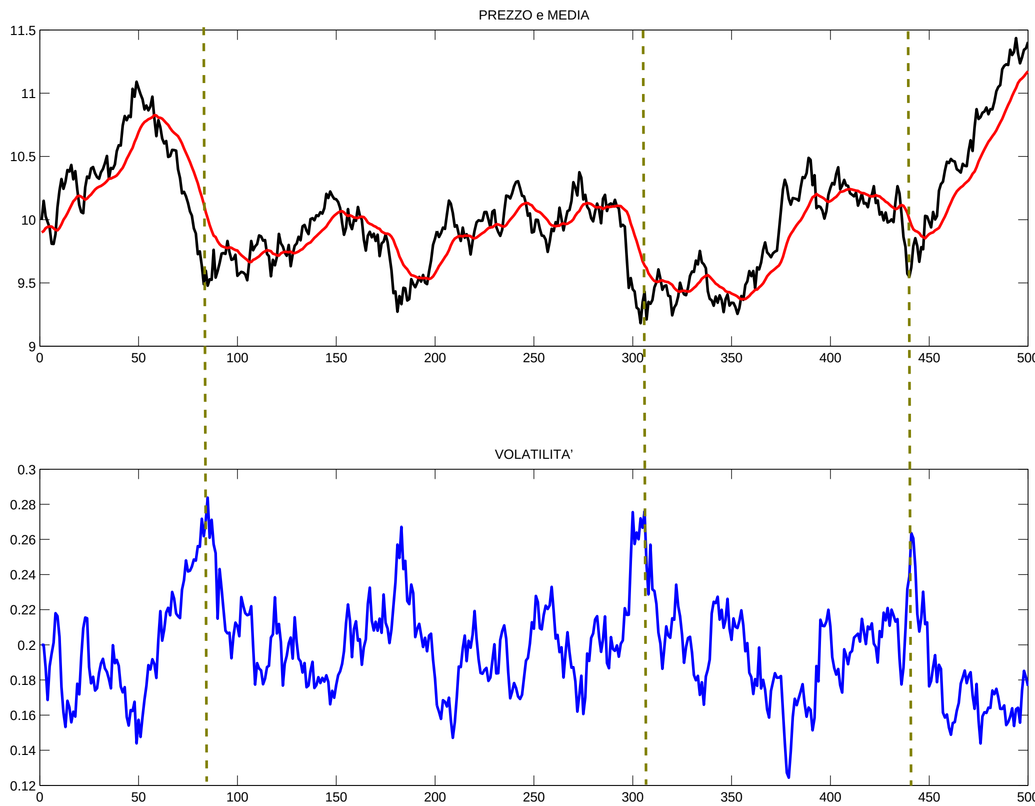
Log-price: $$dZ_t = \mu(Z_t - A_t)dt + \sigma(Z_t - A_t)dW_t$$



- **Hobson&Rogers model**: $\psi(t) = e^{-\lambda t}$

PROs

- realistic asset/volatility dynamics



- market completeness
- with the simple specification of the volatility function

$$\sigma^2(x) = a_1 + a_2(x - a_3)^2,$$

it reproduces observed volatility surfaces

- information on the past: better out of sample and P&L performance

- not more difficult than local volatility

CONs

- completeness is unrealistic: hedging volatility risk
- evidences of volatility driven by autonomous process
- the average

$$A_t = \lambda \int_{-\infty}^{t} e^{-\lambda(t-s)} Z_s ds$$

involves prices from $-\infty$ in time: mathematical and economical problems (cf. Hallulli-Vargiolu)

Preliminary results: fundamental solution

Polidoro (1994-95)

Di Francesco - P. (2006)

(using *the parametrix method*)

$$K = \sum_{i,j=1}^{d} a_{ij}(t,x)\partial_{x_i x_j} + \underbrace{\langle Bx, \nabla \rangle + \partial_t}_{Y} \qquad \text{on } \mathbb{R} \times \mathbb{R}^N$$

