Dislocation dynamics : a non-local moving boundary

P. Cardaliaguet, F. Da Lio, N. Forcadel, and R. Monneau

Abstract. In this article, we present briefly the mathematical study of the dynamics of line defects called dislocations, in crystals. The mathematical model is an eikonal equation describing the motion of the dislocation line with a velocity which is a non-local function of the whole shape of the dislocation. We present some partial existence and uniqueness results. Finally we also show that the self-dynamics of a dislocation line at large scale is asymptotically described by an anisotropic mean curvature motion.

1. Introduction

1.1. What are dislocations ?

The crystal defects called dislocations are lines whose typical length in metallic alloys is of the order of $10^{-6}m$, with thickness of the order of $10^{-9}m$ (see Figure 1 for an example of observations of dislocations by electron microscopy).

In the face centered cubic structure, dislocations move at low temperature in well defined crystallographic planes (the slip planes), at velocities of the order of 10 ms^{-1}. We refer for instance to Hirth and Lothe [17] for a description at the atomic level of these dislocations.

The concept of dislocations has been introduced and developed in the XXth century, as the main microscopic explanation of the macroscopic plastic behaviour of metallic crystals (see for instance the physical monographs Nabarro [20], Hirth and Lothe [17], or Lardner [19] for a mathematical presentation). Since the beginning of the 90's, the research field of dislocations has enjoyed a new boom based on the increasing power of computers, allowing simulations with a large number of dislocations (see for instance Kubin et al. [18]). This simultaneously motivated new theoretical developments for the modelling of dislocations. Recently Rodney, Le Bouar and Finel introduced in [21] a new model that we present and study mathematically in this paper. We also refer the reader to [6] and the references therein for a more detailed introduction to dislocations dynamics. This model has

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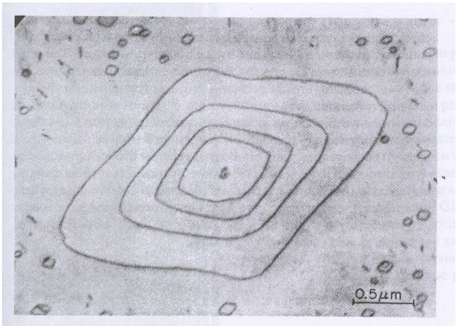

FIGURE 1. Dislocations in a Al-Mg alloy (from [23])

also been numerically studied by Alvarez, Carlini, Monneau and Rouy in [3] and [4]; see also Alvarez, Carlini, Hoch, Le Bouar and Monneau [2]

1.2. Mathematical modelling of dislocations dynamics

An idealization consists in assuming that the thickness of these lines is zero, and in the case of a single line, in assuming that this line is contained and moves in the $x = (x_1, x_2)$ plane. The motion of the line Γ_t (where the subscript t denotes the time) is simply given by the normal velocity c (see Figure 2).

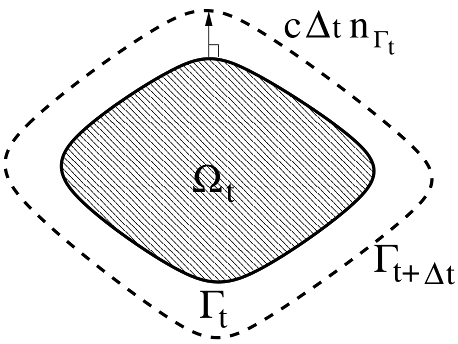

FIGURE 2. Schematic evolution of a dislocation line Γ_t by normal velocity c between the times t and $t + \Delta t$ with unit normal n_{Γ_t}.

