Steady state and long time convergence of spirals moving by forced mean curvature motion

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Abstract

In this paper, we prove the existence and uniqueness of a "steady" spiral moving with forced mean curvature motion. This spiral has a stationary shape and rotates with constant angular velocity. Under appropriate conditions on the initial data, we also show the long time convergence (up to some subsequence in time) of the solution of the Cauchy problem to the steady state. This result is based on a new Liouville result which is of independent interest.

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1 Introduction

In this paper we are interested in curves in $\mathbb{R}^2$ which are half-lines attached at the origin. These lines are assumed to move with normal velocity

$$V_n = 1 + \kappa \tag{1.1}$$

where κ is the curvature of the line. We assume that these curves Γ_t can be parametrized in polar coordinates as follows

$$\Gamma_t = \{(r\cos\theta, r\sin\theta), \quad \text{such that} \quad r \geq 0, \quad \theta = -U(t,r)\}.$$

On the one hand, the Geometric Law (1.1) holds true if U satisfies

$$U_t = (1 + \kappa_U)|\nabla U|.$$

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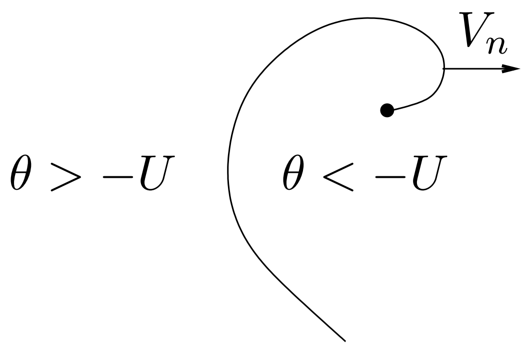

Figure 1: Motion of the spiral. Here θ is the angle in polar coordinates.

On the other hand, it is known (see for instance [10]) that the curvature of the parametrized curve Γ_t has the following form

$$\kappa_U(t,r) = U_r\left(\frac{2+(rU_r)^2}{(1+(rU_r)^2)^{\frac{3}{2}}}\right) + \frac{rU_{rr}}{(1+(rU_r)^2)^{\frac{3}{2}}}. \tag{1.2}$$

Hence, the function U has to satisfy the following quasi-linear parabolic equation in non-divergence form for $(t,r) \in (0,+\infty)\times(0,+\infty)$:

$$rU_t = \sqrt{1+r^2U_r^2} + U_r\left(\frac{2+r^2U_r^2}{1+r^2U_r^2}\right) + \frac{rU_{rr}}{1+r^2U_r^2} \tag{1.3}$$

supplemented with the following initial condition for $r \in (0,+\infty)$

$$U(0,r) = U_0(r). \tag{1.4}$$

1.1 Main results

In [10], we were able to prove an existence and uniqueness result for equation (1.3)-(1.4). We improve it by proving in particular that solutions are regular up to the boundary $r=0$.

Theorem 1.1 (Existence and uniqueness for the Cauchy problem)**.** *Assume that* $U_0 \in W^{2,\infty}_{loc}(0,+\infty)$ *is globally Lipschitz continuous and satisfies*

$$(U_0)_r \in W^{1,\infty}(0,+\infty) \quad or \quad \kappa_{U_0} \in L^\infty(0,+\infty)$$

and that there exists a radius $r_0 > 0$ *such that*

$$|1+\kappa_{U_0}| \le Cr \quad for \quad 0 \le r \le r_0.$$

Then there exists a globally Lipschitz continuous (in space and time) solution U *such that*

$$U \in C^{1+\frac{1}{6},2+\frac{1}{3}}_{t,r}((0,+\infty)\times[0,+\infty)) \cap C^\infty((0,+\infty)\times(0,+\infty)).$$

Moreover, for every $\delta > 0$*,* $R > 0$*, there exists a constant* $C = C(\delta,R)$ *such that for every* $T \ge \delta > 0$*,*

$$\|U - U(T,0)\|_{C^{1+\frac{1}{6},2+\frac{1}{3}}_{t,r}([T,T+\delta]\times[0,R])} \le C.$$

Such a solution is unique in the class of continuous viscosity solutions of (1.3)-(1.4).

Remark 1.2. In view of (1.2) and (1.3), the regularity of U stated in the previous theorem implies in particular that

$$\kappa_U + 1 = 0 \quad \text{at} \quad r = 0 \tag{1.5}$$

holds for $t > 0$.

Remark 1.3. The assumption that U_0 is globally Lipschitz was missing in the statement of Theorem 1.7 in [10]. We will recall below (see Theorem 2.1) the corrected version of this result.

Our second main result is about the existence of a spiral with stationary shape and rotating at constant speed.

Theorem 1.4 (A steady state)**.** *There exists a constant $\lambda \in \mathbb{R}$ and a globally Lipschitz function continuous Φ in $[0, +\infty)$, satisfying*

$$\lambda \geq 0 \quad \textit{and} \quad \Phi_r \leq 0 \quad \textit{on} \quad [0, +\infty)$$

such that $U(t, r) = \lambda t + \Phi(r)$ is a solution of (1.3) in $\mathbb{R} \times (0, +\infty)$. Moreover such a λ is unique and such a function Φ is unique up to addition of a constant. Moreover, the following properties hold true:

i) *we have*

$$\frac{1}{4} \leq \lambda \leq \frac{1}{2}.$$

ii) *$\Phi \in C^\infty(0, +\infty) \cap C^{2+\frac{1}{3}}([0, +\infty))$ satisfies for all $r \in [0, +\infty)$*

$$-\frac{1}{2} \leq \Phi_r \leq -\lambda, \tag{1.6}$$

$$0 \leq 1 + \kappa_\Phi \leq \lambda r. \tag{1.7}$$

Moreover Φ_r and the curvature κ_Φ are non-decreasing and

$$\Phi_r(0) = -\frac{1}{2}, \quad \Phi_r(+\infty) = -\lambda, \qquad \kappa_\Phi(0) = -1, \quad \kappa_\Phi(+\infty) = 0.$$

iii) *There exist some constants $a \in \mathbb{R}$ and $C > 0$, such that Φ satisfies for all $r \in [0, +\infty)$*

$$|\Phi(r) + \lambda r + \lambda \ln(1 + r) - a| \leq \frac{C}{1 + r}.$$

Remark 1.5. Notice that the value of the angular velocity λ have been estimated to be 0.315 by approximation in [5], and computed to be 0.330958961 by a shooting method in [26].

Our third main result is concerned with the large time behaviour of solutions of the Cauchy problem for initial data that are "reasonably close" to the steady state.

Theorem 1.6 (Long time convergence)**.** *Under assumptions of Theorem 1.1, if the initial data U_0 further satisfies*

$$|U_0 - \Phi| \leq C \tag{1.8}$$

and

$$(U_0)_r \leq \Phi_r \leq -\lambda < 0, \tag{1.9}$$

where (λ, Φ) are given in Theorem 1.4, then for any sequence $t_n \to +\infty$, there exists a subsequence (still denoted by t_n) and a constant $a \in \mathbb{R}$ such that

$$U(t + t_n, r) - (\lambda(t + t_n) + \Phi(r)) \to a \quad \textit{locally uniformly in } \mathbb{R} \times [0, +\infty).$$

Remark 1.7. The fact that convergence only happens along a subsequence of times is expected. Indeed a similar fact happens already for the linear heat equation on the real line. It is possible to cook up an initial data which stays between 0 and 1 such that the solution does not converges as times goes to infinity, but such that convergence to a constant (locally uniformly) still happens for subsequences in time.

The proof of Theorem 1.6 is based on the following Liouville result of independent interest.

Theorem 1.8 (Liouville result)**.** *Let $U(t, r)$ be a globally Lipschitz continuous function (in space and time) in $\mathbb{R} \times [0, +\infty)$. We assume that U is a global solution of* (1.3) *in $\mathbb{R} \times (0, +\infty)$ and that there exists a constant $C > 0$ such that the following holds:*

$$|U(t, r) - \lambda t - \Phi(r)| \leq C \quad \textit{on} \quad \mathbb{R} \times [0, +\infty) \tag{1.10}$$

where (λ, Φ) is given by Theorem 1.4. We also assume that there exists some $\delta > 0$ such that

$$U_r \leq -\delta < 0 \quad \textit{in} \quad \mathbb{R} \times [0, +\infty). \tag{1.11}$$

Then

$$U(t, r) = \lambda t + \Phi(r) + a$$

for some constant $a \in \mathbb{R}$.

