Counter-example in 3D and homogenization of geometric motions in 2D
(long version)

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Abstract

In this paper we give a counter-example to the homogenization of the forced mean curvature motion in a periodic setting in dimension $N \geq 3$ when the forcing is positive. We also prove a general homogenization result for geometric motions in dimension $N = 2$ under the assumption that there exists a constant $\delta > 0$ such that every straight line moving with a normal velocity equal to δ is a subsolution for the motion.

We also present a generalization in dimension 2, where we allow sign changing normal velocity and still construct bounded correctors, when there exists a subsolution with compact support expanding in all directions.

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Contents

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1 Introduction

1.1 Setting of the problem

In this paper we are interested in solutions $u^\varepsilon(x,t)$ for $\varepsilon > 0$ of geometric equations that can be written as

$$(1.1) \qquad \begin{cases} u^\varepsilon_t = F(\varepsilon D^2 u^\varepsilon, Du^\varepsilon, \varepsilon^{-1}x) & \text{on} \quad \mathbb{R}^N \times (0,+\infty) \\ u^\varepsilon = u_0 & \text{on} \quad \mathbb{R}^N \times \{0\} \end{cases}$$

for suitable F which are in particular periodic in the variable $\varepsilon^{-1}x$. Under certain assumptions we can show the homogenization as $\varepsilon \to 0$, i.e. that u^ε converges to a function u^0

solution of an equation

$$(1.2)\qquad \begin{cases} u^0_t = \bar{F}(Du^0) & \text{on} \quad \mathbb{R}^N \times (0,+\infty) \\ u^0 = u_0 & \text{on} \quad \mathbb{R}^N \times \{0\}\,. \end{cases}$$

Our starting point is the study of the mean curvature motion forced by a given periodic verlocity c, i.e.

$$(1.3)\qquad \begin{cases} u^\varepsilon_t = \varepsilon\ \mathrm{tr}\left\{D^2u^\varepsilon\cdot(I-\widehat{Du^\varepsilon}\otimes\widehat{Du^\varepsilon})\right\} + c(\varepsilon^{-1}x)|Du^\varepsilon| & \text{on} \quad \mathbb{R}^N\times(0,+\infty) \\ u^\varepsilon = u_0 & \text{on} \quad \mathbb{R}^N\times\{0\} \end{cases}$$

where for $p\in\mathbb{R}^N\setminus\{0\}$, $\widehat{p}=\dfrac{p}{|p|}$. It turns out (see for instance [32, 33]) that the level set

$$\Gamma^\varepsilon_t = \left\{x\in\mathbb{R}^N : u^\varepsilon(x,t)=0\right\}$$

can be seen as a generalized evolution of the set Γ^ε_0 with normal velocity

$$(1.4)\qquad V = \varepsilon\kappa + c(\varepsilon^{-1}x)$$

where κ is the mean curvature of the hypersurface Γ^ε_t where it is smooth, and the normal is by convention the outward normal to the set $\left\{x\in\mathbb{R}^N : u^\varepsilon(x,t)>0\right\}$. When this set is convex, the mean curvarure is non positive. It is known from [25] that equation (1.3) admits Lipschitz correctors for Lipschitz $\mathbb{Z}^N$-periodic function c satisfying moreover the condition

$$(1.5)\qquad \inf_{y\in\mathbb{R}^N}\left(c^2(y)-(N-1)|Dc(y)|\right)>0.$$

But the question was left open to know if (1.5) is necessary for homogenization or whether

$$(1.6)\qquad \inf_{y\in\mathbb{R}^N} c(y)>0$$

is enough, as it is the case when there is no curvature term in (1.4).

1.2 Main results

It turns out that condition (1.6) is not enough to get homogenization in dimension $N\geq 3$ as shows the following counter-example. We use the notation $x=(x_1,...,x_N)\in\mathbb{R}^N$.

Theorem 1.1 (Counter-example to homogenization in dimension $N\geq 3$)
Let $N\geq 3$. Then there exists a function $c\in C^\infty(\mathbb{R}^N)$ which is $\mathbb{Z}^N$-periodic, satisfying (1.6) and moreover with $c(x)$ independent on the variable x_N, such that the following holds. For the initial data $u_0(x)=-x_N$, the solution u^ε of (1.3) satisfies for some constants $\overline{c}>\underline{c}$

$$(1.7)\qquad \limsup_{(\tilde{x},\tilde{t},\varepsilon)\to(x,t,0)} u^\varepsilon(\tilde{x},\tilde{t}) \geq \overline{c}t - x_N > \underline{c}t - x_N \geq \liminf_{(\tilde{x},\tilde{t},\varepsilon)\to(x,t,0)} u^\varepsilon(\tilde{x},\tilde{t}) \quad \textit{for all} \quad t>0$$

i.e. there is no strong limit, and hence homogenization does not take place.

On the contrary in dimension $N = 2$, condition (1.6) is sufficient to get homogenization as we will see below (see Theorem 1.4). Indeed in dimension $N = 2$, homogenization holds for general equation (1.1) with F safistying certain conditions.

Let us define

$$\mathcal{D}_0 := S^N \times (\mathbb{R}^N \setminus \{0\}) \times \mathbb{R}^N,$$

where S^N denotes the set of real symmetric $N \times N$ matrices. We assume that $F(X, p, y)$ has arguments $(X, p, y) \in \mathcal{D}_0$ and satisfies the following properties:

Assumption (A)
(A1) Degenerate ellipticity: *$F \in C(\mathcal{D}_0)$ and for all $(X, p, y) \in \mathcal{D}_0$, we have*

$$F(X + Q, p, y) \geq F(X, p, y) \quad \text{for all} \quad Q \geq 0 \quad \text{with} \quad Q \in S^N$$

(A2) F **is geometric**: *for all $(X, p, y) \in \mathcal{D}_0$, we have*

$$F(\lambda X + \mu p \otimes p, \lambda p, y) = \lambda F(X, p, y) \quad \text{for all} \quad \lambda > 0, \mu \in \mathbb{R}$$

(A3) $\mathbb{Z}^N$**-Periodicity**: *for all $(X, p, y) \in \mathcal{D}_0$, we have*

$$F(X, p, y + k) = F(X, p, y) \quad \text{for all} \quad k \in \mathbb{Z}^N$$

(A4) Regularity: *this technical assumption is given in Subsection 3.2.*

We will also assume the following

Assumption (B): **Bound from below**:
There exists $\delta > 0$ such that for all arguments $(0, p, y) \in \mathcal{D}_0$, we have

$$F(0, p, y) \geq \delta |p|. \tag{1.8}$$

In order to keep simple the presentation, we chose not to give the details of the classical (but technical) regularity assumption (A4) in this introduction. Under assumption (A), a comparison principle holds (see Theorem 3.3).

Remark 1.2 *Notice that assumptions (A1), (A3), and (B) imply that there exist constants $C_0, c_0 > 0$ and $R > \sqrt{2}/2$ such that for all $(p, y) \in \mathbb{S}^{N-1} \times \mathbb{R}^N$, we have*

$$C_0 \geq F(0, p, y) \geq F\left(-\frac{1}{R} I, p, y\right) \geq c_0 > 0. \tag{1.9}$$

Then we have the following result.

Theorem 1.3 (The cell problem in 2D)
Assume that $N = 2$ and that (A) and (B) hold. Then for any $p \in \mathbb{R}^N$, there exists a unique

real number $\bar{F}(p)$ (with $\bar{F}(p) > 0$ if $p \neq 0$ and $\bar{F}(0) = 0$) such that there exists a bounded $\mathbb{Z}^N$-periodic function $v : \mathbb{R}^N \to \mathbb{R}$ solution of

$$\bar{F}(p) = F(D^2 v, p + Dv, y) \quad \text{on} \quad \mathbb{R}^N. \tag{1.10}$$

We can choose v such that

$$\sup v - \inf v \leq \kappa_0 |p| \quad \text{with} \quad \kappa_0 := 100\ R\frac{C_0}{c_0} \tag{1.11}$$

where R, C_0, c_0 are given in (1.9).

Moreover the map $p \mapsto \bar{F}(p)$ is continuous and positively 1-homogeneous, i.e. for any $p \in \mathbb{R}^N$

$$\bar{F}(\lambda p) = \lambda \bar{F}(p) \quad \text{for any} \quad \lambda \geq 0.$$

Let us mention that under assumptions (A) and (B), in the case where $F(X, p, y_1, y_2)$ is independent on y_2, the existence and uniqueness (up to addition of constants) of a corrector v when $p \in \mathbb{R}^2 \backslash \mathbb{R}e_1$, has been established in Lou [28] (see also Lou, Chen [29] and Chen, Namah [14], for particular cases).

As a consequence, we can show the following homogenization result (with an Ansatz that looks like $p \cdot x + t\bar{F}(p) + \varepsilon v(\varepsilon^{-1}x)$, but contrarily to the common belief, is much more involved than the classical perturbed test function method due to Evans. The main difficulty is created by the discontinuity of the Hamiltonian F when the gradient vanishes):

Theorem 1.4 (Homogenization of geometric motions in 2D)
Assume that $N = 2$ and that (A) and (B) hold. Let us consider the solution u^ε of (1.1) with initial data u_0 which is uniformly continuous on $\mathbb{R}^N$. Then u^ε converges locally uniformly on compact sets of $\mathbb{R}^N \times [0, +\infty)$ towards the unique solution u^0 of (1.2) with the function $\bar{F}$ given by Theorem 1.3.

Indeed, Theorem 1.4 appears to be a corollary of a more general result in any dimension (Theorem 1.5), for which we need to introduce the following assumption:

Assumption (B'): **Perturbed correctors**:
We set for $\eta > 0$:

$$F^\eta(X, p, x) = \sup_{|y-x| \leq \eta} F(X, p, y), \quad \left(\text{resp.} \quad F_\eta(X, p, x) = \inf_{|y-x| \leq \eta} F(X, p, y)\right).$$

For all $p \in \mathbb{R}^N$, there exists $\eta_0 > 0$ and $\bar{\kappa}_0 > 0$ such that for all $\eta \in [0, \eta_0)$, there exists a corresponding $\mathbb{Z}^N$-periodic function v^η (resp. v_η) and a real number $\bar{F}^\eta \geq \eta_0 |p|$ (resp. $\bar{F}_\eta \geq \eta_0 |p|$) such that

$$\bar{F}^\eta = F^\eta(D^2 v^\eta, p + Dv^\eta, y) \quad \big(\text{resp.} \quad \bar{F}_\eta = F^\eta(D^2 v_\eta, p + Dv_\eta, y)\big) \quad \text{on} \quad \mathbb{R}^N$$

such that for $v = v^\eta, v_\eta$, we have

$$\sup v - \inf v \leq \bar{\kappa}_0.$$

Then we have:

Theorem 1.5 (Conditional homogenization in dimension $N \geq 2$ when perturbed correctors do exist)
Assume that $N \geq 2$ and that (A) and (B') hold. Let us consider the solution u^ε of (1.1) with initial data u_0 which is uniformly continuous on $\mathbb{R}^N$. Then u^ε converges locally uniformly on compact sets of $\mathbb{R}^N \times [0, +\infty)$ towards the unique solution u^0 of (1.2) with the function $\bar{F} = \bar{F}^0 = \bar{F}_0$ given by assumption (B').

With an assumption weaker than (B) allowing negative normal velocities, namely assumption (B") in Section 10, it is still possible to get a corrector (Theorem 10.3). As an interesting application of Theorem 1.5, it is for instance possible to get homogenization results in $2D$ of equation (1.3) with certain sign changing velocities (see Theorem 11.2).

1.3 Brief review of the literature

The first results of uniqueness for the mean curvature motion, were obtained by Evans, Spruck [21] and Chen, Giga, Goto [13]. For general presentations of viscosity approaches to the motion of fronts, see Giga [22], Souganidis [32, 33], Ambrosio [1], Soner [31]. One of the main difficulty with the evolution of fronts is the possibility of fattening (see Barles, Soner, Souganidis [4]).

The homogenization of Hamilton-Jacobi equations was pionered in Lions, Papanicolaou, Varadhan [26], and then extended to the fully non linear uniformly elliptic case in Evans [19, 20]. The case of geometric equations was studied only recently. In Lions, Souganidis [25], in any dimensions $N \geq 1$, a Lipschitz bound on the correctors associated to forced MCM equation (1.3) is shown under assumption (1.5) (and also for more general equations under suitable assumptions).

In Cardaliaguet, Lions, Souganidis [6], it is in particular shown that in dimension $N = 2$, if $c(y) = g(y_1)$ with

$$\int_{[0,1]} g > 0, \quad \text{and} \quad 0 \leq \int_{[0,1]} g - \min_{[0,1]} g < 2$$

then for any $p \in \mathbb{R}^2$, there exists a Lipschitz continuous corrector v (only depending on y_1) solution of (1.10). Moreover $\bar{F}(p) > 0$ if $p \in \mathbb{R}^2 \backslash \mathbb{R}e_1$, and $\bar{F}(p) = 0$ if $p \in \mathbb{R}e_1$. Among other things, in dimension $N = 2$, a counter-example to homogenization is also given in a case where $\int_{[0,1]} g = 0$ (see also Remark 4.3).

In Cesaroni, Novaga [9], still in dimension $N = 2$, it is in particular shown that for $p = e_2$, there exists a Lipschitz continuous corrector v (only depending on y_1) if

$$\int_{[0,1]} g > 0, \quad \text{and} \quad \min_{[0,1]} g \leq 0 \quad \text{and} \quad \max_{[0,1]} g - \min_{[0,1]} g < 2^{\frac{3}{2}}$$

or if

$$g > 0.$$

More generally, it is shown in dimension $N \geq 1$, that if $c(y) = g(y_1, ..., y_{N-1})$ (this is the case of a laminate), and if

$$\exists A \subset \mathbb{T}^{N-1}, \quad \int_A g \quad > \quad \text{Per}(A, \mathbb{T}^{N-1})$$

then there exists a (pseudo) corrector v (only dependending on $y' = (y_1, ..., y_{N-1})$) and an open set $E \subset \mathbb{T}^{N-1}$ such that v is locally bounded on E and $v = -\infty$ on $\mathbb{T}^{N-1} \backslash E$. The (pseudo) corrector is a kind of (pseudo) travelling wave. Notice that our counter-example (Theorem 1.1) provides an example of a case where such a (pseudo) corrector is not a true corrector in the case $g > 0$ in dimension $N \geq 3$. Under certain assumptions, the homogenization result of [9] has been extended in [10] to the case with an additional drift term given by a gradient vector field.

Let us mention Craciun, Bhattacharya [15], where a formal assymptotics of $\bar{F}(p)$ is given in the limit $\lambda \to +\infty$ for a geometric motion given by

$$V = \lambda\kappa + c.$$

On the other hand, it is shown in Dirr, Karali, Yip [18], that for a geometric motion

$$V = \kappa + \delta c$$

with $c \in C^2(\mathbb{T}^N)$ (without any sign condition on c), if $\delta > 0$ is small enough, then for any $p \in \mathbb{R}^N$, there exists a Lipschitz continuous corrector v solution of (1.10), which is moreover unique if $\bar{F}(p) \neq 0$. Part of the method of proof is based on the arguments of Caffarelli, De La Llave [5] for the construction of minimal surfaces in a periodic setting. See also Chambolle, Thouroude [12] for a BV approach of the result in [5]. It is shown in particular in [12], that if

$$\int_{\mathbb{T}^N} c = 0 \quad \text{and} \quad \exists \mu \in (0,1), \quad \forall A \subset \mathbb{T}^N, \quad \int_A c \quad \leq \quad \mu \operatorname{Per}(A, \mathbb{T}^N) \tag{1.12}$$

then, for any $p \in \mathbb{R}^N$, there exists a corrector v and $\bar{F}(p) = 0$. Let us mention that the homogenization of geometric motions

$$V = \kappa + \varepsilon^{-1} c(\varepsilon^{-1} x)$$

has been done in Barles, Cesaroni, Novaga [3] under the assumption that $c(y) = g(y')$ and that (1.12) holds with $(c, \mathbb{T}^N)$ replaced by $(g, \mathbb{T}^{N-1})$. The case of a geometric motion in dimension $N = 2$

$$V = \kappa + c(\varepsilon^{-1} x)$$

with $c(y) = g(y_1)$, has been studied in Cesaroni, Novaga, Valdinoci [11].

1.4 Organization of the paper

The paper is organized as follows. In Section 2, we present the strategies of our main proofs. In Section 3, we recall basic properties of viscosity solutions. In Section 4, we do the proof of Theorem 1.1 about the counter-example to homogenization in dimension $N \geq 3$. In Section 5, we present preliminary results on the evolution of the front, including the connectedness property (Proposition 5.9) and the black ball barrier (Proposition 5.10). In Section 6, we prove the flatness of the front using Section 5. In Section 7, we prove Theorem 1.3, i.e. we show the existence of a corrector for the cell problem. In Section 8, we prove Theorem 1.5 about the conditional homogenization in any dimension. In Section 9, we prove Theorem 1.4 about the homogenization in 2D. In Section 10, we prove the existence of correctors in 2D (Theorem 10.3) under a general assumption (B") which allows sign changing normal velocities. In Section 11, we present some examples and applications, both for the forced MCM and the G-equation.

Finally the appendix (Section 12) contains three subsections, respectively about barriers, inf-convolutions and the proof of the comparison principle (Theorem 3.3). We did not find precisely in the literature the result we need for the comparison principle, even if its proof is essentially based on [23]. We expect that this detailed proof will be of future use for other authors.

2 Strategy of the proofs

We discuss here the ideas underlying the proofs of the main results.

2.1 The counter-example in 3D (Theorem 1.1)

The basic idea is that in dimension 3 (or higher dimension) we can find unbounded convex sets (with negative curvature) which are invariant by the geometric motion given by the normal velocity

$$V = \kappa + c$$

with $c = 1$. This is the case of **cylinders** whose section are circles. Then we can perturb the velocity c inside the cylinder and outside the cylinder in order to allow the propagation of fronts (which are asymptotic to the cylinder) with different velocities inside the cylinder and outside the cylinder. Considering periodic copies of the cylinder, we can construct a periodic velocity c (which does not depend on the coordinate along the cylinder). Then the two fronts (inside and outside the cylinders) can be used as barriers to show that homogenization can not occur (at least in a strong sense). Notice that the analogue in dimension 2 of the cylinder is simply a circle, which does not allow the propagation of a front inside the disk!

2.2 The cell problem in 2D (Theorem 1.3)

The idea of the construction of a corrector is purely geometric (even if we find convenient to use a level set formulation to work). Under assumption (B), we can think that the front propagates as a fire. This means that the front never comes back. Therefore we can distinguish the burnt region (black region) and the unburnt region (white region). Moreover if our initial black region is a half plane, then (at least in some weak sense), we can show that the black region stays connected for all time. The basic phenomenon to avoid is the creation of a very thick transition region between the black and the white region like on Figure 1.

A bounded connected component of the white region (like in the bottom of Figure 1) can exist, but has to be thin enough. Indeed, it can not contain a unit square, otherwise by an integer translation argument (the Birkhoff property), it will contain infinitely many such squares. Notice that all such bounded white components will diseapear, because they are contained in a white ball (surrounded by the black region) that will itself disappear in finite time (depending on its size). This remark is not sufficient, because we need to bound the time after which they have disappeared. The situation is even worth if we have very long fingers like in Figure 1. We have to show that these fingers will disappear sufficiently quickly.

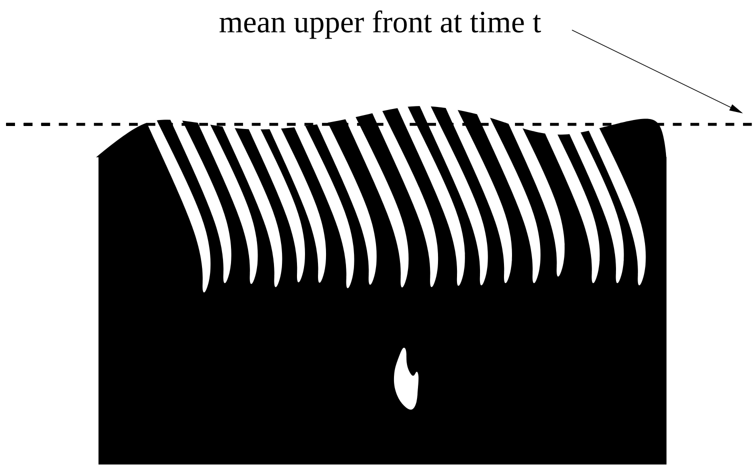

Figure 1: A typical situation to avoid: thick transition with long fingers

The fundamental remark is that assumption (B) also implies the existence of a "black ball" of sufficiently large radius $R > 0$, which can increase or propagate in any direction. This black ball can be used as a **barrier** that will clean the white region remaining pinned in the black region. This black ball can be used to show that afer a fixed time $T > 0$, the new picture will be necessarily like on Figure 2, with a bounded thickness of the transition region between the black and white parts.

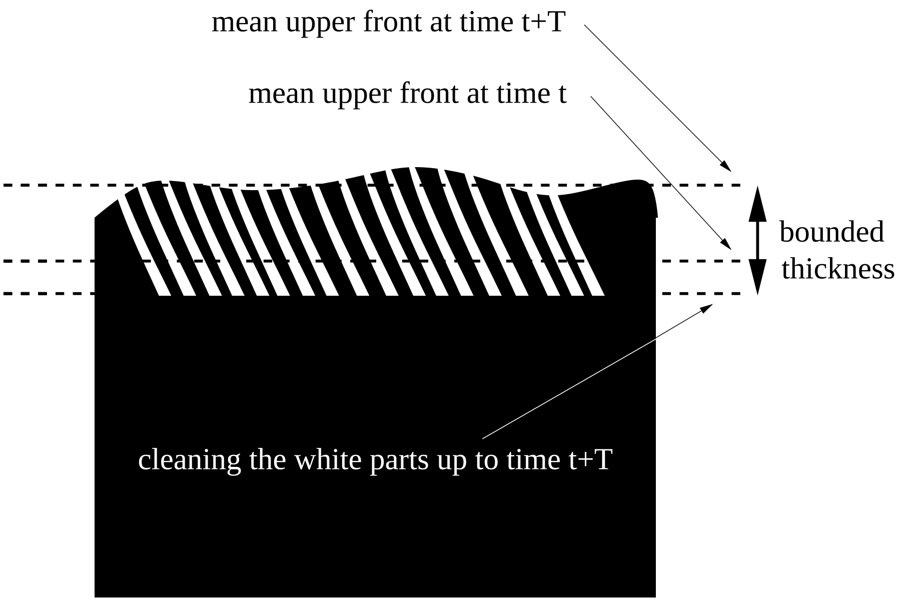

Figure 2: Cleaning the picture after a fixed time $T > 0$

The cleaning phenomenon is possible because the boundary of the white long fingers is connected and then in 2D locally separates the plane in two big parts W (for white) and B (for black), like locally two half planes if the white finger is straight enough. Then we can introduce (see Figure 3) the black ball in the part B (which is no longer true in higher dimensions, like it is shown in the counter-example with cylinders in dimension 3 for instance) and propagate the black ball in the direction of the part W. This process cleans the white part W and make disappear the white finger in a fixed finite time (at least in the direction of the thickness of the finger).

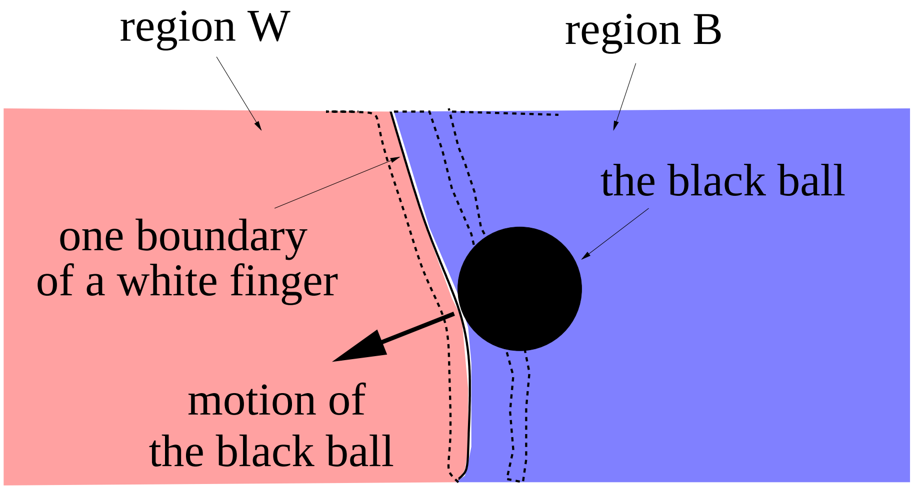

Figure 3: Cleaning the white part with the black ball barrier

Once we are able to show that the thickness of the transition region between the white and black region is bounded uniformly in time, this shows that the front is "roughly flat". This property is sufficient to show that the "flat front" propagates with a well-defined velocity. Passing to the limit as the time goes to infinity, it is then possible to define a corrector which describes the periodic propagation of the "flat front" in the periodic framework.

2.3 Homogenization (Theorem 1.5)

The goal of this subsection is to give some heuristic explanations of the difficulties arising in the homogenization of geometric equations, and the main arguments that we have introduced in our proof of Theorem 1.5.

Try 1: the naive approach and the difficulty when the gradient vanishes
The naive try is the following perturbed test function (for a corrector w)

$$\tilde{\varphi}^\varepsilon(x,t) = \varphi(x,t) + \varepsilon w(x/\varepsilon). \tag{2.1}$$

It is a common belief (see for instance [15] and [25]) that once we are able to show the existence of correctors, then the homogenization result is a corollary obtained using Evans perturbed test function method (see [19]). The point is that this belief is false when we want to homogenize equations like mean curvature motion, because the Hamiltonian is discontinuous when the gradient vanishes.
More precisely, in the following, we recall the classical Evans method and then present the difficulty we have to face.
1.1) The classical Evans method
If φ is a test function touching $\overline{u} := \limsup_{\varepsilon\to 0}{}^* u^\varepsilon$ from above and which does not satisfy the subsolution viscosity inequality, i.e.

$$\varphi_t > \bar{F}(D\varphi) \quad \text{at some point} \quad P_0 \tag{2.2}$$

then given a (super) corrector w associated to $p = D\varphi(P_0)$, we hope that the perturbed test function $\tilde{\varphi}^\varepsilon$ given by (2.1) satisfies

$$\tilde{\varphi}^\varepsilon_t \geq F(\varepsilon D^2\tilde{\varphi}^\varepsilon, D\tilde{\varphi}^\varepsilon, x/\varepsilon) \quad \text{in a neighborhood of the point} \quad P_0 \tag{2.3}$$

(in order to get later a contradiction with the fact that $\varphi(P_0) = \overline{u}(P_0)$).

1.2) The difficulty

Inequality (2.3) means for $y = x/\varepsilon$ and $P = (x,t)$

$$\varphi_t(P) \geq F_*(\varepsilon D^2\varphi(P) + (D^2w)(y), D\varphi(P) + (Dw)(y), y) \quad \text{with} \quad P \quad \text{close to} \quad P_0, \tag{2.4}$$

i.e. for a point P in a neighborhood of P_0, which is independent on ε for ε small enough. Notice that (2.2) means for all y:

$$\varphi_t(P_0) > \bar{F}(D\varphi(P_0)) \geq F_*((D^2w)(y), D\varphi(P_0) + (Dw)(y), y) \tag{2.5}$$

but (2.5) does not imply in general (2.4) for small quantities $|\varphi_t(P) - \varphi_t(P_0)|$, $|D\varphi(P) - D\varphi(P_0)|$ and $\varepsilon|D^2\varphi(P)|$. The difficulty comes from the fact that, even if $\bar{F}(D\varphi(P_0)) > 0$, it may happen that

$$\exists y_0 \quad \text{such that} \quad D\varphi(P_0) + (Dw)(y_0) = 0 \tag{2.6}$$

and

$$\text{the “curvature of the level set”} \ \frac{(D^2w)(y)}{|D\varphi(P_0) + (Dw)(y)|} \ \text{blows up when } y \to y_0. \tag{2.7}$$

Then, it is not clear (at least for us) how to avoid the case where a small perturbation $q = D\varphi(P) + (Dw)(y_0)$ would satisfy $q \neq 0$ with q in a direction such that

$$F((D^2w)(y_0), q, y_0) \simeq F^*((D^2w)(y_0), 0, y_0) > \varphi_t(P_0) > F_*((D^2w)(y_0), 0, y_0).$$

Try 2: New ingredients

Given a parameter $\eta > 0$, the new perturbed test function is the following

$$\bar{\varphi}^\varepsilon(x,t) = \inf_{z \in \overline{B_{\varepsilon\eta}(x)}} \tilde{\varphi}^\varepsilon(z,t). \tag{2.8}$$

From Try 1, it is clear that we can not really hope to construct a perturbed test function like $\tilde{\varphi}^\varepsilon$ which is a supersolution in a neighborhood of a point P_0. We show here that, in order to get a contradiction, we only need this perturbed test function to be a strict supersolution at a **one contact point** with u^ε, which is much easier to check. Of course, we also need to control the curvature of the level set at that contact point. In what follows we present some ideas to reach our goal.

2.1) A pointwise Evans method

The first main idea is to replace the standard Evans method, by the following “Pointwise Evans method”. Let us define for some general function u:

$$F[\varepsilon, u] := F(\varepsilon D^2 u, Du, x/\varepsilon).$$

We just consider a local maximum point $\tilde{P}_\varepsilon$ (close to P_0 as ε goes to zero) of $u^\varepsilon - \tilde{\varphi}^\varepsilon$. We formally have at $\tilde{P}_\varepsilon$

$$u^\varepsilon_t = F[\varepsilon, u^\varepsilon] \leq F[\varepsilon, \tilde{\varphi}^\varepsilon] < \tilde{\varphi}^\varepsilon_t$$

because we expect $\tilde{\varphi}^\varepsilon$ to be a **strict supersolution**. We then get a contradiction from the fact that $u^\varepsilon_t = \tilde{\varphi}^\varepsilon_t$ at the point $\tilde{P}_\varepsilon$.
Notice that in order to have a strict supersolution, we still need to control the curvature of the interface and this is the goal of the next idea.

2.2) Geometric inf-convolution by balls

We now introduce an argument of inf-convolution by balls, in order to bound the curvature (from one side) and then to avoid difficulty (2.7). We recall that, given a corrector w associated to a gradient p, the planar-like function

$$l(y,\tau) = \lambda\tau + p\cdot y + w(y), \quad \text{with} \quad \lambda = \bar{F}(p) > 0$$

solves

$$l_\tau = F(D^2 l, Dl, y). \tag{2.9}$$

We then define the inf-convolution by balls of radius $\eta > 0$:

$$l_\eta(y,\tau) = \inf_{z\in \overline{B_\eta(y)}} l(z,\tau).$$

Then (for any $a \in \mathbb{R}$) each upper level set $\{l_\eta > a\}$ has exterior tangent balls of radius η at each point of its boundary, which implies that its curvature matrix is bounded from above by $\frac{1}{\eta}I$ (see Figure 4).

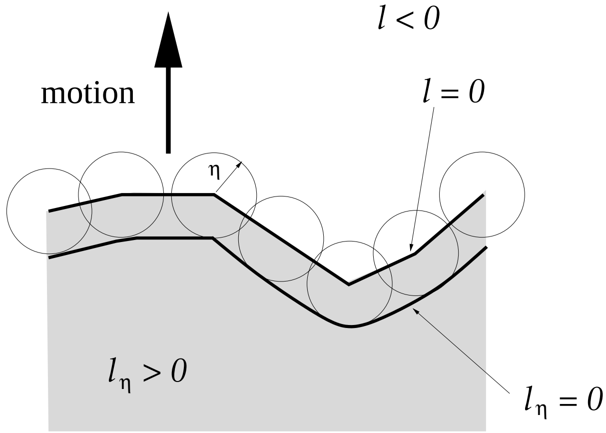

Figure 4: The new interface after inf-convolution by balls

This implies in particular that

$$F(D^2 l_\eta, Dl_\eta, y) = |Dl_\eta| F\left(\frac{D^2 l_\eta}{|Dl_\eta|}, \frac{Dl_\eta}{|Dl_\eta|}, y\right) \le |Dl_\eta| F\left(\frac{1}{\eta}I, \frac{Dl_\eta}{|Dl_\eta|}, y\right) \le c_\eta |Dl_\eta|$$

for some constant $c_\eta > 0$.

