A non local free boundary problem arising in a theory of financial bubbles

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Abstract

We consider an evolution non local free boundary problem that arises in the modeling of speculative bubbles. The solution of the model is the speculative component in the price of an asset. In the framework of viscosity solutions, we show the existence and uniqueness of the solution. We also show the convexity in space of the solution, and various monotonicity properties of the solution and of the free boundary with respect to parameters of the problem. In order to study the free boundary, we use in particular the fact that the odd part of the solution solves a more standard obstacle problem. We show moreover that the free boundary is Lipschitz continuous, and describe the asymptotics of the free boundary as c, the cost of transacting the asset, goes to zero.

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1 Introduction

1.1 Main results

The goal of this paper is to study an evolution model where financial bubbles appear as a result of differences in beliefs among traders in a financial market. This model is briefly presented in Subsection 1.3, and a more precise derivation can be found in [4]. The stationary version of this model (i.e. for infinite horizon) was introduced and solved by Scheinkman and Xiong [18]. The article [18] uses a different approach from the one presented here and provides an explicit stationary solution, based on Kummer functions. Chen and Kohn [8, 9] study a stationary model that is related to the one in [18], and construct an explicit solution in terms of Weber-Hermite functions. A natural motivation for the evolution problem treated in the present paper is that a finite horizon model is necessary to deal with finite-horizon

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assets, such as many fixed-income securities. As will be seen, it leads to more involved mathematical problems.

In what follows we let

$$r, c > 0 \quad \text{and} \quad \sigma, \rho, \lambda \geq 0 \tag{1.1}$$

denote given constants. The parameter σ is the volatility, r is the rate of interest, c represents the transaction cost, and ρ and λ are relaxation parameters. We define the following parabolic operator (possibly degenerate when $\sigma = 0$)

$$\mathcal{L}u = u_t + \mathcal{M}u \quad \text{with} \quad \mathcal{M}u = -\frac{1}{2}\sigma^2 u_{xx} + \rho x u_x + ru$$

and the obstacle

$$\psi(x,t) = x\alpha(t) - c \quad \text{with} \quad \alpha(t) = \frac{1 - e^{-(r+\lambda)t}}{r + \lambda}.$$

We consider the following (non local) obstacle problem:

$$\begin{cases} \min\left(\mathcal{L}u, \quad u(x,t) - u(-x,t) - \psi(x,t)\right) = 0 & \text{for} \quad (x,t) \in \mathbb{R} \times (0, +\infty) \\ u(x,0) = 0 & \text{for} \quad x \in \mathbb{R}. \end{cases} \tag{1.2}$$

In the economic interpretation, the (indeed non negative) quantity u can be seen as the speculative component of the price of an asset, due to disagreement among investors. The larger is u, the larger is the financial bubble.

We also introduce the stationary problem (formally for $t = +\infty$) with $\psi_\infty(x) := \psi(x, +\infty) = \dfrac{x}{r+\lambda} - c$:

$$\min\left(\mathcal{M}u_\infty, \quad u_\infty(x) - u_\infty(-x) - \psi_\infty(x)\right) = 0 \quad \text{for} \quad x \in \mathbb{R}. \tag{1.3}$$

This is the problem which has been studied in [18]. The present paper deals with resolution and qualitative properties of problems (1.2) and (1.3). We establish here rigorous results in the framework of viscosity solutions.

Our first main result is:

Theorem 1.1 (Existence and uniqueness of a solution) *Assume* (1.1).
i) Evolution equation. *There exists a unique viscosity solution u of* (1.2) *(in the sense of Definition* 2.1 *below) satisfying*

$$|u - \max(0, \psi)| \leq C \quad \text{on} \quad \mathbb{R} \times [0, +\infty).$$

ii) Stationary equation. *Moreover, there exists a unique viscosity solution u_∞ of* (1.3) *(in the sense of Definition* 2.3*) satisfying*

$$|u_\infty - \max(0, \psi_\infty)| \leq C \quad \text{on} \quad \mathbb{R}.$$

It is easy to see that if $\sigma = 0$, then $u = \max(0, \psi)$ and $u_\infty = \max(0, \psi_\infty)$. In the general case, we only have inequalities as in the next result. We also list a series of qualitative properties such as monotonicity, convexity, asymptotics, and large time behavior that are related to the economic motivation of the problem. A precise derivation of the model from assumptions on the behavior of investors, as well as a discussion of the economic significance of the qualitative properties will be provided in our forthcoming work [4]).

Theorem 1.2 (Properties of the solution)
Assume (1.1) *and let* u *be the solution given in Theorem* 1.1. *Then* u *is continuous. In addition,*
i) Asymptotics. *There exists a function* ϕ *such that*

$$\phi(y) \geq \max(0,y) \quad \textit{and} \quad \lim_{|y|\to+\infty} |\phi(y) - \max(0,y)| = 0$$

and such that

$$\text{(1.4)} \quad \max(0,\psi(x,t)) = \alpha(t)\max(0,x-d(t)) \leq u(x,t) \leq \alpha(t)\phi(x-d(t)), \ \textit{with}\ d(t) = \frac{c}{\alpha(t)}.$$

ii) Monotonicity and convexity: $u_t \geq 0,\ 0 \leq u_x \leq \alpha(t),\ u_{xx} \geq 0.$
iii) Convergence in long time: $u(x,t) \to u_\infty(x)$ *as* $t \to +\infty$ *locally uniformly in* x.
iv) Monotonicity with respect to the parameters r, c, λ, σ**.** *The following properties hold for* $r, c > 0,\ \sigma, \lambda \geq 0$*:* $\frac{\partial u}{\partial c} \leq 0,\ \frac{\partial u}{\partial r} \leq 0,\ \frac{\partial u}{\partial \lambda} \leq 0$ *and* $\frac{\partial u}{\partial \sigma} \geq 0$.
v) The limit $c \to 0$**:** *When* $c \to 0$, $u \to u_0$ *where* u_0 *is the minimal solution of* (1.2) *for* $c = 0$ *satisfying* $|u_0(x,t) - \max(0, x\alpha(t))| \leq C$ *on* $\mathbb{R} \times [0,+\infty)$*, for some constant* $C > 0$.
vi) The w**-problem.** *Set*

$$\text{(1.5)} \qquad w(x,t) = u(x,t) - u(-x,t).$$

Then in the viscosity sense (see Definition 2.4 *below)* w *solves:*

$$\text{(1.6)} \qquad \begin{cases} \min(\mathcal{L}w, \quad w - \psi) = 0 & \textit{for} \quad (x,t) \in (0,+\infty) \times (0,+\infty), \\ w(0,t) = 0 & \textit{for} \quad t \in [0,+\infty), \\ w(x,0) = 0 & \textit{for} \quad x \in [0,+\infty). \end{cases}$$

As we will see (Proposition 10.1), for $c = 0$ the solutions of (1.2) are not unique. This is why the limit u_0 of solutions u as $c \to 0$ is only characterized as the minimal solution.

We also show that w defined in (1.5) satisfies properties similar to those in Theorem 1.2. They will be stated in Section 7. Clearly, problem (1.6) is a free boundary problem where the exercise region is defined as the set $\{w = \psi\}$. We now make this precise and list some properties.

Theorem 1.3 (Properties of the free boundary)
Assume (1.1) *and let* u *be the solution given in Theorem* 1.1. *There exists a lower semi-continuous function* $a : (0,+\infty) \to [0,+\infty)$ *such that for all* $t > 0$*:*

$$\{x \in [0,+\infty), \quad u(x,t) - u(-x,t) = \psi(x,t)\} = \{x \geq a(t)\}.$$

The following properties hold:
i) Bounds on the free boundary. *For* $\sigma \geq 0$*, we have*

$$\text{(1.7)} \qquad \frac{c}{\alpha(t)} \leq a(t) \leq \frac{c}{\alpha(t)} + \frac{\sigma}{2\sqrt{r}}\sqrt{3 + \frac{\left(1+\frac{\rho}{r}\right)^2}{\left(1+\frac{2\rho}{r}\right)}}.$$

ii) Lipschitz regularity of the free boundary. *The lower semi-continuous function* a *satisfies:*

$$-a\frac{\alpha'}{\alpha} \leq a'.$$

Moreover if $\rho \geq \lambda$, then $a \in W^{1,\infty}_{loc}(0, +\infty)$ and the function a is nonincreasing: $a'(t) \leq 0$.

iii) Monotonicity with respect to the parameters $\rho, c, r, \lambda, \sigma$. *The following properties hold: $\frac{\partial a}{\partial \rho} \leq 0$, $\frac{\partial a}{\partial c} \geq 0$, and $\frac{\partial a}{\partial \sigma} \geq 0$.*
Moreover, if $\rho \geq \lambda$, then

$$\frac{\partial a}{\partial r} \geq 0, \quad \frac{\partial a}{\partial \lambda} \geq 0. \tag{1.8}$$

iv) Convergence of the rescaled free boundary when $c \to 0$. *Assume that $\sigma > 0$ and $\lambda \leq 3r + 4\rho$. Then the following convergence of the rescaled free boundary holds true when $c \to 0$:*

$$\bar{a} \leq \frac{a}{c^{\frac{1}{3}}} \longrightarrow \bar{a} \quad \textit{locally uniformly on any compact sets of} \quad (0, +\infty), \quad \textit{as} \quad c \to 0$$

where

$$\bar{a}(t) = \left(\frac{3\sigma^2}{2(1 + (\rho - \lambda)\alpha(t))} \right)^{\frac{1}{3}}. \tag{1.9}$$

Remark 1.4 In the models of equilibrium asset-pricing derived in [18] or [4] starting from assumptions on the the behavior of investors, the condition $\rho \geq \lambda$ is always satisfied. Note that the expression of $\bar{a}(t)$ in (1.9) shows that for $c << 1$, the free boundary $a(t)$ can not be non-increasing in time when $\rho < \lambda$. Therefore the argument proving that $a(t)$ is nonincreasing in time when $\rho \geq \lambda$ is optimal. Similarly, it is possible to see from (1.9) that the monotonicities in (1.8) do not hold for $\rho < \lambda$ and $c << 1$.

1.2 Comments on an alternative approach

As we have seen, the non-local problem we study here, namely (1.2), is closely related to a somewhat classical *local obstacle* problem (1.6). This problem is not straightforward either. Indeed, it is set on the whole real line and it will be seen that the free boundary starts from infinity at $t = 0$. Nevertheless, it is tempting to approach the non-local u-problem by first solving the local w-problem (1.6). As a matter of fact, to derive further qualitative properties, we will study this w-problem in Section 7. However, the w-problem does not yield the solution of the u-problem that is of interest in a straightforward fashion. By solving the w-problem, we indeed get the free boundary but we then need to show that it is of the form $\{x = a(t)\} = \partial \{w > \psi\}$. We further need to recover u from w and this does not follow immediately from the local obstacle problem. Indeed, we have to solve the equation for u in the domain $\{x < a(t)\}$ with $a(t) > 0$, and this equation reads:

$$\begin{cases} \mathcal{L}u = 0 & \text{on} \quad \{(x,t) \in \mathbb{R} \times (0, +\infty), \quad x < a(t)\}, \\ u(a(t), t) = u(-a(t), t) + \psi(a(t), t) & \text{for} \quad t \in (0, +\infty), \\ u(x, 0) = 0 & \text{for} \quad x \in \mathbb{R} \end{cases} \tag{1.10}$$

This equation too is non local because of the boundary condition. One way to solve the u-problem then is to rewrite problem (1.10) with the coordinates $y = x - a(t)$. For this, we need to first prove regularity of the free boundary $a(t)$, that is not known in general. But we actually derive such a property here, for $\rho \geq \lambda$. Then, we could solve this problem by using a fixed point procedure. Furthermore, to reconstruct u from w in the region $x > a(t)$, we can use the obstacle condition $u(x,t) - u(-x,t) = \psi(x,t)$ for $x > a(t)$.

However, even if we succeed with this procedure, the best we can get is the existence of one solution u to the u-problem. It does not solve the question of uniqueness of the solution (and more generally the question of the comparison principle). In particular, if equality were to hold in the obstacle condition for some $x < 0$, that is $u(x,t) - u(-x,t) = \psi(x,t)$, then, $w(x,t) = u(x,t) - u(-x,t)$ would not satisfy the local obstacle problem (1.6). Rather, in this case, at least formally, it would satisfy a double obstacle problem with $\psi(x,t) \leq w(x,t) \leq -\psi(-x,t))$. Such a situation therefore has to be ruled out.

Lastly, our aim here is to establish several qualitative properties of the solution u related to the economic motivation of the problem (see the forthcoming paper [4]). We will also derive some properties of w but the properties for u do not follow immediately from w. Our *direct approach* of proving a comparison principle for the u-problem, in the framework of viscosity solutions, allows us to prove the properties of u (uniqueness, comparison, convexity in x, monotonicity in t) that are of interest.

1.3 A brief description of the economic model

We refer the reader to [18] for a detailed derivation of the stationary model, starting with postulates on the behavior of investors. Here we present a self-contained and heuristic introduction to the evolution model. In [4], we give a complete and detailed derivation of the finite horizon model. The description we give here is a simplification with respect to the machinery developed in [18] and [4]. In particular, we do not consider aspects of the model that deal with the optimal filtering of information.

We consider a market with a single risky asset and two groups of investors A and B. The risky asset provides dividends up to maturity $T > 0$. Investors disagree about the future evolution of the cumulative dividends D_t. Under the belief of investors in group $C \in \{A, B\}$ the process of dividends is given by the following pair of diffusions:

$$dD_t = \hat{f}_t^C dt + \sigma_D dW_t^{C,D} \tag{1.11}$$

$$d\hat{f}_t^C = -\lambda(\hat{f}_t^C - \bar{f})dt + \sigma_{\hat{f}} dW_t^{C,\hat{f}} \tag{1.12}$$

where for each $C \in \{A, B\}$, $W^{C,D}$ and $W^{C,\hat{f}}$ are Brownian motions (under C's beliefs) that are possibly contemporaneously correlated, $\lambda > 0$ is a rate of mean reversion and $\bar{f}$ is the (common) long run mean value of $\bar{f}^C$. When $\hat{f}_t^A > \hat{f}_t^B$, investors in group A are relatively optimistic.

To complete the model we need to consider the views that investors in group $C \in \{A, B\}$ have of the evolution of beliefs of the investors in the complementary group. We write $\bar{C}$ the complementary group of investors (i.e. $\bar{C} = B$ if $C = A$, and $\bar{C} = A$ if $C = B$), and $g^C = \hat{f}^{\bar{C}} - \hat{f}^C$. We assume that from the viewpoint of agents in group C, g^C satisfies:

$$dg_t^C = -\rho g_t^C dt + \sigma_g dW_t^g \tag{1.13}$$

where $\rho > \lambda$, $\sigma_g > 0$, and W^g is a Brownian motion from the point of view of *both* group of investors. Assuming that investors agree on the evolution of differences in beliefs amounts to assuming that investors in one group know the model used by investors in the other group and agree to disagree.

The model developed in [18] postulates a particular information structure and derives equations (1.11 -1.13) using results on optimal filtering (see also [12]).

We assume that Investors are *risk-neutral* - that is they value a random payoff according to its expected value - and discount the future at a continuously compounded rate r. Short-sales are not allowed, that is every investor must hold a non-negative amount of the asset. In addition, we assume that the supply of the asset is finite and that investors can borrow resources at the risk-free interest rate r. As a result, competition among buyers will guarantee that buyers of the asset will always pay their *reservation price*, the maximum price they are willing to pay for the asset.

We now characterize the price p_t^C that investors in each group are willing to pay for the asset at time t. Assets are traded ex-dividend, that is a buyer of the asset at time s gains the right only to the dividends that accumulate after time s. Since there are no dividends payable after time T, $p_T^C = 0$. We assume that there is a cost $c > 0$ per unit for any transaction. We also assume that if an investor holds the asset to the maturity T, he can dispose the worthless asset for free. Note that since every agent in a group has exactly the same valuation for the asset and transaction costs are positive, every transaction must necessarily involve a seller in a group and a buyer in the complementary group which values the asset more. Thus if an investor buys the asset at time $s = t + \tau < T$ at price $p_{t+\tau}^{\bar{C}}$ then the seller, who is necessarily a member of group C, gets the amount (positive or negative)

$$p_{t+\tau}^{\bar{C}} - c1_{\{t+\tau<T\}}. \tag{1.14}$$

Note that *we do not use the positive part* in (1.14), because we assume unlimited liability in the model which greatly simplifies the analysis. Indeed an investor would sell (even if the amount (1.14) is negative), if he thinks that this is better than waiting to sell at a later time. In turn, the buyer must believe that, on average, she will get more than what she pays for the asset as future dividends and perhaps the proceeds of a future sale.

Since $\sigma_g > 0$, a buyer of the asset at t knows that in the future, investors in the other group may become more optimistic than her and give her an opportunity to sell at a price that will exceed her reservation price at the time of the sale.

Hence one should expect that:

$$p_t^C = \sup_{\tau\in[0,T-t]} \mathbb{E}_t^C \left\{ e^{-r\tau}(p_{t+\tau}^{\bar{C}} - c1_{\{t+\tau<T\}}) + \int_t^{t+\tau} e^{-r(s-t)} dD_s \right\} \tag{1.15}$$

Here the second term is the discounted cumulative dividends over the period $(t, t+\tau)$, while the first term represents the discounted payoff of a sale at time $t + \tau$. The price is thus computed by maximizing the expectation over random selling times. Note that equations (1.15) for $C \in \{A, B\}$ form a pair of optimal stopping times where the reward for stopping in one problem is related to the value function of the other problem.

Given the assumptions concerning the laws of motion (1.11)–(1.13) one can rigorously show that there is a solution given by

$$p_t^C = \mathbb{E}_t^C \left\{ \int_t^T e^{-r(s-t)} dD_s \right\} + u(g_t^C, T-t)$$

where u solves (1.2) with $\sigma = \sigma_g$. (See [18] or [4] for a derivation.)

The function $u(g_t^C, T-t)$ can be interpreted as the speculative value contained in the price of the asset, i.e. the amount that an investor in group C is willing to pay for the asset in addition to her valuation of the future flow of dividends. This amounts reflects the option value of resale and is a result of *fluctuating* differences in beliefs among investors. Since a buyer of the asset at a time t would be a member of the most optimistic group, the amount by which the purchase price exceeds his valuation, u, can be legitimately called a *bubble*.

1.4 Organisation of the paper

In Section 2, we recall the definition of viscosity solutions and the stability properties of these solutions for the evolution problem, the stationary problem and the w-problem. In Section 3, we prove the comparison principle for the u-problem. Section 4 is devoted to the proof of Theorem 1.1 which states existence and uniqueness of the solution u. In Section 5, we prove some properties of the solution u, and we establish further properties of u in Section 6, by introducing a modified problem (problem (6.3)) which allows us to show that w solves an obstacle problem. As a consequence, we give the proof of Theorem 1.2 at the beginning of Section 6. In Section 7, we study the w-problem, following the lines of proof used previously for the u-problem. In Section 8, we establish a Lipschitz estimate for the free boundary. We study the asymptotics of the free boundary in the limit $c \to 0$ in Section 9. As a consequence, we obtain the proof of Theorem 1.3. In Section 10 we show that the comparison principle does not hold for $c = 0$ (and $\sigma > 0$).

In the Internet Appendix we show in a first subsection a Jensen-Ishii Lemma (Lemma 11.1) which is used to prove the comparison principle for the obstacle problem in the case $c > 0$. In a second subsection, we derive the equation satisfied by w (Lemma 11.2), where w is the antisymmetric (or odd) part of u. The comparison principle for w (Theorem 7.5) is proved in a third subsection. The purpose of the fourth subsection, is to prove the convexity of w in x (Proposition 7.10). In the fifth (and last) subsection, we show a comparison principle for a stationary obstacle problem without zero order terms (Lemma 9.4), which is used in the proof of the asymptotics of the free boundary.

