On a Liouville type theorem for isotropic homogeneous fully nonlinear elliptic equations in dimension two

Jean Dolbeault* and Régis Monneau †

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Abstract

In this paper we establish a Liouville type theorem for fully nonlinear elliptic equations related to a conjecture of De Giorgi in $\mathbb{R}^2$. We prove that if the level lines of a solution have bounded curvature, then these level lines are straight lines. As a consequence, the solution is one-dimensional. The method also provides a result on free boundary problems of Serrin type.

Keywords. Fully nonlinear elliptic equations – Semilinear elliptic equations – Quasilinear elliptic equations – One-dimensional symmetry - Conjecture of De Giorgi – Liouville Theorem – Bernstein's Problem – Serrin's Problem

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*CEREMADE, U.M.R. C.N.R.S. no. 7534, Univ. Paris IX-Dauphine, Pl. de Lattre de Tassigny, 75775 Paris Cédex 16, France. E-mail: dolbeaul@ceremade.dauphine.fr
Phone: 01 44 05 46 78 & 01 43 37 86 94 - Fax: 01 44 05 45 99
†CERMICS, E.N.P.C., 6 & 8 avenue Blaise Pascal, Cité Descartes, Champs-sur-Marne, 77455 Marne-La-Vallée, France. E-mail: monneau@cermics.enpc.fr
Phone: 01 64 15 35 80 & 01 45 85 19 44 - Fax: 01 64 15 35 86

1 Introduction

In this paper, we are interested in qualitative properties of solutions of nonlinear elliptic equations in $\mathbb{R}^2$. The kind of results we are going to prove are *Liouville type theorems*. The original Liouville theorem states that any bounded harmonic function on the whole space is constant. In our case we will prove that if the curvature of the level lines of a solution of a nonlinear elliptic equation is bounded, then this curvature is zero. This means that the level lines are straight lines and the solution is one-dimensional.

In the formulation we shall use, however, our result is one of the variants of the famous *conjecture of De Giorgi* [19] (see below for a list of recent reference on this subject), in the sense that we prove that the solution is one-dimensional. It is also related to *Serrin type problems* [37] (for an unknown open set, overdetermined conditions on its boundary mean that the boundary is a circle, or, in some cases, a straight line).

We are going to state three results corresponding respectively to a De Giorgi type problem for $\Delta u + u - u^3 = 0$, to a more general free boundary problem of Serrin type for the equation $\Delta u + f(u) = 0$, and to a general fully nonlinear elliptic equation, under some appropriate bounds. All these results are actually consequences of a more technical result of Liouville type in the fully nonlinear case, that will be exposed in Section 2. The key tool of our proof uses estimates on the curvature of the level lines that have been introduced by the authors in [20, 21] in the context of free boundaries.

Definition [Bounded curvature of the level set] *We shall say that the level set $\{u = \lambda\}$ of a function u has a bounded curvature if there exists a constant $\delta > 0$ such that for every point X of the level set, there exist two points X_-, X_+ such that*

$$X \in \partial B_\delta(X_+) \cap \partial B_\delta(X_-) \text{ and } u \geq \lambda \text{ on } B_\delta(X_+),\ u \leq \lambda \text{ on } B_\delta(X_-).$$

The smallest possible $K = 1/\delta$ will be called the curvature.

Later, we will also consider a notion of signed curvature that will be well defined at any point such that $\nabla u \neq 0$. Here we denote by $B_\delta(X)$ the ball of center X and radius δ.

1.1 The semi-linear case

We first state a Liouville type theorem: if the curvature of the level sets is bounded on $\mathbb{R}^2$, then these level sets are straight lines (and the curvature is identically zero).

Theorem 1 *Let u be a bounded solution of*

$$\Delta u + u - u^3 = 0 \quad in \quad \mathbb{R}^2 .$$

If $\nabla u \neq 0$ on $\mathbb{R}^2$ and if each level set $\{u = \lambda\}$ has a bounded curvature, then these level sets are straight lines.

Theorem 1 is actually a special case of our second result. Let f be such that

$$\left\{ \begin{array}{l} f < 0 \quad \text{in} \quad (-1, \lambda_0) \\ f > 0 \quad \text{in} \quad (\lambda_0, 1) \\ f'(\lambda_0) > 0 \end{array} \right. \tag{1}$$

and consider a solution of

$$\left\{ \begin{array}{l} u_0'' + f(u_0) = 0 \quad \text{in} \quad I\!\!R \\ u_0(-\infty) = -1 \ , \quad u_0(0) = \lambda_0 \ , \quad u_0(+\infty) = 1 \\ u_0' > 0 \quad \text{in} \quad I\!\!R \end{array} \right. \tag{2}$$

Then we have the following Serrin type result.

Theorem 2 *Assume that f is an analytic function satisfying (1) such that (2) has a solution u_0. Let u be a solution of*

$$\Delta u + f(u) = 0 \quad in \quad \Omega$$

where $\Omega \subset I\!\!R^2$ is a C^2 connected and simply connected domain. If $\Omega \neq I\!\!R^2$, we assume moreover that $\partial\Omega$ has a bounded curvature and satisfies the boundary condition

$$u = \lambda_1 \quad and \quad -\frac{\partial u}{\partial n} = u_0'(u_0^{-1}(\lambda_1)) \quad on \quad \partial\Omega$$

for some $\lambda_1 \in [-1, 1)$. Assume that $\lambda_1 < u < 1$ on Ω and

$$\nabla u(x) \neq 0 \quad \forall\, x \in \Omega \ .$$

If each level set $\{u = \lambda\}$ has a bounded curvature, then it is a straight line. Moreover, if $\Omega \neq I\!\!R^2$, then Ω is a half plane.

