Monotone approximation schemes for integro-PDEs arising in optimal control theory

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Outline

Controlled Levy process $X_t = X_t^{x,\vartheta.}$,

$$dX_t = \sigma^{\vartheta_t}(X_{t-})dW_t + b^{\vartheta_t}(X_{t-})dt + \int_{\mathbb{R}^M} j^{\vartheta_t}(X_{t-}, z)\Pi(dt, dz), \quad t > 0,$$
$$X_0 = x \in \mathbb{R}^N,$$

where

- $\vartheta.$: admissible control process, values in Θ (compact)
- Π : Poisson random measure, intensity $ds \times \pi(dz)$.
- π: Levy measure, $\int_{|z|<1} |z|^2 \pi(dz) < \infty$.

Cost: $$J(x, \vartheta.) = E^{x,\vartheta.}\left[\int_0^\infty e^{-cs} f^{\vartheta_s}(X_s)ds\right], \quad c > 0.$$

Control problem:

Find minimizing (admissible) control $\theta^*_.$ such that

$$J(x, \vartheta^*_.) = \inf_{\vartheta.} J(x, \vartheta.)$$

Value function: $v(x) = \inf_{\vartheta.} J(x, \vartheta.)$

Value function v satisfies (formally) non-linear integro-PDE

$$\sup_{\vartheta\in\Theta}\left\{-\mathcal{L}^{\vartheta}u-\mathcal{I}^{\vartheta}u+cu-f^{\vartheta}(x)\right\}=0 \quad \text{in} \quad \mathbb{R}^{N}, \qquad \text{(HJB)}$$

where

$$\mathcal{L}^{\vartheta}u(x)=\frac{1}{2}\operatorname{tr}\left[\sigma^{\vartheta}(x)\sigma^{\vartheta}(x)^{T}D^{2}u(x)\right]+b^{\vartheta}(x)Du(x),$$

$$\mathcal{I}^{\vartheta}u(x)=\int_{\mathbb{R}^{M}}\left[u(t,x+j^{\vartheta}(x,z))-u(x)-\mathbf{1}_{|z|<1}\,j^{\vartheta}(x,z)Du(x)\right]\pi(dz).$$

Remarks

- A Hamilton-Jacobi-Bellman (HJB) equation.
- Convex, fully non-linear, degenerate (σ, j may be zero), and non-local equation.
- π may be singular at $z=0$ [Infinite activity]: E.g. CGMY model in Finance

$$\pi(dz)=m(z)dz, \qquad m(z)=\frac{c_{-}e^{-\lambda_{-}|z|}}{|z|^{1+\alpha}}1_{z<0}+\frac{c_{+}e^{-\lambda_{+}|z|}}{|z|^{1+\alpha}}1_{z>0}.$$

- Fractional Laplacian: $\mathcal{I}u=-(-\Delta)^{\frac{\alpha}{2}}u$ if $j\equiv z$, $\pi(dz)=|z|^{-N-\alpha}dz$, $\alpha\in(0,2)$.

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Assume:

(A1) π positive Radon measure, $\pi(\{0\}) = 0$, for some $l > 0$,

$$\int_{|z|<1} |z|^2 \pi(dz) + \int_{|z|>1} e^{l|z|} \pi(dz) < \infty.$$

(A2) For K independent of ϑ and $|\phi|_1 = \|D_x\phi\|_{L^\infty} + \|\phi\|_{L^\infty}$,

$$|\sigma|_1 + |b|_1 + |f|_1 \leq K \quad \text{and} \quad |j(\cdot, z)|_1 \leq Ke^{l|z|}(|z| \wedge 1).$$

Theorem

(a) For any $c > 0$, (HJB) has unique bounded Hölder continuous viscosity solution u.
(b) If c big enough, then u Lipschitz continuous.
(c) Comparison principle: $u \leq v$ whenever u, v bounded viscosity sub-, supersolutions of (HJB).

Proof: Standard: Doubling of variables + IPDE maximum principle for semicontinuous functions (the matrix lemma) + ideas of Ishii and Soner. *[Barles-Buckdahn-Pardoux 94, Pham 98, J-Karlsen 05]*
Note: IPDE version of the matrix lemma: *[J-Karlsen 06].*

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Introduction: Well-posedness for HJB equation

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Numerical methods:

- Monte Carlo (high dimensions)
- Finite Difference, Finite Element, semi Lagrangian Methods (low dimensions)
- Quadrature, FFT.