2.3) Bound from below on the gradient

Notice that $\tilde{\varphi}^\varepsilon(x,t)$ looks like $\varepsilon l\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right)$, and then its natural to replace $\tilde{\varphi}^\varepsilon$ by $\bar{\varphi}^\varepsilon$ given in (2.8), and to look at a local maximum point $\bar{P}_\varepsilon$ (close to P_0) of $u^\varepsilon - \bar{\varphi}^\varepsilon$. Therefore we have at $\bar{P}_\varepsilon$:

$$0 < \frac{1}{2}\bar{F}(D\varphi(P_0)) \le \bar{\varphi}^\varepsilon_t = u^\varepsilon_t = F[\varepsilon, u^\varepsilon] \le F[\varepsilon, \bar{\varphi}^\varepsilon] \le c_\eta |D\bar{\varphi}^\varepsilon|$$

which shows that the gradient $|D\bar{\varphi}^\varepsilon|$ is bounded from below by a constant depending only on η.

2.4) Difficulty for checking that $\bar{\varphi}^\varepsilon$ is a strict supersolution at $\bar{P}_\varepsilon$
We have for $\bar{P}_\varepsilon = (\bar{x}_\varepsilon, \bar{t}_\varepsilon)$:

$$\bar{\varphi}^\varepsilon(\bar{P}_\varepsilon) = \tilde{\varphi}^\varepsilon(\tilde{P}_\varepsilon) \quad \text{for some point} \quad \tilde{P}_\varepsilon = (\tilde{x}_\varepsilon, \bar{t}_\varepsilon) \quad \text{with} \quad \tilde{x}_\varepsilon \in \overline{B_{\varepsilon\eta}(\bar{x}_\varepsilon)}.$$

On the one hand, we get

$$\bar{\varphi}^\varepsilon_t(\bar{P}_\varepsilon) = \tilde{\varphi}^\varepsilon_t(\tilde{P}_\varepsilon) = \varphi_t(\tilde{P}_\varepsilon) > \bar{F}(D\varphi(P_0))$$

for ε small enough. On the other hand, we have with $\bar{y}_\varepsilon = \dfrac{\bar{x}_\varepsilon}{\varepsilon}$, $\tilde{y}_\varepsilon = \dfrac{\tilde{x}_\varepsilon}{\varepsilon}$

$$\begin{aligned} F[\varepsilon, \bar{\varphi}^\varepsilon](\bar{P}_\varepsilon) &= F(\varepsilon D^2\bar{\varphi}^\varepsilon(\bar{P}_\varepsilon), D\bar{\varphi}^\varepsilon(\bar{P}_\varepsilon), \frac{\bar{x}_\varepsilon}{\varepsilon}) \\ &= F(\varepsilon D^2\tilde{\varphi}^\varepsilon(\tilde{P}_\varepsilon), D\tilde{\varphi}^\varepsilon(\tilde{P}_\varepsilon), \bar{y}_\varepsilon) \\ &= F(\varepsilon D^2\varphi(\tilde{P}_\varepsilon) + D^2 w(\tilde{y}_\varepsilon), D\varphi(\tilde{P}_\varepsilon) + Dw(\tilde{y}_\varepsilon), \bar{y}_\varepsilon) \end{aligned}$$

for which we have

$$\begin{cases} |D\bar{\varphi}^\varepsilon(\bar{P}_\varepsilon)| = |D\varphi(\tilde{P}_\varepsilon) + Dw(\tilde{y}_\varepsilon)| & \text{is bounded from below,} \\ |\bar{y}_\varepsilon - \tilde{y}_\varepsilon| \le \eta. \end{cases}$$

And in order to conclude that $\bar{\varphi}^\varepsilon$ is a strict supersolution at $\bar{P}_\varepsilon$, it is enough to show that
(2.10)
$$F(\varepsilon D^2\varphi(\tilde{P}_\varepsilon)+D^2w(\tilde{y}_\varepsilon), D\varphi(\tilde{P}_\varepsilon)+Dw(\tilde{y}_\varepsilon), \bar{y}_\varepsilon) \quad \simeq \quad F(D^2w(\tilde{y}_\varepsilon), D\varphi(P_0)+Dw(\tilde{y}_\varepsilon), \tilde{y}_\varepsilon) = \bar{F}(D\varphi(P_0)).$$

We consider here a small perturbation of the arguments of F. Because F is not uniformly continuous on the set where the gradients are bounded from below, we still need the following property:

$$|\varepsilon D^2\tilde{\varphi}^\varepsilon(\tilde{P}_\varepsilon)|, |D\tilde{\varphi}^\varepsilon(\tilde{P}_\varepsilon)| \le C \tag{2.11}$$

which is not true in general.

Try 3: Further regularization
Given a parameter $\rho > 0$, the new perturbed test function is the following

$$\varphi^\varepsilon(x,t) = \inf_{z \in \mathbb{R}^N} \left(\bar{\varphi}^\varepsilon(z,t) + \frac{|x-z|^4}{4\varepsilon^3\rho} \right) = \left(\bar{\varphi}^\varepsilon(z,t) + \frac{|x-z|^4}{4\varepsilon^3\rho} \right)_{|z=z_x}. \tag{2.12}$$

3.1) Classical regularization
In order to control the quantities in (2.11), this is natural to introduce the inf-convolution (2.12). Classically, this kind of inf-convolution is convenient for mean curvature type PDE, because the function $|\cdot|^4$ has zero second derivatives when its gradient is zero. Notice that here we could have taken another inf-convolution, because the case where the gradient

vanishes is already avoided by the bound from below on the gradient.
For (2.12), we can show that (for ρ small)

$$|z_x - x| \le \varepsilon O(\rho^{\frac{1}{4}})$$

which implies

$$|\varepsilon D^2\varphi^\varepsilon| \le O(\rho^{-\frac{1}{2}}), \quad |D\varphi^\varepsilon| \le O(\rho^{-\frac{1}{4}})$$

which will give (2.11).

3.2) Difficulty

The drawback of this regularization by inf-convolution is that in (2.10), it will make move the contact points $\bar{y}_\varepsilon$ into points y_ε where now we have the estimate:

$$|y_\varepsilon - \tilde{y}_\varepsilon| \le |y_\varepsilon - \bar{y}_\varepsilon| + |\bar{y}_\varepsilon - \tilde{y}_\varepsilon| \le O(\rho^{\frac{1}{4}}) + \eta.$$

We still have to face the lack of uniform continuity of F (as ρ goes to zero).

Try 4: Our definitive choice

We consider the test function φ^ε given by (2.12) where the corrector w appearing in $\tilde{\varphi}^\varepsilon$ (see (2.1)) has to be replaced by $w^{2\eta}$ associated to the Hamiltonian:

$$F^{2\eta}(X,p,x) = \sup_{|y-x|\le 2\eta} F(X,p,y).$$

We choose ρ small enough satisfying

$$|y_\varepsilon - \tilde{y}_\varepsilon| \le O(\rho^{\frac{1}{4}}) + \eta \le 2\eta$$

and the adjustment of the parameter η is done such that the associated effective Hamiltonian $\bar{F}^{2\eta}$ is close enough to $\bar{F} = \bar{F}^0$ in order to satisfy

$$\varphi_t(P_0) > \bar{F}^{2\eta}(D\varphi(P_0)).$$

This last choice allows us to conclude the raisonning.

The previous method is used to show that $\limsup_{\varepsilon\to 0}{}^* u^\varepsilon$ is a subsolution of the limit equation. A similar (but adapted method because we may have $\varphi_t(P_0) < 0$) is used to show that $\liminf_{\varepsilon\to 0}{}_* u^\varepsilon$ is a supersolution.

Remark 2.1 (Link between (1.5) and inf-convolution by small balls)

It is possible to see that assumption (1.5) implies that if a characteristic function $\chi(x,t)$ is a solution of (1.3) in dimension $N \ge 2$ (with $\varepsilon = 1$ to fix the ideas), then

$$\chi^\eta(x,t) = \sup_{y\in B_\eta(x)} \chi(y,t) \quad \textit{and} \quad \chi_\eta(x,t) = \inf_{y\in B_\eta(x)} \chi(y,t)$$

are respectively sub and supersolutions for $\eta > 0$ small enough, when the total mean curvature of the smooth moving boundary satisfies $\kappa^2 \ge |Dc|$ which is the case if $\kappa \le -c$.

3 Properties of viscosity solutions

3.1 Viscosity solutions

Let $\Omega \subset \mathbb{R}^N$ be an open set and let $T \in (0, +\infty]$. We consider solutions u of the following equation

$$u_t = F(D^2u, Du, y) \quad \text{on} \quad \Omega \times (0,T) =: \Omega_T \tag{3.1}$$

with boundary - initial data

$$u = g \quad \text{on} \quad (\Omega \times \{0\}) \cup (\partial\Omega \times [0,T)) =: \partial_p\Omega_T. \tag{3.2}$$

For a general function $u : \overline{\Omega} \times [0,T) \to [-\infty, +\infty]$, we recall the definition of the upper (resp. lower) semi-continuous envelope u^* (resp. u_*) of u:

$$u^*(x,t) = \limsup_{(y,s)\to(x,t)} u(t,s) \quad \left(\text{resp.} \quad u_*(x,t) = \liminf_{(y,s)\to(x,t)} u(t,s)\right).$$

We also recall that we say that u is upper (resp. lower) semi-continuous if and only if $u = u^*$ (resp. $u = u_*$). Given a function F continuous on $\mathcal{D}_0 = S^N \times (\mathbb{R}^N \setminus \{0\}) \times \mathbb{R}^N$, we also define for all $(X, p, x) \in S^N \times \mathbb{R}^N \times \mathbb{R}^N$:

$$\begin{cases} F^*(X,p,x) = \lim\limits_{\varepsilon\to 0} \quad \sup\{F(Y,q,y), \quad (Y,q,y) \in \mathcal{D}_0, \quad |X-Y|, |p-q|, |x-y| \le \varepsilon\} \\ F_*(X,p,x) = \lim\limits_{\varepsilon\to 0} \quad \inf\{F(Y,q,y), \quad (Y,q,y) \in \mathcal{D}_0, \quad |X-Y|, |p-q|, |x-y| \le \varepsilon\}. \end{cases}$$

Because of the continuity of $F(X,p,x)$ for $p \neq 0$, we have in particular $F^*(X,p,x) = F(X,p,x) = F_*(X,p,x)$ if $p \neq 0$.

We are now ready to recall the definition of a viscosity solution

Definition 3.1 (Viscosity solution)
We use the previous notation.
i) Sub/super/solution of (3.1)
We say that $u : \overline{\Omega} \times [0,T) \to [-\infty, +\infty]$ is a subsolution (resp. a supersolution) of (3.1) if $u < +\infty$ (resp. $u > -\infty$) and u is upper (resp. lower) semi-continuous and if for any $P_0 = (x_0, t_0) \in \Omega_T$, if there exists some $r_0 > 0$ such that $B_{r_0}(P_0) \subset \Omega_T$ and a function $\varphi \in C^2(B_{r_0}(P_0))$ such that

$$\begin{cases} u \le \varphi & \text{on} \quad B_{r_0}(P_0) \\ u = \varphi & \text{at} \quad P_0 \end{cases} \quad \left(\text{resp.} \quad \begin{cases} u \ge \varphi & \text{on} \quad B_{r_0}(P_0) \\ u = \varphi & \text{at} \quad P_0 \end{cases}\right)$$

then we have at P_0

$$\varphi_t \le F^*(D^2\varphi, D\varphi, x_0) \quad \left(\text{resp.} \quad \varphi_t \ge F_*(D^2\varphi, D\varphi, x_0)\right).$$

We say that u is a viscosity solution of (3.1) if u^* is a subsolution and if u_* is a supersolution.
ii) Sub/super/solution of (3.1)-(3.2)
A function $u : \overline{\Omega} \times [0,T) \to [-\infty, +\infty]$ is said to be a subsolution (resp. a supersolution) of (3.1)-(3.2) if it is a subsolution (resp. supersolution) of (3.1) and if furthermore it satisfies

$$u \le g^* \quad \text{on} \quad \partial_p\Omega_T \quad (\text{resp.} \quad u \ge g_* \quad \text{on} \quad \partial_p\Omega_T).$$

We say that u is a viscosity solution of (3.1)-(3.2) if u^* is a subsolution and u_* is a supersolution.

3.2 The technical assumption (A4)

We give below the precise assumption (A4).

(A4) Regularity:

$$\begin{cases} \textbf{i) Boundedness close to } p=0\textbf{:} \\ \textit{For all } R>0, \textit{ there exists a constant } C_R>0 \textit{ such that for all } \quad y\in\mathbb{R}^N \\ |F(X,p,y)|\le C_R \quad \textit{for all} \quad |X|\le R, \quad 0<|p|\le R \\ \\ \textbf{ii) Variations in } (X,x)\textbf{:} \\ \textit{There exists } K\ge 9 \textit{ and } \sigma_K:[0,+\infty)\to[0,+\infty) \textit{ satisfying } \sigma_K(0^+)=0, \textit{ such that we have} \\ \\ F^*(X,\alpha(x-y),x)-F_*(Y,\alpha(x-y),y)\le \sigma_K\left\{|x-y|(1+\alpha|x-y|)\right\} \\ \\ \textit{for all } \alpha\ge 0 \textit{ and } X,Y\in S^N,\ x,y\in\mathbb{R}^N \textit{ satisfying} \\ -K\alpha\begin{pmatrix} I & 0 \\ 0 & I\end{pmatrix}\le\begin{pmatrix} X & 0 \\ 0 & -Y\end{pmatrix}\le K\alpha\begin{pmatrix} I & -I \\ -I & I\end{pmatrix} \\ \textit{with } \alpha=0 \textit{ if } x=y \end{cases}$$

This kind of regularity assumptions are given (partially) page 443 in [4].

Remark 3.2
Notice that condition (A4)ii) joint to the geometric property of F (assumption (A2)) imply

$$F^*(0,0,y)=0=F_*(0,0,y). \tag{3.3}$$

Notice that we also have

$$F(X,p,y)=F(\Pi(p)\cdot X\cdot\Pi(p),p,x) \quad \textit{with} \quad \Pi(p)=I-\hat{p}\otimes\hat{p} \quad \textit{if} \quad p\neq 0 \tag{3.4}$$

which follows from assumptions (A1)-(A2) and Theorem 1.6.12 (page 48) of [22].

Then we have the following result.

Theorem 3.3 (Comparison principle)
Assume that either $\Omega=\mathbb{R}^N$ or that Ω is a bounded open set of $\mathbb{R}^N$, and assume (A). If u is a subsolution of (3.1) and v is a supersolution of (3.1) such that
(3.5)

$$\begin{cases} \lim_{\theta\to 0}\ \sup\left\{u(x,0)-v(y,0), \quad x,y\in\mathbb{R}^N, \quad |x-y|\le\theta\right\}\le 0 & \textit{if} \quad \Omega=\mathbb{R}^N \\ \\ u\le v \quad \textit{on} \quad \partial_p\Omega_T & \textit{if } \Omega \textit{ is a bounded open set} \end{cases}$$

then

$$u\le v \quad \textit{on} \quad \Omega_T.$$

3.3 An example

Let us consider for instance the following natural example

(3.6)

$$F(X,p,y) = \text{tr}\left\{\Sigma^T(\widehat{p},y)\cdot\Sigma(\widehat{p},y)\cdot X\right\} + H(p,y) \quad \text{with} \quad \left\{\begin{array}{l} H(p,y) \text{ positively 1-homogeneous in } p \\ \Sigma(\widehat{p},y)\cdot(I-\widehat{p}\otimes\widehat{p}) = \Sigma(\widehat{p},y)\end{array}\right.$$

where $H(p,y)$ and $\Sigma(\widehat{p},y)$ are $\mathbb{Z}^N$-periodic in y. We assume the following regularity

$$H \in C(\mathbb{R}^N \times \mathbb{R}^N;\mathbb{R}) \quad \text{and} \quad \Sigma \in C(\mathbb{S}^{N-1}\times\mathbb{R}^N;\mathbb{R}^{N\times N}),$$

and that there exists a constant $L > 0$ such that

$$(3.7) \qquad |H(p,x)-H(p,y)| \le L|x-y||p| \quad \text{and} \quad |\Sigma(\widehat{p},x)-\Sigma(\widehat{p},y)| \le L|x-y|.$$

Notice that equation (1.3) corresponds to the particular subcase

$$\Sigma(p,y) = I - \widehat{p}\otimes\widehat{p} \quad \text{and} \quad H(p,y) = c(y)|p|$$

and assumption (B) means

$$c(y) \ge \delta > 0.$$

More generally, if we assume moreover that H satisfies the bound from below (1.8), then F also satisfies (B).

Checking (A) for F given by (3.6).
We claim that F given by (3.6)) satisfies (A). The only thing non trivial to check is (A4)ii) in the special case where $H \equiv 0$. We consider X, Y satisfying with $\alpha \ge 0$

$$(3.8) \qquad -K\alpha\begin{pmatrix} I & 0 \\ 0 & I\end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & -Y\end{pmatrix} \le K\alpha\begin{pmatrix} I & -I \\ -I & I\end{pmatrix}.$$

Then for $p = \alpha(x-y) \neq 0$, we multiply on the left by $(\Sigma(\widehat{p},x),\Sigma(\widehat{p},y))$ and on the right by $(\Sigma(\widehat{p},x),\Sigma(\widehat{p},y))^T$, and we get

$$\begin{aligned} &\text{tr}\left\{\Sigma^T(\widehat{p},x)\Sigma(\widehat{p},x)X - \Sigma^T(\widehat{p},y)\Sigma(\widehat{p},y)Y\right\} \\ (3.9) \qquad &\le K\alpha|\Sigma(\widehat{p},x)-\Sigma(\widehat{p},y)|^2 \\ &\le KL^2\alpha|x-y|^2 \end{aligned}$$

where we have used (3.7) in the second line. If $x = y$, then $\alpha = 0$ in (3.8) implies $X = Y = 0$ and then (3.9) still holds. This shows (A4)ii) with $\sigma_K(a) = KL^2 a$.

3.4 More properties on viscosity solutions

The following results are classical in the theory of viscosity solutions (see [17]).

Proposition 3.4 (Stability)
i) semi-limits
If $(u^\varepsilon)_\varepsilon$ is a sequence of subsolutions (resp. supersolutions) of (3.1), let

$$\overline{u}(x,t) = \limsup_{(\tilde{x},\tilde{t},\varepsilon)\to(x,t,0)} u^\varepsilon(\tilde{x},\tilde{t}), \quad \underline{u}(x,t) = \liminf_{(\tilde{x},\tilde{t},\varepsilon)\to(x,t,0)} u^\varepsilon(\tilde{x},\tilde{t}).$$

If $\overline{u} < +\infty$ (resp. $\underline{u} > -\infty$), then $\overline{u}$ is a subsolution (resp. $\underline{u}$ is a supersolution) of (3.1).
ii) suppremum/infimum
Let S be a set of functions w such that w^* is a subsolution (resp. a set of functions w such that w_* is a supersolution) of (3.1), and

$$\overline{u} = \left(\sup_{w\in S} w\right)^* \quad \left(\textit{resp.} \quad \underline{u} = \left(\inf_{w\in S} w\right)_*\right).$$

If $\overline{u} < +\infty$ (resp. $\underline{u} > -\infty$), then $\overline{u}$ is a subsolution (resp. $\underline{u}$ is a supersolution) of (3.1).

Proposition 3.5 (Perron's method)
Let u^+ (resp. u^-) be a supersolution (resp. subsolution) of (3.1) satisfying $u^- \le u^+$. Then there exists a viscosity solution u of (3.1) satisfying

$$u^- \le u \le u^+.$$

Proof of Proposition 3.5
The proof is essentially based on [13] and [24]. We repeat it for completness.
We call

$$S = \{w : \quad \text{such that } w^* \text{ is subsolution of (3.1)}, \quad w \le u^+\} \ni u^-.$$

We define

$$u(x,t) = \sup_{w\in S}\ w(x,t).$$

From the stability result Proposition 3.4 ii), we know that u^* is a subsolution. Assume that u_* is not a supersolution and let us get a contradiction. Then there exists a point $P_0 = (x_0, t_0)$ and a test function $\varphi \in C^2(B_{r_0}(P_0))$ for some r_0 small enough such that $B_{r_0}(P_0) \subset \Omega_T$ and such that

$$\begin{cases} u_* \ge \varphi \quad \text{on} \quad B_{r_0}(P_0) \\ \\ u_* = \varphi \quad \text{on} \quad P_0 \end{cases}$$

and

$$\varphi_t = -\theta + F_*(D^2\varphi, D\varphi, x_0), \quad \text{with} \quad \theta > 0. \tag{3.10}$$

Up to replace $\varphi(P)$ by $\varphi(P) - |P - P_0|^4$, we can assume that

$$(u_* - \varphi)(P) \ge |P - P_0|^4. \tag{3.11}$$

Notice that $u_*(P_0) < u^+(P_0)$ because otherwise φ would be a test function for u^+ and (3.10) would be in contradiction with the fact that u^+ is a supersolution. Therefore there exists some small $\delta \in (0, r_0/2)$ such that

$$\left\{\begin{array}{l} \varphi_t \le F_*(D^2\varphi, D\varphi, x) \\ \\ \varphi(P) + \delta^4/2 \le u^+(P) \end{array}\right| \quad \text{for} \quad P \in \overline{B_{2\delta}(P_0)}. \tag{3.12}$$

From (3.11), we deduce that

$$u(P) \ge u_*(P) - \delta^4/2 \ge \varphi(P) + \delta^4/2 \quad \text{for} \quad P \in B_{2\delta}(P_0)\backslash B_\delta(P_0). \tag{3.13}$$

We now define

$$w(P) = \begin{cases} \max(\varphi(P) + \delta^4/2, u^*(P)), & P \in B_\delta(P_0) \\ u^*(P), & P \in (\overline{\Omega} \times [0,T)) \backslash B_\delta(P_0). \end{cases}$$

Notice that from (3.13), we have

$$w(P) = \max(\varphi(P) + \delta^4/2, u^*(P)) \quad \text{for} \quad P \in B_{2\delta}(P_0)$$

and then w is a subsolution as the maximum of two subsolutions on $B_{2\delta}(P_0)$ (see Proposition 3.4 ii)). This implies that w is a subsolution everywhere and from (3.12) that $w \in S$. On the other hand, we have

$$0 = (u_* - \varphi)(P_0) = \lim_{\eta \to 0} \inf \{(u - \varphi)(P), \quad |P - P_0| \leq \eta\}.$$

Therefore there exists some $P_1 \in B_\delta(P_0)$ such that $(u - \varphi)(P_1) < \delta^4/2$, which implies that $u(P_1) < w(P_1)$. This is in contradiction with the definition of u as the "maximal subsolution". This ends the proof of the Proposition.

3.5 Subdifferentials

For later use, we recall here the definitions of sub/superdifferentials.

Definition 3.6 (Sub/superdifferentials)
Let $(x,t) \mapsto u(x,t)$ be a upper semicontinuous (resp. lower semi-continuous) function defined on an open set. For $P_0 = (x_0, t_0)$, we say that

$$(\tau, p, X) \in \mathcal{P}^{2,+}u(P_0) \qquad \big(resp. \quad (\tau, p, X) \in \mathcal{P}^{2,-}u(P_0)\big)$$

if there exists a C^2 test function φ such that

$$u \leq \varphi \quad (\ resp. \quad u \geq \varphi) \qquad with\ equality\ at\ P_0$$

and

$$(\tau, p, X) = (\varphi_t, D\varphi, D^2\varphi) \quad at\ P_0.$$

Remark 3.7 *If $u(x,t)$ is independent on t, we say that*

$$(p, X) \in \mathcal{D}^{2,\pm} \quad if\ an\ only\ if \quad (0, p, X) \in \mathcal{P}^{2,\pm}.$$

Definition 3.8 (Limit sub/superdifferentials)
Let $(x,t) \mapsto u(x,t)$ be a upper semicontinuous (resp. lower semi-continuous) function defined on an open set. For $P_0 = (x_0, t_0)$, we say that

$$(\tau, p, X) \in \overline{\mathcal{P}}^{2,+}u(P_0) \qquad \Big(resp. \quad (\tau, p, X) \in \overline{\mathcal{P}}^{2,-}u(P_0)\Big)$$

if there exists exists sequences such that

$$(\tau_k, p_k, X_k) \in \mathcal{P}^{2,+}u(P_k) \quad \big(\ resp. \quad (\tau_k, p_k, X_k) \in \mathcal{P}^{2,-}u(P_k)\big)$$

such that

$$(\tau_k, p_k, X_k, u(P_k)) \to (\tau, p, X, u(P_0)).$$

4 Counter-example in dimension $N \geq 3$

We set $x' = (x_1, ..., x_{N-1})$ such that $x = (x', x_N)$. We now consider solutions U of the first line of (1.3) in the case $\varepsilon = 1$ for a velocity $c(x)$ replaced by some general velocity $c_\infty(x')$ which is independent on x_N, i.e. U solution of

$$U_t = \text{tr}\left\{D^2U \cdot (I - \widehat{DU} \otimes \widehat{DU})\right\} + c_\infty(x')|DU|.$$

We look for particular solutions

$$U(x,t) = u(x',t) - x_N$$

which means that u solves (at least for smooth solutions u)

$$\frac{u_t}{\sqrt{1+|Du|^2}} = c_\infty(x') + \text{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right). \tag{4.1}$$

Then we have

Proposition 4.1 (Traveling profiles with different velocities for $N \geq 3$)
Let $N \geq 3$. There exists a radial function $c_\infty \in C^\infty(\mathbb{R}^{N-1})$ which is positive, and radial functions $u_+ : \mathbb{R}^{N-1} \to [-\infty, +\infty)$, $u_- : \mathbb{R}^{N-1} \to (-\infty, +\infty]$ and constants c_+, c_- satisfying

$$u_+ < u_- \quad \textit{and} \quad c_+ > c_-$$

such that the profiles

$$c_\pm t + u_\pm(x')$$

are solutions of (4.1). We have

$$u_+(x') = \begin{cases} u_0(x') & \textit{if} \quad |x'| < 1 \\ -\infty & \textit{if} \quad |x'| \geq 1 \end{cases} \quad \textit{and} \quad u_-(x') = \begin{cases} +\infty & \textit{if} \quad |x'| \leq 1 \\ u_0(x') & \textit{if} \quad |x'| > 1 \end{cases} \tag{4.2}$$

with $u_0 \in C^\infty(B_1)$ and $u_0 \in C^\infty(\mathbb{R}^{N-1} \backslash \overline{B}_1)$. Moreover there exists $r_0 > 1$ such that

$$\begin{cases} c_\infty(x') = c_- & \textit{for} \quad |x'| \geq r_0 \\ u_-(x') = \textit{constant} & \textit{for} \quad |x'| \geq r_0. \end{cases}$$

The profiles of Proposition 4.1 are illutrated on Figure 5.

Proof of Theorem 1.1
Using Proposition 4.1, we first define for $R > 2r_0$ a velocity defined on the centered square

$$c^R(x') = c_\infty(x') \quad \text{for} \quad x' \in [-R/2, R/2]^{N-1}.$$

Similarly we define

$$U^R_\pm(x,t) = c_\pm t + u_\pm(x') - x_N \quad \text{for} \quad x' \in [-R/2, R/2]^{N-1}.$$

Moreover, up to add a suitable constant to u_+ (resp. u_-), we can assume that

$$u_+ \leq 0 \leq u_-. \tag{4.3}$$

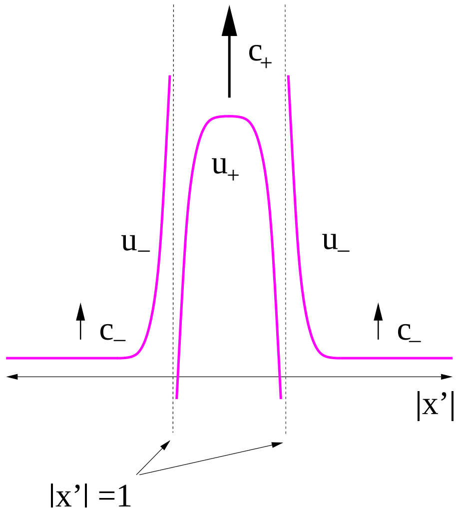

Figure 5: Profiles $u_\pm$ with velocities $c_+ > c_-$

We then extend by periodicity $c^R(x')$ and $U^R_\pm(x', x_N, t)$ as $(R\mathbb{Z})^{N-1}$-periodic functions for $x' \in \mathbb{R}^{N-1}$. We get that $U^R_\pm$ are both solutions of

$$U_t = \operatorname{tr}\left\{D^2U \cdot (I - \widehat{DU} \otimes \widehat{DU})\right\} + c^R(x')|DU|.$$

Then the new functions

$$\bar{U}^\varepsilon_\pm(x,t) := R^{-1}\varepsilon\; U^R_\pm\left(R\varepsilon^{-1}x, R^2\varepsilon^{-1}t\right)$$

are solutions of (1.3) with the velocity

$$c(x) = Rc^R(Rx')$$

which is a positive smooth $\mathbb{Z}^N$-periodic function independent on x_N. Using (4.3), we see that we have

$$\bar{U}^\varepsilon_+(x,t) \le u^\varepsilon(x,t) \le \bar{U}^\varepsilon_-(x,t) \tag{4.4}$$

at time $t = 0$. From the comparison principle, we deduce that (4.4) holds true for all time $t \ge 0$. Setting

$$\bar{c} = Rc_+, \quad \underline{c} = Rc_-$$

we deduce (1.7) from the fact that for $t > 0$

$$\limsup_{(\tilde{x},\tilde{t},\varepsilon)\to(x,t,0)} \bar{U}^\varepsilon_+(\tilde{x},\tilde{t}) \ge \bar{c}t - x_N > \underline{c}t - x_N \ge \liminf_{(\tilde{x},\tilde{t},\varepsilon)\to(x,t,0)} \bar{U}^\varepsilon_-(\tilde{x},\tilde{t}).$$

This ends the proof of the theorem.

Proof of Proposition 4.1
Step 1: preliminary
For the radial functions u_0 and c_∞, we make the abuse of notation $u_0(x') = u_0(r)$ and $c_\infty(x') = c_\infty(r)$ for $r = |x'|$ with $x' \in \mathbb{R}^n$ and

$$n = N - 1.$$

We define the function ζ by the relation

$$\nabla u_0 = \zeta(r)\frac{x'}{|x'|}. \tag{4.5}$$

Then we easily see that a function $c_* t + u_0(r)$ is solution of (4.1) if and only if

$$c_\infty(r) = \frac{c_*}{\sqrt{1+\zeta^2}} - \kappa \quad \text{with} \quad \kappa = \left\{ \left(\frac{\zeta}{\sqrt{1+\zeta^2}} \right)' + \frac{n-1}{r} \left(\frac{\zeta}{\sqrt{1+\zeta^2}} \right) \right\}. \tag{4.6}$$

We look for a function ζ which blows up at $r = 1$ and is smooth for $r \neq 1$ such that we can take

$$c_* = \begin{cases} c_+ & \text{if} \quad r \in [0,1) \\ c_- & \text{if} \quad r > 1 \end{cases} \tag{4.7}$$

and we want to check that c_∞ given by (4.6) is nevertheless smooth (and positive).

Step 2: first computation

As a first candidate for ζ, we propose

$$\tilde{\zeta}(r) = -\sqrt{e^{2\left(\frac{1}{|1-r^2|}-1\right)} - 1} \quad \text{for} \quad r \in [0, \sqrt{2}). \tag{4.8}$$

After some computations, we get

(4.9)

$$c_\infty(r) = c_* e^{-\left(\frac{1}{|1-r^2|}-1\right)} + \frac{2\ \mathrm{sign}(1-r^2) e^{-2\left(\frac{1}{|1-r^2|}-1\right)}}{(1-r^2)^2} \frac{r}{\sqrt{1-e^{-2\left(\frac{1}{|1-r^2|}-1\right)}}} + (n-1)\frac{\sqrt{1-e^{-2\left(\frac{1}{|1-r^2|}-1\right)}}}{r}$$

and

$$\frac{\sqrt{1-e^{-2\left(\frac{1}{|1-r^2|}-1\right)}}}{r} = \sqrt{2 - \frac{2}{3}r^4 + O(r^6)}$$

which is then a smooth function up to $r = 0$ (analytic close to $r = 0$). With the choice (4.7) for any constants $c_\pm$, this shows that c_∞ is C^∞ for $r < \sqrt{2}$.

