



Coercive Hamilton-Jacobi equations in domains: the twin blow-ups method

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Abstract. In this note, we consider an evolution coercive Hamilton-Jacobi equation posed in a domain and supplemented with a boundary condition. We are interested in proving a comparison principle in the case where the time and the (normal) gradient variables are strongly coupled at the boundary. We elaborate on a method introduced by P.-L. Lions and P. Souganidis (*Atti Accad. Naz. Lincei*, 2017) to extend their comparison principle to more general boundary conditions and to Hamiltonians that are not globally Lipschitz continuous in the time variable. Their argument relies on a single blow-up procedure after rescaling the semi-solutions to be compared. In this work, two blow-ups are performed simultaneously, one for each variable of the doubling variable method. We show a key one-sided Lipschitz estimate satisfied by a combination of the two blow-up limits. Both blow-up limits are a priori allowed to be infinite separately. For expository reasons, the result is presented here in the framework of space dimension one and the general case is treated in a companion paper.

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1. Introduction: a comparison principle

Given $T > 0$, we consider viscosity solutions of a standard evolutive Hamilton-Jacobi equation posed in the geometric setting of a domain $\Omega := (0, +\infty)$,

$$u_t + H(X, u_x) = 0 \quad \text{in } (0, T) \times \Omega \quad (1.1)$$

where $X := (t, x)$, supplemented with the boundary condition,

$$u_t + F(X, u_x) = 0 \quad \text{in } (0, T) \times \partial\Omega$$

and the initial condition,

$$u(0, \cdot) = u_0 \quad \text{in } \{0\} \times \overline{\Omega}.$$

Since the boundary condition can be lost when characteristics reach $\partial\Omega$, it has to be imposed in a weak sense. When working with viscosity solutions, a classical way to handle this discrepancy is to impose that either the boundary condition or the equation is satisfied (in the viscosity sense) at the boundary,

$$\begin{cases} u_t + \min\{F, H\}(X, u_x) \leq 0 & \text{in } (0, T) \times \partial\Omega \quad (\text{for subsolutions}), \\ u_t + \max\{F, H\}(X, u_x) \geq 0 & \text{in } (0, T) \times \partial\Omega \quad (\text{for supersolutions}). \end{cases} \quad (1.2)$$

Comparison principles are strong uniqueness results for Hamilton-Jacobi equations. In the case of the previous initial boundary value problem, it is known (see [1–3, 7, 8]) that it is difficult to treat the case when tangential coordinates, such as the time variable t , and the normal derivative u_x of the solution, are strongly coupled reaching the boundary $(0, T) \times \partial\Omega$. It is standard to make the (strong) assumption of uniform continuity in time t , uniformly in the gradient u_x . Such an assumption is not satisfied by the following simple example,

$$\begin{cases} u_t + a(t, x)|u_x| = 0 & \text{in } (0, T) \times \Omega, \\ u_t + \max\{0, -b(t, x)u_x\} = 0 & \text{in } (0, T) \times \partial\Omega \end{cases} \quad (1.3)$$

when $a, b \geq 1$ are bounded Lipschitz continuous functions (here with $b(t, x) = b(t, 0)$).

In this note, we choose structural assumptions on H and F that encompass a large variety of examples (including (1.3)) but we do not seek for generality to avoid technicalities in proofs as much as possible. We assume that H and F are continuous in X , Lipschitz continuous and (semi-) coercive in p , with a time dependence allowing strong coupling with the gradient variable. Precisely, we assume that there exists a constant $C > 0$ such that,

$$\begin{cases} H \text{ is continuous and } |H(X, 0)| \leq C \text{ and } |H(X, p) - H(X, q)| \leq C|p - q| \\ H(X, p) \rightarrow +\infty \text{ as } |p| \rightarrow \infty \text{ uniformly in } X \\ |H(s, x, p) - H(t, x, p)| \leq C|t - s|(1 + \max(0, H)(t, x, p)) \end{cases} \quad (1.4)$$

$$\begin{cases} F \text{ is continuous and } |F(X, 0)| \leq C \text{ and } |F(X, p) - F(X, q)| \leq C|p - q|, \\ F \text{ is nonincreasing w.r.t. } p, \\ F(X, p) \rightarrow +\infty \text{ as } p \rightarrow -\infty \text{ uniformly in } X, \\ |F(s, x, p) - F(t, x, p)| \leq C|t - s|(1 + \max(0, F, H)(t, x, p)). \end{cases} \quad (1.5)$$

We make artificially appear the dependence of F on $x \in \overline{\Omega}$ in order to unify the presentation of assumptions for H and F . Note that condition (1.4) is classical and already appears for instance in [7, p. 44]. On the contrary condition (1.5) seems to be new.

We do not regularize the sub/supersolution by sup/inf-convolutions. In particular, we cannot assume without loss of generality that the subsolution is Lipschitz continuous. On the contrary, we only use a pure tangential doubling of variables, and we are able to prove the following new comparison principle.

Theorem 1 (A comparison principle with strong time coupling). *Let $T > 0$, assume that (1.4) and (1.5) hold true and that u_0 is bounded and Lipschitz continuous. Let $u : [0, T) \times \overline{\Omega} \rightarrow \mathbb{R}$ (resp. v) be a bounded upper semi-continuous viscosity subsolution (resp. bounded lower semi-continuous viscosity supersolution) of (1.1)-(1.2). If there exists $C_T > 0$ such that $u(0, \cdot) \leq u_0 \leq v(0, x)$ in $\overline{\Omega}$, then $u \leq v$ in $[0, T) \times \overline{\Omega}$.*

More important than the result is the new method introduced to prove Theorem 1, that can be used and adapted in numerous other situations.