Remarks:

- Linear problems correspond to option pricing in Finance.
  Large and rappidly increasing litterature.
- Non-linear problems: Few papers.
- Problems with singular Levy measures π: Few papers.
  In many modern stock marked models π is singular!
  When π singular at $z = 0$ one often truncate π to get bounded measure.

Some references:

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Reduction to bounded Levy measure with compact support

Truncate Levy measure, compensate by adding small diffusion:

$$\sup_{\vartheta\in\Theta}\Big\{-\operatorname{tr}[\bar a^{\vartheta}(x)D^2u]-\bar{\mathcal{I}}^{\vartheta}u(x)-\bar b^{\vartheta}(x)Du+cu-f^{\vartheta}(x)\Big\}=0 \quad \text{in} \quad \mathbb{R}^N, \quad (\text{HJB}_T)$$

where $\pi_{r,R}(dz):=1_{r<|z|<R}\,\pi(dz)$,

$$\bar a^{\vartheta}(x)=\frac{1}{2}\sigma^{\vartheta}(x)\sigma^{\vartheta}(x)^T+\frac{1}{2}\int_{0<|z|\le r}j^{\vartheta}(x,z)j^{\vartheta}(x,z)^T\,\pi(dz),$$

$$\bar b^{\vartheta}(x)=b^{\vartheta}(x)+\int 1_{|z|<1}j^{\vartheta}(x,z)\pi_{r,R}(dz),$$

$$\bar{\mathcal{I}}^{\vartheta}u(x)=\int\Big[\phi(x+j^{\vartheta}(x,z))-\phi(x)\Big]\,\pi_{r,R}(dz).$$

Error bound: $|\bar{\mathcal{I}}\phi+(\bar b-b)D\phi+\operatorname{tr}[(\bar a-a)D^2\phi]-\mathcal{I}\phi|\le K\|D^3\phi\|_{L^\infty}\|\int_{|z|<r}|j|^3\pi\|_{L^\infty}$

Assume (A1), (A2),

(A3) $\pi(dz)=m(z)dz$ for $0\le m(z)\le K(|z|^{-\alpha-N}1_{|z|<1}+e^{-(l+\varepsilon)|z|}1_{|z|>1})$, $\alpha\in(0,2)$

Theorem: (J-Karlsen-LaChioma)

If $u,\bar u$ Lipschitz and solve (HJB), (HJB_T), then $|u-\bar u|_0\le C(r^{1-\alpha/3}+e^{-\ell R})$.

Remark: Adding small diffusion improves rate from $1-\alpha/2$ to $1-\alpha/3$.

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Unless otherwise stated, from now on:

- Levy measures are bounded with compact support
- HJB equation is (HJB$_T$)

Define:

$\lambda = \pi(\mathbb{R}^M) < \infty$ total mass
h= time step
$\{N_n\}_n$= jump times (i.i.d., $h\lambda$ exponential distribution)
$\{Z_n\}_n$= jump size and direction (i.i.d., distribution π)
$\{\xi_n\}_n$= random walk, (i.i.d., $\xi_n = \pm e_k, k = 1, \ldots, d$ with probability $1/2d$).

Approximative discrete time control problem:

$$\begin{cases} X_0 = x & n = 0 \\ X_n = X_{n-1} + hb^{\vartheta_{n-1}}(X_{n-1}) + \sqrt{h}\sum_{m=1}^{d} \sigma_m^{\vartheta_{n-1}}(X_{n-1})\xi_{n-1}^m & n \neq N_i, i \in \mathbb{N} \\ X_n = X_{n-1} + j^{\vartheta_{n-1}}(X_{n-1}, Z_i) & n = N_i, i \in \mathbb{N}. \end{cases}$$

Cost: $J_h(x, \vartheta.) = E^x\left[\sum_{n=0}^{\infty} he^{-nhc} f^{\vartheta_{n-1}}(X_n)\right]$