It turns out that this result itself can be extended to fully nonlinear elliptic equations.

1.2 The fully nonlinear case

Consider an open set $\Omega \subset I\!\!R^2$ with C^2 boundary. Let u be an analytic function on Ω which is a solution of a fully nonlinear elliptic equation defined by

$$\mathcal{G}(D^2u, \nabla u, u) = 0 \ , \tag{3}$$

where $\mathcal{G}$ is an *analytic* function. This equation is homogeneous. Assume moreover that it is *isotropic:*

$$\mathcal{G}(D^2u, \nabla u, u) = \mathcal{F}(D_{nn}u, D_{\tau\tau}u, D_{n\tau}u, |\nabla u|^2, u) \ .$$

Here n and τ are respectively the unit normal and tangential (to the level lines) vectors, which are defined when $\nabla u \neq 0$ by $n = \frac{\nabla u}{|\nabla u|}$, $\tau = -n^\perp$. We take the direct trigonometric orientation, so that (τ, n) is a direct basis of $I\!R^2$. Moreover, when $\nabla u = 0$, we assume *(compatibility condition)* that there exists a function $\tilde{\mathcal{F}}$ such that

$$\mathcal{G}(D^2u, 0, u) = \tilde{\mathcal{F}}(\Delta u, \det D^2 u, u)\ . \tag{4}$$

For simplicity, we shall say that $\mathcal{G}$ *and* u *satisfy Assumption* $(F0)$ *if all the above conditions are fulfilled.*

Let $[\alpha, \beta] \subset I\!R$. Regarding the boundary conditions, we assume that

$$\left\{ \begin{array}{l} u = \lambda_1 \\ -\frac{\partial u}{\partial n} = u_0'(u_0^{-1}(\lambda_1)) \end{array} \right| \quad \text{on} \quad \partial\Omega\ , \tag{BC}$$

for some $\lambda_1 \in [\alpha, \beta)$. Here u_0 is a one-dimensional solution satisfying

$$\left\{ \begin{array}{l} \mathcal{F}(u_0'', 0, 0, u_0'^2, u_0) = 0 \quad \text{in} \quad I\!R \\ u_0(-\infty) = \alpha, \quad u_0(+\infty) = \beta \\ u_0' > 0 \quad \text{in} \quad I\!R \end{array} \right. \tag{F1}$$

We assume that the function $\mathcal{F} = \mathcal{F}(a, b, c, v, \lambda)$ is analytic and satisfies the following strong ellipticity assumption. *There exists a constant* $\mathcal{B} \geq \|u_0\|_{C^2(I\!R)}$ *and a constant* $C >$ *such that*

$$\begin{pmatrix} \mathcal{F}_a' & \frac{1}{2}\mathcal{F}_c' \\ \frac{1}{2}\mathcal{F}_c' & \mathcal{F}_b' \end{pmatrix} \xi \cdot \xi \geq C\,|\xi|^2 \quad \forall \xi \in I\!R^2\ , \quad \forall (a, b, c, v, \lambda) \in [-\mathcal{B}, \mathcal{B}]^3 \times [0, \mathcal{B}^2] \times [\alpha, \beta]\ . \tag{F2}$$

We also assume that *there exists* $\lambda_0 \in (\alpha, \beta) \subset I\!R$ *such that*

$$\begin{array}{l} \forall \lambda \in (\alpha, \beta)\ , \quad \mathcal{G}(0, 0, \lambda) = 0 \Longrightarrow \lambda = \lambda_0 \\ \text{and} \quad \mathcal{G}_\lambda'(0, 0, \lambda_0) > 0 \end{array} \tag{F3}$$

Theorem 3 *We consider a C^2 solution u of (3) on a connected and simply connected open domain Ω in $I\!R^2$. We assume the existence of real numbers $\alpha < \beta$, $\mathcal{B}$, $\lambda_1 \in [\alpha, \beta)$ such that (F0), (F1), (F2), (F3) and (BC) are satisfied, $\|u\|_{C^2(\overline{\Omega})} \leq \mathcal{B}$ and $\lambda_1 < u < \beta$ on Ω. If $\nabla u(x) \neq 0$ for any $x \in \Omega$ and each level set $\{u = \lambda\}$ has bounded curvature, then all level sets are straight lines. Moreover, if $\Omega \neq I\!R^2$, then Ω is a half plane.*

Theorem 3 is a consequence of a slightly more technical result, Theorem 4, which will be stated in the next section. In Theorem 4, we assume for instance that the solution is *analytic*. It is actually sufficient to impose like in Theorem 3 that the solution is of class C^2 on $\overline{\Omega}$. Using a regularity result due to Nirenberg [36], the solution is then of class $C^{2+\varepsilon}$ for some $\varepsilon > 0$. Since $\mathcal{G}$ is analytic, u is then also analytic according to Morrey's regularity theory [35]. The assumption that $\mathcal{G}$ is analytic cannot be removed, since we use the analyticity of both $\mathcal{G}$ and u in the proof of Theorem 3, unless a density result is available. To our knowledge, this has been proved only if (3) has at most one solution. Such uniqueness results are not available in the general case.

