

# On the continuity of the time derivative of the solution to the parabolic obstacle problem with variable coefficients

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## Abstract

This paper is devoted to continuity results of the time derivative of the solution to the one-dimensional parabolic obstacle problem with variable coefficients. Under regularity assumptions on the obstacle and on the coefficients, we prove that the time derivative of the solution is continuous for almost every time. When the solution is nondecreasing in time this result holds for every time. We also give an energy criterion which characterizes the continuity of the time derivative of the solution at a point of the free boundary. Such a problem arises in the pricing of american options in generalized Black-Scholes models of finance. Our results apply in financial mathematics.

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## 1. Introduction

Let  $\alpha \in (0, 1)$  and consider a domain  $D$  of  $\mathbb{R}^2$ . We denote by  $\mathcal{H}^\alpha$  the Banach space of Hölder functions

$$\mathcal{H}^\alpha(D) := \left\{ f \in C^0 \cap L^\infty(D) : \|f\|_{\alpha;D} < \infty \right\}$$

where  $\|f\|_{\alpha;D} = \|\cdot\|_{L^\infty(D)} + [f]_{\alpha;D}$ ,

$$[f]_{\alpha;D} := \sup_{\substack{(x,t),(y,s) \in D \\ (x,t) \neq (y,s)}} \frac{|f(x,t) - f(y,s)|}{(|x-y|^2 + |t-s|)^{\alpha/2}}$$

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(see [12], Chap. 3, Sec. 2). For all  $q \in [1, \infty]$  we also define the Sobolev space

$$W_{x,t}^{2,1;q}(D) := \left\{ u \in L^q(D) : \left( \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial u}{\partial t} \right) \in (L^q(D))^3 \right\}.$$

To  $P_0 = (x_0, t_0) \in \mathbb{R}^2$  and  $R \in (0, \infty)$ , we associate the open parabolic cylinder

$$Q_R(P_0) := \{ (x, t) \in \mathbb{R}^2 : |x - x_0| < R \text{ and } |t - t_0| < R^2 \},$$

and the lower half parabolic cylinder

$$Q_R^-(P_0) := \{ (x, t) \in \mathbb{R}^2 : |x - x_0| < R \text{ and } 0 < t_0 - t < R^2 \}.$$

Such notations for parabolic problems are standard. See [20,12,18] for more details. On  $W^{2,1;q}(Q_R(P_0))$ , consider now the parabolic operator

$$Lu := a(x, t) \frac{\partial^2 u}{\partial x^2} + b(x, t) \frac{\partial u}{\partial x} + c(x, t) u - \frac{\partial u}{\partial t},$$

where  $a$ ,  $b$  and  $c$  are variable coefficients which depend on  $x$  and  $t$ .

This paper is devoted to regularity properties of the solutions to the one-dimensional parabolic obstacle problem

$$\begin{cases} Lu(x, t) = f(x, t) \mathbb{1}_{\{u>0\}}(x, t) \\ u(x, t) \geq 0 \end{cases} \quad (x, t) \in Q_R(P_0) \text{ a.e.} \quad (1.1)$$

The function  $\mathbb{1}_{\{u>0\}}$  denotes the characteristic function of the set  $\{u > 0\} := \{(x, t) \in Q_R(P_0) : u(x, t) > 0\}$ :

$$\mathbb{1}_{\{u>0\}}(x, t) = \begin{cases} 1 & \text{if } u(x, t) > 0, \\ 0 & \text{if } u(x, t) = 0. \end{cases}$$

Our main assumption is the following assumption on uniform parabolicity and non degeneracy and regularity of the coefficients and of the function  $f$ :

$$\begin{cases} a, b, c \text{ and } f \text{ belong to } \mathcal{H}^\alpha(Q_R(P_0)) \text{ for some } \alpha \in (0, 1), \\ \text{there exists a constant } \delta_0 > 0 \text{ such that for any } (x, t) \in Q_R(P_0), a(x, t) \geq \delta_0 \text{ and } f(x, t) \geq \delta_0. \end{cases} \quad (1.2)$$

By [14], under Assumption (1.2), (1.1) has a unique solution for suitable initial datum and boundary conditions. From standard regularity theory for parabolic equations, [20,12,18], it is known that any solution  $u$  belongs to  $W_{x,t}^{2,1;q}(Q_r(P_0))$  for any  $r < R$  and  $q < +\infty$ . As a consequence of Sobolev's embeddings,  $u$  is continuous. The set  $\{u = 0\}$  is then closed in  $Q_R(P_0)$ .

**Definition** *The sets  $\{u = 0\}$  and  $\Gamma := Q_R(P_0) \cap \partial\{u = 0\}$  are respectively called the coincidence set and the free boundary of the parabolic obstacle problem (1.1).*

*Notations.* We will use  $u_t$ ,  $u_x$  and  $u_{xx}$  respectively for  $\frac{\partial u}{\partial t}$ ,  $\frac{\partial u}{\partial x}$  and  $\frac{\partial^2 u}{\partial x^2}$ . By  $|A|$  we denote the volume of the set  $A \subset \mathbb{R}^2$  with respect to the Lebesgue measure, and by  $\mathcal{D}(\mathbb{R})$  the set of smooth functions with compact support. For any domain  $D \subset \mathbb{R}^2$ , we will write  $u \in W_{x,t}^{2,1;q}(D)$  if and only if  $u \in W_{x,t}^{2,1;q}(K)$  for all compact  $K \subset\subset D$ . The heat operator will be abbreviated to  $H$ ,  $Hu := u_{xx} - u_t$ . The *parabolic boundary* of  $Q_r^-(P_0)$  is the set  $\partial^p Q_r^-(P_0) := [x_0-r, x_0+r] \times \{t_0-r^2\} \cup \{x_0-r, x_0+r\} \times [t_0-r^2, t_0]$ . We define the parabolic distance  $\text{dist}_p$  between two points  $P = (x, t)$  and  $P' = (x', t')$  by  $\text{dist}_p(P, P') := \sqrt{(x - x')^2 + |t - t'|}$ .

By standard parabolic estimates  $u_t$  is continuous in a neighborhood of any point  $P$  such that  $u(P) > 0$ . If  $P$  is in the interior of the region  $\{u = 0\}$ ,  $u_t$  is obviously continuous. The key issue is therefore the regularity of  $u_t$  on the free boundary  $\Gamma$ . Our first result states that  $u$  is almost never discontinuous.

**Theorem 1.1 (Continuity of  $u_t$  for almost every  $t$ )** *Let  $u$  be a solution of (1.1) and assume (1.2). For almost any  $t_1 \in (t_0 - R, t_0 + R)$ , if  $P_1 = (x_1, t_1)$  is a point on the free boundary  $\Gamma$ , then*

$$\lim_{P \rightarrow P_1} \frac{\partial u}{\partial t}(P) = 0 .$$

As far as the authors know, this result is new, even in the case of constant coefficients. The continuity of  $u_t$  cannot be obtained everywhere in  $t$ , as shown by the following example. Let  $u(x, t) = \max\{0, -t\}$ . It satisfies  $u_{xx} - u_t = \mathbb{1}_{\{u > 0\}}$  and its time derivative is obviously discontinuous at  $t = 0$ . If we additionally assume that  $u_t \geq 0$ , we achieve a more precise result:

**Theorem 1.2 (Continuity of  $u_t$  for all  $t$  when  $u_t \geq 0$ )** *Under the assumptions of Theorem 1.1, if  $u_t$  is nonnegative, then  $u_t$  is continuous everywhere, and satisfies*

$$\frac{\partial u}{\partial t} = 0 \quad \text{on} \quad \Gamma .$$

The assumption that  $u_t$  is nonnegative can be established in some special cases (special initial conditions, boundary conditions, and time independent coefficients). See for example the results of Friedman [13], for further results on the one-dimensional parabolic obstacle problem with particular initial conditions.

When we are not assuming that  $u$  is nondecreasing in time, it is useful to have some criteria to determine the points where the time derivative of the solution is continuous. We begin with a *density criterion* based on the density  $\theta(P_1)$  of the coincidence set  $\{u = 0\}$  at the point  $P_1 \in Q_R(P_0)$ :

$$\theta(P_1) := \liminf_{r \rightarrow 0} \frac{|\{u = 0\} \cap Q_r(P_1)|}{|Q_r(P_1)|}$$

and on the lower density  $\theta^-(P_1)$  of  $\{u = 0\}$  at  $P_1$ :

$$\theta^-(P_1) := \liminf_{r \rightarrow 0} \frac{|\{u = 0\} \cap Q_r^-(P_1)|}{|Q_r^-(P_1)|} .$$

**Theorem 1.3 (Density criterion: continuity of  $u_t$ )** *Let  $u$  be a solution of (1.1), assume (1.2) and consider a point  $P_1 \in Q_R(P_0)$ . If either  $\theta(P_1) = 0$ , or  $\theta(P_1) \neq 0$  and  $\theta^-(P_1) \neq 0$ , then we have*

$$\lim_{P \rightarrow P_1} \lim_{P \in Q_R(P_0) \setminus \Gamma} \frac{\partial u}{\partial t}(P) = 0 .$$

*Otherwise, if  $\theta(P_1) \neq 0$  and  $\theta^-(P_1) = 0$ , then  $u_t$  is not continuous at  $P_1$ .*

The second criterion is an *energy criterion* based on a monotonicity formula. Consider a nonnegative cut-off function  $\psi \in \mathcal{D}(\mathbb{R})$  such that  $\psi \equiv 1$  on  $\left(-\frac{r}{2} \sqrt{\frac{f(P_1)}{a(P_1)}}, \frac{r}{2} \sqrt{\frac{f(P_1)}{a(P_1)}}\right)$  and  $\psi \equiv 0$  on  $\left(-\infty, -r \sqrt{\frac{f(P_1)}{a(P_1)}}\right] \cup \left[r \sqrt{\frac{f(P_1)}{a(P_1)}}, \infty\right)$ . Let  $Q_r(P_1) \subset Q_R(P_0) \subset \mathbb{R}^2$ . With  $P_1 = (x_1, t_1)$ , and  $a, f$  the functions involved respectively in the definition of the operator  $L$  and in Equation (1.1), define the function  $v_{P_1}$  for all  $(x, t) \in \mathbb{R} \times (-r^2 f(P_1), r^2 f(P_1))$  by

$$v_{P_1}(x, t) := u \left( x_1 + x \sqrt{\frac{a(P_1)}{f(P_1)}}, t_1 + \frac{t}{f(P_1)} \right) \cdot \psi(x) \text{ if } |x| \leq r \sqrt{\frac{f(P_1)}{a(P_1)}}, \quad v_{P_1} \equiv 0 \text{ otherwise} . \quad (1.3)$$

For all  $t \in (-r^2 f(P_1), 0)$ , let

$$\mathcal{E}(t; v) := \int_{\mathbb{R}} \left\{ \left[ \frac{1}{-t} \left( \left| \frac{\partial v}{\partial x} \right|^2 + 2v \right) - \frac{v^2}{t^2} \right] G \right\} (x, t) dx - \int_t^0 \frac{1}{s^2} \int_{\mathbb{R}} \{(Hv - 1)(\mathcal{L}v)G\} (x, s) dx ds ,$$

with  $Hv := v_{xx} - v_t$ ,  $\mathcal{L}v := -2v + x \cdot v_x + 2tv_t$  and  $G(x, t) := (2\pi(-t))^{-1/2} \exp(-x^2/(-4t))$ .

**Theorem 1.4 (Energy criterion: continuity of  $u_t$ )** *Under the assumptions of Theorem 1.1,*

- (i) *either  $\lim_{t \rightarrow 0, t < 0} \mathcal{E}(t; v_{P_1}) = \sqrt{2}$ ,*
- (ii) *or there exists some  $t \in (-r^2 f(P_1), 0)$  such that  $\mathcal{E}(t; v_{P_1}) < \sqrt{2}$ . In that case,  $\lim_{t \rightarrow 0, t < 0} \mathcal{E}(t; v_{P_1}) = \sqrt{2}/2$  and  $u_t$  is continuous in a neighborhood of  $P_1$ .*

The one-dimensional parabolic obstacle problem for differential operators with variable coefficients is a generalisation to the case of an operator with variable coefficients of Stefan's problem (case where the parabolic operator is  $Lu = u_{xx} - u_t$ ). Stefan's problem describes the interface of ice and water (see [17,25,14]). The problem with variable coefficients arises in the pricing of american options in mathematical finance (see [4,2,26,19,16,27,1,3,22,23]). The regularity of  $u_t$  is a natural question to apply the "smooth-fit principle" which amounts to require the  $C^1$  continuity of the solution at the free boundary. This principle is often assumed in numerical methods (see for instance [10]).

In [26] Van Moerbeke studied a special case where he proved that  $u_t$  is continuous except at one point and gave some asymptotics of the free boundary at this point. In [13], Friedman specifically studied the case of an american option and proved that  $u_t$  is continuous on some subsets of the free boundary. Using the maximum principle, he also proved for a special class of initial data that the free boundary is piecewise monotone. Then until recently the theory of the obstacle problem has essentially been studied in the stationary case (see [17,25,14] and references therein). Variational inequalities have been related to probabilistic methods in [2,16,19], and also to viscosity solutions methods [27,24]. Also see [1] for a recent paper revisiting variational inequalities and raising questions on the regularity of the solution and of the free boundary.

Recently in [6], Caffarelli, Petrosyan and Shahgholian considered the case with constant coefficients in any dimension and without any sign assumptions on the solution. They developed a nice theory of the regularity of the free boundary, based on Liouville type results and monotonicity formulas, like the one introduced by Weiss in [29]. As we shall see below, such tools are extremely useful for our purpose.

This paper is organized as follows. In Section 2 we obtain  $W_{x,t}^{2,1;\infty}$  a priori estimates on the solution and prove a non-degeneracy lemma. As a consequence the free boundary is a closed subset of zero measure. In Section 3 we introduce the notion of blow-up sequences which are a kind of zooming at a point of the free boundary. We will use them to study the regularity of the solution. These sequences converge, up to the extraction of sub-sequences, to the blow-up limit which is a solution in the whole space of the obstacle problem with constant coefficients. Using a monotonicity formula we prove in Section 3 that the blow-up limit is scale-invariant. This allows us to classify all possible blow-up limits in a Liouville theorem. The energy also gives a criterion to distinguish regular and singular points of the free boundary, see Section 4. In Section 5 we prove the uniqueness of the blow-up limit at each singular point. The last section is devoted to the completion of the proofs of all results stated in Section 1 and some additional results on the time derivative of the solution.

## 2. Regularity estimates and properties of the free boundary

### 2.1. A priori regularity estimates

Assume that (1.2) holds and consider a solution  $u$  of (1.1). By a bootstrap argument,  $u$  is bounded in  $W_{x,t}^{2,1;q}(Q_{R/2}(P_0))$  for all  $q \in (1, \infty)$ . In particular, by Sobolev imbeddings,  $u$  is continuous. Further regularity estimates require more sophisticated methods. To this end, let us consider a function  $u \in$

$W_{x,t;\text{loc}}^{2,1;1}(Q_R(P_0)) \cap C(Q_R(P_0))$  solution of

$$\begin{cases} Lu \leq f & \text{a.e. } Q_R(P_0), \\ Lu = f & \text{a.e. } \{u > 0\}, \\ u \geq 0 & \text{a.e. } Q_R(P_0). \end{cases} \quad (2.1)$$

**Theorem 2.1** (*A priori regularity estimates*) *Assume (1.2) and consider a solution  $u$  of (2.1). For all  $R' < R$ ,  $u$  is bounded in  $W_{x,t}^{2,1;\infty}(Q_{R'}(P_0))$ .*

The  $W_{x,t}^{2,1;\infty}$  regularity is essentially optimal. Consider indeed in the case  $a \equiv 1$ ,  $b \equiv 0$  and  $c \equiv 0$  the function

$$u(x, t) = \theta(-t)_+ + \frac{1}{2}(1 - \theta)x_+^2$$

where  $x_+ := \max\{0, x\}$ . For  $\theta = 0$  or  $1$ ,  $u$  is a solution in  $W_{x,t;\text{loc}}^{2,1;\infty}(\mathbb{R}^2)$  to  $u_{xx} - u_t = \mathbf{1}_{\{u > 0\}}$ , but it is neither in  $W_{x,t;\text{loc}}^{2,2;1}(\mathbb{R}^2)$  for  $\theta = 1$ , nor in  $W_{x,t;\text{loc}}^{3,1;1}(\mathbb{R}^2)$  for  $\theta = 0$ .

The proof of [6] can be adapted to our case. This last fact has been pointed to us by H. Shahgholian correcting a mistake in the previous version of the manuscript.

**Lemma 2.2** *Let  $u$  be a solution of (1.1), assume (1.2) and consider a point  $0 \in \Gamma$ . Then there exists a constant  $C$  such that*

$$\sup_{Q_r(0)} |u| \leq Cr^2$$

for any  $r \in [0, 1]$ .