2 Definition of viscosity solutions

2.1 Viscosity solutions for the u-problem

2.1.1 The evolution problem

Definition 2.1 (Viscosity sub/super/solution of equation (1.2)**)** *Let* $T \in (0, +\infty]$.
i) (Viscosity sub/supersolution on $\mathbb{R} \times (0,T)$**)**
A function $u : \mathbb{R} \times [0,T) \to \mathbb{R}$ *is a viscosity subsolution (resp. supersolution) of* (1.2) *on* $\mathbb{R} \times (0,T)$*, (that is, of the first equation in* (1.2)*), if* u *is upper semi-continuous (resp. lower semi-continuous), and if for any function* $\varphi \in C^{2,1}(\mathbb{R} \times (0,T))$ *and any point* $P_0 = (x_0, t_0) \in \mathbb{R} \times (0,T)$ *such that* $u(P_0) = \varphi(P_0)$ *and*

$$u \le \varphi \quad \textit{on} \quad \mathbb{R} \times (0,T) \qquad (\textit{resp.} \quad u \ge \varphi \quad \textit{on} \quad \mathbb{R} \times (0,T))$$

then

$$\min\left\{(\mathcal{L}\varphi)(x_0,t_0), \quad u(x_0,t_0) - u(-x_0,t_0) - \psi(x_0,t_0)\right\} \le 0,$$

$$(\textit{resp.} \qquad \min\left\{(\mathcal{L}\varphi)(x_0,t_0), \quad u(x_0,t_0) - u(-x_0,t_0) - \psi(x_0,t_0)\right\} \ge 0).$$

ii) (Viscosity sub/supersolution on $\mathbb{R}\times[0,T)$)
A function $u:\mathbb{R}\times[0,T)\to\mathbb{R}$ is a viscosity subsolution (resp. supersolution) of (1.2) *on $\mathbb{R}\times[0,T)$, (that is, of the initial value problem), if u is a viscosity subsolution (resp. supersolution) of* (1.2) *on $\mathbb{R}\times(0,T)$ and satisfies moreover $u(x,0)\le 0$ (resp. $u(x,0)\ge 0$) for all $x\in\mathbb{R}$.*

iii) (Viscosity solution on $\mathbb{R}\times[0,T)$)
A function $u:\mathbb{R}\times[0,T)\to\mathbb{R}$ is a viscosity solution of (1.2) *on $\mathbb{R}\times[0,T)$, if and only if u^* is a viscosity subsolution and u_* is a viscosity supersolution on $\mathbb{R}\times[0,T)$ where*

$$u^*(x,t)=\limsup_{(y,s)\to(x,t)} u(y,s) \qquad \textit{and} \qquad u_*(x,t)=\liminf_{(y,s)\to(x,t)} u(y,s).$$

A key property of the viscosity sub/supersolutions is their stability which is stated next.

Proposition 2.2 (Stability of sub/supersolutions)
For any $\varepsilon\in(0,1)$, let $\mathcal{F}_\varepsilon$ be a non empty familly of subsolutions (resp. supersolutions) of (1.2) *on $\mathbb{R}\times(0,T)$. Let*

$$\underline{u}(x,t)=\limsup_{(y,s,\varepsilon)\to(x,t,0)}\left(\sup_{v\in\mathcal{F}_\varepsilon} v(y,s)\right), \quad \left(\textit{resp.}\quad \overline{u}(x,t)=\liminf_{(y,s,\varepsilon)\to(x,t,0)}\left(\inf_{v\in\mathcal{F}_\varepsilon} v(y,s)\right)\right).$$

If $|\underline{u}|<+\infty$ (resp. $|\overline{u}|<+\infty$), then $\underline{u}$ is a subsolution (resp. $\overline{u}$ is a supersolution) of (1.2) *on $\mathbb{R}\times(0,T)$.*

Proof of Proposition 2.2
The proof of Proposition 2.2 is classical, except for the new term $u(x,t)-u(-x,t)$. In fact, Barles and Imbert gave related definitions of viscosity solution and established stability results for a general class of non local operators in [2]. Here, we simply check this property, proving that if for all functions $v\in\mathcal{F}_\varepsilon$, we have

$$v(x,t)-v(-x,t)-\psi(x,t)\le 0 \tag{2.1}$$

in the viscosity sense, then $\underline{u}$ still satisfies (2.1) (the proof beeing similar for $\overline{u}$).
Indeed, by definition of $\underline{u}$, there exists $(y_\varepsilon,s_\varepsilon,\varepsilon)\to(x,t,0)$ and $v_\varepsilon\in\mathcal{F}_\varepsilon$ such that $\underline{u}(x,t)=\lim_{\varepsilon\to 0} v_\varepsilon(y_\varepsilon,s_\varepsilon)$ and $v_\varepsilon(y_\varepsilon,s_\varepsilon)-v_\varepsilon(-y_\varepsilon,s_\varepsilon)-\psi(y_\varepsilon,s_\varepsilon)\le 0$. Since ψ is continuous,

$$\underline{u}(x,t)-\psi(x,t)\le\limsup_{\varepsilon\to 0} v_\varepsilon(-y_\varepsilon,s_\varepsilon)\le\underline{u}(-x,t)$$

which ends the proof.

2.1.2 The stationary problem

Similarly, we define viscosity solutions for the stationary problem:

Definition 2.3 (Viscosity sub/super/solution of equation (1.3))
i) (Viscosity sub/supersolution)
A function $u:\mathbb{R}\to\mathbb{R}$ is a viscosity subsolution (resp. supersolution) of (1.3)*, if u is upper*

semi-continuous (resp. lower semi-continuous), if for any function $\varphi \in C^2(\mathbb{R})$ *and any point* $x_0 \in \mathbb{R}$ *such that* $u(x_0) = \varphi(x_0)$ *and* $u \leq \varphi$ *on* $\mathbb{R}$ *(resp.* $u \geq \varphi$ *on* $\mathbb{R}$*), then*

$$\begin{array}{ll} & \min\{(\mathcal{M}\varphi)(x_0), \quad u(x_0) - u(-x_0) - \psi_\infty(x_0)\} \leq 0, \\ (\textit{resp.} & \min\{(\mathcal{M}\varphi)(x_0), \quad u(x_0) - u(-x_0) - \psi_\infty(x_0)\} \geq 0). \end{array}$$

ii) (Viscosity solution)
A function $u : \mathbb{R} \to \mathbb{R}$ *is a viscosity solution of* (1.3)*, if and only if* u^* *is a viscosity subsolution and* u_* *is a viscosity supersolution.*

An analogue of the stability property (see Proposition 2.2), holds, but we do not state it explicitly.

2.2 Viscosity solutions for the w-problem

Similarly to the u-problem, we have the following definition of viscosity solutions for the evolution w-problem (we skip the definition for the stationary w-problem, which is similar).

Definition 2.4 (Viscosity sub/super/solution of equation (1.6)**)** *Let* $T \in (0, +\infty]$.
i) (Viscosity sub/supersolution on $(0, +\infty) \times (0, T)$**)**
A function $w : [0, +\infty) \times [0, T) \to \mathbb{R}$ *is a viscosity subsolution (resp. supersolution) of* (1.6) *on* $(0, +\infty) \times (0, T)$*, if* w *is upper semi-continuous (resp. lower semi-continuous), if for any function* $\varphi \in C^{2,1}((0, +\infty) \times (0, T))$ *and any point* $P_0 = (x_0, t_0) \in (0, +\infty) \times (0, T)$ *such that* $w(P_0) = \varphi(P_0)$ *and* $w \leq \varphi$ *on* $(0, +\infty) \times (0, T)$ *(resp.* $w \geq \varphi$ *on* $(0, +\infty) \times (0, T)$*) then*

$$\begin{array}{ll} & \min\{(\mathcal{L}\varphi)(x_0, t_0), \quad w(x_0, t_0) - \psi(x_0, t_0)\} \leq 0, \\ (\textit{resp.} & \min\{(\mathcal{L}\varphi)(x_0, t_0), \quad w(x_0, t_0) - \psi(x_0, t_0)\} \geq 0). \end{array}$$

ii) (Viscosity sub/supersolution on $[0, +\infty) \times [0, T)$**)**
A function $w : [0, +\infty) \times [0, T) \to \mathbb{R}$ *is a viscosity subsolution (resp. supersolution) of* (1.6) *on* $[0, +\infty) \times [0, T)$*, if* w *is a viscosity subsolution (resp. supersolution) of* (1.6) *on* $(0, +\infty) \times (0, T)$ *and satisfies moreover*

$$w(x, t) \leq 0 \qquad (\textit{resp.} \quad w(x, t) \geq 0) \qquad \textit{for all } (x, t) \in ([0, +\infty) \times \{0\}) \cup (\{0\} \times [0, +\infty)).$$

iii) (Viscosity solution on $[0, +\infty) \times [0, T)$**)**
A function $w : [0, +\infty) \times [0, T) \to \mathbb{R}$ *is a viscosity solution of* (1.6) *on* $[0, +\infty) \times [0, T)$*, if and only if* w^* *is a viscosity subsolution and* w_* *is a viscosity supersolution on* $[0, +\infty) \times [0, T)$.

3 Comparison principle for the u-problem

3.1 Comparison principle for the original u-problem

We consider the following equation

$$(3.1) \qquad \left\{ \begin{array}{lll} \min(\mathcal{L}u, \quad u(x, t) - u(-x, t) - \psi(x, t)) = 0 & \text{for} & (x, t) \in \mathbb{R} \times (0, +\infty), \\ u(x, 0) = 0 & \text{for} & x \in \mathbb{R}. \end{array} \right.$$

Theorem 3.1 (Comparison principle for the evolution problem)
Assume (1.1)*, in particular that* $c > 0$*. Let* u *(resp.* v*) be a subsolution (resp. supersolution) of* (3.1) *on* $\mathbb{R} \times [0, T)$ *for some* $T > 0$*, satisfying for some constant* $C_T > 0$*:*

$$u(x,t) \le C_T(1+\max(0,x)) \quad \textit{and} \quad v(x,t) \ge -C_T(1+\max(0,x)) \quad \textit{for all} \quad (x,t) \in \mathbb{R}\times[0,T).$$

Then $u \le v$ *on* $\mathbb{R} \times [0, T)$*.*

Notice that the comparison principle does not hold when $c = 0$, as it is shown in the appendix in subsection 10.

We start by explaining the heuristic idea that underlies the proof.

Quick heuristic proof of the comparison principle
Let u be a subsolution and v a supersolution of (3.1). If

$$M = \sup(u - v) = (u - v)(x_0, t_0) > 0$$

then formally we have at the point (x_0, t_0):

$$\text{(3.2)} \qquad \left\{ \begin{array}{l} \mathcal{L}u \le 0 \\ \text{or} \\ u(x_0,t_0) - u(-x_0,t_0) - \psi(x_0,t_0) \le 0 \end{array} \right.$$

and

$$\text{(3.3)} \qquad \left\{ \begin{array}{l} \mathcal{L}v \ge 0 \\ \text{and} \\ v(x_0,t_0) - v(-x_0,t_0) - \psi(x_0,t_0) \ge 0. \end{array} \right.$$

i) case $\mathcal{L}u \le 0$. We get the usual comparison principle using $\mathcal{L}v \ge 0$.
ii) case $\mathcal{L}u > 0$. In this case, we have

$$\text{(3.4)} \qquad u(x_0,t_0) - u(-x_0,t_0) - \psi(x_0,t_0) \le 0.$$

Substracting the second line of (3.3) from this inequality, we deduce that $M = (u - v)(x_0, t_0) \le (u - v)(-x_0, t_0) = M$ and we can apply the same reasoning at the point $(-x_0, t_0)$. Again, case i) for $(-x_0, t_0)$ is excluded, and it remains case ii) for $(-x_0, t_0)$, i.e. $u(-x_0, t_0) - u(x_0, t_0) - \psi(-x_0, t_0) \le 0$. Summing this inequality to (3.4), we get:

$$2c = -\psi(x_0,t_0) - \psi(-x_0,t_0) \le 0$$

which yields a contradiction.

Proof of Theorem 3.1
We now give a rigorous proof using the doubling of variable technique.
Step 1: preliminaries
Let

$$M = \sup_{(x,t)\in\mathbb{R}\times[0,T)} u(x,t) - v(x,t)$$

and let us assume by contradiction that $M > 0$. Then for small parameters $\varepsilon, \beta, \eta > 0$ and $\delta \geq 0$, let us consider

$$M_{\varepsilon,\beta,\eta,\delta} = \sup_{x,y\in\mathbb{R},\ t\in[0,T)} \Phi_\delta(x,y,t)$$

with

$$\Phi_\delta(x,y,t) := u(x,t) - v(y,t) - \frac{(x-y)^2}{2\varepsilon} - \beta\frac{x^2}{2} - \frac{\eta}{T-t} - \delta\left(\frac{1}{4}|x-z_0|^4 + \frac{1}{2}|t-s_0|^2\right)$$

for a point $(z_0, s_0) \in \mathbb{R} \times [0,T)$ to be fixed later. Clearly, Φ_δ satisfies:

$$\begin{aligned} \Phi_\delta(x,y,t) \quad &\leq 2C_T + 2C_T|x| + C_T|x-y| - \frac{(x-y)^2}{2\varepsilon} - \beta\frac{x^2}{2} - \frac{\eta}{T-t} \\ &\leq 2C_T + 4\frac{(C_T)^2}{\beta} + \varepsilon(C_T)^2 - \frac{(x-y)^2}{4\varepsilon} - \beta\frac{x^2}{4} - \frac{\eta}{T-t} \end{aligned}$$

which shows that the suppremum in $M_{\varepsilon,\beta,\eta,\delta}$ is reached at some point $(\bar{x}, \bar{y}, \bar{t}) \in \mathbb{R}^2 \times [0,T)$. Because of the zero initial data, it must be the case that $\bar{t} > 0$. Moreover, for β, η, δ small enough, we have

$$\Phi_\delta(\bar{x}, \bar{y}, \bar{t}) = M_{\varepsilon,\beta,\eta,\delta} \geq M/2 > 0$$

and we see, in particular, that the following penalization terms are bounded:

$$\frac{(\bar{x}-\bar{y})^2}{4\varepsilon} + \beta\frac{\bar{x}^2}{4} + \frac{\eta}{T-\bar{t}} \leq 2C_T + 4\frac{(C_T)^2}{\beta} + \varepsilon(C_T)^2 - M/2.$$

Step 2: viscosity inequalities
Set

$$\tilde{u}(x,t) = u(x,t) - \beta\frac{x^2}{2} - \frac{\delta}{4}|x-z_0|^4.$$

Then

$$\Phi_\delta(x,y,t) = \tilde{u}(x,t) - v(y,t) - \varphi_\delta(x,y,t) \quad \text{with} \quad \varphi_\delta(x,y,t) := \frac{(x-y)^2}{2\varepsilon} + \frac{\eta}{T-t} + \frac{\delta}{2}|t-s_0|^2.$$

We can then apply the adapted (parabolic) version of Jensen-Ishii Lemma to this obstacle problem, namely Lemma 11.1 in the appendix. This Lemma implies that we have

$$A \leq 0 \quad \text{or} \quad (B \leq 0 \quad \text{and} \quad B_1 \leq 0)$$

with

$$\begin{cases} A := & \dfrac{\eta}{(T-\bar{t})^2} + \delta(\bar{t}-s_0) - \dfrac{1}{2}\sigma^2(X - Y + \beta + 3\delta|\bar{x}-z_0|^2) \\ & +\rho\left(\dfrac{(\bar{x}-\bar{y})^2}{\varepsilon} + \beta\bar{x}^2 + \delta\bar{x}(\bar{x}-z_0)^3\right) + r\left(u(\bar{x},\bar{t}) - v(\bar{y},\bar{t})\right), \\ B := & u(\bar{x},\bar{t}) - v(\bar{y},\bar{t}) - (u(-\bar{x},\bar{t}) - v(-\bar{y},\bar{t})) - (\psi(\bar{x},\bar{t}) - \psi(\bar{y},\bar{t})) \\ B_1 := & u(\bar{x},\bar{t}) - u(-\bar{x},\bar{t}) - \psi(\bar{x},\bar{t}) \end{cases}$$

and

$$X \leq Y. \tag{3.5}$$

Case $A \le 0$ for $\delta \ge 0$
From (3.5) and the fact that $u(\bar{x}, \bar{t}) - v(\bar{y}, \bar{t}) \ge M_{\varepsilon,\alpha,\eta} \ge M/2 > 0$, we deduce that

$$\frac{\eta}{T^2} + r\frac{M}{2} \le \frac{1}{2}\sigma^2\beta + \delta\left(\frac{3}{2}\sigma^2|\bar{x} - z_0|^2 - \rho\bar{x}(\bar{x} - z_0)^3 - (\bar{t} - s_0)\right)$$

which gives a contradiction for $\beta > 0$ small enough and $\delta \ge 0$ small enough with $\delta \le \delta_0(\beta, z_0)$.
Case $A > 0$ for $\delta \ge 0$
In this case, we have $B, B_1 \le 0$, i.e.

$$\left\{\begin{array}{l} u(\bar{x}, \bar{t}) - v(\bar{y}, \bar{t}) - (u(-\bar{x}, \bar{t}) - v(-\bar{y}, \bar{t})) \le \psi(\bar{x}, \bar{t}) - \psi(\bar{y}, \bar{t}) = \alpha(\bar{t})(\bar{x} - \bar{y}), \\ u(\bar{x}, \bar{t}) - \psi(\bar{x}, \bar{t}) \le u(-\bar{x}, \bar{t}). \end{array}\right. \tag{3.6}$$

In the limit $\varepsilon \to 0$ and up to extract a subsequence, we have for $(\bar{x}_{\varepsilon,\delta}, \bar{y}_{\varepsilon,\delta}, \bar{t}_{\varepsilon,\delta}) := (\bar{x}, \bar{y}, \bar{t})$

$$(\bar{x}_{\varepsilon,\delta}, \bar{y}_{\varepsilon,\delta}, \bar{t}_{\varepsilon,\delta}) \to (\bar{x}_\delta, \bar{y}_\delta, \bar{t}_\delta) \quad \text{with} \quad \bar{x}_\delta = \bar{y}_\delta.$$

It is also classical that

$$\left\{\begin{array}{l} \lim_{\varepsilon\to 0} u(\bar{x}_{\varepsilon,\delta}, \bar{t}_{\varepsilon,\delta}) = u(\bar{x}_\delta, \bar{t}_\delta), \\ \lim_{\varepsilon\to 0} v(\bar{y}_{\varepsilon,\delta}, \bar{t}_{\varepsilon,\delta}) = v(\bar{x}_\delta, \bar{t}_\delta), \\ M_{0,\beta,\eta,\delta} := \sup_{x\in\mathbb{R},\ t\in[0,T)} \Phi_\delta(x, x, t) = \Phi_\delta(\bar{x}_\delta, \bar{x}_\delta, \bar{t}_\delta). \end{array}\right. \tag{3.7}$$

Passing to the limit in (3.6), using (3.7) and the semi-continuities of u and v, we get

$$\left\{\begin{array}{l} u(\bar{x}_\delta, \bar{t}_\delta) - \psi(\bar{x}_\delta, \bar{t}_\delta) \le u(-\bar{x}_\delta, \bar{t}_\delta), \\ u(\bar{x}_\delta, \bar{t}_\delta) - v(\bar{x}_\delta, \bar{t}_\delta) \le u(-\bar{x}_\delta, \bar{t}_\delta) - v(-\bar{x}_\delta, \bar{t}_\delta). \end{array}\right. \tag{3.8}$$

For the special case $\delta = 0$, and from the fact that

$$\Phi_0(\bar{x}_0, \bar{x}_0, \bar{t}_0) \ge \Phi_0(-\bar{x}_0, -\bar{x}_0, \bar{t}_0)$$

we deduce that

$$u(\bar{x}_0, \bar{t}_0) - v(\bar{x}_0, \bar{t}_0) = u(-\bar{x}_0, \bar{t}_0) - v(-\bar{x}_0, \bar{t}_0)$$

i.e.

$$\Phi_0(\bar{x}_0, \bar{x}_0, \bar{t}_0) = \Phi_0(-\bar{x}_0, -\bar{x}_0, \bar{t}_0).$$

We also recall (from (3.8)) that

$$u(\bar{x}_0, \bar{t}_0) - u(-\bar{x}_0, \bar{t}_0) - \psi(\bar{x}_0, \bar{t}_0) \le 0 \tag{3.9}$$

Case $A > 0$ in the case $\delta > 0$ with the choice $(z_0, s_0) = (-\bar{x}_0, \bar{t}_0)$
Notice that

$$\begin{aligned} \Phi_0(-\bar{x}_0, -\bar{x}_0, \bar{t}_0) = \Phi_\delta(-\bar{x}_0, -\bar{x}_0, \bar{t}_0) \quad &\le \Phi_\delta(\bar{x}_\delta, \bar{x}_\delta, \bar{t}_\delta) \\ &= -\delta\left(\frac{1}{4}|\bar{x}_\delta - z_0|^4 + \frac{1}{2}|\bar{t}_\delta - s_0|^2\right) + \Phi_0(\bar{x}_\delta, \bar{x}_\delta, \bar{t}_\delta) \\ &\le -\delta\left(\frac{1}{4}|\bar{x}_\delta - z_0|^4 + \frac{1}{2}|\bar{t}_\delta - s_0|^2\right) + \Phi_0(-\bar{x}_0, -\bar{x}_0, \bar{t}_0) \end{aligned}$$

which shows that $\bar{x}_\delta = z_0 = -\bar{x}_0$ and $\bar{t}_\delta = s_0 = \bar{t}_0$. Then from (3.8), we get

$$u(-\bar{x}_0, \bar{t}_0) - u(\bar{x}_0, \bar{t}_0) - \psi(-\bar{x}_0, \bar{t}_0) \le 0$$

and from (3.9), we get

$$u(\bar{x}_0, \bar{t}_0) - u(-\bar{x}_0, \bar{t}_0) - \psi(\bar{x}_0, \bar{t}_0) \le 0.$$

Summing these two inequalities, we get

$$0 < 2c = -\psi(\bar{x}_0, \bar{t}_0) - \psi(-\bar{x}_0, \bar{t}_0) \le 0$$

which gives the desired contradiction. The proof of Theorem 3.1 is thereby complete.

