NONLINEAR DIFFUSION OF DISLOCATION DENSITY AND SELF-SIMILAR SOLUTIONS

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Abstract. We study a nonlinear pseudodifferential equation describing the dynamics of dislocations. The long time asymptotics of solutions is described by the self-similar profiles.

1. Introduction

In this paper, we study the following initial value problem for the nonlinear and nonlocal equation

$$u_t = -|u_x|\ \Lambda^\alpha u \quad \text{on} \quad \mathbb{R}\times(0,+\infty), \tag{1.1}$$

$$u(x,0) = u_0(x) \quad \text{for} \quad x\in\mathbb{R}, \tag{1.2}$$

where the assumptions on the initial datum u_0 will be precised later. Here, for $\alpha\in(0,2)$, $\Lambda^\alpha = \left(\partial^2/\partial x^2\right)^{\alpha/2}$ is the pseudodifferential operator defined *via* the Fourier transform

$$\widehat{(\Lambda^\alpha w)}(\xi) = |\xi|^\alpha \widehat{w}(\xi). \tag{1.3}$$

For $\alpha = 1$, equation (1.1) is an integrated form of a model studied by Head [9] for the self-dynamics of a dislocation density represented by u_x. This model is a mean field model that has been derived rigorously in [7] as the limit of a system of particles in interactions with forces in $\frac{1}{x}$. In this model, dislocations can be of two types, $+$ or $-$, depending on the sign of their Burgers vector (see the book by Hirth and Lothe [11] for a physical definition of the Burgers vector). Here, the density u_x means the positive density $|u_x|$ of dislocations of type of the sign of u_x. Moreover, the occurrence of the absolute value $|u_x|$ in the equation allows the vanishing of dislocation particles of opposite sign. In the present paper, we study the general

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case $\alpha \in (0,2)$ that could be seen as a mean field model of particles with repulsive interactions in $\frac{1}{x^\alpha}$.

Recall that the operator Λ^α has the Lévy-Khintchine integral representation for every $\alpha \in (0,2)$

$$-\Lambda^\alpha w(x) = C(\alpha) \int_{\mathbb{R}} \left\{ w(x+z) - w(x) - z w'(x) \mathbf{1}_{\{|z| \le 1\}} \right\} \frac{\mathrm{d}z}{|z|^{1+\alpha}}, \tag{1.4}$$

where $C(\alpha) > 0$ is a constant. This formula (discussed in, e.g., [6, Th. 1] for functions w in the Schwartz space) allows us to extend the definition of Λ^α to functions which are bounded and sufficiently smooth, however, not necessarily decaying at infinity.

1.1. **Main results.** First note that equation (1.1) is invariant under the scaling

$$u^\lambda(x,t) = u(\lambda x, \lambda^{\alpha+1} t) \tag{1.5}$$

for each $\lambda > 0$ which means that if $u = u(x,t)$ is a solution to (1.1), then $u^\lambda = u^\lambda(x,t)$ is so. Hence, our first goal is to construct self-similar solutions of equation (1.1), i.e. solutions which are invariant under the scaling (1.5). By a standard argument, any self-similar solution should have the following form

$$u_\alpha(x,t) = \Phi_\alpha(y) \quad \text{with} \quad y = \frac{x}{t^{1/(\alpha+1)}}, \tag{1.6}$$

where the self-similar profile Φ_α has to satisfy the following equation

$$-(\alpha+1)^{-1}\, y\, \Phi_\alpha'(y) = -(\Lambda^\alpha \Phi_\alpha(y))\, \Phi_\alpha'(y) \quad \text{for all} \quad y \in \mathbb{R}. \tag{1.7}$$

In our first theorem, we construct solutions to equation (1.7).

Theorem 1.1 (Existence of self-similar profile)**.** *Let $\alpha \in (0,2)$. There exists a nondecreasing function Φ_α of the regularity $C^{1+\alpha/2}$ at each point and analytic on $(-y_\alpha, y_\alpha)$ for some $y_\alpha > 0$, which satisfies*

$$\Phi_\alpha = \begin{cases} 0 & \text{on} \quad (-\infty, -y_\alpha), \\ 1 & \text{on} \quad (y_\alpha, +\infty), \end{cases}$$

and

$$(\Lambda^\alpha \Phi_\alpha)(y) = \frac{y}{\alpha+1} \quad \text{for all} \quad y \in (-y_\alpha, y_\alpha).$$

Remark 1.2. We can obtain the self-similar solutions corresponding to different boundary values at infinity, simply considering for any $\gamma > 0$ and $b \in \mathbb{R}$ the profiles $\gamma \Phi_\alpha\left(\gamma^{-1/(\alpha+1)} y\right) + b$ which are also solutions of equation (1.7).

Remark 1.3. The fact that $\partial_y \Phi_\alpha$ has compact support reveals a finite velocity propagation of the support of the solution which is typical for solutions the porous medium equation, cf. Remark 1.7, below.

At least formally, the function Φ_α is the solution of (1.7), and the self-similar function u_α given by (1.6) is a solution of equation (1.1) with the initial datum being the Heaviside function

$$u_0(x) = H(x) = \begin{cases} 0 & \text{if} \quad x < 0, \\ 1 & \text{if} \quad x > 0. \end{cases} \tag{1.8}$$

In order to check that u_α given by (1.6) solves (1.1), we introduce a suitable notion of viscosity solutions to the initial value problem (1.1)–(1.2), see Section 3. In this setting, we show in Theorem 3.7 the existence and the uniqueness of a solution for any initial condition u_0 in $BUC(\mathbb{R})$, i.e. the space of bounded and uniformly continuous functions on $\mathbb{R}$. Although, the initial datum (1.8) is not continuous, we have the following result.

Theorem 1.4 (Uniqueness of the self-similar solution)**.** *Let $\alpha \in (0,2)$. Then the function u_α defined in (1.6) with the profile Φ_α constructed in Theorem 1.1 is the unique viscosity solution of equation (1.1) with the initial datum (1.8).*

In Theorem 1.4, the uniqueness holds in the sense that if u is another viscosity solution to (1.1), (1.8), then $u = u_\alpha$ on $(\mathbb{R} \times [0, +\infty)) \setminus \{(0,0)\}$.

The self-similar solutions are not only unique, but are also stable in this framework of viscosity solutions, as the following result shows.

Theorem 1.5 (Stability of the self-similar solution)**.** *Let $\alpha \in (0,2)$. For any initial data $u_0 \in BUC(\mathbb{R})$ satisfying*

$$\lim_{x \to -\infty} u_0(x) = 0 \quad \text{and} \quad \lim_{x \to +\infty} u_0(x) = 1, \tag{1.9}$$

let us consider the unique viscosity solution $u = u(x,t)$ of (1.1)–(1.2) and, for each $\lambda > 1$, its rescaled version $u^\lambda = u^\lambda(x,t)$ given by equation (1.5). Then, for any compact set $K \subset (\mathbb{R} \times [0, +\infty)) \setminus \{(0,0)\}$, we have

$$u^\lambda(x,t) \to \Phi_\alpha\left(\frac{x}{t^{1/(\alpha+1)}}\right) \quad \text{in} \quad L^\infty(K) \quad \text{as} \quad \lambda \to +\infty. \tag{1.10}$$

Theorem 1.5 contains a result on the long time behaviour of solution because, first, choosing $t = 1$ in (1.10) and, next, substituting $\lambda = t^{1/(\alpha+1)}$ we obtain the convergence of $u\left(xt^{1/(\alpha+1)}, t\right)$ toward the self-similar profile $\Phi_\alpha(x)$.

On the other hand, convergence (1.10) can be seen as a stability result when we consider initial data which are perturbations of the Heaviside function. This is a nonstandard stability result in the framework of discontinuous viscosity solutions. It shows that the approach by viscosity solutions is a good one in the sense of Hadamard, even if we consider here initial conditions which are perturbations of the Heaviside function.

Finally, we have the following result of independent interest.