*Proof.* This proof follows the first part of the proof of Lemma 4.2 in [6] which was adapted from [8]. We introduce

$$S_j(u) := \sup_{Q_{2^{-j}}^-(0)} |u| \quad \text{and} \quad N(u) := \{k \in \mathbb{N} : 2^2 S_{j+1}(u) \geq S_j(u)\}.$$

By a recurrence argument we easily see that it is sufficient to prove that there exists  $C_0 \geq 1$  such that

$$S_{j+1}(u) \leq C_0 M 2^{-2j} \quad \text{for } j \in N(u) \quad (2.2)$$

where  $M := \sup_{Q_R(P_0)} |u|$  to complete the result with  $C := 16 M C_0$ .

Assume by contradiction that (2.2) is false and that there exists  $k_j \in N(u)$  such that

$$S_{k_j+1}(u) \geq j 2^{-2k_j}. \quad (2.3)$$

We define

$$u_j(x, t) := \frac{1}{S_{k_j+1}(u)} u(2^{-k_j}x, 2^{-2k_j}t) \quad \text{in } Q_1^-.$$

By regularity of  $a, b, c$  and  $f$  and by (2.3), the functions  $u_j$  satisfy

$$\lim_{j \rightarrow \infty} \sup_{Q_1^-} |Lu_j| = 0 \quad \text{and} \quad \sup_{Q_1^-} u \leq 4.$$

By definition of  $S_j$  and  $u_j$

$$\sup_{Q_{1/2}^-} u_j = 1. \quad (2.4)$$

Furthermore  $u_j$  is non-negative and  $u_j(0, 0) = 0$ . By standard  $L^p$  parabolic estimates, up to the extraction of a sub-sequence,  $(u_j)_{j \in \mathbb{N}}$  converges to a function  $u^0$ . The function  $u^0$  satisfies  $a(0, 0) \frac{\partial^2 u^0}{\partial x^2} - \frac{\partial u^0}{\partial t} = 0$ , is bounded in  $Q_1^-$  and achieves its minimum at 0. By the strong maximum principle  $u^0$  is constant and equal

to zero in  $Q_1^-$  which contradicts  $\sup_{Q_{1/2}^-} u^0 = 1$  as a consequence of (2.4). We then deduce the bound on the whole cylinder  $Q_r(0)$ , considering a suitable supersolution in the upper cylinder  $Q_r(0) \setminus Q_r^-(0)$  (see Lemma 4.3 in [6]).

*Proof of Theorem 2.1.* This proof is the same that the proof of Theorem 4.1 in [6].  $\square$

## 2.2. Non-degeneracy lemma

The non-degeneracy lemma is an important tool which has first been introduced by Caffarelli in [7] for the elliptic obstacle problem. It can be interpreted as the fact that the free boundary can not appear or disappear suddenly, or is not “blurred”. It has been for instance proved for the parabolic problem with constant coefficient in [6]. Here we extend it to the case of variable coefficients.

**Lemma 2.3 (Non-degeneracy lemma)** *Under Assumption (1.2), consider a solution  $u$  of (2.1) in  $Q_R(P_0)$ . Let  $R' \in (0, R)$ ,  $P_1 \in \{\overline{u > 0}\}$  be such that  $Q_{\bar{r}}^-(P_1) \subset Q_{R'}(P_0)$  for some  $r > 0$  small enough. There exist two positive constants  $\bar{C}$  and  $\bar{r} > 0$  such that if  $Q_{\bar{r}}^-(P_1) \cap \{u = 0\} \neq \emptyset$ :*

$$r \leq \bar{r} \implies \sup_{Q_{\bar{r}}^-(P_1)} u \geq \bar{C} r^2.$$

The constants  $\bar{C}$  and  $\bar{r}$  only depend on  $R'$  and  $L$ .

*Proof.* Consider first  $P' = (x', t') \in \{u > 0\} \cap Q_r(P_1)$ . For some positive constant  $\bar{C}$  to be fixed later, we set for all  $(x, t) \in Q_r(P') \subset Q_{R'}(P_0)$

$$w(x, t) := u(x, t) - u(P') - \bar{C}((x - x')^2 + |t - t'|).$$

By Assumption (1.2),  $Lu = f \geq \delta_0$  in  $\{u > 0\}$ . For all  $(x, t) \in Q_r(P') \cap \{u > 0\}$ , we have

$$\begin{aligned} Lw(x, t) - c(x, t)w(x, t) &\geq Lu(x, t) - c(x, t)u(x, t) - \bar{C}(2a(x, t) + 1) - 2\bar{C}b(x, t) \cdot (x - x') \\ &\geq \delta_0 - \bar{C}|c(x, t)|(2r)^2 - \bar{C}(2a(x, t) + 1) - 2\bar{C}|b(x, t)|(2r) \end{aligned}$$

according to Lemma 2.1. With  $\bar{C} := \frac{\delta_0}{4} (2 \|a\|_{L^\infty(Q_R(P_0))} + 1)^{-1}$  and

$$\bar{r} := \frac{\delta_0}{8} \min \left\{ (4\bar{C} \|b\|_{L^\infty(Q_R(P_0))})^{-1}, \left(4\bar{C} \|c\|_{L^\infty(Q_R(P_0))}\right)^{-1/2} \right\},$$

we obtain

$$Lw(x, t) - c(x, t)w(x, t) \geq 0 \quad \text{in } Q_{\bar{r}}^-(P') \cap \{u > 0\}.$$

Notice that  $w(P') = 0$ . Applying the parabolic maximum principle in  $Q_{\bar{r}}^-(P') \cap \{u > 0\}$  for  $\rho \leq \bar{r}$  (cf. [20] Theorem 2.9 (p. 13), or [12] Theorem 1, Chap. 2, Sec. 1 (p. 34)) we get that the maximum of  $w$  is nonnegative and achieved in  $\{(x, t) \in \overline{Q_{\bar{r}}^-(P')} : u(x, t) > 0, t < t'\}$ . On  $\partial\{u = 0\} \cap \overline{Q_{\bar{r}}^-(P')}$ ,  $u = 0$  implies that  $w$  is negative then there exists  $P_2 = (x_2, t_2) \in \partial^p Q_{\bar{r}}^-(P') \cap \{u > 0\}$  such that

$$\sup_{Q_{\bar{r}}^-(P') \cap \{u > 0\}} w = w(P_2) = u(P_2) - u(P') - \bar{C}((x_2 - x')^2 + |t_2 - t'|) \geq 0.$$

This means that when there exists  $P_1 \in \Gamma$  such that  $P' \in Q_{\bar{r}}^-(P_1) \cap \{u > 0\}$ , then for  $\rho \leq \bar{r}$  we have

$$\sup_{Q_{\bar{r}}^-(P')} u \geq u(P_2) \geq u(P') + \bar{C}\rho^2 \geq \bar{C}\rho^2$$

and by continuity of  $u$ , the estimate remains true when  $P'$  tends to  $P_1 \in \Gamma$ .  $\square$

### 2.3. Properties of the free boundary

**Theorem 2.4** *Under Assumption (1.2), the free boundary  $\Gamma$  associated to a solution  $u$  of (2.1) is a closed set of zero Lebesgue measure.*

The proof is based on several results which are consequences of Lemma 2.1 and Lemma 2.3.

**Lemma 2.5 (Cube property of the free boundary)** *Under Assumption (1.2), consider a solution  $u$  of (2.1) in  $Q_R(P_0)$ . There exists a constant  $\lambda \in (0, \frac{1}{2})$  such that for any  $r > 0$  small enough, for any  $P_1 \in \Gamma \cap Q_{3R/4}(P_0)$  such that  $Q_r(P_1) \subset Q_{3R/4}(P_0)$ , there exists  $P_2 \in Q_{r/2}^-(P_1)$  such that  $Q_{\lambda r}(P_2) \subset \{u > 0\} \cap Q_r(P_1)$ .*

*Proof.* By Lemma 2.3, there exists  $P_2 = (x_2, t_2) \in Q_{r/2}^-(P_1)$  such that

$$u(P_2) \geq \frac{1}{4} \bar{C} r^2.$$

On the other hand, according to Lemma 2.1 applied to  $\rho = r$ , there exists a positive constant  $\tilde{C}$  such that for all  $P = (x, t) \in Q_{\lambda r}(P_2)$ ,

$$|u(P) - u(P_2)| \leq |u(x, t) - u(x_2, t)| + |u(x_2, t) - u(x_2, t_2)| \leq \frac{1}{2} \tilde{C} r |x - x_2| + \tilde{C} |t - t_2| \leq (\lambda \tilde{C} + \lambda^2 \tilde{C}) r^2. \quad (2.5)$$

Collecting these two estimates, we obtain

$$u(P) \geq \frac{1}{4} \bar{C} r^2 - (\lambda \tilde{C} + \lambda^2 \tilde{C}) r^2,$$

which is positive if  $\lambda$  is chosen small enough.  $\square$

Recall now the following result on measurable sets.

**Lemma 2.6 (Density in a point of a measurable set)** *Let  $A$  be a measurable subset in  $\mathbb{R}^2$ . If  $A$  has non-zero Lebesgue measure, then for almost every  $P_1 = (x_1, t_1) \in A$ , we have*

$$\limsup_{n \rightarrow \infty} \frac{|A \cap C_n(P_1)|}{|C_n(P_1)|} = 1,$$

where  $C_n(P_1) := [x_1 - \frac{1}{n}, x_1 + \frac{1}{n}] \times [t_1 - \frac{1}{n}, t_1 + \frac{1}{n}]$ .

See [11], Theorem 2.9.11 (p. 158), Remark 2.9.12 (p. 158), Theorem 2.8.18 (p. 152) and Remark 2.8.9 (p. 145).

*Proof of Theorem 2.4.* For the convenience of the reader, we recall here a proof that can be found in [6]. Let us suppose by contradiction that the measure of  $\Gamma$  is non-zero. By Lemma 2.6 there exists at least one point  $P_1$  such that

$$\limsup_{n \rightarrow \infty} \frac{|\Gamma \cap C_n(P_1)|}{|C_n(P_1)|} = 1.$$

Divide the euclidean cylinder  $C_n(P_1)$  into  $n$  parabolic cylinders  $Q_{i,n} := Q_{\frac{1}{n}}(x_1, t_i)$ ,  $t_i := t_1 - \frac{1}{n} + \frac{2i+1}{n^2}$ ,  $i \in \{0, \dots, n-1\}$ . If  $Q_{i,n} \cap \Gamma = \emptyset$ , we set  $E_{i,n} := Q_{i,n}$ . Otherwise, by Lemma 2.5 there exists  $E_{i,n}$  in  $Q_{i,n} \cap \{u > 0\}$  with  $|E_{i,n}| \geq \lambda^3 |Q_{i,n}|$ .

Let us set  $E_n := \cup_{i=0}^{n-1} E_{i,n}$ . We have

$$\limsup_{n \rightarrow \infty} \frac{|\Gamma \cap C_n(P_1)|}{|C_n(P_1)|} \leq 1 - \liminf_{n \rightarrow \infty} \frac{|E_n|}{|C_n(P_1)|} \leq 1 - \lambda^3 < 1$$

which contradicts Lemma 2.6.  $\square$

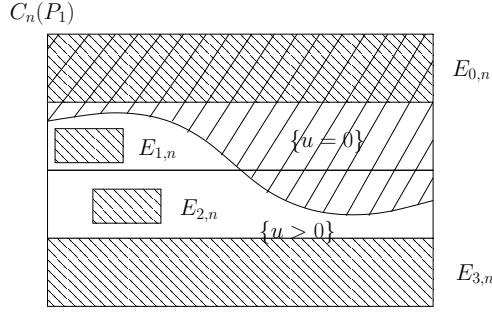


Figure 1. Construction of the sets  $E_{i,n}$ .

A straightforward consequence of Theorems 2.1 and 2.4 is the following result:

**Proposition 2.7** *Let  $D$  be a domain of  $\mathbb{R}^2 \times \mathbb{R}$ . If  $U^0 \in W_{x,t;\text{loc}}^{2,1;\infty}(D)$  is a function satisfying*

$$\begin{cases} U_{xx}^0 - U_t^0 \leq 1 & \text{a.e. } D \\ U_{xx}^0 - U_t^0 = 1 & \text{a.e. } \{U^0 > 0\} \\ U^0 \geq 0 & \text{a.e. } D \end{cases}$$

then

$$|\partial \{U^0 > 0\}| = 0 \quad \text{and} \quad U_{xx}^0 - U_t^0 = \mathbb{1}_{\{U^0 > 0\}}.$$

### 3. Properties of blow-up limits

#### 3.1. Reduction to the constant coefficient case

The reduction of a general operator  $L$  to the heat operator  $H$  is done by a classical transformation which goes as follows. Assume (1.2) and consider a solution of (1.1). Let  $P_1 = (x_1, t_1) \in \Gamma$  and take  $r > 0$  such that  $Q_r(P_1) \subset\subset Q_R(P_0)$ . For all  $P = (x, t) \in Q_r(P_1) \cap \{u > 0\}$ , Equation (1.1) can be rewritten as

$$a(P_1) \frac{\partial^2 u}{\partial x^2}(P) - \frac{\partial u}{\partial t}(P) = f(P_1) + (f(P) - f(P_1)) - (a(P) - a(P_1)) \frac{\partial^2 u}{\partial x^2}(P) - b(P) \frac{\partial u}{\partial x}(P) - c(P) u(P).$$

Consider the affine change of variables

$$(x, t) \mapsto \left( X := \sqrt{\frac{f(P_1)}{a(P_1)}} (x - x_1), \quad T := f(P_1) (t - t_1) \right) \quad (3.1)$$

and define

$$\begin{aligned} U(X, T) &:= u(x, t), \\ g(X, T) &:= \frac{1}{f(P_1)} \left( (f(P) - f(P_1)) - (a(P) - a(P_1)) \frac{\partial^2 u}{\partial x^2}(P) - b(P) \frac{\partial u}{\partial x}(P) - c(P) u(P) \right). \end{aligned}$$

In the  $(X, T)$  variables, the function  $U$  is a solution in  $W_{x,t}^{2,1;1}(Q)$  of the parabolic obstacle problem

$$\frac{\partial^2 U}{\partial X^2} - \frac{\partial U}{\partial T} = (1 + g) \mathbb{1}_{\{U > 0\}}, \quad U \geq 0 \quad \text{a.e. in } Q \quad (3.2)$$



such that  $\partial\{U = 0\} \ni 0$  where

$$Q := \left( -r\sqrt{\frac{f(P_1)}{a(P_1)}}, r\sqrt{\frac{f(P_1)}{a(P_1)}} \right) \times (-r^2 f(P_1), r^2 f(P_1)) .$$

By construction,  $g(0) = 1$ .

**Important remark** *To avoid further tedious notations and up to make a previous reduction of the problem, we will assume (except when we will have to move the point  $P_1$ ) from now on and in the whole paper that  $f(P_1) = a(P_1) = 1$  and  $r = 1$ . Then  $U$  satisfies*

$$\frac{\partial^2 U}{\partial X^2} - \frac{\partial U}{\partial T} = (1 + g) \mathbb{1}_{\{U > 0\}} , \quad U \geq 0 \quad \text{a.e. in } Q_1(0) \quad (3.3)$$

From Assumption (1.2) and Theorem 2.1, we deduce that there exist an  $\alpha \in (0, 1)$  and a positive constant  $C$  such that for  $r > 0$ , small enough,

$$\frac{1}{2} \leq |g(X, T)| \leq C(X^2 + |T|)^{\alpha/2} \quad \forall (X, T) \in Q_1(0) . \quad (3.4)$$

**Proposition 3.1** *Under Assumption (1.2), consider a solution  $u$  of (1.1). With the above notations,  $U \in W_{x,t}^{2,1;\infty}(Q_1(0))$  and there exist a positive constant  $\bar{C}$  such that for any  $P \in \overline{\{U > 0\}} \cap Q_1(0)$ ,*

$$Q_r(P) \subset Q_1(0) \quad \implies \quad \sup_{Q_r^-(0)} U \geq \bar{C} r^2 .$$

Moreover,  $\partial\{U = 0\}$  has zero Lebesgue measure.

*Proof.* This result is a straightforward consequence of Lemma 2.3 and Theorem 2.4 using the change of variables (3.1).  $\square$

### 3.2. Localization, localized energy

Let us first rephrase in terms of  $U$  the energy which has been introduced in Section 1. We need to localize the solution first.