- K has a fundamental solution/transition density

$$u(t,x) = \int_{\mathbb{R}^N} \Gamma(t,x;T,y)\varphi(y)dy$$

is solution to the Cauchy problem

$$\begin{cases} Ku = 0, & \text{in }]t,T] \times \mathbb{R}^N, \\ u(T,\cdot) = \varphi. \end{cases}$$

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- Gaussian **upper** bounds

$$\Gamma(z,\zeta) \le C\ \Gamma_0(z,\zeta)$$

  Γ_0 fundamental solution of $K_0 = \triangle_{\mathbb{R}^d} + Y$

Sobolev and Hölder spaces

$$K = \sum_{i,j=1}^{d} a_{ij}(t,x)\partial_{x_i x_j} + \underbrace{\langle Bx, \nabla \rangle + \partial_t}_{Y} \qquad \text{on } \mathbb{R} \times \mathbb{R}^N$$

Sobolev spaces

$$\|u\|_{\mathcal{S}^p(\Omega)} = \|u\|_{L^p(\Omega)} + \sum_{i=1}^{d} \|\partial_{x_i} u\|_{L^p(\Omega)}$$
$$+ \sum_{i,j=1}^{d} \|\partial_{x_i x_j} u\|_{L^p(\Omega)} + \|Yu\|_{L^p(\Omega)}$$

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Hölder spaces

$$\|u\|_{C_B^\alpha(\Omega)} = \sup_{\Omega} |u| + \sup_{\substack{z,\zeta,\in\Omega \\ z\neq\zeta}} \frac{|u(z) - u(\zeta)|}{\|z^{-1} \circ z\|_B^\alpha}$$

$$\|u\|_{C_B^{2,\alpha}(\Omega)} = \|u\|_{C_B^\alpha(\Omega)} + \sum_{i=1}^{d} \|\partial_{x_i} u\|_{C_B^\alpha(\Omega)} + \sum_{i,j=1}^{d} \|\partial_{x_i x_j} u\|_{C_B^\alpha(\Omega)} + \|Yu\|_{C_B^\alpha(\Omega)}$$

Strong solution

$$\max\{Ku, \varphi - u\} = 0$$

where

$$K = \sum_{i,j=1}^{d} a_{ij}(t,x)\partial_{x_i x_j} + \underbrace{\langle Bx, \nabla\rangle + \partial_t}_{Y} \qquad \text{on } \mathbb{R} \times \mathbb{R}^N$$

$u \in \mathcal{S}^1_{\text{loc}}$ that is

$$u, \quad \partial_{x_i} u, \quad \partial_{x_i x_j} u, \quad Yu \in L^1_{\text{loc}}$$

for $i, j = 1, \dots, d$

A priori estimates

- **Embedding theorem**

$$\|u\|_{C_B^{1,\alpha}(O)} \leq c\|u\|_{\mathcal{S}^p(\Omega)}$$

for

$$p \geq Q+2, \qquad \alpha = 1 - \frac{Q+2}{p}$$

where $O \subset\subset \Omega$ and $c = c(K, O, \Omega, p, \alpha)$

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- **Schauder and Sobolev type a-priori estimates**

Bramanti, Cerutti e Manfredini 1996

Di Francesco e Polidoro 2006

Di Francesco, P. e Polidoro 2007

Obstacle problem on a bounded cylinder

$$\max\{Ku, \varphi - u\} = 0$$

- on a bounded cylindrical domain $H(T) :=]0, T[\times H$
- Cauchy-Dirichlet boundary conditions

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any point ζ of the parabolic boundary admits a *barrier*

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any point ζ of the parabolic boundary admits a *barrier*

