Free boundary regularity close to initial state and applications to finance

Teitur Arnarson
KTH, Stockholm

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Outline

- ⋆ Problem and setup
- ⋆ History
- ⋆ The blow-up technique
- ⋆ Application to indifference pricing

American option

- A contract on one or several underlying assets that can be exercised during some predetermined period $[t, T]$.

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- Payoff $g : \mathbb{R}^n \to \mathbb{R}$ at exercise $\tau \in \mathcal{T}_{[t,T]}$.

Example: American put option

Gives you the right, but not the obligation, to sell the underlying stock X_s for a predetermined price K any time $s \in [t, T]$.

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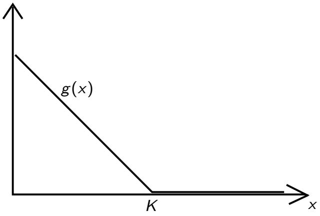
At exercise τ the payoff is $g(X_\tau) = \max(K - X_\tau, 0)$.



Complete markets

The market consists of

- non-risky asset

$$
\begin{aligned}
dB_s &= \rho B_s ds \\
B_t &= B.
\end{aligned}
$$

- traded asset

$$
\begin{aligned}
dX_s &= \mu X_s ds + \sigma X_s dW_s \\
X_t &= x
\end{aligned}
$$

W_s is Brownian motion.

Option price

The price h of an American option with payoff g is given by

Theorem (Risk-neutral valuation formula)

$$h(x,t) = \sup_{\tau \in [t,T]} E(e^{-\rho(\tau - t)} g(X_\tau) | X_t = x).$$

Variational inequality

h solves the following linear variational inequality

$$
\begin{aligned}
\min\Big(-h_t-\frac{1}{2}\sigma^2x^2h_{xx}-\rho x h_x+\rho h,& \\
h(x,t)-g(x)\Big) &= 0 \quad \text{in } \mathbb{R}\times[0,T) \\
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A *free boundary* Γ separates the sets

$$\begin{aligned}\mathscr{C} &= \{-h_t - \frac{1}{2}\sigma^2 x^2 h_{xx} - \rho x h_x + \rho h = 0\} \\ \mathscr{E} &= \{h - g = 0\}.\end{aligned}$$

History

Results for the American put.

- Kuske & Keller (1998)
- Bunch & Johnsson (2000)
- Stamicar, Sevcovic & Chadam (1999)

Chen, Chadam: Reformulation

In dimensionless variables the price function $\tilde{h}(x,t)$ solves

$$\begin{aligned}
\tilde{h}_t - \tilde{h}_{xx} - (k-1)\tilde{h}_x + k\tilde{h} &= 0 \qquad \text{for } x > \tilde{\beta}(t) \\
\tilde{h} &= 1 - e^x \qquad \text{for } x < \tilde{\beta}(t) \\
\tilde{h}(0,x) &= (1-e^x)^+,
\end{aligned}$$

where $x = \tilde{\beta}(t)$ is a parameterization of the free boundary Γ.

Fundamental solution

Find the fundamental solution for the PDE

$$\Phi(x,t) = \frac{1}{2\sqrt{\pi t}} \exp\left\{ -\frac{(x+(k-1)t)^2}{4t} \right\}$$

and get the following integral representation

$$\tilde{h}(x,t) = \int_{-\infty}^{0} (1-e^y)\Phi(x-y,t)dy + k\int_0^t \int_{-\infty}^{\tilde{\beta}(t-\theta)} \Phi(x-y,\theta)dyd\theta.$$

ODE for the free boundary

Derive an ODE for the free boundary

$$\dot{\tilde{\beta}} = -\frac{2\Phi_x(\tilde{\beta}(t), t)}{k} - 2\int_0^t \Phi_x(\tilde{\beta}(t) - \tilde{\beta}(t-\theta), \theta)\dot{\tilde{\beta}}(t-\theta)d\theta.$$

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Asymptotic expansion

$$\frac{\tilde{\beta}^2}{4t} = -\xi - \frac{1}{2\xi} + \frac{1}{8\xi^2} + \frac{17}{24\xi^3} + \dots$$

where $\xi = \sqrt{4\pi k^2 t}$.