To a nonnegative cut-off function  $\psi \in \mathcal{D}(\mathbb{R})$  such that  $\psi \equiv 1$  on  $(-1/2, 1/2)$  and  $\psi \equiv 0$  on  $(-\infty, -1] \cup [1, \infty)$ , we associate the function

$$v(x, t) = v_{P_1}(x, t) := U(x, t) \psi(x) , \quad (x, t) \in \mathbb{R} \times (-1, 0) .$$

To simplify the notations, we shall drop the index  $P_1$  whenever there is no ambiguity. The energy now takes the form:

$$\mathcal{E}(v; t) := \int_{\mathbb{R}} \left\{ \left[ \frac{1}{-t} \left( \left| \frac{\partial v}{\partial x} \right|^2 + 2v \right) - \frac{v^2}{t^2} \right] G \right\} (x, t) dx - \int_t^0 \frac{1}{s^2} \int_{\mathbb{R}} \{(Hv - 1) \mathcal{L}v G\} (x, s) dx ds , \quad (3.5)$$

with

$$Hv := v_{xx} - v_t ,$$

$$\mathcal{L}v := -2v + x \cdot v_x + 2t v_t ,$$

$$G(x, t) := \frac{e^{-\frac{|x|^2}{(-4t)}}}{\sqrt{2\pi(-t)}} .$$

The function  $G$  satisfies the backward heat equation:

$$G_{xx} + G_t = 0 \quad \text{in } \mathbb{R} \times (\infty, 0) .$$

The kernel of  $\mathcal{L}$  is spanned by the space of scale-invariant functions:

$$\mathcal{L}v \equiv 0 \iff v(x, t) = \lambda^{-2} v(\lambda x, \lambda^2 t) =: v_\lambda(x, t), \quad \forall (x, t) \in \mathbb{R} \times (-\infty, 0), \quad \forall \lambda > 0. \quad (3.6)$$

This is easily proved by writing  $v_\lambda(x, t) - v(x, t) = \int_1^\lambda \mu^{-3} (\mathcal{L}v)(\mu x, \mu^2 t) d\mu$ .

### 3.3. Notion of blow-up

In [7] Caffarelli introduces the notion of blow-up sequences in order to study the free boundary of the elliptic obstacle problem. Such a tool is convenient as long as only a *a priori*  $W_{x,t;loc}^{2,1;\infty}$  estimates of the solution is known. Here we adapt such a notion of blow-up sequences to the parabolic obstacle problem.

**Definition (Blow-up sequence)** *Let  $(\epsilon_n)_{n \in \mathbb{N}}$  be a sequence which converges to 0. The blow-up sequence  $(U^{\epsilon_n})_{n \in \mathbb{N}}$  associated to a function  $U : Q_1(0) \rightarrow \mathbb{R}$  around 0 is the sequence defined by*

$$U^{\epsilon_n}(x, t) := \epsilon_n^{-2} U(\epsilon_n x, \epsilon_n^2 t) \quad \forall (x, t) \in Q_{1/\epsilon_n}(0), \quad \forall n \in \mathbb{N}.$$

The parabolic scaling  $(x, t) \mapsto (\epsilon x, \epsilon^2 t)$  transforms the parabolic cylinder  $Q_\epsilon(0)$  into the parabolic cylinder  $Q_1(0)$  and preserves the heat operator  $H$ , in the sense that, for any  $\epsilon > 0$ ,

$$(HU^\epsilon)(x, t) = (HU)(\epsilon x, \epsilon^2 t) \quad \forall (x, t) \in Q_{1/\epsilon}(0).$$

**Proposition 3.2 (Blow-up limit)** *Assume (3.4) and consider a blow-up sequence  $(U^{\epsilon_n})_{n \in \mathbb{N}}$  associated to a solution  $U$  of (3.3). There exist a subsequence  $(\epsilon_{n_k})_{k \in \mathbb{N}}$  and a function  $U^0 \in W_{x,t;loc}^{2,1;\infty}(\mathbb{R}^2)$  such that*

- (i) *For any compact set  $K$  in  $\mathbb{R}^2$ ,  $\lim_{k \rightarrow \infty} \|U^{\epsilon_{n_k}} - U^0\|_{L^\infty(K)} = 0$ ,*
- (ii) *The limit  $U^0$  is nonnegative almost everywhere and it is a solution of*

$$\frac{\partial^2 U^0}{\partial x^2} - \frac{\partial U^0}{\partial t} = \mathbb{1}_{\{U^0 > 0\}},$$

- (iii) *0 belongs to the free boundary of the limit,  $\partial\{U^0 = 0\}$ .*

*Proof.* By Proposition 3.1 and Ascoli-Arzelà theorem (see for instance [5], Theorem IV.24 p. 72), up to the extraction of a subsequence that we still denote by  $(\epsilon_n)_{n \in \mathbb{N}}$ ,  $(U^{\epsilon_n})_{n \in \mathbb{N}}$  uniformly converges to a nonnegative function  $U^0 \in W_{x,t;loc}^{2,1;\infty}(\mathbb{R}^2)$  in any compact set  $K \subset \subset \mathbb{R}^2$ . Let  $P' \in \{U^0 > 0\}$ . There exists  $r > 0$  such that  $U^0 > U^0(P')/2$  in  $Q_r(P')$ . Because of the uniform convergence, there exists  $N \in \mathbb{N}$  such that for all  $n \in \mathbb{N}$ ,  $n > N$  implies

$$U^{\epsilon_n}(P) \geq \frac{1}{4} U^0(P) > 0 \quad \forall P \in Q_r(P').$$

In other words,  $Q_r(P') \subset \{U^{\epsilon_n} > 0\}$  for  $n > N$  and we can pass to the limit in the equation:

$$\frac{\partial^2 U^0}{\partial x^2} - \frac{\partial U^0}{\partial t} = 1 \quad \text{in } Q_r(P').$$

The equation  $U_{xx}^0 - U_t^0 = \mathbb{1}_{\{U^0 > 0\}}$  holds both in  $\{U^0 > 0\}$  and in the interior of  $\{U^0 = 0\}$ . Moreover  $U_{xx}^0 - U_t^0 \leq 1$  in  $\mathbb{R}^2$ . Then from Proposition 2.7, we deduce that  $\partial\{U^0 = 0\}$  has zero Lebesgue measure which proves Assertion (ii).

To prove that  $0 \in \partial\{U^0 = 0\}$  we first notice that  $U^0(0) = 0$  by uniform convergence. Because of Proposition 3.1 there exists a positive constant  $\bar{C}$  such that for all  $r > 0$  small enough,

$$\begin{aligned}\bar{C} r^2 &\leq \sup_{(x,t) \in Q_r^-(0)} U(x,t) = \sup_{(x,t) \in Q_r^-(0)} \epsilon_n^2 U^{\epsilon_n} \left( \frac{x}{\epsilon_n}, \frac{t}{\epsilon_n^2} \right), \\ \bar{C} \left( \frac{r}{\epsilon_n} \right)^2 &\leq \sup_{(x,t) \in Q_{r/\epsilon_n}^-(0)} U^{\epsilon_n}(x,t).\end{aligned}$$

Replacing  $\epsilon_n r$  by  $\tilde{r}$ , we obtain

$$\bar{C} \tilde{r}^2 \leq \sup_{Q_{\tilde{r}}^-(0)} U^{\epsilon_n} \rightarrow \sup_{Q_{\tilde{r}}^-(0)} U^0 \quad \text{as } n \rightarrow \infty,$$

which proves that  $0 \in \partial\{U^0 = 0\}$ . □

Lemma 2.3 gives a much more detailed result than the statement of Proposition 3.2, (iii).

**Proposition 3.3** *Under the assumptions of Proposition 3.2,*

$$\mathbb{1}_{\{U^{\epsilon_n} > 0\}} \rightarrow \mathbb{1}_{\{U^0 > 0\}} \quad \text{in } \mathbb{R}^2 \quad \text{a.e. as } n \rightarrow \infty,$$

where  $(U^{\epsilon_n})_{n \in \mathbb{N}}$  is a convergent blow-up sequence associated to  $U$ , with blow-up limit  $U^0$ .

*Proof.* From the proof of Proposition 3.2 if  $P \in \{U^0 > 0\}$  there exists  $N$  such that, if  $\mathbb{N} \ni n > N$ , then  $P \in \{U^{\epsilon_n} > 0\}$ . Assume now by contradiction that  $P \in \text{Int}\{U^0 = 0\}$  is such that  $P \in \{U^{\epsilon_n} > 0\}$  for all  $n \in \mathbb{N}$ . By Proposition 3.1,  $\sup_{Q_r^-(P)} U^0 \geq \bar{C} r^2$ , which means that  $P \in \partial\{U^0 = 0\}$ , and is a contradiction. To conclude we apply Proposition 2.7 to  $U^0$ . □

### 3.4. A monotonicity formula and application to blow-up limits

Some monotonicity formulas have been introduced by G. Weiss in [28] to study the elliptic obstacle problem and also by Giga and Kohn in [15], in a different context.

**Proposition 3.4 (Local monotonicity formula)** *Under Assumption (3.4), if  $U$  is a solution of (3.3), then the function  $t \mapsto \mathcal{E}(t; v)$  is a nonincreasing function, which is bounded from below and bounded in  $W^{1,\infty}(-1, 0)$ , and such that for almost every  $t \in (-1, 0)$*

$$\frac{d}{dt} \mathcal{E}(t; v) = -\frac{1}{2(-t)^3} \int_{\mathbb{R}} |\mathcal{L}v(x, t)|^2 G(x, t) dx.$$

Before to prove Proposition 3.4, let us remark that a simple change of variable gives

$$\mathcal{E}(\lambda^2 t; v) = \mathcal{E}(t; v^\lambda) \quad \forall t \in (-\lambda^{-2}, 0) \tag{3.7}$$

where  $v^\lambda(x, t) := \lambda^{-2} v(\lambda x, \lambda^2 t)$ . Using (3.6), we obtain a characterization of the functions which are invariant under the scaling  $v \mapsto v^\lambda$ .

**Corollary 3.5 (Scale invariance of  $\mathcal{E}$ )** *Let  $v \in W_{x,t}^{2,1;\infty}(\mathbb{R} \times \mathbb{R}_-)$ . Then*

$$\mathcal{E}v \equiv 0 \Leftrightarrow \mathcal{E}(t; v) = \mathcal{E}(t; v^\lambda) \quad \forall t < 0, \quad \forall \lambda > 0.$$

*Proof of Proposition 3.4.* We split it into two main steps.

*First Step.* Exactly as in [6], we can evaluate the time derivative of the first term in the expression of  $\mathcal{E}$ . Assume that  $v \in \mathcal{D}(\mathbb{R} \times [-1, 0])$ , let

$$\mathfrak{e}(t; v) := \int_{\mathbb{R}} \left\{ \frac{1}{-t} \left( \left| \frac{\partial v}{\partial x}(x, t) \right|^2 + 2v(x, t) \right) - \frac{1}{t^2} v^2(x, t) \right\} G(x, t) dx$$

and compute  $\frac{d}{d\lambda} \mathbf{e}(t; v_\lambda)$  at  $\lambda = 1$  using  $\frac{d}{d\lambda} v_\lambda = \mathcal{L}v$  at  $\lambda = 1$ , and  $\mathbf{e}(\lambda^2 t; v) = \mathbf{e}(t; v_\lambda)$  :

$$\frac{d\mathbf{e}}{dt}(t; v) = \frac{1}{2t} D_v \mathbf{e}(t; v) \cdot \mathcal{L}v(x, t),$$

where  $D_v \mathbf{e}$  is defined for all  $\phi$  in  $\mathcal{C}^\infty(\mathbb{R} \times (-1, 0))$  by

$$D_v \mathbf{e}(t; v) \cdot \phi := \int_{\mathbb{R}} \left\{ \frac{1}{-t} \left( 2 \frac{\partial v}{\partial x} \cdot \frac{\partial \phi}{\partial x} + 2 \phi \right) \right\} G dx - \int_{\mathbb{R}} \frac{2}{t^2} v \phi G dx.$$

To compute  $D_v \mathbf{e}(t; v) \cdot \mathcal{L}v$ , we integrate by parts.

$$D_v \mathbf{e}(t; v) \cdot \mathcal{L}v(x, t) = \int_{\mathbb{R}} \left\{ \frac{2}{-t} \left( 1 - Hv(x, t) \right) + \frac{1}{t^2} \mathcal{L}v(x, t) \right\} \mathcal{L}v(x, t) G(x, t) dx.$$

This proves

$$\frac{d}{dt} \mathbf{e}(t; v) = \int_{\mathbb{R}} \left\{ \frac{1}{2t^3} |\mathcal{L}v(x, t)|^2 + \frac{1}{t^2} \mathcal{L}v(x, t) \left( Hv(x, t) - 1 \right) \right\} G(x, t) dx.$$

By density, the above expression also holds for a.e. time for any  $v \in W_{x,t}^{2,1;\infty}(\mathbb{R} \times [-1, 0])$ , with compact support, and the function  $t \mapsto \mathbf{e}(t; v)$  is bounded from below and bounded in  $W_{\text{loc}}^{1,\infty}(-1, 0)$ .

*Second Step.* We prove that the function

$$s \mapsto \mathbf{r}(v; s) := \frac{1}{s^2} \int_{\mathbb{R}} (Hv(x, s) - 1) \mathcal{L}v(x, s) G(x, s) dx$$

is integrable. The integral  $\int_t^0 \frac{1}{s^2} \int_{\mathbb{R}} |(Hv(x, s) - 1) \mathcal{L}v(x, s) G(x, s)| dx ds$  can indeed be bounded by (I) + (II), with

$$(I) := \int_t^0 \frac{1}{s^2} \int_{\mathbb{R}} |(Hv(x, s) - (1 + g(x, s))) \mathcal{L}v(x, s) G(x, s)| dx ds,$$

$$(II) := \int_t^0 \frac{1}{s^2} \int_{\mathbb{R}} |g(x, s) \mathcal{L}v(x, s) G(x, s)| dx ds.$$

By definition of  $v(x, t) := U(x, t) \psi(x)$ ,  $(Hv - (1 + g)) \mathcal{L}v$  vanishes on  $(-1/2, 1/2)$  because  $U$  is a solution of (3.3), and on  $(-\infty, -1) \cup (1, +\infty)$  because of  $\psi$ . As a consequence of Theorem 2.1, there exists a constant  $C > 0$  such that  $|\mathcal{L}v(x, t)| \leq C(x^2 + |t|)$ . For  $t \in (-1, 0)$ , with  $c := C'(\|\psi_{xx}\|_{L^\infty} \|U\|_{L^\infty(Q_1(0))} + \|\psi_x\|_{L^\infty} \|U_x\|_{L^\infty(Q_1(0))} + \|\psi\|_{L^\infty} (\|U_{xx}\|_{L^\infty(Q_1(0))} + \|U_t\|_{L^\infty(Q_1(0))} + \|1+g\|_{L^\infty(Q_1(0))})$ , we get

$$(I) \leq c \int_0^{|t|} \frac{ds}{s^2} \int_{1/2}^1 \frac{e^{-\rho^2/4s}}{\sqrt{2\pi s}} d\rho \leq \frac{c}{2\sqrt{2\pi}} \int_0^{|t|} \frac{e^{-1/16s}}{s^{5/2}} ds.$$

With the change of variable

$$(s, x) \mapsto \left( \beta := \sqrt{\frac{x^2 - s}{-s}}, \theta := \sqrt{x^2 - s} \right)$$

we get

$$(II) \leq \text{Const} \int_1^{+\infty} \frac{\beta^3}{\sqrt{\beta^2 - 1}} e^{-\beta^2/4} \left( \int_0^{\theta_{\max}(\beta, |t|)} \frac{\sigma(\theta)}{\theta} d\theta \right) d\beta$$

where  $\theta_{\max}(\beta, |t|) := \min(\beta \sqrt{|t|}, \sqrt{1 + |t|})$  and  $\sigma(\theta) := \sup_{\sqrt{x^2 + |t|} \leq \theta} g(x, t)$  is the modulus of continuity of  $g$  at the origin. By (3.4),  $\sigma$  is Dini-integrable, i.e.  $\theta \mapsto \theta^{-1} \sigma(\theta)$  is integrable, which ends the proof.  $\square$

**Remark 3.6** An inspection of the proof shows that Proposition 3.4 holds under the following weaker conditions:  $U \in W_{x,t}^{2,1;\infty}(Q_1(0))$  is a solution of (3.3) and  $\sigma$ , defined as above, is Dini-integrable.

**Lemma 3.7** Under Assumption (1.2), consider a solution  $u$  of (1.1). Then for any  $t_0 < 0$ ,  $\Gamma \ni P \mapsto \mathcal{E}(v_P, t_0)$  is continuous.

*Proof.* This is a straightforward consequence of the dominated convergence theorem of Lebesgue and the *a priori* bounds on the solution.  $\square$

Proposition 3.4 applies to blow-up limits.