Step 3: conclusion

In order to define a function $c_\infty(r)$ for all r, we simply set

$$\zeta = \tilde{\zeta}\psi$$

where $\psi \in C^\infty[0, +\infty)$ is a cut-off function satisfying

$$\psi(r) = \begin{cases} 1 & \text{for} \quad 0 \le r \le 1 + \eta \\ 0 & \text{for} \quad r \ge 1 + 2\eta \end{cases}$$

where $\eta > 0$ is small enough such that $1 + 2\eta < \sqrt{2}$. We conclude choosing $c_+ > c_- > 0$ large enough such that c_∞ is positive. We finally get the profiles u_0 integrating (4.7) which provides the profiles $u_\pm$ given by (4.2). Notice in particular that ζ is not integrable in any (positive or negative) neighborhood of $r = 1$. This implies that

$$\lim_{r\to 1^-} u_0(r) = -\infty \quad \text{and} \quad \limsup_{r\to 1^+} u_0(r) = +\infty.$$

This implies that $c_\pm t + u_\pm(x')$, even if they are unbounded, are solutions of (4.1) in the sense of Definition 3.1. This ends the proof of the proposition.

Remark 4.2 *In our example, we can deduce from (4.8) that*

$$|1-r| \sim \frac{1}{\ln |u_0(r)|} \quad \textit{as} \quad r \to 1.$$

Notice also that in (4.6), we have

$$c_\infty(1) = -\kappa = n-1$$

which corresponds to the **negative mean curvature of the tube of equation** $r = 1$*. On the contrary in dimension* $N = 2 = n+1$*, this curvature vanishes and then the velocity* c_∞ *too.*

Remark 4.3 (Example of non homogenization in 2D with sign changing velocity) *In the case* $N = 2$*, i.e.* $n = 1$ *in (4.9), we can take any* $c_+ > 0$ *and* $c_- < 0$ *and the construction of Proposition 4.1 and Theorem 1.1 provides a non homogenization result for a sign changing velocity* $c_\infty(x_1)$ *which is R-periodic.*

5 Preliminaries in any dimension

We consider a solution $u(x,t)$ of

$$u_t = F(D^2u, Du, y) \quad \text{on} \quad \mathbb{R}^N \times (0,+\infty) \tag{5.1}$$

with initial data

$$u(x,0) = u_0(x) = x \cdot \nu \quad \text{for} \quad x \in \mathbb{R}^N. \tag{5.2}$$

Proposition 5.1 (Existence and properties of the solution)
Assume (A). Let $\nu \in \mathbb{S}^{N-1}$ *and* $u_0(x) = \nu \cdot x$*. Then there exists a unique solution* u *of (5.1)-(5.2). Moreover* u *is continuous and* $u(x,t) - \nu \cdot x$ *is* $\mathbb{Z}^N$*-periodic in* x*, and there exists a constant* $C > 0$ *such that*

$$|u_t| \le C \quad \textit{on} \quad \mathbb{R}^N \times [0,+\infty).$$

For any $0 < T < +\infty$*, there exists a modulus of continuity* m_T *such that*

$$|u(x,t) - u(y,t)| \le m_T(|x-y|) \quad \textit{for all} \quad x, y \in \mathbb{R}^N, \quad t \in [0,T]. \tag{5.3}$$

If we assume moreover (B), then we have

$$u_t \ge \delta > 0 \quad \textit{on} \quad \mathbb{R}^N \times [0,+\infty). \tag{5.4}$$

Proof of Proposition 5.1
Step 1: barriers, existence, uniqueness
We set the barriers sub/supersolutions

$$u^\pm(x,t) = u_0(x) + C^\pm t \quad \text{with} \quad \pm C^\pm = \sup_{x \in \mathbb{R}^N} \pm F(0, \nu, x).$$

Then by Perron's method (Proposition 3.5), there exists a solution u of (5.1) satisfying

$$u_- \leq u \leq u^+$$

which implies in particular

$$u(\cdot, 0) = u_0.$$

Then u solves (5.1)-(5.2). Furthermore we deduce from the comparison principle (Theorem 3.3) that this solution is unique and is then continuous.

Step 2: periodicity

For any $k \in \mathbb{Z}^N$, we have

$$u_0(x+k) = u_0(x) + \nu \cdot k.$$

The comparison principle implies that

$$u(x+k, t) = u(x,t) + \nu \cdot k$$

i.e. $u(x,t) - \nu \cdot x$ is $\mathbb{Z}^N$-periodic in the x variable.

Step 3: time regularity

Let $h \geq 0$. Then we have

$$u(x, t+h) \leq u(x,t) + C^+ h \tag{5.5}$$

for $t = 0$ and the comparison principle implies that (5.5) holds for every time. Similarly, we get that

$$u(x,t) + C^- h \leq u(x, t+h). \tag{5.6}$$

Then (5.5) and (5.6) show that

$$C^- \leq u_t \leq C^+.$$

Notice that this result joint to the continuity of u and to the periodicity of $u(x,t) - \nu \cdot x$ implies the existence of a modulus of continuity as in (5.3).

Step 4: further result under assumption (B)

Then we have $C^- \geq \delta$ and this implies (5.4).

This ends the proof of the proposition.

Proposition 5.2 (No fattening)

Assume (A) and (B) and let u be the function given in Proposition 5.1.

i) (No fattening)

Then u satisfies for all $t \geq 0$:

$$Int \ \left\{x \in \mathbb{R}^N, \quad u(x,t) = 0\right\} = \emptyset. \tag{5.7}$$

As a consequence, the sets

$$E_t = \left\{x \in \mathbb{R}^N, \quad u(x,t) \geq 0\right\} \quad and \quad E_t^o = \left\{x \in \mathbb{R}^N, \quad u(x,t) > 0\right\}$$

only differ on a set of empty interior and

$$\partial E_t^o, \partial E_t \subset \left\{x \in \mathbb{R}^N, \quad u(x,t) = 0\right\}.$$

ii) (monotonicity)
We have

$$E_t \subset E_s \quad (\textit{resp.} \quad E_t^o \subset E_s^o) \quad \textit{for all} \quad s \geq t \geq 0. \tag{5.8}$$

iii) (stability of E)
We have

$$\bigcap_{s>t} E_s = E_t \quad \textit{and} \quad \bigcup_{s<t} E_s^o = E_t^o. \tag{5.9}$$

Remark 5.3 *Notice that $E_t^o \subset Int\ E_t$, but we may have $Int\ E_t \neq E_t^o$ (if for instance $u(\cdot,t)$ is positive on $B_1(0)\backslash\{0\}$ and vanishes at $x=0$). Similarly, we have $\overline{E_t^o} \subset E_t$, but we may have $E_t \neq \overline{E_t^o}$ (if for instance $u(\cdot,t)$ is negative on $B_1(0)\backslash\{0\}$ and vanishes at $x=0$).*

Remark 5.4 *We can even show that*

$$\bigcap_{s>t} E_s^o = E_t \quad \textit{and} \quad \bigcup_{s<t} E_s = E_t^o. \tag{5.10}$$

Proof of Proposition 5.2
Proof of i)
Assume that there exists $t_0 > 0$ such that there exists x_0 and $r > 0$ such that

$$\overline{B_r(x_0)} \subset \left\{x \in \mathbb{R}^N, \quad u(x,t_0) = 0\right\}. \tag{5.11}$$

Given such $r > 0$ and some $\varepsilon \in (0,1)$, we consider the test function

$$\phi^\varepsilon(x,t) = A_r|x-x_0|^4 + \bar{A}_\varepsilon|t-t_0|^2 \quad \text{for} \quad x \in \overline{B_r(x_0)}, \quad |t-t_0| \leq \varepsilon$$

where $A_r > 0$ and $\bar{A}_\varepsilon$ are constants that we will fix later. If $x \in \overline{B_r(x_0)}$ and $t \in [0, t_0+1]$ recall that

$$|u(x,t) - u(x_0,t)| \leq m_{t_0+1}(r)$$

where the modulus of continuity m_{t_0+1} is given in (5.3). Moreover, for $|t-t_0| \leq \varepsilon$, we have

$$|u(x,t) - u(x,t_0)| \leq C\varepsilon.$$

Therefore for

$$\begin{cases} A_r := 2r^{-4}m_{t_0+1}(r) > r^{-4}m_{t_0+1}(r), \\ \\ \bar{A}_\varepsilon := 2\dfrac{C}{\varepsilon} > \dfrac{C\varepsilon}{\varepsilon^2} \end{cases}$$

and

$$Q_\varepsilon = \overline{B_r(x_0)} \times [t_0 - \varepsilon, t_0 + \varepsilon]$$

we have

$$\sup_{Q_\varepsilon}(u - \phi^\varepsilon) > \sup_{\partial Q_\varepsilon}(u - \phi^\varepsilon).$$

In particular there exists $(x_\varepsilon, t_\varepsilon) \in \text{Int } Q_\varepsilon$ such that

$$\sup_{Q_\varepsilon}(u - \phi^\varepsilon) = (u - \phi^\varepsilon)(x_\varepsilon, t_\varepsilon)$$

and then

$$\delta \leq \phi_t^\varepsilon \leq F^*(D^2\phi^\varepsilon, D\phi^\varepsilon, x_\varepsilon) \quad \text{at} \quad (x_\varepsilon, t_\varepsilon) \tag{5.12}$$

where we have used that $u_t \geq \delta$ (see (5.4)). We pass to the limit $(x_\varepsilon, t_\varepsilon) \to (\bar{x}_0, t_0)$ as $\varepsilon \to 0$ with $\bar{x}_0 \in \overline{B_r(x_0)}$. In particular we get that

$$\sup_{Q_0}(u - \phi^0) = (u - \phi^0)(\bar{x}_0, t_0) \quad \text{with} \quad \phi^0(x,t) = A_r|x - x_0|^4.$$

Therefore (5.11) implies that $\bar{x}_0 = x_0$. Passing also to the limit in (5.12), we get

$$0 < \delta \leq F^*(0, 0, x_0) = 0$$

where we have used (3.3) to identify to zero the right hand side. Contradiction. This implies (5.7).

Proof of ii)

The monotonicity of u (see (5.4)) implies (5.8).

Proof of iii)

The continuity of u implies (5.9).

This ends the proof of the proposition.

Proposition 5.5 (Birkhoff property)

Using the notation of Proposition 5.2, let us define the set

$$A = \left\{k \in \mathbb{Z}^N, \quad k + E_0 \subset E_0\right\}.$$

Then

$$A = \left\{k \in \mathbb{Z}^N, \quad k \in E_0\right\}.$$

Moreover, if $k \in A$, then for all $t \geq 0$

$$k + E_t \subset E_t. \tag{5.13}$$

Proof of Proposition 5.5

We simply notice that

$$E_0 = \left\{x \in \mathbb{R}^N, \quad \nu \cdot x \geq 0\right\}$$

and $k \in A$ if and only if

$$\nu \cdot k \geq 0.$$

We also notice that

$$u_0(x + k) \geq u_0(x).$$

Therefore from the comparison principle (and the invariance of the equation by integer translations), we deduce that

$$u(x + k, t) \geq u(x, t).$$

This implies (5.13) and ends the proof of the proposition.

Proposition 5.6 (Characteristic functions)
Assume (A) and (B). Let us consider the sets E_t and E_t^0 defined in Proposition 5.1. Then the following two functions

$$\chi_E(\cdot,t) := \chi_{E_t} \quad \textit{and} \quad \chi_{E^o}(\cdot,t) := \chi_{E_t^o} \quad \textit{for all} \quad t \geq 0$$

are solutions of (5.1) and

$$(\chi_E)_* = \chi_{E^o} \quad \textit{and} \quad (\chi_{E^o})^* = \chi_E. \tag{5.14}$$

Proof of Proposition 5.6
Step 1: Proof of $(\chi_{E^o})^* = \chi_E$
Let us consider a point $P_0 = (x_0, t_0)$. If $u(P_0) \neq 0$, then by continuity of u, we conclude that

$$(\chi_{E^o})^*(P_0) = \chi_{E^o}(P_0) = \chi_E(P_0).$$

Now if $u(P_0) = 0$, then $u(x_0, t_0 + h) \geq \delta h$ for all $h > 0$ and then $P_h = (x_0, t_0 + h) \in E_{t_0+h}$. This implies that

$$(\chi_{E^o})^*(P_0) \geq \limsup_{h\to 0} \chi_{E^o}(P_h) = 1$$

and then

$$(\chi_{E^o})^*(P_0) = 1 = \chi_E(P_0).$$

Step 2: Proof of $(\chi_E)_* = \chi_{E^o}$
Similarly, if a point $P_0 = (x_0, t_0)$ is such that $u(P_0) \neq 0$, then by continuity of u, we get that

$$(\chi_E)_*(P_0) = \chi_{E^o}(P_0).$$

Because $u_t \geq \delta$, we deduce that if $u(P_0) = 0$, then for all $h \geq 0$ such that $t_0 - h \geq 0$, we have $u(x_0, t_0) - u(x_0, t_0 - h) \geq \delta h$, and then $P_{-h} = (x_0, t_0 - h)$ is such that

$$u(P_{-h}) < 0 \quad \text{for all} \quad h > 0.$$

Therefore if $t_0 > 0$,

$$(\chi_E)_*(P_0) \leq \liminf_{h\to 0} \chi_E(P_{-h}) = 0$$

which means

$$(\chi_E)_*(P_0) = 0 = \chi_{E^o}(P_0). \tag{5.15}$$

If $u(P_0) = 0$ with $t_0 = 0$, we simply check that $(\chi_{E_0})_* = \chi_{E_0^o}$, which again implies (5.15).
Step 3: Sub/supersolutions
We use an idea of [4]. Let us define for $\varepsilon > 0$

$$u^\varepsilon(x,t) = \beta_\varepsilon(u(x,t)) \quad \text{with} \quad \beta_\varepsilon(a) = \frac{1}{2}\left\{1 + \tanh\left(\frac{a}{\varepsilon}\right)\right\}.$$

Notice that β_ε is smooth and then, using the fact that the equation is geometric (assumption (A2)), it is easy to check that u^ε is also solution of (5.1). Let us define

$$\overline{u} := \limsup_{\varepsilon\to 0}{}^* u^\varepsilon \quad \text{and} \quad \underline{u} := \liminf_{\varepsilon\to 0}{}_* u^\varepsilon.$$

Then we have (using the pointwise limit of u^ε as ε goes to zero)

$$\chi_{E^o} \le \underline{u} \le \overline{u} \le \chi_E.$$

Because by construction, $\underline{u}$ is lower semicontinuous and $\overline{u}$ is upper semicontinuous, we deduce from (5.14) that

$$\overline{u} = \chi_E \quad \text{and} \quad \underline{u} = \chi_{E^o}.$$

By stability of viscosity solutions (see Proposition 3.4 i)), we deduce that $\overline{u} = \chi_E$ is a subsolution and $\underline{u} = \chi_{E^o}$ is a supersolution. This ends the proof of the Proposition.

Proposition 5.7 (Bound from inside on the expansion of E_t)
Let u be the solution given in Proposition 5.1.
If $u(x_0, t_0) \ge 0$, then for each $\tau > 0$

$$u(x, t_0 + \tau) \ge 0 \quad \textit{for} \quad x \in B_r(x_0) \quad \textit{with } r \textit{ such that} \quad m_{t_0+\tau}(r) \le \delta\tau.$$

In particular this implies that

$$\bigcup_{x_0 \in E_{t_0}} B_r(x_0) \quad \subset \quad E_{t_0+\tau}.$$

Proof of Proposition 5.7
Let $\alpha = u(x_0, t_0) \ge 0$. We have for $\tau \ge 0$

$$u(x_0, t_0 + \tau) \ge \alpha + \delta\tau$$

and for $t \ge t_0$

$$|u(x,t) - u(x_0,t)| \le m_t(|x - x_0|).$$

Therefore for $\tau \ge 0$, we get

$$u(x, t_0 + \tau) \ge \alpha + \delta\tau - m_{t_0+\tau}(|x - x_0|).$$

This implies the result.

Corollary 5.8 (Bound from outside for the forward evolution of E_t)
Let u be the solution given in Proposition 5.1.
If $x_0 \in \overline{\{u(\cdot, t_0) < 0\}}$, then for $\tau > 0$ such that $t_0 - \tau \ge 0$, we have

$$u(x, t_0 - \tau) < 0 \quad \textit{for} \quad x \in B_\rho(x_0) \quad \textit{with } \rho \textit{ such that} \quad m_{t_0}(\rho) \le \delta\tau.$$

In particular this implies that

$$E_{t_0-\tau} \quad \subset \quad E_{t_0} \backslash \left(\bigcup_{x_0 \in \partial E_{t_0}} B_\rho(x_0) \right). \tag{5.16}$$

Proof of Corollary 5.8
Just consider a sequence $x_n \to x_0$ with $x_n \in \{u(\cdot, t_0) < 0\}$ and apply Proposition 5.7 assuming by contradiciton that $u(x, t_0 - \tau) \geq 0$ with $x \in B_\rho(x_n)$, in order to get a contradiction. This means

$$\bigcup_{x_0 \in \overline{\mathbb{R}^N \backslash E_{t_0}}} B_\rho(x_0) \quad \subset \quad \mathbb{R}^N \backslash E_{t_0 - \tau}$$

which implies in particular (5.16).

Proposition 5.9 (Arbitrarily long arc-connected components of the set Int E_t)
Let $x_0 \in$ Int E_{t_0} with $t_0 \geq 0$ and ω_0 be the arc-connected component of Int E_{t_0} containing x_0. Then for any $r > 0$, we have

$$\omega_0 \cap \partial B_r(x_0) \neq \emptyset. \tag{5.17}$$

Proof of Proposition 5.9
Notice that ω_0 is an open set (because Int E_{t_0} is open). Assume that for some $r > 0$, (5.17) does not hold. Then this means that

$$\omega_0 \subset B_r(x_0).$$

In particular, we also have

$$\omega_0 \cap \text{Int } E_0 = \emptyset \quad \text{and} \quad t_0 > 0$$

using the fact that Int E_0 is arc-connected and unbounded. Let us define

$$t_* = \inf \{s \in [0, t_0], \quad E_s \cap \omega_0 \neq \emptyset\}.$$

From (5.9), we deduce that

$$E_{t_*} \cap \omega_0 = \bigcap_{s > t_*} (E_s \cap \omega_0). \tag{5.18}$$

Case 1: $t_* < t_0$
Notice that

$$\partial \omega_0 \subset \partial E_{t_0}$$

(indeed if $(\partial \omega_0) \cap \text{Int } E_0 \neq \emptyset$, then we get a contradiction with the definition of the arc-connected component ω_0). Because of Corollary 5.8, we deduce that for $\tau > 0$ with $t_0 - \tau \geq 0$

$$(E_{t_0 - \tau} \cap \omega_0) \quad \subset \quad \omega_0 \backslash \left(\bigcup_{x_1 \in \partial \omega_0} B_\rho(x_1) \right) \quad \text{with} \quad m_{t_0}(\rho) \leq \delta \tau$$

and then

$$(E_s \cap \omega_0) \subset\subset \omega_0 \quad \text{if} \quad s < t_0.$$

Therefore, for $s < t_0$, the set $E_s \cap \omega_0$ is a closed set and from (5.18) and the monotonicity of the sets, we deduce that

$$E_{t_*} \cap \omega_0 = \bigcap_{n \in \mathbb{N} \backslash \{0\}} \left(E_{t_* + 1/n} \cap \omega_0 \right).$$

There exists $x_n \in E_{t_*+1/n} \cap \omega_0$, from which we can extract a convergent subsequence with

$$x_n \to x_\infty \in \omega_0.$$

This shows that $x_\infty \in E_{t_*+1/k} \cap \omega_0$ for all $k > 0$. Then

$$x_\infty \in E_{t_*} \cap \omega_0.$$

Let us consider the function

$$\phi(x,t) = t - t_* + 1$$

which is a test function (from above) for χ_E on $\omega_0 \times (t_* - \varepsilon, t_* + \varepsilon)$. We get at (x_∞, t_*):

$$1 = \phi_t \leq F^*(0,0,x_\infty) = 0.$$

Contradiction.
Case 2: $t_* = t_0$
We get the same contradiction at any point (x, t_0) with $x \in \omega_0$.

Proposition 5.10 (The self-propagating ball barrier)
Assume that (1.9) holds and let us consider some $\xi \in \mathbb{S}^{N-1}$ and $z_0 \in \mathbb{R}^N$. For $t \geq 0$, let us define the function

$$(\chi_G)(\cdot,t) = \chi_{G_t} \quad \textit{with} \quad G_t = \bigcup_{0 \leq s \leq t} \overline{B_R(z_0 + c_0 s\xi)}.$$

Then χ_G is a subsolution of (5.1) on $\mathbb{R}^N \times (0,+\infty)$.

Proof of Proposition 5.10
Let us consider a test function φ satisfying for some $r_0 > 0$

$$\chi_G \leq \phi \quad \text{on} \quad B_{r_0}(P_0) \quad \text{with equality at some point} \quad P_0 = (x_0, t_0) \in \mathbb{R}^N \times (0,+\infty).$$

We want to check the viscosity inequality satisfied by φ. In particular, there exists a unique $s_0 \in [0, t_0]$ such that $x_0 \in \partial B_R(z_0 + c_0 s_0 \xi)$ and then

$$x_0 = z_0 + c_0 s_0 \xi - R p_0 \quad \text{for some} \quad p_0 \in \mathbb{S}^{N-1}.$$

Step 1: time derivative
For $\tau \in \mathbb{R}$ small, let us define

$$x_\tau = z_0 + c_0(s_0 + \tau)\xi - R p_0.$$

Then we have for τ small enough

$$\varphi(x_\tau, t_0 + \tau) \geq \chi_G(x_\tau, t_0 + \tau) = 1 \quad \text{with equality for} \quad \tau = 0.$$

This implies

$$\partial_t \varphi + c_0 \xi \cdot D\varphi = 0 \quad \text{at} \quad P_0. \tag{5.19}$$

Step 2: gradient estimate
Let us set $y_0 = z_0 + c_0 s_0 \xi$. We have

$$\phi(\cdot, t_0) \geq 1 \quad \text{on} \quad \overline{B_R(y_0)} \quad \text{with equality at some point} \quad x_0 = y_0 - Rp_0. \tag{5.20}$$

This implies that

$$D\varphi(x_0, t_0) = p_0 |D\varphi(x_0, t_0)|. \tag{5.21}$$

Step 3: curvature estimate
From (5.20), we also deduce that there exists a C^2 function β satisfying $\beta(0) = 0$ and

$$\beta'(0) = |D\varphi(x_0, t_0)| \tag{5.22}$$

(when $\beta'(0) > 0$ it is enough to take any $\beta''(0) < D^2\varphi(x, t_0) \cdot (p, p)$ for all $x = y_0 - Rp$ with $p \in \mathbb{S}^{N-1}$ close to p_0) such that

$$\varphi(x, t_0) \geq \beta(|x - y_0| - R) \quad \text{in a neighborhood of } x_0.$$

This implies that

$$D^2\varphi(x_0, t_0) \geq -\frac{\beta'(0)}{R} I + \beta''(0)\ p_0 \otimes p_0. \tag{5.23}$$

Step 4: conclusion
We get

$$\begin{aligned}
\varphi_t &= -c_0 \xi \cdot D\varphi \\
&= -c_0 \xi \cdot p_0 |D\varphi| \\
&\leq c_0 |D\varphi| \\
&\leq |D\varphi| F(-\frac{1}{R} I, p_0, x_0) \\
&\leq F^*(D^2\varphi, D\varphi, x_0)
\end{aligned}$$

where we have used (5.19) in the first line, (5.21) in the second line, (1.9) in the fourth line, and (5.23), (A1), (A2) in the last line. This shows that χ_G is a subsolution and ends the proof of the Proposition.

Remark 5.11 *Notice that the function*

$$u(x, t) = R + c_0 t - |x|$$

is a subsolution in $(\mathbb{R}^N \backslash \overline{B_R(0)}) \times (0, +\infty)$ *and this can also be used to check that Proposition 5.10 holds true.*

6 Flatness of E_t for $N = 2$

In order to simplify the description, we will use the analogy with the propagation of a fire. We call E_t the burnt (or black) region and its complement $\mathbb{R}^N \backslash E_t$ is called the unburnt (or white) region.

Proposition 6.1 (Black cubes)
Let us assume (A) and (B) and consider $t_0 \geq 0$. If $x_0 \in$ Int E_{t_0}, then

$$x_0 + \left\{x \in \mathbb{R}^2, \quad \nu \cdot x \geq \bar{R}\right\} \subset E_{t_0+\tau}$$

with $\tau = 5R/c_0$ and $\bar{R} = \sqrt{2}/2 + 2R$.

Proof of Proposition 6.1
Step 1: choice of a ball
Let ω_0 be the connected component of Int E_{t_0} containing x_0. From Proposition 5.9, we know that there exists a point $y_0 \in (\partial B_{4R}(x_0)) \cap \omega_0$ and a continuous path $\gamma : [0,1] \to \overline{B_{4R}(x_0)} \cap \omega_0$ with $\gamma(0,1] \subset B_{4R}(x_0)$, such that

$$\gamma(1) = x_0, \quad \gamma(0) = y_0.$$

We set $\xi := \dfrac{(y_0 - x_0)^\perp}{|y_0 - x_0|}$. Let us call t_* the smallest t such that

$$(\gamma(t) - x_0) \cdot (y_0 - x_0) = 0.$$

Then $\gamma([0, t_*])$ splits the half disk

$$D^+ := \{x \in B_{4R}(x_0), \quad (x - x_0) \cdot (y_0 - x_0) > 0\}$$

in two open connected components ω_σ for $\sigma = +, -$ with

$$\partial\omega_\pm \supset \left\{x \in (\partial D^+) \cap \partial B_{4R}(x_0), \quad \pm(x - x_0) \cdot \xi > 0\right\}.$$

See Figure 6. We also define the strip

$$S_{x_0,y_0} = \left\{x \in \mathbb{R}^2, \quad 0 < (x - x_0) \cdot (y_0 - x_0) < |y_0 - x_0|^2 = (4R)^2\right\}$$

and the extensions of the sets $\omega_\pm$ as

$$\hat{\omega}_\pm = \omega_\pm \cup \{x \in S_{x_0,y_0} \backslash B_{4R}(x_0), \quad \pm(x - x_0) \cdot \xi > 0\}.$$

The sets $\hat{\omega}_\pm$ are also two connected open sets and we have the partition of the strip:

$$S_{x_0,y_0} = \hat{\omega}_+ \cup \hat{\omega}_- \cup (\gamma([0, t_*]) \cap S_{x_0,y_0}).$$

Step 2: Using a self-propagating ball barrier
Let $z_0 = (y_0 + x_0)/2$. For $t \geq t_0$, the characteristic function of the set

$$\bigcup_{0 \leq \tau \leq t - t_0} \overline{B_R(z_0 + \xi(-5R + c_0\tau))}$$

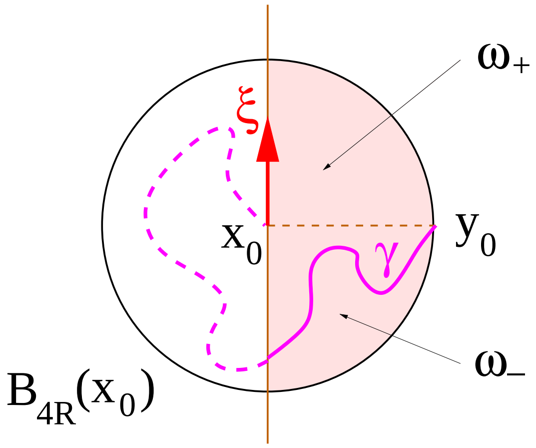

Figure 6: Construction of ω_+ and ω_-

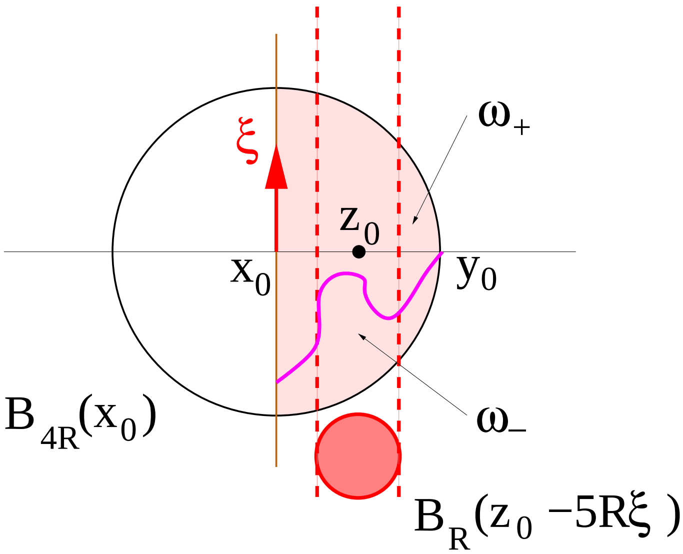

Figure 7: The ball barrier propagating on ω_+

is a subsolution on $\hat{\omega}_+ \times [t_0, +\infty)$ (see Proposition 5.10 and Figure 7).

Similarly, the characteristic function of the set

$$\bigcup_{0 \le \tau \le t - t_0} \overline{B_R(z_0 + \xi(5R - c_0\tau))}$$

is a subsolution on $\hat{\omega}_- \times [t_0, +\infty)$, and we deduce that for $\tau = 5R/c_0$, we have

$$E_{t_0+\tau} \supset \left\{ x \in B_{4R}(x_0), \quad -R \le (x - z_0) \cdot \frac{y_0 - x_0}{|y_0 - x_0|} \le R \right\} \supset \overline{B_R(z_0)} \supset \overline{B_{\sqrt{2}/2}(z_0)}$$

where we have used the fact that $R \ge \sqrt{2}/2$.

Step 3: Using Birkhoff property

From the Birkhoff property (Proposition 5.5), we deduce that for any $k \in \mathbb{Z}^2$ such that $k \in E_0$, we have

$$\overline{B_{\sqrt{2}/2}(z_0 + k)} \subset k + E_{t_0+\tau} \subset E_{t_0+\tau}.$$

Notice that

$$\bigcup_{k\in\mathbb{Z}^2\cap E_0} \overline{B_{\sqrt{2}/2}(k)} \supset \left\{x\in\mathbb{R}^2, \quad \nu\cdot x \geq \sqrt{2}/2\right\}.$$

Indeed if $\nu\cdot x\geq \sqrt{2}/2$ and $y\in \overline{B_{\sqrt{2}/2}(x)}$, then $\nu\cdot y\geq 0$. But there exists $k\in\mathbb{Z}^2$ such that $k\in x+[-\frac{1}{2},\frac{1}{2}]^2\subset \overline{B_{\sqrt{2}/2}(x)}$ and then $\nu\cdot k\geq 0$, i.e. $k\in E_0$. This shows that $x\in \overline{B_{\sqrt{2}/2}(k)}$ for some $k\in\mathbb{Z}^2\cap E_0$.
Therefore

$$z_0+\left\{x\in\mathbb{R}^2, \quad \nu\cdot x\geq \sqrt{2}/2\right\}\subset E_{t_0+\tau}$$

and then

$$x_0+\left\{x\in\mathbb{R}^2, \quad \nu\cdot x\geq \sqrt{2}/2+2R\right\}\subset E_{t_0+\tau}.$$

This ends the proof of the proposition.

Remark 6.2 *In the proof of Proposition 6.1, we use strongly the topology in dimension 2, which is no longer possible in higher dimensions.*

Corollary 6.3 (Black cubes (bis))
Let us assume (A) and (B) and consider $t_0\geq 0$. If $x_0\in E_{t_0}$, then

$$x_0+\left\{x\in\mathbb{R}^2, \quad \nu\cdot x\geq \bar{R}\right\}\subset E_{t_0+\tau} \tag{6.1}$$

with $\tau=5R/c_0$ and $\bar{R}=\sqrt{2}/2+2R$.