The velocity c is proportional to the shear stress in the material. This stress can be computed solving the equations of linearized elasticity where the shape of the dislocation line appears as a source term. This gives a coupled system where the dislocation line evolution is a function of the velocity c, and the velocity c is a function of the dislocation line Γ_t itself. In the case of a single dislocation line it is possible to write the velocity c as a non-local quantity depending on the whole shape of the dislocation line (see Alvarez et al. [6]):

$$c(x,t) = (c_0 \star \rho(\cdot,t))\,(x) + c_1(x,t)$$

where ρ is the characteristic function of an open set $\Omega_t \subset \mathbb{R}^2$ whose the boundary is the dislocation line $\Gamma_t = \partial\Omega_t$:

$$\rho(x,t) = 1_{\Omega_t} := \begin{cases} 1 & \text{if} \quad x \in \Omega_t \\ 0 & \text{if} \quad x \in \mathbb{R}^2 \backslash \Omega_t, \end{cases}$$

and $c_0(x)$ is a given kernel depending on the material. Here the convolution is only done in space on $\mathbb{R}^2$.

It can be easily checked (at least formally), that the evolution on the time interval $(0,T)$ of the dislocation line Γ_t is described by the equation of dislocations dynamics:

$$(1.1) \qquad \begin{cases} \dfrac{\partial \rho}{\partial t} = (c_0 \star \rho + c_1)\ |D\rho| \quad \text{on} \quad \mathbb{R}^2 \times (0,T) \\ \\ \rho(\cdot,0) = \rho^0(\cdot) := 1_{\Omega_0} \quad \text{on} \quad \mathbb{R}^2 \end{cases}$$

where Ω_0 is an open set whose boundary $\Gamma_0 = \partial\Omega_0$ is the position of the dislocation line at initial time $t = 0$.

In what follows, we will study this equation in the framework of discontinuous voscosity solutions (see Barles [7] for an introduction to this notion). To simplify the presentation we will state results in dimension $n = 2$, assuming smooth (C^∞) regularity of the initial position Γ_0 of the dislocation line, of the kernel c_0, and of the velocity c_1.

We also assume the following behaviour of the kernel at infinity (for some function g)

$$(1.2) \qquad c_0(x) = \frac{1}{|x|^3}\ g\left(\frac{x}{|x|}\right) \quad \text{for} \quad |x| \geq 1$$

which is a natural assumption for dislocations.

For considerably weakened assumptions and in any dimensions n, we refer the reader to the original articles cited in the references.

1.3. Organization of the paper

Altought equation (1.1) seems very simple, general results of existence and uniqueness are unkown up to our knowledge. Technically, the main difficulty comes from the fact that we have no sign conditions on the kernel c_0, and then that there is no inclusion principle for this evolution.

In this paper we present some partial results. In section 2, we give a short time existence (and uniqueness) result for a smooth initial dislocation loop. In section 3, we give a long time existence (and uniqueness) result for a smooth initial curve with non-negative velocity. Finally in section 4, we consider the “monotone case” where the kernel satisfies $c_0 \geq 0$. In this particular case, a Slepčev “level sets” formulation of equation (1.1) is available. In this framework, we show that at large scales, the dislocation dynamics is asymptotically described by an (anisotropic) mean curvature motion related to the behaviour of the kernel $c_0(x)$ as $|x| \to +\infty$.

2. Short time existence results in the general case

We will make the following global assumptions on the smooth velocity $c_1(x,t)$ and the smooth kernel $c_0(x,t) := c_0(x)$, for $i = 0, 1$ and some constants M, L_0, L_1:

(2.1)
$$\left\{\begin{array}{llll} i) & |c_i(y,t)| \leq M & & \forall (y,t) \in \mathbb{R}^2 \times [0,+\infty) \\ ii) & |c_i(y_2,t) - c_i(y_1,t)| & \leq L_0|y_2 - y_1| & \forall (y_1,y_2,t) \in \mathbb{R}^2 \times \mathbb{R}^2 \times [0,+\infty) \\ iii) & |Dc_i(y_2,t) - Dc_i(y_1,t)| & \leq L_1|y_2 - y_1| & \forall (y_1,y_2,t) \in \mathbb{R}^2 \times \mathbb{R}^2 \times [0,+\infty) \end{array}\right.$$

To state our results, we first need to recall the notion of discontinuous viscosity solution. We recall that for a function ρ locally bounded on $\mathbb{R}^2 \times [0,T)$, the function ρ^* designates its upper-semicontinuous envelope (i.e. the smallest u.s.c. function $\geq \rho$), and the function ρ_* its lower semi-continuous envelope.