1.2 Review of the literature

Spirals appear in several applications. Our main motivation comes from continuum mechanics. In a two dimensional space, the seminal paper of Burton, Cabrera and Frank [5] studies the growth of crystals with vapor. When a screw dislocation line reaches the boundary of the material, atoms are adsorbed on the surface in such a way that a spiral is generated; moreover, under appropriate physical assumptions, these authors prove that the geometric law governing the dynamics of the growth of the spiral is precisely given by (1.1). We mention that there is an extensive literature in physics dealing with crystal growth in spiral patterns (see for instance [31, 30]). We also want to point out that motion of spirals appear in other applications like in the modeling of the Belousov-Zhabotinsky reagent [23]. To model the appearence of such shapes, the reagent is modeled in [16] by a system of semi-linear parabolic equations; so-called spiral wave fronts satisfying the geometric law (1.1) can be formally derived. The interested reader is also referred to e.g. [22, 21, 13].

There exist different mathematical approaches to describe the motion of spirals. As far as we know, it appeared first in geometry in [1]. It was also used in order to study singularity formation [2, 3]. Other approaches have been used; for instance, a phase-field approach was proposed in [15] and the reader is also referred to [9, 25, 27]. In [11], spirals moving in (compact) annuli with homogeneous Neumann boundary condition are constructed. From a technical point of view, the classical parabolic theory is used to construct smooth solutions of the associated partial differential equation; in particular, gradient estimates are derived. We point out that in [11], the geometric law is anisotropic, and is thus more general than (1.1). In [32, 28, 29, 12], the geometric flow is studied by using the level-set approach. As in [11], the authors of [28, 29] consider spirals that typically move inside a (compact) annulus and reaches the boundary perpendicularly.

Concerning the existence of "steady" spirals (in the case where the exterior stress is zero), we refer to [14] where the construction is done by studying an ordinary differential equation and to [6] where the authors consider a two-point free boundary problem for the curvature flow equation. We also refer to [11] where they construct a steady state on an annulus using classical parabolic theory. In [26], a numerical computation of the angular velocity λ of the spirals is done. The authors find that the angular velocity is approximatively $0,330958961$ (recall that we find that $\frac{1}{4} \leq \lambda \leq \frac{1}{2}$).

1.3 Organization of the article

In Section 2, we prove that the solution has a certain smoothness up to the boundary $r = 0$, namely Theorem 1.1. In Section 3, we construct the steady state, first on an annulus and then on the whole space. In Section 4, we prove some asymptotics of any profile, and then deduce the uniqueness of the profile (and of its angular velocity λ) as a consequence of the asymptotics. In Section 5, we provide some additional qualitative properties of the profile solution, including monotonicity of its gradient and of its curvature. We also give a bound from below on λ. In Section 6, we prove Liouville theorem 1.8. In Section 7, we prove the long time convergence of the solution to the steady state (up to addition of a constant), namely Theorem 1.6. This result follows from Liouville Theorem and a gradient bound on the solution (Proposition 7.1) that is proven in Section 7. Finally, Section A is an appendix where we recall standard materials, like strong maximum principle, Hopf lemma, Interior Schauder estimates. We also prove a technical lemma (Lemma A.4) which is used in Section 2, and also prove a result of independent interest which is not used in the rest of the paper: the equation satisfied by the curvature of the graph of the solution of the evolution problem.

Notation. For a real number $a \in \mathbb{R}$, a^+ denotes $\max(a, 0)$ and a^- denotes $\max(-a, 0)$. The ball of radius r centered at x are denoted $B(x, r)$. If $x = 0$, we simply write B_r.

2 Regular solutions up to the origin

This section is devoted to the proof of Theorem 1.1. This theorem improves [10, Theorem 1.7] by establishing regularity of solutions up to the origin. As we pointed out previously, the assumption that U_0 is globally Lipschitz was missing in the statement of [10, Theorem 1.7]. This is the reason why we first state a corrected version of this theorem.

Theorem 2.1 (Existence and uniqueness of smooth solutions for $r > 0$, [10])**.** *Assume that* $U_0 \in W^{2,\infty}_{loc}(0,+\infty)$ *is globally Lipschitz continuous and satisfies*

$$(U_0)_r \in W^{1,\infty}(0,+\infty) \quad or \quad \kappa_{U_0} \in L^\infty(0,+\infty)$$

and that there exists a radius $r_0 > 0$ *such that*

$$|1+\kappa_{U_0}| \leq Cr \quad for \quad 0 \leq r \leq r_0.$$

Then there exists a unique viscosity solution U *of* (1.3),(1.4) *which is globally Lipschitz in space and time. Moreover this solution* U *belongs to* $C^\infty((0,+\infty)\times(0,+\infty))$*.*

In view of this result, proving Theorem 1.1 amounts to prove the following proposition.

Proposition 2.2 (Space-time Lipschitz implies uniform regularity up to $r = 0$)**.** *Assume that* U *is a globally Lipschitz continuous (in space and time) solution of* (1.3) *in* $(0,+\infty)\times(0,+\infty)$*. Then* $U(t,r)$ *belongs to* $C^{1+\frac{1}{6},2+\frac{1}{3}}_{t,r}((0,+\infty)\times[0,+\infty))$*. Moreover, for every* $\delta > 0$*,* $R > 0$*, there exists a constant* $C = C(\delta, R)$ *such that we have the following uniform bound for every* $T \geq \delta > 0$*:*

$$\|U - U(T,0)\|_{C^{1+\frac{1}{6},2+\frac{1}{3}}_{t,r}([T,T+\delta]\times[0,R])} \leq C. \tag{2.1}$$

Before proving this proposition, we get some useful a priori estimates on the solution.

Lemma 2.3 (A priori estimates)**.** *Assume that* U *is a globally Lipschitz continuous (in space and time) solution of* (1.3) *in* $(0,+\infty)\times(0,+\infty)$*, with Lipschitz constant* $L > 0$*. Then* $U \in C^\infty((0,+\infty)\times(0,+\infty))$ *and there exists a constant* $C = C(L) > 0$ *such that for every* $(t,r) \in (0,\infty)\times(0,\infty)$*, we have*

$$|U_r(t,r) + \frac{1}{2}| \leq Cr \quad \text{and} \quad |U_{rr}(t,r)| \leq C(1+r^2). \tag{2.2}$$

Proof. We recall that we already proved in [10, Theorem 1.7] that $U \in C^\infty((0,+\infty)\times(0,+\infty))$. We also recall that U_t and U_r are bounded, and that U solves

$$rU_t = (1+\kappa_U)\sqrt{1+r^2U_r^2}.$$

We deduce that

$$|1+\kappa_U| \leq Cr \tag{2.3}$$

for some constant C. Remarking that

$$1 + 2U_r + rU_{rr} = (1+\kappa_U)(1+r^2U_r^2)^{\frac{3}{2}} - r^2U_r^3 - \left((1+r^2U_r^2)^{\frac{3}{2}} - 1\right),$$

and using the bound on U_r and (2.3), we deduce that

$$\begin{aligned} |1+2U_r+rU_{rr}| &\leq C(r+r^2+r^3+r^4) \\ &\leq C(r+r^4). \end{aligned} \tag{2.4}$$

For fixed $t > 0$, we set $\psi(r) = U(t,r) + r/2$ which satisfies $(r^2\psi_r)_r = r(1 + 2U_r + rU_{rr})$, and deduce that

$$|(r^2\psi_r)_r| \leq C(r^2 + r^5).$$

This implies $|r^2\psi_r| \leq C(r^3 + r^6)$ and we finally get

$$|U_r + \frac{1}{2}| = |\psi_r| \leq C(r + r^4). \tag{2.5}$$

Injecting this estimate in (2.4), we finally get for all $r \in (0,+\infty)$, $t \in (0,+\infty)$

$$|U_{rr}(t,r)| \leq C(1 + r^3). \tag{2.6}$$

Because U_r and U_t are bounded, we can use (1.3) to get for large r that $|U_{rr}| \leq Cr^2$. We can then improve (2.5) and (2.6) to get (2.2). This ends the proof of the lemma. □

Proof of Proposition 2.2. The idea of the proof is to see U as a radial solution of a partial differential equation in three dimensions and to use the interior regularity theory in 3D in order to deduce the boundary regularity up to $r = 0$.

More precisely, we set

$$V(t,X) := U(t,|X|) + \frac{|X|}{2} \quad \text{for } X \in \mathbb{R}^3,$$

where we see that V is smooth for $X \neq 0$. Here we have to add the term $\frac{|X|}{2}$ in the definition of V, in order to cancel the term $\nabla V(\cdot,0)$. Indeed, remember that $U_r(t,0) = -\frac{1}{2}$. If we do not add that term, this would make appear a bad term like $\frac{1}{X}$ in the coefficient of the PDE satisfied by V which would not allow us to control the regularity of the solution up to $X = 0$.