Remark 2. Here, we assume for simplicity that the initial data u_0 is Lipschitz continuous. With some additional (classical) technicalities, it is possible to deal with uniformly continuous u_0 's and to relax the boundedness assumption on u, v, u_0 by imposing that they grow at most linearly.

Remark 3. Lipschitz continuity of F with respect to p is only used to get barriers in the proof of Theorem 1.

Remark 4. Notice that, given (1.4), we can always define the state constraint boundary function

$$H^-(t, x, p) := \sup_{q \leq p} H(t, x, q) \quad \text{for } (t, x) \in [0, T] \times \partial\Omega \quad \text{and } p \in \mathbb{R}$$

and it satisfies (1.5). Up to our knowledge, the comparison principle was also an open problem for $F = H^-$ in the generality of this note.

Remark 5. Notice that in Theorem 1, semi-coercivity of F in condition (1.5) can be replaced by the weak continuity of the subsolution u on the boundary $(0, T) \times \partial\Omega$, using [4, Proposition 3.12]) and replacing F by $F_1 := \max(F, H^-)$.

Comparison with known results. J. Guerand [5] proved a comparison principle in our geometric setting in the case where H and F are independent of (t, x) . She also proved a comparison principle for non-coercive Hamiltonians.

P.-L. Lions and P. Souganidis [8] introduced a new method to prove comparison principle for junctions with $N \geq 1$ branches (or half-spaces) between bounded uniformly continuous sub/supersolutions. They use a blow-up argument that reduces the study to a 1D problem. The authors show the comparison principle in the case of Kirchoff-type boundary conditions and non-convex Hamiltonians. As far as (t, x) dependence is concerned, these authors can handle Hamiltonians that are Lipschitz in t , see [8, Assumption (4)].

This result is generalized by G. Barles and E. Chasseigne [3, Theorem 15.3.7, page 295] to the case of bounded semi-continuous sub/supersolutions under three different junction conditions. Even if they are presented for $N = 2$ branches, we present their results in our geometric setting: a junction reduced to a single branch $N = 1$ in 1D. The three cases are the following: F is constant in (1.2), the Neumann problem, and general nonincreasing continuous F . In the third case, the normal derivative is not coupled with the tangential coordinates in F (see also the very end of [3, §13.2.2 and condition (GA-G-FLT) p. 247]).

As explained above, we improve these results in the case where the functions H and F imply a strong coupling of the time variable with the normal derivative of the solution.

Organization of the note. In Section 2 we recall the definition of (limiting) semi-differentials and we state and prove a technical lemma relating some slopes (that we call critical) at the boundary with semi-differentials (Lemma 1). We deduce from this technical lemma a corollary about critical slopes of stationary semi-solutions of the boundary value problem (Corollary 1). In Section 3, we state a barrier result (Lemma 2) that helps us dealing with the initial time condition; we also state a result (Lemma 3) about regularized subsolutions, that will be only used as an auxiliary result. In Section 4, the comparison principle (Theorem 1) is proved.

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2. Boundary lemmas

Definition 6 ((Limiting) semi-differentials). For a set $A \subset \overline{\Omega} = [0, +\infty)$ and a point $x_0 \in A$, we define the (first order) super/subdifferential at x_0 of a function u in A as

$$D_A^\pm u(x_0) = \{p \in \mathbb{R}, \text{ such that } 0 \leq \pm \{u(x_0) + p(x - x_0) + o(x - x_0) - u(x)\} \text{ in } A\}$$

and the limiting (first order) super/subdifferential at the boundary $x_0 = 0$ of u in Ω as

$$\bar{D}_\Omega^\pm u(0) = \{p \in \mathbb{R}, \text{ there exists a sequence } p_n \in D_\Omega^\pm u(x_n) \text{ with } x_n \in \Omega \text{ and } (x_n, p_n) \rightarrow (0, p)\}.$$

Remark 7. Note that if $p \in \bar{D}_\Omega^+ u(0)$ and u is a subsolution of $H(u_x) \leq 0$ in Ω , then $H(p) \leq 0$.

Lemma 1 (Critical slopes and semi-differentials). Let $\Omega := (0, +\infty)$ and $u, v : \overline{\Omega} \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ with u upper semi-continuous and v lower semicontinuous such that $u(0) = 0 = v(0)$ and $u \leq v$ in $\overline{\Omega}$. The critical slopes defined by

$$\bar{p} := \limsup_{\Omega \ni x \rightarrow 0} \frac{u(x)}{x}, \quad \underline{p} := \liminf_{\Omega \ni x \rightarrow 0} \frac{v(x)}{x} \quad (2.1)$$

satisfy the following (limiting) semi-differentials inclusions

$$\mathbb{R} \cap [\underline{p}, \bar{p}] \subset \bar{D}_\Omega^+ u(0) \cap \bar{D}_\Omega^- v(0) \quad \text{if } \bar{p} \geq \underline{p} \quad (2.2)$$

$$\mathbb{R} \cap [\bar{p}, \underline{p}] \subset D_\Omega^+ u(0) \cap D_\Omega^- v(0) \quad \text{if } \bar{p} \leq \underline{p} \quad (2.3)$$

$$\begin{cases} \bar{p} \in \bar{D}_\Omega^+ u(0) & \text{if } \bar{p} \in \mathbb{R} \\ \underline{p} \in \bar{D}_\Omega^- v(0) & \text{if } \underline{p} \in \mathbb{R} \end{cases} \quad (2.4)$$

Proof. We first notice that (2.3) is a straightforward consequence of the definition of semi-differentials.