**Proposition 3.8 (Scale invariance of the blow-up limit for  $t < 0$ )** Under Assumption (3.4), consider a solution  $U$  of (3.3), and  $U^0$  a blow-up limit corresponding to a blow-up sequence associated to  $U$ . Then  $U^0$  is scale-invariant for  $t < 0$ :

$$U^0(\lambda x, \lambda^2 t) = \lambda^2 U^0(x, t) \quad \forall (x, t) \in \mathbb{R} \times (-\infty, 0), \quad \forall \lambda > 0.$$

*Proof.* Consider as above  $v(x, t) := U(x, t)\psi(x)$ . Let  $(v^{\epsilon_n})_{n \in \mathbb{N}}$  be a blow-up sequence associated to  $v$ , and  $v^0$  a blow-up limit. By (3.7) we have

$$\mathcal{E}(\epsilon_n^2 t; v) = \mathcal{E}(t; v^{\epsilon_n}) \quad \forall t \in (-\epsilon_n^{-2}, 0). \quad (3.8)$$

Since  $\mathcal{E}$  is monotone nonincreasing and bounded from below by Proposition 3.4, we may pass to the limit in (3.8) and obtain

$$\lim_{n \rightarrow \infty} \mathcal{E}(\epsilon_n^2 t; v) = \mathcal{E}(t; v^0) \quad \forall t < 0. \quad (3.9)$$

Note that because of the monotonicity of  $\mathcal{E}$ ,  $\lim_{n \rightarrow \infty} \mathcal{E}(\epsilon_n^2 t; v)$  does not depend on the subsequence. As a consequence,

$$0 = \frac{d}{dt} \mathcal{E}(t; v^0) = \frac{1}{2t^3} \int_{\mathbb{R}} |\mathcal{L}v^0(x, t)|^2 G(x, t) dx \quad \forall t < 0$$

and  $v^0$  is scale invariant by (3.6). Since  $U^{\epsilon_n}(x, t) = v^{\epsilon_n}(x, t)$  for any  $x, t$  such that  $|\epsilon_n x| \leq 1/2$ ,  $-1 < \epsilon_n^2 t < 0$ , we have:  $U^0 \equiv v^0$ , which ends the proof.  $\square$

### 3.5. Classification of the blow-up limits

According to Proposition 3.2, blow-up limits are solutions in  $\mathbb{R}^2$  of the parabolic obstacle problem with constant coefficients:

$$\begin{cases} H v^0(x, t) = \mathbb{1}_{\{v^0 > 0\}}(x, t) & (x, t) \in \mathbb{R}^2 \text{ a.e.} \\ v^0(x, t) \geq 0 & (x, t) \in \mathbb{R}^2 \text{ a.e.} \\ 0 \in \partial\{v^0 > 0\} \end{cases} \quad (3.10)$$

which are scale-invariant in  $\mathbb{R} \times (-\infty, 0)$  by Proposition 3.8. For all  $(x, t) \in \mathbb{R}^2$ , define the functions:

$$v_+^0(x, t) := \frac{1}{2} (\max\{0, x\})^2 ,$$

$$v_-^0(x, t) := \frac{1}{2} (\max\{0, -x\})^2 ,$$

$$v_m^0(x, t) := \begin{cases} m t + \frac{1+m}{2} x^2 & \text{if } t < 0 , \\ \max \left\{ 0, t V \left( \frac{|x|}{\sqrt{t}} \right) \right\} & \text{if } t \geq 0 , \end{cases}$$

where  $m \in [-1, 0]$  and  $V(\xi) = -1 + C_1(a) (\xi^2 + 2) + C_2(a) \left( 2\xi e^{-\xi^2/4} + (\xi^2 + 2) \int_0^\xi e^{-s^2/4} ds \right)$ . The constants  $C_1(a)$  and  $C_2(a)$  are given by

$$C_1(a) = -\frac{1}{4} \left( 2 + e^{a^2/4} \int_0^a e^{-s^2/4} ds \right) , \quad C_2(a) = \frac{a}{4} e^{a^2/4} .$$

where the parameter  $a \in [0, +\infty]$  is uniquely determined in terms of  $m$  by the equation

$$1 + m = 2(C_1(a) + \sqrt{\pi} C_2(a)) . \quad (3.11)$$

The limiting cases correspond to

$$\begin{aligned} m = -1 , \quad a = 0 , \quad v_{-1}^0(x, t) &= \max\{0, -t\} , \\ m = 0 , \quad a = +\infty , \quad v_0^0(x, t) &= \frac{1}{2} x^2 . \end{aligned}$$

We have the following classification result.

**Theorem 3.9 (A Liouville type result)** *Consider a solution  $v^0 \in W_{x,t,\text{loc}}^{2,1;\infty}(\mathbb{R}^2)$  of (3.10) with  $v_{xx}^0$  and  $v_t^0$  bounded. If  $v^0$  is such that*

$$v^0(\lambda x, \lambda^2 t) = \lambda^2 v^0(x, t) \quad \forall (x, t) \in \mathbb{R} \times (-\infty, 0) , \quad \forall \lambda \in (0, +\infty) ,$$

then  $v^0 = v_+^0$ ,  $v^0 = v_-^0$  or  $v^0 = v_m^0$  for some  $m \in [-1, 0]$ .

*Proof.* We first classify the solutions in  $\mathbb{R} \times (-\infty, 0)$ . Then we extend the solutions to  $\mathbb{R}^2$ .

*First Step: Classification in  $\mathbb{R} \times (-\infty, 0)$ .* This result is given in [6]. We reproduce it for completeness.

(1) Assume first that the interior of  $\{v^0 = 0\} \cap \{t < 0\}$  is non-empty. Because of the self-similarity property, the function  $V(\xi) := v^0(\xi, -1)$  is such that  $v(x, t) = |t| V(x/\sqrt{-t})$  and it is solution in  $\{u > 0\}$  of

$$V''(\xi) + V(\xi) - \frac{\xi}{2} V'(\xi) = 1 .$$

A direct computation gives  $V(\xi) = 1 + C_1 (\xi^2 - 2) + C_2 \left( -2\xi e^{\xi^2/4} + (\xi^2 - 2) \int_0^\xi e^{s^2/4} ds \right)$ . Because of the regularity of  $v^0$ , we have to choose  $a \in \mathbb{R}$  such that  $V(a) = V'(a) = 0$ . The functions  $\xi \mapsto \xi^2 - 2 =: V_1(\xi)$  and  $\xi \mapsto -2\xi e^{\xi^2/4} + (\xi^2 - 2) \int_0^\xi e^{s^2/4} ds =: V_2(\xi)$  are respectively even and odd, so there is no restriction to take  $a \geq 0$ , up to a sign change of  $C_1$  and  $C_2$ . This amounts to

$$C_1 = \frac{1}{2} - \frac{a}{4} e^{-a^2/4} \int_0^a e^{s^2/4} ds \quad \text{and} \quad C_2 = \frac{a}{4} e^{-a^2/4} .$$

Note that  $V''(a) = 1$  and  $V'''(\xi) = 2C_2 e^{\xi^2/4}$ . If  $a \neq 0$ , this clearly contradicts the nonnegativity of  $V$  and we have therefore  $a = 0$ ,  $C_1 = 1/2$ :  $V(\xi) = \xi^2/2$  in  $\{v^0 > 0\}$ , or, equivalently,  $v^0 = v_\pm^0$  since

$$\text{either } V(\xi) = \frac{1}{2} (\max\{0, \xi\})^2 \quad \text{or} \quad V(\xi) = \frac{1}{2} (\max\{0, -\xi\})^2 .$$

(2) Assume now that  $\{v^0 = 0\} \cap \{t < 0\}$  is of empty interior: by Theorem 2.4,  $\Gamma$  has zero Lebesgue measure and for almost all  $(x, t) \in \mathbb{R} \times (-\infty, 0)$ ,  $Hv^0(x, t) = 1$ . As a consequence,  $Hv_t^0 = 0$  in  $\mathbb{R} \times (-\infty, 0)$ . Since  $v_t^0$  is bounded,  $m := v_t^0$  has to be a constant by Liouville's principle (see for instance [30], Chapter XIV, Theorem 1.2). Integrating with respect to  $t$ , we get:  $v^0(x, t) = mt + v^0(x, 0)$  with  $v_{xx}^0(x, 0) = 1 + m$ . Taking into account the conditions  $v^0 \geq 0$  and  $v^0(0) = 0$ , an integration with respect to  $x$  gives  $v^0(x, 0) = (1 + m)x^2/2$ . Therefore  $v^0(x, t) = v_m^0(x, t) := mt + (1 + m)x^2/2$  in  $\mathbb{R} \times (-\infty, 0)$ . Since  $v^0$  is nonnegative, this implies that  $m \in [-1, 0]$ .

*Second Step: Classification in  $\mathbb{R}^2$ .* The solution of (3.10) is uniquely extended to the domain corresponding to  $t > 0$ , once it is known for  $t < 0$ .

(1) If  $v^0 = v_{\pm}^0$  in  $\mathbb{R} \times (-\infty, 0)$  a.e., by unique continuation  $v^0 = v_{\pm}^0$  in  $\mathbb{R}^2$ .

(2) If  $v^0 = v_m^0$  for some  $m \in [-1, 0]$ , in  $\mathbb{R} \times (-\infty, 0)$  a.e., as in the first step of the proof, we may use the scale invariance. In the interior of  $\{v^0 > 0\} \cap \{t > 0\}$ , the function  $V(\xi) := v^0(\xi, 1)$  is such that  $v^0(x, t) = tV(x/\sqrt{t})$  is solution of

$$V''(\xi) - V(\xi) + \frac{\xi}{2} V'(\xi) = 1 .$$

A direct computation gives  $V(\xi) = -1 + C_1(\xi^2 + 2) + C_2 \left( 2\xi e^{-\xi^2/4} + (\xi^2 + 2) \int_0^\xi e^{-s^2/4} ds \right)$ . The free boundary condition  $V(a) = V'(a) = 0$  allows to parametrize  $C_1$  and  $C_2$  in terms of  $a$ :  $C_1(a) = -\frac{1}{4} \left( 2 + e^{a^2/4} \int_0^a e^{-s^2/4} ds \right)$  and  $C_2(a) = \frac{a}{4} e^{a^2/4}$ . Taking the limit  $t \rightarrow 0$ ,  $t > 0$ , we get

$$v^0(x, 0) = (C_1 + \sqrt{\pi} C_2) x^2$$

that we have to identify with  $\lim_{t \rightarrow 0, t < 0} v_m^0(x, t) = \frac{1}{2}(1 + m)x^2$ . The point  $\xi = a$  corresponds to  $t = x^2/a^2$  it remains to characterize the solution in  $(-\infty, -a)$ . As  $V_1(\xi) := \xi^2 + 2$  is even and  $V_2(\xi) := 2\xi e^{-\xi^2/4} + (\xi^2 + 2) \int_0^\xi e^{-s^2/4} ds$  is odd we can keep the same  $C_1$  and  $C_2$  by replacing  $V_2$  by  $-V_2$ . This provides (3.11) and completes the proof of Theorem 3.9.  $\square$

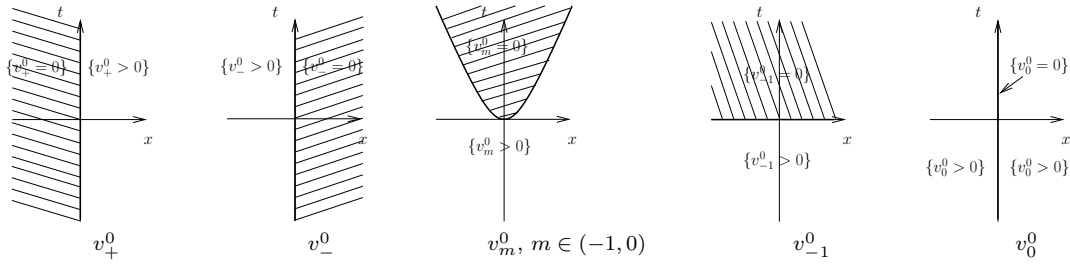


Figure 2. Solutions of Theorem 3.9. The equation of the free boundary associated to  $v_m^0$  for  $m \in (-1, 0)$  is  $t(x) = a^{-2}x^2$ , where  $a$  and  $m$  are related by (3.11).

## 4. Regular and singular points of the free boundary

### 4.1. An energy characterisation

As in Section 3.2, to a nonnegative cut-off function  $\psi \in \mathcal{D}(\mathbb{R})$  such that  $\psi \equiv 1$  on  $(-1/2, 1/2)$  and  $\psi \equiv 0$  on  $(-\infty, -1] \cup [1, \infty)$ , we associate the function  $v(x, t) := U(x, t) \psi(x)$ ,  $(x, t) \in \mathbb{R} \times (-1, 0)$  where  $U$  is given in terms of a solution of (1.1) as in Section 3.1 for some  $P_1 \in \Gamma$ , and solves (3.3) (also see Equation (1.3)). The localized energy is defined by (3.5). As in Section 3.2, we omit the index  $P_1$  whenever there is no ambiguity. Otherwise, we write  $v_{P_1} = v$ . We refer to Section 3.5 for the definition of  $v_{\pm}^0$  and  $v_m^0$ .

**Proposition 4.1 (Energy characterisation of the points of  $\Gamma$ )** *Let  $u$  be a solution of (1.1) and consider  $P_1 \in \Gamma$  such that  $Q_r(P_1) \subset Q_R(P_0)$  for some  $r > 0$ . With the above notations and under Assumption (1.2), if  $v^0$  is a blow-up limit associated to  $v$ , then*

$$\Lambda(v) := \lim_{\substack{\tau \rightarrow 0 \\ \tau < 0}} \mathcal{E}(\tau; v) \in \{\sqrt{2}, \sqrt{2}/2\} = \mathcal{E}(t; v^0) \quad \forall t < 0.$$

If  $\Lambda(v) = \sqrt{2}/2$ , then  $v^0 = v_{\pm}^0$ . If  $\Lambda(v) = \sqrt{2}$ , then  $v^0 = v_m^0$  for some  $m \in [-1, 0]$ .

*Proof.* The uniqueness of the limit of the energy is a consequence of the monotone decay of  $\mathcal{E}$ , according to Proposition 3.4, and of (3.9).

Since a blow-up limit is scale invariant by Proposition 3.8, by (3.6) and Proposition 3.4,  $\mathcal{E}(t; v^0)$  does not depend on  $t < 0$ . By Theorem 3.9, the only possible values of  $\Lambda(v)$  are  $\mathcal{E}(t; v_{\pm}^0)$  and  $\mathcal{E}(t; v_m^0)$ ,  $m \in [-1, 1]$ . Using  $\mathcal{L}v^0 = 0$  and integrating by parts with respect to  $x$ , we get

$$\begin{aligned} \mathcal{E}(t; v^0) &= \int_{\mathbb{R}} \left\{ \frac{1}{-t} \left( \left| \frac{\partial v^0}{\partial x} \right|^2 + 2v^0 \right) - \frac{1}{t^2} (v^0)^2 \right\} G(x, t) dx \\ &= \int_{\mathbb{R}} \left\{ \frac{1}{-t} \left( -\frac{\partial^2 v^0}{\partial x^2} - \frac{x}{2t} \frac{\partial v^0}{\partial x} + 2 \right) v^0 - \frac{1}{t^2} (v^0)^2 \right\} G(x, t) dx \\ &= \int_{\mathbb{R}} \left\{ \frac{1}{-t} (-Hv^0 + 2) + \frac{1}{2t^2} \mathcal{L}v^0 \right\} v^0 G(x, t) dx \end{aligned}$$

Using again  $\mathcal{L}v^0 = 0$  and Equation (3.3), we get  $\mathcal{E}(t; v^0) = \int_{\mathbb{R}} \frac{1}{-t} (-\mathbb{1}_{\{v^0 > 0\}} + 2) v^0 G(x, t) dx$ . Taking into account that  $\mathcal{E}(t; v^0) = \mathcal{E}(1; v^0)$ , this amounts to

$$\mathcal{E}(t; v^0) = \frac{1}{\sqrt{2}\pi} \int_{\mathbb{R}} (-\mathbb{1}_{\{v^0 > 0\}} + 2) v^0(x, -1) e^{-x^2/4} dx.$$

We easily conclude that

$$\mathcal{E}(t; v_+^0) = \int_0^{\infty} \frac{x^2}{2} \frac{e^{-x^2/4}}{\sqrt{2}\pi} dx = \frac{\sqrt{2}}{2}, \quad \mathcal{E}(t; v_m^0) = \int_{\mathbb{R}} \left( -m + \frac{1+m}{2} x^2 \right) \frac{e^{-x^2/4}}{\sqrt{2}\pi} dx = \int_{\mathbb{R}} \frac{e^{-x^2/4}}{\sqrt{2}\pi} dx = \sqrt{2}.$$

□

Proposition 4.1 allows to divide the free boundary in two sets, depending on the value of  $\Lambda(v)$ . Recall that according to the notations of Section 3.2, the function  $v$  depends on  $P_1 \in \Gamma$ . When there is no ambiguity on the blow-up point, we will denote the blow-up limit by  $v_{m(P_1)}^0$ , consistently with the notations of Section 3.5, and by  $v_{P_1}^0$  when the point of blow-up  $P_1$  is not fixed.