Proof of Corollary 6.3
We simply notice that from Proposition 5.7, we have

$$E_{t_0}\subset \text{Int } E_s \quad \text{for all} \quad s>t_0.$$

Then Proposition 6.1 implies that

$$x_0+\left\{x\in\mathbb{R}^2, \quad \nu\cdot x\geq \bar{R}\right\}\subset E_{s+\tau} \quad \text{for all} \quad s>t_0$$

and using (5.9), we deduce (6.1). This ends the proof of the corollary.

Proposition 6.4 (Uniform flatness property of E_t)
Let us assume (A) and (B). Then there exists

$$L:=8R\frac{C_0}{c_0} \tag{6.2}$$

such that for any $t\geq 0$, there exists $\underline{c}_t\in\mathbb{R}$ such that

$$\left\{x\in\mathbb{R}^2, \quad x\cdot\nu\geq \underline{c}_t+L\right\}\subset E_t\subset \left\{x\in\mathbb{R}^2, \quad x\cdot\nu\geq \underline{c}_t\right\}. \tag{6.3}$$

Proof of Proposition 6.4
Step 1: definition of $\bar{c}_{t_0}$
First, for any $t_0 \geq 0$, let us define $\bar{c}_{t_0}$ as the biggest constant such that:

$$E_{t_0} \subset \{x \cdot \nu \geq \bar{c}_{t_0}\}. \tag{6.4}$$

Recall that $E_0 \subset E_{t_0}$ and then $\bar{c}_{t_0} \leq 0$. Notice also that this constant $\bar{c}_{t_0}$ is well defined because of the barriers of Step 1 of the proof of Proposition 5.1.
Step 2: consequences
From the definition of $\bar{c}_{t_0}$ as the biggest constant satisfying (6.4), we deduce that for any $\varepsilon > 0$, there exists $x_\varepsilon \in E_{t_0} \backslash \{x \cdot \nu \geq \bar{c}_{t_0} + \varepsilon\}$, and Corollary 6.3 implies that

$$x_\varepsilon + \left\{x \in \mathbb{R}^2, \quad \nu \cdot x \geq \bar{R}\right\} \subset E_{t_0+\tau}.$$

Because $\bar{c}_{t_0} + \varepsilon > x_\varepsilon \cdot \nu$, we deduce that for all $\varepsilon > 0$

$$\left\{x \in \mathbb{R}^2, \quad \nu \cdot x \geq \bar{R} + \bar{c}_{t_0} + \varepsilon\right\} \subset \left\{x \in \mathbb{R}^2, \quad \nu \cdot x \geq \bar{R} + \nu \cdot x_\varepsilon\right\} \subset E_{t_0+\tau}$$

and then (using the fact that $E_{t_0+\tau}$ is closed)

$$\left\{x \in \mathbb{R}^2, \quad \nu \cdot x \geq \bar{R} + \bar{c}_{t_0}\right\} \subset E_{t_0+\tau}. \tag{6.5}$$

On the other hand, we can easily check that, for any $\varepsilon > 0$, the characteristic function of the set

$$\left\{x \in \mathbb{R}^2, \quad x \cdot \nu \geq \bar{c}_{t_0} - C_0\tau - \varepsilon\right\}$$

is a supersolution. From the comparison principle, we deduce that for any $\varepsilon > 0$

$$E_{t_0+\tau} \subset \left\{x \in \mathbb{R}^2, \quad x \cdot \nu \geq \bar{c}_{t_0} - C_0\tau - \varepsilon\right\}$$

which implies

$$E_{t_0+\tau} \subset \left\{x \in \mathbb{R}^2, \quad x \cdot \nu \geq \bar{c}_{t_0} - C_0\tau\right\}. \tag{6.6}$$

Therefore this implies (6.3) for $t = t_0 + \tau \geq \tau$ with $\underline{c}_{t_0+\tau} = \bar{c}_{t_0} - C_0\tau$ and

$$L = \bar{R} + C_0\tau. \tag{6.7}$$

For $t \in [0, \tau]$, we have

$$\{x \cdot \nu \geq 0\} = E_0 \subset E_t \subset E_\tau \subset \{x \cdot \nu \geq -C_0\tau\}$$

which implies (6.3) still with L given in (6.7).
Step 3: conclusion
Using the fact that $R \geq \sqrt{2}/2$ and $C_0 \geq c_0$ (see Remark 1.2), we deduce (6.3) with L given by (6.2). This ends the proof of the proposition.

7 Existence of a corrector: proof of Theorem 1.3

Proof of Theorem 1.3
Step 1: Control of the space oscillations of u
We will prove the following estimate

$$-L \le u(x,t) - (x\cdot\nu - \underline{c}_t) \le 0. \tag{7.1}$$

Let us first explain heuristically why we can get such an estimate. Given some $x_0 \in \mathbb{R}^2$ and some time $t \ge 0$, let us assume that there exists some $b \in \mathbb{Z}^2$ such that:

$$u(x_0,t) = \nu\cdot b.$$

Then we have for all $x \in \mathbb{R}^2$:

$$u(x-b,t) = u(x,t) - \nu\cdot b$$

which shows that

$$u(x_0-b,t) = 0. \tag{7.2}$$

Then (6.3) implies that

$$\nu\cdot(x_0-b) \ge \underline{c}_t. \tag{7.3}$$

If we assume that (7.2) is now replaced by

$$u(x_0-b,t) < 0 \tag{7.4}$$

we see that $\nu\cdot(x_0-b) \ge \underline{c}_t + L$ would imply $u(t,x_0-b) \ge 0$ which is in contradiction with (7.4). Therefore $\nu\cdot(x_0-b) < \underline{c}_t + L$, which in the limit case (7.2) should give

$$\nu\cdot(x_0-b) \le \underline{c}_t + L.$$

Joint to (7.3), this gives

$$-L \le u(x_0-b,t) - (\nu\cdot(x_0-b) - \underline{c}_t) \le 0$$

i.e.

$$-L \le u(x_0,t) - (\nu\cdot x_0 - \underline{c}_t) \le 0$$

which is (7.1). We now do the (rigorous) proof of (7.1).
Case A: $\nu \in \mathbb{R}^2\backslash(\mathbb{R}\cdot\mathbb{Q}^2)$
In that case, $\nu\cdot\mathbb{Z}^2$ is a dense subgroup of $\mathbb{R}$, and then for any $x_0 \in \mathbb{R}^N$ and $t \ge 0$, for any $\varepsilon > 0$, there exists $b_\pm \in \mathbb{Z}^2$ such that

$$\nu\cdot b_- < u(t,x_0) < \nu\cdot b_+ \quad \text{and} \quad |\nu\cdot(b_+-b_-)| \le \varepsilon.$$

Because we have for any $x \in \mathbb{R}^2$

$$u_0(x-b_\pm) = u_0(x) - \nu\cdot b_\pm$$

we deduce that

$$u(x_0 - b_\pm, t) = u(x_0, t) - \nu \cdot b_\pm.$$

Therefore $\varepsilon \geq u(x_0 - b_-, t) > 0$, and as above, this implies that

$$\nu \cdot (x_0 - b_-) \geq \underline{c}_t$$

and then

$$u(x_0 - b_-, t) - (\nu \cdot (x_0 - b_-) - \underline{c}_t) \leq \varepsilon$$

i.e.

$$u(x_0, t) - (\nu \cdot x_0 - \underline{c}_t) \leq \varepsilon. \tag{7.5}$$

On the other hand $-\varepsilon \leq u(x_0 - b_+, t) < 0$ also implies (as above) that

$$\nu \cdot (x_0 - b_+) < \underline{c}_t + L$$

and then

$$-\varepsilon - L \leq u(x_0 - b_+, t) - (\nu \cdot (x_0 - b_+) - \underline{c}_t)$$

i.e.

$$-\varepsilon - L \leq u(x_0, t) - (\nu \cdot x_0 - \underline{c}_t). \tag{7.6}$$

Because (7.5) and (7.6) are true for any $\varepsilon > 0$, we deduce (7.1).
Case B: $\nu \in \mathbb{R} \cdot \mathbb{Q}^2$
We simply deduce the result from case A, considering a sequence $\nu_k \to \nu$ with $\nu_k \in \mathbb{R}^2 \backslash (\mathbb{R} \cdot \mathbb{Q}^2)$, and using the stability of viscosity solutions which implies (locally uniformly) the continuity dependence of the solution with respect to the initial data.

Step 2: Global bound in time
We show in this step that

$$|u(x,t) - (x \cdot \nu + \lambda t)| \leq 3L. \tag{7.7}$$

For any $T > 0$ and $t \geq 0$, let us define

$$\lambda(t, T) = \frac{u(0, t+T) - u(0,t)}{T}$$

and

$$\begin{cases} \lambda^+(T) = \sup_{t \geq 0} \lambda(t, T) \\ \\ \lambda^-(T) = \inf_{t \geq 0} \lambda(t, T). \end{cases}$$

Because u is globally Lipschitz in time, we know that those quantities are bounded.
Step 2.1: first estimate on $\lambda^+(T) - \lambda^-(T)$
For any $\tau_+, \tau_- \geq 0$, let us define

$$\begin{cases} w^+(x,t) = u(x, t + \tau_+) - (x \cdot \nu - \underline{c}_{\tau_+}) \\ \\ w^-(x,t) = u(x, t + \tau_-) - (x \cdot \nu - \underline{c}_{\tau_-}) + L. \end{cases}$$

Then we have

$$-L \leq w^+(x,0) \leq 0 \leq w^-(x,0) \leq L.$$

And the comparison principle for the solution u, gives

$$w^+(x,T) \leq w^-(x,T)$$

which implies (using $-2L \leq w^+(x,0) - w^-(x,0)$)

$$w^+(x,T) - w^+(x,0) \leq w^-(x,T) - w^-(x,0) + 2L$$

and then

$$\lambda(\tau_+, T) \leq \lambda(\tau_-, T) + \frac{2L}{T}.$$

This implies

$$\lambda^-(T) \leq \lambda^+(T) \leq \lambda^-(T) + \frac{2L}{T}. \tag{7.8}$$

Step 2.2: refined estimate
Let us consider $T_1, T_2 > 0$ such that there exists $P, Q \in \mathbb{N}\setminus \{0\}$ such that

$$PT_1 = QT_2.$$

Then we have

$$\lambda^+(T_1) \geq \lambda^+(PT_1) = \lambda^+(QT_2) \geq \lambda_-(QT_2) \geq \lambda_-(T_2) \geq \lambda^+(T_2) - \frac{2L}{T_2}$$

which shows that

$$\lambda^+(T_2) - \lambda^+(T_1) \leq \frac{2L}{T_2}$$

and then by symmetry

$$|\lambda^+(T_2) - \lambda^+(T_1)| \leq \max\left(\frac{2L}{T_1}, \frac{2L}{T_2}\right).$$

Doing the same raisoning with λ^-, we get finally

$$|\lambda^\pm(T_2) - \lambda^\pm(T_1)| \leq \max\left(\frac{2L}{T_1}, \frac{2L}{T_2}\right) \tag{7.9}$$

for T_2/T_1 rational. When T_2/T_1 is not rational, the result still holds by continuity of the map $T \mapsto \lambda^\pm(T)$, which follows from the fact that the solution u is uniformly continuous in time (because it is globally Lipschitz in time).
Step 2.3: conclusion
This shows that $(\lambda^\pm(T))_T$ is a Cauchy sequence, which has then a limit $\lambda^\pm$, with $\lambda^\pm = \lambda$ becaiuse of (7.8). Passing to the limit $T_2 \to +\infty$ in (7.9), we get

$$|\lambda - \lambda^\pm(T_1)| \leq \frac{2L}{T_1}.$$

This implies with $T = T_1$ and any $t \geq 0$:

$$|u(0, t+T) - u(0, t) - \lambda T| \leq 2L.$$

On the other hand, (7.1) implies that

$$|u(x, t+T) - u(0, t+T) - x \cdot \nu| \leq L.$$

This shows that

$$|u(x, t+T) - u(0, t) - (x \cdot \nu + \lambda T)| \leq 3L \tag{7.10}$$

which implies (7.7), taking $t = 0$ in (7.10).

Step 3: Shifting at infinity in time and Perron's method

For any $n \in \mathbb{N}$, let us define

$$u_n(x, t) = u(x, t+n) - u(0, n).$$

Then u_n satisfies

$$\begin{cases} |u_n(x,t) - (x \cdot \nu + \lambda t)| \leq 3L \\ \\ \text{the map} \quad x \mapsto u_n(x,t) - x \cdot \nu \quad \text{is } \mathbb{Z}^2\text{-periodic.} \end{cases} \tag{7.11}$$

Then we define

$$\begin{cases} \overline{u}(x,t) = \displaystyle\limsup_{(x',t',n)\to(x,t,+\infty)} u_n(x',t') \\ \\ \underline{u}(x,t) = \displaystyle\liminf_{(x',t',n)\to(x,t,+\infty)} u_n(x',t'). \end{cases}$$

Then $\overline{u}$ and $\underline{u}$ are respectively a subsolution and a supersolution of (5.1) on $\mathbb{R}^2 \times \mathbb{R}$, and moreover they satisfy (7.11). Then we define

$$\begin{cases} \overline{\overline{u}}(x,t) = \displaystyle\sup_{a \in \mathbb{R}} \left(\overline{u}(x, t+a) - \lambda a\right) \\ \\ \underline{\underline{u}}(x,t) = \displaystyle\inf_{a \in \mathbb{R}} \left(\underline{u}(x, t+a) - \lambda a\right). \end{cases}$$

Then $\overline{\overline{u}}^*$ and $\underline{\underline{u}}_*$ are still respectively subsolution and a supersolution of (5.1) on $\mathbb{R}^2 \times \mathbb{R}$, and still satisfy (7.11). They satisfy moreover that

$$\overline{\overline{u}}^*(x,t) = \overline{\overline{u}}^*(x,0) + \lambda t \quad \text{and} \quad \underline{\underline{u}}_*(x,t) = \underline{\underline{u}}_*(x,0) + \lambda t.$$

Let us define

$$\begin{cases} \overline{v}(x) := \overline{\overline{u}}^*(x,0) - x \cdot \nu \geq -3L, \\ \\ \underline{v}(x) := 6L + \underline{\underline{u}}_*(x,0) - x \cdot \nu \leq 9L. \end{cases}$$

Then $\overline{v}$ and $\underline{v}$ are $\mathbb{Z}^2$-periodic, they are respectively sub and supersolution of

$$\lambda = F(D^2 v, p + Dv, x) \quad \text{on} \quad \mathbb{R}^2 \tag{7.12}$$

and satisfy

$$\overline{v} \leq \underline{v}.$$

Then, we can apply Perron's method to show the existence of a solution v (for instance as a suppremum of subsolutions) of (7.12) such that

$$-3L \leq \overline{v} \leq v \leq \underline{v} \leq 9L.$$

Therefore

$$\sup v - \inf v \leq 12L$$

which implies

$$\sup v - \inf v \leq 96\ R\frac{C_0}{c_0}$$

and then implies (1.11).

Step 4: Continuity and homogeneity of $\overline{F}$

For any $p \in \mathbb{R}^2 \setminus \{0\}$, let us call u_p the solution of (5.1) with initial data

$$u_p(x,0) = p \cdot x.$$

Then, from (A2) and the uniqueness of the solution, we get that

$$u_p = |p| u_{\frac{p}{|p|}}.$$

On the other hand

$$u_p = 0 \quad \text{for} \quad p = 0.$$

This implies (using for instance (7.7)) that

$$\overline{F}(\lambda p) = \lambda \overline{F}(p) \quad \text{for any} \quad \lambda \geq 0.$$

Finally, the continuity of $\overline{F}$ follows classically from (7.7) and the stability of viscosity solutions with respect to initial data.

This ends the proof of Theorem 1.3.

8 Conditional homogenization in any dimension: proof of Theorem 1.5

This section is fully devoted to the proof of Theorem 1.5.

Let us first define for $0 < \rho < R$ the set:

$$\mathcal{D}_{\rho,R} = \left\{ (X,p,x) \in S^N \times \left(\mathbb{R}^N \setminus \{0\}\right) \times \mathbb{R}^N, \quad |X|, |p| < R, \quad |p| > \rho \right\}.$$

Let us denote by $\omega_{\rho,R}$ a modulus of continuity for F on $\mathcal{D}_{\rho,R}$, i.e. such that we have

$$(8.1) \quad |F(X,p,x) - F(X',p',x')| \leq \omega_{\rho,R}(a) \quad \text{if} \quad \left\{ \begin{array}{ll} & (X,p,x), (X',p',x') \in \mathcal{D}_{\rho,R} \\ \text{and} & \\ & |X - X'|, |p-p'|, |x-x'| \leq a. \end{array} \right.$$

In the proof of Theorem 1.5, we will also need quite classical results recalled in the appendix about barriers (see Subsection 12.1, Lemma 12.1 and Corollary 12.2) and about inf-convolutions (see Subsection 12.2, Lemmata 12.3 and 12.4).

Proof of Theorem 1.5
Step 1: barriers
We notice that the functions v^+_{K,x_0} (resp. v^-_{K,x_0}) given in Lemma 12.1 is independent on ε but is a supersolution (resp. subsolution) of (1.1) for any $\varepsilon \in (0, 1]$. We now recall that u_0 is uniformly continuous. Therefore the function u^+ (resp. u^-) given in Corollary 12.2 is still independent on ε, and is a supersolution (resp. subsolution) of (1.1) for $\varepsilon \in (0, 1]$. Then from Perron's method, there exists a solution u^ε satisfying

$$u^- \leq u^\varepsilon \leq u^+$$

and

$$\lim_{t\to 0^+} \sup_{x\in\mathbb{R}^N} |u^\pm(x,t) - u_0(x)| = 0.$$

Moreover from the comparison principle, we deduce that u^ε is unique and continuous.
Step 2: definition of $\overline{u}, \underline{u}$
As usual, we define the half relaxed limits:

$$\begin{cases} \overline{u} = \limsup_{\varepsilon\to 0}{}^* u^\varepsilon \\ \\ \underline{u} = \liminf_{\varepsilon\to 0}{}_* u^\varepsilon. \end{cases}$$

By construction we have

$$u^- \leq \underline{u} \leq \overline{u} \leq u^+$$

which shows that the initial data is satisfied by the limits:

$$\underline{u}(x, 0) = \overline{u}(x, 0) = u_0(x).$$

We now have the
Claim: *$\overline{u}$ (resp. $\underline{u}$) is a subsolution (resp. a supersolution) of the limit equation (1.2).*
Then the comparison principle for the limit equation (1.2) implies that

$$\overline{u} \leq \underline{u}$$

which shows that

$$\overline{u} = \underline{u} = u$$

where u is the unique solution of (1.2). This finally implies the convergence (locally uniformly) of u^ε towards u.
The rest of the proof is then devoted to prove the Claim. We will first prove that $\overline{u}$ is a subsolution of (1.2) on $\mathbb{R}^N \times (0, +\infty)$ (and the proof is similar to show that $\underline{u}$ is a supersolution, with some adaptations).

Step 3: if $\overline{u}$ is not a subsolution
We assume by contradiction that $\overline{u}$ is not a subsolution of (1.2) on $\mathbb{R}^N \times (0,+\infty)$. Then there exists a test function φ and a point $P_0 = (x_0, t_0)$ (with $t_0 > 0$) such that

$$\begin{cases} \overline{u}(P_0) = \varphi(P_0) \\ \overline{u} \leq \varphi \quad \text{on} \quad Q_r(P_0) = B_r(x_0) \times (t_0 - r, t_0 + r) \subset \mathbb{R}^N \times (0,+\infty) \\ \varphi_t(P_0) - \bar{F}(D\varphi(P_0)) = \theta > 0. \end{cases}$$

Let us set

$$p = D\varphi(P_0), \quad \bar{\lambda} = \bar{F}(p), \quad \lambda = \varphi_t(P_0).$$

Up to replace φ by $|(x,t) - P_0|^4 + \varphi$, we can moreover assume that there exists $\delta_1 = \delta_1(r) > 0$ such that

$$\begin{cases} \overline{u} + 2\delta_1 \leq \varphi & \text{on} \quad \overline{Q_r(P_0)} \backslash Q_{r/2}(P_0), \\ \overline{u} < \varphi & \text{on} \quad \overline{Q_r(P_0)} \backslash \{P_0\}. \end{cases}$$

Step 3.1: $p = 0$
Then for $\varepsilon > 0$ small enough, we have also

$$u^\varepsilon + \delta_1 \leq \varphi \quad \text{on} \quad \overline{Q_r(P_0)} \backslash Q_{r/2}(P_0)$$

and for ε small enough

$$S^\varepsilon = \sup_{\overline{Q_r(P_0)}} (u^\varepsilon - \varphi) = (u^\varepsilon - \varphi)(P_\varepsilon) \quad \text{with} \quad P_\varepsilon = (x_\varepsilon, t_\varepsilon) \in Q_{r/2}(P_0)$$

and

$$P_\varepsilon \to P_0, \quad S^\varepsilon \to 0, \quad \text{as} \quad \varepsilon \to 0.$$

On the one hand $p = 0$ implies $\bar{\lambda} = 0$ and on the other hand we have

$$\varphi_t(P_\varepsilon) \leq F(\varepsilon D^2\varphi(P_\varepsilon), D\varphi(P_\varepsilon), x_\varepsilon).$$

Passing to the limit $\varepsilon \to 0$, we get

$$0 < \theta = \varphi_t(P_0) \leq F^*(0, 0, x_0) = 0.$$

Contradiction.
Step 3.2: $p \neq 0$
Notice that

$$\varphi(x,t) = \varphi(P_0) + p \cdot (x - x_0) + \lambda(t - t_0) + \psi(x,t) \tag{8.2}$$

with ψ and its derivatives small in $\overline{Q_r(P_0)}$, i.e.

$$|D\psi|, |D^2\psi|, |\psi_t| \leq \mu \leq 1 \quad \text{on} \quad \overline{Q_r(P_0)} \quad \text{with} \quad \mu = o_1(r). \tag{8.3}$$

We also extend by continuity (keeping the same notation) φ and ψ outside $Q_r(P_0)$ in order to keep the relation (8.2) and such that there exists a $\mu_0 > 0$ (not necessarily small) with

$$|D\psi| \leq \mu_0 \quad \text{in} \quad \mathbb{R}^N \times [0,+\infty).$$

Step 3.2.1: Regularizations and definition of the perturbed test function φ^ε
Then for $\eta > 0$, we consider $w^{2\eta} = w_*^{2\eta}$ a supercorrector associated to the approximate Hamiltonian $F^{2\eta}$ given in assumption (B') for the gradient p and let us define

$$\bar{\varphi}^\varepsilon(x,t) = \inf_{z\in \overline{B_{\varepsilon\eta}(x)}} \tilde{\varphi}^\varepsilon(z,t) \quad \text{with} \quad \tilde{\varphi}^\varepsilon(x,t) = \varphi(x,t) + \varepsilon w^{2\eta}(x/\varepsilon) \tag{8.4}$$

where $\bar{\varphi}^\varepsilon$ is lower semi-continuous (from Lemma 12.3). The reason for introducing the regularization given by (8.4), is that it will allow us to control the curvature of the level sets of $\bar{\varphi}^\varepsilon$ from one side, and then to bound the gradient from below.

We consider the following perturbed test function for some $\rho > 0$ (to fix later)

$$\varphi^\varepsilon(x,t) = \inf_{z\in\mathbb{R}^N}\left(\bar{\varphi}^\varepsilon(z,t) + \frac{|x-z|^4}{4\varepsilon^3\rho}\right)$$

which is also lower semi-continuous (fact similar to the proof of Lemma 12.3). The reason for introducing this further regularization (given by φ^ε) is that it will help us later to control the second derivatives from above and from below, which will be important in the perturbation argument (in order to get a contradiction).

We recall that (from assumption (B'))

$$\sup w^{2\eta} - \inf w^{2\eta} \le \bar{\kappa}_0$$

and choose $\eta > 0$ small enough such that

$$|\bar{F}^{2\eta}(p) - \bar{\lambda}| \le \theta/4 \tag{8.5}$$

for later use. Then, using A)vii) of Lemma 12.4 with $L = |p| + \mu_0$, we get for ε small enough

$$u^\varepsilon + \delta_1 \le \varphi^\varepsilon \quad \text{on} \quad \overline{Q_r(P_0)}\backslash Q_{r/2}(P_0) \tag{8.6}$$

and

$$S^\varepsilon = \sup_{\overline{Q_r(P_0)}} (u^\varepsilon - \varphi^\varepsilon) = (u^\varepsilon - \varphi^\varepsilon)(P_\varepsilon) \to 0 \quad \text{as} \quad \varepsilon \to 0 \tag{8.7}$$

for some point $P_\varepsilon = (x_\varepsilon, t_\varepsilon) \in Q_{r/2}(P_0)$.

In general, there is no reason for φ^ε to be locally a supersolution of the PDE satisfied by u^ε (i.e. equation (1.1)), because the viscosity supersolution inequality may not be satisfied at points where the gradient of φ^ε vanishes. This is due to the "instability" of the discontinuous Hamiltonian F (or $F^{2\eta}$), by perturbation of its arguments and this is the reason why the classical perturbed test function of Evans does not apply directly to this case of mean curvature type equations involving hamiltonians F. The point is that, even if φ^ε would not be a supersolution, we will show that φ^ε still satisfies the strict viscosity supersolution inequality (for the PDE satisfied by u^ε) at the point P_ε, which is enough to get a contradiction.

Step 3.2.2: Controlling the distance between points
Still using A)vii) of Lemma 12.4 with $L = |p| + \mu_0$, we see that there exists a point $\bar{x}_\varepsilon \in \mathbb{R}^N$ such that

$$|\bar{x}_\varepsilon - x_\varepsilon| \le \varepsilon C_{\bar{\kappa}_0,L,\rho} \quad \text{with} \quad C_{\bar{\kappa}_0,L,\rho} \to 0 \quad \text{as} \quad \rho \to 0$$

and

$$\varphi^\varepsilon(x_\varepsilon,t_\varepsilon)=\bar\varphi^\varepsilon(\bar x_\varepsilon,t_\varepsilon)+\frac{|x_\varepsilon-\bar x_\varepsilon|^4}{4\varepsilon^3\rho}.$$

We choose ρ small enough such that $C_{\bar\kappa_0,L,\rho}\le\eta$ and then

$$|\bar x_\varepsilon-x_\varepsilon|\le\varepsilon\eta. \tag{8.8}$$

Step 3.2.3: Framework to apply Ishii's Lemma
Then we have

$$\begin{aligned}S^\varepsilon &= \sup_{(x,t)\in\overline{Q_r(P_0)}}\left(u^\varepsilon(x,t)-\inf_{y\in\mathbb{R}^N}\left(\bar\varphi^\varepsilon(y,t)+\frac{|x-y|^4}{4\varepsilon^3\rho}\right)\right)\\ &= \sup_{(x,t)\in\overline{Q_r(P_0)},\ y\in\mathbb{R}^N}\left(u^\varepsilon(x,t)-\bar\varphi^\varepsilon(y,t)-\frac{|x-y|^4}{4\varepsilon^3\rho}\right)\\ &= u^\varepsilon(x_\varepsilon,t_\varepsilon)-\bar\varphi^\varepsilon(\bar x_\varepsilon,t_\varepsilon)-\frac{|x_\varepsilon-\bar x_\varepsilon|^4}{4\varepsilon^3\rho}.\end{aligned}$$

From Ishii's Lemma (see Theorem 8.3 in the Users's Guide [17], and Theorem 7 in [16]), we deduce with

$$\Phi(x,y,t)=\frac{|x-y|^4}{4\varepsilon^3\rho}$$

that for every $\gamma>0$, there exists

(8.9)

$$\begin{cases}(b_1,q,X)\in\overline{\mathcal{P}}^{2,+}u^\varepsilon(x_\varepsilon,t_\varepsilon)\\ (b_2,q,Y)\in\overline{\mathcal{P}}^{2,-}\bar\varphi^\varepsilon(\bar x_\varepsilon,t_\varepsilon)\\ b_1-b_2=0=\Phi_t(x_\varepsilon,\bar x_\varepsilon,t_\varepsilon)\\ -\left(\frac{1}{\gamma}+||A||\right)\begin{pmatrix}I&0\\0&I\end{pmatrix}\le\begin{pmatrix}X&0\\0&-Y\end{pmatrix}\le A+\gamma A^2\\ \text{with}\quad q=D_x\Phi(x_\varepsilon,\bar x_\varepsilon,t_\varepsilon)=-D_y\Phi(x_\varepsilon,\bar x_\varepsilon,t_\varepsilon)=\delta(x_\varepsilon-\bar x_\varepsilon)\quad\text{with}\quad\delta=\frac{|x_\varepsilon-\bar x_\varepsilon|^2}{\varepsilon^3\rho}\\ \text{with}\quad A=D^2\Phi(x_\varepsilon,\bar x_\varepsilon,t_\varepsilon)=\delta\begin{pmatrix}I+2\widehat q\otimes\widehat q&-I-2\widehat q\otimes\widehat q\\-I-2\widehat q\otimes\widehat q&I+2\widehat q\otimes\widehat q\end{pmatrix}\in S^{2N}\quad\text{with}\quad\widehat q=\frac{q}{|q|}\quad(\text{if}\quad q\ne0)\end{cases}$$

where $||A||=\sup_{|\xi|=1}<A\xi,\xi>$.

From A)v)-vi) of Lemma 12.4, we see that there exists a point $\tilde x_\varepsilon\in\mathbb{R}^N$ such that

$$(b_2,q,Y)\in\overline{\mathcal{P}}^{2,-}\tilde\varphi^\varepsilon(\tilde x_\varepsilon,t_\varepsilon)\quad\text{and}\quad\bar\varphi^\varepsilon(\bar x_\varepsilon,t_\varepsilon)=\tilde\varphi^\varepsilon(\tilde x_\varepsilon,t_\varepsilon)\quad\text{with}\quad|\tilde x_\varepsilon-\bar x_\varepsilon|\le\varepsilon\eta \tag{8.10}$$

and then we have at $\tilde P_\varepsilon=(\tilde x_\varepsilon,t_\varepsilon)$

$$\begin{cases}(b_2-(\lambda-\bar F^{2\eta}(p)+\psi_t(\tilde P_\varepsilon)),q-D\psi(\tilde P_\varepsilon),Y-D^2\psi(\tilde P_\varepsilon))\in\overline{\mathcal{P}}^{2,-}\tilde l^\varepsilon(\tilde x_\varepsilon,t_\varepsilon)\\ \text{with}\qquad\tilde l^\varepsilon(x,t)=\varphi(P_0)+p\cdot(x-x_0)+\bar F^{2\eta}(p)(t-t_0)+\varepsilon w^{2\eta}(x/\varepsilon).\end{cases} \tag{8.11}$$

We have in particular the viscosity inequalities

$$\begin{cases}b_1\le F^*\left(\varepsilon X,q,\frac{x_\varepsilon}{\varepsilon}\right)\\ b_2-(\lambda-\bar F^{2\eta}(p)+\psi_t(\tilde P_\varepsilon))\ge F_*^{2\eta}\left(\varepsilon(Y-D^2\psi(\tilde P_\varepsilon)),q-D\psi(\tilde P_\varepsilon),\frac{\tilde x_\varepsilon}{\varepsilon}\right).\end{cases} \tag{8.12}$$

Recall that to be able to apply Ishii's Lemma, we need to be able to bound $b_1 \le C$ and $b_2 \ge -C$ for general $(b_1, q_1, X) \in \overline{\mathcal{P}}^{2,+} u^\varepsilon(x,t)$ and $(b_2, q_2, Y) \in \overline{\mathcal{P}}^{2,-} \bar{\varphi}^\varepsilon(y,t)$ for (x,t) close to $(x_\varepsilon, t_\varepsilon)$, (y,t) close to $(\bar{x}_\varepsilon, t_\varepsilon)$, and bounded $q_1, q_2, X, Y, u^\varepsilon(x,t), \bar{\varphi}^\varepsilon(y,t)$. Indeed this is true and comes from the viscosity inequalities similar to (8.12), using in particular assumption (A4)i).

