Convexity Theory for the Term Structure Equation

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Convexity Theory for the Black-Scholes Equation

Let $F(x,t) = e^{-r(T-t)} E_{x,t} g(X_T)$, where

$$dX_t = rX_t\,dt + \sigma(X_t,t)\,dW$$

Alternatively, consider the Black-Scholes equation

$$\begin{cases} F_t + \frac{1}{2}\sigma^2 F_{xx} + rxF_x - rF = 0 \\ F(x,T) = g(x). \end{cases}$$

General result:
If r is deterministic and $\sigma = \sigma(x,t)$ is Hölder(1/2) in x, measurable in t, then

$$g(x) \text{ convex} \implies F(x,t) \text{ convex in } x \text{ for any } t < T.$$

Several references:

- Merton: Theory of rational option pricing (1973).
- Bergman, Grundy, Wiener: General properties of option prices (1996).
- El Karoui, Jeanblanc-Picque, Shreve: Robustness of the Black-Scholes formula (1998).
- Hobson: Volatility misspecification, option pricing and superreplication via coupling (1998).
- Janson, Tysk: Volatility time and properties of option prices (2003).

Why this interest in convexity?

1. Convexity is a fundamental property.
2. Parameter monotonicity: if convexity is preserved, then the price is increasing in the volatility.
3. Robustness: a delta hedger overestimating the volatility obtains a superhedge for the claim.

The Term Structure Equation

Consider

$$u(x,t) = E_{x,t}\left[e^{-\int_t^T X_s\,ds} g(X_T)\right],$$

where the interest rate X is modelled under the pricing measure as

$$dX = \beta(X_t,t)\,dt + \sigma(X_t,t)\,dW.$$

The corresponding *term structure equation* is

$$\begin{cases} F_t + \frac{1}{2}\sigma^2 F_{xx} + \beta F_x - xF = 0 \\ F(x,T) = g(x) \end{cases}$$

($g \equiv 1$ in the case of bonds).

Is convexity preserved?

Only one reference:

- Alvarez: On the form and risk-sensitivity of zero coupon bonds for a class of interest rate models (2001).

Reason 3 above to study convexity is no longer directly applicable since the short rate is not a traded asset. However, 1 and 2 remain valid.

Log-Convexity and Log-Concavity

Convexity properties of the logarithm of the bond prices are also natural to consider. They are connected with the notion of duration:

$$\text{duration} = -\frac{u_x}{u} = -(\ln u)_x.$$

(The analogous concept for options is elasticity.)
The price is log-convex if the logarithm of the price is convex, and analogously for log-concavity.

Log-convexity: The relative decline of bond prices decreases when x grows, which corresponds to a decreasing duration in x.

Log-concavity: The relative decline of bond prices increases when x grows, which corresponds to an increasing duration in x.

Statements of the Main Results

Recall that $dX = \beta(X,t)\,dt + \sigma(X,t)\,dW$ under the pricing measure.

- If $\beta_{xx} \leq 2$, then the bond prices are convex in the current short rate x, increasing in the volatility and decreasing in the drift.
- Similar results hold for call options written on bonds.
- For models with regular coefficients, the condition $\beta_{xx} \leq 2$ is also necessary for preservation of convexity.
- If β is concave and σ^2 is convex, then bond prices are log-convex.
- If β is convex and σ^2 is concave, then bond prices are log-concave.

If we demand log-concavity and log-convexity we recover the condition β and σ^2 being linear for admitting an affine term structure.

Some well-known models

Model	Dynamics	C	LCV	LCC
V	$dX = k(\theta - X)\,dt + \sigma\,dB$	Yes	Yes	Yes
CIR	$dX = k(\theta - X)\,dt + \sigma\sqrt{X}\,dB$	Yes	Yes	Yes
D	$dX = bX\,dt + \sigma X\,dB$	Yes	Yes	No
EV	$dX = X(\eta - a\ln X)\,dt + \sigma X\,dB$	Yes	Yes	No
HW	$dX = k(\theta_t - X)\,dt + \sigma\,dB$	Yes	Yes	Yes
BK	$dX = X(\eta_t - a\ln X)\,dt + \sigma X\,dB$	Yes	Yes	No
MM	$dX = X(\eta_t - (\lambda - \frac{\gamma}{1+\gamma t})\ln X)\,dt + \sigma X\,dB$	Yes	Yes	No

Table: Vasicek, Cox-Ingersoll-Ross, Dothan, Exponential Vasicek, Hull-White, Black-Karinski, Mercurio-Moraleda. The parameters are positive and $\lambda \geq \gamma$.