Summary of the expansion method

Advantage

- Good precision

Drawback

- One-dimensional, linear setting.

The blow-up technique (only 4 slides)

A general obstacle problem

Obstacle problem with a non-linear, $n+1$-dimensional, parabolic operator

$$\begin{aligned} \min(D_t u - F(D^2 u, Du, u, x, t), u - g) &= 0 \quad \text{in } B_1 \times (0,1) \\ u(x,0) &= g(x) \quad \text{in } B_1 \end{aligned}$$

where B_1 is the unit ball in $\mathbb{R}^n$.

Scaling in the point $(0,0)$

For simplicity assume: $u(0,0) = g(0) = 0$.

Scaled function

$$u_r(x,t) = \frac{u(rx, r^2 t)}{\alpha_r}$$

Scaled operator

$$F_r(D^2u, Du, u, x, t) = F(D^2u, rDu, r^2u, rx, r^2t).$$

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Choose α_r so that $0 < \lim_{r\to 0} u_r < \infty$.

Scaled obstacle problem

Under standard assumptions on F the scaled function u_r solves

$$\begin{aligned} \min(D_t u_r - F_r(D^2 u_r, Du_r, u_r, x, t), \\ u_r - g_r) &= 0 \quad \text{in } B_{1/r} \times (0, \frac{1}{r^2}) \\ u_r(x,0) &= g_r(x) \quad \text{in } B_{1/r}. \end{aligned}$$

Blow-up limit

Take the so called *blow-up limit* by letting $r \to 0$.

If we have the right growth and continuity of u the limit function $u_0 = \lim_{r\to 0} u_r$ will solve

$$\begin{aligned} \min(D_t u_0 - F_0(D^2 u_0, Du_0, u_0, x, t),& \\ u_0 - g_0) &= 0 \quad \text{in } \mathbb{R}\times\mathbb{R}^+ \\ u_0(x,0) &= g_0(x) \quad \text{in } \mathbb{R}. \end{aligned}$$

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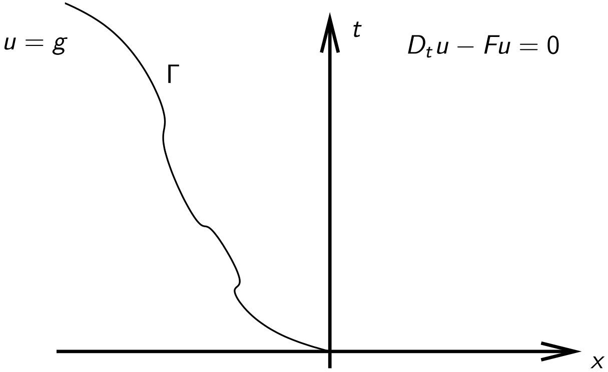
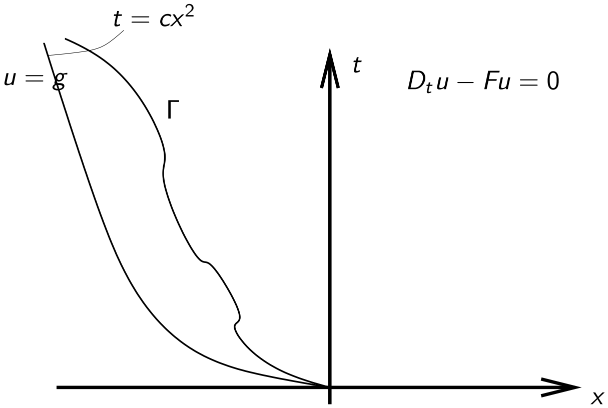
Free boundary regularity

Assume we have a free boundary.