Definition 2.1. i)We say that a function $\rho \in C\left([0,T); L^1(\mathbb{R}^2)\right) \cap L^\infty\left(\mathbb{R}^2 \times (0,T)\right)$ is a discontinuous viscosity subsolution (resp. supersolution) of (1.1), if

$$\rho^*(\cdot,0) \leq (\rho^0)^* \quad (\text{resp.} \quad \rho_*(\cdot,0) \geq (\rho^0)_*)$$

and for every point $(\overline{x},\overline{t}) \in \mathbb{R}^2 \times (0,T)$ and every test function $\phi \in C^1(\mathbb{R}^2 \times (0,T))$ satisfying

$$\rho^* \leq \phi \quad (\text{resp.} \quad \rho_* \geq \phi) \quad \text{in} \quad \mathbb{R}^2 \times (0,T) \quad \text{and} \quad \rho^*(\overline{x},\overline{t}) = \phi(\overline{x},\overline{t}),$$

we have with $c = c_0 \star \rho + c_1$:

$$\frac{\partial \phi}{\partial t}(\overline{x},\overline{t}) \leq c(\overline{x},\overline{t})|D\phi(\overline{x},\overline{t})| \quad \left(\text{resp.} \ \frac{\partial \phi}{\partial t}(\overline{x},\overline{t}) \leq c(\overline{x},\overline{t})|D\phi(\overline{x},\overline{t})|\right)$$

ii) We say that ρ is a discontinuous viscosity solution of (1.1), if it is a discontinuous viscosity subsolution and a discontinuous viscosity supersolution.

We are now able to state the first result

Theorem 2.2. *[***Short time existence and uniqueness,** [5], [6]*]*
Let us assume (1.2)-(2.1), and that Ω_0 is a smooth bounded open set in $\mathbb{R}^2$. Then there exists a time $T^* > 0$ and let us consider functions $\rho \in C\left([0,T^*); L^1(\mathbb{R}^2)\right)$ with $0 \leq \rho \leq 1$, solutions of equation (1.1) on the interval of time $(0,T^*)$ with initial data $\rho(\cdot,0) = 1_{\Omega_0}$. Then
i) (existence) : There exists such a solution ρ.
ii) (uniqueness) : The solution is unique, where the uniqueness has the following meaning: if ρ_1 and ρ_2 are two such solutions, then $(\rho_1)^* = (\rho_2)^*$, $(\rho_1)_* = (\rho_2)_*$ and for every $t \in [0,T^*)$, $\rho_1(\cdot,t) = \rho_2(\cdot,t)$ a.e. on $\mathbb{R}^2$.

Let us remark that on the time interval $(0, T^*)$ where the Theorem is proved to hold, the solution can be written $\rho(\cdot, t) = 1_{\Omega_t}$ where Ω_t is a Lipschitz open set. Theorem 2.2 says nothing when Ω_t ceases to be a Lipschitz open set. This is for instance the case when the topology of Ω_t changes.

The proof of Theorem 2.2 is based on the application of a fixed point theorem in the framework of viscosity solutions.

Up to our knowledge, existence and uniqueness for all times (excepted in the case of non-negative velocities (see Theorem 3.1 below)) is still an open problem in general.

3. Long time existence for non-negative velocities

In this section we make the following assumption

$$c_1(y,t) \geq ||c_0||_{L^1(\mathbb{R}^2)} \quad \forall (y,t) \in \mathbb{R}^2 \times [0,+\infty) \tag{3.1}$$

Because we are interested in solutions ρ satisfying $0 \leq \rho \leq 1$, we see that condition (3.1) implies that $c = c_0 \star \rho + c_1 \geq 0$.