In the framework of Theorem 2, it is sufficient to impose an L^∞ bound instead of a C^2 bound. The L^∞ bound implies a uniform C^2 bound as it is shown by Danielli & Garofalo in [18].

1.3 A short review of the literature and outline of the paper

The *conjecture of De Giorgi* has been formulated in [19] as follows.

Conjecture (De Giorgi [19]). *Let u be a solution of $\Delta u = u^3 - u$ in the whole space $\mathbb{R}^n$, such that $|u| < 1$ and $\frac{\partial u}{\partial x_n} > 0$. All level sets $\{u = \lambda\}$ of u are hyperplanes if $n \leq 8$.*

An equivalent formulation is that *u depends only on one variable.*

The first result concerning the conjecture of De Giorgi has been given by Modica & Mortola [34] for $n = 2$ under the condition that the level sets of u are the graphs of a family of functions with a uniform Lipschitz estimate. An extension of this result has recently been proved by Barlow, Bass & Gui [6] in any dimensions, using probabilistic methods (see [11, 22] for further results in this direction). Several related results rely on local bounds and monotonicity properties (see for instance [32, 33, 17, 40, 18]).

The conjecture has been proved by Ghoussoub & Gui in [28] for $n = 2$ in connection with results obtained by Berestycki, Caffarelli & Nirenberg in [9], and by Ambrosio & Cabré in [3] for $n = 3$ (also see [2], and [29] for a recent result if $n = 4, 5$). The method used in [28] relies on the existence of negative spectrum for linear Schrödinger operators, which is true if $n = 1$ or 2 [9] but false if $n \geq 3$ [28, 5] (also see [31] for directions towards a counter-example in high dimensions).

Recently, Farina proved in [25] the conjecture in dimension $n = 2$ in the quasilinear case (also see [22, 23, 24, 26, 27] for related results) and independently Danielli & Garofalo in [18] proved the conjecture in the quasilinear case both in dimension $n = 2$ and in dimension $n = 3$. These results are very close to ours, except that we use a completely different method based on an evaluation of the curvature of the level lines, which was introduced in [20, 21] in the context of free boundary problems involving fully nonlinear elliptic equations. The main drawback of the method is that it is apparently limited to the case $n = 2$.

The conjecture of De Giorgi is heuristically related with Bernstein's problem: see for instance [14] for a short survey on recent results and [12, 39, 13, 30, 2, 15, 16] for more details.

Many other problems are related to the conjecture of De Giorgi and it is out of the scope of this introduction to cite all of them. They are of course strongly connected with other symmetry questions. For instance, when the domain is not the whole space, free boundary conditions may impose special symmetry properties for the solution and the domain like in *Serrin's problem* [37]. One-dimensional questions that have been studied by moving planes techniques and sliding methods [10, 7, 8, 11, 23] are very close to our results. Some extensions, for instance to systems [1], have also already been studied.

Section 2 is devoted to a statement of Theorem 4, which is the key of our approach, and its proof. It relies on estimates of the curvature at the points of the level lines which maximize or minimize the gradient, following an idea of [20, 21]. The technical part of the proof has been rejected in an appendix. In Section 3, we prove Theorem 3. Theorem 1 and 2 are easy consequence of Theorem 3.

2 The key result in the fully nonlinear case

In this section, we are going to state and prove the key result of this paper. Note that since we deal with unbounded domains, we need to introduce an approximation procedure (Proposition 7, proof in the appendix).

Theorem 4 *Let $\alpha, \beta \in I\!R$, $\alpha < \beta$, and $\Omega \subset I\!R^2$ be a connected and simply connected domain. Under Assumptions (F0), (F1) and (F2), consider an analytic solution u of (3) with boundary conditions (BC) such that $\lambda_1 < u < \beta$, $\|u\|_{C^2(\overline{\Omega})} \leq \mathcal{B}$. Assume that*

$$|\nabla u|_{\{u=\lambda\}} - u_0'(\lambda) \to 0 \quad as \quad \lambda \to \lambda_1 \text{ or } \beta \;. \tag{5}$$

If there exists a continuous function g such that

$$|\nabla u(x)| \geq g(u(x)) > 0 \quad \forall\, x \in \Omega \;,$$

then the level sets $\{u = \lambda\}$ are straight lines.

Exactly as in Theorem 3, we could simply assume that u belongs to $C^2(\overline{\Omega})$ (analyticity follows from the same reasons). The assumptions on the boundedness of the curvature in Theorem 3 are contained in the regularity hypotheses and the lower estimate of the gradient. The main difference with Theorem 3 is that the assumption on the lower bound of the gradient is uniform on any level line. The explicit form of the function g in Theorem 4 plays no role.

In this section, we are going to use the extrema

$$m(t) = \inf_{u(x)=t} |\nabla u(x)| \quad \text{and} \quad M(t) = \sup_{u(x)=t} |\nabla u(x)|$$

which are functions of the level t. We shall also use the sets on which they are reached:

$$\begin{array}{l} X_m^t = \{x \in I\!\!R^2 \ : \quad u(x) = t, \quad |\nabla u(x)| = m(t)\} \ , \\ X_M^t = \{x \in I\!\!R^2 \ : \quad u(x) = t, \quad |\nabla u(x)| = M(t)\} \ . \end{array}$$

Like in [20, 21], consider the minimal curvatures of the level lines on these sets:

$$k(t) = \inf_{y \in X_m^t} \frac{D_{\tau\tau} u}{|\nabla u|}(y) \quad \text{and} \quad K(t) = \inf_{y \in X_M^t} \frac{D_{\tau\tau} u}{|\nabla u|}(y) \ .$$

Note that K could also be defined as a supremum, but this would not change the following results, because these two kinds of curvature are equal for almost every t. The rest of this section is devoted to the proof of Theorem 4. For simplicity, we decompose it into two steps.