A similar proof gives the following result:

Theorem 3.2 (Comparison principle for the stationary problem)
Assume (1.1). *Let u (resp. v) be a subsolution (resp. supersolution) of* (1.3) *satisfying for some constant $C > 0$:*

$$u(x) \le C(1 + \max(0, x)) \quad \textit{and} \quad v(x) \ge -C(1 + \max(0, x)) \quad \textit{for all} \quad x \in \mathbb{R}.$$

Then $u \le v$ on $\mathbb{R}$.

3.2 Comparison principle for a modified u-problem

We now consider the following modified problem for some positive constant $\varepsilon_0 > 0$:

$$(3.10) \quad \begin{cases} \mathcal{L}u = 0 & \text{for} \quad (x,t) \in (-\infty, \varepsilon_0) \times (0, +\infty), \\ \min\left(\mathcal{L}u,\ u(x,t) - u(-x,t) - \psi(x,t)\right) = 0 & \text{for} \quad (x,t) \in (0, +\infty) \times (0, +\infty), \\ u(x, 0) = 0 & \text{for} \quad x \in \mathbb{R}. \end{cases}$$

Similarly to Section 2, we can introduce the notion of viscosity sub and supersolutions. Then, adpating the proof of Theorem 3.1, we get easily the following result

Theorem 3.3 (Comparison principle for the modified evolution problem)
Assume (1.1) and $\varepsilon_0 > 0$. Let u (resp. v) be a subsolution (resp. supersolution) of (3.10) on $\mathbb{R} \times [0, T)$ for some $T > 0$, satisfying for some constant $C_T > 0$:

$$u(x,t) \le C_T(1+\max(0, x)) \quad \textit{and} \quad v(x,t) \ge -C_T(1+\max(0, x)) \quad \textit{for all} \quad (x,t) \in \mathbb{R}\times[0, T).$$

Then $u \le v$ on $\mathbb{R} \times [0, T)$.

4 Existence by sub/supersolutions

The goal of this section is to prove Theorem 1.1 on existence and uniqueness of the solution to the u-problem. This result will be proven by the method of sub and supersolutions. We start with two lemmata respectively on sub and supersolutions.

Lemma 4.1 (Subsolution)
The function $\underline{u} = \max(0, \psi)$ is a subsolution of (1.2).

Proof of Lemma 4.1
We have $\mathcal{L}\underline{u} = 0$ in the region $\{\psi < 0\}$ and $\underline{u}(x,t) - \underline{u}(-x,t) - \psi(x,t) = 0$ in the region $\{\psi \geq 0\}$.

Remark 4.2 Note that $\max(0,\psi)$ is the solution of the problem for $\sigma = 0$.

The obstacle ψ depends on t, and for this reason, the function $u_\infty(x) = u(x,+\infty)$ is not a natural supersolution of the evolution problem (indeed $u_\infty(x)$ is not a supersolution for $x < 0$, because $\psi(x,t)$ has the wrong monotonicity in time for $x < 0$). Actually, a direct computation shows that the function

$$\frac{\alpha(t)}{\alpha(+\infty)} u_\infty(x)$$

is indeed a supersolution of the evolution problem (1.2), where u_∞ is the stationary solution of (1.3). We could thus use the result of [18] which proves existence of u_∞. In order to keep a self-contained proof, we indicate in the following lemma a direct construction of a supersolution $\overline{u}$ (see Figure 1).

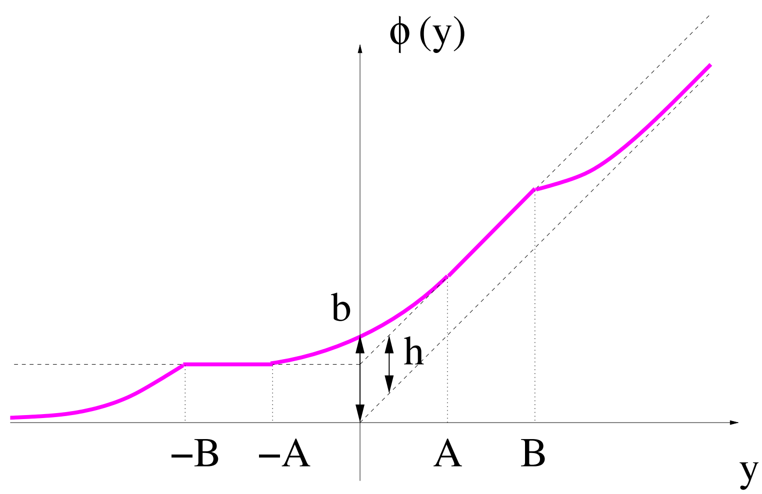

Figure 1: Graph of ϕ defined in Lemma 4.3, with supersolution $\overline{u}(x,t) = \alpha(t)\phi\left(x - \dfrac{c}{\alpha(t)}\right)$

Lemma 4.3 (Supersolution)
Set

$$\overline{u}(x,t) = \alpha(t)\phi(x - d(t)) \quad \textit{with} \quad d(t) = \frac{c}{\alpha(t)} \quad \textit{and} \quad \phi(y) = \zeta(y) + \frac{y}{2}$$

where for $A > 0$:

$$\zeta(y) = \zeta(-y) = \begin{cases} \dfrac{y^2}{4A} + b & \textit{for} \quad |y| < A, \\ \dfrac{|y|}{2} + h \cdot \min\left(1, \dfrac{B^q}{|y|^q}\right) & \textit{for} \quad |y| \geq A, \qquad \textit{with} \quad h := b - \dfrac{A}{4} \geq 0. \end{cases}$$

We choose the positive constants b, B and q to satisfy the following inequalities:

$$b \geq \frac{\sigma^2}{4rA} + \frac{A}{4}\frac{\left(1 + \frac{\rho}{r}\right)^2}{\left(1 + 2\frac{\rho}{r}\right)} \tag{4.1}$$

(4.2)
$$B \geq A, \quad B \geq qh,$$

(4.3)
$$B \geq \left(\frac{\sigma^2 q(q+1)}{2r\left(1 - q\frac{\rho}{r}\right)} \right)^{\frac{1}{2}} \quad \textit{with} \quad 0 < q < \frac{r}{\rho},$$

and

(4.4)
$$B \geq \left(\frac{\sigma^2 hq(q+1)}{2r\left(1 + \frac{\rho}{r}\right)} \right)^{\frac{1}{3}}.$$

Then, for $\sigma \geq 0$ the function $\overline{u}$ is a supersolution of (1.2).

Proof of Lemma 4.3
We first notice that $\phi \in Lip(\mathbb{R})$ and ϕ is C^1 except for $|y| = B$, and C^2 except for $|y| = B, A$. We also check that condition (4.2) implies that ϕ is non decreasing, which also implies that $\phi \geq 0$, because $\phi(-\infty) = 0$.
On the one hand, we have with $y = x - d(t)$

$$\begin{aligned}
\overline{u}(x,t) - \overline{u}(-x,t) &= \alpha(t)\left(\phi(x - d(t)) - \phi(-x - d(t))\right) \\
&= \alpha(t)\left(\phi(y) - \phi(-y - 2d(t))\right) \\
&\geq \alpha(t)\left(\phi(y) - \phi(-y)\right) \\
&\geq \alpha(t) y = \psi(x,t)
\end{aligned}$$

where we have used in the third line the fact that ϕ is non decreasing.
On the other hand, we want to check that

(4.5)
$$\overline{u}_t + \mathcal{M}\overline{u} \geq 0.$$

Notice that this inequality is automatically satisfied in the viscosity sense at points corresponding to $|y| = B$ (because there is no test functions from below at those points). Outside that set, we have
$$\overline{u}_t = -d'(t)\alpha(t)\phi'(y) + \alpha'(t)\phi(y) \geq 0$$
because $\phi \geq 0$, $\phi' \geq 0$ and
$$-d'(t) = \frac{c\alpha'(t)}{\alpha^2(t)} \geq 0.$$
Therefore it is enough to show that $\mathcal{M}\overline{u} \geq 0$ which means
$$r\overline{u} \geq \frac{1}{2}\sigma^2 \overline{u}_{xx} - \rho x \overline{u}_x$$
i.e.
$$\rho d(t)\overline{u}_x + r\overline{u} \geq \frac{1}{2}\sigma^2 \overline{u}_{xx} - \rho(x - d(t))\overline{u}_x.$$

Using the fact that $\overline{u}_x \geq 0$, it is enough to show that

$$\overline{u} \geq \frac{\sigma^2}{2r}\overline{u}_{xx} - \frac{\rho}{r}y\overline{u}_x$$

i.e.

$$\phi \geq \frac{\sigma^2}{2r}\phi'' - \frac{\rho}{r}y\phi'. \tag{4.6}$$

Case 1: $|y| < A$
Then (4.6) means

$$\frac{y}{2} + \frac{y^2}{4A} + b \geq \frac{\sigma^2}{2r}\frac{1}{2A} - \frac{\rho}{r}\left(\frac{y}{2} + \frac{y^2}{2A}\right)$$

i.e.

$$f(y) \geq 0 \qquad \text{for} \quad f(y) := \frac{y^2}{4A}\left(1 + 2\frac{\rho}{r}\right) + \frac{y}{2}\left(1 + \frac{\rho}{r}\right) + b - \frac{\sigma^2}{4rA}. \tag{4.7}$$

The minimum of f is reached for

$$y_0 = -A\frac{\left(1 + \frac{\rho}{r}\right)}{\left(1 + 2\frac{\rho}{r}\right)} \in [-A, A], \quad \text{with} \quad f(y_0) = -\frac{A}{4}\frac{\left(1 + \frac{\rho}{r}\right)^2}{\left(1 + 2\frac{\rho}{r}\right)} + b - \frac{\sigma^2}{4rA}$$

and then (4.7) is satisfied if and only if (4.1) holds true, which also implies $h = b - \frac{A}{4} \geq 0$.

Case 2: $A < |y| < B$
Then (4.6) means

$$h \geq 0, \quad \text{for} \quad -B < y < -A$$

and

$$h + y \geq -\frac{\rho}{r}y, \quad \text{for} \quad A < y < B$$

which are obviously true.

Case 3: $B < |y|$
Case 3.1: $y < -B$
Then (4.6) means

$$\left(1 - q\frac{\rho}{r}\right)\frac{1}{|y|^q} \geq q(q+1)\frac{\sigma^2}{2r}\frac{1}{|y|^{q+2}}$$

which is true if (4.3) holds true.

Case 3.2: $y > B$
Then (4.6) means

$$y + h\frac{B^q}{y^q} \geq \frac{\sigma^2}{2r}hq(q+1)\frac{B^q}{y^{q+2}} - \frac{\rho}{r}y\left(1 - hq\frac{B^q}{y^{q+1}}\right)$$

i.e.

$$y\left(1 + \frac{\rho}{r}\right) \geq \frac{\sigma^2}{2r}hq(q+1)\frac{B^q}{y^{q+2}} + h\frac{B^q}{y^q}\left(q\frac{\rho}{r} - 1\right)$$

which is implied by (because $q < r/\rho$)

$$y\left(1+\frac{\rho}{r}\right) \geq \frac{\sigma^2}{2r} hq(q+1)\frac{B^q}{y^{q+2}}$$

which is true if

$$y^3\left(1+\frac{\rho}{r}\right) \geq \frac{\sigma^2}{2r} hq(q+1)$$

i.e. if (4.4) holds true.

Thus (4.5) holds in all the previous cases and then by continuity also for $|y| = A$. Therefore (4.5) holds in the viscosity sense everywhere, what concludes the proof of the lemma.

Proof of Theorem 1.1
We only prove the i) (the proof of ii) for the stationary problem being similar, replacing $\underline{u}$ and $\overline{u}$, respectively by $\underline{u}_\infty(x) = \underline{u}(x, +\infty)$ and $\overline{u}_\infty(x) = \overline{u}(x, +\infty)$).
Step 1: definition of S
We easily check that

$$\underline{u} \leq \overline{u}$$

with $\underline{u}$ and $\overline{u}$ respectively defined in Lemmata 4.1 and 4.3.
Indeed $\phi \geq 0$, and then it is sufficient to check that $\phi(y) \geq y$ for $y \geq 0$. Moreover, for $B > A$, ϕ is C^1 and convex on $(-B, B)$ and then it is easy to check that $\phi(y)$ is above $|y|$ on this interval. It is also straightforward to check that this is true on its complement. By continuity, it stays true in the limit case $B = A$.
We define the set of functions

$$S = \{w : \mathbb{R} \times [0, +\infty) \to \mathbb{R}, \quad w \text{ subsolution of (1.2)}, \quad \underline{u} \leq w \leq \overline{u}\} \neq \emptyset.$$

Step 2: existence by Perron's method
As usual we define

$$u = W^* \quad \text{with} \quad W(x,t) = \sup_{w \in S} w(x,t).$$

From the stability property (Proposition 2.2), we can deduce that u is automatically a subsolution. We now check that u_* is a supersolution. Because

$$0 = \underline{u}(x, 0) \leq u_*(x, 0) \leq \overline{u}(x, 0) = 0$$

we only have to check that if

$$u_* \geq \varphi \quad \text{with} \quad u_* = \varphi \quad \text{at} \quad (x_0, t_0) \in \mathbb{R} \times (0, +\infty)$$

then

$$\min\left((\mathcal{L}\varphi)(x_0, t_0), \quad u_*(x_0, t_0) - u_*(-x_0, t_0) - \psi(x_0, t_0)\right) \geq 0.$$

If $(\mathcal{L}\varphi)(x_0, t_0) < 0$, we get a contradiction with the optimality of u as usual (see Ishii [15], or for instance Chen, Giga, Goto [7]). If $u_*(x_0, t_0) - u_*(-x_0, t_0) - \psi(x_0, t_0) < 0$, we can write it as follows for some $\eta > 0$:

$$\varphi(x_0, t_0) - \psi(x_0, t_0) \leq u_*(-x_0, t_0) - \eta \quad \text{and} \quad x_0 \neq 0.$$

As usual, up to change φ in $\varphi(x,t) - |(x,t)-(x_0,t_0)|^4$, we can assume that

$$u_*(x,t) > \varphi(x,t) \quad \text{for} \quad (x,t) \neq (x_0,t_0).$$

We then check that

$$\tilde{u}_\delta = \max(u, \varphi + \delta)$$

satisfies

$$\tilde{u}_\delta = u \quad \text{on} \quad (\mathbb{R} \times (0,+\infty)) \backslash B_{R_\delta}(x_0,t_0)$$

with $R_\delta \to 0$ as $\delta \to 0$. And if $\tilde{u}_\delta = \varphi + \delta$ at some point $(y,s) \in B_{R_\delta}(x_0,t_0)$, then we have

$$\tilde{u}_\delta(y,s) - \psi(y,s) \leq u_*(-x_0,t_0) - \eta/2 \leq u(-y,s) = \tilde{u}_\delta(-y,s)$$

for $\delta > 0$ small enough, where the last equality holds (for $\delta > 0$ small enough) because $x_0 \neq 0$. This implies that $\tilde{u}_\delta$ is a subsolution for $\delta > 0$ small enough, i.e. $\tilde{u}_\delta \in S$. On the other hand, it is classical to check that we do not have $\tilde{u}_\delta \leq u$ everywhere, which gives a contradiction with the optimality of u. This shows that u is a viscosity solution of (1.2).
Step 3: uniqueness
We just apply the comparison principle (Theorem 3.1), which proves the uniqueness of u among solutions satisfying $|u - \max(0,\psi)| \leq C$ This completes the proof of the theorem.

5 First properties of the solution u

The main result of this section is:

Theorem 5.1 (Properties of the solution)
Assume (1.1) *and let u be the solution given in Theorem* 1.1. *Then u is continuous. There exists a function ϕ such that*

$$\phi(y) \geq \max(0,y) \quad \textit{and} \quad \limsup_{|y|\to+\infty} |\phi(y) - \max(0,y)| = 0$$

and the following properties hold:
i) asymptotics

$$\max(0,\psi(x,t)) = \alpha(t)\max(0, x-d(t)) \leq u(x,t) \leq \alpha(t)\phi(x-d(t)), \ \textit{with}\ d(t) = \frac{c}{\alpha(t)}. \tag{5.1}$$

ii) monotonicity and convexity*:* $0 \leq u_x \leq \alpha(t)$, $u_{xx} \geq 0$.
iii) convergence in long time*:* $u(x,t) \to u_\infty(x)$ *as* $t \to +\infty$.
iv) monotonicity with respect to the parameters c, σ*: For* $c > 0, \sigma \geq 0$, *we have* $\dfrac{\partial u}{\partial c} \leq 0$, *and* $\dfrac{\partial u}{\partial \sigma} \geq 0$.
v) The limit $c \to 0$*:* $u \to u_0$ *as* $c \to 0$ *where* u_0 *is the minimal solution of* (1.2) *for* $c = 0$ *satisfying* $|u_0(x,t) - \max(0, x\alpha(t))| \leq C$ *on* $\mathbb{R} \times [0,+\infty)$*, for some constant* $C > 0$.

Proposition 5.2 (Convexity of the solution)
The solution u of (1.2) given by Theorem 3.1 *i), is convex in x, for all time $t \geq 0$.*

Proof of Proposition 5.2
In the literature, we find a few proofs of the convexity of solutions (see for instance Alvarez, Lasry, Lions [1], Imbert [14], Giga [13], Rapuch [17]), but none of these approaches seem to apply directly to our problem. For this reason, we provide a new approach: our proof is based on a scheme obtained by an implicit discretization in time of the problem. This allows us to come back (at each time step) to a stationary problem that we can analyse more easily.
Step 1: the implicit scheme
Given a time step $\varepsilon > 0$, consider an approximation $u^n(x)$ of $u(x, n\varepsilon)$ defined for $n \in \mathbb{N}$ as a solution of the following implicit scheme:
(5.2)
$$\begin{cases} u^0 = 0, \quad \text{and, for} \quad n \in \mathbb{N}, \\ \min\left(\dfrac{u^{n+1} - u^n}{\varepsilon} + \mathcal{M}u^{n+1}, \quad u^{n+1}(x) - u^{n+1}(-x) - \psi(x, (n+1)\varepsilon)\right) = 0 \quad \text{for} \quad x \in \mathbb{R}. \end{cases}$$

Step 2: subsolution $\underline{u}^{n+1}$
As in the proof of Lemma 4.1, we check that

$$\underline{u}^{n+1}(x) = \max(0, \psi(x, (n+1)\varepsilon))$$

is a subsolution of the scheme (5.2), distinguishing for $\underline{u}^{n+1}$ the regions $\psi^{n+1} \geq 0$ and $\psi^{n+1} < 0$ with $\psi^{n+1}(x) = \psi(x, (n+1)\varepsilon)$ (and using the fact that ψ is non decreasing in time).
Step 3: supersolution $\overline{u}^{n+1}$
Set
$$\overline{u}^{n+1}(x) = \overline{u}(x, (n+1)\varepsilon)$$
and as in the proof of Lemma 4.3, we easily check that $\overline{u}^{n+1}$ is a supersolution of the scheme (5.2). To this end, we have in particular to notice that $u^{n+1} - u^n \geq 0$ and we already checked that $\mathcal{M}\overline{u} \geq 0$ which implies
$$\mathcal{M}\overline{u}^{n+1} \geq 0.$$
Step 4: existence of a unique solution for the scheme
We can then apply Perron's method as in Step 2 of the proof of Theorem 1.1 and also prove a comparison principle similar to Theorem 3.1. This shows that there exists a unique solution $(u^n)_n$ to the scheme. Moreover the comparison principle implies that for each n, the function u^n is continuous.
Step 5: convexity of u^{n+1}
We prove by recurrence that u^{n+1} is convex, assuming that u^n is convex (and noticing that $u^0 = 0$ is obviously convex).
Substep 5.1: definition of the convex envelope U^{n+1}
Define the convex envelope of u^{n+1} as

$$U^{n+1}(x) = \sup_{l \in E} l(x)$$

with the set E of affine functions below u^{n+1} defined as

$$E = \left\{ l = l_{a,b}, \quad \text{such that} \quad l_{a,b}(x) = ax + b \leq u^{n+1}(x) \right\}.$$

By construction, we have
$$U^{n+1} \leq u^{n+1}.$$

Our goal is to show that U^{n+1} is a supersolution. Then the comparison principle will imply

$$U^{n+1} = u^{n+1}$$

which will show that u^{n+1} is convex.