Step 1: Estimate on D^2V. We make the following pointwise computation of the second derivatives

$$\begin{aligned} D^2_{ji}V =& D_j(D_iV) = D_j\left(\frac{X_i}{|X|}U_r + \frac{1}{2}\frac{X_i}{|X|}\right) \\ =& U_{rr}\frac{X_iX_j}{|X|^2} + \left(U_r + \frac{1}{2}\right)\left(\frac{\delta_{ij}}{|X|} - \frac{X_iX_j}{|X|^3}\right). \end{aligned}$$

For $R > 0$ fixed and $0 < r \leq R$, we deduce from Lemma 2.3 that there exists a constant $C_R > 0$ such that

$$|U_{rr}| \leq C_R, \quad |U_r + \frac{1}{2}| \leq rC_R.$$

This implies that $D^2V \in L^\infty((0,+\infty) \times (B_R \setminus \{0\}))$.

Moreover for all $\phi \in C_c^\infty((0,T) \times B_R)$, we have in the distribution sense

$$\begin{aligned} -\langle D^2_{ji}V, \phi\rangle &= \lim_{\varepsilon\to 0}\int_{(0,T)\times(B_R\setminus B_\varepsilon)} (D_jV)(D_i\phi) \\ &= \lim_{\varepsilon\to 0}\left\{\int_{(0,T)\times(B_R\setminus B_\varepsilon)} -(D^2_{ji}V)\phi + \int_{(0,T)\times\partial B_\varepsilon} \phi(n\cdot e_i)D_jV\right\} \end{aligned}$$

where n is the outward nomal to $B_R\backslash B_\varepsilon$ on the boundary ∂B_ε, and e_i is a unit vector of the canonical basis of $\mathbb{R}^3$. Since ∇V is bounded, we recover that

$$(D^2V)_{j,i} = U_{rr}\frac{X_iX_j}{|X|^2} + \left(U_r + \frac{1}{2}\right)\left(\frac{\delta_{ij}}{|X|} - \frac{X_iX_j}{|X|^3}\right)$$

in the distribution sense on $(0,+\infty)\times B_R$. This implies that the distribution D^2V satisfies $D^2V \in L^\infty((0,+\infty)\times B_R)$.

Step 2: Estimate on ∇V. Moreover, since $V_r, V_t \in L^\infty((0,+\infty)\times B_R)$, we get that for every $\delta > 0$ and for every $1 < p < +\infty$, there exists a constant $C = C(\delta, R, p) > 0$ such that for every $T \geq \delta$, we have

$$\|V - V(T,0)\|_{W^{1,2;p}((T-\delta,T+\delta)\times B_R)} \leq C.$$

Using parabolic Sobolev Embedding in parabolic Hölder spaces (see [18, Lemma 3.3]), we get, for every $0 < \alpha < 1$ and a suitable constant $C = C(\delta, R, \alpha) > 0$, that

$$\|V - V(T,0)\|_{C^{\frac{1+\alpha}{2},1+\alpha}_{t,X}((T-\delta,T+\delta)\times B_R)} \leq C \tag{2.7}$$

which implies that

$$\|\nabla V\|_{C^{\frac{\alpha}{2},\alpha}_{t,X}((T-\delta,T+\delta)\times B_R)} \leq C. \tag{2.8}$$

Step 3: Equation satisfied by V. A computation gives that V is solution (at least in the distributional sense) of

$$V_t = A(X,\nabla V)\Delta V + B(X,\nabla V) \quad \text{for} \quad (t,x) \in (0,T)\times B_R(0)$$

where

$$\begin{aligned} A(X,p) &= \frac{1}{1 + (X\cdot p - \frac{|X|}{2})^2}, \\ B(X,p) &= \frac{(X\cdot p)}{|X|^2}\frac{q^2}{1+q^2} + \frac{q^2}{|X|}G_0(q) \end{aligned}$$

with $q = X\cdot p - \frac{|X|}{2}$ and

$$G_0(q) = \frac{1}{q^2}\left(\sqrt{1+q^2} - \frac{1}{2}\left(\frac{2+q^2}{1+q^2}\right)\right) = 1 + O(q^2).$$

Let us set

$$\tilde{X} = |X|^\alpha \frac{X}{|X|} \quad \text{with} \quad \alpha = 1/3.$$

In particular, we can easily check that the map $X \mapsto \tilde{X}$ is in C^α (see Lemma A.4). Then we can write

$$B(X,p) = (\tilde{X}\cdot p)\frac{(\tilde{X}\cdot p - \frac{|\tilde{X}|}{2})^2}{1+q^2} + |\tilde{X}|\left(\tilde{X}\cdot p - \frac{|\tilde{X}|}{2}\right)^2 G_0(q) \quad \text{with} \quad q = X\cdot p - \frac{|X|}{2}.$$

Therefore on the set $\{|X| \le R, |p| \le R\}$, we see that the function B is Lipschitz continuous both in p and in $\tilde{X}$, i.e. satisfies

$$|B(X',p') - B(X,p)| \le C_R \left(|\tilde{X}' - \tilde{X}| + |p'-p| \right).$$

Using Lemma A.4, this implies (increasing C_R if necessary) that

$$|B(X',p') - B(X,p)| \le C_R \left(|X' - X|^\alpha + |p'-p| \right)$$

i.e. B is locally Lipschitz in p and C^α in X. Similarly

$$|A(X',p') - A(X,p)| \le C_R \left(|X' - X| + |p'-p| \right)$$

i.e. A is locally Lipschitz in p and X.

Denoting by

$$\tilde{A}(t,X) = A(X, \nabla V(t,X)) \quad \text{and} \quad \tilde{B}(t,X) = B(X, \nabla V(t,X)),$$

and using (2.8) for the regularity of ∇V, we get that there exists a constant $C > 0$ such that

$$||\tilde{A}||_{C^{\frac{1}{6},\frac{1}{3}}_{t,X}((T-\delta,T+\delta)\times B_R)}, ||\tilde{B}||_{C^{\frac{1}{6},\frac{1}{3}}_{t,X}((T-\delta,T+\delta)\times B_R)} \le C. \tag{2.9}$$

Because $\tilde{V}(t,x) := V(t,x) - V(T,0)$ solves

$$\tilde{V}_t = \tilde{A} \Delta \tilde{V} + \tilde{B} \quad \text{in} \quad (T-\delta, T+\delta) \times B_R,$$

we can use interior Schauder estimates (see Proposition A.3 in the appendix), and deduce that

$$\begin{aligned} &\|V - V(T,0)\|_{C^{2+\frac{1}{3},1+\frac{1}{6}}_{x,t}([T,T+\delta]\times B_{R/2})} \\ &\qquad \le C \left\{ ||\tilde{B}||_{C^{\alpha,\frac{\alpha}{2}}_{x,t}((T-\delta,T+\delta)\times B_R)} + |V - V(T,0)|_{L^\infty((T-\delta,T+\delta)\times B_R)} \right\} \le C \end{aligned}$$

where we have used (2.9) and (2.7) for the last inequality. This implies in particular (2.1) (changing $R/2$ in R), and ends the proof of the proposition. □

3 Existence of a steady state

The main result of this section is the following proposition.

Proposition 3.1 (Existence of a steady state)**.** *There exists a constant $\lambda \ge 0$ and a function $\Phi \in C^\infty(0,+\infty)$, satisfying*

$$-1/2 \le \Phi_r \le 0 \quad \text{on} \quad (0,+\infty) \tag{3.1}$$

such that $U(t,r) = \lambda t + \Phi(r)$ is a solution of (1.3) on $\mathbb{R} \times (0,+\infty)$.

In a first subsection, we build a solution on an annulus $R^{-1} < r < R$, and in a second subsection we pass to the limit $R \to +\infty$.

3.1 Steady state in a annulus

In the following, we will frequently work in log coordinates with the function $u(t,x) = U(t,e^x)$. The function U solves (1.3) if and only if u solves the following equation

$$u_t = F(x, u_x, u_{xx}) := e^{-x}\sqrt{1+u_x^2} + e^{-2x}u_x + e^{-2x}\frac{u_{xx}}{1+u_x^2}. \tag{3.2}$$

See for instance [10].

For $R > 1$, we consider the annulus $R^{-1} \le r \le R$, we study the following problem with Neumann boundary condition on the boundary of the annulus:

$$\begin{cases} rU_t = \sqrt{1+r^2U_r^2} + U_r\left(\dfrac{2+r^2U_r^2}{1+r^2U_r^2}\right) + \dfrac{rU_{rr}}{1+r^2U_r^2} & \text{on} \quad (0,+\infty)\times(R^{-1},R), \\ U_r = 0 & \text{on} \quad (0,+\infty)\times\left\{R^{-1},R\right\}, \end{cases} \tag{3.3}$$

with initial data

$$U(0,r) = U_0(r) \quad \text{for all} \quad r \in [R^{-1},R] \tag{3.4}$$

Then we have the following result.