<u>First step:</u> *Assume that the following system holds*

$$\left\{ \begin{array}{l} \mathcal{F}\left(\frac{d}{dt}\left(\frac{m^2}{2}\right), mk, 0, m^2, t\right) = 0 \\ \\ \frac{dk}{dt} \geq -\frac{k^2}{m} \end{array} \right. \quad \textit{and} \quad \left\{ \begin{array}{l} \mathcal{F}\left(\frac{d}{dt}\left(\frac{M^2}{2}\right), MK, 0, M^2, t\right) = 0 \\ \\ \frac{dK}{dt} \leq -\frac{K^2}{M} \end{array} \right. \tag{6}$$

for almost all $t \in (\lambda_1, \beta)$. Here the above inequalities have to be understood in the sense of distributions. A similar statement has been justified in a previous paper, [21], in the case of bounded level sets.

Lemma 5 *Under the assumptions of Theorem 4, if there exists some t_0 for which $k(t_0) = 0$ (resp. $K(t_0) = 0$), then*

$$k(t)(t - t_0) \geq 0$$

(resp. $K(t)(t - t_0) \leq 0$) for any t in the range of u.

Proof. We know that $\frac{dk}{dt} \geq -\frac{k^2}{m}$ and thus $\frac{d}{dt}\left(\frac{1}{k}\right) \leq \frac{1}{m}$: for any $t_1 < t_2$,

$$\left(\frac{1}{k}\right)_{|t=t_2} \leq \int_{t_1}^{t_2} \frac{dt}{m(t)} + \left(\frac{1}{k}\right)_{|t=t_1} .$$

Because k can have positive jumps but not negative ones, this implies that if $k(t_1) > 0$, then k stays positive as long as m is nonzero. □

Thus k can only cross zero from negative to positive values and K can only cross zero from positive to negative values, so that we have to consider only two cases:
Case 1: k and K keep the same sign with $sgn(k) = sgn(K)$.
Case 2: either k and K keep the same sign with $sgn(k) = -sgn(K)$, or at least k or K changes of sign, which means that

i) either $k \leq 0 \leq K$ on an interval (λ_1, γ),

ii) or $K \leq 0 \leq k$ on an interval (γ, β).

Lemma 6 *Under the assumptions of Theorem 4, if $k \leq 0 \leq K$ on (λ_1, γ), then $k = K = 0$ and $m = M$ on (λ_1, γ).*

Proof. Because of the ellipticity assumption (F2), by the inverse function theorem, there exists a function $\mathcal{H}$ such that

$$\frac{dm^2}{dt} = \mathcal{H}(m^2, mk, t) \quad \text{and} \quad \frac{dM^2}{dt} = \mathcal{H}(M^2, MK, t) .$$

Then

$$\begin{aligned} \tfrac{d}{dt}\left(M^2 - m^2\right) &= \mathcal{H}(M^2, MK, t) - \mathcal{H}(M^2, 0, t) + \mathcal{H}(m^2, 0, t) - \mathcal{H}(m^2, mk, t) \\ &\quad + J\left(M^2 - m^2\right) \\ &\leq J\left(M^2 - m^2\right) , \quad \text{where} \quad J = \tfrac{\mathcal{H}(M^2,0,t) - \mathcal{H}(m^2,0,t)}{M^2 - m^2} \in L^\infty(\lambda_1, \gamma) . \end{aligned}$$

We use the fact that the first two terms on the right hand side on the first line are nonpositive, because $\mathcal{H}$ is nonincreasing in its second argument, as a consequence of (F2). This gives the following Gronwall estimate:

$$\frac{d}{dt}\left(\left(M^2 - m^2\right) e^{-\int_{\lambda_1}^t J(s)ds}\right) \leq 0 .$$

By definition, $M \geq m$ and $M(\lambda_1) = m(\lambda_1)$, so that $M \equiv m$ and $k \equiv K \equiv 0$ on (λ_1, γ). □

Let us go back to the proof of Theorem 4 under the assumption that (6) holds.

Proof in Case 2 : there are two cases:

1) Applying Lemma 6, we conclude in case i) that $k \equiv K \equiv 0$ and $m \equiv M$ on (λ_1, γ). But from Lemma 5, we have $0 \leq k$ on (γ, β), and, similarly, $K \leq 0$ on (γ, β). If $\gamma < \beta$, the problem is reduced to case ii) on (γ, β).

2) In case ii), $k \equiv K \equiv 0$, $m \equiv M$ on (γ, β) by a result analog to the one of Lemma 6, and if $\gamma > \lambda_1$ then case i) holds on (λ_1, γ).

In both cases, we obtain $k \equiv K \equiv 0$ and $m \equiv M$ on (λ_1, β). Using the fully nonlinear partial differential equation, this is possible if and only if the curvature is constant on each level line. Level lines are therefore circles with the same center. Because $|\nabla u| > 0$ and Ω is simply connected, this center is at infinity and all level lines are straight lines.