Step 3.2.4: Bound on the second derivatives and the gradient
Notice that

$$0 \le A \le 3\delta E \quad \text{with} \quad E = \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

We also have $E^2 = 2E$, and then $A^2 \le 18\delta^2 E$. Because $||A|| = 6\delta$, setting $\gamma = \dfrac{1}{3\delta}$ in (8.9), we get

$$(8.13) \qquad -9\delta \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le 9\delta \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

Therefore

$$(8.14) \qquad -9\delta I \le Y \le 9\delta I \quad \text{and} \quad |q| \le \frac{\eta^3}{\rho}.$$

where the last bound on q follows from (8.9) and (8.8).
Step 3.2.5: Gradient estimate from below
Notice that (8.11) also implies that

$$(8.15) \qquad b_2 = \lambda + \psi_t(\tilde{P}_\varepsilon).$$

Therefore, for

$$(8.16) \qquad |\psi_t| \le \mu \le \theta/2$$

and using the fact that $\lambda = \bar{\lambda} + \theta$ and $\bar{\lambda} = \bar{F}(p) \ge 0$, we deduce from (8.9), (8.15) and (8.12) that

$$\theta/2 \le \lambda + \psi_t(\tilde{P}_\varepsilon) = b_2 = b_1 \le F^* \left(\varepsilon X, q, \frac{x_\varepsilon}{\varepsilon}\right) \le F^* \left(\varepsilon Y, q, \frac{x_\varepsilon}{\varepsilon}\right)$$

where we have used $X \le Y$ which is implied by (8.13).
We deduce that

$$\theta/2 \le F^* \left(\varepsilon Y, q, \frac{x_\varepsilon}{\varepsilon}\right) \le c_\eta |q|$$

where we have used Lemma 12.3 (and in particular (12.8)) for the last inequality. This shows that the gradient is bounded from below:

$$\frac{\theta}{2c_\eta} \le |q|.$$

Step 3.2.6: Perturbation and contradiction
Notice that from (8.8), we have

$$(8.17) \qquad \delta \le \frac{\eta^2}{\varepsilon\rho}.$$

Now, we deduce from (8.12) and (8.15) that

$$\begin{aligned}\bar F^{2\eta}(p) &\ge F_*^{2\eta}\left(\varepsilon(Y-D^2\psi(\tilde P_\varepsilon)),q-D\psi(\tilde P_\varepsilon),\frac{\tilde x_\varepsilon}{\varepsilon}\right)\\ &= F^{2\eta}\left(\varepsilon(Y-D^2\psi(\tilde P_\varepsilon)),q-D\psi(\tilde P_\varepsilon),\frac{\tilde x_\varepsilon}{\varepsilon}\right)\\ &\ge F\left(\varepsilon(Y-D^2\psi(\tilde P_\varepsilon)),q-D\psi(\tilde P_\varepsilon),\frac{x_\varepsilon}{\varepsilon}\right)\\ &\ge -\omega_{\rho_0,R}(\mu)+F\left(\varepsilon Y,q,\frac{x_\varepsilon}{\varepsilon}\right)\end{aligned}$$

where we have used (8.8) and (8.10) in the third line. In the last line, we have used the bounds (8.14) on the second derivatives (with (8.17)) and the gradient and the modulus of continuity $\omega_{\rho_0,R}$ defined in (8.1) with (for $\varepsilon\le 1$)

$$\rho_0=\frac{\theta}{4c_\eta},\quad R=1+\max\left(\frac{9\eta^2}{\rho},\frac{\eta^3}{\rho}\right)$$

and choosing

$$|D\psi|\le\mu\le\frac{\theta}{4c_\eta},\quad |D^2\psi|\le\mu\le 1.$$

Therefore, choosing moreover μ such that

$$\omega_{\rho_0,R}(\mu)\le\theta/4$$

we get

$$\bar F^{2\eta}(p)+\theta/4\ge F\left(\varepsilon Y,q,\frac{x_\varepsilon}{\varepsilon}\right)\ge F\left(\varepsilon X,q,\frac{x_\varepsilon}{\varepsilon}\right)\ge b_1=\lambda+\psi_t(\tilde P_\varepsilon)\ge\bar\lambda+\theta+\psi_t(\tilde P_\varepsilon)\ge\bar F^{2\eta}(p)+\frac{3\theta}{4}+\psi_t(\tilde P_\varepsilon)$$

where we have used the control (8.5) on $\bar F^{2\eta}(p)$ and $\lambda=\bar\lambda+\theta$. This gives a contradiction for the choice

$$|\psi_t|\le\mu\le\theta/4.$$

Therefore we conclude that we can not have $\theta>0$, and then $\overline{u}$ is a subsolution.

Step 4: if $\underline{u}$ is not a supersolution

We assume by contradiction that $\underline{u}$ is not a supersolution of (1.2) on $\mathbb{R}^N\times(0,+\infty)$. Then there exists a test function φ and a point $P_0=(x_0,t_0)$ (with $t_0>0$) such that

$$\begin{cases}\underline{u}(P_0)=\varphi(P_0)\\ \underline{u}\ge\varphi\quad\text{on}\quad Q_r(P_0)=B_r(x_0)\times(t_0-r,t_0+r)\subset\mathbb{R}^N\times(0,+\infty)\\ \varphi_t(P_0)-\bar F(D\varphi(P_0))=-\theta<0.\end{cases}$$

Let us set

$$p=D\varphi(P_0),\quad\bar\lambda=\bar F(p),\quad\lambda=\varphi_t(P_0).$$

Up to replace φ by $-|(x,t)-P_0|^4+\varphi$, we can moreover assume that there exists $\delta_1=\delta_1(r)>0$ such that

$$\begin{cases}\underline{u}-2\delta_1\ge\varphi & \text{on}\quad\overline{Q_r(P_0)}\backslash Q_{r/2}(P_0),\\ \underline{u}>\varphi & \text{on}\quad\overline{Q_r(P_0)}\backslash\{P_0\}.\end{cases}\tag{8.18}$$

Step 4.1: $p = 0$
Then for $\varepsilon > 0$ small enough, we have also

$$u^\varepsilon - \delta_1 \geq \varphi \quad \text{on} \quad \overline{Q_r(P_0)} \backslash Q_{r/2}(P_0)$$

and for ε small enough

$$S^\varepsilon = \inf_{\overline{Q_r(P_0)}} (u^\varepsilon - \varphi) = (u^\varepsilon - \varphi)(P_\varepsilon) \quad \text{with} \quad P_\varepsilon = (x_\varepsilon, t_\varepsilon) \in Q_{r/2}(P_0)$$

and

$$P_\varepsilon \to P_0, \quad S^\varepsilon \to 0, \quad \text{as} \quad \varepsilon \to 0.$$

On the one hand $p = 0$ implies $\bar{\lambda} = 0$ and on the other hand we have

$$\varphi_t(P_\varepsilon) \geq F(\varepsilon D^2 \varphi(P_\varepsilon), D\varphi(P_\varepsilon), x_\varepsilon).$$

Passing to the limit $\varepsilon \to 0$, we get

$$0 > -\theta = \varphi_t(P_0) \geq F_*(0, 0, x_0) = 0.$$

Contradiction.
Step 4.2: $p \neq 0$
Notice that

$$\varphi(x,t) = \varphi(P_0) + p \cdot (x - x_0) + \lambda(t - t_0) + \psi(x,t) \tag{8.19}$$

with ψ and its derivatives small in $\overline{Q_r(P_0)}$, i.e.

$$|D\psi|, |D^2\psi|, |\psi_t| \leq \mu \leq 1 \quad \text{on} \quad \overline{Q_r(P_0)} \quad \text{with} \quad \mu = o_1(r). \tag{8.20}$$

We also extend by continuity (keeping the same notation) φ and ψ outside $Q_r(P_0)$ in order to keep the relation (8.19) and such that there exists a $\mu_0 > 0$ (not necessarily small) with

$$|D\psi| \leq \mu_0 \quad \text{in} \quad \mathbb{R}^N \times [0, +\infty).$$

Step 4.2.1: Regularizations and definition of the perturbed function $\bar{u}^\varepsilon$
Then for $\eta > 0$, we consider $w_{2\eta} = w_*^{2\eta}$ a subcorrector associated to the approximate Hamiltonian $F_{2\eta}$ given in assumption (B') for the gradient p and we define

$$\tilde{l}^\varepsilon(x,t) = \varphi(P_0) + p \cdot (x - x_0) + \bar{F}_{2\eta}(p)(t - t_0) + \varepsilon w_{2\eta}(x/\varepsilon)$$

where we recall that (from assumption (B'))

$$\sup w_{2\eta} - \inf w_{2\eta} \leq \bar{\kappa}_0.$$

We choose $\eta > 0$ small enough such that (using the fact that $\bar{\lambda} > 0$)

$$|\bar{F}_{2\eta}(p) - \bar{\lambda}| \leq \min(\bar{\lambda}/2, \theta/4) \tag{8.21}$$

for later use. We define

$$\tilde{u}^\varepsilon(x,t) = u^\varepsilon(x,t) + (\bar{F}_{2\eta}(p) - \lambda)(t - t_0) - \psi(x,t)$$

and

$$\bar{u}^\varepsilon(x,t) = \underline{\inf_{z\in B_{\varepsilon\eta}(x)}}\, \tilde{u}^\varepsilon(z,t) \tag{8.22}$$

which is lower semi-continuous (from Lemma 12.3). The reason for introducing the regularization given by (8.22), is that it will allow us to control the curvature of the level sets of $\bar{u}^\varepsilon$ from one side, and then to bound the gradient from below.
Notice that we have

$$\liminf_{\varepsilon\to 0}{}_*\; \bar{u}^\varepsilon(x,t) = \underline{u}(x,t) + (\bar{F}_{2\eta}(p)-\lambda)(t-t_0) - \psi(x,t). \tag{8.23}$$

We also consider the following kind of perturbed test function for some $\rho > 0$ (to fix later)

$$l^\varepsilon(x,t) = \sup_{z\in\mathbb{R}^2}\left(\tilde{l}^\varepsilon(z,t) - \frac{|x-z|^4}{4\varepsilon^3\rho}\right)$$

which is also upper semi-continuous (fact similar to the proof of Lemma 12.3). The reason for introducing this further regularization (given by $\tilde{l}^\varepsilon$) is that it will help us later to control the second derivatives from above and below, which will be important in the perturbation argument (in order to get a contradiction).

Then, using A)vii) of Lemma 12.4 with $L = |p|$, we get for ε small enough

$$\bar{u}^\varepsilon - \delta_1 \geq l^\varepsilon \quad \text{on} \quad \overline{Q_r(P_0)}\backslash Q_{r/2}(P_0) \tag{8.24}$$

and

$$S^\varepsilon = \underline{\inf_{Q_r(P_0)}}(\bar{u}^\varepsilon - l^\varepsilon) = (\bar{u}^\varepsilon - l^\varepsilon)(P_\varepsilon) \to 0 \quad \text{as} \quad \varepsilon \to 0 \tag{8.25}$$

for some point $P_\varepsilon = (x_\varepsilon, t_\varepsilon) \in Q_{r/2}(P_0)$.

In general, there is no reason for $\bar{u}^\varepsilon$ to be locally a supersolution of a modification of the PDE satisfied by u^ε (i.e. equation (1.1)), because the viscosity supersolution inequality may not be satisfied at points where the gradient of u^ε vanishes.

Step 4.2.2: Controlling the distance between points
Still using A)vii) of Lemma 12.4 with $L = |p|$, we see that there exists a point $\bar{x}_\varepsilon \in \mathbb{R}^N$ such that

$$|\bar{x}_\varepsilon - x_\varepsilon| \leq \varepsilon C_{\bar{\kappa}_0,L,\rho} \quad \text{with} \quad C_{\bar{\kappa}_0,L,\rho} \to 0 \quad \text{as} \quad \rho \to 0$$

and

$$l^\varepsilon(x_\varepsilon,t_\varepsilon) = \tilde{l}^\varepsilon(\bar{x}_\varepsilon,t_\varepsilon) - \frac{|x_\varepsilon - \bar{x}_\varepsilon|^4}{4\varepsilon^3\rho}.$$

We choose ρ small enough such that $C_{\bar{\kappa}_0,L,\rho} \leq \eta$ and then

$$|\bar{x}_\varepsilon - x_\varepsilon| \leq \varepsilon\eta. \tag{8.26}$$

Step 4.2.3: Framework to apply Ishii's Lemma
Then we have

$$\begin{aligned} S^\varepsilon &= \underline{\inf_{(x,t)\in Q_r(P_0)}}\left(\bar{u}^\varepsilon(x,t) - \sup_{y\in\mathbb{R}^N}\left(\tilde{l}^\varepsilon(y,t) - \frac{|x-y|^4}{4\varepsilon^3\rho}\right)\right) \\ &= \underline{\inf_{(x,t)\in Q_r(P_0),\ y\in\mathbb{R}^N}}\left(\bar{u}^\varepsilon(x,t) - \tilde{l}^\varepsilon(y,t) + \frac{|x-y|^4}{4\varepsilon^3\rho}\right) \\ &= \bar{u}^\varepsilon(x_\varepsilon,t_\varepsilon) - \tilde{l}^\varepsilon(\bar{x}_\varepsilon,t_\varepsilon) + \frac{|x_\varepsilon - \bar{x}_\varepsilon|^4}{4\varepsilon^3\rho} \end{aligned}$$

i.e.

$$\sup_{(y,t)\in\overline{Q_r(P_0)},\ x\in\mathbb{R}^N}\left(\tilde{l}^\varepsilon(x,t)-\bar{u}^\varepsilon(y,t)-\frac{|x-y|^4}{4\varepsilon^3\rho}\right)=\tilde{l}^\varepsilon(\bar{x}_\varepsilon,t_\varepsilon)-\bar{u}^\varepsilon(x_\varepsilon,t_\varepsilon)-\frac{|x_\varepsilon-\bar{x}_\varepsilon|^4}{4\varepsilon^3\rho}.$$

From Ishii's Lemma (see Theorem 8.3 in the Users's Guide [17], and Theorem 7 in [16]), we deduce with

$$\Phi(x,y,t)=\frac{|x-y|^4}{4\varepsilon^3\rho}$$

that for every $\gamma>0$, there exists
(8.27)

$$\left\{\begin{array}{l}(b_1,q,X)\in\overline{\mathcal{P}}^{2,+}\tilde{l}^\varepsilon(\bar{x}_\varepsilon,t_\varepsilon)\\ (b_2,q,Y)\in\overline{\mathcal{P}}^{2,-}\bar{u}^\varepsilon(x_\varepsilon,t_\varepsilon)\\ b_1-b_2=0=\Phi_t(x_\varepsilon,\bar{x}_\varepsilon,t_\varepsilon)\\ -\left(\frac{1}{\gamma}+||A||\right)\left(\begin{array}{cc}I&0\\0&I\end{array}\right)\le\left(\begin{array}{cc}X&0\\0&-Y\end{array}\right)\le A+\gamma A^2\\ \text{with}\quad q=D_x\Phi(x_\varepsilon,\bar{x}_\varepsilon,t_\varepsilon)=-D_y\Phi(x_\varepsilon,\bar{x}_\varepsilon,t_\varepsilon)=\delta(x_\varepsilon-\bar{x}_\varepsilon)\quad\text{with}\quad\delta=\frac{|x_\varepsilon-\bar{x}_\varepsilon|^2}{\varepsilon^3\rho}\\ \text{with}\quad A=D^2\Phi(x_\varepsilon,\bar{x}_\varepsilon,t_\varepsilon)=\delta\left(\begin{array}{cc}I+2\widehat{q}\otimes\widehat{q}&-I-2\widehat{q}\otimes\widehat{q}\\-I-2\widehat{q}\otimes\widehat{q}&I+2\widehat{q}\otimes\widehat{q}\end{array}\right)\in S^{2N}\quad\text{with}\quad\widehat{q}=\frac{q}{|q|}\quad(\text{if}\quad q\neq 0)\end{array}\right.$$

where $||A||=\sup_{|\xi|=1}<A\xi,\xi>$.

From A)v)-vi) of Lemma 12.4, we see that there exists a point $\tilde{x}_\varepsilon\in\mathbb{R}^N$ such that

$$(8.28)\qquad (b_2,q,Y)\in\overline{\mathcal{P}}^{2,-}\tilde{u}^\varepsilon(\tilde{x}_\varepsilon,t_\varepsilon)\quad\text{and}\quad\bar{u}^\varepsilon(x_\varepsilon,t_\varepsilon)=\tilde{u}^\varepsilon(\tilde{x}_\varepsilon,t_\varepsilon)\quad\text{with}\quad|\tilde{x}_\varepsilon-x_\varepsilon|\le\varepsilon\eta$$

and then we have at $\tilde{P}_\varepsilon=(\tilde{x}_\varepsilon,t_\varepsilon)$

$$(8.29)\qquad (b_2-(\bar{F}_{2\eta}(p)-\lambda-\psi_t(\tilde{P}_\varepsilon)),q+D\psi(\tilde{P}_\varepsilon),Y+D^2\psi(\tilde{P}_\varepsilon))\in\overline{\mathcal{P}}^{2,-}u^\varepsilon(\tilde{x}_\varepsilon,t_\varepsilon).$$

We have in particular the viscosity inequalities

$$(8.30)\qquad\left\{\begin{array}{l}b_2-(\bar{F}_{2\eta}(p)-\lambda-\psi_t(\tilde{P}_\varepsilon))\ge F_*\left(\varepsilon(Y+D^2\psi(\tilde{P}_\varepsilon)),q+D\psi(\tilde{P}_\varepsilon),\frac{\tilde{x}_\varepsilon}{\varepsilon}\right)\\ b_1\le(F_{2\eta})^*\left(\varepsilon X,q,\frac{\bar{x}_\varepsilon}{\varepsilon}\right).\end{array}\right.$$

Recall that to be able to apply Ishii's Lemma, we need to be able to bound $b_1\le C$ and $b_2\ge -C$ for general $(b_1,q_1,X)\in\overline{\mathcal{P}}^{2,+}\tilde{l}^\varepsilon(x,t)$ and $(b_2,q_2,Y)\in\overline{\mathcal{P}}^{2,-}\bar{u}^\varepsilon(y,t)$ for (x,t) close to $(\bar{x}_\varepsilon,t_\varepsilon)$, (y,t) close to $(x_\varepsilon,t_\varepsilon)$, and bounded $q_1,q_2,X,Y,\tilde{l}^\varepsilon(x,t),\bar{u}^\varepsilon(y,t)$. Indeed this is true and comes from the viscosity inequalities similar to (8.30), using in particular assumption (A4)i).

Step 4.2.4: Bound on the second derivatives and the gradient
As in Step 3.2.4, we get for $\gamma=\frac{1}{3\delta}$:

$$(8.31)\qquad -9\delta\left(\begin{array}{cc}I&0\\0&I\end{array}\right)\le\left(\begin{array}{cc}X&0\\0&-Y\end{array}\right)\le 9\delta\left(\begin{array}{cc}I&-I\\-I&I\end{array}\right).$$

Therefore

$$-9\delta I \le Y \le 9\delta I \quad \text{and} \quad |q| \le \frac{\eta^3}{\rho}. \tag{8.32}$$

where the last bound on q follows from (8.27) and (8.26).

Step 4.2.5: Gradient estimate from below

Notice that (8.27) also implies that

$$b_1 = \bar{F}_{2\eta}(p). \tag{8.33}$$

Using (8.21) and the fact that $\bar{\lambda} = \bar{F}(p) > 0$, we deduce from (8.30) that

$$0 < \bar{\lambda}/2 \le \bar{F}_{2\eta}(p) = b_1 \le (F_{2\eta})^* \left(\varepsilon X, q, \frac{\bar{x}_\varepsilon}{\varepsilon}\right) \le (F_{2\eta})^* \left(\varepsilon Y, q, \frac{\bar{x}_\varepsilon}{\varepsilon}\right) \tag{8.34}$$

where we have used $X \le Y$ which is implied by (8.27).
We deduce that

$$0 < \bar{\lambda}/2 \le (F_{2\eta})^* \left(\varepsilon Y, q, \frac{\bar{x}_\varepsilon}{\varepsilon}\right) \le c_\eta |q|$$

where we have used Lemma 12.3 (and in particular (12.8)) for the last inequality. This shows that the gradient is bounded from below:

$$\frac{\bar{\lambda}}{2c_\eta} \le |q|.$$

Step 4.2.6: Perturbation and contradiction

From (8.26), we see that

$$\delta \le \frac{\eta^2}{\varepsilon\rho}. \tag{8.35}$$

Now, we deduce from (12.23) that

$$\begin{aligned}
\bar{F}_{2\eta}(p) &\le (F_{2\eta})^* \left(\varepsilon Y, q, \frac{\bar{x}_\varepsilon}{\varepsilon}\right) \\
&= F_{2\eta} \left(\varepsilon Y, q, \frac{\bar{x}_\varepsilon}{\varepsilon}\right) \\
&\le F \left(\varepsilon Y, q, \frac{\tilde{x}_\varepsilon}{\varepsilon}\right) \\
&\le \omega_{\rho_0,R}(\mu) + F \left(\varepsilon(Y + D^2\psi(\tilde{P}_\varepsilon)), q + D\psi(\tilde{P}_\varepsilon), \frac{\tilde{x}_\varepsilon}{\varepsilon}\right)
\end{aligned}$$

where we have used (8.26) and (8.28) in the third line. In the last line, we have used the bounds (8.32) on the second derivatives (with (8.35)) and the gradient and the modulus of continuity $\omega_{\rho_0,R}$ defined in (8.1) with (for $\varepsilon \le 1$)

$$\rho_0 = \frac{\bar{\lambda}}{4c_\eta}, \quad R = 1 + \max\left(\frac{9\eta^2}{\rho}, \frac{\eta^3}{\rho}\right)$$

and choosing

$$|D\psi| \le \mu \le \frac{\bar{\lambda}}{4c_\eta}, \quad |D^2\psi| \le \mu \le 1.$$

Therefore, choosing moreover μ such that

$$\omega_{\rho_0,R}(\mu) \leq \theta/4$$

we get using (8.21):

$$\begin{aligned}
\bar{F}_{2\eta}(p) - \theta/4 &\leq F\left(\varepsilon(Y + D^2\psi(\tilde{P}_\varepsilon)), q + D\psi(\tilde{P}_\varepsilon), \tfrac{\tilde{x}_\varepsilon}{\varepsilon}\right) \\
&= F_*\left(\varepsilon(Y + D^2\psi(\tilde{P}_\varepsilon)), q + D\psi(\tilde{P}_\varepsilon), \tfrac{\tilde{x}_\varepsilon}{\varepsilon}\right) \\
&\leq b_2 - (\bar{F}_{2\eta}(p) - \lambda - \psi_t(\tilde{P}_\varepsilon)) \\
&= \lambda + \psi_t(\tilde{P}_\varepsilon) \\
&= \bar{\lambda} - \theta + \psi_t(\tilde{P}_\varepsilon) \\
&\leq \bar{F}_{2\eta}(p) - 3\theta/4 + \psi_t(\tilde{P}_\varepsilon)
\end{aligned}$$

where we have used the control (8.21) on $\bar{F}_{2\eta}(p)$ and $\lambda = \bar{\lambda} + \theta$. This gives a contradiction for the choice

$$|\psi_t| \leq \mu \leq \theta/4.$$

Therefore we conclude that we can not have $\theta > 0$, and then $\underline{u}$ is a supersolution.
This ends the proof of the Theorem 1.5.

9 Homogenization in 2D: proof of Theorem 1.4

In order to do the proof of Theorem 1.4, we will need the following result

Proposition 9.1 (Approximate hamiltonians)
We assume that F satisfies (A) and (B). For any $\eta > 0$, let us define and $(M, p, x) \in \mathcal{D}_0$, let us define

$$F^\eta(M, p, x) = \sup_{|e| \leq \eta} F(M, p, x + e), \quad F_\eta(M, p, x) = \inf_{|e| \leq \eta} F(M, p, x + e).$$

Then F^η, F_η satisfy assumptions (A), (B) and (1.9) with the same constants.
Let $\bar{F}^\eta(p)$ (resp. $\bar{F}_\eta(p)$) is the effective hamiltonian given by Theorem 1.3 associated to F^η (resp. F_η). Then we have as $\eta \to 0$:

$$\bar{F}^\eta(p) \to \bar{F}(p), \quad \bar{F}_\eta(p) \to \bar{F}(p).$$

Proof of Proposition 9.1
We do the proof for F^η (the proof for F_η being similar).
Step 1: cheking $F^\eta \in C(\mathcal{D}_0)$
Notice that $F^\eta \in C(\mathcal{D}_0)$, because F is locally uniformly continuous on $\mathcal{D}_0$ and $\mathcal{D}_0$ is periodic in the last variable on which we take the suppremum defining F^η.
Step 2: The approximate hamiltonian F^η
Because F satisfies (A),(B) and (1.9), it is easy to check that it is also the case for F^η (with the same constants and the same function σ_K). Let us give the details for checking (A4)ii) (skipping the verification of the other assumptions which are easier). Let us consider $x, y \in \mathbb{R}^N$, $\alpha \geq 0$, and $X, Y \in S^N$ as in (A4)ii). We set $p = \alpha(x - y)$.

Case $p \neq 0$
We write with $|e_x|, |e_y| \leq \eta$:

$$F^\eta(X,p,x) = F(X,p,x+e_x), \quad F^\eta(Y,p,y) = F(Y,p,y+e_y)$$

and then

$$\begin{aligned}
&(F^\eta)^*(X,p,x) - (F^\eta)_*(Y,p,y)\\
&= \{F(X,p,x+e_x) - F(Y,p,y+e_x)\} + \{F(Y,p,y+e_x) - F(Y,p,y+e_y)\}\\
&\leq F(X,p,x+e_x) - F(Y,p,y+e_x)\\
&\leq \sigma_K \{|x-y|(1+\alpha|x-y|)\}.
\end{aligned}$$

Case $p = 0$
If $x = y$, this is assumed that $\alpha = 0$. Therefore $p = 0$ always implies that $\alpha = 0$ which implies $X = Y = 0$. We then have to check that

$$(F^\eta)^*(0,0,x) - (F^\eta)_*(0,0,y) \leq \sigma_K(|x-y|). \tag{9.1}$$

But by definition, we have

$$(F^\eta)^*(0,0,x) = \limsup_{(M,q,z)\to(0,0,x)} F^\eta(M,q,z) = \limsup_{(M,q,z)\to(0,0,x)} F(M,q,z+e_z) \leq F^*(0,0,x+e) = 0$$

where e is a limit (up to extract a subsequence) of some e_z with $|e_z| \leq \eta$. Similarly, we show that

$$(F^\eta)^*(0,0,y) \geq 0.$$

This implies (9.1).
Step 3: the effective hamiltonian $\bar{F}^\eta$
From Step 1 and Theorem 1.3, there exists a $\mathbb{Z}^2$-periodic corrector w^η solution of

$$\bar{F}^\eta(p) = F^\eta(D^2 w^\eta, p + Dw^\eta, y) \quad \text{on} \quad \mathbb{R}^2 \tag{9.2}$$

such that

$$\sup w^\eta - \inf w^\eta \leq \kappa_0 |p| \quad \text{with} \quad \kappa_0 = 100\ R\frac{C_0}{c_0}.$$

It is easy to check (as usual) that

$$\bar{F}^\eta(p) \to \bar{F}(p) \quad \text{as} \quad \eta \to 0. \tag{9.3}$$

This ends the proof of the proposition.

Remark 9.2 *We will not use that fact, but we can check that F^η and F_η (given in Proposition 9.1) still satisfy (8.1) with the same modulus of continuity $\omega_{\rho,R}$.*

Proof of Theorem 1.4
We simply apply Proposition 9.1 which shows that assumption (B') is satisfied. Then Theorem 1.4 follows from Theorem 1.5. This ends the proof of the theorem.

10 The cell problem in 2D with sign changing normal velocity

A natural question is: do we still have existence of correctors for geometric motions with normal velocity:

$$V = \kappa + c(x) \quad \text{when } c \text{ changes sign.}$$

More generally, in this section, we show the existence of a corrector when assumption (B) is replaced by:

Assumption (B”): **Barrier from below**:
There exists a (upper semi-continuous) subsolution U of (5.1) for F satisfying (A) for $N = 2$, that we can write

$$U(x,t) = \chi_{K_t}(x)$$

for a familly $(K_t)_{t\geq 0}$ of compact sets $K_t \subset \mathbb{R}^2$ satisfying for some $T_0 > 0$

$$K_t = K_{T_0} \quad \text{for} \quad t \geq T_0$$

and

$$K_{T_0} \supset K_0 + e \quad \text{for all} \quad e \in \{0, e_1, -e_1, e_2, -e_2\}.$$

We also assume that there exists $R_0 \geq 8$ such that

$$\begin{cases} K_0 \supset [0,1]^2, \\ \\ \overline{B_{R_0}} \supset K_t \quad \text{for all} \quad t \geq 0. \end{cases}$$

Remark 10.1 *Notice that assumptions (A1) and (A3) imply that there exist constants $C_0, c_1 > 0$ such that for all $(p,y) \in \mathbb{S}^{N-1} \times \mathbb{R}^N$, we have*

$$C_0 \geq F(0,p,y) \geq -c_1. \tag{10.1}$$

Remark 10.2 *Notice that assumption (B”) is still satisfied for equation (1.3) and certain velocities $c(y)$ which can change sign (see Subsection 11.1).*

Then we have the following result:

Theorem 10.3 (The cell problem in 2D under assumption (B”))
Assume that $N = 2$ and that (A) and (B”) hold. Then for any $p \in \mathbb{R}^N$, there exists a unique real number $\bar{F}(p)$ (with $\bar{F}(p) > 0$ if $p \neq 0$ and $\bar{F}(0) = 0$) such that there exists a bounded $\mathbb{Z}^N$-periodic function $v : \mathbb{R}^N \to \mathbb{R}$ solution of

$$\bar{F}(p) = F(D^2v, p + Dv, y) \quad \text{on} \quad \mathbb{R}^N. \tag{10.2}$$

We can choose v such that

$$\sup v - \inf v \leq \kappa_1 |p| \quad \text{with} \quad \kappa_1 = 400\, C_0 T_0 \,(R_0 + c_1 T_0). \tag{10.3}$$

Moreover the map $p \mapsto \bar{F}(p)$ is continuous and positively 1-homogeneous, i.e. for any $p \in \mathbb{R}^N$

$$\bar{F}(\lambda p) = \lambda \bar{F}(p) \quad \text{for any} \quad \lambda \geq 0.$$

In order to prove Theorem 10.3, we follow the plan of the proof proposed in Sections 5, 6, 7.

10.1 New preliminary results

For $N = 2$, let us consider u solution of

$$u_t = F(D^2u, Du, y) \quad \text{on} \quad \mathbb{R}^N \times (0, +\infty) \tag{10.4}$$

with initial data

$$u(0, x) = u_0(x) = x \cdot \nu \quad \text{for} \quad x \in \mathbb{R}^N \tag{10.5}$$

with $\nu \in \mathbb{S}^{N-1}$.