To emphasize the dependence of  $v$  on the point  $P_1 \in \Gamma$ , we will write explicitly the index and note  $v_{P_1}$  in the rest of this section.



**Definition (Regular and singular points)** *Under the assumptions of Proposition 4.1, a point  $P_1 \in \Gamma$  is said to be regular (respectively singular) if  $\Lambda(v_{P_1}) = \sqrt{2}/2$  (respectively if  $\Lambda(v_{P_1}) = \sqrt{2}$ ). We will denote by  $\mathcal{R}$  the set of regular points, and by  $\mathcal{S}$  the set of singular points.*

4.2. *First topological properties of the regular and singular sets*

**Lemma 4.2 (Topological properties of  $\mathcal{R}$  and  $\mathcal{S}$ )** *Under Assumption (1.2),  $\mathcal{S}$  is a closed set, and  $\mathcal{R} = \Gamma \setminus \mathcal{S}$  is open in  $\Gamma$ .*

*Proof.* Let  $P_1, P_2 \in \Gamma$  and take  $t_0 < t < 0$ . We may write

$$\mathcal{E}(t; v_{P_2}) - \mathcal{E}(t; v_{P_1}) = \mathcal{E}(t_0; v_{P_2}) - \mathcal{E}(t_0; v_{P_1}) + \mathcal{E}(t; v_{P_2}) - \mathcal{E}(t_0; v_{P_2}) + \mathcal{E}(t_0; v_{P_1}) - \mathcal{E}(t; v_{P_1}).$$

Since the function  $t \mapsto \mathcal{E}(t; v_{P_1})$  is monotone nonincreasing,  $\mathcal{E}(t; v_{P_2}) - \mathcal{E}(t_0; v_{P_2}) \leq 0$ . Passing to the limit  $t \rightarrow 0$ , we get

$$\Lambda(v_{P_2}) - \Lambda(v_{P_1}) \leq \mathcal{E}(t_0; v_{P_2}) - \mathcal{E}(t_0; v_{P_1}) + \mathcal{E}(t_0; v_{P_1}) - \Lambda(v_{P_1}).$$

We fix  $P_1$  and will move  $P_2$  close to  $P_1$ . For  $|t_0|$  small enough,  $\mathcal{E}(t_0; v_{P_1}) - \Lambda(v_{P_1})$  can be chosen arbitrarily small. Now, from Lemma 3.7 for a fixed  $t_0$ ,  $P_2 \mapsto \mathcal{E}(t_0; v_{P_2})$  is continuous, so that  $\mathcal{E}(t_0; v_{P_2}) - \mathcal{E}(t_0; v_{P_1})$  can also be chosen arbitrarily small for  $P_2$  close enough to  $P_1$ . Then  $\limsup_{P_2 \rightarrow P_1} \Lambda(v_{P_2}) \leq \Lambda(v_{P_1})$ , i.e. the function  $\Gamma \ni P \mapsto \Lambda(v_P)$  is upper semi-continuous. If  $\Lambda(v_{P_1}) = \sqrt{2}/2$ , then  $\Lambda(v_{P_2}) = \sqrt{2}/2$  for  $P_2$  in a neighborhood of  $P_1$ . This proves that  $\mathcal{R}$  is an open set in  $\Gamma$ .  $\square$

## 5. Study of the singular points of the free boundary

### 5.1. A monotonicity formula for singular points

We adapt a monotonicity formula for the elliptic obstacle problem [21] to the parabolic case. As in the second step of the proof of Proposition 3.4, let

$$\mathbf{r}(v; s) := \frac{1}{s^2} \int_{\mathbb{R}} (Hv(x, s) - 1) \mathcal{L}v(x, s) G(x, s) dx.$$

With the notations of Sections 3.2 and 3.4 and 3.5 consider  $v = v_{P_1}$  given by  $v_{P_1}(x, t) := U(x, t) \psi(x)$ , for some fixed point  $P_1 \in \mathcal{S}$  and  $v_m^0$  one of the blow-up limit of  $v_{P_1}$ . We define the functional

$$\Phi_m(t; v) := \frac{1}{t^2} \int_{\mathbb{R}} |v - v_m^0|^2 G dx - \int_t^0 \frac{2}{s^2} \int_{\mathbb{R}} (Hv - 1) (v - v_m^0) G dx ds + \int_t^0 \frac{2}{s} \int_s^0 \mathbf{r}(\theta; v) d\theta ds.$$

**Proposition 5.1 (Local monotonicity formula for singular points)** *Under Assumption (3.4), let  $U$  be a solution of (3.3). With the above notations the function  $t \mapsto \Phi_m(t; v)$  is nonincreasing, bounded in  $W^{1,1}(-1, 0)$ .*

*Proof.* By density, it is sufficient to prove the result for a smooth function  $v$  as in the proof of Proposition 3.4. Let  $w := v - v_m^0$ . Using the change of variable  $x = \sqrt{-t}y$ , since  $\frac{1}{t^2} \int_{\mathbb{R}} w^2(x, t) G(x, t) dx = \int_{\mathbb{R}} \frac{1}{t^2} w^2(\sqrt{-t}y, t) G(y, 1) dx$  and  $\frac{d}{dt} w(\sqrt{-t}y, t) = -\frac{y}{2\sqrt{-t}} \frac{\partial w}{\partial x}(\sqrt{-t}y, t) + \frac{\partial w}{\partial t}(\sqrt{-t}y, t)$ , we get

$$\frac{d}{dt} \left[ \frac{1}{t^2} \int_{\mathbb{R}} w^2(x, t) G(x, t) dx \right] = \frac{1}{t^3} \int_{\mathbb{R}} \mathcal{L}v(x, t) w(x, t) G(x, t) dx.$$

Let  $\mathbf{e}(t; v) := -\int_{\mathbb{R}} \left\{ \frac{1}{t} (|v_x|^2 + 2v) + \frac{1}{t^2} v^2 \right\} G(x, t) dx$  be as in the first step of the proof of Proposition 3.4.

$$\mathbf{e}(t; v) - \mathbf{e}(t; v_m^0) = -\int_{\mathbb{R}} \left[ \frac{1}{t} \left( \frac{\partial}{\partial x} (v + v_m^0) \frac{\partial w}{\partial x} + 2w \right) + \frac{1}{t^2} (v + v_m^0) w \right] G dx .$$

Integrating by parts with respect to  $x$  and using  $Hv_m^0 = 1$  and  $\mathcal{L}v_m^0 = 0$  for every  $t < 0$ , we get

$$\begin{aligned} \mathbf{e}(t; v) - \mathbf{e}(t; v_m^0) &= \int_{\mathbb{R}} \left[ \frac{1}{t} \left( \frac{\partial^2}{\partial x^2} (v + v_m^0) + \frac{x}{2t} \frac{\partial}{\partial x} (v + v_m^0) - 2 \right) w - \frac{1}{t^2} (v + v_m^0) w \right] G dx \\ &= \int_{\mathbb{R}} \left[ \frac{1}{t} (Hv - 1) + \frac{1}{2t^2} \mathcal{L}v \right] w G dx . \end{aligned}$$

Thus

$$\begin{aligned} \frac{d}{dt} \left[ \frac{1}{t^2} \int_{\mathbb{R}} w^2(x, t) G(x, t) dx \right] &= \frac{2}{t} [\mathbf{e}(t; v) - \mathbf{e}(t; v_m^0)] - \frac{2}{t^2} \int_{\mathbb{R}} (Hv - 1) w G dx , \\ \frac{d}{dt} \Phi_m(t; v) &= \frac{2}{t} [\mathbf{e}(t; v) - \mathbf{e}(t; v_m^0)] - \frac{2}{t} \int_t^0 \mathbf{r}(s; v) ds . \end{aligned}$$

Recall that  $\mathcal{E}(t; v) = \mathbf{e}(t; v) - \int_t^0 \mathbf{r}(v; s) ds$  by definition of  $\mathcal{E}$ ,  $\mathbf{e}$  and  $\mathbf{r}$ , and for any  $t < 0$ ,  $\mathbf{e}(t; v_m^0) = \sqrt{2} = \lim_{\tau \rightarrow 0} \mathcal{E}(\tau; v)$  according to Proposition 4.1. Thus

$$\frac{d}{dt} \Phi_m(t; v) = \frac{2}{t} \left[ \mathcal{E}(t; v) - \lim_{\tau \rightarrow 0} \mathcal{E}(\tau; v) \right] \quad (5.1)$$

is nonpositive by Proposition 3.4. It remains to prove that

$$(I) = \frac{2}{t} \int_t^0 \mathbf{r}(s; v) ds \quad \text{and} \quad (II) = \frac{-2}{t^2} \int_{\mathbb{R}} (Hv - 1) w G dx ,$$

are integrable. (I) can be evaluated as in the second step of the proof of Proposition 3.4, using the integrability of

$$t \mapsto \frac{1}{|t|} \int_0^{|t|} \frac{e^{-1/16s}}{s^{5/2}} ds \quad \text{and} \quad t \mapsto \frac{1}{|t|} \int_1^{+\infty} \frac{\beta^3}{\sqrt{\beta^2-1}} e^{-\beta^2/4} \left( \int_0^{\min(\beta\sqrt{|t|}, \sqrt{1+|t|})} \sigma(\theta) \frac{d\theta}{\theta} \right) d\beta ,$$

where  $\sigma(\theta) := \sup_{\sqrt{x^2+|t|} \leq \theta} g(x, t) \leq \text{Const} \cdot \theta^\alpha$  by Assumption (3.4).

As for (II), in  $\{v > 0\}$ ,  $Hv - 1 = g$ , and  $v \equiv 0$  in  $(\mathbb{R} \setminus (-1, 1)) \times (-1, 0)$ , so we have:

$$-\frac{1}{2} \times (II) = \frac{1}{t^2} \int_{|x| < 1/2} g \mathbb{1}_{\{v > 0\}} w G dx + \frac{1}{t^2} \int_{1/2 < |x| < 1} (Hv - 1) w G dx + \frac{1}{t^2} \int_{|x| > 1} v_m^0 G dx .$$

The last term is integrable: a trivial change of variable shows that the exponential decay is the dominant factor. The second term is integrable because of the gaussian weight, as in the second step of the proof of Proposition 3.4: the function  $t \mapsto |t|^{-5/2} \int_{1/2}^1 e^{-s^2/(-4t)} ds$  is indeed integrable. The first term  $|t|^{-2} \int_{|x| < r/2} g \mathbb{1}_{\{v > 0\}} w G dx$  is also integrable: using the change of variables  $(s, x) \mapsto (\beta, \theta)$  as in the second step of the proof of Proposition 3.4 again, we can conclude as above.  $\square$

**Remark 5.2** *As for the local monotonicity formula for  $\mathcal{E}$  studied in Proposition 3.4, an inspection of the proof shows that a sufficient condition for the proof of Proposition 5.1 is that the map  $\alpha \mapsto \frac{1}{\alpha} \int_0^\alpha \frac{\sigma(\theta)}{\theta} d\theta$  is integrable, i.e.  $\sigma$  is twice Dini-integrable.*

As a consequence, we can state the following result.

**Corollary 5.3** *Under Assumption (1.2), consider a solution  $u$  of (1.1). Let us fix  $P_1 \in \mathcal{S}$ . Note  $v_{m(P_1)}^0$  a blow-up limit in  $P_1$ . Then for any  $r \in (0, R)$  there exists  $t_r < 0$  and a continuous function  $\mathfrak{s} : (t_r, 0] \times Q_r(P_0) \cap \mathcal{S} \rightarrow \mathbb{R}$  with  $\mathfrak{s}(0, P_1) = 0$  such that for any  $P_2 \in Q_r(P_0) \cap \mathcal{S}$  and  $t \in (t_r, 0)$  we have*

$$\Phi_{m(P_1)}(t; v_{P_2}) \leq \Phi_{m(P_1)}(t; v_{P_1}) + \mathfrak{s}(t, P_2) .$$

Similarly there exists a continuous function  $\tilde{\mathfrak{s}}$  satisfying  $\tilde{\mathfrak{s}}(0, P_1) = 0$ , such that

$$\int_{\mathbb{R}} \frac{1}{t^2} \left| v_{P_2}(x, t) - v_{m(P_1)}^0(x, t) \right|^2 G(x, t) dx \leq \int_{\mathbb{R}} \frac{1}{t^2} \left| v_{P_1}(x, t) - v_{m(P_1)}^0(x, t) \right|^2 G(x, t) dx + \tilde{\mathfrak{s}}(t, P_2)$$

*Proof.* The point  $P_1$  is fixed and we write for  $t_0 < t < 0$ ,

$$\Phi(t; v_{P_2}) - \Phi(t; v_{P_1}) = \Phi(t; v_{P_2}) - \Phi(t_0; v_{P_2}) + \Phi(t_0; v_{P_2}) - \Phi(t_0; v_{P_1}) + \Phi(t_0; v_{P_1}) - \Phi(t; v_{P_1}) .$$

By the monotonicity formula, the first term satisfies  $\Phi(t; v_{P_2}) - \Phi(t_0; v_{P_2}) \leq 0$ . There exists a modulus of continuity  $\omega_{t_0}(d)$ , continuous in  $(t_0, d)$  such that  $\omega_{t_0}(0) = 0$  and

$$|\Phi(t_0; v_{P_2}) - \Phi(t_0; v_{P_1})| \leq \omega_{t_0}(|P_2 - P_1|) .$$

Finally there exists a monotone modulus of continuity  $\omega$  such that

$$|\Phi(t; v_{P_1}) - \Phi(0; v_{P_1})| \leq \omega(|t|) .$$

Therefore we get

$$\Phi(t; v_{P_2}) - \Phi(t; v_{P_1}) \leq \mathfrak{s}(t, P_2)$$

with

$$\mathfrak{s}(t; P_2) = \inf_{t_0, t_r < t_0 < t} (\omega_{t_0}(|P_2 - P_1|) + 2\omega(|t_0|))$$

We now prove the second inequality. A careful investigation of the proof of Proposition 5.1 shows that the estimates on (I) and (II) are uniform with respect to the point  $P_2 \in \mathcal{S}$ . So there exists  $t \mapsto \tilde{c}_1(t)$  which tends to zero when  $t$  tends to zero such that uniformly in  $P_2 \in \mathcal{S}$ , we have

$$\left| \Phi_{m(P_1)}(t; v_{P_2}) - \frac{1}{t^2} \int_{\mathbb{R}} \left| v_{P_2} - v_{m(P_1)}^0 \right|^2 G dx \right| \leq \tilde{c}_1(t)$$

This implies the result with  $\tilde{\mathfrak{s}}(t, P_2) = \mathfrak{s}(t, P_2) + 2\tilde{c}_1(t)$ .  $\square$

## 5.2. Scale invariance and blow-up limits

A simple change of variable gives

$$\Phi_m(\lambda^2 t; v) = \Phi_m(t; v^\lambda) \quad \forall t \in (-\lambda^{-2}, 0), \quad \forall \lambda > 0, \quad (5.2)$$

where  $v^\lambda(x, t) := \lambda^{-2} v(\lambda x, \lambda^2 t)$ . If we replace  $\mathcal{E}$  by  $\Phi_m$ , we have a result which is similar to Corollary 3.5 and Proposition 4.1.

**Proposition 5.4 (Scale invariance of  $\Phi_m$  and consequences)** *Under Assumption (1.2), consider a solution  $u$  of (1.1). For some  $P_1 \in \mathcal{S}$  define  $v$  as in Section 3.2 and take  $m \in [-1, 0]$ . Consider a blow-up limit  $v^0$  associated to  $v$ . Then*

$$\lim_{\substack{\tau \rightarrow 0 \\ \tau < 0}} \Phi_m(\tau; v) = \Phi_m(t; v^0) \quad \forall t < 0 .$$

with

$$\Phi_m(t; v^0) = \frac{1}{t^2} \int_{\mathbb{R}} |v^0 - v_m^0|^2 G dx \quad \forall t < 0 .$$

In the particular case where we choose  $v_m^0 = v^0$ , we get  $\lim_{\tau \rightarrow 0, \tau < 0} \Phi_m(\tau; v) = \Phi_m(t; v_m^0) = 0$  for all  $t < 0$ .