We now focus on the proof of

$$\mathbb{R} \cap [\underline{p}, \bar{p}] \subset \bar{D}_\Omega^+ u(0) \quad \text{in case } \bar{p} > \underline{p} \quad (2.5)$$

and will even show the following better result

$$\mathbb{R} \cap [q, \bar{p}] \subset \bar{D}_\Omega^+ u(0) \quad \text{in case } \bar{p} > q := \liminf_{\Omega \ni x \rightarrow 0} \frac{u(x)}{x} \quad (2.6)$$

where the last inequality is assumed because $u \leq v$ implies $q \leq \underline{p}$. This claim is a variant of [8, Eq. (18)], and its proof is a variant of [3, Lemma 15.3.1]. We give the details for the sake of completeness.

Notice that $p \in (q, \bar{p})$ means

$$\limsup_{\Omega \ni x \rightarrow 0} \frac{u(x)}{x} = \bar{p} > p > q = \liminf_{\Omega \ni x \rightarrow 0} \frac{u(x)}{x}$$

We deduce that for any $\varepsilon > 0$, there exists $y_\varepsilon \in (0, \varepsilon)$ and $z_\varepsilon \in (0, y_\varepsilon)$ such that

$$\frac{u(z_\varepsilon)}{z_\varepsilon} > p > \frac{u(y_\varepsilon)}{y_\varepsilon}$$

Hence the function $\zeta(x) := u(x) - px$ satisfies $\zeta(0) = 0 > \zeta(y_\varepsilon)$. Let $M := \sup_{[0, y_\varepsilon]} \zeta \geq \zeta(z_\varepsilon) > 0$. Hence at a point $x_\varepsilon \in (0, y_\varepsilon)$ of maximum of ζ , we see that $p \in D_\Omega^+ u(x_\varepsilon)$. In the limit $\varepsilon \rightarrow 0$, we recover $p \in \bar{D}_\Omega^+ u(0)$. Then (2.6) follows from the fact that $\bar{D}_\Omega^+ u(0)$ is closed. Then (2.5) holds true for u . The similar inclusion for v implies (2.2) in the special case where $\bar{p} > \underline{p}$. On the other hand, notice that (2.4) implies (2.2) in the case $\bar{p} = \underline{p}$. Hence it remains to show (2.4).

We now explain why the following fact holds true,

$$\underline{p} \in \bar{D}_\Omega^- v(0) \quad \text{if } \underline{p} \in \mathbb{R} \quad (2.7)$$

This result is a property of the critical slope for any lower semi-continuous functions. Its proof follows exactly the lines of the proof of [6, Lemma 2.9]. A similar result holds for u and this proves (2.4). \square

Definition 8 (Coercive and semi-coercive functions). *Consider a function $G : \mathbb{R} \rightarrow \mathbb{R}$. Then G is coercive if $\lim_{|p| \rightarrow +\infty} G(p) = +\infty$, and semi-coercive if $\lim_{p \rightarrow -\infty} G(p) = +\infty$.*

As a consequence of Lemma 1, we have

Corollary 1 (Boundary viscosity inequalities). *Let Ω and u, v be as in statement of Lemma 1. For $\gamma = \alpha, \beta$, consider continuous functions $H_\gamma, F_\gamma : \mathbb{R} \rightarrow \mathbb{R}$ with H_α coercive and F_α semi-coercive. Assume that we have the following viscosity inequalities for some $\eta > 0$*

$$\left\{ \begin{array}{ll} H_\alpha(u_x) \leq 0 & \text{on } \Omega \cap \{|u| < +\infty\} \\ \min\{F_\alpha, H_\alpha\}(u_x) \leq 0 & \text{on } \{0\} \cap \{|u| < +\infty\} \\ \\ H_\beta(v_x) \geq \eta & \text{on } \Omega \cap \{|v| < +\infty\} \\ \max\{F_\beta, H_\beta\}(v_x) \geq \eta & \text{on } \{0\} \cap \{|v| < +\infty\} \end{array} \right. \quad (2.8)$$

For \underline{p}, \bar{p} defined in (2.1), we set $a := \min(\underline{p}, \bar{p})$ and $b := \max(\underline{p}, \bar{p})$. Then $\bar{p} \in [a, b] \cap \mathbb{R}$ and there exists a real number $p \in [a, b]$ such that

$$\text{either } H_\alpha(p) \leq 0 < \eta \leq (H_\beta - H_\alpha)(p) \quad \text{or} \quad \max(F_\alpha, H_\alpha)(p) \leq 0 < \eta \leq (F_\beta - F_\alpha)(p)$$

Remark 9. Corollary 1 can very easily be extended to the case of junctions where the Hamiltonians H_α 's are coercive on each branch and the junction function is semi-coercive in the sense of [6, Eq. (2.2)].

Proof of Corollary 1. We sketch the proof that $\bar{p} \in \mathbb{R}$. Because H_α is coercive and F_α is semi-coercive, we know from [4, Lemma 3.8] that u is weakly continuous at $x = 0$, i.e.

$$0 = u(0) = \limsup_{\Omega \ni x \rightarrow 0^+} u(x). \quad (2.9)$$

Then the proof [6, Lemma 2.10] shows additionally that $\bar{p} > -\infty$. Now we claim that we also have $\bar{p} < +\infty$. Indeed, it can be seen by contradiction, leaving fall down above the graph of u on $[0, y]$, some straight lines of slopes $s = \frac{u(y)}{y}$ for large positive s and using the equation satisfied by u . We conclude that $\bar{p} \in \mathbb{R} \cap [a, b]$.