Substep 5.2: U^{n+1} is a supersolution

Consider a test function φ such that

$$\varphi \le U^{n+1} \quad \text{with equality at} \quad x_0 \in \mathbb{R}$$

We want to show that

(5.3)
$$\min\left(\frac{U^{n+1}(x_0) - u^n(x_0)}{\varepsilon} + (\mathcal{M}\varphi)(x_0), \quad U^{n+1}(x_0) - U^{n+1}(-x_0) - \psi(x_0,(n+1)\varepsilon)\right) \ge 0.$$

Because u^{n+1} is continuous, we see that the set E is closed, and then the suppremum defining $U^{n+1}(x_0)$ is a maximum, i.e. there exists $l_0 \in E$ such that we have

$$U^{n+1}(x_0) = l_0(x_0) \quad \text{and} \quad l_0 \le u^{n+1}.$$

Let us write

$$l_0(x) = p(x - x_0) + d_0 \quad \text{with} \quad d_0 = U^{n+1}(x_0)$$

and

$$l_0^\pm(x) = p^\pm(x - x_0) + d_0 \le u^{n+1}(x)$$

the extremal affine functions below u^{n+1} with p^+ maximal and p^- minimal. Then we have

$$\inf_{x \le x_0} (u^{n+1} - l_0^-) = 0 \quad \text{and} \quad \inf_{x \ge x_0} (u^{n+1} - l_0^+) = 0.$$

If $U^{n+1}(x_0) = u^{n+1}(x_0)$, then φ is a test function for u^{n+1} which implies that (5.3) is satisfied. Let us therefore assume that $U^{n+1}(x_0) < u^{n+1}(x_0)$. This implies that $p^+ = p^- = p$ and then $l_0^+ = l_0^- = l_0$. Hence,

$$\inf_{x \le x_0} (u^{n+1} - l_0) = 0 = (u^{n+1} - l_0)(x_-) \quad \text{for some} \quad x_- \in [-\infty, x_0) \tag{5.4}$$

and

$$\inf_{x \ge x_0} (u^{n+1} - l_0) = 0 = (u^{n+1} - l_0)(x_+) \quad \text{for some} \quad x_+ \in (x_0, +\infty] \tag{5.5}$$

and moreover

$$U^{n+1} = l_0 \quad \text{in a neighborhood of } x_0. \tag{5.6}$$

Because of the assymptotics given by the inequalities

$$\underline{u}^{n+1} \le u^{n+1} \le \overline{u}^{n+1} \tag{5.7}$$

we deduce that

$$\begin{cases} p = 0 = d_0 & \text{if} \quad x_- = -\infty, \\ p = \alpha((n+1)\varepsilon) \quad \text{and} \quad d_0 = p x_0 - c & \text{if} \quad x_+ = +\infty. \end{cases} \tag{5.8}$$

We distinguish several cases.
Case 1: x_- and x_+ finite. Note that l_0 is a test function from below for (the supersolution) u^{n+1} both at $x = x_-$ and $x = x_+$. This implies that

$$(5.9)\quad \frac{l_0(x_\pm) - u^n(x_\pm)}{\varepsilon} + (\mathcal{M}l_0)(x_\pm) \geq 0 \quad \text{and} \quad u^{n+1}(x_\pm) - u^{n+1}(-x_\pm) - \psi(x_\pm, (n+1)\varepsilon) \geq 0.$$

We can write $x_0 = \lambda x_- + (1-\lambda)x_+$ for some $\lambda \in (0,1)$. Using the fact that l_0 and $\psi(\cdot, (n+1)\varepsilon))$ are affine, we deduce that

$$\begin{aligned} l_0(x_0) - \psi(x_0, (n+1)\varepsilon)) &\geq \lambda u^{n+1}(-x_-) + (1-\lambda)u^{n+1}(-x_+) \\ &\geq \lambda U^{n+1}(-x_-) + (1-\lambda)U^{n+1}(-x_+) \\ &\geq U^{n+1}(-x_0) \end{aligned}$$

where we used the convexity of U^{n+1} to obtain the last inequality. This implies

$$U^{n+1}(x_0) - U^{n+1}(-x_0) - \psi(x_0, (n+1)\varepsilon)) \geq 0.$$

We also compute

$$(\mathcal{M}l_0)(x) = \rho p x + r l_0(x)$$

which is affine in x. Using the convexity of u^n, we then see that (5.9) implies

$$(5.10)\qquad \frac{l_0(x_0) - u^n(x_0)}{\varepsilon} + (\mathcal{M}l_0)(x_0) \geq 0.$$

Finally, we see that this implies (5.3), since

$$-\frac{1}{2}\sigma^2 \varphi''(x_0) \geq 0$$

follows from the fact that φ is tangent from below to the affine function l_0 (because of (5.6)).
Case 2: x_- finite and $x_+ = +\infty$. We consider a sequence of points $x_+^k \to +\infty$. We first compute for $\delta > 0$

$$l_0(x_+^k) - u^{n+1}(-x_+^k) - \psi(x_+^k, (n+1)\varepsilon) = -u^{n+1}(-x_+^k) \geq -\delta$$

for k large enough depending on δ (using the asymptotics (5.7)). This shows that

$$x_0 = \lambda^k x_- + (1-\lambda^k)x_+^k \quad \text{for some} \quad \lambda^k \in (0,1).$$

This implies as in case 1 that

$$(5.11)\qquad U^{n+1}(x_0) - U^{n+1}(-x_0) - \psi(x_0, (n+1)\varepsilon)) \geq -\delta(1-\lambda^k) \geq -\delta.$$

Similarly we compute (using the asymptotics (5.7) at the level n):

$$\frac{l_0(x) - u^n(x)}{\varepsilon} + (\mathcal{M}l_0)(x) = D_n x - rc + \varepsilon^{-1}(\alpha(n\varepsilon)x - c - u^n(x)) \geq D_n x - rc - \delta$$

for x large enough with

$$D_n = \frac{\alpha((n+1)\varepsilon) - \alpha(n\varepsilon)}{\varepsilon} + (\rho + r)\alpha((n+1)\varepsilon) \geq r\alpha((n+1)\varepsilon) > 0.$$

Therefore

$$\frac{l_0(x_+^k) - u^n(x_+^k)}{\varepsilon} + (\mathcal{M}l_0)(x_+^k) \geq 0$$

for k large enough. As in case 1, this implies (5.10). Taking the limit $\delta \to 0$ in (5.11), this implies (5.3) as in case 1.

Case 3: $x_- = -\infty$ **and** x_+ **finite**. This case is similar to case 2 and we omit the details.

Case 4: $x_- = -\infty$ **and** $x_+ = +\infty$. This case is excluded by (5.8).

This ends step 5 and shows that U^{n+1} is a supersolution. We then conclude that $u^{n+1} = U^{n+1}$ is convex.

Step 6: convergence towards u as ε tends to zero. We set

$$\underline{\underline{u}}(x,t) = \limsup_{(y,n\varepsilon,n)\to(x,t,+\infty)} u^n(y) \qquad \text{and} \qquad \overline{\overline{u}}(x,t) = \liminf_{(y,n\varepsilon,n)\to(x,t,+\infty)} u^n(y).$$

Using the asymptotics (5.7) and adpating the stability property (Proposition 2.2) to this framework, it is then standard to show (see Barles, Souganidis [3]) that $\underline{\underline{u}}$ is a subsolution of (1.2) and $\overline{\overline{u}}$ is a supersolution of (1.2). The comparison principle then implies that

$$\underline{\underline{u}} = u = \overline{\overline{u}}$$

and u is convex in x as a limit of convex (in x) functions. This concludes the proof of the proposition.

Proof of Theorem 5.1

The continuity of u follows from the comparison principle.

Proof of i)

Estimate (5.1) follows from inequality

$$\underline{u} \leq u \leq \overline{u} \tag{5.12}$$

with $\underline{u}$ and $\overline{u}$ given in Lemmata 4.1 and 4.3.

Proof of ii)

The convexity follows from Proposition 5.2, and the asymptotics (5.12) implies

$$0 \leq u_x \leq \alpha(t).$$

Proof of iii)

Locally in x, we see that u is uniformly bounded in time (because of the asymptotics (5.12)) and is non decreasing in time. Therefore we have

$$u(x,t) \to U(x)$$

and U is a viscosity solution of the stationary problem (1.3). Moreover, we have

$$\underline{u}(x,+\infty) \leq U(x) \leq \overline{u}(x,+\infty).$$

Then the comparison for the stationary problem (Theorem 3.2) implies that $U = u_\infty$ i.e.

$$u(x,t) \to u_\infty(x) \quad \text{as} \quad t \to +\infty$$

which shows in particular that u_∞ is also convex.

Proof of iv)

Wew start by showing that $\dfrac{\partial u}{\partial c} \le 0$. Let $c_2 > c_1 > 0$ and the corresponding solutions u^2, u^1. We notice that u_2 is a subsolution for the problem satisfied by u_1. The comparison principle implies that $u^2 \le u^1$, what implies the result.

Next we show that $\dfrac{\partial u}{\partial \sigma} \ge 0$. Similarly, let $\sigma_2 \ge \sigma_1 \ge 0$ and the corresponding solutions u^2, u^1. We recall that $u^2_{xx} \ge 0$. This implies that u_2 is a supersolution for the problem satisfied by u_1 and then $u^2 \ge u^1$.

Proof of v)

For $c > 0$, consider the solution u given by Theorem 1.1. Chhose any solution u^0 of (1.2) for $c = 0$ satisfying $|u^0(x,t) - \max(0, x\alpha(t))| \le C$ for some constant $C > 0$. Then u^0 is a supersolution of the equation satisfied by u. The comparison principle implies that $u^0 \ge u \ge 0$. The monotonicity of u with respect to c implies that u has a limit u_0 as c goes to zero, which satisfies

$$0 \le u_0 \le u^0. \tag{5.13}$$

By stability of viscosity solutions and by (5.13), it is straightforward that u_0 is a viscosity solution of (1.2) for $c = 0$. Therefore u_0 is the minimal solution.

This completes the proof of the theorem.

6 Further properties of the solution u

The main result of this section is:

Theorem 6.1 (Further properties of the solution)
Assume (1.1) *and let u be the solution given in Theorem* 1.1*. Then, in the standard viscosity sense,*

$$\mathcal{L}u = 0 \quad in \quad \left\{ (x,t) \in \mathbb{R} \times (0,+\infty), \quad x < \frac{c}{\alpha(t)} \right\}. \tag{6.1}$$

Moreover, $u_t \ge 0$ and the following monotonicities with respect to the parameters $r > 0$ and $\lambda \ge 0$ hold: $\dfrac{\partial u}{\partial r} \le 0$ and $\dfrac{\partial u}{\partial \lambda} \le 0$. Set $w(x,t) := u(x,t) - u(-x,t)$. Then, in the viscosity sense (see Definition 2.4 *above), w solves:*

$$\begin{cases} \min\left(\mathcal{L}w, \quad w - \psi\right) = 0 & for \quad (x,t) \in (0,+\infty) \times (0,+\infty), \\ w(0,t) = 0 & for \quad t \in [0,+\infty), \\ w(x,0) = 0 & for \quad x \in [0,+\infty). \end{cases} \tag{6.2}$$

Proof of Theorem 1.2

Theorem 1.2 just combines Theorems 5.1 and 6.1.

In order to get the further properties of the solution u stated in Theorem 6.1 (including in particular the monotonicity with respect to the parameter r), it is convenient to consider the following *modified equation*:

$$
(6.3) \qquad \begin{cases} \min\left(\mathcal{L}u, \quad u(x,t) - u(-x,t) - \psi(x,t)\right) = 0 & \text{for} \quad (x,t) \in \mathbb{R} \times (0,+\infty), \\ (\mathcal{L}u)(x,t) \le 0 & \text{for} \quad x < \dfrac{c}{\alpha(t)} \quad \text{and} \quad t > 0, \\ u(x,0) = 0 & \text{for} \quad x \in \mathbb{R}. \end{cases}
$$

Similarly to Definition 2.1, we can introduce a notion of viscosity solution for this equation. The only difference, is that for a viscosity subsolution u such that

$$
u \le \varphi \quad \text{with equality at} \quad (x_0, t_0) \in \mathbb{R} \times (0, T)
$$

we require both

$$
\min\left\{(\mathcal{L}\varphi)(x_0,t_0), \quad u(x_0,t_0) - u(-x_0,t_0) - \psi(x_0,t_0)\right\} \le 0
$$

and

$$
(\mathcal{L}\varphi)(x_0,t_0) \le 0 \quad \text{if} \quad x_0 < \frac{c}{\alpha(t_0)}.
$$

Proposition 6.2 (Existence and uniqueness of a solution of the modified equation (6.3))
Assume (1.1)*. Then, there exists a unique solution u of* (6.3)*. Moreover this solution u is the same as the one given by Theorem* 1.1*.*

Proof of Proposition 6.2
We can check that the notion of viscosity solution for (6.3) is stable (as in Proposition 2.2). It is straightforward to verify that the function $\underline{u}$ given in Lemma 4.1 is a subsolution of (6.3). Since the definition of a supersolution is unchanged for (6.3) in comparison to (1.2), the function $\overline{u}$ given in Lemma 4.3 is still a supersolution of (6.3). Thus we can apply Perron's method that shows the existence of a solution $\tilde{u}$ of (6.3). Finally, notice that any viscosity solution of (6.3) is also a viscosity solution of (1.2). Therefore we can apply the comparison principle for equation (1.2) which shows that the solution $\tilde{u}$ is the same as the one given by Theorem 1.1. This ends the proof of the proposition.

Proof of Theorem 6.1
The first part of the Theorem, viz. (6.1), follows from proposition 6.2. To show the monotonicity in time of u, we simply check that $u_h(x,t) := u(x,t+h)$ is a supersolution of (3.10) for $h > 0$, because $u_h(x,0) \ge 0 = u(x,0)$ and the obstacle satisfies $\psi_t \ge 0$ for $x > 0$. Then the comparison principle (Theorem 3.3) yields $u_h \ge u$ for any $h > 0$. This implies that $u_t \ge 0$.

Proof of monotonicity with respect to parameters r and λ
For $r > 0$, define the set

$$
E_r = \left\{(x,t) \in \mathbb{R} \times (0,+\infty), \quad x \ge \frac{c}{\alpha(r+\lambda,t)}\right\}
$$

where the notation is explicit of the dependence on r:

$$
\alpha(r+\lambda,t) = \frac{1 - e^{-(r+\lambda)t}}{r+\lambda}
$$

and set

$$\psi^r(x,t) := x\alpha(r+\lambda,t) - c$$

and note the dependence in r by writing

$$\mathcal{L}^r u := u_t - \frac{1}{2}\sigma^2 u_{xx} + \rho x u_x + ru.$$

We have

$$\frac{\partial \alpha}{\partial r}(r+\lambda,t) = \frac{e^{-(r+\lambda)t}}{(r+\lambda)^2}\left(1+(r+\lambda)t - e^{(r+\lambda)t}\right) \leq 0. \tag{6.4}$$

Let $r^2 > r^1 > 0$ and the corresponding solutions u^2 and u^1 of (1.2) (or equivalently (6.3)). Because of (6.4),

$$E_{r^2} \subset E_{r^1}$$

and

$$u^2(x,t) - u^2(-x,t) - \psi^{r^2}(x,t) \geq u^2(x,t) - u^2(-x,t) - \psi^{r^1}(x,t) \quad \text{on} \quad E_{r^2}.$$

On the other hand,

$$\mathcal{L}^{r^2} u^2 \geq \mathcal{L}^{r^1} u^2$$

because $u^2 \geq 0$. Since u^2 is a solution of (6.3) for $r = r^2$, for any test point (x,t) (tested from above), either

$$(\mathcal{L}^{r^2} u^2)(x,t) \leq 0$$

or

$$(\mathcal{L}^{r^2} u^2)(x,t) > 0 \quad \text{and} \quad u^2(x,t) - u^2(-x,t) - \psi^{r^2}(x,t) \leq 0 \quad \text{and} \quad (x,t) \in E_{r^2}.$$

This implies that

$$\min((\mathcal{L}^{r^1} u^2)(x,t), \quad u^2(x,t) - u^2(-x,t) - \psi^{r^1}(x,t)) \leq 0$$

which shows that u^2 is a subsolution for the equation satisfied by u^1. Therefore $u^2 \leq u^1$ which implies the expected monotonicity in r of the solution. The proof of monotonicity with respect to the parameter λ is similar.

Equation satisfied by w
Set

$$w(x,t) = u(x,t) - u(-x,t) \quad \text{for} \quad (x,t) \in [0,+\infty) \times [0,+\infty)$$

The fact that w solves (6.2) in the viscosity sense follows from Lemma 11.2 in the appendix.

7 The obstacle problem satisfied by w

Recall that

$$w(x,t) = u(x,t) - u(-x,t)$$

solves the problem:

$$\text{(7.1)} \qquad \begin{cases} \min(\mathcal{L}w, w-\psi) = 0 & \text{for} \quad (x,t) \in (0,+\infty) \times (0,+\infty), \\ w(0,t) = 0 & \text{for} \quad t \in (0,+\infty), \\ w(x,0) = 0 & \text{for} \quad x \in (0,+\infty) \end{cases}$$

and define the stationary problem (for $t = +\infty$) with $\psi_\infty(x) = \psi(x,+\infty)$:

$$\text{(7.2)} \qquad \begin{cases} \min(\mathcal{M}w_\infty, w_\infty - \psi_\infty) = 0 & \text{for} \quad x \in (0,+\infty), \\ w_\infty(0) = 0. \end{cases}$$

7.1 Main results of this section

Theorem 7.1 (Properties of the solution w)
Assume (1.1). *Then there exists a unique solution w to equation* (7.1) *satisfying*

$$|w(x,t)| \leq C(1+|x|) \quad on \quad [0,+\infty) \times [0,+\infty).$$

Moreover w is continuous and there exists a function $\tilde{\phi}$ satisfying

$$\tilde{\phi}(y) \geq \max(0,y) \quad and \quad \limsup_{|y|\to+\infty} |\tilde{\phi}(y) - \max(0,y)| = 0$$

such that the following properties hold:
i) asymptotics

$$\text{(7.3)} \quad \max(0,\psi(x,t)) = \alpha(t)\max(0,x-d(t)) \leq w(x,t) \leq \alpha(t)\min(x,\tilde{\phi}(x-d(t))), \quad with \quad d(t) = \frac{c}{\alpha(t)}.$$

ii) monotonicity and convexity: $w_t \geq 0,\ 0 \leq w_x \leq \alpha(t),\ w_{xx} \geq 0.$
iii) convergence in long time: $w(x,t) \to w_\infty(x)$ *as* $t \to +\infty$, *where* w_∞ *is the unique solution of* (7.2) *satisfying* $|w_\infty(x)| \leq C(1+|x|)$ *on* $[0,+\infty)$.
iv) monotonicity with respect to the parameters c,ρ,r,λ,σ : $\frac{\partial w}{\partial c} \leq 0,\ \frac{\partial w}{\partial \rho} \leq 0,\ \frac{\partial w}{\partial r} \leq 0,\ \frac{\partial w}{\partial \lambda} \leq 0,$ *and* $\frac{\partial w}{\partial \sigma} \geq 0$.

Notice that $w = \max(0,\psi)$ if $\sigma = 0$.

Theorem 7.2 (Properties of a and $\tilde{w}$)
Assume (1.1) *and let w be the solution given in Theorem* 7.1. *Then there exists a lower semi-continuous function $a : (0,+\infty) \to [0,+\infty)$ such that for all $t > 0$:*

$$\{x \in [0,+\infty), \quad w(x,t) = \psi(x,t)\} = \{x \geq a(t)\}.$$

Let

$$\tilde{w}(x,t) = w(x,t) - \psi(x,t).$$

Then the following properties hold:
i) bounds for $\sigma \geq 0$
We have

$$\text{(7.4)} \qquad \frac{c}{\alpha(t)} \leq a(t) \leq \frac{c}{\alpha(t)} + \frac{\sigma}{2\sqrt{r}}\sqrt{3 + \frac{\left(1+\frac{\rho}{r}\right)^2}{\left(1+\frac{2\rho}{r}\right)}}$$

and

$$0 \leq \tilde{w} \leq c. \tag{7.5}$$

ii) time monotonicity
If $\rho \geq \lambda$, then we have

$$\tilde{w}_t \leq 0 \quad \textit{and} \quad a'(t) \leq 0. \tag{7.6}$$

iii) monotonicity with respect to the parameters $\rho, c, r, \lambda, \sigma$

$$\frac{\partial \tilde{w}}{\partial \rho} \leq 0, \quad \textit{and} \quad \frac{\partial \tilde{w}}{\partial c} \geq 0, \quad \frac{\partial \tilde{w}}{\partial \sigma} \geq 0 \tag{7.7}$$

and

$$\frac{\partial a}{\partial \rho} \leq 0, \quad \textit{and} \quad \frac{\partial a}{\partial c} \geq 0, \quad \frac{\partial a}{\partial \sigma} \geq 0. \tag{7.8}$$

Moreover, if $\rho \geq \lambda$, then

$$\frac{\partial \tilde{w}}{\partial r} \geq 0, \quad \frac{\partial \tilde{w}}{\partial \lambda} \geq 0, \quad \textit{and} \quad \frac{\partial a}{\partial r} \geq 0, \quad \frac{\partial a}{\partial \lambda} \geq 0. \tag{7.9}$$

Remark 7.3 The condition $\rho \geq \lambda$ is always satisfied for the model derived in [18].