Lemma 3.2 (The Cauchy problem in an annulus)**.** *Let $R > 1$ and $\alpha \in (0,1)$ and assume that $U_0 \in C^{2+\alpha}([R^{-1},R])$ and that U_0 satisfies*

$$\begin{cases} (U_0)_r(R^{-1}) = 0 = (U_0)_r(R) \\ -M \le (U_0)_r(r) \le 0 \quad \textit{for all} \quad r \in (R^{-1},R). \end{cases} \tag{3.5}$$

Then there exists a unique solution $U \in C^{1+\frac{\alpha}{2},2+\alpha}([0,+\infty)\times[R^{-1},R])$ of (3.3), (3.4). Moreover U satisfies

$$-\max(1/2, M) \le U_r(t,r) \le 0 \quad \textit{for all} \quad (t,r) \in (0,+\infty)\times(R^{-1},R). \tag{3.6}$$

Proof of Lemma 3.2. The proof proceeds in several steps.

Step 1: Existence of a smooth solution As it is explained in [11], the classical theory allows to construct a unique solution $U \in C^{2+\alpha,1+\frac{\alpha}{2}}([0,+\infty)\times[R^{-1},R])$ of (3.3). Moreover, from the classical parabolic regularity theory, we can bootstrap and get that $U \in C^\infty((0,+\infty)\times[R^{-1},R])$.

Step 2: Gradient bound from above. We first recall that $u(t,x) = U(t,e^x)$ solves (3.2). Let

$$w = u_x$$

Then by derivation of (3.2), we easily get that w solves in $(0,+\infty)\times(-a,a)$ (with $a = \ln R$),

$$\begin{aligned} w_t = \quad & -e^{-x}\sqrt{1+w^2} + e^{-x}\frac{ww_x}{\sqrt{1+w^2}} - 2e^{-2x}w + e^{-2x}w_x \\ & -2e^{-2x}\frac{w_x}{1+w^2} + e^{-2x}\left(\frac{w_{xx}}{1+w^2} - \frac{2w(w_x)^2}{(1+w^2)^2}\right), \end{aligned} \tag{3.7}$$

and

$$w(t, \pm a) = 0 \quad \text{for all} \quad t \in (0, +\infty)$$

and

$$w(0, x) = e^x (U_0)_r(e^x) \quad \text{for all} \quad x \in [-a, a]. \tag{3.8}$$

Notice that $\overline{w} = 0$ is a supersolution of (3.7), (3.8), where we use (3.5) to check the initial condition inequality. Therefore the classical comparison principle implies that

$$w \leq 0. \tag{3.9}$$

Step 3: Gradient bound from below. We now define the function

$$z(t, x) = e^{-x} w(t, x) = U_r(t, e^x).$$

It is easy to check that z satisfies

$$z_t = -e^{-2x} \left(\frac{1}{\sqrt{1 + w^2}} + z \right) - e^{-3x} \left(\frac{w}{1 + w^2} + \frac{2w^3}{(1 + w^2)^2} \right) + e^{-2x} \frac{z_{xx}}{1 + w^2} + O(z_x). \tag{3.10}$$

Because we already know that $z \leq 0$, we deduce that:

$$e^{2x} z_t \geq -g(x, z) + \frac{z_{xx}}{1 + w^2} + O(z_x) \tag{3.11}$$

with

$$g(x, z) = \frac{1}{\sqrt{1 + e^{2x} z^2}} + z + \frac{z}{1 + e^{2x} z^2}.$$

Let us set

$$h(\gamma) = \gamma + z + \gamma^2 z.$$

Then we have

$$g(x, z) = h(\gamma) \quad \text{with} \quad \gamma = \frac{1}{\sqrt{1 + e^{2x} z^2}} \in (0, 1].$$

Remark that the maximum of $h(\gamma)$ is reached at $\gamma = -\frac{1}{2z}$ if $z < 0$. Therefore

$$\sup_{\gamma \in (0,1]} h(\gamma) \leq h\left(-\frac{1}{2z} \right) = z - \frac{1}{4z} \leq 0 \quad \text{if} \quad z \leq -\frac{1}{2}.$$

Therefore

$$g(x, z) \leq 0 \quad \text{if} \quad z \leq -\frac{1}{2}.$$

Remark now that $\underline{z} = -\max(1/2, M)$ is then a subsolution of the equation with equality in (3.11) (with zero boundary conditions). This implies that $\underline{z}$ is a subsolution of (3.10) with zero boundary condition. Again, the comparison principle for z implies that

$$-\max(1/2, M) \leq z. \tag{3.12}$$

Finally (3.12) and (3.9) implies (3.6) which ends the proof of the proposition. □

Lemma 3.3 (Periodic solution in an annulus)**.** *For $R > 1$, there exists a solution U_R of* (3.3) *in* $(0, +\infty) \times (R^{-1}, R)$ *such that*

$$U_R(t + T_R, r) = U_R(t, r) + 2\pi \tag{3.13}$$

for some $T_R > 0$.

Proof of Lemma 3.3. Let I denote the interval (R^{-1}, R). In view of [11, Remark 2.1] and the discussion preceding [11, Proposition 4.3], we know that for all $U_0 \in C^{2+\alpha}(\bar{I})$ for some $\alpha \in (0, 1)$ such that $(U_0)_r \leq 0$, there exists a solution U_R of (1.3) in $(0, \infty) \times I$. Moreover, for all $t > 0$, we have $U_R(t, \cdot) \in C^\infty(\bar{I})$. We then choose $U_0 \in C^{2+\alpha}(\bar{I})$ satisfying (3.5) with $M = 1/2$ and we denote by U_R the corresponding solution. Thanks to Lemma 3.2, we know that

$$-1/2 \leq (U_R)_r(t, r) \leq 0.$$

Moreover, by [11, Proposition 4.3], there exists a period $T_R > 0$ and U_0 such that (3.13) holds true. This achieves the proof of Lemma 3.3. □

Lemma 3.4 (Steady state in an annulus)**.** *For $R > 1$, there exists $\lambda_R > 0$ and $\Phi_R \in C^\infty([R^{-1}, R])$ satisfying*

$$-1/2 \leq (\Phi_R)_r \leq 0 \quad in \quad [R^{-1}, R]$$

such that $\lambda_R t + \Phi_R(r)$ is a solution of (3.3)*.*

Proof of Lemma 3.4. Remark first (with $\lambda_R = \frac{2\pi}{T_R}$) that $v(t, r) = U_R(t, r) - \lambda_R t$ is T_R-periodic with respect to the time variable. We want to prove that it is constant. To do so, we prove that it is non-decreasing.

Consider $\varepsilon, \delta > 0$ and define

$$v^{\varepsilon,\delta}(t, r) = v(t + \delta, r) - \varepsilon.$$

We have $v^{\varepsilon,0} < v^{0,0}$ and since U_R is Lipschitz continuous, $v^{\varepsilon,\delta} < v^{0,0}$ for δ small enough. We then define for $\varepsilon > 0$

$$\delta_\varepsilon = \sup\{\delta > 0 : v^{\varepsilon,\delta} < v^{0,0}\} > 0.$$

Assume that $\delta_\varepsilon < +\infty$. Remark that $v^{\varepsilon,\delta}$ and $v^{0,0}$ are both solutions of (1.3) and the optimality of δ_ε implies that

$$\max_{t \in [0, T_R], r \in \bar{I}} \{v^{\varepsilon,\delta_\varepsilon} - v^{0,0}\} = 0.$$

By Lemma A.2 and the Neumann boundary condition, we deduce that the maximum is attained for some inner point $r_0 \in I$. Since the function is T_R-periodic with respect to the time variable, the strong maximum principle (Theorem A.1) written for the difference function $v^{\varepsilon,\delta_\varepsilon} - v^{0,0}$ implies that $v^{\varepsilon,\delta_\varepsilon} \equiv v^{0,0}$. Then for all $k \in \mathbb{N}$, we have

$$v(t + k\delta_\varepsilon, r) = v(t, r) + k\varepsilon.$$

It is now enough to choose $k \in \mathbb{N}$ such that $k\varepsilon > \operatorname{osc} v$ to get the desired contradiction.