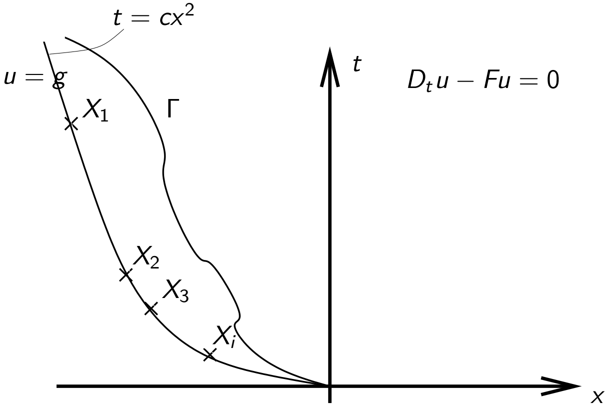
Free boundary regularity

Assume that the free boundary stays above $t = cx^2$.



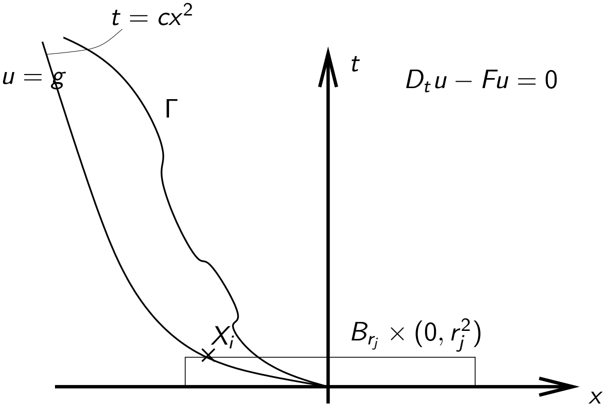
Free boundary regularity

Pick a sequence $X_1, X_2 \ldots \in \{t = cx^2\}$,where $X_j = (x_j, t_j)$.



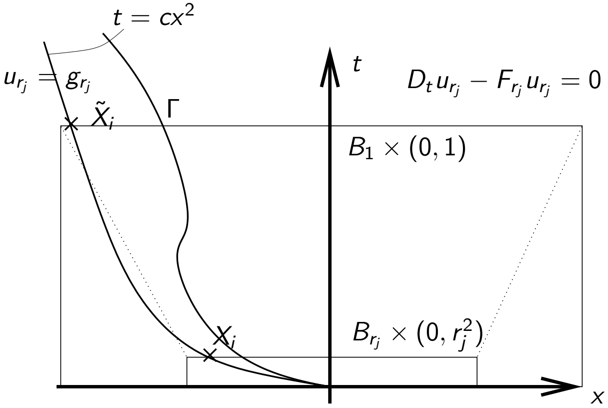
Free boundary regularity

Set $r_j = |X_j|\ldots$



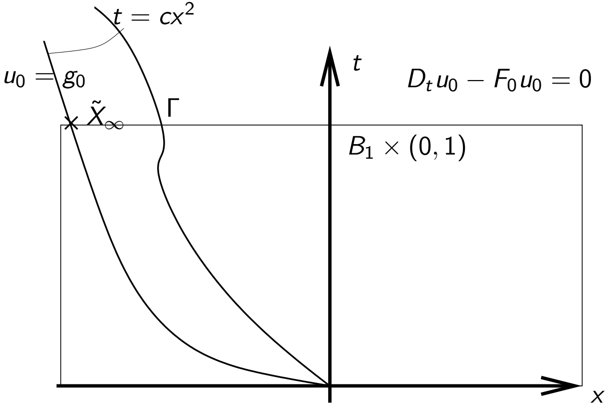
Free boundary regularity

...and scale the problem by r_j. $\tilde{X}_j = (x_j/r_j, t_j/r_j^2)$.



Free boundary regularity

Take the limit as $j \to \infty$. Note $|\tilde{X}_\infty| = 1$.



The blow-up limit problem

- For the limit problem no lower order terms occur in the PDE.
- The limit obstacle g_0 is possibly simpler than the original g.