$$w : V \cap \overline{H(T)} \to \mathbb{R},$$

such that

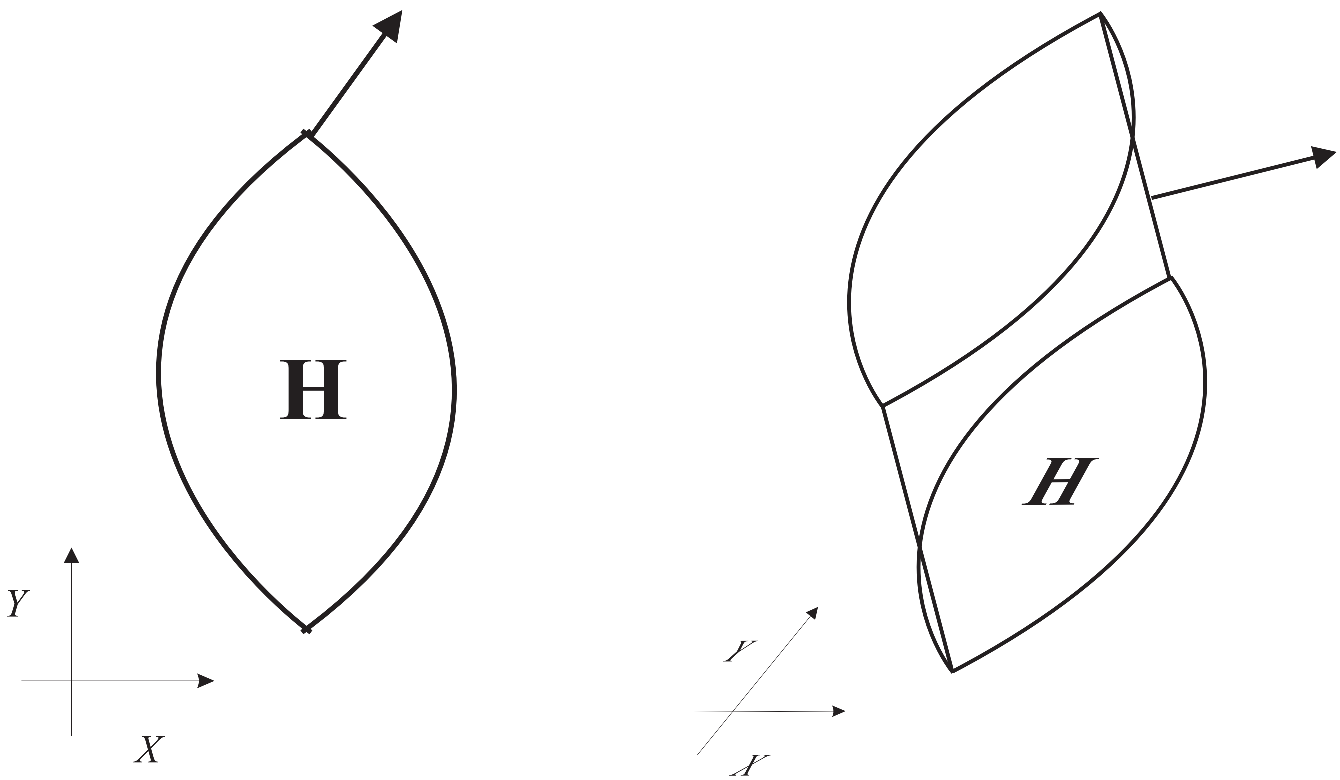
i) $Kw \leq -1$ in $V \cap H(T)$;

ii) $w(z) > 0$ in $V \cap \overline{H(T)} \setminus \{\zeta\}$ and $w(\zeta) = 0$.

Maximum principle $\Longrightarrow$
uniform bounds at the boundary

Regular cylinder $]0, T[\times H$: an example

$$K = \partial_{xx} + x\partial_y + \partial_t$$



Obstacle / Payoff function

$\varphi \in \mathrm{Lip}(\overline{H(T)})$ and

$$\sum_{i,j=1}^{d} \xi_i \xi_j \partial_{x_i x_j} \varphi \geq \widetilde{C} |\xi|^2 \qquad \text{in } H(T),\ \xi \in \mathbb{R}^d$$

in the distributional sense, $\widetilde{C} \in \mathbb{R}$ (possibly $\widetilde{C} < 0$)

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Examples:

- C^2 functions

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Examples:

- C^2 functions
- Lipschitz continuous functions, *convex w.r.t.* $x_1, \dots, x_d$
- call and put options

$$\varphi(x) = (E - x)^+ \qquad \Longrightarrow \qquad \varphi'' = \delta_E \geq 0$$

Obstacle problem

Theorem [Di Francesco, P., Polidoro]

Problem

$$\begin{cases} \max\{Ku, \varphi - u\} = 0, & \text{in } H(T) \\ u|_{\partial_P H(T)} = g \end{cases}$$

admits a strong solution u such that

$$\|u\|_{\mathcal{S}^p(O)} \leq C,$$

for any $p \geq 1$ and $O \subset\subset H(T)$, with

$$C = C(K, O, H(T), p, \varphi, \|g\|_\infty)$$

In particular $u \in C_B^{1,\alpha}$.