Assume first that $\underline{p} \leq \bar{p}$. Here (2.2) shows that $H_\alpha(p) \leq 0 < \eta \leq H_\beta(p)$ and then

$$\eta \leq (H_\beta - H_\alpha)(p) \quad \text{for all } p \in [\underline{p}, \bar{p}] \cap \mathbb{R}$$

which implies in particular the desired conclusion.

If $\underline{p} > \bar{p}$, then we have $[a, b] \subset (-\infty, +\infty]$ with $a < b$ and

$$\left\{ \begin{array}{ll} H_\alpha(a) \leq 0 & \text{because } a \in \mathbb{R} \\ 0 < \eta \leq H_\beta(b) & \text{if } b \in \mathbb{R} \\ \min\{H_\alpha, F_\alpha\} \leq 0 < \eta \leq \max\{H_\beta, F_\beta\} & \text{on } [a, b] \cap \mathbb{R} \end{array} \right. \quad (2.10)$$

where the last line follows from (2.3), and the first two lines follow from (2.4).

INTERMEDIATE CLAIM. Now we assume by contradiction that there exists $\varepsilon > 0$ (small enough) such that

$$\left\{ \begin{array}{l} \text{i) } H_\beta - H_\alpha < \eta - \varepsilon \quad \text{or} \quad \varepsilon < H_\alpha \\ \text{and} \\ \text{ii) } F_\beta - F_\alpha < \eta \quad \text{or} \quad \varepsilon < \max(F_\alpha, H_\alpha) \end{array} \right. \quad \text{for all } p \in [a, b] \cap \mathbb{R} \quad (2.11)$$

Recall that the coercivity of H_α means $H_\alpha(\pm\infty) := \liminf_{p \rightarrow \pm\infty} H_\alpha(p) = +\infty$.

Case 1: $H_\alpha(b) > \varepsilon$

Here b can be finite or equal to $+\infty$. We get

$$H_\alpha(b) > \varepsilon > 0 \geq H_\alpha(a)$$

Therefore by continuity, there exists $p \in (a, b)$ such that $H_\alpha(p) = \varepsilon$. Hence in the last line of (2.10), the first inequality implies that $F_\alpha(p) \leq 0$. Because (2.11) i) and ii) hold true for p , we get

$$H_\beta(p) < \eta \quad \text{and} \quad F_\beta(p) < \eta$$

which leads to a contradiction in the last line of (2.10), the second inequality.

Case 2: $H_\alpha(b) \leq \varepsilon$

Then $b \in \mathbb{R}$ and (2.11) i) implies for $p = b$ that $H_\beta(b) < \eta$, which is in contradiction with the second line of (2.10).

CONCLUSION. We just proved that (2.11) does not hold true. This implies that for all $\varepsilon > 0$ small enough, there exists some $p_\varepsilon \in [a, b] \cap \mathbb{R}$ such that we have at p_ε

$$\text{i) } H_\alpha \leq \varepsilon < \eta - \varepsilon \leq H_\beta - H_\alpha \quad \text{or} \quad \text{ii) } \max(F_\alpha, H_\alpha) \leq \varepsilon < \eta \leq F_\beta - F_\alpha$$

Because H_α is coercive, we see in both cases i) or ii), that we can always extract a subsequence as $\varepsilon \rightarrow 0$ such that $p_\varepsilon \rightarrow p \in [a, b] \cap \mathbb{R}$. As a consequence, we get that p satisfies i) or ii) for $\varepsilon = 0$, which is the desired conclusion. \square

3. Barriers and auxiliary tangential regularization

In the proof of the comparison principle, two standard results about the construction of barriers and regularization of subsolutions by sup-convolution are needed.

Lemma 2 (Barriers). *Assume (1.4) and (1.5) and that the initial data u_0 is bounded and Lipschitz continuous. Assume that u (resp. v) is a bounded upper semi-continuous subsolution (resp. a bounded lower semi-continuous supersolution) of (1.1)-(1.2). Then there exists some constant $\lambda > 0$ such that the functions*

$$u^\pm(t, x) = u_0(x) \pm \lambda t$$

satisfy the following barrier properties:

- if $u \leq u_0$ in $\{0\} \times \overline{\Omega}$, then $u \leq u^+$ in $[0, T) \times \overline{\Omega}$,
- if $v \geq u_0$ in $\{0\} \times \overline{\Omega}$, then $v \geq u^-$ in $[0, T) \times \overline{\Omega}$.

The previous lemma is a direct consequence of the definition of viscosity solutions if u_0 is C^1 . In the general case, it follows by a standard approximation procedure.

We now turn to an auxiliary result about the regularization of subsolutions with respect to tangential variables. Even if the proof is very standard, we will give below a short sketch of it.

Lemma 3 (Auxiliary tangential regularization of subsolutions by sup-convolution). *Assume that H satisfies (1.4). Let $u : [0, T) \times \overline{\Omega} \rightarrow \mathbb{R}$ be an upper semi-continuous subsolution of (1.1) which satisfies*

$$|u - u_0| \leq C_T \quad \text{in} \quad [0, T) \times \overline{\Omega}.$$

We extend u to $t = T$ by $u(T, x) := \limsup\{u(s, y) : (s, y) \rightarrow (T, x), (s, y) \in [0, T) \times \overline{\Omega}\}$ and to $\mathbb{R} \times \overline{\Omega}$ by,

$$\begin{cases} u(t, x) = u(T, x) & \text{for } t \geq T, \\ u(t, x) = u(0, x) & \text{for } t \leq 0. \end{cases}$$

Then for $\nu > 0$, we consider the tangential sup-convolution

$$u^\nu(s, x) := \sup_{t \in \mathbb{R}} \left\{ u(t, x) - \frac{|t - s|^2}{2\nu} \right\} = u(\bar{t}, x) - \frac{|\bar{t} - s|^2}{2\nu}$$

and any such \bar{t} (depending on $(s, x) \in \mathbb{R} \times \bar{\Omega}$) satisfies $\bar{t} \in [s - \theta^\nu, s + \theta^\nu]$ with $\theta^\nu := 2\sqrt{\nu C_T} < T/2$.