Theorem 3.1. **[existence and uniqueness for all time for non-negative velocity,** [1]**]** *Let us assume (1.2)-(2.1)-(3.1), and that* Ω_0 *is a smooth bounded open set in* $\mathbb{R}^2$*. Then there exists a unique function* $\rho \in C\left([0,+\infty); L^1(\mathbb{R}^2)\right)$ *with* $0 \leq \rho \leq 1$*, solution of equation (1.1) on the interval of time* $(0,+\infty)$ *with initial data* $\rho(\cdot, 0) = 1_{\Omega_0}$.

In [1], Alvarez *et al.* used a geometrical proof. A similar result was also proved by Barles and Ley [8] using a level sets approach and arguments based on a nice L^1 estimate on the level sets of the solution. We also refer to Cardaliaguet, Marchi [11] for a geometrical study of a similar problem on a bounded set in the plane with Neumann boundary conditions. The proof of Theorem 3.1 in [1] uses strongly the following monotonicity formula that we state in any dimension N:

Theorem 3.2. **[Monotonicity formula,** [1]**]**
Let K *be a compact subset of* $\mathbb{R}^N$*, and* d_K *the distance to the set* K*. Then for any* $t_2 > t_1 > 0$*, we have*

$$\frac{1}{t_2^{N-1}}\mathcal{H}^{N-1}\left(\{d_K(x) = t_2\}\right) \leq \frac{1}{t_1^{N-1}}\mathcal{H}^{N-1}\left(\{d_K(x) = t_1\}\right)$$

Here $\mathcal{H}^{N-1}$ *stands for the* $(N-1)$*-dimensional Hausdorff measure.*

Main formal arguments in the proof of Theorem 3.1
Argument 1 : interior ball condition: Let us call $R(t) > 0$ the radius of the largest ball included in Ω_t and tangent at any point of the boundary $\partial\Omega_t$. Then we can easily check (at least formally) that this radius satisfies the following ODE:

$$\dot{R} = c - R\,(n \cdot Dc) + R^2\left(D^2_{\tau\tau} c\right)$$

where n is the outward unit normal to Ω_t and τ is a tangent unit vector to $\Gamma_t = \partial\Omega_t$. Using the fact that $c \geq 0$, we deduce that

$$R(t) \geq C_1 e^{-\gamma t}$$

for some constants $C_1, \gamma > 0$.

Argument 2 : length of the dislocation: We denote by $|\Gamma_t|$ the length of Γ_t. Then using the fact that the curvature K of Γ_t satisfies $K \leq 1/R(t)$, we deduce

$$\frac{d}{dt}|\Gamma_t| = \int_{\Gamma_t} cK \leq \int_{\Gamma_t} \frac{c}{R(t)} \leq \frac{||c||_{L^\infty}}{R(t)}|\Gamma_t|$$

which gives an estimate $|\Gamma_t| \leq l(t) < +\infty$.

Argument 3 : error estimate: based on the monotonicity formula Theorem 4.2, this is possible to show that if ρ_i satisfy for $i = 1, 2$

$$\text{(3.2)} \qquad \begin{cases} \dfrac{\partial \rho_i}{\partial t} = c_i\ |D\rho_i| \quad \text{on} \quad \mathbb{R}^2 \times (0,T) \\ \\ \rho_i(\cdot, 0) = 1_{\Omega_0} \quad \text{on} \quad \mathbb{R}^2 \end{cases}$$

then we have for any t small enough and some constant $C_2 > 0$:

$$||\rho_2(\cdot,t) - \rho_2(\cdot,t)||_{L^\infty(\mathbb{R}^2)} \leq C_2\ l(t)||c_2 - c_1||_{L^\infty(\mathbb{R}^2\times(0,T))} \left(\frac{e^{L_0 t} - 1}{L_0}\right)$$

Combined with the fact that for dislocation dynamics $c_i = c_0 \star \rho_i + c_1$, we get

$$||\rho_2(\cdot,t) - \rho_2(\cdot,t)||_{L^1(\mathbb{R}^2)} \leq \alpha(t)||\rho_2 - \rho_1||_{L^\infty((0,T);L^1(\mathbb{R}^2))}$$

with $\alpha(t) = C_2\ l(t)||c_0||_{L^\infty(\mathbb{R}^2)} \left(\frac{e^{L_0 t}-1}{L_0}\right)$. This shows in particular the uniqueness of the solution for small time, which can also be used as a contraction argument for a fixed point theorem.