Lemma 10.4 (Bound from below)
Assume (A) and let u be the unique solution of (10.4)-(10.5) given by Proposition 5.1. Under assumption (B"), we have moreover

$$u(x,t) \geq u_0(x) - \left(R_0 + \frac{5\sqrt{2}}{2} + c_1 T_0\right) + t\frac{\sqrt{2}}{2T_0} \quad \text{for all} \quad x \in \mathbb{R}^2, \quad t \geq 2T_0. \tag{10.6}$$

We also recall that $u(x,t) - x \cdot \nu$ is $\mathbb{Z}^N$-periodic in x, that

$$-c_1 \leq u_t \leq C_0 \quad \text{on} \quad \mathbb{R}^N \times [0, +\infty) \tag{10.7}$$

with $C_0, c_1 > 0$ given in (10.1), and that for any $0 < T < +\infty$, there exists a modulus of continuity m_T such that

$$|u(x,t) - u(y,t)| \leq m_T(|x-y|) \quad \text{for all} \quad x, y \in \mathbb{R}^N, \quad t \in [0, T].$$

Proof of Lemma 10.4
Step 0: Basic properties
Notice that (10.7) follows from (10.1), and that all the other properties of u (except the bound from below (10.6)) follow from Proposition 5.1.
Step 1: $\chi_{\{u>a\}}$ is a solution
We proceed as in Step 3 of the proof of Proposition 5.6 and define

$$u^\varepsilon(x,t) = \beta_\varepsilon(u(x,t) - a) \quad \text{with} \quad \beta_\varepsilon(b) = \frac{1}{2}\left\{1 + \tanh\left(\frac{b}{\varepsilon}\right)\right\}.$$

Here u^ε is a viscosity solution of (10.4) and we have

$$u^\varepsilon \to u^0 \quad \text{with} \quad u^0(x,t) = \begin{cases} 1 & \text{if} \quad u(x,t) > a, \\ \frac{1}{2} & \text{if} \quad u(x,t) = a, \\ 0 & \text{if} \quad u(x,t) < a \end{cases}$$

where u^0 is a still a viscosity solution of (10.4), by stability of viscosity solutions. Let us now define

$$v^\varepsilon = \beta_\varepsilon\left(u^0 - \frac{3}{4}\right)$$

which is still a viscosity solution of (10.4) and converges

$$v^\varepsilon \to v^0 = \chi_{\{u>a\}}.$$

This shows that $\chi_{\{u>a\}}$ is a viscosity solution of (10.4).

Step 2: estimate from below on the growth of the burnt region

Moreover, by assumption (B”), if the level set $\{u(\cdot,0)>a\}$ contains the ball $\overline{B_{R_0}} \supset K_0 \supset [0,1]^2$, then from the comparison principle applied to $\chi_{\{u>a\}}$ and U, we deduce that

$$\{u(\cdot,T_0)>a\} \quad \supset \quad \bigcup_{e\in S_1} \left(e+[0,1]^2\right) \quad \text{with} \quad S_1=\{0,e_1,-e_1,e_2,-e_2\}\,.$$

Defining the sequence:

$$\text{(10.8)} \qquad \left\{\begin{array}{l} S_0=\{0\}\,,\\ S_{n+1}=S_n+S_1 \quad \text{for} \quad n\ge 0 \end{array}\right.$$

it is straightforward to check that

$$S_n=\left\{(x_1,x_2)\in\mathbb{Z}^2, \quad |x_1|+|x_2|\le n\right\}$$

and then we get for any $n\in\mathbb{N}$:

(10.9)

$$\{u(\cdot,nT_0)>a\} \quad \supset \quad \bigcup_{e\in S_n}\left(e+[0,1]^2\right) \quad \supset \quad \overline{B_{n\frac{\sqrt{2}}{2}}} \qquad \text{if} \qquad \{u(\cdot,0)>a\} \quad \supset \quad \overline{B_{R_0}}.$$

Step 3: consequence at time nT_0

More generally, let us define the set

$$A_a=\left\{k\in\mathbb{Z}^2, \quad k\cdot\nu>a+R_0\right\}$$

and

$$\widehat{A}_a=\bigcup_{k\in A_a}\left(k+[0,1]^2\right)$$

which has the property that

$$\left\{x\cdot\nu>a+R_0+\sqrt{2}\right\} \quad \subset \quad \widehat{A}_a \quad \subset \quad \bigcup_{k\in A_a}\left(k+\overline{B_{R_0}}\right) \quad \subset \quad \{x\cdot\nu>a\}=\{u_0(x)>a\}\,.$$

This implies (using the natural generalization of (10.9)) that (for $n \geq 2$)

$$\begin{aligned}
\{u(\cdot, nT_0) > a\} &\supset \bigcup_{k \in A_a + S_n} \left(k + [0,1]^2\right) \\
&= A_a + \bigcup_{e \in S_n} \left(e + [0,1]^2\right) \\
&\supset A_a + \overline{B_{n\frac{\sqrt{2}}{2}}} \\
&\supset A_a + [0,1]^2 + \overline{B_{(n-2)\frac{\sqrt{2}}{2}}} \\
&= \widehat{A}_a + \overline{B_{(n-2)\frac{\sqrt{2}}{2}}} \\
&\supset \left\{x \cdot \nu > a + R_0 + \sqrt{2}\right\} + \overline{B_{(n-2)\frac{\sqrt{2}}{2}}} \\
&\supset \left\{x \cdot \nu > a + R_0 + (4-n)\frac{\sqrt{2}}{2}\right\}.
\end{aligned}$$

Step 4: conclusion
Now for any time $t \in [nT_0, (n+1)T_0]$, we deduce from estimate (10.1) that

$$\{u(\cdot, t) > a\} \quad \supset \quad \left\{x \cdot \nu > a + R_0 + (4-n)\frac{\sqrt{2}}{2} + c_1(t - nT_0)\right\}.$$

Let us now consider a point $x \in \mathbb{R}^2$ such that $u_0(x) = b$. Then for any $a \in \mathbb{R}$ such that

$$u_0(x) = b > a + R_0 + (4-n)\frac{\sqrt{2}}{2} + c_1(t - nT_0)$$

we deduce that

$$u(x,t) > a.$$

This implies for $t \in [nT_0, (n+1)T_0]$ and $n \geq 2$

$$u(x,t) \geq b - R_0 + (n-4)\frac{\sqrt{2}}{2} - c_1(t - nT_0)$$

and then

$$u(x,t) \geq b - (R_0 + 2\sqrt{2} + c_1 T_0) + (nT_0)\frac{\sqrt{2}}{2T_0}$$

which implies

$$u(x,t) \geq b - \left(R_0 + \frac{5\sqrt{2}}{2} + c_1 T_0\right) + t\frac{\sqrt{2}}{2T_0} \quad \text{for} \quad t \geq 2T_0.$$

and then more generally this shows (10.6) and ends the proof of the lemma.

Lemma 10.5 (An increasing subsolution)
Assume (A) and (B"). Let us fix

$$t_0^* > T_0\left(R_0\sqrt{2}+5+c_1T_0\sqrt{2}\right) \tag{10.10}$$

and define

$$\overline{u}(x,t) = \delta_1 t + \sup_{s\in[0,t]} (u(x,s)-\delta_1 s) \quad \textit{with} \quad \delta_1 := \frac{\sqrt{2}}{2T_0} - \frac{R_0 + \frac{5\sqrt{2}}{2} + c_1T_0}{t_0^*} > 0$$

where u is the solution given in Lemma 10.4.
Then $\overline{u}(x,t)$ is a subsolution of (10.4) for $t\in(t_0^*,+\infty)$.
Moreover $\overline{u}(x,t)-x\cdot\nu$ is $\mathbb{Z}^N$-periodic in x, and with the notation of Lemma 10.4, we have

$$\delta_1 \le \overline{u}_t \le C_0 \quad \textit{on} \quad \mathbb{R}^N\times[0,+\infty) \tag{10.11}$$

and for all $0<T<+\infty$

$$|\overline{u}(x,t)-\overline{u}(y,t)| \le m_T(|x-y|) \quad \textit{for all} \quad x,y\in\mathbb{R}^N, \quad t\in[0,T]. \tag{10.12}$$

Proof of Lemma 10.5
Step 1: Subsolution
For $t>t_0^*$, we can rewrite (10.6) as

$$u(x,t) \ge u_0(x) + (t-t_0^*)\frac{\sqrt{2}}{2T_0} + \delta_1 t_0^*.$$

Therefore we get

$$\begin{aligned}
\overline{u}(x,t)-\delta_1 t &= \sup_{s\in[0,t]}(u(x,s)-\delta_1 s)\\
&\ge u(x,t)-\delta_1 t\\
&\ge u_0(x) + (t-t_0^*)\frac{\sqrt{2}}{2T_0} + \delta_1 t_0^* - \delta_1 t\\
&\ge u_0(x) + (t-t_0^*)\left(\frac{\sqrt{2}}{2T_0}-\delta_1\right)\\
&> u_0(x).
\end{aligned}$$

This shows that

$$\overline{v}(x,t) := \overline{u}(x,t)-\delta_1 t = \sup_{s\in[0,t]}(u(x,s)-\delta_1 s) = (u(x,s)-\delta_1 s)_{|s=s_t} \quad \text{with} \quad s_t\in(0,t].$$

Recall that $v(x,s)=u(x,s)-\delta_1 t$ solves

$$\delta_1 + v_t = F(D^2v, Dv, x) \quad \text{on} \quad \mathbb{R}^N\times(0,+\infty). \tag{10.13}$$

Because $\overline{v}$ appears to be a suppremum of solutions, it is a subsolution. The argument is classical and is now repeated for convenience for the reader.

Assume that the there exists a test function φ satisfing

$$\overline{v} \le \varphi \quad \text{on} \quad B_r(P_0) \quad \text{with equality at} \quad P_0 = (x_0, t_0)$$

with $0 < r < s_{t_0}$, then (because $v \le \overline{v}$)

$$v \le \overline{\varphi} \quad \text{on} \quad B_r(\bar{P}_0) \quad \text{with equality at} \quad \bar{P}_0 = (x_0, s_{t_0})$$

where

$$\overline{\varphi}(x, s) = \varphi(x, t_0 + s - s_{t_0}).$$

Because $\overline{\varphi}$ is a test function for v at $\bar{P}_0$, this implies that $\overline{\varphi}$ satisfies the subsolution viscosity inequality at $\bar{P}_0$, and then φ satisfies the subsolution viscosity inequality P_0, which shows that $\overline{v}$ is a subsolution of (10.13) and then $\overline{u}$ is a subsolution of (10.4).

Step 2: Monotonicity
By construction, we have

$$\overline{v}_t \ge 0$$

which implies the lower bound in (10.11).
Step 3: Other properties of $\overline{u}$
By construction (as a suppremum of functions), we deduce the upper bound in (10.11) with the same constant C_0 as in (10.7). The suppremum in time, of $\mathbb{Z}^N$-periodic in space solutions is also $\mathbb{Z}^N$-periodic in space. Finally, we have (with obvious notation)

$$\overline{u}(x, t) = u(x, s_{(x,t)}), \quad \overline{u}(y, t) = u(y, s_{(y,t)})$$

and then for $t \in [0, T]$

$$\overline{u}(x, t) - \overline{u}(y, t) = \overline{u}(x, t) - u(y, s_{(y,t)}) \ge u(x, s_{(y,t)}) - u(y, s_{(y,t)}) \ge -m_T(|x - y|).$$

Similarly, we get the symmetric inequality (exchanging x and y) which implies (10.12).
This ends the proof of the lemma.

Proposition 10.6 (A nice increasing solution)
Assume (A) and (B"). Let us consider the initial condition

$$\tilde{u}_0(x) = \overline{u}(x, t_0^*)$$

for $\overline{u}$ and t_0^* (> 0) given in Lemma 10.5. We have

$$\delta_1 t_0^* \le \tilde{u}_0(x) - u_0(x) \le C_0 t_0^* \quad \text{with} \quad u_0(x) = \nu \cdot x. \tag{10.14}$$

Then there exists a unique solution $\tilde{u}$ of (10.4) on $\mathbb{R}^N \times (0, +\infty)$ with initial data $\tilde{u}_0$.

Moreover, $\tilde{u}(x, t) - x \cdot \nu$ is $\mathbb{Z}^N$-periodic in x, and with the notation of Lemma 10.5, we have

$$0 < \delta_1 \le \tilde{u}_t \quad \text{on} \quad \mathbb{R}^N \times [0, +\infty) \tag{10.15}$$

and for all $0 < T < +\infty$, there exists a modulus of continuity $\tilde{m}_T$ such that

$$|\tilde{u}(x, t) - \tilde{u}(y, s)| \le \tilde{m}_T(|x - y| + |t - s|) \quad \text{for all} \quad x, y \in \mathbb{R}^N, \quad t, s \in [0, T]. \tag{10.16}$$

Proof of Proposition 10.6
The proof is similar to the proof of Proposition 5.1, using the subsolution $\overline{u}$ (given in Lemma 10.5) as a lower barrier. Notice that Corollary 12.2 provides a supersolution u^+ with the same initial data $\tilde{u}_0$. Then the comparison principle implies that

$$\overline{u}(x, t + t_0) \leq u^+(x, t) \quad \text{for all} \quad x \in \mathbb{R}^N, \quad t \geq 0.$$

Therefore Perron's method implies the existence of a solution, which is then continuous and $\mathbb{Z}^N$-periodic, and then uniformly continuous on $\mathbb{R}^N \times [0, T]$ for every $T > 0$. This ends the proof of the proposition.

10.2 Revisiting Section 5

The function $\tilde{u}$ given in Proposition 10.6, satisfies all the properties of Proposition 5.1, except the fact that we have no bound from above on $\tilde{u}_t$, and the initial data $\tilde{u}_0$ is not linear. We now have to revisit Section 5, dealing with $\tilde{u}$ as for u, but without the bound from above on u_t replaced by a modulus of continuity in time, and the linear intial data u_0 replaced by estimate (10.14) and assumption (B) replaced by (B"). We also have to replace the modulus of continuity in space m_t by $\tilde{m}_t$ and δ by δ_1.

We notice that Proposition 5.2 (with (B) replaced by (B")) is still true, because the proof also works well with a general modulus of continuity in time, instead of a Lipschitz in time estimate.

The Birkhoff property (Proposition 5.5) has also to be adapted as follows

Proposition 10.7 (Birkhoff property)
Assume (A) and (B"). Using the notation of Proposition 5.2 for the function $u = \tilde{u}$ given in Proposition 10.6, we set

$$E_t = \left\{ x \in \mathbb{R}^N, \quad \tilde{u}(x, t) \geq 0 \right\}.$$

Let us define the set

$$A = \left\{ k \in \mathbb{Z}^N, \quad \nu \cdot k \geq (C_0 - \delta_1) t_0^* \right\}.$$

If $k \in A$, then for all $t \geq 0$

$$k + E_t \subset E_t. \tag{10.17}$$

Proof of Proposition 10.7
If $k \in A$, then we have

$$\tilde{u}_0(x + k) \geq \nu \cdot (x + k) + \delta_1 t_0^* \geq \nu \cdot x + C_0 t_0^* \geq \tilde{u}_0(x).$$

We deduce from the comparison principle that

$$\tilde{u}(x + k, t) \geq \tilde{u}(x, t)$$

which implies (10.17). This ends the proof of the proposition.

Then, up to the previous mentionned changes, Propositions 5.6, 5.7, 5.9 and Corollary 5.8 still hold.

We now present an analogue of Proposition 5.10.

Proposition 10.8 (The self-propagating barrier)
Assume (A) and (B"). Let us consider some $\xi \in \mathbb{S}^{N-1}$ *and* $z_0 \in \mathbb{R}^N$ *and an integer* $n_0 \geq 3$. *Then there exists a familly* $G_t^\xi \subset \mathbb{R}^2$ *for* $t \geq 0$ *of compact sets such that*

$$(\chi_{G^\xi})(\cdot,t) = \chi_{G_t^\xi}$$

is a subsolution of (10.4) on $\mathbb{R}^N \times (0,+\infty)$, *satisfying for all* $t \geq 0$*:*
(10.18)

$$\bigcup_{s\in[0,s_-]} \overline{B_{(n_0-2)\frac{\sqrt{2}}{2}}(s\xi)} \quad \subset \quad G_t^\xi \quad \subset \quad \bigcup_{0\leq \bar{t}\leq s_+} \overline{B_{\bar{R}_0}(\bar{t}\xi)} \quad \textit{with} \quad \left\{\begin{array}{l} s_- = \dfrac{t}{T_0\sqrt{2}}, \\ \\ s_+ = \sqrt{2}\left(\dfrac{t}{T_0}+1\right), \\ \\ \bar{R}_0 = R_0 + n_0 + 1 + \sqrt{2}. \end{array}\right.$$

Proof of Proposition 10.8
Step 1: definition of a sequence of points on the grid
Up to simple changes of coordinates (by rotation and reflection), we can reduce the analysis to the case $\xi = (\xi_1,\xi_2) \in (0,1]\times[0,1)$ (the other cases being easily deduced from that case). For any $s \geq 0$, we define

$$n(s) = (n_1(s), n_2(s)) \quad \text{with} \quad n_1(s) = \lfloor s\xi_1 \rfloor, \quad n_2(s) = \lfloor s\xi_2 \rfloor$$

such that we always have

$$s\xi \in n(s) + [0,1]^2.$$

We also define for $(x_1,x_2) \in \mathbb{R}^2$:

$$|(x_1,x_2)|_1 = |x_1| + |x_2|.$$

We define a sequence of times $(s_l)_{l\in\mathbb{N}}$

$$\left\{\begin{array}{l} s_0 = 0 \\ s_{l+1} = \inf\{s > s_l, \quad n(s) \neq n(s_l)\}, \quad l \geq 0. \end{array}\right.$$

From this sequence, we define a new sequence of times $t_k = kT_0$ and of points $(P_k)_{k\in\mathbb{N}}$ as follows (see also Figure 8)

$$\left\{\begin{array}{l} P_0 = (0,0), \\ \text{if} \quad |n(s_l)|_1 = k \geq 1 \quad \text{and} \quad |n(s_{l-1})|_1 = k-1, \quad \text{then} \quad P_k = n(s_l), \\ \text{if} \quad |n(s_l)|_1 = k \geq 1 \quad \text{and} \quad |n(s_{l-1})|_1 = k-2, \quad \text{then} \quad \left\{\begin{array}{l} P_k = n(s_l), \\ P_{k-1} = n(s_l) - (1,0). \end{array}\right. \end{array}\right.$$

Then we have
(10.19)

$$\left\{\begin{array}{l} |P_{k+1} - P_k|_1 = 1, \quad P_k \in \mathbb{Z}^2, \quad \text{for all} \quad k \geq 0, \\ \\ s\xi \in P_k + [0,1]^2 \quad \text{for} \quad k = |n(s)|_1, \quad \text{for all} \quad s \geq 0, \\ \\ \text{for} \quad s > 0, \quad s\xi = P_k \quad \text{if and only if} \quad |n(s)|_1 = k \quad \text{and} \quad |n(s)|_1 - |n(s+0^-)|_1 = 2 \end{array}\right.$$

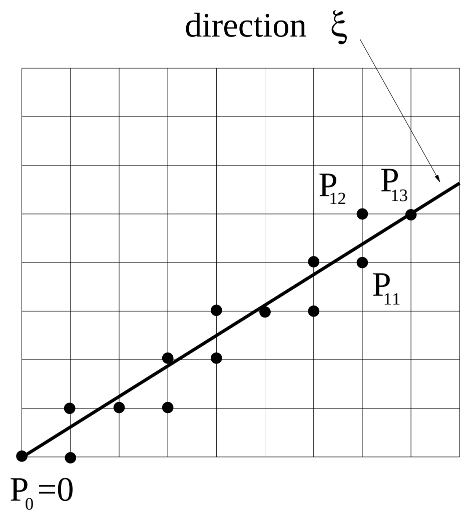

Figure 8: The sequence of points P_k

and

$$(10.20)\qquad \begin{cases} P_k \cdot \xi \le s \quad \text{for all } s \text{ such that} \quad |n(s)|_1 \ge k, \\ \\ \mathrm{dist}(P_k, \xi\mathbb{R}^+) \le \sqrt{2} \quad \text{for all} \quad k \ge 0. \end{cases}$$

Step 2: Definition of $\widehat{K}_t$
We define for $n_0 \ge 1$ and $t \ge 0$:

$$\widehat{K}_t = \left(\bigcup_{e \in S_{n_0}} (e + K_{T_0}) \right) \cup \left(\bigcup_{e \in S_{n_0+1}} (e + K_t) \right) \quad \text{with the sequence } (S_n)_n \text{ defined in (10.8)}$$

which satisfies

$$\widehat{K}_t = \widehat{K}_{T_0} \quad \text{for} \quad t \ge T_0$$

and

$$(10.21)\qquad \widehat{K}_0 \subset \widehat{K}_t \subset \overline{B_{R_0+n_0+1}} \quad \text{for all} \quad t \ge 0$$

and $\chi_{\widehat{K}}(\cdot, t) := \chi_{\widehat{K}_t}$ is a subsolution, as a suppremum of subsolutions.
We moreover have

$$(10.22)\qquad \overline{B_{\frac{n_0\sqrt{2}}{2}}} \subset \widehat{K}_0.$$

Step 3: Definition of G_t^ξ and subsolution
We then define the following familly of compact sets for $t \ge 0$

$$G_t^\xi = (P_k + \widehat{K}_{t-k}) \cup \left(\bigcup_{j=0}^{k-1} (P_j + \widehat{K}_{T_0}) \right) \quad \text{for} \quad k = \lfloor \frac{t}{T_0} \rfloor.$$

Notice that the first line of (10.19) and (B") imply

$$P_k + \widehat{K}_0 \subset P_{k-1} + \widehat{K}_{T_0} \quad \text{for} \quad k \ge 1.$$

Therefore

$$\chi_{G^\xi}(\cdot,t) = \chi_{G^\xi_t}$$

appears to be a suppremum of subsolutions and is then a subsolution.
Step 4: Estimate on G^ξ_t
From the second line of (10.19), we deduce that

$$s\xi \in \overline{B_{\sqrt{2}}(P_{k_s})} \quad \text{for} \quad k_s = |n(s)|_1.$$

Using moreover (10.22) this implies

$$\overline{B_{\frac{(n_0-2)\sqrt{2}}{2}}(s\xi)} \subset \overline{B_{\frac{n_0\sqrt{2}}{2}}(P_{k_s})} \subset P_{k_s} + \widehat{K}_0 \quad \text{for} \quad k_s = |n(s)|_1. \tag{10.23}$$

This implies that for all $t \geq 0$:

$$\overline{B_{\frac{(n_0-2)\sqrt{2}}{2}}(s\xi)} \subset G^\xi_t \quad \text{for all} \quad s \geq 0 \quad \text{such that} \quad |n(s)|_1 \leq k = \lfloor \frac{t}{T_0} \rfloor. \tag{10.24}$$

Notice that we have the following estimate

$$s\frac{\sqrt{2}}{2} - 1 \leq |n(s)|_1 \leq \lfloor (\xi_1+\xi_2)s \rfloor \leq \lfloor s\sqrt{2} \rfloor. \tag{10.25}$$

More generally, this implies (10.18), i.e. for all $t \geq 0$

$$\bigcup_{s\in[0,s_-]} \overline{B_{\frac{(n_0-2)\sqrt{2}}{2}}(s\xi)} \quad \subset \quad G^\xi_t \quad \subset \quad \bigcup_{0\leq \bar{t}\leq s_+} \overline{B_{\bar{R}_0}(\bar{t}\xi)} \quad \text{with} \quad \left\{ \begin{array}{l} s_- = \dfrac{t}{T_0\sqrt{2}}, \\[2ex] s_+ = \sqrt{2}\left(\dfrac{t}{T_0}+1\right), \\[2ex] \bar{R}_0 = R_0 + n_0 + 1 + \sqrt{2} \end{array} \right.$$

where the first inclusion follows from (10.24)-(10.25), and the last inclusion follows from (B"), (10.20), (10.21) and (10.25).
This ends the proof of the proposition.

10.3 Revisiting Section 6

Proposition 6.1 has to be adapted as follows:

Proposition 10.9 (Black cubes)
Assume (A) and (B"). Let us consider any time $t_0 \geq 0$ and

$$R \geq R_0 + 6 + \sqrt{2}. \tag{10.26}$$

If $x_0 \in Int\ E_{t_0}$, then

$$x_0 + \left\{ x \in \mathbb{R}^2, \quad \nu\cdot x \geq \bar{R} \right\} \subset E_{t_0+\tau}$$

with $\tau = 5\sqrt{2}\ RT_0$ and $\bar{R} = \sqrt{2}/2 + 2R + (C_0-\delta_1)t_0^*$.

Proof of Proposition 10.9
We follow the proof of Proposition 6.1. We choose $R \geq \bar{R}_0$ with $\bar{R}_0$ given in (10.18).
Step 1: Choice of a ball
Unchanged.
Step 2: Using the self-propagating barrier
We replace z_0 by a point $z_0^\pm \in \overline{B_{\sqrt{2}}(z_0)}$ such that

$$z_0^\pm \mp 5R\xi \in \mathbb{Z}^2$$

and then consider

$$z_0^\pm \mp 5R\xi + G_{t-t_0}^{\pm\xi}$$

which shows that for

$$\tau = 5\sqrt{2}RT_0$$

we have for $n_0 \geq 5$:

$$\overline{B_{\frac{\sqrt{2}}{2}}(z_0)} \quad \subset \overline{B_{(n_0-4)\frac{\sqrt{2}}{2}}(z_0)} \quad \subset \quad \bigcup_{\pm} \left\{ \left(z_0^\pm \mp 5R\xi + G_\tau^{\pm\xi}\right) \cap \overline{\omega_\pm} \right\} \quad \subset \quad E_{t_0+\tau}.$$

Let us choose

$$n_0 = 5.$$

Then using definition (10.18) of $\bar{R}_0$, this gives (10.26).
Step 3: Using Birkhoff property
Using the Birkhoff property (Proposition 10.7), let us recall that

$$A = \left\{ k \in \mathbb{Z}^N, \quad \nu \cdot k \geq (C_0 - \delta_1)t_0^* \right\}.$$

Then we have for every $k \in A$:

$$\overline{B_{\frac{\sqrt{2}}{2}}(z_0 + k)} \subset k + E_{t_0+\tau} \subset E_{t_0+\tau}.$$

Notice that

$$\bigcup_{k\in A} \overline{B_{\frac{\sqrt{2}}{2}}(k)} \quad \supset \quad \left\{ x \in \mathbb{R}^2, \quad \nu \cdot x \geq \frac{\sqrt{2}}{2} + (C_0 - \delta_1)t_0^* \right\}$$

which implies that

$$x_0 + \left\{ x \in \mathbb{R}^2, \quad \nu \cdot x \geq 2R + \frac{\sqrt{2}}{2} + (C_0 - \delta_1)t_0^* \right\} \subset E_{t_0+\tau}.$$

This ends the proof of the proposition.

The statement and the proof of Corollary 6.3 is unchanged, except for the values of τ and $\bar{R}$ which are the ones given in Proposition 10.9.

Before to continue, we will need the following result

Lemma 10.10 (Bound from below on T_0)
Under assumption (A) and (B"), we have

$$C_0 T_0 \geq 1. \tag{10.27}$$

Proof of Lemma 10.10
Let us consider the supersolution barrier of the form

$$\tilde{K}_t^a := \left\{x \in \mathbb{R}^N, \quad x \cdot e_1 \geq a - C_0 t\right\}$$

for some constant $a \in \mathbb{R}$ that we choose such that

$$K_0 \subset \tilde{K}_0^a \qquad \text{and} \quad K_0 \not\subset \tilde{K}_0^{a+\varepsilon} \quad \text{for any } \varepsilon > 0.$$

In particular, there exists a point $x_0 = (a, a') \in K_0$ for some $a' \in \mathbb{R}$. From the comparison principle, we get that for any $\varepsilon > 0$

$$K_t \subset \tilde{K}_t^{a+\varepsilon}$$

which implies

$$K_t \subset \tilde{K}_t^a.$$

We also have

$$(a+1, a') = e_1 + x_0 \in e_1 + K_0 \subset K_{T_0} \subset \tilde{K}_{T_0}^a = \left\{x \in \mathbb{R}^N, \quad x \cdot e_1 \geq a - C_0 T_0\right\}$$

which implies (10.27). This ends the proof of the lemma.

Proposition 6.4 and its proof have to be adapted as follows:

Proposition 10.11 (Uniform flatness property of E_t)
Let us assume (A) and (B"). Then there exists

$$L := 18\sqrt{2}\ C_0 T_0 \left(R_0 + c_1 T_0\right) \tag{10.28}$$

such that for any $t \geq 0$, there exists $\underline{c}_t \in \mathbb{R}$ such that

$$\left\{x \in \mathbb{R}^2, \quad x \cdot \nu \geq \underline{c}_t + L\right\} \subset E_t \subset \left\{x \in \mathbb{R}^2, \quad x \cdot \nu \geq \underline{c}_t\right\}. \tag{10.29}$$

Proof of Proposition 10.11
Step 1: definition of $\bar{c}_{t_0}$
First, for any $t_0 \geq 0$, let us define $\bar{c}_{t_0}$ as the biggest constant such that:

$$E_{t_0} \subset \left\{x \cdot \nu \geq \bar{c}_{t_0}\right\}. \tag{10.30}$$

We recall (10.14), namely

$$\delta_1 t_0^* \leq \tilde{u}_0(x) - \nu \cdot x \leq C_0 t_0^*.$$

Using (10.15) and (10.1), we see that the comparison principle implies for all $t \geq 0$

$$\delta_1 t_0^* + \delta_1 t \leq \tilde{u}(x,t) - \nu \cdot x \leq C_0 t_0^* + C_0 t.$$

This shows that $\bar{c}_{t_0}$ is well defined. Moreover $\{x \cdot \nu \geq 0\} \subset E_{t_0}$ and then $\bar{c}_{t_0} \leq 0$.
Step 2: consequences
As in the proof of Proposition 6.4, the same arguments imply (10.29) for $t \geq \tau$ with

$$L = \bar{R} + C_0 \tau. \tag{10.31}$$

For $t \in [0, \tau]$, we have

$$\{x \cdot \nu \geq 0\} \subset E_t \subset E_\tau \subset \{x \cdot \nu \geq -C_0(t_0^* + \tau)\}$$

which implies (10.29) with L given by

$$L = C_0(t_0^* + \tau).$$

Step 3: conclusion
We have to choose

$$L \geq \max\left(\bar{R} + C_0\tau, C_0(t_0^* + \tau)\right)$$

and we estimate

$$\max\left(\bar{R} + C_0\tau, C_0(t_0^* + \tau)\right) \leq C_0 t_0^* + C_0\tau + \frac{\sqrt{2}}{2} + 2R$$

and

$$C_0\tau + \frac{\sqrt{2}}{2} + 2R \leq 5\sqrt{2}\ C_0T_0R + \frac{\sqrt{2}}{2} + 2R \leq 8\sqrt{2}C_0T_0R \leq 16\sqrt{2}C_0T_0R_0$$

using (10.27) and choosing $R = 2R_0$ with $R_0 \geq 8$.
From (10.26), recall that we have to choose:

$$t_0^* > T_0\left(R_0\sqrt{2} + 5 + c_1T_0\sqrt{2}\right)$$

and we have

$$R_0\sqrt{2} + 5 + c_1T_0\sqrt{2} < 2\sqrt{2}R_0 + c_1T_0\sqrt{2}.$$

Therefore for the choice

$$C_0t_0^* = C_0T_0\left(2\sqrt{2}R_0 + c_1T_0\sqrt{2}\right)$$

we get

$$\max\left(\bar{R} + C_0\tau, C_0(t_0^* + \tau)\right) \leq 16\sqrt{2}C_0T_0R_0 + C_0T_0\left(2\sqrt{2}R_0 + c_1T_0\sqrt{2}\right) \leq L := 18\sqrt{2}C_0T_0(R_0 + c_1T_0).$$

This ends the proof of the proposition.

10.4 Revisiting Section 7 and proof of Theorem 10.3

Proof of Theorem 10.3
The proof is similar to the proof of Theorem 1.3. We only have to make the following changes.
At the begining of Step 2.
From (10.14), we have

$$|\tilde{u}_0 - u_0| \leq C_0t_0^*$$

which implies

$$|\tilde{u} - u| \leq C_0t_0^*.$$

On the other hand, from (10.7) we have

$$|u_t| \leq \max(C_0, c_1).$$

This implies

$$|\lambda^\pm(T)| \le \frac{2C_0 t_0^*}{T} + \max(C_0, c_1)$$

which shows in particular that $\lambda^\pm(T)$ are well-defined.

In Step 2.2.

We do not have the uniform continuity of $\tilde{u}$ in time, and for this reason we get (7.10) for $T > 0$ only for $T \in \mathbb{Q}$, i.e.

$$|\tilde{u}(x,T) - \tilde{u}(0,0) - (\nu \cdot x + \lambda T)| \le 3L.$$

Moreover, up to shift the origin, we can assume that $\tilde{u}(0,0) = 0$, and from the continuity of $\tilde{u}$ in time, we recover that

$$|\tilde{u}(x,t) - (\nu \cdot x + \lambda t)| \le 3L \quad \text{for all} \quad (x,t) \in \mathbb{R}^N \times [0,+\infty).$$

Conclusion

Then Step 3 gives the bound with L given in (10.31)

$$\sup v - \inf v \le 12L = 216\sqrt{2}\ C_0 T_0 (R_0 + c_1 T_0).$$

This implies (10.3).