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$$\Downarrow$$

When analysing the limit problem we might arrive at different conclusios:

- We might find an analytic solution.
- We might be able to contradict the picture above.

The obstacle is a strict subsolution

g_0 is a strict subsolution if

$$-F(D^2 g_0, 0, 0, 0, 0) < 0 \text{ in } B_1 \times (0, 1).$$

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$D_t u_0 - F(u_0, 0, 0, 0, 0) \geq 0$ in $B_1 \times (0,1)$ and the maximum principle

$$\Downarrow$$

$$u_0 > g_0 \text{ in } B_1 \times (0,1).$$

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$$\Downarrow$$

No free boundary exists for the limit problem, i.e.

$$\Gamma \in \{t < x^2 \cdot \sigma(x)\}$$

for some modulus of continuity $\sigma(x)$.

Incomplete markets: Market components

The market consists of

- non-risky asset (zero interest rate for simplicity)

$$B_s = B.$$

- traded asset

$$\begin{aligned} dX_s &= \mu X_s ds + \sigma X_s dW_s \\ X_t &= x \end{aligned}$$

- non-traded asset

$$\begin{aligned} dY_s &= b(Y_s, s)ds + a(Y_s, s)dW'_s \\ Y_t &= y \end{aligned}$$

W_s and W'_s are correlated with correlation $\rho \in (-1, 1)$.

Aim

Define the *indifference price* h of a call option written on the non-traded asset Y_s.

Investment alternatives

Alternative 1: Invest in stock X_s and bond B_s

- Allocation in traded stock X_s: π_s
  Allocation in bond: π_s^0
- Wealth: $Z_s = \pi_s^0 + \pi_s$.

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Alternative 2: Invest in stock X_s, bond B_s and buy a call option on non-traded asset Y_s at time t for price h

- American call payoff: $g(y) = (y - K)^+$.

Indifference pricing

- **Alternative 1** (Stock and bond only)
  Initial wealth: z
  Terminal wealth: Z_T
  Value function:

$$V_1(z,t) = \sup_{\pi} E(U(Z_T)|Z_t = z).$$

where $U(z) = -e^{-\gamma z}$.

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- **Alternative 2** (Stock, bond and call option)
  Initial wealth: $z - h$
  Wealth at exercise time τ: $Z_\tau + g(Y_\tau)$
  Value function:

$$V_2(z,y,t) = \sup_{\pi,\tau} E(V_1(Z_\tau + g(Y_\tau),\tau)|Z_\tau = z, Y_\tau = y)$$

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  $$V_2(z,y,t) = \sup_{\pi,\tau} E(V_1(Z_\tau + g(Y_\tau),\tau)|Z_\tau = z, Y_\tau = y)$$
- Definition: The indifference price h satisfies
  $$V_1(z,t) = V_2(z-h,y,t)$$

Variational inequality

$$
\begin{aligned}
\min(\mathcal{H}h, h-g) &= 0 \qquad \text{in } \mathbb{R}\times[0,T) \\
h(y,T) &= g(y) \qquad \text{in } \mathbb{R}
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{H}u &= D_t u - \frac{1}{2}a^2(y,t)D_y^2 u - \left(b(y,t) - \rho\frac{\mu}{\sigma}a(y,t)\right) D_y u \\
&\quad + \frac{1}{2}\gamma(1-\rho^2)a^2(y,t)(D_y u)^2.
\end{aligned}
$$

Free boundary at expiry

- Parameterization of free boundary: $\Gamma = (\beta(t), t)$
- Location at expiry: $\beta_0 = \lim_{t\to 0} \beta(t)$
- $A(y,t) = -\mathcal{H}g \overset{\text{call}}{=} b - \rho\frac{\mu}{\sigma}a - \frac{1}{2}\gamma(1-\rho^2)a^2$