Proof in Case 1 : Assume for instance that $sgn(k) = sgn(K) \leq 0$ on (λ_1, β). Let $m_0(t) = u_0'(u_0^{-1}(t))$:

$$\frac{dm_0^2}{dt} = \mathcal{H}(m_0^2, 0, t) \; .$$

Since on the other hand

$$\frac{dm^2}{dt} = \mathcal{H}(m^2, mk, t) \geq \mathcal{H}(m, 0, t) \; ,$$

we obtain as before an inequality:

$$\frac{d}{dt}\left(m_0^2 - m^2\right) \leq \left(m_0^2 - m^2\right) \tilde{J}$$

for some $\tilde{J}$ in $L^\infty(\lambda_1, \beta)$. The functions m and m_0 have the same value at $t = \lambda_1$. Then by a Gronwall argument, we get

$$\frac{d}{dt}\left(\left(m_0^2 - m^2\right) e^{-\int_{\lambda_1}^t \tilde{J}(s)ds}\right) \leq 0 \; ,$$

and deduce that $m_0^2 - m^2 \leq 0$. Integrating this equation from β in place of λ_1, we get the reverse inequality, which implies that $m \equiv m_0$ on (λ_1, β). Similarly we get $M = m_0$ on (λ_1, β). We conclude as above that all level lines are straight lines.

Second step: The same results hold without (6) as a consequence of the following approximation result which will be proved in the Appendix.

Proposition 7 *For every $\varepsilon > 0$ small enough, we can find continuous functions $m^\varepsilon, M^\varepsilon$ defined on $[\lambda_1, \beta]$ which satisfy*

$$\begin{array}{l} m^\varepsilon(\lambda_1) = M^\varepsilon(\lambda_1) = m_0(\lambda_1) \ , \quad m^\varepsilon(\beta) = M^\varepsilon(\beta) = m_0(\beta) \\ \textit{and} \quad |m^\varepsilon - m| < \varepsilon \ , \quad |M^\varepsilon - M| < \varepsilon \ . \end{array}$$

We can also find functions $k^\varepsilon, K^\varepsilon$ defined on (λ_1, β) which satisfy

$$\left\{ \begin{array}{l} \mathcal{F}(m^\varepsilon \frac{dm^\varepsilon}{dt}, m^\varepsilon k^\varepsilon, 0, (m^\varepsilon)^2, t) = 0 \\ \\ \frac{dk^\varepsilon}{dt} \geq -\frac{(k^\varepsilon)^2}{m^\varepsilon} \end{array} \right. \quad \textit{and} \quad \left\{ \begin{array}{l} \mathcal{F}(M^\varepsilon \frac{dM^\varepsilon}{dt}, M^\varepsilon K^\varepsilon, 0, (M^\varepsilon)^2, t) = 0 \\ \\ \frac{dK^\varepsilon}{dt} \leq -\frac{(K^\varepsilon)^2}{M^\varepsilon} \end{array} \right. \tag{7}$$

where the equations hold for $t \in (\lambda_1, \beta)$ a.e. and the inequalities are satisfied in $\mathcal{D}'(\lambda_1, \beta)$.

Applying the first part of the proof to $m^\varepsilon, k^\varepsilon, M^\varepsilon, K^\varepsilon$, we conclude that $m^\varepsilon = M^\varepsilon$. The fact that it is true for every $\varepsilon > 0$ small enough proves by continuity that $m = M$, and then $k = K = 0$ as previously. The proof of Proposition 7 is by no way difficult but it is quite technical and for this reason it has been rejected in the appendix.

This ends the proof of Theorem 4. □

3 Proof of Theorem 3

In this Section, we are going to prove Theorem 3. Proofs of Theorems 1 and 2 are straightforward consequences of Theorem 3, since the required C^2 bound is automatically satisfied in the quasilinear case for bounded solutions [18].

Let us prove that if the gradient is zero at some level $\lambda \in (\lambda_1, \beta)$, then $\lambda = \lambda_0$. Because λ_0 is an unstable critical value by Assumption (F3), we will get a contradiction. Thus we get a positive lower bound on the gradient, which is uniform on each level line. We may then conclude by applying Theorem 4.

Step 1 : Consider a sequence $(X_n)_{n\in\mathbb{N}}$ such that $|\nabla u(X_n)| \to 0$ and $u(X_n) \to \lambda \in (-1, 1)$. *We claim that $\mathcal{G}(0, 0, \lambda) = 0$.*

By assumption, u is uniformly bounded in C^2. According to [36], u also has a uniform bound in $C^{2+\varepsilon}$. Thus we can extract a converging subsequence of $u(X + X_n)$ with limit u_∞, locally on compact sets, such that $u_\infty(0) = \lambda$. Because of the boundedness of the curvature of the level set $\{u = \lambda\}$ (in the sense of Definition given in Section 1), we deduce that $D^2 u_\infty(0) = 0$. The curvature can indeed be written $\frac{D_{\tau\tau} u_\infty}{|\nabla u_\infty|}$ whenever $\nabla u \neq 0$, so that $\lim_{n\to\infty} D_{\tau\tau} u(X_n) = 0$. By Taylor expansion, it is then clear that we need both $D_{nn}u_\infty(0) = 0$ and $D_{n\tau}u_\infty(0) = 0$ to avoid a contradiction with the definition of the curvature given in Section 1 applied to u_∞ at 0. This implies that $\lambda = \lambda_0$ according to (F3).