This ends the proof of the theorem.

11 Examples and applications

In this section we consider examples in 2D for geometric motions whose normal velocity can change sign.

11.1 The case where c is not positive

In this subsection, we focus on the case of normal velocity given by

$$V = \kappa + c(x) \tag{11.1}$$

where κ is the curvature. This means that we consider solutions of the following level sets equation

$$u_t = \operatorname{tr}\left\{D^2 u \cdot (I - \widehat{Du} \otimes \widehat{Du})\right\} + c(x)|Du| \quad \text{on} \quad \mathbb{R}^N \times (0,+\infty). \tag{11.2}$$

We consider the following assumption (see Figure 9):

<u>Assumption (C)</u>: Non positive velocity

We assume that c is a $\mathbb{Z}^2$-periodic function defined as follows. Let $0 \le r_0 < \frac{1}{4}$. For a point $x \in \mathbb{R}^2$, let us define

$$c(x) = c_0(r) \quad \textit{with} \quad r = \mathit{dist}(x, \mathbb{Z}^2)$$

where c_0 is a Lipschitz-continuous function satisfying for some $\delta > 0$:

$$(11.3) \qquad \begin{cases} c_0(r) + \dfrac{1}{r} \geq \delta > 0 & \text{if} \quad r \leq r_0, \\ \\ c_0(r) \geq \dfrac{1}{r_0} + \delta & \text{if} \quad r_0 \leq r \leq \dfrac{\sqrt{2}}{2}. \end{cases}$$

Remark 11.1 *Notice that assumption (C) allows the velocity c to be negative in part of the ball B_{r_0}.*

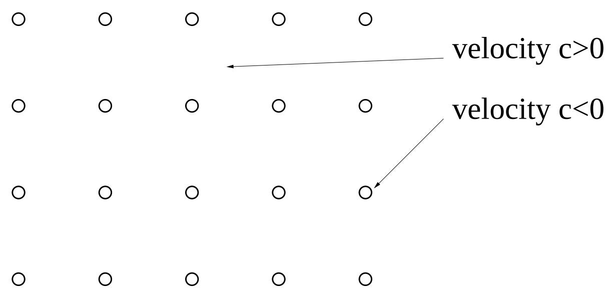

Figure 9: Illustration of assumption (C)

We have

Theorem 11.2 (Homogenization in 2D with non positive c)
Assume that $N = 2$ and that (C) holds with

$$F(X, p, x) = tr\{X \cdot (I - \widehat{p} \otimes \widehat{p})\} + c(x)|p|.$$

Then (B") holds. Moreover, let us consider the solution u^ε of (1.1) with initial data u_0 which is uniformly continuous on $\mathbb{R}^N$. Then u^ε converges locally uniformly on compact sets of $\mathbb{R}^N \times [0, +\infty)$ towards the unique solution u^0 of (1.2) with the function $\bar{F}$ given by Theorem 10.3.

In order to prove Theorem 11.2, we first need the following result:

Lemma 11.3 (A barrier subsolution for the forced MCM under assumption (C))
Assume (C). Then there exists a non-decreasing familly of compact sets $(E_t)_{t \geq 0}$ and a time $T_0 = \dfrac{1 + (\pi - 1)r_0}{\delta} > 0$ such that

$$(11.4) \qquad \begin{cases} E_0 = \overline{B_{r_0}}, \\ E_t = E_{T_0} \quad \text{for} \quad t \geq T_0 > 0, \\ \overline{B_{1+3r_0}} \supset E_{T_0} \supset \left[-\dfrac{1}{2}, \dfrac{1}{2}\right]^2 \cup \left(\displaystyle\bigcup_{e \in \{0, \pm e_1, \pm e_2\}} (e + E_0) \right) \end{cases}$$

and such that

$$u(\cdot, t) = \chi_{E_t}$$

is a subsolution of (11.2) with $N = 2$.

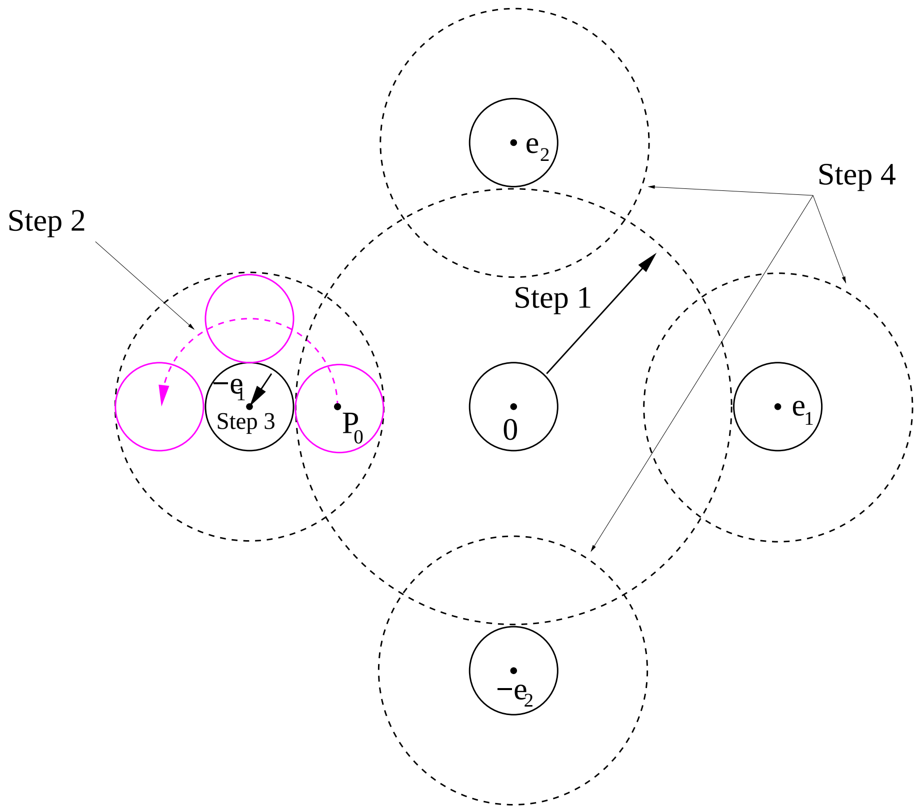

Figure 10: Illustration of the proof of Lemma 11.3

Proof of Lemma 11.3
With an abuse of terminology, we say that a time dependent compact set E_t is a subsolution (resp. solution), if χ_{E_t} is a subsolution (resp. solution) of (11.2).
Step 1: increasing the initial ball
When the solution is a ball $\overline{B_{r(t)}}$ of radius $r_0 \leq r(t) \leq 1 - r_0$, then equation (11.1) means

$$r' = -\frac{1}{r} + c_0(r) \geq -\frac{1}{r_0} + c_0(r).$$

Because of the second line of (11.3), we see that the ball $\overline{B_{r_0+\delta t}}$ is a subsolution for $t \in (0, t_1]$ with $r_0 + \delta t_1 = 1 - r_0$.
Step 2: rotating a ball tangent to $B_{r_0}(e)$ with $e = -e_1$
We do the raisonning with $e = -e_1$ (we will consider the other cases later). At time $t = t_1$, we know that $B_{r(t)}$ is tangent to $B_{r_0}(-e_1)$. Setting $P_0 = -(1 - 2r_0)e_1$, this implies that $B_{r_0}(P_0)$ is tangent to $B_{r_0}(-e_1)$ at $-(1 - r_0)e_1$. We define

$$K_\tau = R_{\theta(\tau)}(\overline{B_{r_0}(P_0)})$$

where R_θ is the rotation of center $-e_1$ and of angle θ. Notice that $\overline{B_{r_0}(P_0)}$ is a stationary subsolution. On the other hand, for any point $P = (x_1, x_2) \in \partial B_{r_0}(P_0)$ with $x_2 \geq 0$, we have

$$\frac{d}{d\tau} R_{\theta(\tau)}(P) = \theta'(t) R_{\frac{\pi}{2}+\theta(\tau)}(P)$$

which implies that the normal velocity satisfies:

$$V \leq \left| \frac{d}{d\tau} R_{\theta(\tau)}(P) \right| \leq r_0 |\theta'(\tau)|.$$

From the second line of (11.3), we can easily deduce (using test functions) that

$$K_\tau^+ := \bigcup_{0\le s\le \tau} K_s \quad \text{is a subsolution if} \quad 0 \le \theta'(s) \le \frac{\delta}{r_0}.$$

Therefore, choosing

$$\theta(\tau) = \frac{\delta}{r_0}\tau$$

and defining K_τ^- as the symmetric of K_τ^+ with respect to the axis $x_2 = 0$, we get that

$$K_\tau := K_\tau^+ \cup K_\tau^- \quad \text{is a subsolution for} \quad \tau \in (0, \tau_2) \quad \text{with} \quad \frac{\delta}{r_0}\tau_2 = \pi$$

and

$$K_{\tau_2} = \overline{B_{3r_0}(-e_1)} \backslash B_{r_0}(-e_1).$$

Step 3: filling the hole
We then define for $\tau \ge 0$

$$K_{\tau_2+\tau} = \overline{B_{3r_0}(-e_1)} \backslash B_{r(\tau)}(-e_1) \quad \text{with} \quad r(0) = r_0.$$

Then equation (11.1) means on $\partial B_{r(\tau)}(-e_1)$

$$-r' = \frac{1}{r} + c_0(r).$$

We then get a subsolution for the choice

$$r(\tau) = r_0 - \delta\tau \quad \text{for} \quad \tau \in [0, \tau_3] \quad \text{with} \quad \delta\tau_3 = r_0$$

with

$$K_{\tau_2+\tau_3} = \overline{B_{3r_0}(-e_1)}.$$

Step 4: getting a subsolution using $e = \pm e_1, \pm e_2$
We have for $e = \pm e_1, \pm e_2$

$$e = R_{\theta_e}(-e_1) \quad \text{for some angle} \quad \theta_e.$$

Then we set

$$K_\tau^e = e + e_1 + R_{\theta_e} K_\tau.$$

We then define the subsolution $(E_t)_t$ as follows

$$E_t = \begin{cases} \overline{B_{r_0+\delta t}} & \text{for} \quad 0 \le t \le t_1, \\ \overline{B_{r_0+\delta t_1}} \cup \left(\displaystyle\bigcup_{e\in\{\pm e_1, \pm e_2\}} \left(K_{t-t_1}^e\right) \right) & \text{for} \quad t_1 < t \le t_1 + \tau_2 + \tau_3 =: T_0, \\ E_{T_0} & \text{for} \quad t > T_0. \end{cases}$$

This is then easy to check the last inclusion of the last line of (11.4). It is also easy to check that $(E_t)_t$ is still a subsolution at times $t^* = t_1, t_1+\tau_2, T_0$, because

$$\tilde{E}_t = \begin{cases} E_t & \text{for} \quad 0 \le t \le t^*, \\ E_{t^*} & \text{for} \quad t > t^* \end{cases}$$

is still a subsolution for any $t^* \ge 0$, which follows from the fact that the sets E_t are non-decreasing with t.
This ends the proof of the lemma.

Proof of Theorem 11.2
Step 1: checking assumption (B")
We simply use Lemma 11.3 to define:

$$K_t = E_{T_0} \cup \left(\bigcup_{e \in \{\pm e_1, \pm e_2\}} (e + E_t) \right) \quad \text{for} \quad t \ge 0$$

which satisfies

$$\overline{B_{R_0}} \supset \overline{B_{2+3r_0}} \supset K_t \quad \text{for all} \quad t \ge 0 \quad \text{with} \quad R_0 = 8.$$

This familly of sets is associated to the subsolution:

$$U(\cdot, t) = \chi_{K_t}.$$

We also introduce constants $C_0, c_1 > 0$ such that

$$C_0 \ge c(x) \ge -c_1 \qquad \text{for all} \quad x \in \mathbb{R}^N.$$

Notice that up to a translation in the direction $(\frac{1}{2}, \frac{1}{2})$, the solution U satisfies (B"), and then we can apply Theorem 10.3 which shows, for every gradient p, the existence of a bounded corrector v such that

$$\sup v - \inf v \le \kappa_1 |p| \quad \text{with} \quad \kappa_1 = 400\ C_0 T_0 (R_0 + c_1 T_0). \tag{11.5}$$

Step 2: checking assumption (B') and conclusion
Using the definition of assumption (B'), we have

$$F^\eta(X,p,x) = \operatorname{tr}\{X \cdot (I - \widehat{p} \otimes \widehat{p})\} + c^\eta(x)|p|, \quad F_\eta(X,p,x) = \operatorname{tr}\{X \cdot (I - \widehat{p} \otimes \widehat{p})\} + c_\eta(x)|p|$$

with

$$c^\eta(x) = \sup_{|y-x| \le \eta} c(y), \quad c_\eta(x) = \inf_{|y-x| \le \eta} c(y).$$

Using the fact that assumption (C) is an open condition, it is easy to check that c^η and c_η still satisfy (C) for $\eta > 0$ small enough (for some perturbed δ and the same r_0). We see that it does not change C_0, c_1, R_0, but only changes slightly T_0 and then the bound κ_1 in (11.5). Therefore, we can apply Theorem 1.5, which ends the proof of the theorem.

Remark 11.4 (Change of topology of the front for certain positive c)
Notice that even if c is potivive, we can have a change of topology of the front. This is for instance the case, if $c_0(r)$ is large and positive for $r > r_0$ and if $c_0(r) + \frac{1}{r}$ is positive but very close to zero for $r < r_0$. This can be checked, using sub and supersolutions.

11.2 The G-equation with large divergence vector field

We recall that the G-equation is the motion with normal velocity

$$V = 1 + a \cdot n$$

where n is the ouward normal to the moving set $\left\{x \in \mathbb{R}^N, \quad u(x,t) > 0\right\}$ and $a(x)$ is a vector field. The level set formulation of this motion is the following:

$$u_t = |Du| - a \cdot Du \quad \text{on} \quad \mathbb{R}^N \times (0, +\infty). \tag{11.6}$$

The difficulty arises when $|a| > 1$ for which the Hamiltonian ceases to be coercive. A recent literature exists about homogenization results for this equation: see [27, 34, 7] (see also the very recent extension in random environments [8]). A typical condition that is assumed in order to construct correctors, is that the divengence of the vector field a is sufficiently small. See also [2] for some other results about non-coercive Hamiltonians.

Here we consider the assumption:
Assumption (D): the vector field a on $\mathbb{R}^2$
We assume that a is a $\mathbb{Z}^2$-periodic vector field defined as follows. Let $0 \le r_0 < \frac{1}{4}$. Let f be a scalar Lipschitz function satisfying for some $\delta_0 \in (0, r_0)$:

$$f(x) = 0 \quad \text{if} \quad |x| \le \delta_0 \quad \text{or} \quad |x| \ge r_0. \tag{11.7}$$

For a point $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]^2$, let us define

$$a(x) = f(x)b(x)$$

with

$$\begin{cases} i) \quad b(x) \equiv \dfrac{x}{|x|^2} \quad \text{with } f \le 0 \text{ and } f \not\equiv 0, \\ OR \\ ii) \quad b(x) \equiv \dfrac{x^\perp}{|x|^2} \quad \text{with } f \text{ not radial for } |x| < r_0. \end{cases}$$

Remark 11.5 *Notice that we have*

$$div\ a = \nabla f \cdot b$$

and then, up to multiply f by a positive constant, the divergence of a can be taken as big as we want.

Then we have the following result

Lemma 11.6 (A barrier subsolution for the G-equation under assumption (D)) *Assume (D). Then there exists a non-decreasing familly of compact sets $(E_t)_{t \ge 0}$ and a time $T_0 = 1 + (\pi - 1)r_0 > 0$ such that*

$$\begin{cases} E_0 = \overline{B_{r_0}}, \\ E_t = E_{T_0} \quad \text{for} \quad t \ge T_0 > 0, \\ \overline{B_{1+3r_0}} \supset E_{T_0} \supset \left[-\frac{1}{2}, \frac{1}{2}\right]^2 \cup \left(\bigcup_{e \in \{0, \pm e_1, \pm e_2\}} (e + E_0) \right) \end{cases} \tag{11.8}$$

and such that

$$u(\cdot,t)=\chi_{E_t}$$

is a subsolution of (11.6) with $N=2$.

Proof of Lemma 11.6
We follow the proof of Lemma 11.3. Steps 1,2 and 4 are unchanged with the choice $\delta=1$. For Step 3 (filling the hole), we define for $\tau\ge 0$

$$K_{\tau_2+\tau}=\overline{B_{3r_0}(-e_1)}\backslash B_{r(\tau)}(-e_1)\quad\text{with}\quad r(0)=r_0.$$

Then equation (11.1) means on $\partial B_{r(\tau)}(-e_1)$

$$-r'=1+fb\cdot n\ge 1\quad\text{with}\quad n=-\frac{x}{|x|}.$$

We then get a subsolution for the choice

$$r(\tau)=r_0-\delta\tau\quad\text{for}\quad \tau\in[0,\tau_3]\quad\text{with}\quad \delta\tau_3=r_0$$

and again $\delta=1$. This ends the proof of the lemma.

Conclusion
Similarly to Subsection 11.1, we can get existence of correctors (and of perturbed correctors), and then an homogenization result in 2D associated to equation (11.6), under assumption (D), which is an illustrative case without bound on the divergence of the vector field a.

12 Appendix

In this appendix, in a first subsection we give classical barriers associated to initial data, in a second subsection we present some technical lemmata about inf-convolutions used in the proof of Theorem 1.5, and in a third subsection, we give the proof of the comparison principle (Theorem 3.3).

12.1 Barriers from the initial data

We look for solutions of

$$u_t=F(D^2u,Du,x)\quad\text{on}\quad\mathbb{R}^N\times(0,+\infty). \tag{12.1}$$

We start with the following result (borrowed from Lemmata 3.1.3 and 4.3.3 in [22], that we recall here for the convenience of the reader):

Lemma 12.1 (Fundamental barrier)
Assume (A). Let us consider any C^2 function $f:[0,+\infty)\to[0,+\infty)$ satisfying $f(0)=0$ and for some constant $L_0>0$

$$0\le\frac{f'(r)}{\min(r,1)}\le L_0\quad\textit{for all}\quad r\ge 0.$$

We consider a constant $C_1 > 0$ such that

$$\sup_{|X|\le 1,\ |p|\le 1,\ y\in\mathbb{R}^N} |F(X,p,y)| \le C_1. \tag{12.2}$$

Then there exists a constant $M = C_1 L_0 > 0$ such that for any $K \ge 0$ and any $x_0 \in \mathbb{R}^N$, the following functions v^+_{K,x_0} and v^-_{K,x_0} are respectively super and subsolutions of (12.1):

$$v^{\pm}_{K,x_0}(x,t) = \pm K\left(Mt + f(|x-x_0|)\right).$$

Proof of Lemma 12.1
We set $z = x - x_0$ and $r = |z|$. We compute

$$Dv^+_{1,x_0}(x,t) = f'(r)\frac{z}{r}, \quad D^2 v^+_{1,0}(x,t) = f''(r)\frac{z}{r}\otimes\frac{z}{r} + \frac{f'(r)}{r}\left(I - \frac{z}{r}\otimes\frac{z}{r}\right).$$

Using the fact that F is geometric (assumption (A2)), we deduce that

$$\partial_t v^{\pm}_{K,x_0} - F(D^2 v^{\pm}_{K,x_0}, Dv^{\pm}_{K,x_0}, x) = \pm K\left(M \mp \frac{f'(r)}{r} F(\pm I, \pm z, x)\right).$$

Notice that by assumption (A4)i), there exists a constant C_1 such that (12.2) holds.
We also have for $r \ge 1$ (using again (A2), and also (A1))

$$\left|\frac{1}{r}F(\pm I, \pm z, x)\right| = \left|F(\pm\frac{I}{r}, \pm\frac{z}{r}, x)\right| \le \sup_{\pm}\left|F(\pm I, \pm\frac{z}{r}, x)\right| \le C_1.$$

Therefore we get

$$\left|\frac{f'(r)}{r}F(\pm I, \pm z, x)\right| \le \begin{cases} C_1\left(\sup_{r\in[0,1]}\left|\frac{f'(r)}{r}\right|\right) & \text{if} \quad r\in[0,1],\\[2ex] C_1\left(\sup_{r\ge 1}|f'(r)|\right) & \text{if} \quad r \ge 1.\end{cases}$$

If we choose

$$M = C_1 L_0$$

we then conclude that v^+_{K,x_0} and v^-_{K,x_0} are respectively super and subsolutions of (12.1), which ends the proof of the lemma.

Then we have the following consequence (see also Lemma 4.3.4 in [22]):

Corollary 12.2 (barriers from uniformly continuous initial data)
Let $u_0 : \mathbb{R}^N \to \mathbb{R}$ be a uniformly continuous initial data. Then there exists u^+, u^-, respectively super and subsolutions of (12.1), such that

$$u^-(x,t) \le u_0(x) \le u^+(x,t) \quad \textit{for all} \quad x\in\mathbb{R}^N, \quad t\ge 0$$

such that

$$\lim_{t\to 0^+}\ \sup_{x\in\mathbb{R}^N} |u^{\pm}(x,t) - u_0(x)| = 0.$$

Proof of Corollary 12.2
We do the construction for u^+ (the construction of u^- being similar).
Step 1: First estimate
Because u_0 is uniformly continuous, there exists a (non-decreasing) modulus of continuity ω such that

$$|u_0(x) - u_0(y)| \leq \omega(|x-y|) \quad \text{for all} \quad x, y \in \mathbb{R}^N.$$

We choose f as in Lemma 12.1, satisfying moreover for some $\eta > 0$:

$$0 < \eta \leq \frac{f'(r)}{\min(r,1)} \leq L_0 \quad \text{for all} \quad r \geq 0.$$

Therefore, for any $\delta > 0$, there exists $K_\delta > 0$ such that

$$u_0(x) \leq u_0(x_0) + \delta + K_\delta f(|x - x_0|). \tag{12.3}$$

Indeed, given some $r_\delta > 0$, we have

$$\begin{cases} \omega(r) \leq \lceil \dfrac{r}{r_\delta} \rceil \omega(r_\delta) \leq \dfrac{\omega(r_\delta)}{r_\delta}(r + r_\delta) & \text{for} \quad r \geq 0, \\ \\ f'(r) \geq \eta \min(r_\delta, 1) =: \eta_\delta > 0 & \text{for} \quad r \geq r_\delta. \end{cases}$$

Therefore for $r \geq 0$:

$$\delta + K_\delta f(r) \geq \delta + K_\delta \eta_\delta (r - r_\delta) \geq \omega(r)$$

with the choice of K_δ and r_δ such that

$$\begin{cases} K_\delta \eta_\delta = \dfrac{\omega(r_\delta)}{r_\delta}, \\ \\ \delta = 2\omega(r_\delta). \end{cases}$$

Step 2: Consequence
From (12.3) and the comparison principle, we deduce that:

$$u(x,t) \leq u_0(x_0) + \delta + v^+_{K_\delta, x_0}(x,t) =: w^+_{\delta, x_0}(x,t) \quad \text{with} \quad w^+_{\delta,x_0}(x,t) \geq w^+_{\delta,x_0}(x,0) \geq u_0(x)$$

where $v^+_{K_\delta, x_0}$ is defined in Lemma 12.1. Then the following function

$$u^+(x,t) = \inf_{\delta \in (0,1),\ x_0 \in \mathbb{R}^N} w^+_{\delta,x_0}(x,t)$$

is a supersolution (as an infimum of supersolutions) and satisfies

$$u_0(x_0) \leq u^+(x_0, t) \leq w^+_{\delta,x_0}(x,t) \leq u_0(x_0) + \delta + MK_\delta t.$$

Defining

$$\bar{\omega}(t) = \inf_{\delta \in (0,1)} (\delta + MK_\delta t)$$

we get

$$0 \leq u^+(x_0, t) - u_0(x_0) \leq \bar{\omega}(t) \quad \text{for all} \quad x_0 \in \mathbb{R}^N, \quad t \geq 0 \tag{12.4}$$

where

$$\bar{\omega}(t) \to 0 \quad \text{as} \quad t \to 0.$$

Therefore (12.4) implies the result. This ends the proof of the corollary.

12.2 Technical lemmata used in the proof of Theorem 1.5

In this subsection, we present results about inf-convolutions that seem quite classical, but that we did not find (in the precise form we need) in the literature. The results are presented in Lemmata 12.3 and 12.4 that are used in the proof of Theorem 1.5.

Lemma 12.3 (Inf/Sup-convolution using balls)
i) Inf-convolution
Let us consider a lower semi-continuous function $\underline{u} > -\infty$ in an open set Ω. Let us define for $\eta > 0$:

$$\Omega_{-\eta} = \left\{(x,t) \in \mathbb{R}^n \times \mathbb{R}, \quad \overline{B_\eta(x)} \times \{t\} \subset \Omega\right\}$$

and

$$\underline{u}_\eta(x,t) = \inf_{y \in \overline{B_\eta(x)}} \underline{u}(y,t).$$

Then $\underline{u}_\eta$ is lower semi-continuous.
If $\varphi \in C^2$ is a test function tangent to $\underline{u}_\eta$ from below at $P_0 = (x_0,t_0) \in \Omega_{-\eta}$, then

$$(12.5) \qquad (\varphi_t(P_0), D\varphi(P_0), D^2\varphi(P_0)) \in \mathcal{P}^{2,-}\underline{u}(z_0,t_0) \quad \textit{for some} \quad z_0 \in \overline{B_\eta(x_0)}$$

and

$$(12.6) \qquad \xi^T \cdot D^2\varphi(P_0) \cdot \xi \quad \leq \quad |\xi|^2 \frac{|D\varphi(P_0)|}{\eta} \quad \textit{for all} \quad \xi \perp D\varphi(P_0)$$

and also

$$(12.7) \qquad D^2\varphi(P_0) \leq 0 \quad \textit{if} \quad D\varphi(P_0) = 0.$$

Moreover, if F satisfies assumption (A), then for any $\varepsilon > 0$, there exists a constant $c_{\frac{\eta}{\varepsilon}} > 0$ (depending only on $\frac{\eta}{\varepsilon}$ and F) such that for any $y_0 \in \mathbb{R}^N$

$$(12.8) \qquad F^*\left(\varepsilon D^2\varphi(P_0), D\varphi(P_0), y_0\right) \leq |D\varphi(P_0)|\ c_{\frac{\eta}{\varepsilon}}.$$

ii) Sup-convolution
We have similar results (with reversed and adapted inequalities) for upper semi-continuous functions $\overline{u} < \infty$ in an open set Ω and

$$\overline{u}^\eta(x,t) = \sup_{y \in \overline{B_\eta(x)}} \overline{u}(y,t).$$

Proof of Lemma 12.3
Step 0: $\underline{u}_\eta$ is lower semi-continuous
Consider a sequence $(x_k,t_k)_{k\in\mathbb{N}}$ converging to a point (x_∞,t_∞) such that

$$(\underline{u}_\eta)_*(x_\infty,t_\infty) = \lim_{k\to+\infty} \underline{u}_\eta(x_k,t_k).$$

Then there exists a point (y_k,t_k) such that

$$\underline{u}_\eta(x_k,t_k) = \underline{u}(y_k,t_k) \quad \text{with} \quad |x_k - y_k| \leq \eta$$

and

$$y_k \to y_\infty \quad \text{with} \quad |x_\infty - y_\infty| \le \eta.$$

Therefore we have

$$\underline{u}_\eta(x_\infty, t_\infty) \le \underline{u}(y_\infty, t_\infty) \le \liminf_{k\to+\infty} \underline{u}(y_k, t_k) = (\underline{u}_\eta)_*(x_\infty, t_\infty)$$

which shows that $\underline{u}_\eta$ is lower semi-continuous.

Step 1: proof of (12.5)

Let us define the function

$$\zeta(x) = \begin{cases} 0 & \text{if} \quad x \in \overline{B_\eta(0)}, \\ +\infty & \text{otherwise.} \end{cases}$$

Then for (x_0, t_0) as in the statement of the lemma,

$$\varphi(x_0, t_0) = \underline{u}_\eta(x_0, t_0) = \inf_{y\in\mathbb{R}^N} (\underline{u}(x_0 - y, t_0) + \zeta(y)) = \underline{u}(x_0 - y_0, t_0) + \zeta(y_0)$$

with $\zeta(y_0) = 0$. Moreover for all $(x, t) \in \Omega_{-\eta}$:

$$\varphi(x, t) \le \underline{u}_\eta(x, t) = \inf_{y\in\mathbb{R}^N} (\underline{u}(x - y, t) + \zeta(y)) \le \underline{u}(x - y_0, t) + \zeta(y_0) = \underline{u}(x - y_0, t).$$

This shows that

$$\varphi(\cdot + y_0, \cdot) \le \underline{u} \quad \text{with equality at } (z_0, t_0) \text{ with } z_0 = x_0 - y_0$$

which implies (12.5).

Step 2: proof of (12.6)

Up to set

$$\underline{u}(x, t) = +\infty \quad \text{if} \quad (x, t) \notin \Omega$$

we can also rewrite (setting $z = x_0 - y$)

$$\varphi(x_0, t_0) = \underline{u}_\eta(x_0, t_0) = \inf_{z\in\mathbb{R}^N} (\underline{u}(z, t_0) + \zeta(x_0 - z)) = \underline{u}(z_0, t_0) + \zeta(x_0 - z_0)$$

with $z_0 = x_0 - y_0$. We also have

$$\varphi(x, t) \le \underline{u}_\eta(x, t) \le \underline{u}(z_0, t) + \zeta(x - z_0) \quad \text{with equality at } (x_0, t_0).$$

Then either $y_0 \in B_\eta(0)$ and

$$D\varphi(P_0) = 0 \ge D^2\varphi(P_0) \tag{12.9}$$

or $y_0 \in \partial B_\eta(0)$ and

$$\varphi(x, t_0) \le \varphi(x_0, t_0) \quad \text{for all} \quad x \in \partial B_\eta(z_0) \ni x_0.$$

Up to change the coordinates, we can assume that $x_0 = 0$, $z_0 = -\eta e_N$ and then $\partial B_\eta(z_0)$ is locally parametrized by

$$x_N = -\eta + \sqrt{\eta^2 - x'^2} \quad \text{with} \quad x' = (x_1, ..., x_{N-1})$$

and then

$$\psi(x') = \varphi(x', -\eta + \sqrt{\eta^2 - x'^2}, t_0)$$

satisfies

$$\psi(x') \leq \psi(0).$$

This implies

$$D\psi(0) = 0, \quad D^2\psi(0) \leq 0$$

i.e.

$$D_i\varphi(P_0) = 0 \quad \text{for} \quad i = 1, ..., N-1$$

which implies

$$D\varphi(P_0) = D_N\varphi(P_0)e_N \quad \text{with} \quad D_N\varphi(P_0) = |D\varphi(P_0)|$$

and

$$D^2_{ij}\psi(0) = D^2_{ij}\varphi(P_0) - \delta_{ij}\frac{D_N\varphi(P_0)}{\eta} \leq 0 \quad \text{for} \quad i,j = 1, ..., N-1.$$

This implies (12.6).
Step 3: proof of (12.7)
If $y_0 \in B_\eta(0)$, then (12.7) follows from (12.9). If $y_0 \in \partial B_\eta(0)$ and $D\varphi(P_0) = 0$, then (still with $x_0 = 0$)

$$\varphi(x_0, t_0) \geq \varphi(x, t_0) = \varphi(x_0, t_0) + \frac{1}{2}x^T \cdot D^2\varphi(x_0, t_0) \cdot x + o(|x|^2) \quad \text{locally for} \quad x_N \leq -\eta + \sqrt{\eta^2 - x'^2}.$$

By a rescaling, this implies that

$$0 \geq x^T \cdot D^2\varphi(x_0, t_0) \cdot x \quad \text{for all } x = (x', x_N) \text{ such that } x_N < 0.$$

This implies

$$D^2\varphi(P_0) \leq 0$$

which shows (12.7).
Step 4: proof of (12.8)
We want to prove for any $y_0 \in \mathbb{R}^N$

$$F^*\left(\varepsilon D^2\varphi(P_0), D\varphi(P_0), y_0\right) \leq |D\varphi(P_0)| \; c_{\frac{\eta}{\varepsilon}}.$$

Case a: $D\phi(P_0) = 0$
Then we have

$$F^*(\varepsilon D^2\varphi(P_0), D\varphi(P_0), y_0) \leq F^*(0, 0, y_0) = 0$$

where we have used (12.7) and (3.3).
Case b: $D\phi(P_0) \neq 0$
Then with $p = D\varphi(P_0)$ we have

$$\begin{aligned} F^*(\varepsilon D^2\varphi(P_0), D\varphi(P_0), y_0) &= F(\Pi(p) \cdot \varepsilon D^2\varphi(P_0) \cdot \Pi(p), p, y_0) \\ &\leq F\left(\varepsilon \frac{|p|}{\eta} I, p, y_0\right) \\ &\leq |p| F\left(\frac{1}{\left(\frac{\eta}{\varepsilon}\right)} I, \frac{p}{|p|}, y_0\right) \\ &\leq |p| c_{\frac{\eta}{\varepsilon}} \end{aligned}$$

where we have used (3.4) for the first line, (12.6) for the second line, and assumption (A2) for the third line, and in the last line: assumption (A4)i) for fixing $c_{\frac{\eta}{\varepsilon}} = C_R$ for some $R \geq \max\left(\frac{1}{\left(\frac{\eta}{\varepsilon}\right)}, 1\right)$. This shows (12.8).
This ends the proof of the lemma.