4. Convergence to the mean curvature motion at large scale for nonnegative kernels

In this section we assume that the kernel c_0 satisfies the following condition

$$\text{(4.1)} \qquad c_0(-x) = c_0(x) \geq 0 \quad \forall x \in \mathbb{R}^2$$

and consider solutions ρ of (1.1) with $c_1 = -\frac{1}{2}\int_{\mathbb{R}^2} c_0$. This particular choice of c_1 insures the equilibrium of straight dislocations lines and is physically relevant for the description at large scales of isolated dislocations lines without exterior stress.

In this section, we are interested in the dynamics of dislocations lines of large diameter of the order of $1/\varepsilon$ and in the limit as $\varepsilon \to 0$. To this end, we define for $\varepsilon > 0$ the rescalled characteristic function

$$\rho^\varepsilon(x,t) = \rho\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2|\ln\varepsilon|}\right).$$

which satisfies the following equation

$$\frac{\partial \rho^\varepsilon}{\partial t} = \left(c_0^\varepsilon \star \rho^\varepsilon - \frac{1}{2}\int_{\mathbb{R}^2} c_0^\varepsilon \right) |D\rho^\varepsilon| \tag{4.2}$$

with the rescalled kernel

$$c_0^\varepsilon(x) = \frac{1}{\varepsilon^3 |\ln \varepsilon|}\; c_0\left(\frac{x}{\varepsilon}\right).$$

¿From the fact that $c_0^\varepsilon \geq 0$, it can be seen (at least formally) that equation (4.2) preserves the inclusion principle. In this section we do not study directly equation (4.2), but prefer to use the following Slepčev "level sets" formulation for a continuous function u^ε:

(4.3)

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} = \left(\left(c_0^\varepsilon \star 1_{\{u^\varepsilon(\cdot,t) > u^\varepsilon(x,t)\}} \right)(x) - \dfrac{1}{2}\displaystyle\int_{\mathbb{R}^2} c_0^\varepsilon \right) |Du^\varepsilon| \quad \text{on} \quad \mathbb{R}^2 \times (0,T) \\ \\ u^\varepsilon(\cdot,0) = u_0 \quad \text{on} \quad \mathbb{R}^2 \end{cases}$$

In this new formulation each level set $\{u^\varepsilon = \lambda\}$ represents a dislocation line associated to a function $\rho_\lambda^\varepsilon = 1_{\{u^\varepsilon > \lambda\}}$ which satisfies (4.2) (at least formally).

In the limit $\varepsilon \to 0$, this dynamics is well approximated by the following anisotropic mean curvature motion (see for instance Crandall, Ishii, Lions [12] for a definition of viscosity solutions of the second order equation (4.4)):

$$\begin{cases} \dfrac{\partial u^0}{\partial t} + F(D^2u^0, Du^0) = 0 \quad \text{on} \quad \mathbb{R}^2 \times (0,T) \\ \\ u^0(\cdot,0) = u_0 \quad \text{on} \quad \mathbb{R}^2 \end{cases} \tag{4.4}$$

with

$$F(M,p) = -g\left(\frac{p^\perp}{|p|}\right)\; \text{trace}\left(M \cdot \left(Id - \frac{p}{|p|} \otimes \frac{p}{|p|} \right) \right)$$

where g is introduced in (1.2). In particular we see that equation (4.4) describes the anisotropic mean curvature motion with velocity

$$g(\tau)\ \kappa$$

where κ is the curvature of the level line of u^0 and τ is a unit tangent vector to the level line of u^0.