Theorem 1.6 (Optimal decay estimates)**.** *Let* $\alpha \in (0,1]$*. For any initial condition* $u_0 \in BUC(\mathbb{R})$ *such that* $u_{0,x} \in L^1(\mathbb{R})$*, the unique viscosity solution* u *of (1.1)–(1.2) satisfies*

$$\|u(\cdot,t)\|_\infty \le \|u_0\|_\infty \quad \textit{and} \quad \|u_x(\cdot,t)\|_\infty \le \|u_{0,x}\|_\infty \quad \text{for any} \quad t>0.$$

Moreover, for every $p \in [1,+\infty)$ *we have*

$$\|u_x(\cdot,t)\|_p \le C_{p,\alpha}\|u_{0,x}\|_1^{\frac{p\alpha+1}{p(\alpha+1)}}\ t^{-\frac{(p-1)}{p(\alpha+1)}} \quad \text{for any} \quad t>0, \tag{1.11}$$

with some constant $C_{p,\alpha}>0$ *depending only on* p *and* α*.*

The decay given in (1.11) is optimal in the sense that the self-similar solution satisfies $\|(u_\alpha)_x(\cdot,t)\|_p = \|(\Phi_\alpha)_y(\cdot)\|_p\ t^{-\frac{(p-1)}{p(\alpha+1)}}$.

Remark 1.7. The equation satisfied by $v=u_x$ of the following form

$$v_t = (|v|\Lambda^{\alpha-1}\mathcal{H}v)_x \tag{1.12}$$

(with the Hilbert transform denoted by $\mathcal{H}$) can be treated as the nonlocal counterpart of the porous medium equation. Indeed, for $\alpha=2$ and for nonnegative v, equation (1.12) reduces to $v_t=(vv_x)_x=\left(v^2/2\right)_{xx}$. As in the case of the porous medium equation (see *e.g.* [23] and the references therein), estimates (1.11) show a regularizing effect created by the equation, even for the anomalous diffusion: if $v_0 \in L^1(\mathbb{R})$ then $v \in L^p(\mathbb{R})$ for each $p>1$. Observe also that equation (1.12) has the compactly supported self-similar solution $v(x,t)=t^{-\frac{1}{\alpha+1}}\Phi'_\alpha\left(x/t^{\frac{1}{\alpha+1}}\right)$, where the profile Φ_α was constructed in Theorem 1.1. This function for $\alpha=2$ corresponds to the well-known Barenblatt-Prattle solution of the porous medium equation.

Remark 1.8. For $\alpha\in(1,2)$, we do not know how to define the product $|u_x|\ (\Lambda^\alpha u)$ in the sense of distributions, which is an obstacle for us to prove the result of Theorem 1.6 in this case, see Section 6. Note, however, that the inequalities from Theorem 1.6 are valid for $\alpha\in(1,2]$ as well, provided the solution $u=u(x,t)$ is sufficiently regular.

1.2. **Organization of the paper.** In Section 2, we construct explicitly the self-similar solution. In Section 3, we recall the necessary material about viscosity solutions, which will be used in the remainder of the paper. In Section 4, we prove the uniqueness of the self-similar solution. Under the additional assumption that the solution is confined between its boundary values at infinity, we prove the stability of the self-similar solution, namely Theorem 1.5. In Section 5, we prove further decay properties of a solution with compact support. Applying these estimates, we finish the proof of Theorem 1.5 in the general case. In Section 6, we introduce an ε-regularized equation, for which we prove both the global existence of a smooth solution and the corresponding gradient estimates. Finally in Section 7, we deduce the gradient estimate in the limit case $\varepsilon = 0$, namely Theorem 1.6, using the corresponding estimates for the approximate ε-problem.

2. Construction of self-similar solutions

Proof of Theorem 1.1. The crucial role in the construction of the self-similar profile Φ_α is played by the function

$$v(x) = \begin{cases} K(\alpha)\left(1-|x|^2\right)^{\alpha/2} & \text{for} \quad |x|<1, \\ 0 & \text{for} \quad |x| \geq 1, \end{cases} \tag{2.1}$$

with $K(\alpha) = \Gamma(1/2)\left[2^\alpha \Gamma(1+\alpha/2)\Gamma((1+\alpha)/2)\right]^{-1}$. This function (together with its multidimensional counterparts) has an important probabilistic interpretation. Indeed, if $\{X(t)\}_{t\geq 0}$ denotes the symmetric α-stable process in $\mathbb{R}$ of order $\alpha \in (0,2]$ and if $T = \inf\{t : |X(t)| > 1\}$ is the first passage time of the process to the exterior of the ball $\{x : |x| \leq 1\}$, Getoor [8] proved that $\mathbb{E}^x(T) = v(x)$, where $\mathbb{E}^x$ denotes the expectation under the condition $X(0) = x$.

In particular, it was computed in [8, Th. 5.2] using a purely analytical argument (based on definition (1.3) and on properties of the Fourier transform) that $\Lambda^\alpha v \in L^1(\mathbb{R})$ and

$$\Lambda^\alpha v(x) = 1 \quad \text{for} \quad |x| < 1. \tag{2.2}$$

Now, for the function v, we define the bounded, nondecreasing, $C^{1+\alpha/2}$-function

$$u(x) = \int_0^x v(y)\,\mathrm{d}y$$

which obviously satisfies $u(x) = M(\alpha)$ for all $x \geq 1$ and $u(x) = -M(\alpha)$ for $x \leq -1$ with

$$M(\alpha) = K(\alpha)\int_0^1 \left(1-|y|^2\right)^{\alpha/2}\,\mathrm{d}y = \frac{\pi}{2^\alpha(\alpha+1)\Gamma\left(\frac{1+\alpha}{2}\right)^2}.$$

Then, for any $\varphi \in C_c^\infty(\mathbb{R})$, we can introduce the following duality

$$\langle \Lambda^\alpha u, \varphi\rangle = \int_{\mathbb{R}} u(y)(\Lambda^\alpha \varphi)(y)\,\mathrm{d}y.$$

This defines $\Lambda^\alpha u$ as a distribution, because we can check (using the Lévy-Khintchine formula (1.4)) that there exists a constant $C > 0$ such that

$$|(\Lambda^\alpha\varphi)(x)| \leq \frac{C\|\varphi\|_{W^{2,\infty}(\mathbb{R})}}{1+|x|^{1+\alpha}}.$$

If, moreover, $\operatorname{supp}\varphi \subset (-1,1)$, it is easy to check using the properties of the function $v = v(x)$ that

$$\langle \partial_x(\Lambda^\alpha u), \varphi\rangle = -\langle u, \Lambda^\alpha(\partial_x \varphi)\rangle = -\langle u, \partial_x(\Lambda^\alpha \varphi)\rangle = \langle \Lambda^\alpha(\partial_x u), \varphi\rangle = \langle 1, \varphi\rangle,$$

where the last inequality is a consequence of (2.2). From the symmetry of v, we deduce the antisymmetry of u, and then $(\Lambda^\alpha u)(-x) = -(\Lambda^\alpha u)(x)$. Therefore, we get the equality $(\Lambda^\alpha u)(x) = x$ in $\mathcal{D}'(-1,1)$, however by [12, Cor. 3.1.5], in the classical sense for each $y \in (-1,1)$, too.

Finally, we define the nonnegative function

$$\Phi_\alpha(y) = \frac{\gamma}{\alpha+1}\left\{u\left(\gamma^{-1/(\alpha+1)}y\right) + M(\alpha)\right\} \quad \text{with} \quad \gamma^{-1} = \frac{2M(\alpha)}{\alpha+1}.$$

Now, for $y_\alpha = \gamma^{1/(\alpha+1)} = [2M(\alpha)]^{-1/(\alpha+1)}$, we can check easily that Φ_α is exactly as stated in Theorem 1.1, which ends the proof. □

Let us note that we will not use in the sequel the explicit form of the function Φ_α, but only its properties listed in Theorem 1.1.

Remark 2.1. It is known since the work of Head and Louat [10] (see also [9]) that the function $v(x) = K\left(1-x^2\right)^{1/2}$ (with a suitably chosen constant $K = K(1) > 0$) is the solution of the equation $(\Lambda^1 v)(x) = 1$ on $(-1,1)$. This result is a consequence of an inversion theorem due to Muskhelishvili, see either [19, p. 251] or [22, Sec. 4.3].

3. Notion of viscosity solutions

Here, we consider equation (1.1) and its vanishing viscosity approximation, i.e. the following initial value problem for $\alpha \in (0,2)$ and $\eta \geq 0$

$$u_t = \eta u_{xx} - |u_x|\ \Lambda^\alpha u \quad \text{on} \quad \mathbb{R}\times(0,+\infty), \tag{3.1}$$

$$u(x,0) = u_0(x) \quad \text{for} \quad x \in \mathbb{R}. \tag{3.2}$$

In this section, we present the framework of viscosity solutions to problem (3.1)–(3.2). To this end, we recall briefly the necessary material, which can be either found in the literature or is essentially a standard adaptation of those results. We also refer the reader to Crandall *et al.* [5] for a classical text on viscosity solutions to local (i.e. partial differential) equations.