### 5.3. Uniqueness of the blow-up limit at singular points

**Proposition 5.5** *Under Assumption (3.4) consider a solution  $U$  of (3.3) and  $v = v_{P_1}$  given by  $v_{P_1}(x, t) := U(x, t)\psi(x)$ , for some fixed point  $P_1 \in \mathcal{S}$ . There exists a unique  $m \in [-1, 0]$  such that for any sequence  $(\epsilon_n)_{n \in \mathbb{N}}$  converging to 0, the whole blow-up sequence  $(U^{\epsilon_n})_{n \in \mathbb{N}}$  locally uniformly converges to  $v_m^0$ .*

*Proof.* Let  $(v^{\epsilon_{n,1}})_{n \in \mathbb{N}}$  and  $(v^{\epsilon_{n,2}})_{n \in \mathbb{N}}$  be two blow-up sequences associated to  $v$ , with blow-up limits  $v_{(1)}^0$  and  $v_{(2)}^0$ . Assume that  $v_{(1)}^0 = v_m^0$ . By (5.2),

$$\Phi_m(\epsilon_{n,1}^2 t; v) = \Phi_m(t; v^{\epsilon_{n,1}}) \rightarrow \Phi_m(t; v_{(1)}^0) = 0 \quad \text{as } n \rightarrow \infty.$$

With no restriction, we may assume that  $\epsilon_{n,2} \leq \epsilon_{n,1}$ , so that by Proposition 5.1,

$$\Phi_m(\epsilon_{n,1}^2 t; v) \geq \Phi_m(\epsilon_{n,2}^2 t; v) = \Phi_m(t; v^{\epsilon_{n,2}}).$$

Passing to the limit  $n \rightarrow \infty$ , we get

$$0 \geq \Phi_m(t; v_{(2)}^0) = \frac{1}{t^2} \int_{\mathbb{R}} \left| v_{(2)}^0 - v_m^0 \right|^2 G(x, t) dx \geq 0$$

since  $\mathcal{L}v_{(2)}^0 = 0$ ,  $\mathbf{r}(v_{(2)}^0; t) = 0$  and  $Hv_{(2)}^0 \equiv 1$  for  $t < 0$ , since  $\Lambda(v_{(2)}^0) = \sqrt{2}$  by Proposition 4.1. This proves that  $v_{(2)}^0 = v_m^0 = v_{(1)}^0$ .

For any  $(x, t) \in Q_{1/(2\epsilon_{n,i})}(0)$ ,  $i = 1, 2$ ,  $U^{\epsilon_{n,i}}$  coincides with  $v^{\epsilon_{n,i}}$ . This proves the uniqueness of the blow-up limit of  $U$ .  $\square$

To any  $P_1 \in \Gamma$ , we can therefore associate a unique  $m(P_1) := m \in [-1, 0]$  such that the blow-up limit of a solution at this point is  $v_m^0$ . For any  $m \in [-1, 0]$ , we set

$$\mathcal{S}_m = \{P_1 \in \Gamma : m(P_1) = m\}.$$

### 5.4. Continuity properties of the singular set

**Lemma 5.6 (Continuity of the blow-up limit)** *The function  $P_1 \mapsto m(P_1)$  is continuous on  $\mathcal{S}$ .*

*Proof.* Let  $P_1 \in \mathcal{S}$ . From Corollary 5.3 and the scale invariance of the monotonicity formula, we have with  $v_{P_2}^{|t|}(y, \tau) = \frac{1}{t^2} v_{P_2}(|t|y, t^2\tau)$ :

$$\Phi_{m(P_1)}(-1; v_{P_2}^{|t|}) \leq \Phi_{m(P_1)}(-1; v_{P_1}^{|t|}) + \mathbf{s}(t, P_2).$$

At the limit  $t = 0$ , we get

$$\Phi_{m(P_1)}(-1; v_{m(P_2)}^0) \leq \Phi_{m(P_1)}(-1; v_{m(P_1)}^0) + \mathbf{s}(0, P_2)$$

*i.e.*

$$\int_{\mathbb{R}} \left| v_{m(P_2)}^0 - v_{m(P_1)}^0 \right|^2 G(x, -1) dx \leq \mathbf{s}(0, P_2).$$

The continuity of  $\mathbf{s}$  joint to the fact that  $\mathbf{s}(0, P_1) = 0$  implies that

$$\lim_{P_2 \rightarrow P_1} m(P_2) = m(P_1)$$

$\square$

**Lemma 5.7 (A uniform continuity result)** *For any  $r \in (0, R)$ , there exists  $t_r < 0$  such that for any  $t \in (t_r, 0)$ , if  $v_P$  is given in terms of  $U$  as in Section 3.2, where  $U$  is a solution of (3.3), and if (3.4) is satisfied, then*

$$\lim_{\epsilon \rightarrow 0} \sup_{P \in \mathcal{S} \cap \overline{Q_r(P_0)}} \frac{1}{t^2} \int_{\mathbb{R}} \left| v_P^\epsilon(x, t) - v_{m(P)}^0(x, t) \right|^2 G(x, t) dx = 0 .$$

*Proof.* Consider a monotone decreasing sequence  $(\epsilon_n)_{n \in \mathbb{N}}$  with  $\lim_{n \rightarrow \infty} \epsilon_n = 0$  and a sequence  $(P_n)_{n \in \mathbb{N}}$  of points in  $\mathcal{S} \cap \overline{Q_r(P_0)}$ , and assume by contradiction that

$$\lim_{n \rightarrow \infty} \frac{1}{t^2} \int_{\mathbb{R}} \left| v_{P_n}^{\epsilon_n}(x, t) - v_{m(P_n)}^0(x, t) \right|^2 G(x, t) dx =: l > 0 .$$

We also assume that  $P_n \rightarrow P_\infty \in \mathcal{S} \cap \overline{Q_r(P_0)}$ . We first remark that by the scale invariance we have

$$\frac{1}{t^2} \int_{\mathbb{R}} \left| v_{P_n}^{\epsilon_n}(x, t) - v_{m(P_n)}^0(x, t) \right|^2 G(x, t) dx = \frac{1}{(\epsilon_n t)^2} \int_{\mathbb{R}} \left| v(x, \epsilon_n^2 t) - v_{m(P_n)}^0(x, \epsilon_n^2 t) \right|^2 G(x, \epsilon_n^2 t) dx .$$

Next we estimate this expression by  $2((\text{I})_n + (\text{II})_n)$  where

$$(\text{I})_n = \frac{1}{(\epsilon_n t)^2} \int_{\mathbb{R}} \left| v_{P_n}(x, \epsilon_n^2 t) - v_{m(P_\infty)}^0(x, \epsilon_n^2 t) \right|^2 G(x, \epsilon_n^2 t) dx$$

$$(\text{II})_n = \frac{1}{(\epsilon_n t)^2} \int_{\mathbb{R}} \left| v_{m(P_\infty)}^0(x, \epsilon_n^2 t) - v_{m(P_n)}^0(x, \epsilon_n^2 t) \right|^2 G(x, \epsilon_n^2 t) dx .$$

We also introduce the quantity

$$(\text{III})_n = \frac{1}{(\epsilon_n t)^2} \int_{\mathbb{R}} \left| v_{P_\infty}(x, \epsilon_n^2 t) - v_{m(P_\infty)}^0(x, \epsilon_n^2 t) \right|^2 G(x, \epsilon_n^2 t) dx .$$

From Corollary 5.3, we get

$$(\text{I})_n \leq (\text{III})_n + \tilde{\mathfrak{s}}(\epsilon_n^2 t, P_n)$$

with the particular choice  $P_1 = P_\infty$ . Moreover, still by scaling invariance, we have

$$(\text{III})_n = \int_{\mathbb{R}} \left| v_{P_\infty}^{\epsilon_n \sqrt{|t|}}(x, -1) - v_{m(P_\infty)}^0(x, -1) \right|^2 G(x, -1) dx \longrightarrow 0 \quad \text{as } \epsilon_n \rightarrow 0 .$$

This implies that

$$(\text{I})_n \longrightarrow 0 \quad \text{as } n \rightarrow +\infty .$$

Finally we remark that

$$(\text{II})_n = \int_{\mathbb{R}} \left| v_{m(P_\infty)}^0(x, -1) - v_{m(P_n)}^0(x, -1) \right|^2 G(x, -1) dx \leq C |m(P_\infty) - m(P_n)|^2 \longrightarrow 0 \quad \text{as } P_n \rightarrow P_\infty$$

This gives the contradiction with  $l > 0$ . □

### 5.5. Time projection of the singular set

**Proposition 5.8** *The set  $I := \{t \in [-R^2, R^2] : \exists x \in [-R, R], (x, t) \in \mathcal{S} \setminus \mathcal{S}_0\}$  has zero Lebesgue measure.*

To prove Proposition 5.8, we need several preliminary results.

**Lemma 5.9** *For any  $m_0 \in (-1, 0)$ , the set  $\mathcal{S}_{[-1, m_0]} := \bigcup_{m \in [-1, m_0]} \mathcal{S}_m$  is locally a graph as a function of  $x$ .*

*Proof.* Consider two sequences  $(P_n)_{n \in \mathbb{N}}$  and  $(P'_n)_{n \in \mathbb{N}}$  of points in  $\mathcal{S}_{[-1, m_0]}$  converging to some point  $P_\infty \in \Gamma$ . Since  $\mathcal{S}$  is closed,  $P_\infty \in \mathcal{S}$ , and by Lemma 5.6,  $m(P_\infty) \in [-1, m_0]$ . Assume by contradiction that  $P_n = (x_n, t_n)$  and  $P'_n = (x'_n, t'_n)$ ,  $t'_n > t_n$ . Consistently with the previous notations, we consider the function  $v = v_{P_n}$ , which is associated to the change of coordinates (3.1) where now the point  $P_1 = P_n$  is moving. In the new coordinates the image of  $P_n$  is the origin and the image of  $P'_n$  is a point  $\bar{P}'_n = (0, \varepsilon_n^2)$  with  $\varepsilon_n^2 = f(P_n)(t'_n - t_n)$ . We then consider the sequence of functions

$$v_{P_n}^{\varepsilon_n}(x, t) := \varepsilon_n^{-2} v(\varepsilon_n x, \varepsilon_n^2 t).$$

But at time  $t = -1$ , we have

$$\int_{\mathbb{R}} \left| v_{P_n}^{\varepsilon_n} - v_{m(P_\infty)}^0 \right|^2 G dx \leq 2 \int_{\mathbb{R}} \left| v_{P_n}^{\varepsilon_n} - v_{m(P_n)}^0 \right|^2 G dx + 2 \int_{\mathbb{R}} \left| v_{m(P_n)}^0 - v_{m(P_\infty)}^0 \right|^2 G dx.$$

From lemma 5.7, the sequence  $(v_{P_n}^{\varepsilon_n})_{n \in \mathbb{N}}$  is uniformly close to  $v_{m(P_n)}^0$  and  $m(P_n) \rightarrow m(P_\infty) \in [-1, m_0]$ . Therefore on the one hand  $v_{P_n}^{\varepsilon_n}$  converges to  $v_{m(P_\infty)}^0$ . On the other hand, let us remark that by construction the point  $\bar{P}' = (0, 1)$  belongs to  $\partial \{v_{P_n}^{\varepsilon_n} > 0\}$  and from the non-degeneracy Proposition 3.1 in the rescaled variables, we have for any  $r \in (0, 1)$ :

$$\sup_{Q_r^-(\bar{P}')} v_{P_n}^{\varepsilon_n} \geq \bar{C} r^2$$

At the limit we get for any  $r \in (0, 1)$ :

$$\sup_{Q_r^-(\bar{P}')} v_{m(P_\infty)}^0 \geq \bar{C} r^2$$

This is in contradiction (see Section 3.5) with the fact that  $\bar{P}'$  is in the interior of the coincidence set of  $v_{m(P_\infty)}^0$  when  $m(P_\infty) \in [-1, m_0]$  with  $m_0 < 0$ .  $\square$

Although we will not use it later, we can state the following additional result.

**Corollary 5.10** *For any  $m_0 \in (-1, 0)$ ,  $x_0 \in [-R, R]$ ,  $t_0 \in [-R^2, R^2]$ , the sets  $\{(x, t) \in \mathcal{S}_{[-1, m_0]} : x = x_0\}$  and  $\{(x, t) \in \mathcal{S}_0 : t = t_0\}$  are locally finite. Moreover  $\mathcal{S}_0$  is locally contained in a graph, as a function of  $t$ .*

*Proof.* Taking into account Lemma 5.9, we only have to prove that locally  $\mathcal{S}_0$  is contained in a graph. Let us do it as in Lemma 5.9, by contradiction. Consider two sequences  $(P_n)_{n \in \mathbb{N}}$ , and  $(P'_n)_{n \in \mathbb{N}} \in \mathcal{S}_0^{\mathbb{N}}$  such that  $\lim_{n \rightarrow \infty} P_n = \lim_{n \rightarrow \infty} P'_n = P_\infty$ ,  $P_n = (x_n, t_n)$ ,  $P'_n = (x'_n, t_n)$ ,  $x'_n - x_n > 0$ . By Lemma 5.6,  $P_\infty \in \mathcal{S}_0$ . Consider the sequence  $(v_{P_n}^{\varepsilon_n})_{n \in \mathbb{N}}$  defined by  $v_{P_n}^{\varepsilon_n}(x, t) := \varepsilon_n^{-2} v(\varepsilon_n x, \varepsilon_n^2 t)$  for  $v = v_{P_n}$  and  $\varepsilon_n = \sqrt{\frac{f(P_n)}{a(P_n)}}(x'_n - x_n)$ . The remainder of the proof is the same as above. We end up by noticing that the point  $\bar{P}' = (1, 0)$  needs to satisfy  $v_0^0(\bar{P}') = 0$ , while the limit of  $v_{P_n}^{\varepsilon_n}$  is  $v_0^0(x, t) = x^2/2$  when  $m = 0$ .  $\square$

By Lemma 5.9, locally  $\mathcal{S}_{[-1, m_0]}$  can be described as a graph:  $x \mapsto (x, h(x))$ . To the function  $h : \mathbb{R} \rightarrow \mathbb{R}$ , for any  $\delta > 0$ , we associate the quantity:

$$q_h(x, \delta) := \sup_{x'; |x-x'| \leq \delta} \frac{|h(x') - h(x)|}{|x' - x|}.$$

**Lemma 5.11** *Let  $m_0 \in (-1, 0)$ . With the above notations,  $\lim_{\delta \rightarrow 0} q_h(x, \delta) = 0$ , uniformly in  $x$ .*

*Proof.* If the Lemma is false, we can find two sequences of points  $P_n = (x_n, h(x_n))$  and  $P'_n = (x'_n, h(x'_n))$  such that  $(P_n)_{n \in \mathbb{N}}$  and  $(P'_n)_{n \in \mathbb{N}}$  converge to  $P_\infty = (x, h(x))$ , and such that

$$l_n := \frac{|h(x'_n) - h(x_n)|}{|x'_n - x_n|} \rightarrow l \neq 0.$$

Let us consider  $v = v_{P_n}$  and the corresponding change of coordinates which transforms  $P_n$  in the origin and  $P'_n$  in a point  $\bar{P}'_n = (\bar{x}'_n, \bar{t}'_n)$  with  $\bar{x}'_n = \sqrt{\frac{f(P_n)}{a(P_n)}}(x'_n - x_n)$ ,  $\bar{t}'_n = f(P_n)(h(x'_n) - h(x_n))$ . We define  $\varepsilon_n > 0$  such that

$$\bar{P}'_n \in \partial Q_{\varepsilon_n}(0)$$

and consider the blow-up sequence of functions

$$v_{P_n}^{\varepsilon_n} := \varepsilon_n^{-2} v(\varepsilon_n x, \varepsilon_n^2 t)$$

and define

$$\tilde{P}'_n = (\tilde{x}'_n, \tilde{t}'_n) = \left( \frac{\bar{x}'_n}{\varepsilon_n}, \frac{\bar{t}'_n}{\varepsilon_n^2} \right) \in \partial Q_1(0).$$

Up to the extraction of a subsequence,  $(\tilde{P}'_n)_{n \in \mathbb{N}}$  converges to some  $\tilde{P}' = (\tilde{x}', \tilde{t}') \in \partial Q_1(0)$ . By construction,  $\tilde{P}'_n$  belongs to  $\partial\{v_{P_n}^{\varepsilon_n} = 0\}$ , hence by the non-degeneracy Proposition 3.1, we have

$$\sup_{Q_r(\tilde{P}'_n)} v_{P_n}^{\varepsilon_n} \geq \bar{C}r^2, \quad v_{P_n}^{\varepsilon_n}(\tilde{P}'_n) = 0.$$

Taking the limit as  $n$  goes to infinity, we identify  $v_{m(P_\infty)}^0$  as the limit of  $v_{P_n}^{\varepsilon_n}$  as in the proof of Lemma 5.9, and get

$$\sup_{Q_r(\tilde{P}')} v_{m(P_\infty)}^0 \geq \bar{C}r^2, \quad v_{m(P_\infty)}^0(\tilde{P}') = 0.$$

This implies that  $\tilde{P}' \in \partial\{v_{m(P_\infty)}^0 > 0\} = \{(x, t), t = x^2/a^2\}$  where  $a$  is related to  $m$  by (3.11). In particular there exists  $a_0 > 0$  related to  $m_0 \in (-1, 0)$  by (3.11) such that  $a \in [a_0, +\infty]$ . Therefore we get that  $\tilde{t}' = (\tilde{x}')^2/a^2$ , which, joint to the fact that  $\tilde{P}' = (\tilde{x}', \tilde{t}') \in \partial Q_1(0)$ , implies that  $\tilde{x}' \neq 0$ . We now compute

$$\begin{aligned} l_n &= \frac{|h(x'_n) - h(x_n)|}{|x'_n - x_n|} \\ &= \left( \frac{\bar{t}'_n}{f(P_n)} \right) / \left( \bar{x}'_n \sqrt{\frac{a(P_n)}{f(P_n)}} \right) \\ &= \varepsilon_n \frac{\tilde{t}'_n}{\tilde{x}'_n} \frac{1}{\sqrt{a(P_n)f(P_n)}}. \end{aligned}$$

The fact that  $\tilde{x}'_n \rightarrow \tilde{x}' \neq 0$  and  $|\tilde{t}'_n| \leq 1$  implies that  $l_n \rightarrow 0$ . Contradiction.  $\square$

We will now use the Hausdorff area formula. According to [11], Theorem 3.2.3 (p. 243), we have the following result (also see [11], 2.8.9 (p. 145), Theorem 2.8.18 (p. 152), 2.9.12 (p. 158), 3.2.1 (p. 241), Theorem 3.1.8 (p. 217), Definition 2.8.16 (p. 161), 3.1.2 (p. 211), Theorem 2.10.35 (p. 197), for related results).