Moreover u is a viscosity solution.

Penalization technique

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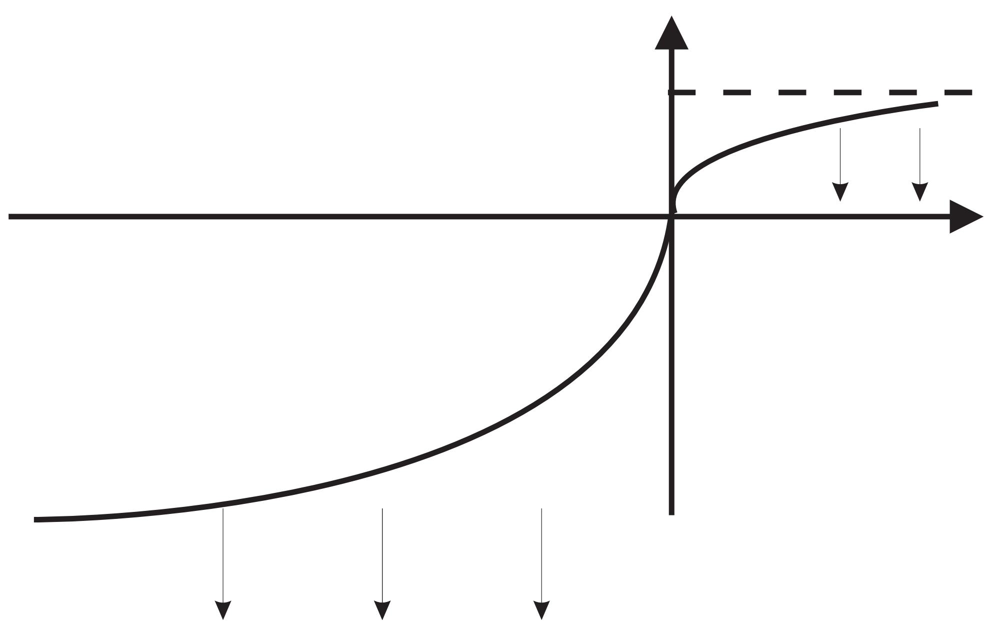
where $(\beta_\varepsilon)_{\varepsilon>0}$ are in $L^\infty \cap C^\infty(\mathbb{R})$ increasing and

- $\beta_\varepsilon(0) = 0$
- if $u > \varphi$ then $\beta_\varepsilon(u - \varphi) \le \varepsilon$
- if $u < \varphi$ then $\beta_\varepsilon(u - \varphi) \longrightarrow -\infty$ as $\varepsilon \to 0$



Quasi-linear problem

If $h = h(z, u) \in \mathrm{Lip}(\overline{H(T)} \times \mathbb{R})$ then

$$\begin{cases} Ku = h(\cdot, u), & \text{in } H(T), \\ u|_{\partial_P H(T)} = g \end{cases}$$

has a classical solution $u \in C_B^{2,\alpha}(H(T)) \cap C(\overline{H(T)})$ s.t.

$$\sup_{H(T)} |u| \le e^{cT}(1 + \|g\|_{L^\infty})$$

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Proof

- solution as limit of (u_j):

$$\begin{cases} Ku_j = h(\cdot, u_{j-1}) & \text{in } H(T), \\ u_j|_{\partial_P H(T)} = g, \end{cases}$$

- interior Schauder estimates

$$\|u\|_{C_B^{2,a}(O)} \le c\left(\|u\|_{L^\infty(\Omega)} + \|Ku\|_{C_B^\alpha(\Omega)}\right)$$