Hence $v(t + \delta, r) \leq v(t, r) + \varepsilon$ for all $\delta > 0$. This implies that for any $s \geq r$, $v(s, r) \leq v(t, r) + \varepsilon$. Since ε is arbitrary, we conclude that $v(s, r) \leq v(t, r)$. Hence v is periodic and non-decreasing with respect to t. It is thus constant. The proof of Lemma 3.4 is now complete. □

3.2 Steady state in the plane

In this subsection, we want to take the limit $R \to 0$ to recover a steady state in the plane. To this end, we first need the following estimate.

Lemma 3.5 (Bound on λ_R)**.** *There exists $\hat{\lambda} \geq 0$ such that for all $R \geq 2$, we have $0 < \lambda_R \leq \hat{\lambda}$.*

Proof of Lemma 3.5. We already know that $\lambda_R > 0$. In order to exhibit $\hat{\lambda} \geq 0$ with the desired property, we are going to construct a super-solution of (1.3) of the type $\hat{u}(t,r) = \hat{\lambda} t + \Psi(r)$.

Let $\theta = -\gamma(r)$ describe the circle (in polar coordinates) of equation $1 + \kappa = 0$ which is tangent from above to the horizontal axis. From an analytical point of view, the reader can check that the right half circle (i.e. for $\theta \in [0, \pi/2]$ and $0 \leq r \leq 2$) corresponds to

$$\gamma(r) = -\arcsin\left(\frac{r}{2}\right)$$

which satisfies $\bar{F}(r, \gamma_r, \gamma_{rr}) = 0$ for $0 < r < 2$, where

$$\bar{F}(r, \gamma_r, \gamma_{rr}) := \frac{1}{r}\left\{\sqrt{1 + r^2\gamma_r^2} + \gamma_r\left(\frac{2 + r^2\gamma_r^2}{1 + r^2\gamma_r^2}\right) + \frac{r\gamma_{rr}}{1 + r^2\gamma_r^2}\right\}.$$

We choose Ψ as follows

$$\Psi(r) = \zeta(r)\gamma(r)$$

where ζ is a smooth cut-off function which is equal to 1 in $[0, \frac{1}{2}]$ and equal to zero for $r \geq 1$.

Now we choose $\hat{\lambda}$ such that

$$\hat{\lambda} \geq \sup_{r>0} \bar{F}(r, \Psi_r, \Psi_{rr}) = \sup_{r \in [1/2, 1]} \bar{F}(r, \Psi_r, \Psi_{rr})$$

We also have for $R \geq 2$:

$$\Psi_r(R) = 0, \quad \text{and} \quad \Psi_r(R^{-1}) = \gamma_r(R^{-1}) < 0.$$

This implies that $\hat{\lambda} t + \Psi(r)$ is a supersolution of (3.3), and the comparison principle with $\lambda_R t + \Phi_R(r)$ implies (for large times) that $\lambda_R \leq \hat{\lambda}$ which ends the proof of the lemma. □

We now want to pass to the limit as $R \to +\infty$ and prove Proposition 3.1.

Proof of Proposition 3.1. Because the functions $\lambda_R t + \Phi_R(r)$ are uniformly Lipschitz continuous in space and time independently on $R \geq 2$, we can pass to the limit $R \to \infty$. We call the limit $\lambda t + \Phi(r)$, which is then a viscosity solution of (1.3) and satisfies:

$$-1/2 \leq \Phi_r \leq 0 \quad \text{and} \quad \hat{\lambda} \geq \lambda \geq 0.$$

Because $\lambda t + \Phi(r)$ is globally Lipschitz continuous in space and time, we can apply Lemma 2.3 and deduce that $\Phi \in C^\infty(0, +\infty)$. This ends the proof of the proposition. □

4 Asymptotics of the steady state and uniqueness

The main result of this section is the following proposition.

Proposition 4.1 (Asymptotics of the steady state and uniqueness)**.** *Assume that* $\lambda t + \Phi(r)$ *is a globally Lipschitz continuous solution of* (1.3) *in* $\mathbb{R} \times (0, +\infty)$ *with* $\Phi \in C^\infty(0, +\infty)$ *satisfying*

$$\lambda \geq 0 \quad \textit{and} \quad \Phi_r \leq 0. \tag{4.1}$$

Then such a λ *is unique and such a* Φ *is unique up to an additive constant. Moreover we have* $\lambda > 0$ *and there exist constants* $a \in \mathbb{R}$ *and* $C > 0$ *such that*

$$|\Phi(r) + \lambda r + \lambda \ln(1 + r) - a| \leq \frac{C}{1 + r}. \tag{4.2}$$

We will do the proof of Proposition 4.1 using several lemmas and propositions.

4.1 Positivity of the angular velocity

We first prove that λ is positive.

Lemma 4.2 (Positivity of λ)**.** *Under the assumptions of Proposition 4.1, we have* $\lambda > 0$.

Proof of Lemma 4.2. Assume by contradiction that $\lambda = 0$. We look for a barrier solution that we will compare to Φ. To this end, let us consider the circle in $\mathbb{R}^2$ of radius 1 (given by the equation $1 + \kappa = 0$) and of center $(0, R)$ for some $R > 1$ in the Cartesian coordinates $X = (x_1, x_2)$. We can parametrize in polar coordinates, the right half circle as follows,

$$\theta = -\gamma_R(r) := \arcsin\left(f(r)\right) \quad \text{for} \quad R - 1 \leq r \leq R + 1$$

with

$$f(r) = \frac{r^2 + R^2 - 1}{2Rr}$$

which satisfies $f(R - 1) = 1 = f(R + 1)$, and $f'(r) = \frac{r^2 - (R^2 - 1)}{2Rr}$ with $f'(R \pm 1) \neq 0$.

This implies in particular that the graph of γ_R has vertical tangents at $r = R \pm 1$. Because γ_R is a stationary solution of (1.3) on $(R - 1, R + 1)$, we can compare it on $(R - 1, R + 1)$ to the stationary solution Φ when $\lambda = 0$. We consider

$$\max_{r \in [R-1, R+1]} \left(\Phi(r) - \gamma_R(r)\right).$$

The maximum can not be achieved at $r = R \pm 1$, because Φ is Lipschitz continuous and γ_R is not at those points. Therefore the maximum is reached at some interior point, and the strong maximum principle implies that

$$\Phi(r) - \gamma_R(r) \text{ is constant in } (R - 1, R + 1).$$

By continuity, this is still true at $r = R \pm 1$ which is again impossible. Finally, we conclude that $\lambda \neq 0$ and then $\lambda > 0$. This ends the proof of the lemma. □

4.2 Asymptotics

In the following proposition, the asymptotics of the profile is stated in Log coordinates. It also contains the asymptotics of the derivative of the profile which will be used later.

Proposition 4.3 (Asymptotics near $r = +\infty$)**.** *Under the assumptions of Proposition 4.1, the function $\varphi(x) = \Phi(e^x)$ satisfies*

$$|\varphi(x) + \lambda e^x + \lambda x - a| \le Ce^{-x} \quad \textit{for} \quad x \ge x_1 \tag{4.3}$$

and

$$\varphi_x(x) = -\lambda e^x - \lambda + O(e^{-x}) \quad \textit{for} \quad x \ge x_1 \tag{4.4}$$

for some constants $a, x_1 \in \mathbb{R}$ and $C > 0$.

Recalling (3.2), we see that φ is a solution of the following second order ODE

$$\lambda = e^{-x}\sqrt{1+\varphi_x^2} + e^{-2x}\varphi_x + e^{-2x}\frac{\varphi_{xx}}{1+\varphi_x^2} \quad \text{for} \quad x \in \mathbb{R}. \tag{4.5}$$

As we shall see it, Proposition 4.3 is a consequence of the study of the ODE satisfied by $v := \varphi_x \le 0$, which is the following

$$v_x = f(v,x) \quad \text{for} \quad x \in \mathbb{R} \tag{4.6}$$

where

$$f(w,x) = e^{2x}(1+w^2)\zeta(w,x) \quad \text{with} \quad \zeta(w,x) = \lambda - e^{-x}\sqrt{1+w^2} - e^{-2x}w.$$

We first need the following result.

Lemma 4.4 (Elementary estimates)**.** *Let $\lambda > 0$. Then there exists a real number $x_0 \ge 0$ such that for $x \ge x_0$, the equation $f(w,x) = 0$ has a single root $w = v_0(x)$ which is non-positive. This function satisfies for $x \ge x_0$*

$$v_0(x) = -\lambda e^x - \lambda + e^{-x}\left(\frac{1}{2\lambda} - \lambda\right) + O(e^{-2x}), \tag{4.7}$$

$$(v_0)_x(x) = -\lambda e^x + O(1) \le 0. \tag{4.8}$$

Moreover we have

$$\frac{\partial f}{\partial w}(w,x) \ge \frac{\lambda^2}{2}e^{3x} \quad \textit{for} \quad w \le v_0(x) \quad \textit{and} \quad x \ge x_0 \tag{4.9}$$

and for all $w_*, y_* \in \mathbb{R}$, we have

$$\left.\begin{array}{l} x \ge y_* \ge x_0 \\ v_0(y_*) < w_* \le w \le 0 \end{array}\right\} \Longrightarrow f(w,x) \ge e^{2y_*}\min(\zeta(w_*,y_*), \lambda/2) > 0. \tag{4.10}$$

Proof of Lemma 4.4. The proof proceeds in several steps.