Vaue function: $u_h(x) = \inf_{\vartheta.} J_h(x, \vartheta.)$

Dynamic programming equation: (Camilli, J)

$$u_h(x) = \inf_{\theta\in\Theta}\Big\{hf(x,\theta) + e^{-hc}\Big[\frac{e^{-h\lambda}}{2d}\sum_{m=1}^{d}\Big(u_h(x + hb(x,\theta) + \sqrt{h}\sigma_m(x,\theta)) + u_h(x + hb(x,\theta) - \sqrt{h}\sigma_m(x,\theta)\Big) + \frac{1 - e^{-h\lambda}}{\lambda}\int u_h(x + j^{\vartheta}(x,z))\pi(dz)\Big]\Big\}. \tag{1}$$

Finite element like control schemes: Semidiscretization

Define:

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Finite element like control schemes: Full discretization

Define:

$\mathcal{T}_k$= non-degenerate triangulation, maximal diameter k.
X^k= vertices of $\mathcal{T}_k$.
W^k= space of continuous piecewise linear functions over $\mathcal{T}_k$.
$\beta_i(x)$= function in W^k, $\beta_i(x_j) = \delta_{ij}$ for $x_j \in X^k$ (Barycentric coordinate).

Observe:

$u \in W^k \Leftrightarrow u(x) = \sum_i \beta_i(x) u(x_i)$
β_i compact support, values in $[0,1]$, $\sum_i \beta_i \equiv 1$

Scheme: (Camilli, J)

Find function $u_{hk} \in W^k$ satisfying (1) at every vertex $x_i \in X^k$:

$$u(x_i) = \inf_{\theta \in \Theta} \Big\{ e^{-hc} \Big[e^{-\lambda h} \sum_j M_{ij}^\theta u(x_j) + (1 - e^{-\lambda h}) \sum_j P_{ij}^\theta u(x_j) \Big] + h f^\theta(x_i) \Big\}, \quad (2)$$

where

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$P_{ij}(v) = \frac{1}{\lambda} \int_{\mathbb{R}^M} \beta_j(x_i + j^\theta(x_i, z)) \pi(dz)$.

Remarks:

M^ϑ stochastic matrix, $2d(N+1)$ non-zero entries in any row.
$P_{ij}^\vartheta \geq 0$, $\neq 0$ only if x_j belongs to simplex intersecting $x_i + \cup_\vartheta j(x_i, \vartheta, \mathrm{supp}(\pi))$.

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Finite element like control schemes: Partial error bound

Assume (A1), (A2),

(A4) $|D_z j^\vartheta(x,z)| \leq Ke^{l|z|}$ and $\pi(dz) = m(z)dz$ for $0 \leq m(z)e^{l|z|} \in W^{1,1}(\mathbb{R}^M)$.

Approximation of integrals:

Replace P_{ij}^ϑ in (2) by

$$P_{ij}^{\vartheta,\Delta z} = \frac{1}{\lambda} Q_{\Delta z}[\beta_j(x_i + j^\theta(x_i,\cdot))m(\cdot)],$$

where $Q_{\Delta z}$ monotone quadrature, $|\int \phi(z)dz - Q_{\Delta z}[\phi]| \leq \bar{K}|D\phi|_{L^1}\Delta z$.

Partiall error bound: (Camilli, J)

If u_h Lipschitz ($|u_h|_1 < \infty$) solve (1), $u_{hk\Delta z}$ solve (2) with $P^{\Delta z}$ replacing P, then

$$|u_h - u_{hk\Delta z}|_0 \leq K|u_h|_1\left(\frac{k}{h} + \Delta z\right).$$

Proof: The final scheme is monotone + a calculation.

Remarks:

- Final scheme is monotone.
- Only linear (or constant) elements preserve monotonicity.
- Gaussian and Newton-Cotes (order < 9) quadratures are monotone/positive.
- Lipschitz regularity of u_h is important!

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Only linear (or constant) elements preserve monotonicity.

Gaussian and Newton-Cotes (order < 9) quadratures are monotone/positive.

Lipschitz regularity of u_h is important!

Finite element like control schemes: Partial error bound

Assume (A1), (A2),

(A4) $|D_z j^\vartheta(x,z)| \le K e^{l|z|}$ and $\pi(dz) = m(z)dz$ for $0 \le m(z)e^{l|z|} \in W^{1,1}(\mathbb{R}^M)$.