**Lemma 5.12 (Hausdorff area formula)** *Let  $\mathcal{A}$  be a measurable set of  $\mathbb{R}$  and consider a function  $h : \mathcal{A} \mapsto \mathbb{R}$  such that, with the above notations, for all  $x \in \mathcal{A}$ ,  $q_h(x, \delta) < \infty$  for some  $\delta > 0$ . If  $N_h(y)$  is the number of elements of  $h^{-1}(t)$ , then*

$$\int_{\mathcal{A}} \left[ \lim_{\delta \rightarrow 0} q_h(x, \delta) \right] dx = \int_{\mathbb{R}} N_h(t) dt.$$

*Proof of Proposition 5.8.* Apply Lemmata 5.11 and 5.12 with  $\mathcal{A} = \{x \in \mathbb{R}, \exists t \in \mathbb{R}, (x, t) \in \mathcal{S}_{[-1, m_0]}\}$ :  $\int_{\mathbb{R}} N_h(t) dt = 0$ . This proves that for any  $m_0 \in (-1, 0)$ , the measure of the set

$$I_{m_0} = \{t \in \mathbb{R} : \exists x \in \mathbb{R}, (x, t) \in \mathcal{S}_{[-1, m_0]}\}$$

is zero. Hence the measure of  $I = \bigcup_{n \in \mathbb{N}} I_{-\frac{1}{n}}$  is also zero.  $\square$

**Remark 5.13** An inspection of the proof of lemma 5.11 shows that  $\frac{h(x')-h(x)}{|x'-x|^2}$  is bounded. This ratio even goes to zero uniformly as  $|x' - x| \rightarrow 0$  and  $(x, h(x)), (x', h(x')) \in \mathcal{S}_{[-1, m_0]}$ , because the two blow-up limits centered in  $P_n$  and in  $P'_n$  need to be the same which implies the limit  $a$  to be equal to  $+\infty$ .

A simple consequence of the boundedness of the ratio  $\frac{h(x')-h(x)}{|x'-x|^2}$  is that the one-dimensional parabolic Hausdorff measure of  $\mathcal{S}_{[-1, m_0]}$ , i.e.  $\mathcal{H}_p^1(\mathcal{S}_{[-1, m_0]})$  is bounded. Let us recall that the parabolic Hausdorff measure is build on the parabolic distance  $\text{dist}_p$  defined for two points  $P = (x, t)$  and  $P' = (x', t')$  by  $\text{dist}_p(P, P') := \sqrt{(x - x')^2 + |t - t'|}$ . At this stage it can be seen that the time projection of  $\mathcal{S}_{[-1, m_0]}$  defined by  $\Pi_{[-1, m_0]} = \{t, \exists(x, t) \in \mathcal{S}_{[-1, m_0]}\}$  satisfies  $\mathcal{H}^{\frac{1}{2}}(\Pi_{[-1, m_0]}) < +\infty$  for the classical euclidian Hausdorff measure. A further inspection shows that the convergence to zero of the ratio  $\frac{h(x')-h(x)}{|x'-x|^2}$  implies that  $\mathcal{H}^{\frac{1}{2}}(\Pi_{[-1, m_0]}) = 0$  As a consequence we get that

$$\mathcal{H}^{\frac{1}{2}}(\Pi_{[-1, 0]}) = 0$$

where

$$\Pi_{[-1, 0]} = \{t, \exists(x, t) \in \mathcal{S} \setminus \mathcal{S}_0\} .$$

This last remark can be of particular interest in higher dimension, especially in space dimension 2.

**Remark 5.14** Using a blow-up argument, it can be easily deduced from this section that any point in  $\mathcal{S} \setminus (\mathcal{S}_0 \cup \mathcal{S}_{-1})$  is an isolated point in  $\mathcal{S}$  and then is only surrounded by regular points from the free boundary.

## 6. On the continuity of $u_t$ and proof of the results of Section 1

In this section using the transformation of Section 3.1, we reduce the problem to the case  $a \equiv 1, b \equiv 0, c \equiv 0$  and  $f(P_1) = 1$  where  $P_1 \in \Gamma$ . After this transformation we have in the new coordinates  $P_1 = 0$ , but we will still keep the notation  $P_1$  to avoid some possible confusions.

### 6.1. Proof of Theorem 1.2

With direct estimates, we first prove the following result.

**Lemma 6.1 (Estimates on the limit of  $u_t$  at the boundary)** Under assumption (1.2), if  $u$  is a solution of (1.1) and  $P_1 \in \Gamma$ , then we have

$$(i) \quad \limsup_{P \rightarrow P_1, P \in \{u > 0\}} \frac{\partial u}{\partial t}(P) \leq 0 \quad \text{and} \quad (ii) \quad \liminf_{P \rightarrow P_1, P \in \{u > 0\}} \frac{\partial u}{\partial t}(P) \geq -1 .$$

Theorem 1.2 is a straightforward consequence of (i).

*Proof.* We first prove (i). Let  $l := \limsup_{P \rightarrow P_1, P \in \{u > 0\}} u_t(P)$ . Assume by contradiction that there exists a sequence  $(P_n = (x_n, t_n))_{n \in \mathbb{N}}$  such that

$$u(P_n) > 0, \quad \lim_{n \rightarrow \infty} P_n = P_1 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\partial u}{\partial t}(P_n) = l > 0 .$$

Define now  $\Pi_n := (\bar{x}_n, \bar{t}_n) \in \Gamma, \eta_n > 0$  such that

$$Q_{\eta_n}(P_n) \subset \{u > 0\}, \quad \Pi_n \in \partial(Q_{\eta_n}(P_n)) \cap \{u = 0\}$$



and  $\nu_n := (\eta_n^{-1}(x_n - \bar{x}_n), \eta_n^{-2}(t_n - \bar{t}_n))$ . Let  $u_{\Pi_n}^{\eta_n}(x, t) := \eta_n^{-2} u(\Pi_n + (\eta_n x, \eta_n^2 t))$ . Up to the extraction of a subsequence,  $(u_{\Pi_n}^{\eta_n})_{n \in \mathbb{N}}$  converges locally uniformly on all compact sets in  $\mathbb{R}^2$  to a function  $u^0 \in W_{x,t;\text{loc}}^{2,1;\infty}(\mathbb{R}^2)$ , and  $(\nu_n)_{n \in \mathbb{N}}$  to some  $\nu \in \partial Q_1(0)$ , such that

$$\begin{cases} H u^0 = 1 \text{ in } \{u^0 > 0\}, & u^0 \geq 0 \text{ a.e. in } \mathbb{R}^2, \\ \frac{\partial u^0}{\partial t}(\nu) = l \text{ and } u^0(\nu) > 0. \end{cases}$$

Here  $u^0(\nu) > 0$  is a consequence of the fact that  $l \neq 0$ .

By standard  $L^p$  parabolic estimates we can pass to the limit in  $u_t$  because  $u_t$  is bounded in  $\mathcal{H}^\alpha$ , and the corresponding bound is uniform under zooming scaling. The function  $u_t^0$  achieves its maximum at  $\nu$ . Otherwise, there would be a point  $P' = (x', t') \in \{u^0 > 0\}$  such that  $u_t^0(P') > l$  and then the point  $T_n = \Pi_n + (\eta_n x', \eta_n^2 t')$  would satisfy

$$\lim_{n \rightarrow \infty} T_n = P_1 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\partial u}{\partial t}(T_n) = \lim_{n \rightarrow \infty} \frac{\partial u_{\Pi_n}^{\eta_n}(P')}{\partial t} = \frac{\partial u^0}{\partial t}(P') > l,$$

a contradiction. Thus  $u_t^0 \leq u_t^0(\nu) = l$ . Moreover  $u_t^0$  satisfies the equation

$$H \left( \frac{\partial u^0}{\partial t} \right) = 0 \quad \text{a.e. in } \{u^0 > 0\}.$$

By the strong maximum principle,  $u_t^0 \equiv l$  in  $Q_r^-(\nu)$  for some  $r > 0$ , small enough, and as a consequence

$$\frac{\partial u^0}{\partial x^2} = 1 + l \quad \text{in } Q_r^-(\nu),$$

which means that there exist  $\bar{x}_0$  and  $k \in \mathbb{R}$  such that

$$u^0(x, t) = l(t - t_\nu) + (1 + l)(x - \bar{x}_0)^2/2 + k \geq 0 \quad \text{in } Q_r^-(\nu).$$

Iterating the method, we may cover the parabolic connected component of  $\{(x, t) \in \mathbb{R}^2 : u^0(x, t) > 0, t < t_\nu\}$  which contains  $\nu$ . Its boundary is given by

$$x \mapsto \phi(x) := \min \{t_\nu, t_\nu - l^{-1}((1 + l)(x - \bar{x}_0)^2/2 + k)\}.$$

For any  $x \in \mathbb{R}$  such that  $\phi(x) < t_\nu$  and  $x \neq \bar{x}_0$ ,  $u_x^0(x, \phi(x)) = (1 + l)(x - \bar{x}_0) \neq 0$  contradicts the continuity of  $u_x^0(\cdot, t)$ . Thus  $l \leq 0$ .

It remains to prove (ii). This is equivalent to prove that

$$q := \limsup_{P \rightarrow P_1, P \in \{u > 0\}} \frac{\partial^2 u}{\partial x^2}(P) \geq 0.$$

Assume by contradiction that  $q < 0$  and as for (i), define  $P_n = (x_n, t_n)$  such that  $\lim_{n \rightarrow \infty} u_{xx}(P_n) = q$ ,  $\Pi_n = (\bar{x}_n, \bar{t}_n)$ ,  $\eta_n$ ,  $\nu_n$  and  $u_{\Pi_n}^{\eta_n}$ . Up to the extraction of a subsequence  $(\nu_n)_{n \in \mathbb{N}}$  and  $(u_{\Pi_n}^{\eta_n})_{n \in \mathbb{N}}$  respectively converge to  $\nu \in \partial Q_1(0)$  and  $u^0 \in W_{x,t;\text{loc}}^{2,1;\infty}(\mathbb{R}^2)$ , which satisfy

$$\begin{cases} H u^0 = 1 \text{ in } \{u^0 > 0\}, & u^0 \geq 0 \text{ a.e. in } \mathbb{R}^2, \\ \frac{\partial^2 u^0}{\partial x^2}(\nu) = q \text{ and } u^0(\nu) > 0. \end{cases}$$

As above, in the parabolic component of  $\{(x, t) \in \mathbb{R}^2 : u^0(x, t) > 0, t < t_\nu\}$  which contains  $\nu$

$$u^0(x, t) = (q - 1)(t - t_\nu) + q(x - \bar{x}_0)^2/2 + k \geq 0.$$

This again contradicts the regularity of  $u_x^0$  on  $\partial\{u^0 = 0\}$ .  $\square$

6.2. A new characterization of some singular points and consequences

**Lemma 6.2** *Under Assumption (1.2) consider a solution  $u$  of (1.1). Let  $l := \liminf_{P \rightarrow P_1, P \in \{u > 0\}} u_t(P)$  be negative. Consider a minimizing sequence  $(P_n = (x_n, t_n))_{n \in \mathbb{N}}$  for  $l$ . Define  $\Pi_n = (\bar{x}_n, \bar{t}_n) \in \Gamma$ ,  $\eta_n > 0$  such that*

$$Q_{\eta_n}(P_n) \subset \{u > 0\}, \quad \Pi_n \in \partial(Q_{\eta_n}(P_n)) \cap \{u = 0\}$$

and  $\nu_n := (\eta_n^{-1}(x_n - \bar{x}_n), \eta_n^{-2}(t_n - \bar{t}_n))$ . Up to the extraction of a subsequence,  $(u_{\Pi_n}^{\eta_n} := \eta_n^{-2} u(\Pi_n + (\eta_n x, \eta_n^2 t)))_{n \in \mathbb{N}}$  converges locally uniformly on all compact sets in  $\mathbb{R}^2$  to a function  $u^0 \in W_{x,t;loc}^{2,1;\infty}(\mathbb{R}^2)$ , and  $(\nu_n)_{n \in \mathbb{N}}$  to some  $\nu = (x_\nu, t_\nu) \in \partial Q_1(0)$ . Moreover there exist  $\bar{x}_0$  and  $k \in \mathbb{R}$  such that

$$u^0(x, t) = l(t - t_\nu) + (1 + l)(x - \bar{x}_0)^2/2 + k \geq 0 \quad \forall (x, t) \in \mathbb{R} \times (-\infty, t_\nu).$$

*Proof.* We proceed as in the proof of Lemma 6.1. The function  $u^0$  and  $\nu$  are such that

$$\begin{cases} Hu^0 = 1 \text{ in } \{u^0 > 0\}, & u^0 \geq 0 \text{ a.e. in } \mathbb{R}^2, \\ \frac{\partial u^0}{\partial t}(\nu) = l \text{ and } u^0(\nu) > 0. \end{cases}$$

The function  $u_t^0$  achieves its minimum at  $\nu$ :  $u_t^0 \geq u_t^0(\nu) = l$ . Moreover  $Hu_t^0 = 0$  almost everywhere in  $\{u^0 > 0\}$ . By the strong maximum principle,  $u_t^0 \equiv l$  in  $Q_r^-(\nu)$  for some  $r > 0$ , small enough, and as a consequence

$$\frac{\partial^2 u^0}{\partial x^2} = 1 + l \quad \text{in } Q_r^-(\nu),$$

which means that there exist  $x_0$  and  $k \in \mathbb{R}$  such that

$$u^0(x, t) = l(t - t_\nu) + (1 + l)(x - \bar{x}_0)^2/2 + k \geq 0 \quad \text{in } Q_r^-(\nu).$$

Iterating the method, we may cover the parabolic connected component of  $\{(x, t) \in \mathbb{R}^2 : u^0(x, t) > 0, t < t_\nu\}$  which contains  $\nu$ . This proves that its boundary is given by

$$x \mapsto \phi(x) := \max \{t_\nu, t_\nu - l^{-1}((1 + l)(x - \bar{x}_0)^2/2 + k)\}.$$

For any  $x \in \mathbb{R}$  such that  $\phi(x) < t_\nu$  and  $x \neq \bar{x}_0$ ,  $u_x^0(x, \phi(x)) = (1 + l)(x - \bar{x}_0) \neq 0$  contradicts the continuity of  $u_x^0(\cdot, t)$  if  $l > -1$ . Thus  $\inf_{\mathbb{R}} \phi \geq t_\nu$  and  $u^0$  is positive in  $\{t < t_\nu\}$ . By unique continuation, we establish the expression of  $u^0$  in  $\mathbb{R} \times (-\infty, t_\nu)$ .  $\square$

As a consequence of this lemma we have

**Lemma 6.3** *Under Assumption (1.2) consider a solution  $u$  of (1.1) and take  $P_1 \in \Gamma$ . If*

$$\liminf_{P \rightarrow P_1, P \in \{u > 0\}} \frac{\partial u}{\partial t}(P) < 0$$

then  $P_1 \in \mathcal{S}$ .