If I^ν denotes the time interval $(\theta^\nu, T - \theta^\nu)$, then the function u^ν is Lipschitz continuous with respect to t in $\mathbb{R} \times \bar{\Omega}$ and with respect to x in $I^\nu \times \Omega$,

$$|\partial_s u^\nu|_{L^\infty(\mathbb{R} \times \bar{\Omega})} \leq \frac{\theta^\nu}{\nu} \quad \text{and} \quad |\partial_x u^\nu|_{L^\infty(I^\nu \times \Omega)} \leq L^\nu$$

with $L^\nu := \sup \left\{ p \in \mathbb{R}, \min_{X \in [0, T] \times \bar{\Omega}} H(X, p) \leq \frac{\theta^\nu}{\nu} \right\}$.

Assume moreover that that u is a subsolution at the boundary $(0, T) \times \partial\Omega$, i.e. satisfies the first line of (1.2), for some F that satisfies (1.5). Then u^ν is Lipschitz continuous in space and time on $I^\nu \times \bar{\Omega}$ of Lipschitz constant $L_\nu := \max(\frac{\theta^\nu}{\nu}, L^\nu)$.

Sketch of the proof. It is easy to check that $u - u^\nu \leq 2C_T$ which gives the bound on $\theta^\nu \geq |\bar{t} - s|$. Moreover the time derivative of u^ν is like $\frac{\bar{t}-s}{\nu}$ which gives the bound on $\partial_s u^\nu$. The PDE inequality satisfied by u^ν gives naturally the bound on $\partial_x u^\nu$. Finally, when F satisfies (1.5), we see using [4, Lemma 3.8]) that u (and then u^ν) is weakly continuous on $I^\nu \times \partial\Omega$, which implies the Lipschitz continuity of u^ν in $I^\nu \times \bar{\Omega}$. This ends the sketch of the proof. \square

4. Proof of the comparison principle

Before proving our comparison principle, we describe the main steps.

We first use the doubling variable technique with respect to time with a parameter $\nu > 0$ (**Step 1**).

We then focus on the case where the supremum of

$$u(t, x) - v(s, x) - \text{correction/penalization}$$

is reached at some $(\bar{t}, \bar{s}, \bar{x})$ with \bar{x} on the boundary of Ω (**Step 2**).

We also show a key one-sided Lipschitz estimate satisfied by the function $u(t, x) - v(s, y)$ (**Step 3**).

We then consider twin blow-ups (**Step 4**): one at (\bar{t}, \bar{x}) for u , and one at (\bar{s}, \bar{x}) for v (up to some correction terms on v). After blow-up, we get half-relaxed limits U^0, V^0 that are globally defined on $\mathbb{R}_t \times \bar{\Omega}$ and satisfy $U^0(0, 0) = 0 = V^0(0, 0)$. As a consequence of Step 3, we show the following one-sided Lipschitz estimate

$$U^0(t, x) - V^0(s, y) \leq L_\nu |x - y| + b(t - s) \quad \text{with} \quad b := \frac{\bar{t} - \bar{s}}{\nu} \quad (4.1)$$

where the Lipschitz constant L_ν is defined in Lemma 3.

Then, considering \bar{u} , the supremum in time of the map $t \mapsto U^0(t, x) - bt$ and \underline{v} , the infimum in time of the map $s \mapsto V^0(s, y) - bs$ (**Step 5**), we see that \bar{u}^* and \underline{v}_* are subsolution and supersolution of a 1D problem with moreover the key one-sided Lipschitz estimate¹

$$\bar{u}^*(x) - \underline{v}_*(y) \leq L_\nu |x - y| \quad \text{with} \quad \bar{u}^*(0) = 0 = \underline{v}_*(0).$$

This procedure reduces the study to a 1D problem that is solved using the boundary viscosity inequalities from Corollary 1 (**Step 6**).

¹Notice that without the one-sided Lipschitz estimate (4.1), we would only get $\bar{u} \leq \underline{v}$ and $\bar{u}(0) = 0 = \underline{v}(0)$, which is not sufficient to conclude, because we need the envelopes.

Proof of Theorem 1. The proof is split into several steps. The first two steps are standard and new ideas appear in the next steps.

STEP 1: APPROXIMATE SUPREMUM.

Let $\eta > 0$. This parameter will be small enough but will not vary until we prove that the following quantity is non-positive,

$$M = \sup_{t \in (0, T), x \in \bar{\Omega}} \left\{ u(t, x) - v(t, x) - \frac{\eta}{T-t} \right\}.$$

It turns out that

$$M = \lim_{\nu \rightarrow 0} \left\{ \lim_{\alpha \rightarrow 0} M_{\nu, \alpha} \right\}. \quad (4.2)$$

with

$$M_{\nu, \alpha} := \sup_{t, s \in (0, T), x \in \bar{\Omega}} \Psi_{\nu, \alpha}(t, s, x) = \Psi_{\nu, \alpha}(\bar{t}, \bar{s}, \bar{x})$$

and (with a careful choice of the penalization term $\frac{\eta}{T-s}$ instead of $\frac{\eta}{T-t}$)

$$\Psi_{\nu, \alpha}(t, s, x) := u(t, x) - v(s, x) - \frac{\eta}{T-s} - \alpha g(x) - \frac{|t-s|^2}{2\nu} \quad \text{with} \quad g(x) := \frac{x^2}{2}.$$