Approximation of integrals:

Replace P_{ij}^ϑ in (2) by

$$P_{ij}^{\vartheta,\Delta z} = \frac{1}{\lambda} Q_{\Delta z}[\beta_j(x_i + j^\theta(x_i,\cdot))m(\cdot)],$$

where $Q_{\Delta z}$ monotone quadrature, $|\int \phi(z)dz - Q_{\Delta z}[\phi]| \le \bar{K}|D\phi|_{L^1}\Delta z$.

Partiall error bound: (Camilli, J)

If u_h Lipschitz ($|u_h|_1 < \infty$) solve (1), $u_{hk\Delta z}$ solve (2) with $P^{\Delta z}$ replacing P, then

$$|u_h - u_{hk\Delta z}|_0 \le K|u_h|_1\left(\frac{k}{h} + \Delta z\right).$$

Proof: The final scheme is monotone + a calculation.

Remarks:

Final scheme is monotone.

Only linear (or constant) elements preserve monotonicity.

Gaussian and Newton-Cotes (order < 9) quadratures are monotone/positive.

Lipschitz regularity of u_h is important!

- To obtain full error bounds for the scheme not so easy.
- We use: Regularization argument + Krylov's method of shaking the coefficients.
- Cruicial, most difficult step: Lipschitz bound for u_h.

Finite element like control scheme: Lipschitz bound

$$u_h = \inf_{\vartheta} \Big\{ hf + e^{-hc}\Big[\frac{e^{-h\lambda}}{2d} \sum_{m=1}^{d} (u_h(\cdot + hb + \sqrt{h}\sigma_m) + u_h(\cdot + hb - \sqrt{h}\sigma_m))$$
$$+ \frac{1-e^{-h\lambda}}{\lambda} \int u_h(\cdot + j)\pi(dz) \Big] \Big\}.$$

Assume (A1), (A2) and (A5) $= \pi(\mathbb{R}^M)h \leq K$.

Lipschitz bound (Camilli, J)

$|u_h(x) - u_h(y)| \leq C|x-y|$ where C depend on K but not on h.

Remarks:

Unbounded meaures: $\pi_{r,R}h \leq K$ if $r \to 0$ slow enough when $h \to 0$.
Direct arguments does not work. Need doubling of variables:

$$\Psi(x,y;h,\varepsilon) = u_h(x) - u_h(y) \underbrace{- K_1\varepsilon - K_2\frac{1}{\varepsilon}|x-y|^2}_{-\phi(x,y)}$$

If $\sup\Psi \leq 0$ for K_1, K_2 big enough, then for all x, y, by minimizing w.r.t ε

$$u_h(x) - u_h(y) \leq K_1\varepsilon + K_2\frac{1}{\varepsilon}|x-y|^2 \sim K|x-y|.$$

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Finite element like control scheme: Proof of Lipschitz bound

Proof of $\sup \Psi \leq 0$ **when** $b \equiv 0$**:**

1. Let $a = \sqrt{h}\sigma_m$, rewrite scheme:

$$(1 - e^{-hc})u =$$

$$\inf_\theta \Big\{ hf + e^{-hc} \Big[\frac{e^{-h\lambda}}{2d} \sum_{m=1}^{d} (u(\cdot + a) - 2u + u(\cdot - a)) + \frac{1 - e^{-h\lambda}}{\lambda} \int (u(\cdot + j) - u)\pi \Big] \Big\}.$$

2. Subtract schemes ($1 - e^{-hc} \approx hc$)

$$hc(u(x) - u(y))$$
$$\leq \sup\{h(f^x - f^y) \ldots [u(x + a^x) - 2u + u(x - a^x)] - [u(y + a^y) - 2u + u(y - a^y)] \ldots\}$$

3. Let $\bar{x}, \bar{y}$ be maximum point of Ψ, then

$$2\Psi(\bar{x}, \bar{y}) \geq \Psi(\bar{x} + a^x, \bar{y} + a^y) + \Psi(\bar{x} - a^x, \bar{y} - a^y).$$