Let us first recall the definition of relaxed *lower semi-continuous* (lsc, for short) and *upper semi-continuous* (usc, for short) *limits* of a family of functions u^ε which is locally bounded uniformly with respect to ε

$$\limsup_{\varepsilon\to 0}{}^* u^\varepsilon(x,t) = \limsup_{\substack{\varepsilon\to 0\\ y\to x, s\to t}} u^\varepsilon(y,s) \quad \text{and} \quad \liminf_{\varepsilon\to 0}{}_* u^\varepsilon(x,t) = \liminf_{\substack{\varepsilon\to 0\\ y\to x, s\to t}} u^\varepsilon(y,s).$$

If the family consists of a single element, we recognize the usc envelope and the lsc envelope of a locally bounded function u

$$u^*(x,t) = \limsup_{y\to x, s\to t} u(y,s) \qquad \text{and} \qquad u_*(x,t) = \liminf_{y\to x, s\to t} u(y,s).$$

Now, we recall the definition of a *viscosity solution* for (3.1)–(3.2). Here, the difficulty is caused by the measure $|z|^{-1-\alpha}\,\mathrm{d}z$ appearing in the Lévy-Khintchine formula (1.4) which is singular at the origin and, consequently, the function has to be at least $C^{1,1}$ in space in order that $\Lambda^\alpha u(\cdot,t)$ makes sense (especially for α close to 2). We refer the reader, for instance, to [20, 3, 16] for the stationary case, and to [15, 14] for the evolution equation where this question is discussed in detail.

Now, we are in a position to define viscosity solutions.

Definition 3.1 (Viscosity solution/subsolution/supersolution)**.** A bounded usc (resp. lsc) function $u : \mathbb{R}\times\mathbb{R}^+ \to \mathbb{R}$ is a *viscosity subsolution* (resp. *supersolution*) of equation (3.1) on $\mathbb{R}\times(0,+\infty)$ if for any point (x_0,t_0) with $t_0 > 0$, any $\tau \in (0,t_0)$, and any test function ϕ belonging to $C^2(\mathbb{R}\times(0,+\infty))\cap L^\infty(\mathbb{R}\times(0,+\infty))$ such that $u-\phi$ attains a maximum (resp. minimum) at the point (x_0,t_0) on the cylinder

$$Q_\tau(x_0,t_0) := \mathbb{R}\times(t_0-\tau, t_0+\tau),$$

we have

$$\partial_t\phi(x_0,t_0) - \eta\phi_{xx}(x_0,t_0) + |\phi_x(x_0,t_0)|\,(\Lambda^\alpha\phi(\cdot,t_0))(x_0) \le 0 \quad (\text{resp. } \ge 0),$$

where $(\Lambda^\alpha\phi(\cdot,t_0))(x_0)$ is given by the Lévy-Khintchine formula (1.4).

We say that u is a *viscosity subsolution* (resp. *supersolution*) of problem (3.1)–(3.2) on $\mathbb{R}\times[0,+\infty)$, if it satisfies moreover at time $t=0$

$$u(\cdot,0) \le u_0^* \quad (\text{resp.} \quad u(\cdot,0) \ge (u_0)_*)\,.$$

A function $u : \mathbb{R}\times\mathbb{R}^+ \to \mathbb{R}$ is a *viscosity solution* of (3.1) on $\mathbb{R}\times(0,+\infty)$ (resp. $\mathbb{R}\times[0,+\infty)$) if u^* is a viscosity subsolution and u_* is a viscosity supersolution of the equation on $\mathbb{R}\times(0,+\infty)$ (resp. $\mathbb{R}\times[0,+\infty)$).

Other equivalent definitions are also natural, see for instance [3].

Remark 3.2. Any bounded function $u \in C^{1+\beta}$ (with some $\beta > \max\{0,\alpha-1\}$) which satisfies pointwisely (using the Lévy-Khintchine formula (1.4)) equation (3.1) with $\eta = 0$, is indeed a viscosity solution.

Theorem 3.3 (Comparison principle)**.** *Consider a bounded usc subsolution u and a bounded lsc supersolution v of (3.1)–(3.2). If $u(x,0) \le u_0(x) \le v(x,0)$ for some $u_0 \in BUC(\mathbb{R})$, then $u \le v$ on $\mathbb{R} \times [0,+\infty)$.*

Proof. Recall that in [14, Th. 5], the comparison principle is proved for $\alpha = 1$ and $\eta = 0$ under the additional assumption that $u_0 \in W^{1,\infty}(\mathbb{R})$. Looking at the proof of that result, the regularity of the initial data u_0 is only used to show that

$$\sup_{x\in\mathbb{R}} \left((u_0)^\varepsilon(x) - (u_0)_\varepsilon(x)\right) \to 0 \quad \text{as} \quad \varepsilon \to 0, \tag{3.3}$$

where $(u_0)^\varepsilon$ and $(u_0)_\varepsilon$ are respectively sup and inf-convolutions. It is easy (and classical) to check that (3.3) is still true for $u_0 \in BUC(\mathbb{R})$. The general case can be done either considering a variation of the proof of [14] taking into account the additional Laplace operator, or applying the "maximum principle" from [16], or following, for instance, the lines of [3]. We skip here the detail of this adaptation. This finishes the proof. □

Theorem 3.4 (Stability)**.** *Let $\{u^\varepsilon\}_{\varepsilon>0}$ be a sequence of viscosity subsolutions (resp. supersolutions) of equation (3.1) which are locally bounded, uniformly in ε. Then $\overline{u} = \limsup^* u^\varepsilon$ (resp. $\underline{u} = \liminf_* u^\varepsilon$) is a subsolution (resp. supersolution) of (3.1) on $\mathbb{R} \times (0,+\infty)$.*

Proof. A counterpart of Theorem 3.4 is proved in [3, Th.1]. Here, the result for the time dependent problem is again a classical adaptation of that argument, so we skip details. □

Remark 3.5. One can generalize directly Theorem 3.4 assuming that $\{u^\varepsilon\}_{\varepsilon>0}$ are solutions to the sequence of equations (3.1) with $\eta = \varepsilon$. Then, in the limit $\varepsilon \to 0^+$, we obtain viscosity subsolutions (resp. supersolutions) of equation (1.1). We use this property in the proof of Theorem 1.6.

Remark 3.6. In Theorem 3.4, we only claim that the limit $\overline{u}$ is a supersolution on $\mathbb{R} \times (0,+\infty)$, but not on $\mathbb{R} \times [0,+\infty)$. In other words, we do not claim that $\overline{u}$ satisfies the initial condition. Without further properties of the initial data u^0, it may happen that $\overline{u}(\cdot,0) \le u_0^*$ is not true.

Theorem 3.7 (Existence)**.** *Consider $u_0 \in BUC(\mathbb{R})$. Then there exists the unique bounded continuous viscosity solution u of (3.1)–(3.2).*

Proof. Applying the argument of [13] (already adapted from the classical arguments), we can construct a solution by the Perron method, if we are able to construct suitable barriers.

Case 1: First, assume that $u_0 \in W^{2,\infty}(\mathbb{R})$. Then the following functions

$$u_\pm(x,t) = u_0(x) \pm Ct \tag{3.4}$$

are barriers for $C > 0$ large enough (depending on the norm $\|u_0\|_{W^{2,\infty}(\mathbb{R})}$), and we get the existence of solutions by the Perron method.