Step 2 : $u_\infty \equiv \lambda_0$

Because the curvature of the level sets is bounded, we can find a small ball $B_\varepsilon(0)$ and a direction $\nu \in \mathbf{S}^1$ such that $w := \partial_\nu u_\infty$ is nonnegative on $B_\varepsilon(0)$. In particular w satisfies

$$\left\{\begin{array}{l} \mathcal{G}'_{D^2u}D^2w + \mathcal{G}'_{Du}Dw + \mathcal{G}'_u w = 0\ , \quad w \geq 0 \quad \text{in } B_\varepsilon(0) \\ w(0) = 0 \end{array}\right.$$

Then the strong maximum principle implies that $w \equiv 0$ on $B_\varepsilon(0)$. Thus $u_\infty \equiv \lambda_0$ on $B_\varepsilon(0)$, and by analyticity of u_∞ we get $u_\infty \equiv \lambda_0$.

Step 3 : *Construction of a subsolution v on B_R*

Let ϕ be a solution of $\Delta\phi + \mu\phi = 0$ on B_R with $\phi > 0$ on B_R and $\phi = 0$ on ∂B_R. For α small enough, we can then build a subsolution $v = \alpha\phi + \lambda_0$ if μ is chosen less than $\frac{\tilde{\mathcal{F}}'_\lambda}{\tilde{\mathcal{F}}'_s}(0, 0, \lambda_0)$, for $\tilde{\mathcal{F}} = \tilde{\mathcal{F}}(s, p, \lambda)$ defined by (4), which requires a radius R large enough and then a curvature $K_0 = \frac{1}{R}$ small enough.

Step 4 : *Contradiction*

Because the gradient does not vanishes on $I\!R^2$, we can introduce natural curvilinear coordinates. We define an abscissa s on the level line $\{u = \lambda_0\}$. Then we note γ_s the integral curve of the vector field ∇u containing the point of abscissa s on $\{u = \lambda_0\}$. Then for every point X of Ω, we define

$$\Psi(X) = (s, t)\ ,$$

where $X \in \gamma_s$ and $t = u(X)$.

In Step 1, we assumed the existence of a sequence of points X_n such that $\Psi(X_n) = (s_n, \lambda_n)$ with $\lambda_n \to \lambda_0$. One can always find (for some λ'_n) a ball $B_R(\Psi^{-1}(s_n, \lambda'_n))$, in which we can define a subsolution $v \leq u$, as above. For a good choice of λ'_n, we can also define

$$u_n^+(X) = u(\Psi^{-1}(s_n, \lambda'_n) + X)$$

such that $u_n^+ \to u_\infty^+$. Because $u_\infty \equiv \lambda_0$, we can assume that $u_\infty^+ \geq \lambda_0$. Now sliding the subsolution v below u_∞^+, we get that $u_\infty^+ \geq \max v > \lambda_0$. In particular this implies that we can find a sequence of points whose gradient tends to zero, but whose level is strictly bigger that λ_0. This is a clear contradiction with Step 1.

Step 5 : From the previous steps we know that $|\nabla u(X_n)|$ does not tend to zero. This proves the existence of a continuous function g such that $|\nabla u(x)| \geq g(u(x)) > 0$ for any $x \in I\!R^2$ and we conclude by applying Theorem 4. $\square$

4 Appendix: an approximation result

This appendix is devoted to the technical part of the proof of Theorem 4 (second step).

4.1 Proof of Proposition 7

We are going to prove Proposition 7 in the case of functions $m^\varepsilon, k^\varepsilon$ (the proof of the existence of $M^\varepsilon, K^\varepsilon$ is similar). This result is based on the following lemma:

Lemma 8 *For every $t_0 \in (\lambda_1, \beta)$, there exist a constant $\varepsilon_{t_0} > 0$ and a sequence of points x^n on the level line $\{u = t_0\}$ such that $u_n(x) = u(x+x^n)$ converges to a function u_∞ which is still a solution of the fully nonlinear elliptic equation, and for which there exists a map*

$$\gamma^{t_0} : (t_0 - \varepsilon_{t_0}, t_0 + \varepsilon_{t_0}) \longrightarrow I\!R^2$$

which has the following properties:

$$\begin{cases} u_\infty(\gamma^{t_0}(t)) = t \quad \text{for} \quad t \in (t_0 - \varepsilon_{t_0}, t_0 + \varepsilon_{t_0}) \\ m^{t_0}(t) := |\nabla u_\infty(\gamma^{t_0}(t))| \quad \text{is continuous}\ , \quad m^{t_0}(t_0) = m(t_0) \quad \text{and} \quad m^{t_0}(t) \geq m(t) \\ k^{t_0}(t) := K\left(\gamma^{t_0}(t)\right)\ , \quad \frac{dk^{t_0}}{dt} \geq -\frac{(k^{t_0})^2}{m^{t_0}} \\ \mathcal{F}(m^{t_0}\frac{dm^{t_0}}{dt}, mk^{t_0}, 0, (m^{t_0})^2, t) = 0 \end{cases}$$

This (possibly non continuous) map γ^{t_0} has furthermore the following properties:

i) either γ^{t_0} is an analytic curve both on $(t_0 - \varepsilon_{t_0}, t_0)$ and $(t_0, t_0 + \varepsilon_{t_0})$ but possibly non continuous at t_0.

ii) or the level line $\{u_\infty = t_0\}$ is a straight line and there exists a sequence of levels $(t_i)_{i\in \mathbb{Z}}$ contained in $(t_0 - \varepsilon_{t_0}, t_0 + \varepsilon_{t_0})$, which accumulate at t_0 and such that γ^{t_0} is analytic on $(t_0 - \varepsilon_{t_0}, t_0 + \varepsilon_{t_0}) \setminus (\cup_i \{t_i\})$.