Lemma 12.4 (General properties of inf/sup-convolution)
A) Inf-convolution
Let us consider three functions $u, v, w : \mathbb{R}^N \times [0, +\infty) \to [0, +\infty]$, We define the space inf-convolution of u and v by

$$(u \Box v)(x,t) = \inf_{y \in \mathbb{R}^N} \left(u(x-y,t) + v(y,t) \right).$$

i) symmetry
We have

$$u \Box v = v \Box u$$

and $u \Box v : \mathbb{R}^N \times [0, +\infty) \to [0, +\infty]$.
ii) associativity
We have

$$(u \Box v) \Box w = u \Box (v \Box w).$$

iii) basic property
Let us consider the following property

$$\text{(12.10)} \qquad w(x,t) \to +\infty \quad \text{as} \quad |x| \to +\infty, \quad \text{uniformly in} \quad t \in [0, +\infty).$$

If u, v are lower semi-continuous and either u or v satisfies (12.10) then $u \Box v$ is also lower semi-continuous. Moreover, if u and v satisfy (12.10), then $u \Box v$ also satisfies (12.10).
iv) subdifferentials
Assume that v is independent on t and that

$$\text{(12.11)} \qquad (u \Box v)(x_0, t_0) = u(x_0 - y_0, t_0) + v(y_0).$$

Then

$$\text{(12.12)} \qquad (\tau, p, X) \in \mathcal{P}^{2,-}(u \Box v)(x_0, t_0) \quad \Longrightarrow \quad \begin{cases} (\tau, p, X) \in \mathcal{P}^{2,-} u(x_0 - y_0, t_0) \\ (p, X) \in \mathcal{D}^{2,-} v(y_0). \end{cases}$$

v) limit subdifferentials
Assume that v is independent on t, that u and v are lower semi-continuous and that either u or v satisfies (12.10). Then

$$\text{(12.13)} \quad (\tau, p, X) \in \overline{\mathcal{P}}^{2,-}(u \Box v)(x_0, t_0) \quad \Longrightarrow \quad \begin{cases} \text{there exists} \quad y_\infty \in \mathbb{R}^N \quad \text{such that} \\ (u \Box v)(x_0, t_0) = u(x_0 - y_\infty, t_0) + v(y_\infty) \\ (\tau, p, X) \in \overline{\mathcal{P}}^{2,-} u(x_0 - y_\infty, t_0) \\ (p, X) \in \overline{\mathcal{D}}^{2,-} v(y_\infty). \end{cases}$$

vi) generalization
Everything stays true if (we weaken the fact that u *has values in* $[0,+\infty]$ *and) require that any function satisfies the following condition (growth at most linear at infinity):*

$$\text{there exists a constant } C>0 \text{ such that } \quad u(x,t)\geq -C(1+|x|+|t|) \tag{12.14}$$

and replace condition (12.10) by the strongest condition (a superlinearity condition):

$$\frac{w(x,t)}{1+|x|}\to +\infty \quad \text{as} \quad |x|\to+\infty, \quad \text{uniformly in} \quad t\in[0,+\infty). \tag{12.15}$$

In the case of three functions u,v,w*, we require that at least two satisfy (12.15) and the third one satisfies (12.14).*
vii) applications
Assume that u *is lower semi-continuous, satisfies (12.14) and that we have*

$$|u(x,t)-g(x,t)|\leq \varepsilon\kappa \quad \text{for some function } g \text{ satisfying} \quad |g(x+a,t)-g(x,t)|\leq L|a|. \tag{12.16}$$

If v *is lower semi-continuous and satisfies (12.15) and* v *is independent on* t*, then* $u\Box v$ *still satisfies (12.16) with* g *replaced by* $g\Box v$ *which has the same Lipschitz constant* L*.*
Moreover if $v(x)=\dfrac{|x|^4}{4\varepsilon^3\rho}$ *and*

$$(u\Box v)(x_0,t_0)=u(x_0-y_0,t_0)+v(y_0)$$

then

$$|y_0|\leq \varepsilon C_{\kappa,L,\rho} \quad \text{with} \quad C_{\kappa,L,\rho}:=\left(16\rho\kappa+3(2\rho L)^{\frac{4}{3}}\right)^{\frac{1}{4}}. \tag{12.17}$$

Moreover we have

$$|u-u\Box v|\leq \varepsilon C'_{\kappa,L,\rho} \quad \text{with} \quad C'_{\kappa,L,\rho}=4\kappa+2LC_{\kappa,L,\rho}. \tag{12.18}$$

B) Sup-convolution
Let us consider two functions $u,v:\mathbb{R}^N\times[0,+\infty)\to[-\infty,0]$*, We define the space sup-convolution of* u *and* v *by*

$$(u\overline{\Box}v)(x,t)=\sup_{y\in\mathbb{R}^N}\left(u(x-y,t)+v(y,t)\right).$$

Then we have

$$-(u\overline{\Box}v)=(-u)\Box(-v) \tag{12.19}$$

which implies for the sup-convolution results similar to the case of the inf-convolution.

Proof of Lemma 12.4
Step 1: proof of A)i)
We have (by a change of variables)

$$(u\Box v)(x,t)=\inf_{y\in\mathbb{R}^N}\left(u(x-y,t)+v(y,t)\right)=\inf_{z\in\mathbb{R}^N}\left(u(z,t)+v(x-z,t)\right)=(v\Box u)(x).$$

Step 2: proof of A)ii)
We have with $z = z' - y$

$$\begin{aligned}
((u\Box v)\Box w)(x) &= \inf_{y\in\mathbb{R}^N} ((u\Box v)(x-y,t) + w(y,t)) \\
&= \inf_{y\in\mathbb{R}^N} \left(\inf_{z\in\mathbb{R}^N} (u(x-y-z,t) + v(z,t)) + w(y,t) \right) \\
&= \inf_{y,z\in\mathbb{R}^N} (u(x-y-z,t) + v(z,t) + w(y,t)) \\
&= \inf_{y,z'\in\mathbb{R}^N} (u(x-z',t) + v(z'-y,t) + w(y,t)) \\
&= \inf_{z'\in\mathbb{R}^N} \left(u(x-z',t) + \inf_{y\in\mathbb{R}^N} (v(z'-y,t) + w(y,t)) \right) \\
&= \inf_{z'\in\mathbb{R}^N} (u(x-z',t) + (v\Box w)(z',t)) \\
&= (u\Box(v\Box w))(x,t).
\end{aligned}$$

Step 3: proof of A)iii)
Let (x_k, t_k) converging to (x_∞, t_∞), and let y_k be a sequence such that

$$(u\Box v)(x_k, t_k) \geq -\frac{1}{k} + u(x_k - y_k, t_k) + v(y_k, t_k).$$

Condition (12.10) implies that y_k stays bounded, and then converges to some y_∞ (up to extraction of some subsequence). Therefore

$$\liminf_{k\to+\infty} (u\Box v)(x_k, t_k) \geq u(x_\infty - y_\infty, t_\infty) + v(y_\infty, t_\infty) \geq (u\Box v)(x_\infty, t_\infty)$$

which shows that $u\Box v$ is lower semi-continuous.
Property (12.10) follows also easily for $u\Box v$, if u and v both satisfy (12.10).
Step 4: proof of A)iv)
Let $P_0 = (x_0, t_0)$. If $(\tau, p, X) \in \mathcal{P}^{2,-}(u\Box v)(P_0)$, then there exists a test function φ which is tangent from below to $u\Box v$ at P_0, such that $\tau = \varphi_t(P_0)$, $p = D\varphi(P_0)$, $X = D^2\varphi(P_0)$. Recall that there exists y_0 such that (12.11) holds. Then we have

$$\varphi(x,t) \leq (u\Box v)(x,t) \leq u(x - y_0, t) + v(y_0)$$

with equality in the inequalities for $(x,t) = (x_0, t_0) = P_0$.
This shows that $(\tau, p, X) \in \mathcal{P}^{2,-}u(x_0 - y_0, t_0)$. Symmetrically, we also have with $z_0 = x_0 - y_0$:

$$(u\Box v)(x_0, t_0) = u(z_0, t_0) + v(x_0 - z_0)$$

and

$$\varphi(x,t) \leq (u\Box v)(x,t) \leq u(z_0, t) + v(x - z_0)$$

with equality for $(x,t) = (x_0, t_0) = P_0$. This implies that $(p, X) \in D^{2,-}v(x_0 - z_0) = D^{2,-}v(y_0)$ as in (12.12).
Step 5: proof of A)v)
If $(\tau, p, X) \in \overline{\mathcal{P}}^{2,-}(u\Box v)(P_0)$, then there exists a sequence $P_k = (x_k, t_k)$ converging to P_0 and $(\tau_k, p_k, X_k) \in \mathcal{P}^{2,-}(u\Box v)(P_k)$ converging to (τ, p, X) such that

$$(u\Box v)(P_k) \to (u\Box v)(P_0).$$

Because either u or v satisfies (12.10) and u and v are lower semi-continuous, we deduce that there exists some $y_k \in \mathbb{R}^N$ such that

$$(u\Box v)(x_k, t_k) = u(x_k - y_k, t_k) + v(y_k). \tag{12.20}$$

From (12.10), we deduce that y_k is bounded, and then convergent (up to some subsequence) to some $y_\infty \in \mathbb{R}^N$. Up to extract a subsequence, we can also assume that there exists l_u, l_v such that

$$u(x_k - y_k, t_k) \to l_u + u(x_0 - y_\infty, t_0), \quad v(y_k) \to l_v + v(y_\infty)$$

where

$$l_u, l_v \geq 0$$

follows from the lower semi-continuity of u and v. From (12.20), we get

$$\begin{aligned}
(u\Box v)(x_0, t_0)) &= \lim_{k\to+\infty}(u\Box v)(x_k)\\
&= \lim_{k\to+\infty} (u(x_k - y_k, t_k) + v(y_k))\\
&= (l_u + l_v) + u(x_0 - y_\infty, t_0) + v(y_\infty)\\
&\geq (u\Box v)(x_0, t_0)
\end{aligned}$$

which implies

$$(u\Box v)(x_0, t_0) = u(x_0 - y_\infty, t_0) + v(y_\infty)$$

and $l_u = 0 = l_v$, i.e.

$$u(x_k - y_k, t_k) \to u(x_0 - y_\infty, t_0), \quad v(y_k) \to v(y_\infty). \tag{12.21}$$

Moreover from A)iv), we have

$$\begin{cases} (\tau_k, p_k, X_k) \in \mathcal{P}^{2,-}u(x_k - y_k, t_k)\\ (p_k, X_k) \in \mathcal{D}^{2,-}v(y_k). \end{cases}$$

Then (12.21) implies that

$$\begin{cases} (\tau, p, X) \in \overline{\mathcal{P}}^{2,-}u(x_0 - y_\infty, t_0)\\ (p, X) \in \overline{\mathcal{D}}^{2,-}v(y_\infty) \end{cases}$$

which proves (12.13).
Step 6: proof of A)vi)
We simply have to prove case a and b (the remaining parts are straightforward to check).
case a: u satisfies (12.14) and v satisfies (12.15)
We simply have to check that $u\Box v$ satisfies (12.14).
We have

$$\begin{aligned}
(u\Box v)(x, t) &= \inf_{y\in\mathbb{R}^N} (u(x - y, t) + v(y, t))\\
&\geq \inf_{y\in\mathbb{R}^N} (-C(1 + |x - y| + |t|) + v(y, t))\\
&\geq -C(1 + |x| + |t|) + \inf_{y\in\mathbb{R}^N} (-C|y| + v(y, t))\\
&\geq -C(1 + |x| + |t|) - C_1 \quad \text{with} \quad C_1 = \max\left(0, -\inf_{y\in\mathbb{R}^N} (-C|y| + v(y, t))\right)\\
&\geq -C'(1 + |x| + |t|) \quad \text{with} \quad C' = C + C_1.
\end{aligned}$$

case b: u satisfies (12.15) and v satisfies (12.15)
We simply have to check that $u\Box v$ satisfies (12.15).
Because of (12.15), there exists a monotone function $h : [0,+\infty) \to [0,+\infty)$ such that $h(+\infty) = +\infty$ and

$$u(x,t), v(x,t) \geq -C + h(|x|) \quad \text{with} \quad \frac{h(\rho)}{1+\rho} \to +\infty \quad \text{as} \quad \rho \to +\infty.$$

Therefore

$$\begin{aligned}(u\Box v)(x,t) &= \inf_{y\in\mathbb{R}^N} (u(x-y,t) + v(y,t)) \\ &\geq -2C + \inf_{y\in\mathbb{R}^N} (h(|x-y|) + h(|y|)) \\ &\geq -2C + \inf_{y\in\mathbb{R}^N} h\left(\max(|x-y|,|y|)\right) \\ &\geq -2C + h(|x|/2)\end{aligned}$$

where we have used in the last line the fact that $|x-y|+|y| \geq |x|$ implies $\max(|x-y|,|y|) \geq |x|/2$. This implies that $u\Box v$ satisfies (12.15).
Step 7: proof of A)vii)
We have

$$-\varepsilon\kappa + g(x,t) \leq u(x,t) \leq \varepsilon\kappa + g(x,t).$$

This implies that

$$-\varepsilon\kappa + \inf_{y\in\mathbb{R}^N} (g(x-y,t) + v(y)) \leq (u\Box v)(x,t) \leq \varepsilon\kappa + \inf_{y\in\mathbb{R}^N} (g(x-y,t) + v(y))$$

i.e.

$$|u - g\Box v| \leq \varepsilon\kappa.$$

If

$$(u\Box v)(x_0,t_0) = u(x_0-y_0,t_0) + v(y_0) \tag{12.22}$$

then

$$u(x_0-y_0,t_0) + v(y_0) \leq u(x_0,t_0) + v(0)$$

i.e.

$$v(y_0) - v(0) \leq u(x_0,t_0) - u(x_0-y_0,t_0).$$

If $v(x) = |x|^4/(4\varepsilon^3\rho)$, then

$$\frac{|y_0|^4}{4\varepsilon^3\rho} \leq 2\varepsilon\kappa + L|y_0| \tag{12.23}$$

i.e. $\bar{y}_0 = y_0/\varepsilon$ satisfies

$$\begin{aligned}\frac{|\bar{y}_0|^4}{4\rho} &\leq 2\kappa + \frac{|\bar{y}_0|}{(2\rho)^{\frac14}} \cdot (2\rho)^{\frac14} L \\ &\leq 2\kappa + \frac14 \left(\frac{|\bar{y}_0|}{(2\rho)^{\frac14}}\right)^4 + \frac34 \left((2\rho)^{\frac14} L\right)^{\frac43}\end{aligned}$$

where in the second line, we have used Young inequality $ab \leq a^4/4 + 3b^{\frac{4}{3}}/4$. This implies

$$\frac{|\bar{y}_0|^4}{4\rho} \leq 4\kappa + \frac{3}{2}\left((2\rho)^{\frac{1}{4}} L\right)^{\frac{4}{3}}$$

and

$$|\bar{y}_0| \leq \left(16\rho\kappa + 3(2\rho L)^{\frac{4}{3}}\right)^{\frac{1}{4}}$$

which implies (12.17). Now for any point $(x_0, t_0) \in \mathbb{R}^N \times [0, +\infty)$, there exists a point $y_0 \in \mathbb{R}^N$ such that (12.22) holds. Therefore, using (12.23), we get

$$|((u\Box v)-u)(x_0,t_0)| = |u(x_0-y_0,t_0)-u(x_0,t_0)+v(y_0)| \leq 2\varepsilon\kappa+L|y_0|+\frac{|y_0|^4}{4\varepsilon^3\rho} \leq 2\left(2\varepsilon\kappa + L|y_0|\right)$$

which implies (12.18).
Step 8: proof of B)
This is straightforward.
This ends the proof of the lemma.

12.3 Proof of the comparison principle Theorem 3.3

Proof of Theorem 3.3
The proof is essentially based on [23] (see also [30]).
We only do the proof for finite T (the limit $T \to +\infty$ gives the result for $T = +\infty$). We also assume that in the case of a bounded open set Ω, condition (3.5) is strenghened as follows

$$u \leq v \quad \text{on} \quad \overline{\partial_p \Omega_T}. \tag{12.24}$$

Indeed, this is always possible to do it, up to replace T by a smaller time $T' < T$. We can then prove the comparison principle for T' and recover it for T in the limit $T' \to T$.
Step 1: reduction to bounded sub/supersolutions
Let $\beta : [-\infty, +\infty] \to [-1, 1]$ continuous with $\beta \in C^2(\mathbb{R}; (-1, 1))$, $\beta' > 0$ on $\mathbb{R}$ and $\beta(\pm\infty) = \pm 1$. We can for instance take $\beta(a) = \tanh a$. Given $u : \overline{\Omega} \times [0, T) \to [-\infty, +\infty]$ a subsolution of (4.1), it is easy to check that $\beta(u)$ is also a subsolution of (4.1).

Indeed, because $u < +\infty$, we have either $\beta(u) \in (-1, 1)$ and we can check Definition 3.1 using (A2). Or $\beta(u) = -1$ and then any test function at such a point satisfies $\varphi \geq \beta(u) \geq -1$ which implies $\varphi_t = 0 = D\varphi$ and $D^2\varphi \geq 0$. Using (A1) and (3.3), this finally implies again the viscosity subsolution inequality.

Similarly we can show that $\beta(v)$ is a supersolution if v is a supersolution. It is then sufficient to show that $\beta(u) \leq \beta(v)$. Therefore, up to replace $\beta(u)$ and $\beta(v)$ by u and v, we can assume that u and v are bounded.
Step 2: reduction to equation with u terms
Let us set

$$\tilde{u}(x,t) = e^{-t}u(x,t), \quad \tilde{v}(x,t) = e^{-t}v(x,t).$$

Then $\tilde{u}$ is a subsolution (resp. $\tilde{v}$ is a supersolution) of

$$\tilde{u}_t + \tilde{u} = \tilde{F}(D^2\tilde{u}, D\tilde{u}, x, t) \tag{12.25}$$

with

$$\tilde{F}(X,p,x,t) = e^{-t}F(e^tX, e^tp, x).$$

Just in order to simplify the notation, we will denote $\tilde{u}, \tilde{v}$ by u, v now respectively sub and supersolutions of (12.25).

Step 3: a priori estimates

Let us set

$$w(x,y,t) = u(x,t) - v(y,t).$$

Let us assume that

(12.26)

$$M = \lim_{\theta\to 0} M_\theta \quad \text{with} \quad M_\theta = \sup\left\{w(x,y,t), \quad x,y\in\overline{\Omega}, t\in[0,T), \quad |x-y|\le\theta\right\} > 0.$$

Then for small parameters $\alpha, \eta, \varepsilon > 0$, we can consider

$$M_{\alpha,\eta,\varepsilon} = \sup_{x,y\in\overline{\Omega},\ t\in[0,T)} \left(u(x,t) - v(y,t) - \frac{|x-y|^4}{4\varepsilon^2} - \alpha x^2 - \alpha y^2 - \frac{\eta}{T-t} \right). \tag{12.27}$$

For $\alpha, \eta > 0$ small enough (independently on ε), we have in particular

$$M_{\alpha,\eta,\varepsilon} \ge M/2 > 0. \tag{12.28}$$

We also know that the suppremum in (12.27) is reached at some points $x^*, y^* \in \overline{\Omega}$, $t^* \in [0,T)$ satisfying in particular

$$\frac{|x^*-y^*|^4}{4\varepsilon^2} + \alpha(x^*)^2 + \alpha(y^*)^2 + \frac{\eta}{T-t^*} \le C_0 := ||u||_\infty + ||v||_\infty. \tag{12.29}$$

Step 4: refined estimate on the penalization term $\frac{|x^*-y^*|^4}{4\varepsilon^2}$

We follow the ideas of Proposition 4.4 in [23]. From the definition of M in (12.26), we know that for any $\delta > 0$, there exists $\theta(\delta) > 0$ (with $\theta(\delta) \to 0$ as $\delta \to 0$) and points $x_\delta, y_\delta \in \overline{\Omega}, t_\delta \in [0,T)$ such that

$$w(x_\delta, y_\delta, t_\delta) \ge M - \delta \quad \text{and} \quad |x_\delta - y_\delta| \le \theta(\delta).$$

Therefore

$$M_{(4\varepsilon^2 C_0)^{\frac14}} - \frac{|x^*-y^*|^4}{4\varepsilon^2} \ge M_{\alpha,\eta,\varepsilon} \ge M - \delta - \frac{|\theta(\delta)^2|^4}{4\varepsilon^2} - \alpha(x_\delta)^2 - \alpha(y_\delta)^2 - \frac{\eta}{T-t_\delta}.$$

And then

$$\limsup_{\alpha,\eta\to 0} \frac{|x^*-y^*|^4}{4\varepsilon^2} \le M_{(4\varepsilon^2 C_0)^{\frac14}} - M + \delta + \frac{|\theta(\delta)^2|^4}{4\varepsilon^2}$$

which implies for $\delta \to 0$

$$\limsup_{\alpha,\eta\to 0} \frac{|x^*-y^*|^4}{4\varepsilon^2} \le M_{(4\varepsilon^2 C_0)^{\frac14}} - M$$

and then

$$\lim_{\varepsilon\to 0} \left(\limsup_{\alpha,\eta\to 0} \frac{|x^*-y^*|^4}{4\varepsilon^2} \right) = 0. \tag{12.30}$$

Step 5: avoiding the parabolic boundary $\partial_p\Omega_T$
Assume that for all $\varepsilon > 0$ small enough, we have $(x^*, t^*) \in \partial_p\Omega_T$ when we pass to the limit $\alpha, \eta \to 0$ for the whole sequence (the case $(y^*, t^*) \in \partial_p\Omega_T$ is similar). We distinguish two cases.
Case 1: $\Omega = \mathbb{R}^N$
Then this means that $t^* = 0$ and we have

$$\begin{aligned} 0 < M/2 \le M_{\alpha,\eta,\varepsilon} &\le \sup_{x,y\in\Omega} \left(u(x,0) - v(y,0) - \frac{|x-y|^4}{4\varepsilon^2} \right) \\ &\le \lim_{\theta\to 0} \sup \left\{ u(x,0) - v(y,0) - \frac{|x-y|^4}{4\varepsilon^2}, \quad |x-y| \le \theta \right\} \le 0 \end{aligned}$$

where we have used (3.5).
Case 2: Ω is bounded
Then we get a contradiction with (12.28), using the fact that

$$0 < M/2 \le \lim_{\alpha,\eta\to 0} M_{\alpha,\eta,\varepsilon} \le \sup_{(x,t)\in\overline{\partial_p\Omega_T}, y\in\overline{\Omega}} \left(u(x,t) - v(y,t) - \frac{|x-y|^4}{4\varepsilon^2} \right).$$

Then taking the limit $\varepsilon \to 0$, we get

$$0 < M/2 \le \lim_{\varepsilon\to 0} \sup_{(x,t)\in\overline{\partial_p\Omega_T}, y\in\overline{\Omega}} \left(u(x,t) - v(y,t) - \frac{|x-y|^4}{4\varepsilon^2} \right) \le \sup_{\overline{\partial_p\Omega_T}} (u - v) \le 0$$

where we have used (12.24).
Step 6: viscosity inequalities
Let us define

$$\tilde{u}(x,t) = u(x,t) - \alpha x^2, \quad \tilde{v}(y,t) = v(y,t) + \alpha y^2,$$

$$\Phi(x,y,t) = M_{\alpha,\eta,\varepsilon} + \frac{|x-y|^4}{4\varepsilon^2} + \frac{\eta}{T-t}. \tag{12.31}$$

From Ishii's Lemma (see Theorem 8.3 in the Users's Guide [17], and Theorem 7 in [16]), we deduce that for every $\gamma > 0$, there exists
(12.32)

$$\begin{cases} (b_1, D_x\Phi(x^*,y^*,t^*), X) \in \overline{\mathcal{P}}^{2,+}\tilde{u}(x^*,t^*) \\ (b_2, -D_y\Phi(x^*,y^*,t^*), Y) \in \overline{\mathcal{P}}^{2,-}\tilde{v}(y^*,t^*) \\ b_1 - b_2 = \Phi_t(x^*,y^*,t^*) \\ -\left(\frac{1}{\gamma} + ||A||\right) \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le A + \gamma A^2 \quad \text{with} \quad A = D^2\Phi(x^*,y^*,t^*) \in S^{2N} \end{cases}$$

where $||A|| = \sup_{|\xi|=1} < A\xi, \xi >$. From (12.32), we have in particular the viscosity inequalities

$$\begin{cases} b_1 + u(x^*,t^*) \le \tilde{F}^*(X + 2\alpha I, D_x\Phi(x^*,y^*,t^*) + 2\alpha x^*, x^*, t^*) \\ b_2 + v(y^*,t^*) \ge \tilde{F}_*(Y - 2\alpha I, -D_y\Phi(x^*,y^*,t^*) - 2\alpha y^*, y^*, t^*). \end{cases} \tag{12.33}$$

Recall that to be able to apply Ishii's Lemma, we need to be able to bound $b_1 \le C$ and $b_2 \ge -C$ for general $(b_1, p_1, X) \in \overline{\mathcal{P}}^{2,+}\tilde{u}(x,t)$ and $(b_2, p_2, Y) \in \overline{\mathcal{P}}^{2,-}\tilde{v}(y,t)$ for (x,t) close to

(x^*,t^*), (y,t) close to (y^*,t^*), and bounded $p_1,p_2,X,Y,\tilde u(x,t),\tilde v(y,t)$. Indeed this is true and comes from the viscosity inequalities similar to (12.33), using in particular assumption (A4)i).

Taking now the difference of the two inequalities in (12.33), we get
(12.34)

$$\begin{aligned} 0<M/2\le M_{\alpha,\eta,\varepsilon} &\le \frac{\eta}{(T-t^*)^2}+u(x^*,t^*)-v(y^*,t^*)\\ &\le \tilde F^*(X+2\alpha I,p+2\alpha x^*,x^*,t^*)-\tilde F_*(Y-2\alpha I,p-2\alpha y^*,y^*,t^*).\end{aligned}$$

We set

$$p=\delta(x^*-y^*),\quad \text{with}\quad \delta=\frac{|x^*-y^*|^2}{\varepsilon^2}.$$

Notice that from (12.29), we deduce that

$$(12.35)\qquad |p|^2\le \frac{(4C_0)^{\frac32}}{\varepsilon},\quad \delta\le\frac{2\sqrt{C_0}}{\varepsilon}\quad\text{and}\quad \alpha|x^*|,\alpha|y^*|\le\sqrt{\alpha C_0}.$$

Case 1: $x^*\neq y^*$
Then we have with $\widehat p=\dfrac{p}{|p|}$ (when $p\neq 0$)

$$0\le A=\delta\begin{pmatrix} I+2\widehat p\otimes\widehat p & -I-2\widehat p\otimes\widehat p\\ -I-2\widehat p\otimes\widehat p & I+2\widehat p\otimes\widehat p\end{pmatrix}\le 3\delta E\quad\text{with}\quad E=\begin{pmatrix} I & -I\\ -I & I\end{pmatrix}.$$

Notice that $E^2=2E$, and then $A^2\le 18\delta^2 E$. Because $||A||=6\delta$, setting $\gamma=\dfrac{1}{3\delta}$ in (12.32), we get

$$-9\delta\begin{pmatrix} I & 0\\ 0 & I\end{pmatrix}\le\begin{pmatrix} X & 0\\ 0 & -Y\end{pmatrix}\le 9\delta\begin{pmatrix} I & -I\\ -I & I\end{pmatrix}.$$

Case 2: $x^*=y^*$
Then $A=0$ and taking the limit $\gamma\to+\infty$, we see from (12.32), that at the limit we get $X=Y=0$.
Step 7: the limit $\alpha,\eta\to 0$
Then from (12.32), we deduce that for $\varepsilon>0$ fixed, the quantities p,X,Y stay bounded as α,η tend to zero. Even if x^*,y^* do not stay bounded, taking advantage of the periodicity of the problem, we know that there exists a sequence $k\in\mathbb{Z}^N$, such that as $\alpha,\eta\to 0$

$$\begin{cases} x^*-k\to\bar x,\\ y^*-k\to\bar y,\\ t^*\to\bar t\in[0,T],\\ \delta\to\bar\delta=\dfrac{|\bar x-\bar y|^2}{\varepsilon^2}\\ p=\delta(x^*-y^*)\to\bar p=\bar\delta(\bar x-\bar y)\\ X\to\bar X,\\ Y\to\bar Y.\end{cases}$$

Therefore at the limit $\alpha,\eta\to 0$ in (12.35), we get

$$0<M/2\le\tilde F^*(\bar X,\bar p,\bar x,\bar t)-\tilde F_*(\bar Y,\bar p,\bar y,\bar t)\quad\text{with}\quad \bar p=\delta(\bar x-\bar y)$$

with (in the general case $\bar{\delta} \geq 0$)

$$-9\bar{\delta}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} \bar{X} & 0 \\ 0 & -\bar{Y} \end{pmatrix} \leq 9\bar{\delta}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

Therefore setting $\tilde{\delta} = e^{\bar{t}}\delta$, we deduce that

$$\begin{aligned} 0 < M/2 \quad & \leq \tilde{F}^*(\bar{X}, \bar{p}, \bar{x}, \bar{t}) - \tilde{F}_*(\bar{Y}, \bar{p}, \bar{y}, \bar{t}) \\ & \leq e^{-\bar{t}}F^*(e^{\bar{t}}\bar{X}, e^{\bar{t}}\bar{p}, \bar{x}) - e^{-\bar{t}}F_*(e^{\bar{t}}\bar{Y}, e^{\bar{t}}\bar{p}, \bar{y}) \\ & \leq e^{-\bar{t}}F^*(e^{\bar{t}}\bar{X}, \tilde{\delta}(\bar{x}-\bar{y}), \bar{x}) - e^{-\bar{t}}F_*(e^{\bar{t}}\bar{Y}, \tilde{\delta}(\bar{x}-\bar{y}), \bar{y}) \\ & \leq e^{-\bar{t}}\sigma_K\left(|\bar{x}-\bar{y}|(1+\tilde{\delta}|\bar{x}-\bar{y}|)\right) \\ & \leq e^{-\bar{t}}\sigma_K\left(|\bar{x}-\bar{y}| + e^{\bar{t}}\frac{|\bar{x}-\bar{y}|^4}{\varepsilon^2}\right) \\ & \to 0 \quad \text{as} \quad \varepsilon \to 0 \end{aligned}$$

where we have used assumption (A4)ii) for $K \geq 9$ in the fourth line, and (12.30) in the last line. Contradiction. This ends the proof of the theorem.

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