Case 2: Let $u_0 \in BUC(\mathbb{R})$. For any $\varepsilon > 0$, we can regularize u_0 by a convolution, and get a function $u_0^\varepsilon \in W^{2,\infty}(\mathbb{R})$ which satisfies, moreover,

$$|u_0^\varepsilon - u_0| \le \varepsilon. \tag{3.5}$$

Let us call u^ε the solution of (3.1)–(3.2) with the initial condition u_0^ε instead of u_0. Then, from the fact that the equation does not see the constants and from the comparison principle (Theorem 3.3), we have for any ε, $\delta > 0$

$$|u^\varepsilon - u^\delta| \le \varepsilon + \delta.$$

Therefore, $\{u^\varepsilon\}_{\varepsilon>0}$ is the Cauchy sequence which converges in $L^\infty(\mathbb{R} \times [0,+\infty))$ to some continuous function u (because all the functions u^ε are continuous). By the stability result (Theorem 3.4), we see that u is a viscosity solution of equation (3.1) on $\mathbb{R} \times (0,+\infty)$. To recover the initial boundary condition, we simply remark that $u^\varepsilon(x,0) = u_0^\varepsilon(x)$ satisfies (3.5), and then passing to the limit, we get $u(x,0) = u_0(x)$. This shows that u is a viscosity solution of problem (3.1)–(3.2) on $\mathbb{R} \times [0,+\infty)$, and ends the proof of Theorem 3.7. □

4. Uniqueness and stability of the self-similar solution

Lemma 4.1 (Comparison with the self-similar solution)**.** *Let v be a subsolution (resp. a supersolution) of equation (1.1) with the Heaviside initial datum given in (1.8). Then we have $v^* \le (u_\alpha)^*$ (resp. $(u_\alpha)_* \le v_*$).*

Proof. Using Remark 3.2 and properties of Φ_α gathered in Theorem 1.1, it is straightforward to check that the self-similar solution $u_\alpha(x,t)$ given in (1.6) is a viscosity solution of equation (3.1)–(3.2) with the initial condition (1.8).

Now, we show the inequality $(u_\alpha)_* \le v_*$. Let v be a viscosity supersolution of (3.1)–(3.2) with the Heaviside initial datum (1.8). Given $a > 0$ and $v^a(x,t) = v(a+x,t)$, we have

$$(u_\alpha)^*(x,0) \le (u_0)^*(x) \le (u_0)_*(a+x) \le v^a(x,0).$$

Because of the translation invariance of the equation (1.1), we see that v^a is still a supersolution. Moreover, for any $a > 0$, we can always find an initial condition $u_a \in BUC(\mathbb{R})$ such that

$$u_\alpha(x,0) \le u_a(x) \le v^a(x,0).$$

Therefore, applying the comparison principle (Theorem 3.3), we deduce that

$$u_\alpha \le v^a.$$

Because this is true for any $a > 0$, we can take the limit as $a \to 0$ and get $(u_\alpha)_* \le v_*$.

For a subsolution v, we proceed similarly to obtain $v^* \le (u_\alpha)^*$. This finishes the proof of Lemma 4.1. □

Proof of Theorem 1.4. We consider a viscosity solution v of equation (1.1) with the Heaviside initial datum (1.8). Using the both inequalities of Lemma 4.1, and the fact that $(u_\alpha)_* = (u_\alpha)^*$ on $(\mathbb{R} \times [0,+\infty)) \setminus \{(0,0)\}$, we deduce the equality $v = u_\alpha$ on $(\mathbb{R} \times [0,+\infty)) \setminus \{(0,0)\}$, which ends the proof of Theorem 1.4. □

We will now prove the following weaker version of Theorem 1.5.

Theorem 4.2 (Convergence for suitable initial data)**.** *The convergence (1.10) in Theorem 1.5 holds true under the following additional assumption*

$$\lim_{y\to-\infty} u_0(y) = 0 \le u_0(x) \le 1 = \lim_{y\to+\infty} u_0(y). \tag{4.1}$$

Proof. Step 1: Limits after rescaling of the solution. Consider a solution u of (1.1)–(1.2) with an initial condition u_0 satisfying (4.1). Recall that for any $\lambda > 0$, the rescaled solution is given by $u^\lambda(x,t) = u(\lambda x, \lambda^{\alpha+1}t)$. Let us define

$$\overline{u} = \limsup_{\lambda\to+\infty}{}^* u^\lambda \quad \text{and} \quad \underline{u} = \liminf_{\lambda\to+\infty}{}_* u^\lambda.$$

From the stability result (Theorem 3.4), we know that $\overline{u}$ (resp. $\underline{u}$) is a subsolution (resp. supersolution) of (1.1) on $\mathbb{R} \times (0,+\infty)$.

Step 2: The initial condition. We now want to prove that

$$\overline{u}(x,0) = \underline{u}(x,0) = H(x) \quad \text{for} \quad x \in \mathbb{R}\setminus\{0\}, \tag{4.2}$$

where H is the Heaviside function. To this end, we remark that u_0 satisfies for some $\gamma > 0$ the inequality $|u_0(x)| \le \gamma$ (note that $\gamma = 1$ under assumption (4.1)), and for each $\varepsilon > 0$, there exists $M > 0$ such that $|u_0(x)| < \varepsilon$ for $x \le -M$.

In particular, we get

$$u_0(x) < \varepsilon + \gamma H(x+M),$$

and then from the comparison principle, we deduce

$$u(x,t) \le \varepsilon + (u_\alpha^\gamma)_*(x+M,t) \tag{4.3}$$

with

$$u_\alpha^\gamma(x,t) = \Phi_\alpha^\gamma\left(\frac{x}{t^{1/(\alpha+1)}}\right) \quad \text{and} \quad \Phi_\alpha^\gamma(y) = \gamma\Phi_\alpha\left(\gamma^{-1/(\alpha+1)}y\right). \tag{4.4}$$

Here Φ_α^γ is the self-similar profile solution of (1.7) with the boundary conditions 0 and γ at infinity. Moreover, because u_α^γ is continuous off the origin, we can simply drop the star $*$, while we are interested in points different from the origin. This implies

$$u^\lambda(x,t) \le \varepsilon + \Phi_\alpha^\gamma\left(\frac{x+M\lambda^{-1}}{t^{1/(\alpha+1)}}\right),$$

and then

$$\overline{u}(x,t) \le \varepsilon + \Phi_\alpha^\gamma\left(\frac{x}{t^{1/(\alpha+1)}}\right).$$

Therefore, for every $x<0$ we have

$$\overline{u}(x,0) \le \varepsilon + \Phi_\alpha^\gamma(-\infty) = \varepsilon.$$

Because this is true for every $\varepsilon>0$, we get $\overline{u}(x,0)\le 0 \quad \text{for every} \quad x<0$. We get the other inequalities similarly, and finally conclude that (4.2) is valid.

Step 3: Initial condition at the origin, using assumption (4.1). We now make use of (4.1) to identify the initial values of the limits $\overline{u}$ and $\underline{u}$. We deduce from the comparison principle that

$$0 \le \underline{u}(x,0) \le \overline{u}(x,0) \le 1,$$

and then for every $x\in\mathbb{R}$ we have

$$\overline{u}(x,0) \le H^*(x) \quad \text{and} \quad \underline{u}(x,0) \ge H_*(x).$$

Step 4: Identification of the limits after rescaling. From Lemma 4.1, we obtain

$$\overline{u} \le (u_\alpha)^* = (u_\alpha)_* \le \underline{u} \quad \text{on} \quad (\mathbb{R}\times[0,+\infty))\setminus\{(0,0)\}.$$

We have by the construction $\underline{u}\le\overline{u}$, hence we infer

$$\overline{u} = \underline{u} = u_\alpha \quad \text{on} \quad (\mathbb{R}\times[0,+\infty))\setminus\{(0,0)\}.$$

Step 5: Conclusion for the convergence. Then for any compact $K\subset(\mathbb{R}\times[0,+\infty))\setminus\{(0,0)\}$, we can easily deduce that

$$\sup_{(x,t)\in K}|u^\lambda(x,t)-u_\alpha(x,t)| \to 0 \quad \text{as} \quad \lambda\to+\infty,$$

which finishes the proof of Theorem 4.2. □

5. Further decay properties and end of the proof of Theorem 1.5

Theorem 5.1 (Decay of a solution with compact support)**.** *Let u be the solution to (1.1)–(1.2) with the initial datum $u_0\in BUC(\mathbb{R})$ satisfying for some $A>0$*

$$u_0(x)\le 0 \quad \text{for} \quad |x|\ge A. \tag{5.1}$$

Let also $\gamma > 0$ be such that $u_0(x) \le \gamma$ for all $x \in \mathbb{R}$. Then, there exist $\beta, \beta' > 0$ (depending on α, but independent of A, γ) such that

$$u(x,t) \le Ct^{-\beta},$$

and

$$u(x,t) \le 0 \quad \text{for} \quad |x| \ge C' t^{\beta'}$$

with some constants $C = C(\alpha, A, \gamma)$ and $C' = C'(\alpha, A, \gamma)$.