Where does the condition $\beta_{xx} \leq 2$ come from?

After a change of variables $t \to T - t$ we have

$$\begin{cases} u_t = \alpha u_{xx} + \beta u_x - xu \\ u(x,0) = g(x) \end{cases} \qquad (\alpha = \frac{1}{2}\sigma^2)$$

Assume all coefficients are regular enough and that convexity is about to be lost, i.e. that $u(x,t)$ is convex for $0 \leq t \leq t_0$ and $u_{xx}(x_0, t_0) = 0$. Then

$$\begin{aligned} \partial_t u_{xx} &= \partial_x^2 u_t = \partial_x^2(\alpha u_{xx} + \beta u_x - xu) \\ &= \alpha u_{xxxx} + (2\alpha_x + \beta) u_{xxx} + (\alpha_{xx} + 2\beta_x - x) u_{xx} + (\beta_{xx} - 2) u_x \end{aligned}$$

Since $x \mapsto u_{xx}(x, t_0)$ has a minimum at $x = x_0$ we have $u_{xxxx} \geq u_{xxx} = u_{xx} = 0$ at (x_0, t_0).

We find

$$\partial_t u_{xx} \geq (\beta_{xx} - 2)u_x \geq 0$$

provided that $\beta_{xx} \leq 0$ (since $u_x \leq 0$).

This suggests that convexity is preserved if $\beta_{xx} \leq 2$. Theorem 5.1of the paper makes this argument rigorous: Convexity is preserved for decreasing pay-off functions if $\beta_{xx} \leq 2$.

Parameter Monotonicity

Assume that $\beta(x,t) \le \tilde{\beta}(x,t)$ and $|\sigma(x,t)| \ge |\tilde{\sigma}(x,t)|$. Let

$$dX_t = \beta(X_t,t)\,dt + \sigma(X_t,t)\,dW,$$

$$d\tilde{X}_t = \tilde{\beta}(\tilde{X}_t,t)\,dt + \tilde{\sigma}(\tilde{X}_t,t)\,dW$$

and define

$$u(x,t) = E_{x,t}\left[e^{-\int_t^T X_s\,ds} g(X_T)\right],$$

$$\tilde{u}(x,t) = E_{x,t}\left[e^{-\int_t^T \tilde{X}_s\,ds} g(\tilde{X}_T)\right].$$

If either $\beta_{xx} \le 2$ or $\tilde{\beta}_{xx} \le 2$ and g is convex and decreasing, then $\tilde{u}(x,t) \le u(x,t)$.

Assume that (x_0, t_0) is a first point where $\tilde{u} \le u$ is about to be lost, i.e. $\tilde{u}(x,t) \le u(x,t)$ for all $0 \le t \le t_0$ and $\tilde{u}(x_0,t_0) = u(x_0,t_0)$. Then

$$\partial_t(u - \tilde{u}) = (\alpha u_{xx} + \beta u_x - xu) - (\tilde{\alpha}\tilde{u}_{xx} + \tilde{\beta}\tilde{u}_x - x\tilde{u}).$$

Since $x \mapsto u(x,t_0) - \tilde{u}(x,t_0)$ has a minimum at $x = x_0$, we have $(u-\tilde{u})_{xx} \ge (u-\tilde{u})_x = u - \tilde{u} = 0$ at (x_0,t_0). Thus

$$\partial_t(u - \tilde{u}) = \alpha u_{xx} - \tilde{\alpha}\tilde{u}_{xx} + (\beta - \tilde{\beta})u_x \ge 0$$

provided u_{xx} or $\tilde{u}_{xx}$ is non-negative (since $u_x \le 0$).

This indicates that the inequality $\tilde{u} \le u$ is preserved. The argument is made precise in Theorem 6.1.

Bond call options

The price of a bond call option is

$$C(x,t;T_1,T_2) = E_{x,t}\left[e^{-\int_t^{T_1} X_s\,ds}(u(X_{T_1},T_1)-K)^+\right],$$

where u is the pricing function of a T_2-bond.