Proof of Proposition 7. With Lemma 8 in hand we may consider a covering of (λ_1, β) by

$$\cup_{t_0\in(\lambda_1,\beta)} \left(t_0 - \frac{1}{4}\varepsilon_{t_0}, t_0 + \frac{1}{4}\varepsilon_{t_0}\right)$$

and extract a locally finite covering, *i.e.* we can find an increasing sequence $(t_0^j)_{j\in\mathbb{Z}}$ such that

$$\lim_{j\to-\infty} t_0^j = \lambda_1\ , \quad \lim_{j\to+\infty} t_0^j = \beta\ ,$$

and such that

$$(\lambda_1, \beta) = \cup_{j\in\mathbb{Z}} \left(t_0^j - \frac{1}{4}\varepsilon_{t_0^j}, t_0^j + \frac{1}{4}\varepsilon_{t_0^j}\right)\ .$$

With the choice $\frac{1}{4}\varepsilon_{t_0}$ in place of the natural choice ε_{t_0}, we can insure that for every $j \in \mathbb{Z}$, we have either $t_0^{j+1} \in (t_0^j, t_0^j + \varepsilon_{t_0^j})$ or $t_0^j \in (t_0^{j+1} - \varepsilon_{t_0^{j+1}}, t_0^{j+1})$. In both cases, $\gamma^{t_0^j}$ and $\gamma^{t_0^{j+1}}$ are defined on the same interval $[t_0^j, t_0^{j+1}]$. From this remark and the fact that $m^{t_0^j}(t_0^j) = m(t_0^j)$ and $m^{t_0^{j+1}}(t_0^{j+1}) = m(t_0^{j+1})$, we deduce that the graphs of $m^{t_0^j}$ and $m^{t_0^{j+1}}$ have to intersect at least in one point in $[t_0^j, t_0^{j+1}]$. Let us choose for a point $s^{j+\frac{1}{2}}$ defined by

$$s^{j+\frac{1}{2}} = \inf\left\{ s \in [t_0^j, t_0^{j+1}] \ : \ m^{t_0^j}(s) = m^{t_0^{j+1}}(s) \right\} .$$

Now we can define the functions m^ε, k^ε on $[t_0^j, t_0^{j+1}]$ as

$$(m^\varepsilon(t), k^\varepsilon(t)) = \begin{cases} (m^{t_0^j}(t), k^{t_0^j}(t)) & \text{if} \quad t \in (t_0^j - \varepsilon_{t_0^j}, s^{j+\frac{1}{2}}] \\ \\ (m^{t_0^{j+1}}(t), k^{t_0^{j+1}}(t)) & \text{if} \quad t \in [s^{j+\frac{1}{2}}, t_0^{j+1} + \varepsilon_{t_0^{j+1}}) \end{cases}$$

Applying Lemma 6 and 7 of [21], we get that $m^\varepsilon, k^\varepsilon$ has to satisfy the system (7) on $(t_0^j - \varepsilon_{t_0^j}, t_0^{j+1} + \varepsilon_{t_0^{j+1}})$.

Repeating this argument on each interval (t_0^j, t_0^{j+1}), we can extend the definition of the functions m^ε, k^ε to the interval (λ_1, β). The fact that $m^\varepsilon(\lambda_1) = m_0(\lambda_1)$ and $m^\varepsilon(\beta) = m_0(\beta)$ is a consequence of (5). The fact that m^ε can be chosen such that $|m^\varepsilon - m| < \varepsilon$, is a simple consequence of the continuity of m, and the continuity of each function m^{t_0} for every $t_0 \in (\lambda_1, \beta)$. This ends the proof of Proposition 7. □

4.2 Proof of lemma 8

The proof of lemma 8 is now quite simple. We only have to distinguish between several different cases.

Let $t_0 \in (\lambda_1, \beta)$ and consider a sequence of points x_1^n belonging to the level line $\{u = t_0\}$ such that $|\nabla u(x_1^n)| \to m(t_0)$ and $u(x + x_1^n) \to u_1(x)$ locally uniformly on compact sets for some limit function u_1. Using the analyticity of u_1, we have to deal with two cases:

Case 1 There exists an $\varepsilon > 0$ s.t. $|\nabla u_1| > |\nabla u_1(0)| = m(t_0)$ on $\{u_1 = t_0\} \cap B_\varepsilon(0) \setminus \{0\}$.

Case 2 $|\nabla u_1| \equiv m(t_0)$ on $\{u_1 = t_0\}$, because of the analyticity.