Moreover all maximisers $(\bar{t}, \bar{s}, \bar{x})$ in the definition of $M_{\nu, \alpha}$ satisfy

$$\lim_{\nu \rightarrow 0} \left\{ \lim_{\alpha \rightarrow 0} \frac{|\bar{t} - \bar{s}|^2}{\nu} \right\} = 0, \quad \lim_{\nu \rightarrow 0} \left\{ \lim_{\alpha \rightarrow 0} \alpha g(\bar{x}) \right\} = 0, \quad \limsup_{\nu \rightarrow 0} \left\{ \limsup_{\alpha \rightarrow 0} \frac{\eta}{T - \bar{s}} \right\} \leq 2C_T. \quad (4.3)$$

STEP 2: REDUCTION TO THE CASE WHERE THE SUPREMUM IS REACHED AT THE BOUNDARY.

Using the doubling variable technique with respect to x for u and v and considering $u(t, x)$, $v(s, y)$ with a further penalization term of the form $\frac{|x-y|^2}{2\delta}$, we can rely on barrier estimates close to $t = 0$ to get estimates on maximum points $(\bar{t}_\delta, \bar{x}_\delta, \bar{s}_\delta, \bar{y}_\delta) \rightarrow (\bar{t}, \bar{x}, \bar{s}, \bar{x})$ as $\delta \rightarrow 0$ (for some subsequence δ and some suitable limit $(\bar{t}, \bar{s}, \bar{x})$). We deduce that in the limit, the following fact holds true,

$$\bar{t}, \bar{s} \in [\tau_\eta, T - \tau_\eta], \quad \bar{x} \in [0, \rho_\alpha], \quad \left(\frac{\bar{t} - \bar{s}}{\nu}, \bar{p} \right) \in \bar{D}_{\bar{t}, \bar{x}}^{1,+} u(\bar{t}, \bar{x}), \quad |\bar{p}| \leq L_\nu \quad (4.4)$$

where L_ν is the Lipschitz constant of u^ν in Lemma 3. Moreover, it is possible to choose τ_η depending on η only and ρ_α depending on α only.²

If $\bar{x} > 0$, then we are in the classical case where we can conclude by writing viscosity inequalities for the sequence $(\bar{t}_\delta, \bar{x}_\delta, \bar{s}_\delta, \bar{y}_\delta)$ and by combining them in the classical way. Classical details for this step are given in the companion paper. We are thus reduced to deal with the case where $\bar{x} = 0$.

STEP 3: THE KEY ONE-SIDED LIPSCHITZ ESTIMATE.

We set

$$V(s, x) = v(s, x) + \frac{\eta}{T-s} + \alpha g(x).$$

Then we have

$$\left\{ \begin{array}{ll} \partial_t u + H(t, x, \partial_x u) \leq 0 & \text{in } (0, T) \times \Omega \\ \partial_t u + \min(F, H)(t, x, \partial_x u) \leq 0 & \text{in } (0, T) \times \partial\Omega \\ -\frac{\eta}{(T-s)^2} + \partial_s V + H(s, x, \partial_x V - \alpha \partial_x g) \geq 0 & \text{in } (0, T) \times \Omega \\ -\frac{\eta}{(T-s)^2} + \partial_s V + \max(F, H)(s, x, \partial_x V - \alpha \partial_x g) \geq 0 & \text{in } (0, T) \times \partial\Omega. \end{array} \right.$$

²Up to increase λ and C_T in the barrier Lemma 2, and to decrease η in the time penalisation term, we can assume that $\lambda T = C_T$ and it is possible to show in this case that we can choose $\tau_\eta := \frac{\eta}{4C_T}$ and $\rho_\alpha = \sqrt{\frac{6C_T}{\alpha}}$.

We now claim the following key one-sided "Lipschitz" estimate

$$u(t, x) - V(s, y) \leq u(\bar{t}, \bar{x}) - V(\bar{s}, \bar{x}) + \frac{|t - s|^2}{2\nu} - \frac{|\bar{t} - \bar{s}|^2}{2\nu} + L_\nu |x - y| \quad (4.5)$$

where equality holds for $t = \bar{t}$, $s = \bar{s}$, $y = x = \bar{x}$. For clarity, the proof of (4.5) is postponed at the end after Step 6.

STEP 4: THE TWIN BLOW-UPS.

We then consider the following twin blow-ups with small parameter $\varepsilon > 0$: one blow-up for u at the point (\bar{t}, \bar{x}) and one blow-up for V at the point (\bar{s}, \bar{x}) ,

$$\begin{cases} U^\varepsilon(\tau, \xi) := \varepsilon^{-1} \{u(\bar{t} + \varepsilon\tau, \bar{x} + \varepsilon\xi) - u(\bar{t}, \bar{x})\}, & U^\varepsilon(0, 0) = 0, \\ V^\varepsilon(\sigma, \zeta) := \varepsilon^{-1} \{V(\bar{s} + \varepsilon\sigma, \bar{x} + \varepsilon\zeta) - V(\bar{s}, \bar{x})\}, & V^\varepsilon(0, 0) = 0. \end{cases} \quad (4.6)$$