Theorem

Assume that $\beta_{xx} \leq 2$. Then the bond option price is convex in x at all times $t \leq T_1$. Moreover, decreasing the drift and increasing the volatility gives a higher option price.

Proof.

It follows from Theorem 5.1 that $u(x,T_1)$ is convex and decreasing. Therefore $C(x,T_1;T_1,T_2) = (u(x,T_1)-K)^+$ is also convex and decreasing. Convexity follows from another application of Theorem 5.1.

Parameter monotonicity is similar. $\square$

Log-Convexity and Log-Concavity

Let $F = \ln u(x,t)$. Then F satisfies the non-linear equation

$$\begin{cases} F_t = \alpha F_{xx} + \alpha F_x^2 + \beta F_x - x \\ F(x,0) = 0. \end{cases}$$

Assume that (x_0, t_0) is a first point where convexity is about to be lost, i.e. $x \mapsto F(x,t)$ is convex for $0 \le t \le t_0$ and $F_{xx}(x_0, t_0) = 0$. Then

$$\begin{aligned} \partial_t F_{xx} &= \partial_x^2 F_t = \partial_x^2(\alpha F_{xx} + \alpha F_x^2 + \beta F_x - x) \\ &= \alpha F_{xxxx} + (2\alpha_x + \beta) F_{xxx} + (\alpha_{xx} + 2\beta_x) F_{xx} + \beta_{xx} F_x \\ &\quad + \alpha_{xx} F_x^2 + 4\alpha_x F_x F_{xx} + 2\alpha(F_x F_{xxx} + F_{xx}^2). \end{aligned}$$

Again, $F_{xxxx} \geq F_{xxx} = F_{xx} = 0$ at (x_0, t_0).

We obtain

$$\partial_t F_{xx} \geq \beta_{xx} F_x + \alpha_{xx} F_x^2,$$

which suggests that β concave and α convex implies F being convex.

Similarly, β convex and α concave implies F being concave.

These results are made precise in Theorems 8.1, 9.1 and 9.3.

More about the precise assumptions

$$dX_t = \beta(X_t, t)\,dt + \sigma(X_t, t)\,dW$$

where $\beta, \sigma : \mathbb{R} \times [0, T] \to \mathbb{R}$ are continuous functions, β is locally Lipschitz in x and σ is locally Hölder(1/2) in x. Moreover,

$$|\sigma(x, t)| \leq D(1 + x^+)$$

$$|\beta(x, t)| \leq D(1 + |x|).$$

The bound on σ implies that bond prices are finite.
The pay-off functions we consider satisfy

$$0 \leq g(x) \leq M \max\{e^{-Kx}, 1\}.$$

- If g is as above, then the corresponding option price is finite.
- Continuity in the model parameters: if $\beta_n \to \beta$ and $\sigma_n \to \sigma$ uniformly on compact sets with uniform bounds on the growth, then

$$\lim_{n\to\infty} u^n(x,t) = u(x,t)$$

  (this follows using a result by Bahlali, Mezerdi, Ouknine (2001)).

Technical Remarks

We often study the function V that solves the parabolic problem

$$\begin{cases} V_t = \alpha V_{xx} + \beta V_x - fV \\ V(x,0) = g(x). \end{cases}$$

This corresponds to the stochastic representation

$$V(x,t) = E_{x,t}\left[e^{-\int_t^T f(X_s)\,ds} g(X_T)\right].$$

Here

$$f(x) = \begin{cases} x & \text{if } x \le K \\ \text{constant} & \text{if } x > K. \end{cases}$$

We then define

$$W(x,t) = e^{f(x)h(t)} V(x,t)$$

where $h(t) = (e^{Dt} - 1)/D + Ke^{Dt}$.

The function W is shown to be a bounded solution of the equation

$$\begin{cases} W_t = \alpha W_{xx} + \hat{\beta} W_x + \gamma W \\ W(x,0) = \hat{g}(x). \end{cases}$$

where

$$\begin{aligned} &\hat{\beta} = \beta - 2 f_x \alpha h, \\ &\gamma = (Df - f_x \beta) h + f_x^2 \alpha h^2 - f_{xx} \alpha h \\ &\hat{g} = e^{Kf(x)} g(x). \end{aligned}$$

Standard theory can then be applied to estimate the derivatives of W.

Thank you for your attention!