Step 1: Definition of v_0. Remark that if $f(w,x)=0$, then w solves the following second order polynomial equation

$$(1-e^{-2x})w^2+2\lambda w+1-\lambda^2 e^{2x}=0\,. \tag{4.11}$$

For some x large enough, there is only one non-positive solution which is given by the following formula

$$v_0(x)=\frac{-\lambda-\sqrt{\lambda^2+(1-e^{-2x})(e^{2x}\lambda^2-1)}}{1-e^{-2x}}.$$

It is now easy to derive the expansion (4.7), and we skip the details.
In order to recover (4.8), we take the x-derivative of equation (4.11) satisfied by v_0, and we get

$$(v_0)_x(v_0(1-e^{-2x})+\lambda)+(v_0)^2e^{-2x}-\lambda^2e^{2x}=0.$$

This implies

$$(v_0)_x(x)=\frac{-(v_0)^2e^{-2x}+\lambda^2e^{2x}}{v_0(1-e^{-2x})+\lambda}.$$

Using (4.7), a Taylor expansion permits to get (4.8).

Step 2: Estimate on $\frac{\partial f}{\partial w}$. Let us now compute

$$\frac{\partial f}{\partial w}(w,x)=2we^{2x}\zeta(w,x)+e^{2x}(1+w^2)\frac{\partial \zeta}{\partial w}(w,x)$$

and

$$\frac{\partial \zeta}{\partial w}(w,x)=-\frac{we^{-x}}{\sqrt{1+w^2}}-e^{-2x}=:g(w,x). \tag{4.12}$$

Remark also that, increasing x_0 if necessary, we have for $x\geq x_0$ both

$$v_0(x)\leq -1$$

and

$$\frac{\partial \zeta}{\partial w}(w,x)\geq \frac{1}{2}e^{-x}\quad \text{for}\quad w(x)\leq v_0(x)\leq -1.$$

But $\zeta(v_0(x),x)=0$, and then the sign of $\frac{\partial \zeta}{\partial w}$ implies

$$\zeta(w(x),x)\leq 0\quad \text{for}\quad w(x)\leq v_0(x)$$

and

$$\frac{\partial f}{\partial w}(w,x)\geq e^{2x}(1+w^2)\frac{\partial \zeta}{\partial w}(w,x).$$

Again, increasing x_0 if necessary, we can assume that $v_0(x)\leq -\lambda e^x$ for $x\geq x_0$ and then

$$\frac{\partial f}{\partial w}(w,x)\geq \frac{\lambda^2}{2}e^{3x}\quad \text{for}\quad w(x)\leq v_0(x).$$

Step 3: Estimate on f. Recall that the function g appears in (4.12). Remark that for $x \geq 0$ we have $g(w,x) = 0$ with $w \leq 0$ if and only if

$$w(x) = -\frac{1}{\sqrt{e^{2x}-1}} =: w_0(x).$$

Moreover we can then deduce that

$$\begin{array}{rcl} g(w,x) & \geq & 0 \quad \text{if} \quad w \leq w_0(x), \\ g(w,x) & \leq & 0 \quad \text{if} \quad w_0(x) \leq w \leq 0. \end{array}$$

Because of (4.12), we deduce that, increasing x_0 if necessary,

$$w_0(x) \leq w \leq 0 \Longrightarrow \zeta(w,x) \geq \zeta(0,x) = \lambda - e^{-x} \geq \lambda/2 \quad \text{if} \quad x \geq x_0$$

and then using the definition of f and a bound from below of $\zeta(w,x)$ for $w \in [w_*, 0]$, we get

$$v_0(x) < w_* \leq w \leq 0 \Longrightarrow f(w,x) \geq e^{2x} \min(\zeta(w_*,x), \lambda/2) > 0 \quad \text{if} \quad x \geq x_0$$

Let us notice that for $w \leq 0$, we have up to increase x_0 if necessary,

$$\frac{\partial \zeta}{\partial x}(w,x) = e^{-x}\sqrt{1+w^2} + 2e^{-2x} w \geq 0 \quad \text{if} \quad x \geq x_0$$

and then this implies (4.10). This ends the proof of the lemma. $\square$

We next prove the following estimate.

Lemma 4.5 (Asymptotics for $v = \varphi_x$)**.** *For any $\mu > 0$, there exists a real number $x_1 \geq x_0$ such that $v = \varphi_x$ satisfies*

$$v_0(x) \geq v(x) \geq v_0(x) - \mu e^{-\frac{3}{2}x} \quad \textit{for} \quad x \geq x_1$$

where v_0 and x_0 are given by Lemma 4.4.

Proof of Lemma 4.5. Recall that $\lambda > 0$ and define

$$\bar{v}(x) := v_0(x) - \mu e^{-\frac{3}{2}x}.$$

The proof proceeds in several steps.

Step 1: $\bar{\mathbf{v}}$ is a super-solution. Remark that, thanks to (4.8),

$$\bar{v}_x(x) = (v_0)_x(x) + \frac{3}{2}\mu e^{-\frac{3}{2}x} = -\lambda e^x + O(1).$$

We also remark that there exists $w(x) \in [\bar{v}(x), v_0(x)]$ such that

$$\begin{aligned} f(\bar{v}(x),x) &= f(v_0(x),x) + \frac{\partial f}{\partial w}(w(x),x)(\bar{v}(x) - v_0(x)) \\ &\leq \frac{\lambda^2}{2} e^{3x}(\bar{v}(x) - v_0(x)) \\ &\leq -\mu \frac{\lambda^2}{2} e^{\frac{3}{2}x} \end{aligned} \tag{4.13}$$

where we used (4.9) in the second line. Therefore there exists $x_1 \geq x_0$ such that

$$\bar{v}_x(x) \geq f(\bar{v}(x),x) \quad \text{for} \quad x \geq x_1.$$

Step 2: Comparison with $\bar{\mathbf{v}}$. Assume by contradiction that $v(x_*) \leq \bar{v}(x_*)$ for some $x_* \geq x_1$. Then, from the comparison principle, we deduce that

$$v(x) \leq \bar{v}(x) \quad \text{for all} \quad x \geq x_*.$$

Then we have

$$v_x(x) = f(v(x), x) \leq f(\bar{v}(x), x) \leq -\mu \frac{\lambda^2}{2} e^{\frac{3}{2}x} \tag{4.14}$$

where we have used the fact that $v \leq \bar{v}$, the monotonicity of $f(w, x)$ in w (see (4.9)) and estimate (4.13). Estimate (4.14) now gives a contradiction with the fact that $\Phi_r(e^x) = e^{-x} v(x)$ is bounded.

Step 3: $\mathbf{v_0}$ is a sub-solution. The inequality $(v_0)_x(x) \leq 0 = f(v_0(x), x)$ for $x \geq x_0$ follows from (4.8).

Step 4: Comparison with $\mathbf{v_0}$. We argue by contradiction. Let us assume that there exists a point $y_* \geq x_0$ such that $v(y_*) > v_0(y_*)$. Then from (4.10), we deduce that there exists a constant $\alpha > 0$ such that

$$f(w, x) \geq \alpha > 0 \quad \text{for} \quad w \in [v(y_*), 0] \quad \text{and} \quad x \geq y_*.$$

But recall that

$$v_x(x) = f(v(x), x).$$

This implies that

$$v_x(x) \geq \alpha \quad \text{for} \quad x \geq y_* \quad \text{while} \quad v(x) \leq 0.$$

Therefore we conclude (using the continuity of f) that there exists a point x_2 such that $v(x_2) > 0$, which is impossible because $v = \varphi_x \leq 0$. We thus get the desired contradiction. This ends the proof of the lemma. □

Proof of Proposition 4.3. It follows from Lemma 4.5 and (4.7). □

4.3 Uniqueness

Proposition 4.6 (Uniqueness)**.** *Under the assumptions of Proposition 4.1, λ is unique and Φ is unique up to addition of constants.*

In order to prove Proposition 4.6, we will need the following space Liouville result which will be proven later in Section 6 as an independent result.