Before to state our convergence result as $\varepsilon \to 0$, we need to give the precise definition of viscosity solutions we use for the non-local equation (4.3) which is less standard. This definition has been introduced by Slepčev [22] (see also Da Lio, Kim, Slepčev [13]).

Definition 4.1. (Viscosity sub/super/solution for the non-local eikonal equation) A locally bounded upper semicontinuous (usc) function u^ε is a viscosity subsolution of (4.3) if it satisfies:

(i) $u^\varepsilon(x,t=0) \le u_0(x)$ in $\mathbb{R}^2$,
(ii) for every $(x_0,t_0) \in \mathbb{R}^2 \times (0,\infty)$ and for every test function $\Phi \in C^\infty\left(\mathbb{R}^2 \times [0,\infty)\right)$, that is tangent from above to u^ε at (x_0,t_0), the following holds:

$$\Phi_t^\varepsilon(x_0,t_0) \le \left((c_0^\varepsilon \star 1_{\{u^\varepsilon(\cdot,t_0)\ge u^\varepsilon(x_0,t_0)\}})(x_0) - \frac{1}{2}\int_{\mathbb{R}^2} c_0^\varepsilon \right) |D\Phi^\varepsilon(x_0,t_0)| \tag{4.5}$$

A locally bounded lower semicontinuous (lsc) function u^ε is a viscosity supersolution of (4.3) if it satisfies:

(i) $u^\varepsilon(x,t=0) \ge u_0(x)$ in $\mathbb{R}^2$,
(ii) for every $(x_0,t_0) \in \mathbb{R}^2 \times (0,\infty)$ and for every test function $\Phi \in C^\infty\left(\mathbb{R}^2 \times [0,\infty)\right)$, that is tangent from below to u^ε at (x_0,t_0), the following holds:

$$\Phi_t^\varepsilon(x_0,t_0) \ge \left((c_0^\varepsilon \star 1_{\{u^\varepsilon(\cdot,t_0)> u^\varepsilon(x_0,t_0)\}})(x_0) - \frac{1}{2}\int_{\mathbb{R}^2} c_0^\varepsilon \right) |D\Phi^\varepsilon(x_0,t_0)| \tag{4.6}$$

A locally bounded continuous function u^ε is a viscosity solution of (4.3) if, and only if, it is a sub and a supersolution of (4.3).

Then the main result of this section is

Theorem 4.2. *[***Convergence of dislocations dynamics to mean curvature motion,** [14]*]*
There exists a constant $C_0 > 0$ only depending on $||c_0||_{L^\infty(\mathbb{R}^2)}$. Given $\varepsilon \in (0,1)$ and a bounded and globally Lipschitz continuous function u_0, there exists a unique viscosity solution $u^\varepsilon \in L^\infty_{loc}\left(\mathbb{R}^2 \times [0,+\infty)\right)$ of problem (4.3). The function u^ε satisfies

$$||Du^\varepsilon||_{L^\infty(\mathbb{R}^2\times[0,+\infty))} \le ||Du_0||_{L^\infty(\mathbb{R}^2)}$$

and for every $\varepsilon \in (0,1/2)$:

$$|u^\varepsilon(x,t+s) - u^\varepsilon(x,s)| \le C_0 ||Du_0||_{L^\infty(\mathbb{R}^2)}\sqrt{t}, \quad \forall (x,s,t) \in \mathbb{R}^2 \times [0,+\infty) \times [0,1]$$

Moreover, the solution u^ε converges locally uniformly in compact sets of $\mathbb{R}^2 \times [0,+\infty)$ to the unique solution u^0 of (4.4) with the same initial condition u_0.

Remark 4.3. In a future work, we will apply this result to propose a numerical scheme for anisotropic mean curvature motion or crystalline motion.

While the proof of this convergent result is quite simple in the case where the gradient of the limit function u^0 is non-zero, the case where the gradient of u^0 vanishes is quite delicate and requires more attention.