First, we need the following

Lemma 5.2 (Decay after the first interaction)**.** *Consider Φ_α and y_α defined in Theorem 1.1. Let $\nu \in (1/2, 1)$ and $\xi_\nu \in (0, y_\alpha)$ be such that $\Phi_\alpha(\xi_\nu) = \nu$. Let $T > 0$ be defined by*

$$\frac{A}{\gamma^{1/(\alpha+1)} T^{1/(\alpha+1)}} = \xi_\nu. \tag{5.2}$$

Then, under the assumptions of Theorem 5.1, we have

$$u(x,t) \le \nu\gamma \quad \text{for all} \quad t \ge T, \quad x \in \mathbb{R}, \tag{5.3}$$

and

$$u(x,t) \le 0 \quad \text{for all} \quad 0 \le t \le T \quad \text{and} \quad |x| \ge A\left(1 + \frac{y_\alpha}{\xi_\nu}\right). \tag{5.4}$$

Proof. Let us denote $\Phi_\alpha^\gamma(y) = \gamma\Phi_\alpha\left(\gamma^{-1/(\alpha+1)} y\right)$. Then we have

$$\gamma H(x+A) \ge u_0(x) \quad \text{for} \quad x \in \mathbb{R},$$

where

$$\gamma H(x+A) = \lim_{t \to 0^+} \Phi_\alpha^\gamma\left(\frac{x+A}{t^{1/(\alpha+1)}}\right) \quad \text{for} \quad x + A \ne 0.$$

Now, we apply the comparison principle to deduce that

$$\Phi_\alpha^\gamma\left(\frac{x+A}{t^{1/(\alpha+1)}}\right) \ge u(x,t) \quad \text{for} \quad (x,t) \in \mathbb{R} \times (0, +\infty).$$

This argument can be made rigorous, simply, by replacing the function $\gamma H(x+A)$ by $\Phi_\alpha^\gamma\left((x+A+\delta)/(t_\varepsilon^{1/(\alpha+1)})\right)$ for $\delta > 0$ and some sequence $t_\varepsilon \to 0^+$, and then taking the limit $\delta \to 0^+$.

Therefore we have

$$\gamma\Phi_\alpha\left(\frac{x+A}{\gamma^{1/(\alpha+1)} t^{1/(\alpha+1)}}\right) \ge u(x,t) \quad \text{for} \quad (x,t) \in \mathbb{R} \times (0, +\infty).$$

From the properties of the support of Φ_α, we also deduce that

$$u(x,t) \le 0 \quad \text{for} \quad x \le -\left(A + y_\alpha(\gamma t)^{1/(\alpha+1)}\right),$$

and then, by symmetry,

$$u(x,t) \le 0 \quad \text{for} \quad |x| \ge A + y_\alpha(\gamma t)^{1/(\alpha+1)}.$$

Moreover, it follows from the monotonicity of Φ_α that

$$\gamma\Phi_\alpha\left(\frac{A}{(\gamma t)^{1/(\alpha+1)}}\right) \geq u(x,t)$$

for $x \leq 0$, and by symmetry we can prove the same property for $x \geq 0$. Then for $T > 0$ defined in (5.2) we easily deduce (5.3) and (5.4). This ends the proof of Lemma (5.2). □

Proof of Theorem 5.1. We apply recurrently Lemma 5.2. Define $A_0 = A$, $\gamma_0 = \gamma$, and

$$A_{n+1} = A_n\left(1 + \frac{y_\alpha}{\xi_\nu}\right), \quad \gamma_{n+1} = \nu\gamma_n, \quad \text{and} \quad \frac{A_n}{(\gamma_n T_n)^{1/(\alpha+1)}} = \xi_\nu.$$

This gives

$$A_n = A_0\left(1 + \frac{y_\alpha}{\xi_\nu}\right)^n, \quad \gamma_n = \nu^n\gamma_0, \quad T_n = K\mu^n,$$

with

$$K = \frac{1}{\gamma_0}\left(\frac{A_0}{\xi_\nu}\right)^{\alpha+1}, \quad 1 < \mu = \frac{1}{\nu}\left(1 + \frac{y_\alpha}{\xi_\nu}\right)^{\alpha+1},$$

and therefore

$$u(x,t) \leq \gamma_n \quad \text{for} \quad t \geq T_0 + ... + T_{n-1} = K\frac{\mu^n - 1}{\mu - 1}.$$

In particular, we get for any $n \in \mathbb{N}$

$$u(x,t) \leq \gamma_0\nu^n \quad \text{for} \quad t \geq K_0\mu^n$$

with $K_0 = K/(\mu - 1)$. This implies

$$u(x,t) \leq \gamma_0 K_0^\beta t^{-\beta} \quad \text{for any} \quad t > 0, \quad x \in \mathbb{R},$$

with

$$\beta = -\frac{\ln\nu}{\ln\mu} > 0.$$

Similarly, we have

$$u(x,t) \leq 0 \quad \text{for} \quad |x| \geq A_n \quad \text{if} \quad t \leq T_0 + ... + T_{n-1} = K\frac{\mu^n - 1}{\mu - 1}.$$

In particular, we get for any $n \in \mathbb{N}\setminus\{0\}$

$$u(x,t) \leq 0 \quad \text{for} \quad |x| \geq A_0\left(1 + \frac{y_\alpha}{\xi_\nu}\right)^n, \quad \text{if} \quad t \leq K_0'\mu^n$$

with $K_0' = K/\mu$. This implies

$$u(x,t) \leq 0 \quad \text{for} \quad |x| \geq A_0(K_0')^{-\beta'}t^{\beta'} \quad \text{for} \quad t \geq 0,$$

with

$$\beta' = \frac{\ln\left(1 + \frac{y_\alpha}{\xi_\nu}\right)}{\ln\mu} > 0.$$

This ends the proof of Theorem 5.1. □

As a corollary, we can now remove assumption (4.1) in Theorem 4.2 and complete the proof of Theorem 1.5.

Proof of Theorem 1.5. We simply repeat Step 3 of the proof of Theorem 4.2, but here without assuming (4.1). Then, for any $\varepsilon > 0$ there exists $A > 0$ such that

$$u_0(x) \le 1 + \varepsilon \quad \text{for} \quad |x| \ge A.$$

By Theorem 5.1 applied to the solution $u(x,t) - 1 - \varepsilon$, this implies that there exists a constant $C > 0$ (depending on ε) such that

$$u(x,t) \le 1 + \varepsilon + Ct^{-\beta}.$$

Therefore, for any for $\lambda > 0$ the following inequality

$$u^\lambda(x,t) \le 1 + \varepsilon + Ct^{-\beta}\lambda^{-\beta}$$

holds true, which implies that $\overline{u} = \limsup_{\lambda \to +\infty}{}^* u^\lambda$ satisfies

$$\overline{u}(x,t) \le 1 + \varepsilon \quad \text{for} \quad (x,t) \in \mathbb{R} \times (0,+\infty).$$

Since this is true for any $\varepsilon > 0$, we deduce that

$$\overline{u}(x,t) \le 1 \quad \text{for} \quad (x,t) \in \mathbb{R} \times (0,+\infty).$$

Let us now define $\tilde{\overline{u}} = \min(1, \overline{u})$. By the construction,

$$\tilde{\overline{u}}(x,t) = \overline{u}(x,t) \quad \text{for} \quad (x,t) \in \mathbb{R} \times [0,+\infty) \setminus \{(0,0)\},$$

and, by (4.2), we have $\tilde{\overline{u}}(x,0) \le H^*(x)$ for all $x \in \mathbb{R}$. Therefore, $\tilde{\overline{u}}$ is a subsolution of (1.1)–(1.2) on $\mathbb{R} \times [0,+\infty)$ with the initial datum being the Heaviside function.

Similarly, we can show that $\underline{u} = \limsup_{\lambda \to +\infty}{}_* u^\lambda$ satisfies

$$\underline{u} \ge 0 \quad \text{for} \quad (x,t) \in \mathbb{R} \times (0,+\infty).$$

Hence, the function $\tilde{\underline{u}} = \max(0, \underline{u})$, which is a supersolution of (1.1)–(1.2) on $\mathbb{R} \times [0,+\infty)$ with the Heaviside initial datum.