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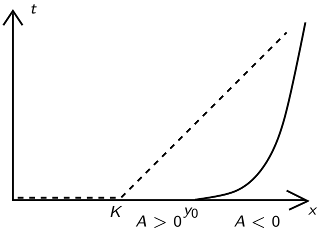
Lemma 1 If $A(y_0, 0) = 0$ and $A(y_0+\delta, 0)A(y_0-\delta, 0) < 0$ for all small δ then either no free boundary exists or

$$\beta_0 = y_0.$$



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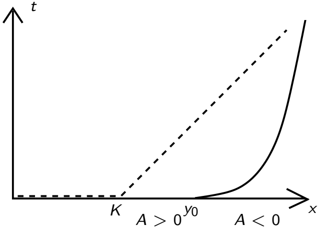
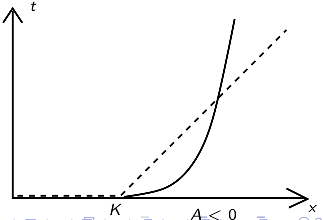


Lemma 2 If $A(y, 0) < -\varepsilon$ for some $\varepsilon > 0$ and all $y \in \{g > 0\}$ then

$$\beta_0 = K.$$



Free boundary regularity: $\beta_0 \neq K$

Theorem 1 There exists ξ_0 and $r > 0$ such that for $\xi_1 < \xi_0^{-2} < \xi_2$ and $t < r$

$$(\beta(t), t) \in \{(y, t) : \xi_1 (y - \beta_0)^2 \leq t \leq \xi_2 (y - \beta_0)^2\}.$$

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ξ_0 solve $u(\xi_0) - \xi_0 u'(\xi_0) = 0$ where

$$u(\xi) = \xi(6a^2(\beta_0, 0) + \xi^2) \int_{-\infty}^{\xi} \frac{\exp\left(\frac{-x^2}{4a^2(\beta_0,0)}\right)}{(6a^2(\beta_0, 0) + x^2)^2 x^2} dx.$$

Proof

- Rewrite equation

$$\hat{\mathcal{H}}u = A(y,t)\chi_{\{u>0\}}$$

where $\hat{\mathcal{H}} = \mathcal{H} + \gamma(1-\rho^2)a^2 g_y D_y$.

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- Scale by r^3

$$u_r(y,t) = \frac{u(ry+\beta_0, r^2 t)}{r^3}$$

and take the limit $r \to 0$

$$D_t u_0 - \frac{1}{2} a_0^2 D_y^2 u_0 = A_0 y \chi_{\{u_0>0\}}.$$

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- Self-similar solution in the variable $\xi = -y/\sqrt{t}$. $\tilde{u}(\xi) = u(y,t)$.

$$-\tilde{u}'' - \frac{1}{2a_0^2}\xi\tilde{u}' + \frac{3}{2a_0^2} = -A_0\xi \quad \text{in } \{\tilde{u} > 0\}$$

Free boundary regularity: $\beta_0 = K$

Theorem 2 There exists a modulus of continuity $\sigma(r)$ such that

$$(\beta(t), t) \in \{(y, t) : t < (y - K)^2 \sigma(y - K)\}.$$

Proof

- Scale by r

$$h_r(y,t) = \frac{h(ry+K, r^2 t)}{r}$$

and take limit $r \to 0$

$$\begin{aligned} \min(D_t h_0 - \frac{1}{2} a_0^2 D_y^2 h_0, h_0 - g_0) &= 0 \\ h_0(y,0) &= g_0(y) \end{aligned}$$

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- $\star$ Scale by r

$$h_r(y,t) = \frac{h(ry+K, r^2 t)}{r}$$

and take limit $r \to 0$

$$\begin{aligned} \min(D_t h_0 - \frac{1}{2} a_0^2 D_y^2 h_0, h_0 - g_0) &= 0 \\ h_0(y,0) &= g_0(y) \end{aligned}$$

- $\star$ $g_0 = y^+$ is a strict subsolution to the limit PDE.

$$\Downarrow$$

The limit problem does not have a free boundary.

$$\Downarrow$$

$$(\beta(t), t) \in \{t < (y-K)^2 \sigma(y-K)\}.$$