We will now present a further property of the limit mean curvature motion. To this end, we need the following:

Definition 4.4. Let $g \in C^\infty(\mathbb{R}^2 \setminus \{0\})$ satisfying $g(\lambda p) = \dfrac{g(p)}{|\lambda|^3}$, $\forall \lambda \in \mathbb{R}\setminus\{0\}$, $\forall p \in \mathbb{R}^2$. We then associate to g a temperate distribution L_g defined by

$$\langle L_g, \varphi\rangle = \int_{\mathbb{R}^2} dx\ \frac{g\left(\frac{x}{|x|}\right)}{|x|^3}\left(\varphi(x) - \varphi(0) - x\cdot D\varphi(0)1_{B_1(0)}(x)\right)$$

for $\varphi \in \mathcal{S}(\mathbb{R}^2)$, where $\mathcal{S}(\mathbb{R}^2)$ is the Schwarz space of test functions on $\mathbb{R}^2$, and $B_1(0)$ denotes the unit ball centered in zero.
We define the Fourier transform

$$\hat{\varphi}(\xi) = \int_{\mathbb{R}^2} dx\ \varphi(x)e^{-i\xi\cdot x}$$

Then we have

Theorem 4.5. *[***Variational origin of the anisotropic mean curvature motion,** [14]*]* *Let $g \in C^\infty(\mathbb{R}^2 \setminus \{0\})$ satisfying $g(\lambda p) = \dfrac{g(p)}{|\lambda|^3}$, $\forall \lambda \in \mathbb{R}\setminus\{0\}$, $\forall p \in \mathbb{R}^2$. Let*

$$G := -\frac{1}{2\pi}\hat{L_g} \tag{4.7}$$

where $\hat{L_g}$ is the Fourier transform of L_g. Then $G(\lambda p) = |\lambda| G(p)$, $\forall \lambda \in \mathbb{R}\setminus\{0\}$, $\forall p \in \mathbb{R}^2$, and

$$g\left(\frac{p^\perp}{|p|}\right)\ \frac{p^\perp}{|p|}\otimes\frac{p^\perp}{|p|} = D^2G\left(\frac{p}{|p|}\right) \tag{4.8}$$

In particular, we see that G is convex if and only if $g \geq 0$. Moreover (4.8) means that in (4.4), we have

$$-F(D^2u^0, Du^0) = \text{div}\ \left(\nabla G\left(\frac{Du^0}{|Du^0|}\right)\right)\ |Du^0|,$$

i.e. this anisotropic mean curvature motion derives from a convex energy $\int G\left(Du^0\right)$.

Remark 4.6. Physically the quantity $\hat{L_g}$ is naturally given, and then the function g can be computed using (4.7)-(4.8) where we can check if g is non negative or not.
In the simplest case of applications for dislocation dynamics, the crystal is described by isotropic elasticity. When the Burgers vector is along the x_1 direction, we have

$$G(p) = \frac{p_2^2 + \frac{1}{1-\nu}\ p_1^2}{|p|} \quad \text{with} \quad \nu \in (-1, \frac{1}{2})$$

where ν is the Poisson ratio of the material, and

$$g(\theta) = \frac{(2\gamma-1)(\theta_1)^2 + (2-\gamma)(\theta_2)^2}{|\theta|^5} \geq 0 \quad \text{with} \quad \gamma = \frac{1}{1-\nu} \in (\frac{1}{2}, 2).$$

Our result is very natural for dislocation dynamics. Indeed, in many references in physics, the authors describe dislocations dynamics by line tension terms deriving from an energy associated to the dislocation line. See for instance Brown [10], Barnet, Gavazza [9] for physical references and Garroni, Müller [16] for a variational approach. We also refer to Forcadel [15] for the study of dislocation dynamics with a mean curvature term. As far as we know, Theorem 4.2, completed by Theorem 4.5, is the first rigourous proof for the convergence of dislocations dynamics to mean curvature motion.

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