A rewrite + parallellogram law

$$[2u(\bar{x}) - u(\bar{x} + a^x) - u(\bar{x} - a^x)] - [2u(\bar{y}) - u(\bar{y} + a^y) - u(\bar{y} - a^y)]$$

$$\leq 2\phi(x, y) - \phi(\bar{x} + a^x, \bar{y} + a^y) - \phi(\bar{x} - a^x, \bar{y} - a^y) = \frac{2}{\varepsilon} K_2 |a^x - a^y|^2$$

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Finite element like control scheme: Proof of Lipschitz bound

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Finite element like control scheme: Proof of Lipschitz bound

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Finite element like control scheme: Difficult error bound

Assume:

- π bounded ($\pi(\mathbb{R}^M) < \infty$), (A1), (A2).
- u solve (HJB), u_h solve semidiscrete scheme (1).
- u, u_h Lipschitz.

Consistency error of (semidiscrete) scheme:

$$K(\|D^2\phi\| + \|D^3\phi\| + \|D^4\phi\|)h$$

Lipschitz bounds and Krylov regularization:

$$|u - u_h| \leq K[(\varepsilon^{-1} + \varepsilon^{-2} + \varepsilon^{-3})h + \varepsilon]$$

Theorem: (Camilli, J)

(a) (General IPDEs) $|u - u_h|_0 \leq Ch^{1/4}$

(b) (1st order IPDEs) If in addition $\sigma \equiv 0$, then $|u - u_h|_0 \leq Ch^{1/2}$

(c) (Pure 2nd order IPDEs) If in addition $b \equiv 0$, $k^2 = h$, then $|u - u_h|_0 \leq Ck^{1/2}$

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Finite element like control scheme: Difficult error bound II

Assume:

- (A1), (A2), (A3) i.e. π unbounded: $\pi = mdz$, $m \leq K|z|^{-\alpha-N}, |z| < 1, \alpha \in (0,2)$
- u solve (HJB), u_h solve semidiscrete scheme (1).
- u, u_h Lipschitz.

Consistency error of (semidiscrete) scheme:

$$C_1 h(\|D^2\phi\| + \|D^3\phi\| + \|D^4\phi\|) + C_2 h \int \pi_{r,R}\big[(1 + \int |z|\pi_{r,R})\|D\phi\| + \|D^2\phi\|\big]$$

where $\pi_{r,R}(dz) = 1_{r<|z|<R}\pi(dz)$. From (A3)

$$\int \pi_{r,R} \leq Kr^{-\alpha} \qquad \text{and} \qquad \int |z|\pi_{r,R} \leq Kr^{1-\alpha}$$

Lipschitz bounds and Krylov regularization:

$$|\bar{u} - u_h| \leq K[h(\varepsilon^{-1} + \varepsilon^{-2} + \varepsilon^{-3}) + h(r^{-\alpha}(1 + \varepsilon^{-1}) + r^{1-2\alpha})' + \varepsilon]$$

where $\bar{u}$ solve truncated equation (HJB_T). Previously

$$|u - \bar{u}| \leq C(r^{1-\alpha/3} + e^{-\ell R}).$$

Theorem: (Camilli, J)

(a) $\alpha \in [0,1)$: There exist $r = r(h)$, $R = R(h)$ such that $|u - u_h| \leq Ch^{1/4}$.

(b) $\alpha \in [1,2)$: There exist $r = r(h)$, $R = R(h)$ such that $|u - u_h| \leq Ch^{\frac{3-\alpha}{3+5\alpha}}$.

Finite element like control scheme: Difficult error bound II

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(a) $\alpha \in [0,1)$: There exist $r = r(h)$, $R = R(h)$ such that $|u - u_h| \leq Ch^{1/4}$.

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Finite element like control scheme: Difficult error bound II

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- To obtain error estimates: Regularization argument + "shaking of coefficients".
- Cruicial, most difficult step: Lipschitz bound for u_h (already considered).
- I will outline general results due to J., Karlsen, and La Chioma.

General error estimates: Upper bound

Equation

$$F(x,u,Du,D^2u,u(\cdot)) := \sup_{\vartheta\in\Theta}\left\{-\mathcal{L}^\vartheta u - \mathcal{I}u + cu - f^\vartheta(x)\right\} = 0 \qquad \text{(HJB)}$$

is approximated by

$$S(h_1,h_2,x,u_h,[u_h]_x) = 0. \qquad \text{(Scheme)}$$

Approximation parameters h_1 [differential], h_2 [integral].