Corollary 7.4 (The exercise region for u is on the right)
Assume (1.1). *Let u be the solution given in Theorem* 1.1. *Then*

$$\{(x,t) \in \mathbb{R} \times [0,+\infty), \quad u(x,t) - u(-x,t) - \psi(x,t) = 0\} \quad = \quad \{x \geq a(t)\} \quad \subset \quad \left\{x \geq \frac{c}{\alpha(t)}\right\}.$$

7.2 Comparison principles

We now consider the following more general problem

$$\left\{\begin{array}{lll} \min(\mathcal{L}w - f, w - g) = 0 & \text{for} & (x,t) \in (0,+\infty) \times (0,+\infty), \\ w(0,t) = h & \text{for} & t \in (0,+\infty), \\ w(x,0) = h & \text{for} & x \in (0,+\infty) \end{array}\right. \tag{7.10}$$

where $f(x,t)$ and $g(x,t)$ are continuous functions and $h \in \mathbb{R}$ is a constant.

Theorem 7.5 (Comparison principle for the w problem)
Assume (1.1). *Let $T > 0$. Suppose w is a subsolution (resp. v a supersolution) of equation* (7.10) *on $[0,+\infty) \times [0,T)$ with $f, g \in C([0,+\infty)^2)$ and $h \in \mathbb{R}$, satisfying*

$$w(x,t) \leq C_T(1+|x|) \quad (\textit{resp.} \quad v(x,t) \geq -C_T(1+|x|)) \quad \textit{for} \quad (x,t) \in [0,+\infty) \times [0,T).$$

Then $w \leq v$.

The proof of Theorem 7.5 is given in the appendix. A proof similar to the one of Theorem 7.5 (but using $r > 0$) gives the following result:

Theorem 7.6 (Comparison principle for the stationary w_∞ problem)
Assume (1.1). *Suppose w is a subsolution (resp. v a supersolution) of equation* (7.2), *satisfying*

$$w(x) \leq C(1+|x|) \quad (\textit{resp.} \quad v(x) \geq -C(1+|x|)).$$

Then $w \leq v$.

7.3 Sub/supersolutions

Lemma 7.7 (Sub/supersolutions)
The function $\underline{w} = \max(0, \psi)$ *is a subsolution of* (7.1). *The function* $\overline{\overline{w}}(x,t) = x\alpha(t)$ *is a supersolution of* (7.1).

Proof of Lemma 7.7
It is straightforward to check that $\underline{w}$ is a subsolution.
For $\overline{\overline{w}}$, we compute

$$(\mathcal{L}\overline{\overline{w}})(x,t) = x(\alpha'(t) + (\rho + r)\alpha(t)) \geq 0.$$

This implies that $\overline{\overline{w}}$ is a supersolution of (7.1), because $\overline{\overline{w}} \geq \psi$.
This ends the proof of the lemma.

Remark 7.8 If $c = 0$, then $w(x,t) = \psi(x,t) = x\alpha(t)$ is a solution of (7.1).

Lemma 7.9 (Refined supersolution)
Assume that $\sigma \geq 0$. *Then for any* $A > 0$, *there exists a function* $\tilde{\phi}$ *such that*

$$\overline{w}(x,t) = \alpha(t)\tilde{\phi}(x - d(t)) \quad \textit{with} \quad d(t) = \frac{c}{\alpha(t)}$$

is a supersolution of (7.1). *More precisely, we have*

$$\tilde{\phi}(y) = \begin{cases} y & \textit{if} \quad y - h > A \\ \phi\,(y-h) & \textit{if} \quad y - h \leq A \end{cases}$$

where ϕ, $h \geq 0$ *are given in Lemma 4.3 for the corresponding chosen* $A > 0$.

Proof of Lemma 7.9
Set $y = x - d(t)$. We start by observing that the map $y \mapsto \tilde{\phi}(y)$ is C^1 except for $y - h = -B$ where it is a supersolution. Also, $\tilde{\phi}' \geq 0$ and, in addition, $\tilde{\phi} \geq 0$ implies that

$$\overline{w}(x,t) \geq 0 \quad \text{for} \quad x = 0 \quad \text{or} \quad t = 0.$$

Furthermore, $\overline{w} \geq \psi$ because $\tilde{\phi}(y) \geq \max(0, y)$.

The computation of the supersolution for u (see the proof of Lemma 4.3) shows that it is enough to check that

$$\tilde{\phi} \geq \frac{\sigma^2}{2r}\tilde{\phi}'' - \frac{\rho}{r}y\tilde{\phi}' \tag{7.11}$$

which is true for $y - h \geq A$. This also holds for $y - h \leq A$, because we already know that ϕ satisfies (7.11), which implies for $h \geq 0$, $\phi(y - h)$ also satisfies (7.11), because $\tilde{\phi}' \geq 0$. This implies that $\overline{w}$ is a supersolution (in the viscosity sense) and ends the proof of the lemma.

7.4 Convexity in x

Proposition 7.10 (Convexity in x of w)
Assume (1.1). *Then the function w defined in Theorem* 7.1 *is convex in x for each $t \geq 0$.*

The proof of Proposition 7.10 is similar to that of Proposition 5.2 and is given in the appendix.

Proof of Theorem 7.1
If $\sigma = 0$, then $u = \max(0, \psi) = w$ and the theorem is true.
If $\sigma \neq 0$, notice that i) follows from

$$\underline{w} \leq w \leq \min(\overline{w}, \overline{\overline{w}}).$$

The remaining part of the proof is similar to the proof of Theorem 1.2 and Theorem 6.1 for the monotonicity of in r. The new monotonicity in ρ follows from the fact that $xw_x \geq 0$. This ends the proof of the theorem.

Proof of Theorem 7.2
If $\sigma = 0$, then we know that $u = \max(0, \psi) = w$ and thus the proposition is true.
Proof of i)
If $\sigma > 0$, then from the explicit supersolution $\overline{w}$ given in Lemma 7.9, we deduce (7.4) with

$$a(t) \leq \frac{c}{\alpha(t)} + A + h$$

where $A + h = b + \dfrac{3A}{4}$ and where we can choose $b = \dfrac{\sigma^2}{4rA} + \dfrac{A}{4} \dfrac{\left(1 + \frac{\rho}{r}\right)^2}{\left(1 + \frac{2\rho}{r}\right)}$. The optimization of $A + h$ as a function of $A > 0$ gives the result.
We also check that the function $\tilde{w}$ solves

$$(7.12) \qquad \begin{cases} \min(\mathcal{L}\tilde{w} + f, \tilde{w}) = 0 & \text{for} \quad (x,t) \in (0,+\infty) \times (0,+\infty), \\ \tilde{w}(0,t) = c & \text{for} \quad t \in (0,+\infty), \\ \tilde{w}(x,0) = c & \text{for} \quad x \in (0,+\infty) \end{cases}$$

with

$$f := \mathcal{L}\psi = x(\alpha'(t) + (\rho + r)\alpha(t)) - rc = x(1 + (\rho - \lambda)\alpha(t)) - rc.$$

We already know that $\tilde{w} \geq \max(0, \psi) - \psi \geq 0$ and on the other hand it is easy check that the constant function equal to c is a supersolution. This implies $\tilde{w} \leq c$ and then (7.5).
Proof of ii)
Clearly,

$$f_t \geq 0 \quad \text{if} \quad \rho \geq \lambda.$$

This implies that for every $h > 0$, the function $\tilde{w}^h(x,t) = \tilde{w}(x, t+h)$ is a subsolution of (7.12) and then $\tilde{w}^h \leq \tilde{w}$. This implies (7.6).
Proof of iii)
We start by establishing monotonicity with respect to c and σ. Notice that

$$\frac{\partial f}{\partial c} \leq 0 \quad \text{and} \quad \frac{\partial f}{\partial \sigma} = 0.$$

On the other hand

$$\tilde{w}_{xx} \geq 0.$$

Therefore the comparison principle implies

$$\frac{\partial \tilde{w}}{\partial c} \geq 0 \quad \text{and} \quad \frac{\partial \tilde{w}}{\partial \sigma} \geq 0. \tag{7.13}$$

Next, we show monotonicity with respect to ρ and r. Notice that ψ is independent on ρ, therefore the monotonicity of $\tilde{w}$ with respect to ρ is the same as for w, i.e.

$$\frac{\partial \tilde{w}}{\partial \rho} \leq 0. \tag{7.14}$$

We also have

$$\mathcal{L}\tilde{w} + f = \tilde{w}_t - \frac{1}{2}\sigma^2 \tilde{w}_{xx} + \rho x \tilde{w}_x + r(\tilde{w} - c) + x(1 + (\rho - \lambda)\alpha(r+\lambda, t))$$

with

$$\alpha(r+\lambda, t) = \frac{1 - e^{-(r+\lambda)t}}{r+\lambda}.$$

Because

$$\tilde{w} - c \leq 0 \quad \text{and} \quad \frac{\partial \alpha}{\partial r}(r+\lambda, t) \leq 0$$

we deduce that

$$\frac{\partial \tilde{w}}{\partial r} \geq 0 \quad \text{if} \quad \rho \geq \lambda. \tag{7.15}$$

Lastly, we derive monotonicity with respect to λ. Similarly to the case of the parameter r,

$$\frac{\partial \tilde{w}}{\partial \lambda} \geq 0 \quad \text{if} \quad \rho \geq \lambda. \tag{7.16}$$

Inequalities (7.13), (7.14), (7.15) and (7.16) imply (7.7) (and then (7.8)) and also (7.9). This ends the proof of the theorem.

Proof of Corollary 7.4
Consider a point $(x,t) \in \mathbb{R} \times [0, +\infty)$ such that $w(x,t) = u(x,t) - u(-x,t)$ satisfies

$$w(x,t) = \psi(x,t).$$

Case $x < 0$
Then this implies that for $-x > 0$

$$w(-x,t) = -w(x,t) = c - x\alpha(t) = 2c + \psi(-x,t) \geq c > 0.$$

This is impossible for $t > 0$, because we know from Theorem 7.2 that

$$\tilde{w} = w - \psi \leq c$$

and this is also impossible for $t = 0$, because $w(\cdot, 0) = 0$.
Case $x \geq 0$
Then we know, still from Theorem 1.3, that

$$\{x \geq 0\} \cap \{w = \psi\} = \{x \geq a(t)\} \subset \left\{ x \geq \frac{c}{\alpha(t)} \right\}$$

which shows the result and ends the proof of the corollary.

8 Lipschitz regularity of the free boundary

Theorem 8.1 (Lipschitz regularity of the free boundary)
With the notation of Theorem 7.2, we have that the map $t \mapsto a(t)\alpha(t)$ is nondecreasing and

$$-a\frac{\alpha'}{\alpha} \le a'. \tag{8.1}$$

Proof of Theorem 8.1
Step 1: change of function
Let us define the function v such that

$$\tilde{w}(x,t) = v(x\alpha(t),t)$$

where $\tilde{w}$ is the solution of (7.12). We have with $y = x\alpha(t)$

$$\begin{aligned} \mathcal{L}\tilde{w} + f \quad &= \tilde{w}_t - \tfrac{1}{2}\sigma^2\tilde{w}_{xx} + \rho x\tilde{w}_x + r(\tilde{w} - c) + x(1 + (\rho - \lambda)\alpha(t)) \\ &= v_t - \frac{1}{2}\sigma^2\alpha^2 v_{yy} + \rho y v_y + r(v - c) + yB(v_y,t) \\ &:= \mathcal{F}(y,t,[v]) \end{aligned}$$

with

$$B(z,t) = \frac{\alpha'(t)}{\alpha(t)}z + \frac{1}{\alpha(t)} + \rho - \lambda.$$

Step 2: monotonicity of the coefficients and v^h supersolution
We recall that $v_y = \dfrac{\tilde{w}_x}{\alpha}$ satisfies $-1 \le v_y \le 0$. Then for $-1 \le z \le 0$, we compute

$$\begin{aligned} \frac{\partial B}{\partial t}(z,t) \quad &= \left(\frac{\alpha'}{\alpha}\right)' z + \left(\frac{1}{\alpha}\right)' \\ &\le \left(-\frac{\alpha'}{\alpha} + \frac{1}{\alpha}\right)' = 0 \end{aligned} \tag{8.2}$$

where in the second line, we have used the fact that $\left(\frac{\alpha'}{\alpha}\right)' \le 0$.
For any $h > 0$, let us define the function

$$v^h(y,t) = v(y,t+h).$$

We see that v^h satisfies

$$\begin{aligned} \mathcal{F}(y,t+h,[v^h]) \quad &= v_t - \tfrac{1}{2}\sigma^2\alpha^2(t+h)v^h_{yy} + \rho y v^h_y + r(v^h - c) + yB(v^h_y, t+h) \\ &\le v_t - \tfrac{1}{2}\sigma^2\alpha^2(t)v^h_{yy} + \rho y v^h_y + r(v^h - c) + yB(v^h_y,t) \\ &= \mathcal{F}(y,t,[v^h]) \end{aligned}$$

where we have used in the second line that $v_{yy} \ge 0$, α is non decreasing and that

$$B(v^h_y, t+h) = B(v^h_y,t) + \int_t^{t+h} ds\ \frac{\partial B}{\partial s}(v^h_y, s) \le B(v^h_y,t)$$

because of (8.2). Therefore v being a supersolution of the equation $\mathcal{F} = 0$, we deduce that v^h is also a supersolution of the same equation.

Step 3: supersolution $\tilde{w}^h$
As a consequence, this implies that

$$w^h(x,t) = v^h(x\alpha(t), t)$$

is also a supersolution of the first line of (7.12). Moreover we have

$$\tilde{w}^h(x,t) = \tilde{w}\left(\frac{x\alpha(t)}{\alpha(t+h)}, t+h\right) \tag{8.3}$$

and thus satisfies

$$\tilde{w}^h(x,0) = c = \tilde{w}^h(0,t) \quad \text{for all} \quad x,t \in [0,+\infty).$$

This shows that $\tilde{w}^h$ is a supersolution of (7.12) (now also including the boundary conditions).
Step 4: conclusion
We can now apply the comparison principle and deduce that for any $h > 0$:

$$\tilde{w}^h(x,t) = \tilde{w}\left(\frac{x\alpha(t)}{\alpha(t+h)}, t+h\right) \geq \tilde{w}(x,t).$$

Fix $t_0 > 0$ and for any $\varepsilon > 0$ (small enough), let us define

$$x_\varepsilon = a(t_0) - \varepsilon > 0.$$

Then we have

$$\tilde{w}\left(\frac{x_\varepsilon\alpha(t_0)}{\alpha(t_0+h)}, t_0+h\right) \geq \tilde{w}(x_\varepsilon, t_0) > 0.$$

This shows that

$$\frac{x_\varepsilon\alpha(t_0)}{\alpha(t_0+h)} < a(t_0+h).$$

Because this is true for any $\varepsilon > 0$ small enough, we deduce that

$$\frac{a(t_0)\alpha(t_0)}{\alpha(t_0+h)} \leq a(t_0+h)$$

which shows that $t \mapsto a(t)\alpha(t)$ is nondecreasing.
Therefore

$$0 \leq (\ln(a\alpha))' = (\ln a)' + (\ln \alpha)' = \frac{a'}{a} + \frac{\alpha'}{\alpha}$$

which implies (8.1). This concludes the proof of the theorem.

9 Convergence of the free boundary as $c \to 0$

The main result of this section is the following

Theorem 9.1 (Convergence of the rescaled free boundary when $c \to 0$)
Assume $\sigma > 0$ and $\lambda \leq 3r + 4\rho$. hen the following convergence of the rescaled free boundary holds: $\bar{a} \leq \frac{a}{c^{\frac{1}{3}}} \to \bar{a}$ locally uniformly on any compact sets of $(0,+\infty)$, as $c \to 0$, where

$$\bar{a}(t) = \left(\frac{3\sigma^2}{2(1+(\rho-\lambda)\alpha(t))}\right)^{\frac{1}{3}}.$$

As a corollary, we can deduce Theorem 1.3.

Proof of Theorem 1.3

Theorem 1.3 follows from Theorems 7.2, 8.1 and 9.1.

9.1 Preliminary results

Lemma 9.2 (Global subsolution and bound from below on the free boundary) *Assume $\sigma > 0$ and $\lambda \leq 3r + 4\rho$. Consider the function*

$$\underline{\tilde{w}}(x,t) = c\,\phi\left(\frac{x}{c^{\frac{1}{3}}\bar{a}(t)}\right)$$

with

$$\bar{a}(t) = \left(\frac{3\sigma^2}{2(1+(\rho-\lambda)\alpha(t))}\right)^{\frac{1}{3}} \tag{9.1}$$

and

$$\phi(\bar{y}) = \begin{cases} \dfrac{\bar{y}^3}{2} - \dfrac{3}{2}\bar{y} + 1 & \text{if} \quad 0 \leq \bar{y} \leq 1, \\ 0 & \text{if} \quad \bar{y} > 1. \end{cases}$$

Then $\underline{\tilde{w}}$ is a subsolution of the equation satisfied by $\tilde{w}$, namely (7.12)*. In particular, we have (for each $c > 0$):*

$$a(t) \geq c^{\frac{1}{3}}\bar{a}(t) \quad \text{for all} \quad t > 0. \tag{9.2}$$

Remark 9.3 Notice that there we have for all $t \geq 0$

$$1 + (\rho - \lambda)\alpha \in \begin{cases} \left[1, \dfrac{r+\rho}{r+\lambda}\right] & \text{if} \quad \rho \geq \lambda, \\ \left[\dfrac{r+\rho}{r+\lambda}, 1\right] & \text{if} \quad \rho \leq \lambda, \end{cases} \tag{9.3}$$

if $r > 0$ and $\rho \geq 0$, $\lambda \geq 0$.

So we have

$$1 + (\rho - \lambda)\alpha \geq \delta_0 = \min\left(1, \frac{r+\rho}{r+\lambda}\right). \tag{9.4}$$

Proof of Lemma 9.2

Our goal is to build a subsolution for $\tilde{w}$ close to the axis $x = 0$.

Step 1: Change of function

Set

$$\tilde{w}(x,t) = cv(y,t) \quad \text{with} \quad y = \frac{x}{c^{\frac{1}{3}}}$$

with

$$\tilde{w}(x,t) > 0 \quad \text{if and only if} \quad x < a(t) =: c^{\frac{1}{3}}\tilde{a}(t).$$

Set

$$F := \mathcal{L}\tilde{w} + f$$

and recall from (7.12) that

$$F = \tilde{w}_t - \frac{1}{2}\sigma^2 \tilde{w}_{xx} + \rho x \tilde{w}_x + r(\tilde{w} - c) + x(1 + (\rho - \lambda)\alpha).$$

This implies that

$$(9.5) \qquad c^{-\frac{1}{3}} F = F_0[v] + c^{\frac{2}{3}} G_0[v] \quad \text{with} \quad \left\{ \begin{array}{l} F_0[v] := -\frac{1}{2}\sigma^2 v_{yy} + y(1 + (\rho - \lambda)\alpha), \\ G_0[v] := v_t + \rho y v_y + r(v - 1). \end{array} \right.$$

Because $\tilde{w}$ satisfies (7.12), v satisfies

$$(9.6) \qquad \left\{ \begin{array}{ll} \min(F_0[v] + c^{\frac{2}{3}} G_0[v], v) = 0 & \text{for all} \quad (y,t) \in \Omega := (0, +\infty)^2, \\ v = 1 & \text{on} \quad \partial\Omega. \end{array} \right.$$

Step 2: Constructing a candidate subsolution

If we neglect completely the term $c^{\frac{2}{3}} G_0[v]$ in (9.6), as a first guess, for each fixed $t > 0$, we can look for a stationary solution v^0 of

$$(9.7) \qquad \left\{ \begin{array}{ll} \min(-\frac{1}{2}\sigma^2 v^0_{yy} + y(1 + (\rho - \lambda)\alpha), v^0) = 0 & \text{for all} \quad y \in (0, +\infty), \\ v^0(y,t) = 1 & \text{for} \quad y = 0 \end{array} \right.$$

with

$$v^0(y,t) > 0 \quad \text{if and only if} \quad y < \bar{a}(t).$$

We can solve this equation explicitly, recalling that for such an obstacle problem, we have

$$v^0_y(\bar{a}(t), t) = 0 = v^0(\bar{a}(t), t).$$

Then we get successively in $\{0 < y \leq \bar{a}\}$

$$(9.8) \qquad v^0_{yy} = 2yA \quad \text{with} \quad A := \frac{(1 + (\rho - \lambda)\alpha)}{\sigma^2},$$

$$v^0_y = (y^2 - \bar{a}^2)A,$$

$$v^0 = \left(\frac{y^3}{3} - y\bar{a}^2 + \frac{2}{3}\bar{a}^3 \right) A.$$

Finally the condition $v^0(0,t) = 1$ implies

$$\bar{a} = \left(\frac{3}{2A} \right)^{\frac{1}{3}} = \left(\frac{3\sigma^2}{2(1 + (\rho - \lambda)\alpha)} \right)^{\frac{1}{3}}$$

which is exactly (9.1). We define $\phi \in C^{1,1}$ by

$$\phi(\bar{y}) = \left\{ \begin{array}{ll} \dfrac{\bar{y}^3}{2} - \dfrac{3}{2}\bar{y} + 1 & \text{if} \quad 0 \leq \bar{y} \leq 1, \\ 0 & \text{if} \quad \bar{y} > 1 \end{array} \right.$$

which satisfies the following properties:

$$\phi(0)=1, \quad \phi(1)=0=\phi'(1), \quad \text{and} \quad \left\{ \begin{array}{c} 1 \geq \phi(\bar{y}) \geq 0, \\ \phi'(\bar{y}) \leq 0 \end{array} \right| \quad \text{for} \quad 0 \leq \bar{y} \leq 1.$$

Then we have

$$v^0(y,t)=\phi(\bar{y}) \quad \text{with} \quad \bar{y}=\frac{y}{\bar{a}(t)}.$$

Step 3: Checking the subsolution property
We set

$$\underline{v}(y,t)=v^0(y,t)=\phi(\bar{y}).$$

By construction, we have

$$\begin{cases} F_0[\underline{v}]=0 & \text{on} \quad \{0<\bar{y}\leq 1\}, \\ 1 \geq \underline{v} \geq 0 & \text{on} \quad (0,+\infty)\times(0,+\infty), \\ \underline{v}(0,t)=1 & \text{for all} \quad t \geq 0. \end{cases}$$

Case 1: $\rho \geq \lambda$
Compute on $\{0<\bar{y}\leq 1\}$:

$$\begin{aligned} G_0[\underline{v}] &= -\frac{\bar{a}'}{\bar{a}}\bar{y}\phi' + \rho\bar{y}\phi' + r(\phi-1) \\ &= \bar{y}\phi'\left(\rho - \frac{\bar{a}'}{\bar{a}}\right) + r(\phi-1) \\ &\leq 0. \end{aligned}$$

In the last line, we have used the fact that $\bar{a}\geq 0, \phi'\leq 0, \phi\leq 1$, and if $\rho\geq\lambda$, then $\bar{a}'\leq 0$.