Theorem 4.7 (Space Liouville theorem)**.** *Let Φ^i for $i = 1, 2$ be two $C^2([0, +\infty))$ functions such that for some $\lambda > 0$, the functions $\lambda t + \Phi^i(r)$ are solutions of (1.3) in $\mathbb{R} \times (0, +\infty)$ for $i = 1, 2$. Assume also that we have for $i = 1, 2$ and $r \geq 0$:*

$$\left|\Phi^i(r) + \lambda r + \lambda \ln(1 + r)\right| \leq \frac{C}{1 + r} \tag{4.15}$$

and

$$\left|\Phi^i_r(r) + \lambda\right| \leq \frac{C}{1 + r}.$$

Then $\Phi^1 = \Phi^2$.

Proof of Proposition 4.6. We already know that Φ satisfies (4.2). From Proposition 2.2, we deduce that $\Phi \in C^2([0, +\infty))$.

Uniqueness of λ. We argue by contradiction by assuming that there exist (Φ^1, λ^1) and (Φ^2, λ^2) two solutions such that

$$\lambda^1 < \lambda^2.$$

Because of (4.2), we deduce that there exists a constant K such that

$$\Phi^1(r) \geq \Phi^2(r) - K \quad \text{for} \quad r > 0.$$

From the comparison principle for (1.3) (see Theorem 1.3 in [10], with Lipschitz continuous initial data $U_0 = \Phi^1$), we deduce

$$\lambda^1 t + \Phi^1(r) \geq \lambda^2 t + \Phi^2(r) - K \quad \text{for all} \quad (t, r) \in (0, +\infty) \times (0, +\infty)$$

which implies (for large times) that $\lambda^1 \geq \lambda^2$. This is the desired contradiction.

Uniqueness of Φ (up to an additive constant). We now consider two profiles Φ^1, Φ^2 with the same $\lambda = \lambda^1 = \lambda^2$. Recall that for $i = 1, 2$, each function Φ^i satisfies (4.2) for some constant a^i. Adding different constants to those two functions if necessary, we can asssume that $a^1 = a^2 = a = 0$, i.e.

$$|\Phi^i(r) + \lambda r + \lambda \ln(1 + r)| \leq \frac{C}{1 + r}, \quad \text{for} \quad i = 1, 2.$$

We then apply Theorem 4.7 to conclude that $\Phi^1 = \Phi^2$. The proof is now complete. □

Proof of Proposition 4.1. It follows from Lemma 4.2 and Propositions 4.3 and 4.6. □

5 Further properties of the steady state

5.1 Monotonicity properties

Proposition 5.1 (Monotonicity of the gradient of the profile)**.** *Let Φ be the profile given in Proposition 4.1. Then we have*

(5.1)
$$\Phi_{rr} \geq 0 \quad \text{in} \quad [0, +\infty)$$

(5.2)
$$-\frac{1}{2} \leq \Phi_r \leq -\lambda$$

and

(5.3)
$$\Phi_r(0) = -\frac{1}{2} \quad \text{and} \quad \Phi_r(+\infty) = -\lambda < 0.$$

Proof of Proposition 5.1. For $\varphi(x) = \Phi(e^x)$, we recall from (3.10) that

$$z(x) := e^{-x}\varphi_x(x) = \Phi_r(e^x)$$

satisfies with $w = \varphi_x$:

$$0 = z_t = -e^{-2x}\left(\frac{1}{\sqrt{1 + w^2}} + z\right) - e^{-3x}\left(\frac{w}{1 + w^2} + \frac{2w^3}{(1 + w^2)^2}\right) + e^{-2x}\frac{z_{xx}}{1 + w^2} + O(z_x)\,.$$

Step 1: case of a local minimum of z. Assume that z has a local minimum at x_0 with value $z_0 = z(x_0)$. Then $z_{xx}(x_0) \geq 0$ and $z_x(x_0) = 0$ which implies,

$$\frac{1}{\sqrt{1 + e^{2x_0} z_0^2}} + z_0 + \frac{z_0}{1 + e^{2x_0} z_0^2} + \frac{2e^{2x_0} z_0^3}{(1 + e^{2x_0} z_0^2)^2} \geq 0.$$

Setting

$$\gamma = \frac{1}{\sqrt{1 + e^{2x_0} z_0^2}} \in (0, 1],$$

we see that this means

$$\gamma + z_0 + \gamma^2 z_0 + 2\gamma^4 z_0 (1/\gamma^2 - 1) \geq 0$$

i.e.

$$\gamma + z_0 (1 + 3\gamma^2 - 2\gamma^4) \geq 0. \tag{5.4}$$

Let

$$g(y) := 1 + 3y - 2y^2$$

Remark that g is maximum at $y = 3/4$ and then

$$\inf_{y \in (0,1]} g(y) \geq \min (g(0), g(1)) = 1.$$

Therefore (5.4) means

$$z_0 \geq -\frac{\gamma}{g(\gamma^2)} =: -K(\gamma).$$

Step 2: Monotonicity of K. Let us compute with $y = \gamma^2$:

$$K'(\gamma) = \frac{1}{g^2(y)} (g(y) - 2yg'(y))$$

with

$$g(y) - 2yg'(y) = 1 + 3y - 2y^2 - 2y(3 - 4y) = 1 - 3y + 6y^2 =: h(y)$$

which is minimal at $y^* = 1/4$ with value $h(y^*) > 0$. Therefore K is increasing.

Step 3: Monotonicity of z. Assume now that z has a local maximum at $\overline{x}$ with value $\overline{z} = z(\overline{x})$. Then we have

$$\overline{z} \leq -K(\overline{\gamma}) \quad \text{with} \quad \overline{\gamma} = \frac{1}{\sqrt{1 + e^{2\overline{x}} \overline{z}^2}}.$$

We already know (see (4.4)) that

$$z(x) = -\lambda - \lambda e^{-x} + o(e^{-2x})$$

which shows that z cannot be non-increasing in $(\overline{x}, +\infty)$ (and satisfies $z(+\infty) = \Phi_r(+\infty) = -\lambda$). Therefore there exists $\underline{x} > \overline{x}$ such that z has a local minimum at $\underline{x}$ with value $\underline{z} = z(\underline{x})$ that we can choose such that

$$\underline{z} \leq \overline{z} \leq 0. \tag{5.5}$$

Moreover we have

$$\underline{z} \geq -K(\underline{\gamma}) \quad \text{with} \quad \underline{\gamma} = \frac{1}{\sqrt{1 + e^{2\underline{x}}\underline{z}^2}} < \overline{\gamma}.$$

The strict monotonicity of K implies

$$\overline{z} \leq -K(\overline{\gamma}) < -K(\underline{\gamma}) \leq \underline{z},$$

which is in contradiction with (5.5). Therefore, we conclude that z has no local maximum.

Step 4: Behaviour at $r = 0$. We recall that $\Phi \in C^2([0, +\infty))$. From the fact that $\lambda t + \Phi(r)$ is a solution of (1.3), we deduce that

$$r\lambda = \sqrt{1 + r^2\Phi_r^2} + \Phi_r \left(\frac{2 + r^2\Phi_r^2}{1 + r^2\Phi_r^2} \right) + \frac{r\Phi_{rr}}{1 + r^2\Phi_r^2}.$$

At $r = 0$, we deduce that

$$1 + 2\Phi_r(0) = 0. \tag{5.6}$$

Close to $r = 0$, we deduce (by Tayor expansion) that

$$\Phi_{rr}(r) = O(r) + \lambda - \frac{1}{r}\left(1 + 2\Phi_r(r) + O(r^2)\right).$$

Using (5.6), we deduce that

$$\Phi_{rr}(0) = \frac{\lambda}{3} > 0.$$

Step 5: Conclusion. Using the fact that $\Phi_{rr}(0) > 0$ and the fact that Φ_r has no local maximum (by Step 3), we deduce that Φ_r is increasing, which in particular implies (5.1) and (5.2). This ends the proof of the proposition. □

Proposition 5.2 (Sign and monotonicity of the curvature)**.** *Let Φ be the profile given in Proposition 4.1. Then the curvature κ_Φ defined in* (1.2) *satisfies,*

$$-1 \leq \kappa_\Phi \leq 0$$

and

$$\kappa_\Phi(0) = -1, \quad \kappa_\Phi(+\infty) = 0.$$

Moreover we have

$$(\kappa_\Phi)_r \geq 0.$$

Proof of Proposition 5.2. We set $\kappa(x) := \kappa_\Phi(e^x)$. Notice that we deduce from (1.2) and (5.3) that

$$\kappa_\Phi(r = 0) = 2\Phi_r(0) = -1.$$

Step 1: $\kappa \in [-1, 0]$. Recall that for the profile, we have,

$$\lambda = e^{-x}\sqrt{1+u_x^2} + e^{-2x}u_x + e^{-2x}\frac{u_{xx}}{1+u_x^2} = e^{-x}(1+\kappa)\sqrt{1+u_x^2} \tag{5.7}$$

where the curvature κ can be written as

$$\kappa := e^{-x}\frac{u_x}{\sqrt{1+u_x^2}} + e^{-x}\frac{u_{xx}}{(1+u_x^2)^{\frac{3}{2}}}. \tag{5.8}$$