Finally, the conclusion of the proof is the same as in the proof of Theorem 4.2 where $\overline{u}$ (resp. $\underline{u}$) is replaced by $\tilde{\overline{u}}$ (resp. $\tilde{\underline{u}}$). This ends the proof of Theorem 1.5. □

6. Approximate equation and gradient estimates

In this section, in order to prove our gradient estimates of viscosity solutions stated in Theorem 1.6, we replace equation (1.1) by an approximate equation for which smooth solutions do exist. Indeed, with $\varepsilon > 0$, we consider the following initial value problem

$$u_t = \varepsilon u_{xx} - |u_x|\Lambda^\alpha u \quad \text{on} \quad \mathbb{R} \times (0,+\infty), \tag{6.1}$$

$$u(x,0) = u_0(x) \quad \text{for} \quad x \in \mathbb{R}. \tag{6.2}$$

We have added to this equation an auxiliary viscosity term which is stronger than $\Lambda^\alpha u$ and u_x. In the case $\alpha \in (0,1]$, we will see later (in Section 7) that it is possible to pass to the limit $\varepsilon \to 0^+$ in $L^\infty(\mathbb{R})$, which is the required convergence for the framework of viscosity solutions. The difficulty in the case $\alpha \in (1,2)$ comes from the fact that, for the limit equation with $\varepsilon = 0$, we are not able to give a meaning to the product $|u_x|\ (\Lambda^\alpha u)$ in the sense of distributions, while it is possible when $\alpha \in (0,1]$.

Our results on qualitative properties of solutions to the regularized problem (6.1)–(6.2) are stated in the following two theorems.

Theorem 6.1 (Approximate equation – existence of solutions)**.** *Let* $\alpha \in (0,1]$ *and* $\varepsilon > 0$*. Given any initial datum* $u_0 \in C^2(\mathbb{R})$ *such that* $u_{0,x} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$*, there exists a unique solution* $u \in C(\mathbb{R} \times [0,+\infty)) \cap C^{2,1}(\mathbb{R} \times (0,+\infty))$ *of (6.1)–(6.2). This solution satisfies*

$$u_x \in C([0,T], L^p(\mathbb{R})) \cap C((0,T];\ W^{1,p}(\mathbb{R})) \cap C^1((0,T], L^p(\mathbb{R})) \tag{6.3}$$

for every $p \in (1,\infty)$ *and each* $T > 0$*.*

Theorem 6.2 (Approximate equation – decay estimates)**.** *Under the assumptions of Theorem 6.1, the solution* $u = u(x,t)$ *of (6.1)–(6.2) satisfies*

$$\|u(\cdot,t)\|_\infty \le \|u_0\|_\infty, \qquad \|u_x(\cdot,t)\|_\infty \le \|u_{0,x}\|_\infty, \tag{6.4}$$

and

$$\|u_x(\cdot,t)\|_p \le C_{p,\alpha} \|u_{0,x}\|_1^{\frac{p\alpha+1}{p(\alpha+1)}} t^{-\frac{1}{\alpha+1}\left(1-\frac{1}{p}\right)}, \tag{6.5}$$

for every $p \in [1,\infty)$*, all* $t > 0$*, and constants* $C_{p,\alpha} > 0$ *(see, (6.20) below), independent of* $\varepsilon > 0$*,* $t > 0$ *and* u_0*.*

6.1. **Existence theory.**

Proof of Theorem 6.1. Note first that

$$\Lambda^\alpha u = -\Lambda^{\alpha-1} \mathcal{H} u_x, \tag{6.6}$$

where $\mathcal{H}$ denotes the Hilbert transform defined in the Fourier variables by $\widehat{(\mathcal{H}v)}(\xi) = i\ \mathrm{sgn}(\xi)\ \widehat{v}(\xi)$. We recall that the Hilbert transform is bounded on the L^p-space for any $p \in (1,+\infty)$ (see [21, Ch. 2, Th. 1]), i.e. it satisfies for any function $v \in L^p(\mathbb{R})$ the following inequality

$$\|\mathcal{H}v\|_p \le C_p \|v\|_p \tag{6.7}$$

with a constant C_p independent of v.

For $\alpha \in (0,1)$, the operator $\Lambda^{\alpha-1}$ defined analogously as in (1.3) corresponds to the convolution with the Riesz potential $\Lambda^{\alpha-1}v = C_\alpha|\cdot|^{-\alpha} * v$. Hence, by [21, Ch. 5, Th. 1], for any $p > 1/\alpha$ with $\alpha \in (0,1]$ and any function $v \in L^q(\mathbb{R})$, we have

$$\|\Lambda^{\alpha-1}v\|_p \le C_{p,\alpha}\|v\|_q \quad \text{with} \quad \frac{1}{q} = \frac{1}{p} + 1 - \alpha. \tag{6.8}$$

Now, if $u = u(x,t)$ is a solution to (6.1)–(6.2), using identity (6.6), we write the initial value problem for $v = u_x$

$$v_t = \varepsilon v_{xx} + (|v|\Lambda^{\alpha-1}\mathcal{H}v)_x \quad \text{on} \quad \mathbb{R}\times(0,+\infty), \tag{6.9}$$

$$v(\cdot,0) = v_0 = u_{0,x} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \tag{6.10}$$

as well as its equivalent integral formulation

$$v(t) = G(\varepsilon t) * v_0 + \int_0^t \partial_x G(\varepsilon(t-\tau)) * (|v|\Lambda^{\alpha-1}\mathcal{H}v)\,\mathrm{d}\tau, \tag{6.11}$$

with the Gauss-Weierstrass kernel $G(x,t) = (4\pi t)^{-1/2}\exp(-x^2/(4t))$.

The next step is completely standard and consists in applying the Banach contraction principle to equation (6.11) in a ball in the Banach space

$$\mathcal{X}_T = C([0,T]; L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))$$

endowed with the usual norm $\|v\|_T = \sup_{t\in[0,T]}(\|v(t)\|_1 + \|v(t)\|_\infty)$. Using well known estimates of the heat semigroup and inequalities (6.7)–(6.8) combined with the imbedding $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \subset L^p(\mathbb{R})$ for each $p \in [1,\infty]$, we obtain a solution $v = v(x,t)$ to equation (6.11) in the space $\mathcal{X}_T$ provided $T > 0$ is sufficiently small. We refer the reader to, e.g., [1, 4] for examples of such a reasoning.

This solution satisfies (6.3) for every $p \in (1,\infty)$ and each $T > 0$, by standard regularity estimates of solutions to parabolic equations. Moreover, following the reasoning from [1], one can show that the solution is regular.

Finally, this local-in-time solution can be extended to global-in-time (i.e. for all $T > 0$) because of the estimates $\|v(t)\|_p \le \|v_0\|_p$ for every $p \in [1,\infty]$ being the immediate consequence of inequalities (6.17), (6.18), and (6.21) below. □

6.2. **Gradient estimates.** In the proof of the decay estimates of u_x, we shall require several properties of the operator Λ^α. First, we recall the Nash inequality for the operator Λ^α.

Lemma 6.3 (Nash inequality)**.** *Let $0 < \alpha$. There exists a constant $C_N > 0$ such that*

$$\|w\|_2^{2(1+\alpha)} \le C_N\|\Lambda^{\alpha/2}w\|_2^2\|w\|_1^{2\alpha} \tag{6.12}$$

for all functions w satisfying $w \in L^1(\mathbb{R})$ and $\Lambda^{\alpha/2} w \in L^2(\mathbb{R})$.

The proof of inequality (6.12) is given, e.g., in [17, Lemma 2.2].

Our next tool is the, so called, Stroock–Varopoulos inequality.

Lemma 6.4 (Stroock–Varopoulos inequality). *Let $0 \le \alpha \le 2$. For every $p > 1$, we have*

$$\int_{\mathbb{R}} (\Lambda^\alpha w)|w|^{p-2} w \, \mathrm{d}x \ge \frac{4(p-1)}{p^2} \int_{\mathbb{R}} \left(\Lambda^{\frac{\alpha}{2}} |w|^{\frac{p}{2}} \right)^2 \mathrm{d}x \tag{6.13}$$

for all $w \in L^p(\mathbb{R})$ such that $\Lambda^\alpha w \in L^p(\mathbb{R})$. If $\Lambda^\alpha w \in L^1(\mathbb{R})$, we obtain

$$\int_{\mathbb{R}} (\Lambda^\alpha w) \operatorname{sgn} w \, \mathrm{d}x \ge 0. \tag{6.14}$$

Moreover, if $w, \Lambda^\alpha w \in L^2(\mathbb{R})$, it follows that

$$\int_{\mathbb{R}} (\Lambda^\alpha w) w^+ \, \mathrm{d}x \ge 0 \quad \text{and} \quad \int_{\mathbb{R}} (\Lambda^\alpha w) w^- \, \mathrm{d}x \ge 0, \tag{6.15}$$

where $w^+ = \max\{0, w\}$ and $w^- = \max\{0, -w\}$.