In the first case, by a continuity argument we deduce that for each level t close enough to t_0, we can find a point on $\{u_1 = t\}$ which is a local minimizer of the gradient on $\{u_1 = t\}$, and which is close enough to 0. As a consequence of the caracterization of analytic sets (see Theorem 4 in [21]), we get the existence of at least two analytic curves $\gamma_+^{t_0}$, $\gamma_-^{t_0}$ whose images are contained in $\left\{F_1(x) := \frac{d}{d\tau}\left(|\nabla u_1|^2/2\right) = 0\right\}$, such that $u_1(\gamma_+^{t_0}(t)) = t$ for $t \in [t_0, t_0+\varepsilon_{t_0})$, $u_1(\gamma_\pm^{t_0}(t_0)) = 0$, and $u_1(\gamma_-^{t_0}(t)) = t$ for $t \in [t_0-\varepsilon_{t_0}, t_0)$. Defining γ^{t_0} as the "union" of $\gamma_+^{t_0}$ and $\gamma_-^{t_0}$, we can define

$$m^{t_0}(t) = |\nabla u_1(\gamma^{t_0}(t))| \quad \text{and} \quad k^{t_0}(t) = K(\gamma^{t_0}(t))$$

and these functions m^{t_0}, k^{t_0} satisfy system (7) on $(t_0-\varepsilon_{t_0}, t_0+\varepsilon_{t_0})$ as a simple consequence of [21].

In the second case we have two subcases:

a) *The curvature K defined on the level line $\{u_1 = t_0\}$ has at least a local minimum x_- and a local maximum x_+.*

Because the gradient is constant on the level line $\{u_1 = t_0\}$, the variation of the gradient with respect to the level t for t close to t_0 is driven by the curvature of the level line $\{u_1 = t_0\}$. A simple analysis like in [21], proves that one can find two analytic curves $\gamma_+^{t_0}$, $\gamma_-^{t_0}$ as in Case 1, but with the only difference that $\gamma_+^{t_0}(t_0)$ may differ from $\gamma_-^{t_0}(t_0)$.

b) *We can find a sequence x_2^n such that $u_1(x+x_2^n) \to u_2(x)$ and the curvature of the level line $\{u_2 = t_0\}$ is identically zero, i.e. $\{u_2 = t_0\}$ is a straight*

In case b), we face again two subcases. Consider the set of rational numbers $D = \mathbb{Q} \cap (t_0-\varepsilon_{t_0}, t_0+\varepsilon_{t_0}) \setminus \{t_0\}$ and denote by $(t_j)_{j\in\mathbb{N}}$ its elements. Assume that we can find an increasing sequence of indices $(j_i)_{i\geq 2}$, and sequences of functions $(u_i)_{i\geq 2}$ which are build as follows. Given a function u_i for some $i \geq 2$, we assume that there exists a sequence of points x_i^n belonging to the level set $\{u_i = t_{j_i}\}$ such that

$$u_{i+1}(x) = \lim_{n\to+\infty} u_i(x+x_i^n)$$

and

$$|\nabla u_{i+1}| \equiv \text{constant} \quad \text{on } \{u_{i+1} = t_{j_i}\}\,, \text{ which is a straight line.}$$

Assume first that the sequence $(t_{j_i})_{i\geq 2}$ has an accumulation point in t_0.

In this case, up to an infinite sequence of limits, we get the existence of a function u_∞ such that

$$|\nabla u_\infty| = c_{t_{j_i}} \quad \text{on} \quad \{u_\infty = t_{j_i}\}$$

where the sequence $(t_{j_i})_i$ has t_0 as an accumulation point. By analyticity of u_∞, we get that $F_\infty(x) = \frac{d}{d\tau}\left(|\nabla u_\infty|^2/2\right)$ is identically zero. This implies that we can take for

γ^{t_0} the straight line perpendicular to the level line $\{u_\infty = t_0\}$. Hence $k^{t_0} = 0$ and m^{t_0} is a solution of

$$\mathcal{F}(m^{t_0}\frac{dm^{t_0}}{dt}, 0, 0, (m^{t_0})^2, t) = 0$$

with $m^{t_0}(t_0) = m(t_0)$. Moreover we have $m^{t_0}(t) \geq m(t)$ by construction.

Assume now that there is no sequence $(t_{j_i})_{i\geq 2}$ with t_0 as above.

In this case we use the same covering argument as in the proof of Proposition 7, with

$$(t_0 - \varepsilon_{t_0}, t_0 + \varepsilon_{t_0}) \subset \cup_{t_j \in D}\left(t_j - \frac{1}{4}\varepsilon_{t_j}, t_j + \frac{1}{4}\varepsilon_{t_j}\right) .$$

We extract a locally finite covering except in t_0 where an accumulation may occur. As in the proof of Proposition 7, by taking for each level a minimum of the gradient on the union of these curves, we build some functions that we note again m^{t_0}, k^{t_0} and which satisfy (7) on $(t_0 - \varepsilon_{t_0}, t_0)$ and on $(t_0, t_0 + \varepsilon_{t_0})$ by construction. Using the uniform convergence of the curvature of the level lines $\{u_2 = t\}$ as t tends to t_0, we get, following Lemma 6 and 7 of [21], that m^{t_0}, k^{t_0} satisfy (7) on $(t_0 - \varepsilon_{t_0}, t_0 + \varepsilon_{t_0})$.

Note that in the statement of Lemma 8, we introduced the limit u_∞ of translations of the function u, namely $u(x + x^n)$. Actually in the above proof we considered not only this limit, but also a sequence of limits of functions given by translations. More precisely, we considered converging sequences of functions defined by $u_k^n(x + x_k^n) \to u_{k+1}(x)$. The result follows from the fact that, even after a enumerable number of extractions of limits, the resulting function can be considered as a simple limit of an appropriate diagonal subsequence.

This ends the proof of lemma 8. □

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