*Proof.* Consider a nonnegative cut-off function  $\psi \in \mathcal{D}(\mathbb{R})$  such that  $\psi = 1$  in a small neighborhood of  $x = 0$  and with small enough compact support. Assume by contradiction that  $P_1$  is regular. For any  $P' = (x', t')$  we define

$$u_{P'}(x, t) = u(x + x', t + t').$$

By Proposition 4.1,  $\lim_{\tau \rightarrow 0} \mathcal{E}(\tau; u_{P_1} \psi) = \sqrt{2}/2$ . By Proposition 3.4, for any  $\delta > 0$ , there exists a  $\tau_0 < 0$  such that

$$\sqrt{2}/2 \leq \mathcal{E}(\tau_0; u_{P_1} \psi) < \sqrt{2}/2 + \delta/2. \quad (6.1)$$

With the notations of Lemma 6.2 and according to Lemma 6.2 the sequence  $(u_{\Pi_n}^{\eta_n})_{n \in \mathbb{N}}$  converges uniformly to  $u^0(x, t) = l(t - t_\nu) + (1 + l)(x - \bar{x}_0)^2/2 + k \geq 0$  in  $\mathbb{R} \times (-\infty, t_\nu)$ . We compute

$$\lim_{t \rightarrow -\infty} \mathcal{E}(t; u^0) = \sqrt{2}.$$

Then for any  $\delta > 0$ , there exists  $t_\infty < 0$  with  $|t_\infty|$  large enough such that for  $t < t_\infty$  we have (using the scaling invariance of the energy):

$$\sqrt{2} - \frac{\delta}{2} \leq \mathcal{E}(t; u^0) = \lim_{n \rightarrow \infty} \mathcal{E}(t; u_{\Pi_n}^{\eta_n} \psi(\eta_n \cdot)) = \lim_{n \rightarrow \infty} \mathcal{E}(\eta_n^2 t; u_{\Pi_n} \psi).$$

So for  $\tau_0$  defined in (6.1) and  $t < t_\infty$  fixed, there exists  $N = N(t, \delta)$  such that

$$\forall n > N, \quad \mathcal{E}(t; u_{\Pi_n}^{\eta_n} \psi(\eta_n \cdot)) > \mathcal{E}(t; u^0) - \frac{\delta}{2} \text{ and } \eta_n^2 t > \tau_0.$$

Proposition 3.4 applies to  $u_{\Pi_n}$ :

$$\mathcal{E}(\eta_n^2 t; u_{\Pi_n} \psi) \leq \mathcal{E}(\tau_0; u_{\Pi_n} \psi).$$

By continuity of the map  $P' \mapsto \mathcal{E}(\tau_0; u_{P'} \psi)$ , we have

$$\mathcal{E}(\tau_0; u_{\Pi_n} \psi) \leq \mathcal{E}(\tau_0; u_{P_1} \psi) + \frac{\delta}{2}.$$

Collecting these estimates, we have for any  $n > N$

$$\sqrt{2} - \delta \leq \mathcal{E}(t; u^0) - \frac{\delta}{2} < \mathcal{E}(\eta_n^2 t; u_{\Pi_n} \psi) \leq \mathcal{E}(\tau_0; u_{\Pi_n} \psi) \leq \mathcal{E}(\tau_0; u_{P_1} \psi) + \frac{\delta}{2} < \frac{\sqrt{2}}{2} + \delta,$$

a contradiction for any  $\delta < \sqrt{2}/4$ . □

As a direct consequence of Lemmata 6.1 and 6.3 we obtain

**Corollary 6.4** *Under Assumption (1.2) consider a solution  $u$  of (1.1). If  $P_1 \in \mathcal{R}$  then*

$$\lim_{P \rightarrow P_1, P \in \{u > 0\}} \frac{\partial u}{\partial t}(P) = 0.$$

**Lemma 6.5** *Let  $u$  be a solution of (1.1) and assume that (1.2) holds. If there exist  $r > 0$  and  $P_1 = (x_1, t_1) \in Q_R(P_0)$  such that  $Q_r(P_1) \subset Q_R(P_0)$  and  $t' := \inf\{t \in (t_P - r^2, t_P + r^2) : \exists x' \in (x_P - r, x_P + r)$  such that  $(x', t) \in \Gamma\}$  is achieved in  $(x_P - r, x_P + r) \times (t_P - r^2, t_P + r^2]$  and  $u$  is positive in  $\{(x, t) \in Q_r(P_1) : t < t'\}$  then  $(x', t')$  is a singular point.*

*Proof.* Assume by contradiction that  $P' \in \mathcal{R}$ . According to Theorem 3.9 and Proposition 5.5, the blow-up limit in  $P'$  corresponding to a blow-up sequence at scale  $\epsilon_n$  is  $u^0 = v_\pm^0$ . There exists therefore some  $\tilde{P} = (x, t)$  with  $t < 0$  such that  $\tilde{P} \in \text{Int}\{u^0 = 0\}$ . By Lemma 2.3, this implies that  $u(P' + \epsilon_n \tilde{P}) = 0$  for  $n$  large enough, a contradiction with the definition of  $P'$ . □

**Theorem 6.6** *Under Assumption (1.2) consider a solution  $u$  of (1.1). For any  $m \in [-1, 0]$ , if  $P_1 \in \mathcal{S}_m$  then*

$$\liminf_{P \rightarrow P_1, P \in \{u > 0\}} \frac{\partial u}{\partial t}(P) = m.$$

*Proof.* Let  $P_1 = (x_1, t_1)$  and  $l := \liminf_{P \rightarrow P_1, P \in \{u > 0\}} u_t(P)$ . By considering a blow-up sequence  $(\epsilon_n)_{n \in \mathbb{N}}$  and by computing  $u_t(x_{P_1}, t_{P_1} - \epsilon_n/2) \rightarrow (v_m^0)_{t(x_{P_1}, t_{P_1} - 1/2)} = m(P_1)$  we get that  $l \leq m = m(P_1)$ . Assume by contradiction that the inequality is strict.

Let  $(P_n)_{n \in \mathbb{N}}$  be a sequence with  $P_n = (x_n, t_n)$  such that  $u(P_n) > 0$ ,  $\lim_n P_n = P_1$  and  $\lim_n u_t(P_n) = l$ . For any  $n \in \mathbb{N}$ , define  $\epsilon_n > 0$ , such that  $P_n \in \partial Q_{\epsilon_n}(P_1)$ . Let us consider a localized blow-up sequence  $(u_{P_n}^{3\epsilon_n})_{n \in \mathbb{N}}$  which converges to  $v_m^0$ . Since  $Q_{\epsilon_n}(P_n) \subset Q_{3\epsilon_n}(P_1)$ , the sequence  $(u_{P_n}^{\epsilon_n} := \epsilon_n^{-2} u(x_n + \epsilon_n x, t_n + \epsilon_n^2 t))_{n \in \mathbb{N}}$  satisfies

$$u_{P_n}^{\epsilon_n} \longrightarrow v_m^0(\cdot + \bar{P}) \quad \text{with} \quad \bar{P} \in \partial Q_1(0).$$

Here  $\bar{P} = (\bar{x}, \bar{t}) = \lim_{n \rightarrow +\infty} \bar{P}_n$  with  $\bar{P}_n = \left( \frac{x_n - x_1}{\epsilon_n}, \frac{t_n - t_1}{\epsilon_n^2} \right)$ . By Lemma 6.2 and using the same notations, for some  $\Pi_n := (\bar{x}_n, \bar{t}_n) \in \Gamma$ ,  $(u_{\Pi_n}^{\eta_n})_{n \in \mathbb{N}}$  uniformly converges to  $u^0(x, t) = l(t - t_\nu) + (1 + l)(x - \bar{x}_0)^2/2 + k$  in  $\mathbb{R} \times (-\infty, t_\nu)$ . Let us define  $\bar{t}_0$  such that  $l(\bar{t}_0 - t_\nu) + k = 0$ . Then for  $\bar{P}_0 = (\bar{x}_0, \bar{t}_0)$ , and by uniqueness of the limit solution  $u^0$ , we have  $u^0 = v_l^0(\cdot + \bar{P}_0)$ . Consequently we have

$$u_{\Pi_n}^{\eta_n} \longrightarrow v_l^0(\cdot + \bar{P}_0).$$

Moreover we have

$$\eta_n \leq \epsilon_n.$$

Now let us consider the sequence  $(u_{\Pi_n}^{\epsilon_n})_{n \in \mathbb{N}}$  which satisfies  $u_{\Pi_n}^{\epsilon_n} = u_{P_n}^{\epsilon_n}(\cdot + \bar{\Pi}_n)$  with  $\bar{\Pi}_n = \left( \frac{\bar{x}_n - x_n}{\epsilon_n}, \frac{\bar{t}_n - t_n}{\epsilon_n^2} \right) \in \partial Q_{\eta_n/\epsilon_n}(0)$ . Up to extraction of a subsequence, we can assume that  $\bar{\Pi}_n \rightarrow \bar{\Pi}$  with  $\bar{\Pi} \in \bar{Q}_1(0)$  and then

$$u_{\Pi_n}^{\epsilon_n} \longrightarrow v_m^0(\cdot + \bar{P} + \bar{\Pi}).$$

Because we assumed that  $l \neq m$ , this implies that  $\eta_n/\epsilon_n \rightarrow 0$  and then  $\bar{\Pi} = 0$ . Given  $\delta > 0$ , we now consider  $\mu > 0$  large enough such that

$$\begin{aligned} |v_m^0((0, -\mu) + \bar{P}) - m\mu| &\leq \delta\mu \\ |v_l^0((0, -\mu) + \bar{P}_0) - l\mu| &\leq \delta\mu. \end{aligned}$$

The function  $\lambda \mapsto u_{\Pi_n}^\lambda := \lambda^{-2} u(\bar{x}_n + \lambda x, \bar{t}_n + \lambda^2 t)$  is continuous: there exists a  $\lambda_n \in (\eta_n, \epsilon_n)$  such that

$$\frac{1}{\mu} u_{\Pi_n}^{\lambda_n}(0, -\mu) = \frac{1}{2} [m + l]$$

for any  $n$  large enough. The sequence  $(u_{\Pi_n}^{\lambda_n})_{n \in \mathbb{N}}$  converges to a function  $\bar{u}$  in  $W_{x,t}^{2,1;\infty}$  which satisfies

$$H\bar{u} = \mathbf{1}_{\{\bar{u} > 0\}}, \quad \bar{u} \geq 0 \quad \text{and} \quad \bar{u}(0, -\mu) = \frac{\mu}{2} [m + l].$$

Consider a nonnegative cut-off function  $\psi \in \mathcal{D}(\mathbb{R})$  such that  $\psi \equiv 1$  on  $(-1/2, 1/2)$ ,  $\text{supp}(\psi) = [-1, 1]$ . On the one hand, there exists  $t_\infty = t_\infty(\delta) < 0$  such that

$$\sqrt{2} - \frac{\delta}{2} \leq \mathcal{E}(t; v_l^0(\cdot + \bar{P}_0)) = \lim_{n \rightarrow \infty} \mathcal{E}(t; u_{\Pi_n}^{\eta_n} \psi(\eta_n \cdot)) = \lim_{n \rightarrow \infty} \mathcal{E}(\eta_n^2 t; u_{\Pi_n} \psi) \quad \forall t < t_\infty.$$

On the other hand by definition of  $\mathcal{S}$ ,  $\lim_{\tau \rightarrow 0} \mathcal{E}(\tau; u_{P_1} \psi) = \sqrt{2}$ . By Proposition 3.4, for any  $\delta > 0$ , there exists  $\tau_0 < 0$ , with  $|\tau_0|$  sufficiently small, such that

$$\mathcal{E}(\tau_0; u_{P_1} \psi) < \sqrt{2} + \delta/2.$$

For any  $\delta > 0$ ,  $t > 0$ , there exists a  $N = N(t, \delta)$  such that

$$n > N \quad \Rightarrow \quad \mathcal{E}(t; u_{\Pi_n}^{\eta_n} \psi(\eta_n \cdot)) > \mathcal{E}(t; v_l^0(\cdot + \bar{P}_0)) - \frac{\delta}{2} \quad \text{and} \quad \eta_n^2 t > \tau_0.$$

Moreover, for a fixed  $\tau_0$ , by continuity of the energy, since  $\Pi_n \in Q_{2\epsilon_n}(P_1)$ ,

$$\mathcal{E}(\tau_0; u_{\Pi_n} \psi) \leq \mathcal{E}(\tau_0; u_{P_1} \psi) + \frac{\delta}{2} \quad \forall n > N$$

for  $N$  large enough. Using Proposition 3.4, for all  $t < t_\infty$ ,  $n > N$ ,  $s \in (\tau_0, \eta_n^2 t) \subset (\tau_0, 0)$ , we get

$$\sqrt{2} - \delta < \mathcal{E}(t; v_l^0(\cdot + \bar{P}_0)) - \frac{\delta}{2} < \mathcal{E}(\eta_n^2 t; u_{\Pi_n} \psi) \leq \mathcal{E}(s; u_{\Pi_n} \psi) \leq \mathcal{E}(\tau_0; u_{\Pi_n} \psi) \leq \mathcal{E}(\tau_0; u_{P_1} \psi) + \frac{\delta}{2} < \sqrt{2} + \delta.$$

For any given  $t < t_\infty$  and  $n > N(t, \delta)$  we define  $s := \lambda_n^2 \tau$ , and  $\tau \in (\tau_0/\lambda_n^2, (\eta_n/\lambda_n)^2 t)$ . As a consequence, the estimate

$$\sqrt{2} - \delta \leq \mathcal{E}(\lambda_n^2 \tau; u_{\Pi_n} \psi) = \mathcal{E}(\tau; u_{\Pi_n}^{\lambda_n} \psi(\eta_n \cdot)) \leq \sqrt{2} + \delta$$

holds true for any  $t < t_\infty(\delta)$  and  $n > N(t, \delta)$ . From our construction we get that  $\lambda_n \rightarrow 0$  and  $\eta_n/\lambda_n \rightarrow 0$ , so that  $|\mathcal{E}(\tau; \bar{u}) - \sqrt{2}| \leq \delta$  for all  $\tau < 0$  and for all  $\delta > 0$ . Therefore

$$\forall \tau \in (-\infty, 0), \quad \mathcal{E}(\tau; \bar{u}) = \sqrt{2}.$$

This means that  $\bar{u}$  is scale-invariant by Corollary 3.5. By Theorem 3.9 there exists an  $\bar{m}$  such that  $\bar{u} = v_{\bar{m}}^0$ . Because of the expression of  $\bar{u}(0, -\mu)/\mu$  we obtain  $\bar{m} = [m + l]/2 \in (-1, 0)$ . From the convergence of  $u_{\Pi_n}^{\lambda_n}$  to  $v_{\bar{m}}^0$  with  $\bar{m} \in (-1, 0)$  where the free boundary  $\partial \{v_{\bar{m}}^0 > 0\}$  is a parabola oriented in the positive time direction, and from the fact that  $P_n \in Q_{\lambda_n}(\Pi_n)$ , we deduce that Lemma 6.5 applies to  $u$  in  $Q_{A\lambda_n}(P_n)$  for some  $A > 0$  large enough, but independent of  $n$ . Then there exists a sequence of singular points  $(Z_n)_{n \in \mathbb{N}}$  in  $Q_{A\lambda_n}(P_n)$  such that  $\lim_{n \rightarrow \infty} m(Z_n) = \bar{m}$ , because  $u_{\Pi_n}^{\lambda_n}$  converges to  $v_{\bar{m}}^0$ . Moreover the sequence  $Z_n$  converges to  $P_1$  and then by Lemma 5.6, we obtain  $\bar{m} = m(P_1)$ , which is impossible.  $\square$

As a very simple consequence, we obtain the following result.

**Corollary 6.7** *Under Assumption (1.2) consider a solution  $u$  of (1.1). If  $P_1 \in \mathcal{S}_0$  then*

$$\lim_{P \rightarrow P_1, P \in \{u > 0\}} \frac{\partial u}{\partial t}(P) = 0.$$

### 6.3. Proofs of the results of Section 1

*Proof of Theorem 1.4.* If for some  $t < 0$ ,  $\mathcal{E}(t; v_{P_1}) < \sqrt{2}$ , then by Proposition 3.4,  $t \mapsto \mathcal{E}(t; v_{P_1})$  is monotone decreasing, and by Proposition 4.1,  $P_1 \in \mathcal{R}$ . By Corollary 6.4,  $u_t$  is continuous at  $P_1$  and in a neighborhood of  $P_1$  by Lemma 4.2.  $\square$

*Proof of Theorem 1.3.* By Proposition 4.1, the limit of  $\mathcal{E}$  is either  $\sqrt{2}$  or  $\sqrt{2}/2$ . In the second case, Corollary 6.4 applies at  $P_1$  and the continuity of  $u_t$  holds because  $\mathcal{R}$  is open in  $\Gamma$  according to Lemma 4.2, which proves (ii).  $\square$

*Proof of Theorem 1.1.* By Proposition 5.8, the set  $I$  has zero Lebesgue measure. If  $(x_1, t_1) = P_1 \in \Gamma$  is such that  $t_1 \notin I$ , then  $P_1 \in \mathcal{S}_0 \cup \mathcal{R}$ , and the result holds by Corollaries 6.4 and 6.7.  $\square$

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