Inequality (6.13) is well known in the theory of sub-Markovian operators and its statement and the proof is given, e.g., in [18, Th. 2.1 combined with the Beurling–Deny condition (1.7)]. Inequality (6.14), called the (generalized) Kato inequality, is used, e.g., in [6] to construct entropy solutions of conservation laws with a Lévy diffusion. It can be easily deduced from [6, Lemma 1] by an approximation argument. The proof of (6.15) can be found, for example, in [18, Prop. 1.6].

Remark 6.5. Remark that inequality (6.14) appears to be a limit case of (6.13) for $p = 1$. Inequality (6.15) for w^+ follows easily from (6.14), by a comparison argument, if for instance $w \in C_c^\infty(\mathbb{R})$. Finally, remark that the constant appearing in (6.13) is the same as for the Laplace operator $\partial^2/\partial x^2 = -\Lambda^2$.

Our proof of the decay of $v(t) = u_x(t)$ is based on the following Gagliardo–Nirenberg type inequality

Lemma 6.6 (Gagliardo–Nirenberg type inequality). *Assume that $p \in (1, \infty)$ and $\alpha > 0$ are fixed and arbitrary. For all $v \in L^1(\mathbb{R})$ such that $\Lambda^{\alpha/2}|v|^{(p+1)/2} \in L^2(\mathbb{R})$, the following inequality is valid*

$$\|v\|_p^a \le C_N \left\| \Lambda^{\alpha/2} |v|^{(p+1)/2} \right\|_2^2 \|v\|_1^b, \tag{6.16}$$

where

$$a = \frac{p(p+\alpha)}{p-1}, \quad b = \frac{p\alpha + 1}{p-1},$$

and C_N is the constant from the Nash inequality (6.12).

Proof. Without loss of generality, we can assume that $\|v\|_1 \neq 0$. Substituting $w = |v|^{(p+1)/2}$ in the Nash inequality (6.12) we obtain

$$\|v\|_{p+1}^{(p+1)(1+\alpha)} \le C_N \left\| \Lambda^{\alpha/2} |v|^{(p+1)/2} \right\|_2^2 \|v\|_{(p+1)/2}^{\alpha(p+1)}.$$

Next, it suffices to apply two particular cases of the Hölder inequality

$$\left(\frac{\|v\|_p}{\|v\|_1^{1/p^2}} \right)^{p^2/(p^2-1)} \le \|v\|_{p+1} \quad \text{as well as} \quad \|v\|_{(p+1)/2} \le \|v\|_p^{p/(p+1)} \|v\|_1^{1/(p+1)},$$

and compute carefully all the exponents which appear on the both sides of the resulting inequality. □

Proof of Theorem 6.2. The first inequality in (6.4) is the immediate consequence of the comparison principle from Theorem 3.3, because classical solutions are viscosity solutions, as well. Maximum principle and an argument based on inequalities (6.15) (cf. [17] for more detail) lead to the second inequality in (6.4). We also discuss this inequality in Remark 6.7 below.

For the proof of the L^1-estimate

$$\|u_x(t)\|_1 \le \|u_{0,x}\|_1 \tag{6.17}$$

(i.e. (6.5) with $p = 1$ and $C_{p,\alpha} = 1$), we multiply equation (6.9) by $\operatorname{sgn} v = \operatorname{sgn} u_x$ and we integrate with respect to x to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} |v| \,\mathrm{d}x = \varepsilon \int_{\mathbb{R}} v_{xx} \operatorname{sgn} v \,\mathrm{d}x + \int_{\mathbb{R}} \big((\Lambda^{\alpha-1}\mathcal{H}v)|v| \big)_x \operatorname{sgn} v \,\mathrm{d}x.$$

The first term on the right hand side is nonpositive by the Kato inequality (i.e. (6.14) with $\alpha = 2$) hence we skip it. Remark that (formally)

$$\begin{aligned} \int_{\mathbb{R}} \big((\Lambda^{\alpha-1}\mathcal{H}v)|v| \big)_x \operatorname{sgn} v \,\mathrm{d}x &= \int_{\mathbb{R}} (\Lambda^{\alpha-1}\mathcal{H}v) v_x (\operatorname{sgn} v)^2 + (\Lambda^{\alpha-1}\mathcal{H}v_x) v \,\mathrm{d}x \\ &= \int_{\mathbb{R}} \big((\Lambda^{\alpha-1}\mathcal{H}v) v \big)_x \,\mathrm{d}x = 0. \end{aligned}$$

Now, approximating the sign function in a standard way by $\operatorname{sgn}_\delta(z) = z/\sqrt{z^2+\delta}$, integrating by parts, and passing to the limit $\delta \to 0^+$, one can show rigorously that the second term on right hand side of the above inequality is nonpositive. This completes the proof of (6.17) with $p = 1$.

Next, we multiply equation in (6.9) by $|v|^{p-2}v$ with $p > 1$ to get

$$\frac{1}{p} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} |v|^p \,\mathrm{d}x = \varepsilon \int_{\mathbb{R}} v_{xx} |v|^{p-2} v \,\mathrm{d}x + \int_{\mathbb{R}} \big((\Lambda^{\alpha-1}\mathcal{H}v)|v| \big)_x |v|^{p-2} v \,\mathrm{d}x.$$

We drop the first term on the right hand side, because it is nonpositive by (6.13) with $\alpha = 2$. Integrating by parts and using the elementary identity

$$|v| \left(|v|^{p-2} v \right)_x = \frac{p-1}{p} \left(|v|^{p-1} v \right)_x,$$

we transform the second quantity on the right hand side as follows

$$\int_{\mathbb{R}} \left((\Lambda^{\alpha-1}\mathcal{H}v)|v|\right)_x |v|^{p-2}v \,\mathrm{d}x = -\frac{p-1}{p}\int_{\mathbb{R}} (\Lambda^\alpha v)|v|^{p-1}v \,\mathrm{d}x.$$

Consequently, by the Stroock–Varopoulos inequality (6.13) (with the exponent p replaced by $p+1$), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|_p^p \le -\frac{4p(p-1)}{(p+1)^2}\left\|\Lambda^{\alpha/2}|v|^{(p+1)/2}\right\|_2^2. \tag{6.18}$$

Hence, the interpolation inequality (6.16) combined with (6.17) lead to the following inequality for $\|v(t)\|_p^p$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|_p^p \le -\frac{4p(p-1)}{(p+1)^2}\left(C_N\|v_0\|_1^{(p\alpha+1)/(p-1)}\right)^{-1}\left(\|v(t)\|_p^p\right)^{(p+\alpha)/(p-1)}. \tag{6.19}$$

Recall now that if a nonnegative (sufficiently smooth function) $f=f(t)$ satisfies, for all $t>0$, the inequality $f'(t)\le -Kf(t)^\beta$ with constants $K>0$ and $\beta>1$, then $f(t)\le (K(\beta-1)t)^{-1/(\beta-1)}$. Applying this simple result to the differential inequality (6.19), we complete the proof of the L^p-decay estimate (6.5) with the constant

$$C_{p,\alpha} = \left(C_N^{-1}\frac{4p(\alpha+1)}{(p+1)^2}\right)^{-\frac{1}{\alpha+1}\left(1-\frac{1}{p}\right)}, \tag{6.20}$$

where C_N is the constant from the Nash inequality (6.12). □

Remark 6.7. Note that, for every fixed α, we have $\lim_{p\to\infty} C_{p,\alpha} = +\infty$. By this reason, we are not allowed to pass directly to the limit $p\to+\infty$ in inequalities (6.5) (as was done in, e.g., [17, Th. 2.3]) in order to obtain a decay estimate of $v(t)$ in the L^∞-norm. Nevertheless, using (6.19) we immediately deduce the inequality $\|v(\cdot,t)\|_p \le \|v_0(\cdot)\|_p$ valid for every $p\in(1,\infty)$. Hence, passing to the limit $p\to+\infty$ we get

$$\|v(\cdot,t)\|_\infty \le \|v_0(\cdot)\|_\infty. \tag{6.21}$$

In general, we cannot hope to get a decay estimate of $\|v(\cdot,t)\|_\infty$ better than that in (6.21), because each constant is a solution of equation (1.1). Moreover, one can show that if v is constant on an interval at the initial time, then it will stay equal to the same constant on an interval depending on t, because of the finite propagation phenomenon that can be seen for the self-similar profile.