- barrier functions

Penalized problem as $\varepsilon \to 0^+$

$$\begin{cases} Ku^\varepsilon = \beta_\varepsilon(u^\varepsilon - \varphi), & \text{in } H(T) \\ u^\varepsilon|_{\partial_P H(T)} = g \end{cases}$$

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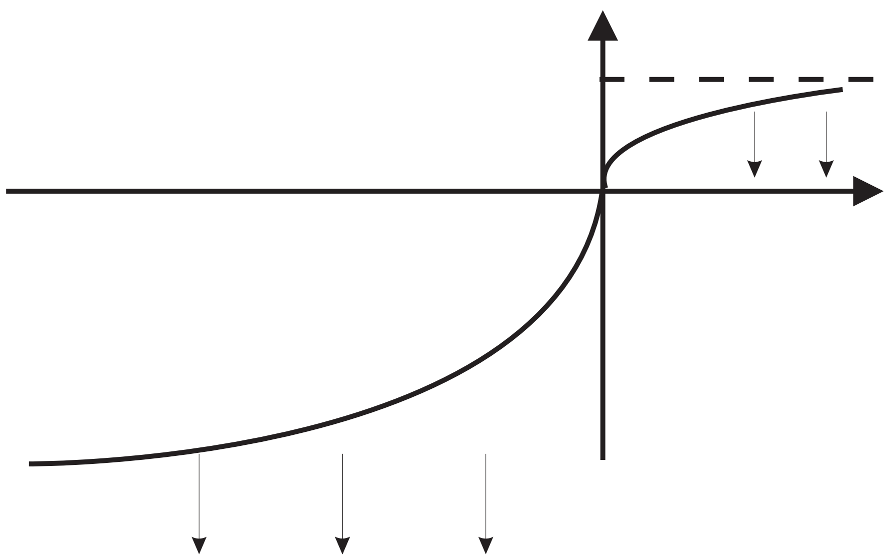
Key estimate:

$$|\beta_\varepsilon(u^\varepsilon - \varphi)| \le C$$



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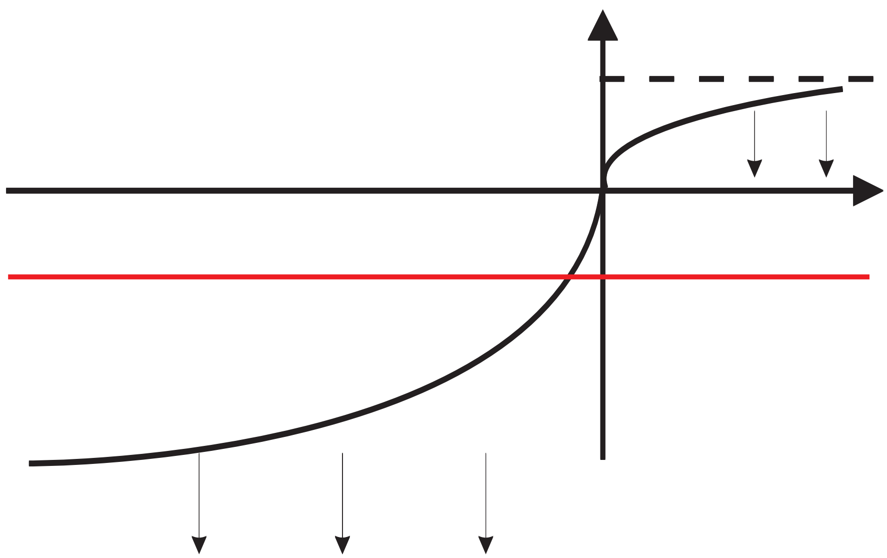