Case 2: $\lambda \leq 3r+4\rho$
When $\lambda<\rho$, we must use a different estimate. Write $\phi-1=\bar{y}\phi'-\bar{y}^3$ and

$$G_0[\underline{v}]=\bar{y}\phi'\left(r+\rho-\frac{\bar{a}'}{\bar{a}}\right)-r\bar{y}^3.$$

Setting

$$\bar{a}(t)=K(A(t))^{-\frac{1}{3}} \quad \text{with} \quad K=\left(\frac{3\sigma^2}{2}\right)^{\frac{1}{3}} \quad \text{and} \quad A(t)=1+(\rho-\lambda)\alpha(t)$$

we get

$$(r+\rho)\bar{a}-\bar{a}'=K\left\{(r+\rho)A^{\frac{1}{3}}+\frac{1}{3}A^{\frac{4}{3}}(\rho-\lambda)\alpha'\right\}$$

i.e.

$$\frac{(r+\rho)\bar{a}-\bar{a}'}{\frac{K}{3}A^{\frac{4}{3}}}=(\rho-\lambda)\alpha'+3(r+\rho)(1+(\rho-\lambda)\alpha)=g(e^{-(r+\lambda)t})$$

with

$$g(z)=(\rho-\lambda)z+3(r+\rho)\left\{1+(\rho-\lambda)\left(\frac{1-z}{r+\lambda}\right)\right\}$$

which is an affine function. It satisfies $g(0) = \dfrac{3(r+\rho)^2}{r+\lambda} \geq 0$ and

$$g(1) = 3r + 4\rho - \lambda \geq 0.$$

This implies that $g(e^{-(r+\lambda)t}) \geq 0$ and then $G_0[\underline{v}] \leq 0$.

Step 4: Conclusion
This implies that on $\{0 < \bar{y} \leq 1\}$

$$F_0[\underline{v}] + c^{\frac{2}{3}} G_0[\underline{v}] \leq 0.$$

This shows that $\underline{v}$ is a subsolution of (9.6). The comparison principle implies the result. This ends the proof of the lemma.

For some constant $b \in \mathbb{R}$, consider the following problem

$$\begin{cases} \min\left(-\dfrac{\sigma^2}{2} v_{xx} + by,\ v\right) = 0 \quad \text{on} \quad (0,+\infty), \\ v(0) = 1. \end{cases} \tag{9.9}$$

Even if there is no zero order term in the PDE part of problem (9.9), we are able to show the following result (see the proof given in the appendix):

Lemma 9.4 (Comparison principle for a stationary obstacle problem without zero order terms)
Assume $\sigma > 0$. If u (resp. v) is subsolution (resp. supersolution) of (9.9), satisfying

$$u \leq 1, \quad v \geq 0. \tag{9.10}$$

Then

$$u \leq v.$$

9.2 Convergence as $c \to 0$

Proposition 9.5 (Convergence of the rescaled solution as $c \to 0$)
Assume $\sigma > 0$ and $\lambda \leq 3r + 4\rho$. Consider the solution $\tilde{w}$ of (7.12) on $\Omega = (0,+\infty)^2$ and set

$$\tilde{w}(x,t) = c\, v^c(y,t) \quad \textit{with} \quad y = \frac{x}{c^{\frac{1}{3}}}. \tag{9.11}$$

Then

$$v^c \to v^0 \quad \textit{locally uniformly on compact sets of } \Omega \textit{ as} \quad c \to 0 \tag{9.12}$$

where

$$v^0(y,t) = \phi(\bar{y}) \quad \textit{with} \quad \bar{y} = \frac{y}{\bar{a}(t)}$$

with ϕ and $\bar{a}$ defined in Lemma 9.2.

Proof of Proposition 9.5
Step 1: The relaxed semi-limits
We know that v^c satisfies on $\Omega = (0, +\infty)^2$:

$$\begin{cases} \min(F_0[v^c] + c^{\frac{2}{3}} G_0[v^c],\ v^c) = 0, \\ v^0 \leq v^c \leq 1 \end{cases}$$

with F_0, G_0 defined in (9.5). Notice that the condition $v^c \leq 1$ follows from (7.5) and $v^c \geq v^0$ follows from Lemma 9.2. We define the relaxed semi-limits:

$$\overline{v} = \limsup_{c\to 0}{}^* \ v^c, \qquad \underline{v} = \liminf_{c\to 0}{}_* \ v^c.$$

By construction,

$$v^0 \leq \underline{v} \leq \overline{v} \leq 1. \tag{9.13}$$

From the stability of viscosity solutions, we deduce that $\overline{v}$ (resp. $\underline{v}$) is a subsolution (resp. supersolution) of

$$\min(F_0[v],\ v) = 0$$

and (9.13) implies that

$$\underline{v}(0,t) = \overline{v}(0,t) = 1 \quad \text{for all} \quad t > 0. \tag{9.14}$$

Step 2: Sub/supersolutions of the stationary problem
We claim that for any fixed $t_0 > 0$, $\overline{v}(\cdot, t_0)$ (resp. $\underline{v}(\cdot, t_0)$) is a subsolution (resp. supersolution) of

$$\begin{cases} \min(-\frac{1}{2}\sigma^2 v_{yy} + yb(t_0),\ v) = 0 \qquad \text{on} \quad (0, +\infty), \\ v(0) = 0 \end{cases} \tag{9.15}$$

with

$$b(t_0) = (1 + (\rho - \lambda)\alpha(t_0)).$$

We check it for $\overline{v}$ (the raisoning being similar for $\underline{v}$).
Step 2.1: Preliminaries
The boundary condition is obvious because of (9.14). Recall that $\overline{v}$ is upper semi-continuous, and then for any $\delta > 0$ small enough, there exists $r_\delta > 0$ such that

$$\overline{v} \leq \overline{v}(y_0, t_0) + \delta \quad \text{on} \quad \overline{Q_{r_\delta}(P_0)} \subset\subset \Omega$$

where $P_0 = (y_0, t_0)$ and

$$Q_{r_\delta}(P_0) := (y_0 - r_\delta, y_0 + r_\delta) \times (t_0 - r_\delta, t_0 + r_\delta).$$

Consider now a test function φ satisfying

$$\overline{v}(\cdot, t_0) \leq \varphi \quad \text{with equality at } y_0 > 0.$$

Up to add $\eta|y - y_0|^2$ to φ (with η large enough), we can assume that

$$\overline{v}(y_0, t_0) + 2\delta \leq \varphi(y) \quad \text{for all} \quad y = y_0 \pm r_\delta$$

and

$$\overline{v}(y,t_0) < \varphi(y) \quad \text{for all} \quad y \neq y_0. \tag{9.16}$$

Step 2.2: The ε-penalization
For $\varepsilon > 0$, let us define

$$\varphi_\varepsilon(y,t) = \varphi(y) + \frac{(t-t_0)^2}{2\varepsilon}.$$

Up to choosing an ε small enough ($\varepsilon \leq \varepsilon_\delta$), we have

$$\varphi_\varepsilon(y, t_0 \pm r_\delta) \geq \overline{v}(y_0,t_0) + 2\delta \quad \text{for all} \quad y \in [x_0 - r_\delta, x_0 + r_\delta].$$

Therefore

$$\varphi_\varepsilon \geq 2\delta + \overline{v}(y_0,t_0) \geq \delta + \overline{v} \quad \text{on} \quad \partial Q_{r_\delta}(P_0)$$

and

$$(\varphi_\varepsilon - \overline{v})(P_\varepsilon) = \min_{Q_{r_\delta}(P_0)} (\varphi_\varepsilon - \overline{v}) \leq (\varphi_\varepsilon - \overline{v})(P_0) = 0 < \delta \leq \min_{\partial Q_{r_\delta}(P_0)} (\varphi_\varepsilon - \overline{v}) \tag{9.17}$$

for some point $P_\varepsilon = (x_\varepsilon, t_\varepsilon) \in Q_{r_\delta}(P_0)$. This implies that we have

$$\min(-\frac{1}{2}\sigma^2 \varphi_{yy} + y_\varepsilon b(t_\varepsilon),\ \overline{v}) \leq 0 \quad \text{at} \quad P_\varepsilon. \tag{9.18}$$

Because $t_\varepsilon \to 0$ as $\varepsilon \to 0$, and up to some subsequence we have $y_\varepsilon \to \bar{y}$, we deduce from (9.17) that

$$\varphi(\bar{y}) - \overline{v}(\bar{y},t_0) \leq \varphi(\bar{y}) - \limsup_{\varepsilon\to 0} \overline{v}(P_\varepsilon) \leq \liminf_{\varepsilon \to 0} (\varphi_\varepsilon - \overline{v})(P_\varepsilon) \leq 0.$$

Hence (9.16) implies that

$$\varphi(\bar{y}) = \overline{v}(\bar{y},t_0) \quad \text{and} \quad \bar{y} = y_0$$

and then

$$\limsup_{\varepsilon \to 0} \overline{v}(P_\varepsilon) = \varphi(\bar{y}) = \overline{v}(P_0)$$

and passing to the limit in (9.18), we get

$$\min(-\frac{1}{2}\sigma^2 \varphi_{yy} + y_0 b(t_0),\ \overline{v}) \leq 0 \quad \text{at} \quad P_0. \tag{9.19}$$

Indeed, either we have $\overline{v}(P_\varepsilon) \leq 0$ for a subsequence, and we define the $\limsup$ along that subsequence which implies that $\overline{v}(P_0) \leq 0$, or we have $-\frac{1}{2}\sigma^2\varphi_{yy}(P_\varepsilon) + y_\varepsilon b(t_\varepsilon) \leq 0$ for a subsequence, and this implies (9.19).
Step 3: Conclusion
We can now apply the comparison principle (Lemma 9.4), to deduce that

$$\overline{v}(\cdot,t_0) \leq \underline{v}(\cdot,t_0).$$

Because we have the reverse inequality by construction, we deduce that

$$\overline{v}(\cdot,t_0) = \underline{v}(\cdot,t_0) = v^0(\cdot,t_0)$$

where $v^0(\cdot,t_0)$ is the explicit solution of (9.15). This implies (9.12).
This ends the proof of the proposition.

In order to conclude to the convergence of the free boundary itself, we need the following result, which is adapted from Caffarelli [6]:

Lemma 9.6 (Non degeneracy)
Assume $\sigma > 0$ and $\lambda \le 3r + 4\rho$. Let $t_0 > 0$ and

$$y_0 = \frac{a(t_0)}{c^{\frac{1}{3}}}.$$

For $d > 0$, define

$$Q_d^-(y_0, t_0) = (y_0 - d, y_0 + d) \times \left(t_0 - \frac{c^{\frac{2}{3}}}{\sigma^2} d^2, t_0\right).$$

Let $\delta_1 > 0$ such that

$$\begin{cases} 1 + (\rho - \lambda)\alpha(t) \ge \delta_1 > 0 & \text{for all} \quad t \ge 0, \\ \bar{a}(t) \ge 2\delta_1 > 0 & \text{for all} \quad t \ge 0. \end{cases}$$

Let v^c defined in (9.11). If

$$\begin{cases} d \le \delta_1, \\ K \le \dfrac{\delta_1^2}{4\sigma^2}, \\ c^{\frac{2}{3}} \le \min\left(\dfrac{K\sigma^2}{r}, \ \dfrac{2\sigma^2}{\rho\delta_1^2}, \dfrac{t_0\sigma^2}{\delta_1^2}\right) \end{cases}$$

then

$$\sup_{\overline{Q_d^-(y_0,t_0)}} v^c \ge K\frac{d^2}{2}.$$

Proof of Lemma 9.6
Step 1: Auxiliary function
Recall that v^c satisfies on $\Omega = (0, +\infty)^2$

$$\begin{cases} \min(F_0[v^c] + c^{\frac{2}{3}} G_0[v^c], \ v^c) = 0, \\ v^0 \le v^c \le 1 \end{cases}$$

with F_0, G_0 defined in (9.5).
In particular, we have

$$F_0[v^c] + c^{\frac{2}{3}} G_0[v^c] = 0 \quad \text{in} \quad \{v^c > 0\}.$$

Given a point $P_0 = (y_0, t_0) \in \Omega$, consider the auxiliary function

$$\xi(y,t) = K\left(\frac{(y-y_0)^2}{2} - \frac{\sigma^2}{2c^{\frac{2}{3}}}(t - t_0)\right).$$

For any $d > 0$ small enough, let us define

$$Q_d^-(P_0) = (y_0 - d, y_0 + d) \times \left(t_0 - \frac{c^{\frac{2}{3}}}{\sigma^2} d^2, t_0\right) \subset \Omega.$$

We have

$$\partial_p Q_d^-(P_0) = \left([y_0 - d, y_0 + d] \times \left\{t_0 - \frac{c^{\frac{2}{3}}}{\sigma^2} d^2\right\}\right) \quad \cup \quad \left(\{y_0 - d, y_0 + d\} \times \left(t_0 - \frac{c^{\frac{2}{3}}}{\sigma^2} d^2, t_0\right]\right)$$

and

$$\xi \geq K\frac{d^2}{2} \quad \text{on} \quad \partial_p Q_d^-(P_0).$$

Step 2: Non degeneracy

Assume (and we will prove it in the next step) that ξ is a supersolution of the linear parabolic operator, namely assume

$$\text{(9.20)} \qquad \begin{cases} F_0[\xi] + c^{\frac{2}{3}} G_0[\xi] \geq 0 \quad \text{in} \quad Q_d^-(P_0) \cap \{v^c > 0\} =: \omega, \\ \\ P_0 \in \{v^c > 0\}. \end{cases}$$

We now apply a non degeneracy argument due to Caffarelli (see [5, 6]). Define

$$M = \sup_{\overline{\omega}} (v^c - \xi) \geq (v^c - \xi)(P_0) = v^c(P_0) > 0.$$

From the local comparison principle (the proof being similar to the usual proof of the comparison principle), we have

$$M = \sup_{(\partial\omega)\setminus\{t=t_0\}} (v^c - \xi).$$

But

$$(\partial\omega)\setminus\{t=t_0\} \subset \Gamma_0 \cup \Gamma_d \quad \text{with} \quad \begin{cases} \Gamma_0 = \Big((\partial\{v^c > 0\}) \cap \overline{Q_d^-(P_0)}\Big) \setminus \{t = t_0\}, \\ \Gamma_d = \Big(\big(\partial Q_d^-(P_0)\big) \cap \overline{\{v^c > 0\}}\Big) \setminus \{t = t_0\}. \end{cases}$$

But

$$v^c - \xi = -\xi < 0 \quad \text{on} \quad \Gamma_0$$

and therefore

$$0 < M = (v^c - \xi)(P) \quad \text{for some point} \quad P \in \Gamma_d.$$

This implies that

$$v^c(P) \geq \xi(P) \geq K\frac{d^2}{2}.$$

This shows that (9.20) implies the following non degeneracy property

$$\text{(9.21)} \qquad \sup_{\overline{Q_d^-(P_0)}} v^c \geq K\frac{d^2}{2}.$$

Step 3: Proof of (9.20)

For the reader convenience, we recall that

$$\begin{cases} F_0[\xi] := -\frac{1}{2}\sigma^2\xi_{yy} + y(1 + (\rho - \lambda)\alpha), \\ G_0[\xi] := \xi_t + \rho y \xi_y + r(\xi - 1). \end{cases}$$

Compute:

$$F_0[\xi] + c^{\frac{2}{3}} G_0[\xi] \quad = -K\sigma^2 + y(1 + (\rho - \lambda)\alpha) + c^{\frac{2}{3}} B$$

with

$$B = K\rho y(y - y_0) + r(\xi - 1) \geq -r + K\rho y(y - y_0).$$

Let $\delta_1 > 0$ such that

$$\begin{cases} 1 + (\rho - \lambda)\alpha \geq \delta_1 > 0 & \text{for all} \quad t \geq 0, \\ \bar{a}(t) \geq 2\delta_1 > 0 & \text{for all} \quad t \geq 0. \end{cases}$$

Then the previous computation shows that for $(y,t) \in Q_d^-(P_0)$

$$\begin{aligned} F_0[\xi] + c^{\frac{2}{3}} G_0[\xi] &\geq y \left\{ 1 + (\rho - \lambda)\alpha - c^{\frac{3}{2}} K \rho d \right\} - \left(K\sigma^2 + c^{\frac{3}{2}} r \right) \\ &\geq y \left(\delta_1 - c^{\frac{2}{3}} K \rho d \right) - \left(K\sigma^2 + c^{\frac{3}{2}} r \right) \\ &\geq \delta_1 \left(\delta_1 - c^{\frac{2}{3}} K \rho \delta_1 \right) - \left(K\sigma^2 + c^{\frac{3}{2}} r \right) \\ &\geq 0 \end{aligned}$$

for $y_0 - d \geq \delta_1$ and $d \leq \delta_1 \leq \dfrac{\bar{a}(t_0)}{2}$, and

$$\begin{cases} c^{\frac{2}{3}} r \leq K\sigma^2, \\ c^{\frac{2}{3}} K \rho \leq \dfrac{1}{2}, \\ 4K\sigma^2 \leq \delta_1^2, \\ c^{\frac{2}{3}} \leq \dfrac{t_0 \sigma^2}{d^2} \end{cases}$$

i.e.

$$\begin{cases} K \leq \dfrac{\delta_1^2}{4\sigma^2}, \\ c^{\frac{2}{3}} \leq \min\left(\dfrac{K\sigma^2}{r}, \ \dfrac{2\sigma^2}{\rho \delta_1^2}, \dfrac{t_0 \sigma^2}{\delta_1^2} \right). \end{cases} \tag{9.22}$$

Step 4: Conclusion
Consider now

$$y_0 := \frac{a(t_0)}{c^{\frac{1}{3}}}$$

which satisfies $y_0 \geq \bar{a}(t_0) \geq 2\delta_1$ because of (9.2). We now consider a sequence of points y_n

$$\begin{cases} y_n < y_0, \\ y_n \to y_0 \end{cases}$$

and a sequence d_n such that for any $d \in (0, \delta_1]$:

$$\begin{cases} y_n - d_n \geq \delta_1, \\ d_n \to d. \end{cases}$$

Then assuming (9.22), and applying the previous steps at the point (y_n, t_0), we get

$$\overline{\sup_{Q_{d_n}^-(y_n,t_0)}} v^c \geq K \frac{d_n^2}{2}.$$

Passing to the limit in n, this implies

$$\overline{\sup_{Q_d^-(y_0,t_0)}} v^c \geq K \frac{d^2}{2}.$$

This ends the proof of the lemma.