7. Passage to the limit and proof of Theorem 1.6

Now, we are in a position to complete the proof of the gradient estimates (1.11). First, we show that form the sequence $\{u^\varepsilon\}_{\varepsilon>0}$ of solutions to the approximate problem (6.1)–(6.2) one can extract, *via* the Ascoli–Arzelà theorem, a subsequence converging uniformly. Theorem 3.4 on the stability and Remark 3.5 imply that the limit function is a viscosity solution to (1.1)–(1.2). Passing to the limit $\varepsilon \to 0^+$ in inequalities (6.4) and (6.5) we complete our reasoning.

Proof of Theorem 1.6. First, let us suppose that $u_0 \in C^\infty(\mathbb{R}) \cap W^{2,\infty}(\mathbb{R})$ with $u_{0,x} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Denote by $u^\varepsilon = u^\varepsilon(x,t)$ the corresponding solution to the approximate problem with $\varepsilon > 0$.

Step 1: Modulus of continuity in space. Under this additional assumption, we have

$$\|u^\varepsilon_x(\cdot,t)\|_p \le C_p t^{-\gamma_p} \tag{7.1}$$

with $C_p = C_{p,\alpha}\|u_{0,x}\|_1^{\frac{p\alpha+1}{p(\alpha+1)}}$ and $\gamma_p = \frac{1}{\alpha+1}\left(1-\frac{1}{p}\right)$. The Sobolev imbedding theorem implies that there exist some $\beta \in (0,1)$ and $C_0 > 0$ such that

$$|u^\varepsilon(x+h,t) - u^\varepsilon(x,t)| \le |h|^\beta C_0 C_p t^{-\gamma_p}. \tag{7.2}$$

Step 2: Modulus of continuity in time. Let us consider a nonnegative function $\varphi \in C^\infty(\mathbb{R})$ with $\operatorname{supp}\varphi \subset [-1,1]$ such that $\int_{\mathbb{R}} \varphi(x)\,\mathrm{d}x = 1$, and for any $\delta > 0$ set $\varphi_\delta(x) = \delta^{-1}\varphi(\delta^{-1}x)$. Then, multiplying (6.1) by φ_δ and integrating in space, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}} u^\varepsilon(x,t)\varphi_\delta(x)\,\mathrm{d}x\right) &= \varepsilon\langle u^\varepsilon(\cdot,t), (\varphi_\delta)_{xx}\rangle \\ &\quad - \int_{\mathbb{R}} \varphi_\delta(x)\ |u^\varepsilon_x(x,t)|(\mathcal{H}\ \Lambda^{\alpha-1}\ u^\varepsilon_x(x,t))\,\mathrm{d}x,\end{aligned}$$

and then with $1/p + 1/p' = 1$

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}} u^\varepsilon(x,t)\varphi_\delta(x)\,\mathrm{d}x\right)\right| &\le \varepsilon\|u^\varepsilon(\cdot,t)\|_\infty\|(\varphi_\delta)_{xx}\|_1 \\ &\quad + \|\varphi_\delta\|_\infty\|u^\varepsilon_x(\cdot,t)\|_p\|\|\mathcal{H}\ \Lambda^{\alpha-1}\ u^\varepsilon_x(\cdot,t)\|_{p'}.\end{aligned} \tag{7.3}$$

Here, we have used relation (6.6). Combining inequalities (6.7) and (6.8) with estimate (7.1), we get for $p' > 1/\alpha$

$$\|\mathcal{H}\ \Lambda^{\alpha-1}\ u^\varepsilon_x(\cdot,t)\|_{p'} \le C_{p'}C_{p',\alpha}C_q t^{-\gamma_q}.$$

Then for any bounded time interval $I \subset (0,+\infty)$ there exists a constant $C_{I,\delta}$ such that for all $t \in I$, we have for any $\varepsilon \in (0,1]$

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}} u^\varepsilon(x,t)\varphi_\delta(x)\,\mathrm{d}x\right)\right| \le C_{I,\delta}.$$

Now, for any $t, t+s \in I$, we get

$$\left|\int_{\mathbb{R}} u^\varepsilon(x,t+s)\varphi_\delta(x)\,\mathrm{d}x - \int_{\mathbb{R}} u^\varepsilon(x,t)\varphi_\delta(x)\,\mathrm{d}x\right| \le |s|C_{I,\delta}.$$

Therefore, the following estimate

$$|u^\varepsilon(0,t+s)-u^\varepsilon(0,t)| \le |s|C_{I,\delta} + \int_{\mathbb{R}} \varphi_\delta(x)\,\mathrm{d}x \cdot \sup_{x\in[-\delta,\delta]} \big(|u^\varepsilon(x,t+s)-u^\varepsilon(0,t+s)| + |u^\varepsilon(x,t)-u^\varepsilon(0,t)|\big)$$

holds true. Using the Hölder estimate (7.2), we deduce that there exists a constant C_I depending on I, but independent of δ and of $\varepsilon \in (0,1]$, such that

$$|u^\varepsilon(0,t+s)-u^\varepsilon(0,t)| \le |s|C_{I,\delta} + C_I\delta^\beta.$$

Since the above inequality is true for any δ, this shows the existence of a modulus of continuity ω_I satisfying

$$|u^\varepsilon(0,t+s)-u^\varepsilon(0,t)| \le \omega_I(|s|) \quad \text{for any} \quad t, t+s \in I.$$

By the translation invariance of the problem, this estimate is indeed true for any $x \in \mathbb{R}$, i.e.

$$|u^\varepsilon(x,t+s)-u^\varepsilon(x,t)| \le \omega_I(|s|) \quad \text{for any} \quad t, t+s \in I, \quad x \in \mathbb{R}. \tag{7.4}$$

Step 3: Convergence as $\varepsilon \to 0^+$. From estimates (7.2) and (7.4), and using the Ascoli–Arzelà theorem and the Cantor diagonal argument, we deduce that there exists a subsequence (still denoted $\{u^\varepsilon\}_\varepsilon$) which converges to a limit $u \in C(\mathbb{R}\times(0,+\infty))$. By the stability result in Theorem 3.4 (see also Remark 3.5), we have that u is a viscosity solution of (1.1) on $\mathbb{R}\times(0,+\infty)$.

Step 4: Checking the initial conditions for u_0 smooth. Remark that for $u_0 \in W^{2,\infty}$ we can use the barriers given in (3.4) with some constant $C>0$ uniform in $\varepsilon \in (0,1]$. This ensures that u is continuous up to $t=0$ and satisfies $u(\cdot,0)=u_0$, so this proves the result under additional assumptions.

Step 5: General case. The proof in the case of less regular initial conditions simply follows by an approximation argument as was in the proof of Theorem 3.7.

Step 6: Gradient estimates. To pass to the limit $\varepsilon \to 0^+$ in estimates (6.5), we use the inequality

$$h^{-1}\|u^\varepsilon(\cdot+h,t)-u^\varepsilon(\cdot,t)\|_p \le \|u^\varepsilon_x(\cdot,t)\|_p \tag{7.5}$$

with fixed $h>0$. Hence, by the Fatou lemma combined with the pointwise convergence of u^ε toward u, we deduce from (7.5) and (6.5) that

$$h^{-1}\|u(\cdot+h,t)-u(\cdot,t)\|_p \le C_{p,\alpha}\|u_{0,x}\|_1^{\frac{p\alpha+1}{p(\alpha+1)}} t^{-\frac{1}{\alpha+1}\left(1-\frac{1}{p}\right)}$$

for all $h>0$. For every fixed $t>0$, the sequence $\{h^{-1}(u(\cdot+h,t)-u(\cdot,t))\}_{h>0}$ is bounded in $L^p(\mathbb{R})$ and converges (up to a subsequence) weakly in $L^p(\mathbb{R})$ toward

$u_x(\cdot,t)$ (see, e.g., [21, Ch. V, Prop. 3]). Using the well-known property of a weak convergence in Banach spaces we conclude

$$\|u_x(\cdot,t)\|_p \le \liminf_{h\to 0^+} h^{-1}\|u(\cdot+h,t)-u(\cdot,t)\|_p \le C_{p,\alpha}\|u_{0,x}\|_1^{\frac{p\alpha+1}{p(\alpha+1)}} t^{-\frac{1}{\alpha+1}\left(1-\frac{1}{p}\right)}.$$

This finishes the proof of Theorem 1.6. □

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