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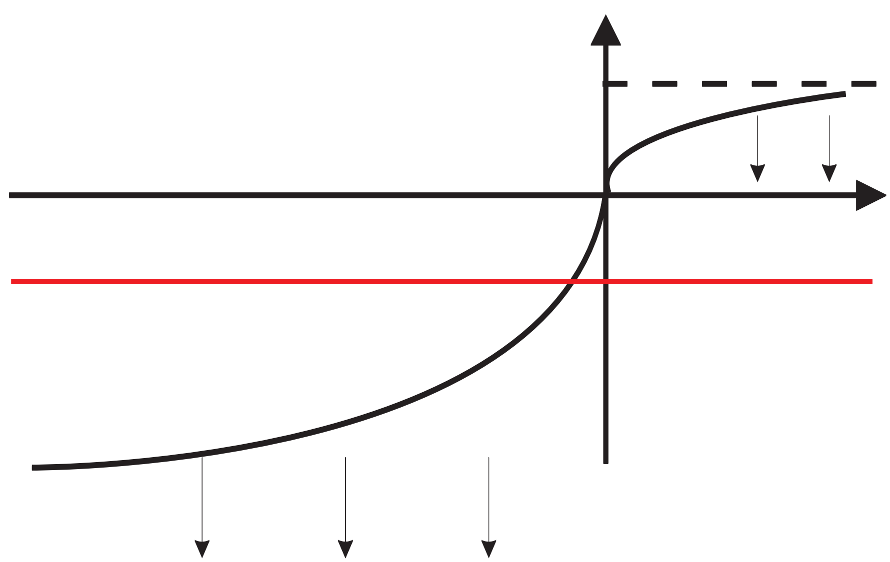

Existence of a strong solution to the obstacle problem:

Estimate (1) + barriers + interior estimates in $\mathcal{S}^p$

$$\|u\|_{\mathcal{S}^p(O)} \le c\left(\|u\|_{L^p(\Omega)} + \|Ku\|_{L^p(\Omega)}\right)$$

Penalized problem as $\varepsilon \to 0^+$

$$|\beta_\varepsilon(u^\varepsilon - \varphi)| \le C \tag{1}$$

Proof of (1):

$$\zeta \text{ interior minimum} \implies K(u^\varepsilon - \varphi)(\zeta) \ge 0$$

$$\beta_\varepsilon(u^\varepsilon - \varphi)(\zeta) = K u^\varepsilon(\zeta) \ge K\varphi(\zeta) \ge \widetilde{C}$$

Obstacle problem on the strip

$$\begin{cases} \max\{Ku, \varphi - u\} = 0, & \text{in }]0, T[\times \mathbb{R}^N \\ u(T, \cdot) = \varphi(T, \cdot), & \text{in } \mathbb{R}^N \end{cases}$$

Theorem [Di Francesco, P., Polidoro]

If a strong super-solution $\bar{u}$ exists

then there exists a strong solution $u \leq \bar{u}$

and $u \in \mathcal{S}^p_{\text{loc}}$ for any $p \geq 1$

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Example: call option, $\varphi(x) = (e^x - K)^+$

$\bar{u}(x) = e^{x+Ct}$ is a super-solution i.e.

$$\max\{K\bar{u}, \varphi - \bar{u}\} \leq 0$$

Optimal stopping

$$dX_t = (BX_t + b(t, X_t))\, dt + \sigma(t, X_t) dW_t$$

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Theorem [P.]
u strong solution to the obstacle problem s.t.

$$|u(t, x)| \le ce^{c|x|^2}$$

Then

$$u(t, x) = \sup_{\tau \in [t,T]} E\left[\varphi(\tau, X_\tau^{t,x})\right]$$

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Proof: $\Gamma(t, x; \cdot, \cdot)$ transition density of $X^{t,x}$

upper Gaussian estimate $\Longrightarrow$

$$\Gamma(t, x; \cdot, \cdot) \in L^q(]t, T[\times \mathbb{R}^N)$$

for some $q > 1$

Itô formula for strong solutions

u^n regularization of u

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Estimate of the deterministic integral

$$E\left[\left|\int_t^\tau \left(Ku(s,X_s^{t,x}) - Ku^n(s,X_s^{t,x})\right) ds\right|\right] \le$$

$$\le \int_{\mathbb{R}^N}\int_t^T |Ku(s,y) - Ku^n(s,y)|\,\Gamma(t,x;s,y) ds dy \le$$

$$\le \|Ku - Ku^n\|_{L^p}\|\Gamma(t,x;\cdot,\cdot)\|_{L^q} \xrightarrow[n\to\infty]{} 0$$

since $u \in \mathcal{S}^p$ and $\Gamma(t,x;\cdot,\cdot) \in L^q$