Assumptions:

(B1) $\mu \geq 0$, $u \leq v \implies S(h,x,r+\mu,u+\mu) \geq S(h,x,r,v) + c\mu$.

(B2) $S(h,x,r,[\phi]_x)$ uniformly continuous in r uniformly in x, cont. in x for cont. ϕ.

(B3) $\|D^i\phi\|_\infty \leq K\varepsilon^{1-i} \implies |F(x,\phi,D\phi,D^2\phi,\phi(\cdot)) - S(h,x,\phi,[\phi])| \leq E(h_1,h_2,\varepsilon^{-1})$

Theorem (J., Karlsen, La Chioma)

If u Lipschitz, then $u - u_h \leq \min_{\varepsilon>0}\left\{C\varepsilon + E(h_1,h_2,\varepsilon^{-1})\right\}$.

Remarks:

- Typical consistency error $E(h_1,h_2,\varepsilon^{-1})$:

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  Then $u - u_h \leq C(h_1^{\min \beta_i/i} + h_2^{\min \bar{\beta}_i/i})$
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Step 3 v Lipschitz $\Rightarrow \|D^i v_\varepsilon\|\le\|Dv\|\varepsilon^{1-i}$. Consistency (B3) + Step 2 implies

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Step 4 Continuous dependence for (HJB) *(J-Karlsen 05)*: $\|u-v\|\le C\varepsilon$.

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$$u-u_h=u-v_\varepsilon+v_\varepsilon-u_h\le C\varepsilon+E(h_1,h_2,\varepsilon^{-1}).$$

General error estimates: Proof of upper bound

Step 1 (B1) and (B2) $\Longrightarrow$ Comparison principle for bounded solutions of (Scheme).

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General error estimates: Lower bound

$$\sup_{\vartheta\in\Theta}\Big\{-\mathcal{L}^{\vartheta}u-\mathcal{I}^{\vartheta}u+\lambda u-f^{\vartheta}(x)\Big\}=0 \qquad \text{(HJB)}$$

$$S(h_1,h_2,x,u_h,[u_h]_x)=0 \qquad \text{(Scheme)}$$

Assume:

(A1) , (A2), (B1) – (B3)

(C1) $S(h_1,h_2,x,r,[u]_x)$ convex in $(r,[u]_x)$, $[u]_x$ commutes with translations in x.

(C2) There is a unique Lipschitz solution v_h satifying $\|u_h-v_h\|_\infty\leq C\varepsilon$ of

$$\sup_{|e|\leq\varepsilon} S(h_1,h_2,x+e,v_h(x),[v_h]_x)=0.$$

Note:

- (C1) + (C2) $\ldots\Longrightarrow v_h*\rho_\varepsilon$ smooth subsolution of (Scheme).
- By (C2) and arguments of proof of upper bound:

Theorem (J-Karlsen-La Chioma)

$$u-u_h\geq -C\min_{\varepsilon>0}\{C\varepsilon+E(h_1,h_2,\varepsilon^{-1})\}.$$

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Prove uniform Lipschitz bound for v_h and $\|u_h - v_h\|_\infty \leq C\varepsilon$ (continuous dependence).

- The two estimates are equivalent, from one you get the other.
- (C2) does not hold for general monotone schemes.
- (C2) hold in the same cases as for pure PDEs:

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  A class of finite difference quadrature schemes – σ indepdendent of x, π bounded. *[J-Karlsen-La Chioma] (submitted)*

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Open postdoc position

Integro-PDEs: Numerical methods, Analysis, and Applications to Finance, Research Council of Norway project 176877/V30.

1 year at *Center of Mathematics and Application* (CMA), Univ. of Oslo, Norway.

1 year at *Norwegian University of Science and Technology* (NTNU), Trondheim.

Very good conditions: Office, salary, money for travel/equipement.

Responsible: Espen R. Jakobsen (NTNU), Kenneth Karlsen (CMA).

Will be announced soon. Interested? Contact me: *erj@math.ntnu.no*