Proof of Theorem 9.1
We already know from (9.2) that

$$\frac{a}{c^{\frac{1}{3}}} \geq \bar{a}.$$

Assume by contradiction that the statement is false.
Then for any $\delta > 0$, there exists $\eta > 0$ and a sequence of times $t_c \in [\delta, 1/\delta]$, such that

$$y_c := \frac{a(t_c)}{c^{\frac{1}{3}}} \geq \eta + \bar{a}(t_c).$$

Applying Lemma 9.6, we get for $c > 0$ small enough that for any $d \in [0, \delta_1]$

$$\overline{\sup_{Q_d^-(y_c,t_c)}} v^c \geq K\frac{d^2}{2}.$$

From Proposition 9.5, we know that $v^c \to v^0$ locally uniformly. Moreover, extracting a subsequence if nenessary, we can assume that

$$(y_c, t_c) \to (y_0, t_0) \quad \text{with} \quad y_0 \geq \eta + \bar{a}(t_0).$$

This implies that for any $d \in (0, \delta_1]$ and $K > 0$

$$\sup_{y \in [y_0 - d, y_0 + d]} v^0(y, t_0) \geq K\frac{d^2}{2} > 0.$$

For $0 < d \leq \eta$, this gives a contradiction, because

$$v^0(y, t_0) = 0 \quad \text{for} \quad y \geq y_0 - d \geq \bar{a}(t_0).$$

This ends the proof of the theorem.

10 No comparison principle for $c = 0$ and $\sigma > 0$

In this section, we discuss the existence of multiple solutions for $c = 0$, i.e. solutions u^0 of

$$(10.1) \qquad \begin{cases} \min\left(\mathcal{L}u^0, u^0(x,t) - u^0(-x,t) - \psi^0(x,t)\right) = 0 \quad \text{for} \quad (x,t) \in \mathbb{R} \times (0, +\infty), \\ u^0(x,0) = 0 \quad \text{for} \quad x \in \mathbb{R}, \end{cases}$$

with $\psi^0(x,t) = \alpha(t)x$. Indeed, for $c = 0$, we can reduce the construction of solutions to a more classical problem.

Proposition 10.1 (Family of solutions u^0 for $c = 0$ and $\sigma > 0$) *For $c = 0$, there exists an infinite familly of viscosity solutions u^0 of (10.1), such that for each u^0, there exists a constant $C > 0$ such that $|u^0 - \max(0, \psi)| \leq C$. More precisely, in case $c = 0$, all viscosity solutions $u^0(x,t)$ of*

$$(10.2) \qquad \begin{cases} u^0(x,t) - u^0(-x,t) \equiv \psi^0(x,t), \quad \text{for all} \quad x \in \mathbb{R},\ t \in (0,T), \\ \mathcal{L}u^0 \geq 0, \quad \text{for} \quad x \in \mathbb{R},\ t \in (0,T), \\ u^0(x,0) = 0 \quad \text{for} \quad x \in \mathbb{R}, \end{cases}$$

are viscosity solutions of (10.1).

Remark 10.2 (No comparison principle for $c = 0$ and $\sigma > 0$)
This shows that the condition $c > 0$ for our comparison principle is sharp, since the comparison principle is not valid when $c = 0$.

Before giving a rigorous proof of Proposition 10.1, let us explain heuristically why we expect to have non uniqueness of solutions of (10.1) in case $c = 0$. First, when $c < 0$ we expect that $u \equiv +\infty$, because we gain $-c$ in each transaction and there are no limits to the amount of trades one can do in any interval. As a consequence, for $c = 0$, we expect to have a transition family of solutions u^0 between the two limit cases $u \equiv +\infty$ for $c < 0$ and $u = u_0$ for $c = 0^+$. Thus, in particular we loose the comparison principle when $c = 0$. Another heuristic argument starts from observing that the functions $w^0(x,t) \equiv u^0(x,t) - u^0(-x,t)$ and $\psi^0(x,t)$ are odd in $x \in \mathbb{R}$, and thus the inequality $u^0(x,t) - u^0(-x,t) \geq \psi^0(x,t)$ on all of $\mathbb{R}$ implies that $u^0(x,t) - u^0(-x,t) \equiv \psi^0(x,t)$. This equality holds pointwise. Hence the system of complementary inequalities (10.1) reduces to $\mathcal{L}u^0 \geq 0$ in the viscosity sense, and one should expect the existence of many solutions.

Proof of Proposition 10.1.
Since we deal with viscosity solutions, a solution of (10.2) is obviously a solution of (10.1). It is clear that system (10.2) admits a large set of solutions. First, we note that it is a convex set. Then, to construct solutions of (10.2) we can reduce the problem to a problem on the half-line in the following manner.

Proposition 10.3 *Let v be a $\mathrm{C}^{2,1}(\mathbb{R}_- \times (0,T))$ solution of the following system:*

$$\begin{cases} \mathcal{L}v \geq 0 & \textit{for} \quad (x,t) \in (-\infty, 0) \times (0,T), \\ v_x(0,t) = \dfrac{\alpha(t)}{2} & \textit{for} \quad t \in (0,T), \\ v(x,0) = 0 & \textit{for} \quad x \in (-\infty, 0). \end{cases} \tag{10.3}$$

Then, u^0 defined by $u^0(x,t) = v(x,t)$ for $x \in \mathbb{R}^-$ and $u^0(x,t) = v(-x,t) + \psi^0(x,t)$ for $x \geq 0$ is a (classical) solution of (10.2).

It is straightforward to check that the construction of u^0 from v yields a function of class $\mathrm{C}^{2,1}$. Further, the function u^0 recovered from a solution v of (10.3) satisfies $\mathcal{L}u^0 \geq 0$. Indeed, this is true on $\mathbb{R}^-$ by the inequality for v. Now, for a function $z = z(x,t)$ denote z^* the function defined by $z^*(x,t) = z(-x,t)$. Observe that $(\mathcal{L}(v))^* = \mathcal{L}(v^*)$. Thus, on $\mathbb{R}^+$ we get

$$\mathcal{L}u^0 = \mathcal{L}v^* + \mathcal{L}\psi^0.$$

We know that on $\mathbb{R}^+$, $\mathcal{L}v^* = (\mathcal{L}v)^* \geq 0$ and

$$\mathcal{L}\psi^0 = x\left(\alpha'(t) + (\rho + r)\alpha(t)\right) \geq 0 \quad \text{for} \quad x > 0 \quad \text{because} \quad \alpha, \alpha' \geq 0.$$

Therefore, we see that $\mathcal{L}u^0 \geq 0$ on $\mathbb{R}^+$ as well. Note also that $\alpha(t) = 2u^0_x(0,t)$.

Now, for instance, a solution (10.3) can be obtained in the form

$$z(x,t) = \alpha(t)\frac{\kappa(x)}{2\kappa'(0)}$$

where κ is a Kummer function (see [18] for the construction and properties of Kummer functions in our setting) satisfying:

$$\mathcal{M}\kappa = 0,\ \kappa > 0 \quad \text{on} \quad (-\infty, 0].$$

We know that $\kappa(-\infty) = 0, \kappa'(x) > 0$ and $\kappa'(0) > 0$ (see [18]). The function z satisfies $\mathcal{L}z = \alpha'(t)\kappa > 0$ since $\alpha'(t) > 0$.

We thus get a solution z of (10.3) such that $z(-\infty) = 0$. As we have seen, such a solution yields a solution u^0 of our original problem (10.1) such that $|u^0 - \max(0, \psi)| \le C$. In fact, it satisfies $\lim_{|x|\to\infty} |u^0 - \max(0, \psi)| = 0$.

Next, for instance, by using the method of super and sub-solution we can construct another solution v that satisfies equality in the first line of (10.3) rather than an inequality. Since the set of solutions of (10.3) is convex, it is clear that because of the inequality in (10.3), there is a very large indeterminacy.

As a further example we can construct a one parameter family of solutions by considering the operators

$$\mathcal{L}_s u = u_t + \mathcal{M}_s u \quad \text{with} \quad \mathcal{M}_s u = -\frac{1}{2}\sigma^2 u_{xx} + \rho x u_x + su.$$

Let κ_s be a Kummer function associated with $\mathcal{M}_s$: $\mathcal{M}_s\kappa_s = 0,\ \kappa_s > 0$ on $(-\infty, 0]$. Then define

$$z_s(x,t) = \alpha(t)\frac{\kappa_s(x)}{2\kappa_s'(0)}.$$

For each s with $0 < s < r$, we get $\mathcal{L}z_s > 0$. Thus, the functions z_s for $0 < s < r$ is a one parameter family of (pairwise distinct) solutions of (10.3). Likewise they yield a one parameter family of solutions of our original problem (10.1) each of which satisfies $\lim_{|x|\to\infty} |u^0 - \max(0, \psi)| = 0$.

This concludes the proof of the proposition.

Remark 10.4 (An explicit supersolution) In the proof of Proposition 10.1, we can also use in place of the Kummer function $\kappa(x)$, the function $\tilde{\kappa}(x) = \alpha(+\infty)\phi(x)$ where ϕ is the function constructed in Lemma 4.3 (for supersolutions of the u-problem). One can verify that $\tilde{\kappa}$ is a supersolution for the stationary problem (1.3) in the case $c = 0$. Proposition 10.3 holds for the function $v(x,t) = \tilde{z}(x,t) = \alpha(t)\frac{\tilde{\kappa}(x)}{2\tilde{\kappa}'(0)}$, because v is $C^{2,1}$ in a neighborhood of $x = 0$. This produces one viscosity solution u^0 for (10.2) for $x \in \mathbb{R}$.

Remark 10.5 (A particular solution u^0) In a recent work, H. Tian [19] constructed a particular solution u^0 of (10.1) from a function v that satisfies (10.3) with equality in the first line (i.e. $\mathcal{L}v = 0$ on the negative half line). One can show that there is a unique function v which satisfies (10.3) and $\mathcal{L}v(x,t) = 0$ for $(x,t) \in (-\infty, 0) \times (0, T)$. This can be verified by setting $z(x,t) = v(x,t) - x\dfrac{\alpha(t)}{2}g(x)$, where g is a smooth function with compact support such that $g'(0) = 1$. Then $z_x(0,t) = 0$ and one can check that $\mathcal{L}z = f(x,t)$ for $x < 0$.

Extending f and z as even functions for positive x (still using the same notation for the extension), we see that z now solves the same PDE for all real x. This can be checked also for $x = 0$ using the boundary regularity of the solution on the half line. We can now apply a comparison principle on the whole real line to derive the uniqueness of the function v. (The proof of this comparison principle is similar to the one of the u-problem, but is much simpler here.)

11 Appendix

11.1 Proof of Jensen-Ishii lemma for the obstacle problem (Lemma 11.1)

Lemma 11.1 (Jensen-Ishii lemma for the obstacle problem)
Let u (resp. v) be a subsolution (resp. a supersolution) of (3.1) on $\mathbb{R} \times [0, T)$ for some $T > 0$, satisfying

$$u(x,t) \leq C_T(1+\max(0,x)) \quad \textit{and} \quad u(x,t) \geq -C_T(1+\max(0,x)) \quad \textit{for all} \quad (x,t) \in \mathbb{R}\times[0,T).$$

Let for $(z_0, s_0) \in \mathbb{R} \times (0, T)$ and $\varepsilon, \beta, \eta > 0$ and $\delta \geq 0$:

$$\tilde{u}(x,t) = u(x,t) - \beta\frac{x^2}{2} - \frac{\delta}{4}|x - z_0|^2$$

and

$$\varphi_\delta(x,y,t) = \frac{(x-y)^2}{2\varepsilon} + \frac{\eta}{T-t} + \frac{\delta}{2}|t - s_0|^2$$

and

$$\Phi_\delta(x,y,t) = \tilde{u}(x,t) - v(y,t) - \varphi_\delta(x,y,t).$$

Assume that there exists a point $(\bar{x}, \bar{y}, \bar{t}) \in \mathbb{R}^2 \times (0, T)$ such that

$$\sup_{(x,y,t)\in\mathbb{R}^2\times[0,T)} \Phi_\delta(x,y,t) = \Phi_\delta(\bar{x},\bar{y},\bar{t}).$$

Then

$$(11.4) \qquad \begin{cases} B_1 \leq 0 \quad \textit{and} \quad B_2 \geq 0, \\ \\ OR \\ \\ A_1 \leq 0 \quad \textit{and} \quad A_2 \geq 0 \quad \textit{and} \quad \begin{cases} \textit{there exist } \tau_1, \tau_2, X, Y \in \mathbb{R} \\ \textit{such that (11.7) holds true} \end{cases} \end{cases}$$

with

$$(11.5) \qquad \begin{cases} X \leq Y, \\ \tau_1 - \tau_2 = (\varphi_\delta)_t = \dfrac{\eta}{(T-\bar{t})^2} + \delta(\bar{t} - s_0) \end{cases}$$

and for $p = \dfrac{\bar{x}-\bar{y}}{\varepsilon}$

$$\begin{cases} A_1 := \tau_1 - \frac{1}{2}\sigma^2(X+\beta+3\delta|\bar{x}-z_0|^2) + \rho\bar{x}(p+\beta\bar{x}+\delta(\bar{x}-z_0)^3) + ru(\bar{x},\bar{t}), \\ B_1 := u(\bar{x},\bar{t}) - u(-\bar{x},\bar{t}) - \psi(\bar{x},\bar{t}), \\ A_2 := \tau_2 - \frac{1}{2}\sigma^2 Y + \rho\bar{y}p + rv(\bar{y},\bar{t}), \\ B_2 := v(\bar{y},\bar{t}) - v(-\bar{y},\bar{t}) - \psi(\bar{y},\bar{t})). \end{cases}$$

In particular we have

$$A \le 0 \quad OR \quad (B \le 0 \quad and \quad B_1 \le 0) \tag{11.6}$$

with

$$\begin{cases} A := A_1 - A_2 = & \dfrac{\eta}{(T-\bar{t})^2} + \delta(\bar{t}-s_0) - \dfrac{1}{2}\sigma^2(X-Y+\beta+3\delta|\bar{x}-z_0|^2) \\ & +\rho\left(\dfrac{(\bar{x}-\bar{y})^2}{\varepsilon} + \beta\bar{x}^2 + \delta\bar{x}(\bar{x}-z_0)^3\right) + r\left(u(\bar{x},\bar{t}) - v(\bar{y},\bar{t})\right), \\ B = B_1 - B_2 = & u(\bar{x},\bar{t}) - v(\bar{y},\bar{t}) - (u(-\bar{x},\bar{t}) - v(-\bar{y},\bar{t})) - (\psi(\bar{x},\bar{t}) - \psi(\bar{y},\bar{t})). \end{cases}$$

Proof of Lemma 11.1
When we can apply Jensen-Ishii Lemma (as stated in Theorem 7 in [10], or Theorem 8.3 in the User's Guide [11]), this shows that for any $\gamma > 0$, we have

$$\begin{cases} (\tau_1, (\varphi_\delta)_x(\bar{x},\bar{y},\bar{t}), X) \in \overline{P}^{2,+}\tilde{u}(\bar{x},\bar{t}), \\ (\tau_2, -(\varphi_\delta)_y(\bar{x},\bar{y},\bar{t}), Y) \in \overline{P}^{2,-}v(\bar{y},\bar{t}), \\ \tau_1 - \tau_2 = (\varphi_\delta)_t = \dfrac{\eta}{(T-\bar{t})^2} + \delta(\bar{t}-s_0), \\ \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le A_0 + \gamma A_0^2 \quad \text{with} \quad A_0 = D^2\varphi_\delta(\bar{x},\bar{y},\bar{t}) = \dfrac{1}{\varepsilon}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \end{cases} \tag{11.7}$$

which implies (11.5). In order to be able to apply Jensen-Ishii Lemma, we need to have the two following bounds

$$\tau_1 \le C \tag{11.8}$$

and

$$\tau_2 \ge -C \tag{11.9}$$

for any point close to $(\bar{x},\bar{t})$ and $(\bar{y},\bar{t})$ with bounded values of the functions, their gradients and their hessians.

However we can not apply directly Jensen-Ishii Lemma, because, while we can obtain bound (11.9), we can not establish bound (11.8).

Instead, we go back to the proof of Jensen-Ishii Lemma based on Lemma 8 in [10]. The idea developed in Lemma 8 in [10] is simply to make a doubling of variable in time, replacing Φ_δ by

$$\tilde{u}(x,t) - v(y,s) - \varphi_\delta(x,y,t) - \frac{(t-s)^2}{2\tilde{\delta}}$$

and to apply the standard elliptic Jensen-Ishii Lemma which does not require (11.8) and (11.9), and finally to pass to the limit $\tilde{\delta} \to 0$.

It suffices to notice that before passing to the limit, we get for some points $(\bar{x}_{\tilde{\delta}}, \bar{t}_{\tilde{\delta}})$, $(\bar{y}_{\tilde{\delta}}, \bar{s}_{\tilde{\delta}})$ and $p_{\tilde{\delta}} = \dfrac{\bar{x}_{\tilde{\delta}} - \bar{y}_{\tilde{\delta}}}{\varepsilon}$ and $(\tau_{1,\tilde{\delta}}, p_{\tilde{\delta}}, X_{\tilde{\delta}}) \in \overline{P}^{2,+}\tilde{u}(\bar{x}_{\tilde{\delta}}, \bar{t}_{\tilde{\delta}})$, $(\tau_{2,\tilde{\delta}}, p_{\tilde{\delta}}, Y_{\tilde{\delta}}) \in \overline{P}^{2,-}v(\bar{y}_{\tilde{\delta}}, \bar{s}_{\tilde{\delta}})$, such that we have the analogue of (11.4), i.e.

$$\min(A_{1,\tilde{\delta}}, B_{1,\tilde{\delta}}) \leq 0 \quad \text{and} \quad \min(A_{2,\tilde{\delta}}, B_{2,\tilde{\delta}}) \geq 0 \tag{11.10}$$

with

$$\begin{cases} A_{1,\tilde{\delta}} := \tau_{1,\tilde{\delta}} - \frac{1}{2}\sigma^2(X_{\tilde{\delta}} + \beta + 3\delta|\bar{x}_{\tilde{\delta}} - z_0|^2) + \rho\bar{x}_{\tilde{\delta}}(p_{\tilde{\delta}} + \beta\bar{x}_{\tilde{\delta}} + \delta(\bar{x}_{\tilde{\delta}} - z_0)^3) + ru(\bar{x}_{\tilde{\delta}}, \bar{t}_{\tilde{\delta}}), \\ B_{1,\tilde{\delta}} := u(\bar{x}_{\tilde{\delta}}, \bar{t}_{\tilde{\delta}}) - u(-\bar{x}_{\tilde{\delta}}, \bar{t}_{\tilde{\delta}}) - \psi(\bar{x}_{\tilde{\delta}}, \bar{t}_{\tilde{\delta}}), \\ A_{2,\tilde{\delta}} := \tau_{2,\tilde{\delta}} - \frac{1}{2}\sigma^2 Y_{\tilde{\delta}} + \rho\bar{y}_{\tilde{\delta}}p_{\tilde{\delta}} + rv(\bar{y}_{\tilde{\delta}}, \bar{s}_{\tilde{\delta}}), \\ B_{2,\tilde{\delta}} := v(\bar{y}_{\tilde{\delta}}, \bar{s}_{\tilde{\delta}}) - v(-\bar{y}_{\tilde{\delta}}, \bar{s}_{\tilde{\delta}}) - \psi(\bar{y}_{\tilde{\delta}}, \bar{s}_{\tilde{\delta}}). \end{cases}$$

Case $A_{1,\tilde{\delta}} \leq 0$
Then we can bound $\tau_{1,\tilde{\delta}}$ (and therefore get (11.8)), which implies the second line of (11.4) in the limit $\tilde{\delta} \to 0$.
Case $A_{1,\tilde{\delta}} > 0$
Then we can not bound $\tau_{1,\tilde{\delta}}$, but we have

$$B_{1,\tilde{\delta}} \leq 0$$

and we always have

$$B_{2,\tilde{\delta}} \geq 0$$

and passing to the limit $\tilde{\delta} \to 0$, we get the first line of (11.4).
This ends the proof of the lemma.