Before passing to the limit $\varepsilon \rightarrow 0$, they satisfy

$$\begin{cases} \partial_\tau U^\varepsilon + H(\bar{t} + \varepsilon\tau, \bar{x} + \varepsilon\xi, \partial_\xi U^\varepsilon) \leq 0 & \text{in } I_{\bar{t}}^\varepsilon \times \Omega \\ \partial_\tau U^\varepsilon + \min(F, H)(\bar{t} + \varepsilon\tau, \bar{x} + \varepsilon\xi, \partial_\xi U^\varepsilon) \leq 0 & \text{in } I_{\bar{t}}^\varepsilon \times \partial\Omega \\ -\bar{\eta}^\varepsilon + \partial_\sigma V^\varepsilon + H(\bar{s} + \varepsilon\sigma, \bar{x} + \varepsilon\zeta, \partial_\xi V^\varepsilon - \alpha \partial_x g(\bar{x} + \varepsilon\zeta)) \geq 0 & \text{in } I_{\bar{s}}^\varepsilon \times \Omega \\ -\bar{\eta}^\varepsilon + \partial_\sigma V^\varepsilon + \max(F, H)(\bar{s} + \varepsilon\sigma, \bar{x} + \varepsilon\zeta, \partial_\xi V^\varepsilon - \alpha \partial_x g(\bar{x} + \varepsilon\zeta)) \geq 0 & \text{in } I_{\bar{s}}^\varepsilon \times \partial\Omega \end{cases} \quad (4.7)$$

with

$$\bar{\eta}^\varepsilon(\sigma) := \frac{\eta}{(T - (\bar{s} + \varepsilon\sigma))^2} \quad \text{and} \quad I_{\bar{r}}^\varepsilon := \left(-\frac{\bar{r}}{\varepsilon}, \frac{T - \bar{r}}{\varepsilon}\right) \quad \text{for } \bar{r} = \bar{t}, \bar{s}.$$

From (4.5), they also satisfy

$$U^\varepsilon(\tau, \xi) - V^\varepsilon(\sigma, \zeta) \leq L_\nu |\xi - \zeta| + b(\tau - \sigma) + \varepsilon \frac{|\tau - \sigma|^2}{2\nu} \quad \text{with} \quad b := \frac{\bar{t} - \bar{s}}{\nu} \quad (4.8)$$

We then define the following half-relaxed limits

$$\begin{cases} U^0 := \limsup_{\varepsilon \rightarrow 0}^* U^\varepsilon, & U^0(0, 0) \geq 0, \\ V^0 := \liminf_{\varepsilon \rightarrow 0}^* V^\varepsilon, & V^0(0, 0) \leq 0. \end{cases}$$

Passing to the limit we get

$$U^0(\tau, \xi) - V^0(\sigma, \zeta) \leq L_\nu |\xi - \zeta| + b(\tau - \sigma), \quad U^0(0, 0) = 0 = V^0(0, 0). \quad (4.9)$$

and from (4.7) thanks to the discontinuous stability of viscosity solutions, we get

$$\begin{cases} \partial_\tau U^0 + H(\bar{t}, \bar{x}, \partial_\xi U^0) \leq 0 & \text{in } (\mathbb{R} \times \Omega) \cap \{|U^0| < +\infty\} \\ \partial_\tau U^0 + \min(F, H)(\bar{t}, \bar{x}, \partial_\xi U^0) \leq 0 & \text{in } (\mathbb{R} \times \partial\Omega) \cap \{|U^0| < +\infty\} \\ -\bar{\eta} + \partial_\sigma V^0 + H(\bar{s}, \bar{x}, \partial_\xi V^0) \geq 0 & \text{in } (\mathbb{R} \times \Omega) \cap \{|V^0| < +\infty\} \\ -\bar{\eta} + \partial_\sigma V^0 + \max(F, H)(\bar{s}, \bar{x}, \partial_\xi V^0) \geq 0 & \text{in } (\mathbb{R} \times \partial\Omega) \cap \{|V^0| < +\infty\} \end{cases} \quad (4.10)$$

with $\bar{\eta} := \frac{\eta}{(T - \bar{s})^2}$. We used the fact that $\alpha \partial_x g(\bar{x}) = \alpha \bar{x} = 0$.

STEP 5: THE 1D PROBLEM.

We now define the following functions as supremum/infimum in time of the functions defined in $\mathbb{R} \times \bar{\Omega}$,

$$\bar{u}(\xi) := \sup_{\tau \in \mathbb{R}} \{U^0(\tau, \xi) - b\tau\}, \quad \underline{v}(\xi) := \inf_{\tau \in \mathbb{R}} \{V^0(\tau, \xi) - b\tau\}.$$

From (4.9), these functions satisfy

$$-\infty \leq -L_\nu |\zeta - \xi| + \bar{u}(\xi) \leq \underline{v}(\zeta) \leq +\infty, \quad 0 \leq \bar{u}(0) \leq \underline{v}(0) \leq 0$$

and then $\bar{u}(0) = 0 = \underline{v}(0)$. Because of this one-sided Lipschitz inequality, this is also the case for their semi-continuous envelopes, *i.e.* we have

$$-\infty \leq -L_\nu |\zeta - \xi| + \bar{u}^*(\xi) \leq \underline{v}_*(\zeta) \leq +\infty, \quad \bar{u}^*(0) = 0 = \underline{v}_*(0). \quad (4.11)$$

From (4.10), we get (again from stability) that these functions satisfy in particular for $\bar{X} := (\bar{t}, \bar{x})$ and $\bar{Y} := (\bar{s}, \bar{x})$

$$\left\{ \begin{array}{ll} b + H(\bar{X}, \partial_{\xi} \bar{u}^*) \leq 0 & \text{in } \Omega \cap \{|\bar{u}^*| < +\infty\} \\ b + \min(F, H)(\bar{X}, \partial_{\xi} \bar{u}^*) \leq 0 & \text{in } \partial\Omega \cap \{|\bar{u}^*| < +\infty\} \\ -\bar{\eta} + b + H(\bar{Y}, \partial_{\xi} \bar{v}_*) \geq 0 & \text{in } \Omega \cap \{|\bar{v}_*| < +\infty\} \\ -\bar{\eta} + b + \max(F, H)(\bar{Y}, \partial_{\xi} \bar{v}_*) \geq 0 & \text{in } \partial\Omega \cap \{|\bar{v}_*| < +\infty\}. \end{array} \right. \quad (4.12)$$

STEP 6: GETTING A CONTRADICTION FROM STRUCTURAL ASSUMPTIONS.