11.2 Proof of the equation satisfied by w (Lemma 11.2)

Lemma 11.2 (Equation satisfied by w)
Assume (1.1) *and let* u *be the solution given in Theorem* 1.1. *Set*

$$w(x,t) = u(x,t) - u(-x,t).$$

Then, in the viscosity sense (see Definition 2.4 *above)* w *solves:*

$$\begin{cases} \min(\mathcal{L}w, \quad w - \psi) = 0 & \text{for} \quad (x,t) \in (0,+\infty) \times (0,+\infty), \\ w(0,t) = 0 & \text{for} \quad t \in [0,+\infty), \\ w(x,0) = 0 & \text{for} \quad x \in [0,+\infty). \end{cases} \tag{11.11}$$

Proof of Lemma 11.2
We want to check that the continuous function w solves (11.11) in the viscosity sense. The boundary conditions are obvious.
Step 1: w is a subsolution
We now check the subsolution property for the PDE . To this end, we consider a test function φ such that

$$w \leq \varphi \quad \text{with equality at} \quad (x_0, t_0) \in (0, +\infty) \times (0, +\infty)$$

and we want to show that

$$\min \{(\mathcal{L}\varphi)(x_0, t_0), \quad w(x_0, t_0) - \psi(x_0, t_0)\} \leq 0. \tag{11.12}$$

Up to change φ in $\varphi + |(x,t) - (x_0, t_0)|^4$, we can assume that

$$(w - \varphi)(x, t) \leq -|(x,t) - (x_0, t_0)|^4. \tag{11.13}$$

We have for some $T > t_0$:

$$M = \sup_{(x,t) \in [0,+\infty) \times [0,T)} (w - \varphi)(x, t) = 0.$$

For $\varepsilon > 0$, we set

$$M_\varepsilon = \sup_{(x,y,t) \in [0,+\infty)^2 \times [0,T)} \Phi_\varepsilon(x, y, t)$$

with

$$\Phi_\varepsilon(x, t, y, s) = \tilde{u}(x, t) - \check{u}(y, t) - \tilde{\varphi}_\varepsilon(x, y) \quad \text{with} \quad \begin{cases} \tilde{u}(x,t) = u(x,t) - \varphi(x,t), \\ \tilde{\varphi}_\varepsilon(x,y) = \dfrac{(x-y)^2}{2\varepsilon}, \\ \check{u}(y,t) := u(-y, t). \end{cases}$$

Then we have

$$M_\varepsilon \geq M = 0.$$

We also recall that

$$0 \leq u(x,t) \leq C(1 + \max(0, x)).$$

This implies that

$$\begin{aligned} \Phi(x, y, t) &= w(x,t) - \varphi(x,t) + u(-x,t) - u(-y,t) - \frac{(x-y)^2}{2\varepsilon} \\ &\leq -|(x,t) - (x_0,t_0)|^4 + C(1 + \max(0,x)) - \frac{(x-y)^2}{2\varepsilon}. \end{aligned}$$

This implies that the suppremum in M_ε is reached at some point $(\bar{x}, \bar{y}, \bar{t}) \in [0, +\infty)^2 \times [0, T]$ and $\bar{x}$ stays bounded as $\varepsilon \to 0$. Here we use the convention that $\bar{t} = T$ if the sequence of points optimizing M_ε converges to T in time. It is then classical that $M_\varepsilon \to M$ as $\varepsilon \to 0$ and

$$(\bar{x}, \bar{y}, \bar{t}) \to (x_0, x_0, t_0)$$

which in particular excludes the case $\bar{t} = T$ for ε small enough.

From an adaptation of Jensen-Ishii Lemma (similar to Lemma 11.1), we get the viscosity inequalities with $p = \dfrac{\bar{x} - \bar{y}}{\varepsilon}$

$$
(11.14) \qquad \begin{cases} B_1 \le 0 \quad \text{and} \quad B_2 \ge 0, \\ \\ \text{OR} \\ \\ A_1 \le 0 \quad \text{and} \quad A_2 \ge 0 \qquad \text{and} \quad \begin{cases} \text{there exist} \\ (\tau_1, (\tilde{\varphi}_\varepsilon)_x(\bar{x},\bar{y}), X) \in \overline{P}^{2,+}\tilde{u}(\bar{x},\bar{t}), \\ (\tau_2, -(\tilde{\varphi}_\varepsilon)_y(\bar{x},\bar{y}), Y) \in \overline{P}^{2,-}\check{u}(\bar{y},\bar{t}), \\ \text{such that (11.15) holds true} \end{cases} \end{cases}
$$

with

$$
(11.15) \qquad \begin{cases} X \le Y, \\ \tau_1 - \tau_2 = (\tilde{\varphi}_\varepsilon)_t = 0 \end{cases}
$$

and using the fact that $\check{u}$ solves (1.2) with ψ replaced by $\check{\psi}(z,t) = \psi(-z,t)$, we get

$$
\begin{cases} A_1 = \tau_1 + \varphi_t(\bar{x},\bar{t}) - \frac{1}{2}\sigma^2(X + \varphi_{xx}(\bar{x},\bar{t})) + \rho\bar{x}(p + \varphi_x(\bar{x},\bar{t})) + ru(\bar{x},\bar{t}), \\ \\ B_1 = u(\bar{x},\bar{t}) - u(-\bar{x},\bar{t}) - \psi(\bar{x},\bar{t}), \\ \\ A_2 = \tau_2 - \frac{1}{2}\sigma^2 Y + \rho\bar{y}p + ru(-\bar{y},\bar{t}), \\ \\ B_2 = u(-\bar{y},\bar{t}) - u(\bar{y},\bar{t}) - \psi(-\bar{y},\bar{t}). \end{cases}
$$

Therefore we have either

$$
(11.16) \qquad B_1 = w(\bar{x},\bar{t}) - \psi(\bar{x},\bar{t}) \le 0
$$

or

$$
A \le 0
$$

with

$$
\begin{aligned} A := A_1 - A_2 = \quad & \varphi_t(\bar{x},\bar{t}) - \frac{1}{2}\sigma^2(X - Y + \varphi_{xx}(\bar{x},\bar{t})) \\ & + \rho\left\{\bar{x}\ \varphi_x(\bar{x},\bar{t}) + \frac{(\bar{x}-\bar{y})^2}{\varepsilon}\right\} + r\left(u(\bar{x},\bar{t}) - u(-\bar{y},\bar{t})\right). \end{aligned}
$$

But we know that $X \le Y$, we see that $A \le 0$ implies

$$
(11.17) \qquad \varphi_t(\bar{x},\bar{t}) - \frac{1}{2}\sigma^2\varphi_{xx}(\bar{x},\bar{t}) + \rho\bar{x}\ \varphi_x(\bar{x},\bar{t}) + r\left(u(\bar{x},\bar{t}) - u(-\bar{y},\bar{t})\right) \le 0.
$$

Passing to the limit in (11.16) and (11.17), we see that we get (11.12).
Step 2: w is a supersolution
Similarly, we consider a test function φ such that

$$
w \ge \varphi \quad \text{with equality at} \quad (x_0,t_0) \in (0,+\infty) \times (0,+\infty)
$$

and we want to show that

$$\min\{(\mathcal{L}\varphi)(x_0,t_0),\quad w(x_0,t_0)-\psi(x_0,t_0)\}\geq 0. \tag{11.18}$$

Up to change φ in $\varphi-|(x,t)-(x_0,t_0)|^4$, we can assume that

$$(w-\varphi)(x,t)\geq |(x,t)-(x_0,t_0)|^4. \tag{11.19}$$

We have for some $T>t_0$:

$$M=\inf_{(x,t)\in[0,+\infty)\times[0,T)}(w-\varphi)(x,t)=0.$$

For $\varepsilon>0$, we set

$$M_\varepsilon=\inf_{(x,y,t)\in[0,+\infty)^2\times[0,T)}\Phi_\varepsilon(x,y,t)$$

with

$$\Phi_\varepsilon(x,t,y,s)=\tilde{u}(x,t)-\check{u}(y,t)-\tilde{\varphi}_\varepsilon(x,y)\quad\text{with}\quad\left\{\begin{array}{l}\tilde{u}(x,t)=u(x,t)-\varphi(x,t),\\[2ex] \tilde{\varphi}_\varepsilon(x,y)=-\dfrac{(x-y)^2}{2\varepsilon},\\[2ex] \check{u}(y,t):=u(-y,t).\end{array}\right.$$

Then we have

$$M_\varepsilon\leq M=0.$$

We also recall that

$$0\leq u(x,t)\leq C(1+\max(0,x)).$$

This implies that

$$\begin{aligned}\Phi(x,y,t)&=w(x,t)-\varphi(x,t)+u(-x,t)-u(-y,t)+\frac{(x-y)^2}{2\varepsilon}\\&\geq|(x,t)-(x_0,t_0)|^4-C(1+\max(0,-y))+\frac{(x-y)^2}{2\varepsilon}.\end{aligned}$$

This implies that the infimum in M_ε is reached at some point $(\bar{x},\bar{y},\bar{t})\in[0,+\infty)^2\times[0,T]$ and $\bar{x}$ stays bounded as $\varepsilon\to 0$. It is then classical that $M_\varepsilon\to M$ as $\varepsilon\to 0$ and

$$(\bar{x},\bar{y},\bar{t})\to(x_0,x_0,t_0).$$

Making an adaptation of Jensen-Ishii Lemma (similar to the proof of Lemma 11.1), we get at a $\tilde{\delta}$ level:

$$\left\{\begin{array}{l}\min(A_{1,\tilde{\delta}},B_{1,\tilde{\delta}})\geq 0,\\[2ex] \min(A_{2,\tilde{\delta}},B_{2,\tilde{\delta}})\leq 0,\\[2ex] A_{2,\tilde{\delta}}\leq 0\quad\text{because}\quad-\bar{y}_{\tilde{\delta}}<\dfrac{c}{\alpha(\bar{t}_{\tilde{\delta}})}\end{array}\right. \tag{11.20}$$

where we have used in (11.20), the modified equation (6.3) satisfied by the solution constructed in Proposition 6.2, which implies additional properties for the subsolution $\check{u}$.
This implies

$$\begin{cases} B_{1,\tilde{\delta}} \geq 0, \\ \\ \text{AND} \\ \\ A_{1,\tilde{\delta}} \geq 0 \quad \text{and} \quad A_{2,\tilde{\delta}} \leq 0. \end{cases}$$

Therefore, at the limit $\tilde{\delta} \to 0$, we get the viscosity inequalities with $-p = \dfrac{\bar{x}-\bar{y}}{\varepsilon}$

$$\begin{cases} B_1 \geq 0, \\ \\ \text{AND} \\ \\ A_1 \geq 0 \quad \text{and} \quad A_2 \leq 0 \qquad \text{and} \quad \begin{cases} \text{there exist } \tau_1, \tau_2, X, Y \in \mathbb{R} \\ \text{such that (11.22) holds true} \end{cases} \end{cases} \tag{11.21}$$

with

$$\begin{cases} X \geq Y, \\ \tau_1 - \tau_2 = (\tilde{\varphi}_\varepsilon)_t = 0 \end{cases} \tag{11.22}$$

and

$$\begin{cases} A_1 = \tau_1 + \varphi_t(\bar{x},\bar{t}) - \frac{1}{2}\sigma^2(X + \varphi_{xx}(\bar{x},\bar{t})) + \rho\bar{x}(p + \varphi_x(\bar{x},\bar{t})) + ru(\bar{x},\bar{t}), \\ \\ B_1 = u(\bar{x},\bar{t}) - u(-\bar{x},\bar{t}) - \psi(\bar{x},\bar{t}), \\ \\ A_2 = \tau_2 - \frac{1}{2}\sigma^2 Y + \rho\bar{y}p + ru(-\bar{y},\bar{t}), \\ \\ B_2 = u(-\bar{y},\bar{t}) - u(\bar{y},\bar{t}) - \psi(-\bar{y},\bar{t}). \end{cases}$$

Therefore we have

$$w(\bar{x},\bar{t}) - \psi(\bar{x},\bar{t}) \geq 0 \tag{11.23}$$

and

$$A \geq 0$$

with

$$\begin{aligned} A := A_1 - A_2 = \;& \varphi_t(\bar{x},\bar{t}) - \frac{1}{2}\sigma^2(X - Y + \varphi_{xx}(\bar{x},\bar{t})) \\ & + \rho\left\{\bar{x}\ \varphi_x(\bar{x},\bar{t}) - \frac{(\bar{x}-\bar{y})^2}{\varepsilon}\right\} + r\left(u(\bar{x},\bar{t}) - u(-\bar{y},\bar{t})\right). \end{aligned}$$

But we know that $X \geq Y$, we see that $A \geq 0$ implies

$$\varphi_t(\bar{x},\bar{t}) - \frac{1}{2}\sigma^2\varphi_{xx}(\bar{x},\bar{t}) + \rho\bar{x}\ \varphi_x(\bar{x},\bar{t}) + r\left(u(\bar{x},\bar{t}) - u(-\bar{y},\bar{t})\right) \geq 0. \tag{11.24}$$

Passing to the limit in (11.23) and (11.24), we see that we get (11.18).
This shows that w is solution of (11.11) and ends the proof of the lemma.

11.3 Proof of the comparison principle for w (Theorem 7.5)

Proof of Theorem 7.5
Step 1: preliminaries
Let

$$M = \sup_{(x,t)\in[0,+\infty)\times[0,T)} w(x,t) - v(x,t)$$

and let us assume by contradiction that

$$M > 0.$$

Then for any small $\varepsilon, \beta, \eta > 0$, we have

$$M_{\varepsilon,\beta,\eta} = \sup_{x,y\in[0,+\infty),\ t\in[0,T)} \Phi(x,y,t)$$

with

$$\Phi(x,y,t) = w(x,t) - v(y,t) - \frac{(x-y)^2}{2\varepsilon} - \beta\frac{x^2}{2} - \frac{\eta}{T-t}.$$

As usual, the suppremum is reached at some point $(\bar{x},\bar{y},\bar{t})$ and

$$\Phi(\bar{x},\bar{y},\bar{t}) = M_{\varepsilon,\beta,\eta} \geq M/2 > 0$$

for β,η small enough. In particular we have $\bar{t} > 0$. Moreover the point $(\bar{x}_\varepsilon,\bar{y}_\varepsilon,\bar{t}_\varepsilon) = (\bar{x},\bar{y},\bar{t})$ satisfies

$$(\bar{x}_\varepsilon,\bar{y}_\varepsilon,\bar{t}_\varepsilon) \to (x_0,x_0,t_0) \quad \text{as} \quad \varepsilon \to 0$$

and then $x_0 > 0$ because of the boundary condition. This also implies that $\bar{x} = \bar{x}_\varepsilon > 0$ and $\bar{y} = \bar{y}_\varepsilon > 0$ and then we have the viscosity inequalities at both points $(\bar{x},\bar{t})$ and $(\bar{y},\bar{t})$.
Step 2: viscosity inequalities
As usual, we get

$$B \leq 0 \qquad \text{or there exists } X \leq Y \text{ such that} \quad A \leq 0$$

with $p = \dfrac{\bar{x}-\bar{y}}{\varepsilon}$ and

$$\begin{cases} A = \dfrac{\eta}{(T-\bar{t})^2} - \dfrac{1}{2}\sigma^2(X - Y + \beta) + \rho\left(\dfrac{(\bar{x}-\bar{y})^2}{\varepsilon} + \beta\bar{x}^2\right) + r(w(\bar{x},\bar{t}) - v(\bar{y},\bar{t})) - (f(\bar{x},\bar{t}) - f(\bar{y},\bar{t})), \\ \\ B = w(\bar{x},\bar{t}) - v(\bar{y},\bar{t}) - (g(\bar{x},\bar{t}) - g(\bar{y},\bar{t})). \end{cases}$$

If $A \leq 0$, then we get a contradiction as usual (in the limit $\varepsilon \to 0$, for β small enough). If $B \leq 0$, then we get

$$0 < M/2 \leq w(\bar{x},\bar{t}) - v(\bar{y},\bar{t}) \leq g(\bar{x},\bar{t}) - g(\bar{y},\bar{t}) \to 0 \quad \text{as} \quad \varepsilon \to 0$$

which gives also a contradiction.
This ends the proof of the theorem.

11.4 Proof of the space convexity of w (Proposition 7.10)

Proof of Proposition 7.10
We follow the proof of Proposition 5.2.
Step 1: the implicit scheme
Given a time step $\varepsilon > 0$, we consider an approximation $w^n(x)$ of $w(x, n\varepsilon)$ defined for $n \in \mathbb{N}$ as a solution of the following implicit scheme:
(11.25)

$$\begin{cases} w^0 = 0, \\ \min\left(\dfrac{w^{n+1} - w^n}{\varepsilon} + \mathcal{M}w^{n+1}, w^{n+1}(x) - \psi(x, (n+1)\varepsilon)\right) = 0 \qquad \text{for} \quad x \in (0, +\infty), \\ w^{n+1}(0) = 0. \end{cases}$$

Step 2: subsolution $\underline{w}^{n+1}$
As in the proof of Lemma 7.7 and Proposition 5.2, we check that

$$\underline{w}^{n+1}(x) = \max(0, \psi(x, (n+1)\varepsilon))$$

is a subsolution of the scheme (11.25).
Step 3: supersolution $\min(\overline{w}^{n+1}, \overline{\overline{w}}^{n+1})$
It is straightworward to check (as in the proof of Lemma 7.7) that

$$\overline{\overline{w}}^{n+1}(x) = \overline{\overline{w}}(x, (n+1)\varepsilon)$$

is a supersolution of the scheme (11.25).
It is also straightforward to check that

$$\overline{w}^{n+1}(x) = \overline{w}(x, (n+1)\varepsilon)$$

is also a supersolution of the scheme (11.25). To this end, we simply notice that $\mathcal{M}\overline{w} \geq 0$ implies

$$\mathcal{M}\overline{w}^{n+1} \geq 0$$

and notice that

$$\overline{w}^{n+1} \geq \overline{w}^n.$$

Finally this implies that

$$\min(\overline{w}^{n+1}, \overline{\overline{w}}^{n+1})$$

is a supersolution of the scheme (11.25).
Step 4: existence of a unique solution to the scheme
We can then apply Perron's method and prove a comparison principle similar to Theorem 7.5. This shows the existence and uniqueness of a unique solution $(w^n)_n$ to the scheme. Moreover the comparison principle implies that for each n, the function w^n is continuous.
Step 5: convexity of w^{n+1}
We proceed as in the proof of Proposition 5.2 to prove by recurrence that w^n is convex starting from $w^0 = 0$.

Substep 5.1: definition of the convex envelope of w^{n+1}
We define

$$W^{n+1}(x) = \sup_{l \in E} l(x)$$

where E is the set of affine functions below w^{n+1}. The fact that

$$w^{n+1} \geq \underline{w}^{n+1} \geq 0$$

shows that

$$W^{n+1}(0) = 0.$$

Our goal is to show that W^{n+1} is a supersolution of (11.25). Then the comparison principle will imply

$$W^{n+1} = w^{n+1}$$

which will show that w^{n+1} is convex.
Substep 5.2: W^{n+1} is a supersolution
We proceed exactly as in Substep 5.2 of the proof of Proposition 5.2. One change is the fact that $u^{n+1}(x) - u^{n+1}(-x) - \psi(x,(n+1)\varepsilon)$ is changed in $w^{n+1}(x) - \psi(x,(n+1)\varepsilon)$ which is even simpler to analyse. The only other change is that for a point $x_0 > 0$, we may have $x_- = 0$ for some affine function which is a linear function. We now claim that
(11.26)

$$\min\left(\frac{l_0(x_-) - w^n(x_-)}{\varepsilon} + (\mathcal{M}l_0)(x_-), \quad l_0(x_-) - \psi(x_-,(n+1)\varepsilon)\right) \geq 0 \quad \text{for} \quad x_- = 0$$

which is straightforward to check. Then, the remaining part of the proof is the same as in the proof of Proposition 5.2 and this shows that W^{n+1} is a supersolution and then $w^{n+1} = W^{n+1}$ is convex.
Step 6: convergence towards w as ε tends to zero
The proof is similar to the one of Proposition 5.2. This ends the proof of Proposition 7.10.

11.5 Proof of a comparison principle for a stationary obstacle problem without zero order terms (Lemma 9.4)

Proof of Lemma 9.4
The proof follows the usual raisoning, with an adaptation here, because there is no zero order term in the equation.
We assume by contradiction that

$$M = \sup_{x \in [0,+\infty)} (u - v)(x) > 0$$

and then consider for $\varepsilon, \eta > 0$:

$$M_{\varepsilon,\eta} = \sup_{x,y \in [0,+\infty)} \Phi(x,y)$$

with

$$\Phi(x,y) = u(x) - v(y) - \frac{(x-y)^2}{2\varepsilon} - \eta\zeta(x), \quad \zeta(x) = \sqrt{1+x} - 1.$$

For $\eta > 0$ small enough, we have

$$M_{\varepsilon,\eta} \geq M/2 > 0$$

and the suppremum is reached at a point $(\bar{x}, \bar{y})$. We deduce moreover from (9.10) that

$$\eta\zeta(\bar{x}) + \frac{(\bar{x} - \bar{y})^2}{2\varepsilon} \leq 1$$

which implies in particular that

$$\bar{x} \leq C_\eta. \tag{11.27}$$

As usual, we can not have $\bar{x} = 0$ or $\bar{y} = 0$ for a sequence $\varepsilon \to 0$, otherwise we get a contradiction from the boundary condition. Therefore $\bar{x}, \bar{y} > 0$ and we have the viscosity inequalities. As usual, we get

$$B \leq 0 \quad \text{or there exists } X \leq Y \text{ such that} \quad A \leq 0$$

with

$$\begin{cases} A = -\dfrac{\sigma^2}{2}(X - Y + \eta\zeta''(\bar{x})) + b(\bar{x} - \bar{y}), \\ \\ B = u(\bar{x}) - v(\bar{y}). \end{cases}$$

Notice that

$$\zeta''(\bar{x}) = -\frac{1}{4(1+\bar{x})^{\frac{3}{2}}} \leq -c_\eta < 0$$

where we have used (11.27). If $A \leq 0$, we get

$$0 < \eta c_\eta \leq -\frac{2b}{\sigma^2}(\bar{x} - \bar{y}) \to 0 \quad \text{as} \quad \varepsilon \to 0$$

which gives a contradiction.
If $B \leq 0$, then

$$0 < M/2 \leq u(\bar{x}) - v(\bar{y}) = B \leq 0$$

which is a contradiction.
This ends the proof of the lemma.

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