We now apply Corollary 1. In order to do so, we consider

$$\bar{p} := \limsup_{\Omega \ni x \rightarrow 0} \frac{\bar{u}^*(x)}{x}, \quad \underline{p} := \liminf_{\Omega \ni x \rightarrow 0} \frac{\bar{v}_*(x)}{x}, \quad a := \min(\underline{p}, \bar{p}), \quad b := \max(\underline{p}, \bar{p})$$

and we get that there exists $p \in [a, b] \cap \mathbb{R} \neq \emptyset$ such that either

$$b + H(\bar{X}, p) \leq 0 < \bar{\eta} \leq H(\bar{Y}, p) - H(\bar{X}, p)$$

or

$$b + \max(F, H)(\bar{X}, p) \leq 0 < \bar{\eta} \leq F(\bar{Y}, p) - F(\bar{X}, p).$$

One of these facts are true along a subsequence $\nu \rightarrow 0$. In the first case, we get from the assumption on the Hamiltonian H , see (1.4) ii), that,

$$\begin{aligned} \bar{\eta} \leq H(\bar{Y}, p) - H(\bar{X}, p) &\leq C|\bar{t} - \bar{s}| \{1 + \max(0, H(\bar{X}, p))\} \\ &\leq C|\bar{t} - \bar{s}| \{1 + \max(0, -b)\} \\ &\leq C \left\{ \frac{|\bar{t} - \bar{s}|^2}{\nu} + |\bar{t} - \bar{s}| \right\} \rightarrow 0 \quad \text{as } \nu \rightarrow 0 \end{aligned}$$

where we have used $b = \frac{\bar{t} - \bar{s}}{\nu}$ in the third line, and (4.3) in the last line. Contradiction because $\bar{\eta} \geq \eta/T^2 > 0$.

From the assumption on the function F , see (1.5) ii), we get a similar contradiction in the second case,

$$\begin{aligned} \bar{\eta} \leq F(\bar{Y}, p) - F(\bar{X}, p) &\leq C|\bar{t} - \bar{s}| \{1 + \max(0, \max(F, H)(\bar{X}, p))\} \\ &\leq C|\bar{t} - \bar{s}| \{1 + \max(0, -b)\} \\ &\leq C \left\{ \frac{|\bar{t} - \bar{s}|^2}{\nu} + |\bar{t} - \bar{s}| \right\} \rightarrow 0 \quad \text{as } \nu \rightarrow 0. \end{aligned}$$

We conclude that $M \leq 0$. Recalling that

$$M = \sup_{t \in [0, T], x \in \bar{\Omega}} \left\{ u(t, x) - v(t, x) - \frac{\eta}{T-t} \right\} \leq 0,$$

it is enough to let $\eta \rightarrow 0$ to get $u \leq v$ as desired.

BACK TO STEP 3: PROOF OF THE KEY ONE-SIDED LIPSCHITZ ESTIMATE (4.5).

We now justify (4.5). Following Lemma 3, we extend u and consider

$$U^\nu(s, x) = \sup_{t \in \mathbb{R}} \left\{ u(t, x) - \frac{|t-s|^2}{2\nu} \right\}$$

and there exists some (possibly non unique) $\bar{t}_s \in [s - \theta^\nu, s + \theta^\nu]$ such that $U^\nu(s, x) = u(\bar{t}_s, x) - \frac{|\bar{t}_s - s|^2}{2\nu}$. If $s \in (\theta^\nu, T - \theta^\nu)$, then we see that $\bar{t}_s \in (0, T)$ and then we also have

$$U^\nu(s, x) := \sup_{t \in [0, T]} \left\{ u(t, x) - \frac{|t-s|^2}{2\nu} \right\}.$$

In particular for $(s, x) = (\bar{s}, \bar{x})$, we can choose $\bar{t}_{\bar{s}} = \bar{t}$ where $\bar{t}_{\bar{s}}$ is given by Lemma 3 and $(\bar{t}, \bar{s}, \bar{x})$ appear in Steps 1 and 2. Now we choose $\nu > 0$ small enough such that $\theta^\nu < \tau_\eta$ and we set $I^\nu := (\theta^\nu, T - \theta^\nu)$. Moreover we have for all $s \in I^\nu$, $y \in \bar{\Omega}$

$$U^\nu(s, y) - V(s, y) = \sup_{t \in [0, T]} \Psi_{\nu, \alpha}(t, s, y) \leq \Psi_{\nu, \alpha}(\bar{t}, \bar{s}, \bar{x}) = U^\nu(\bar{s}, \bar{x}) - V(\bar{s}, \bar{x}).$$

Now from Lemma 3, we also know that U^ν is L_ν -Lipschitz, and then $U^\nu(s, x) - U^\nu(s, y) \leq L_\nu|x - y|$, which implies

$$U^\nu(s, x) - V(s, y) \leq U^\nu(\bar{s}, \bar{x}) - V(\bar{s}, \bar{x}) + L_\nu|x - y|$$

which gives exactly (4.5). This ends the proof of the theorem. \square

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