Equation (5.7) shows that we can find the following other expression for the curvature,

$$\kappa = \frac{\lambda e^x}{\sqrt{1+u_x^2}} - 1. \tag{5.9}$$

Using (4.4), we then deduce that

$$\kappa(x = +\infty) = 0. \tag{5.10}$$

Moreover, using again (5.9), we have

$$\begin{aligned}
\kappa_x &= \frac{\lambda e^x}{\sqrt{1+u_x^2}} - \frac{\lambda e^x}{(1+u_x^2)^{\frac{3}{2}}}u_x u_{xx} \\
&= \frac{\lambda e^x}{\sqrt{1+u_x^2}} - \lambda e^{2x}u_x\left(\kappa - e^{-x}\frac{u_x}{\sqrt{1+u_x^2}}\right) \\
&= \lambda e^x\sqrt{1+u_x^2} - \lambda e^{2x}u_x\kappa.
\end{aligned}$$

Using the fact that $u_x \le 0$, we conclude that

$$\kappa(x_0) > 0 \quad \Longrightarrow \quad \kappa(x) \ge \kappa(x_0) \quad \text{for} \quad x \ge x_0$$

which is in contradiction with (5.10). Therefore $\kappa \le 0$. The fact that $1+\kappa \ge 0$ comes directly from (5.9).

Step 2: κ is non-decreasing. Let us start again from

$$\kappa_x = \lambda e^x\sqrt{1+u_x^2} - \lambda e^{2x}u_x\kappa. \tag{5.11}$$

Then

$$\begin{aligned}
\kappa_{xx} &= \lambda e^x\sqrt{1+u_x^2} - 2\lambda e^{2x}u_x\kappa + \lambda e^x\frac{u_x u_{xx}}{\sqrt{1+u_x^2}} - \lambda e^{2x}u_{xx}\kappa - \lambda e^{2x}u_x\kappa_x \\
&= 2\kappa_x - \lambda e^x\sqrt{1+u_x^2} - \lambda e^{2x}u_x\kappa_x + \frac{u_{xx}}{u_x}\left(\kappa_x - \frac{\lambda e^x}{\sqrt{1+u_x^2}}\right) \\
&= \kappa_x\left(2 - \lambda e^{2x}u_x + \frac{u_{xx}}{u_x}\right) - \frac{\lambda e^x\sqrt{1+u_x^2}}{u_x}\left(u_x + \frac{u_{xx}}{1+u_x^2}\right) \\
&= \kappa_x\left(2 - \lambda e^{2x}u_x + \frac{u_{xx}}{u_x}\right) - \frac{\lambda e^x\sqrt{1+u_x^2}}{u_x}e^x\sqrt{1+u_x^2}\kappa.
\end{aligned}$$

Recall that $u_x < 0$, $\kappa \le 0$ and $\kappa_x = 0$ implies in (5.11) that $u_x\kappa = e^{-x}\sqrt{1+u_x^2} > 0$, which shows that $\kappa < 0$. Therefore we conclude from the above computation that

$$\kappa_{xx} < 0 \quad \text{at any point where} \quad \kappa_x = 0.$$

This implies that κ can not have local minima. Because $-1 \le \kappa(x) \le 0$ and $\kappa(-\infty) = -1$, $\kappa(+\infty) = 0$, we deduce that κ does not have local maxima neither (which would imply the existence of a local minimum). Therefore

$$\kappa_x \ge 0.$$

This ends the proof of the proposition. □

5.2 Bound from below for the angular velocity

We next prove the following lemma.

Lemma 5.3 (Bound from below on λ)**.** *We have* $\lambda \ge 1/4$.

Proof of Lemma 5.3. The proof proceeds in several steps.

Step 1: comparison. The idea is to revisit the proof of the uniqueness of λ. For some $\mu > 0$, we set

$$\varphi_1 := \varphi \quad \text{and} \quad \varphi_2 := -\mu e^x.$$

If

$$\mu > \lambda,$$

then a comparison of the behaviour at $x = +\infty$ implies that

$$\varphi_2 \le \varphi_1 + K \quad \text{on} \quad \mathbb{R}$$

for some suitable constant K. We recall that

$$\lambda = F(x, \varphi_x, \varphi_{xx}),$$

with F defined in (3.2). We then define

$$\begin{aligned} h_\mu(x) \quad &:= F(x, (\varphi_2)_x, (\varphi_2)_{xx}) \\ &= \sqrt{e^{-2x} + \mu^2} - \mu e^{-x}\left(1 + \frac{1}{1+\mu^2 e^{2x}}\right). \end{aligned}$$

If

$$\alpha \le \inf_{x\in\mathbb{R}} h_\mu(x), \tag{5.12}$$

then we can take $\lambda_2 = \alpha$ and we see with $\lambda_1 := \lambda$ that

$$\lambda_2 t + \varphi_2(x) \le \lambda_1 t + \varphi_1(x) + K$$

is true at $t = 0$ and then is true for every time $t \ge 0$, because the left hand side is a subsolution and the right hand side is a solution. Then we conclude that

$$\alpha = \lambda^2 \le \lambda^1 = \lambda$$

i.e.

$$\mu > \lambda \quad \Longrightarrow \quad \lambda \ge \alpha \quad \text{if} \quad \alpha \text{ satisfies (5.12)}. \tag{5.13}$$

Step 2: estimate on α and conclusion. Remark that (5.12) is satisfied for $\alpha \geq 0$ if and only if

$$\left(\alpha + \mu e^{-x}\left(1 + \frac{1}{1+\mu^2 e^{2x}}\right)\right)^2 \leq e^{-2x} + \mu^2 \tag{5.14}$$

Because we have

$$\left(\alpha + \mu e^{-x}\left(1 + \frac{1}{1+\mu^2 e^{2x}}\right)\right)^2 \leq 2\alpha^2 + 2e^{-2x}\mu^2 2^2$$

we see that inequality (5.14) is satisfied in particular if

$$2\alpha^2 \leq \mu^2 \quad \text{and} \quad 8\mu^2 \leq 1.$$

For instance for

$$\mu = 1/(2\sqrt{2}) \quad \text{and} \quad \alpha = 1/4,$$

we conclude from (5.13) that $\lambda \geq 1/4$. This ends the proof of the lemma. □

Proof of Theorem 1.4. Apart from (1.7), Theorem 1.4 is then a consequence of Propositions 3.1, 4.1, 5.1, 5.2 and Lemma 5.3. As far as (1.7) is concerned, it is a simple consequence of

$$0 \leq 1 + \kappa_\Phi = \frac{r\lambda}{\sqrt{1 + r^2\Phi_r^2}}.$$

The proof of Theorem 1.4 is now complete. □

6 A Liouville result

This section is devoted to the proof of a Liouville result (Theorem 1.8) for global solutions of (1.3). This Liouville result will be used in the next section. The Liouville Theorem 1.8 classifies global space-time solutions. Such kind of results have been for instance obtained for certain nonlinear heat equations in [20], where the nonlinearity comes from the source term. On the contrary, the nonlinearity in our problem comes from the geometry itself.

In order to prove Theorem 1.8, we first prove two comparison principles: one for small r's (i.e. in $\mathbb{R} \times [0, r_0^-)$), and one for large r's (i.e. in $\mathbb{R} \times [r_0^+, +\infty)$).

Proposition 6.1 (Comparison principle for small r's)**.** *Given some constant $C > 0$, there exists some $r_0^- = r_0^-(C) > 0$ such that the following holds for every $r_0 \in (0, r_0^-]$. Let $U \in C^{2,1}(\mathbb{R} \times [0, r_0])$ be a subsolution and $V \in C^{2,1}(\mathbb{R} \times [0, r_0])$ be a supersolution of* (1.3) *in $\mathbb{R} \times (0, r_0)$ satisfying*

$$1 + 2V_r(t, 0) \leq 0 \leq 1 + 2U_r(t, 0) \quad \textit{for all} \quad t \in \mathbb{R}. \tag{6.1}$$

Assume moreover that we have

$$\begin{cases} |U_r|, |V_r| \leq C, \\ |rU_{rr}| \leq C, \\ |U - V| \leq C. \end{cases} \tag{6.2}$$

If $U \leq V$ in $\mathbb{R} \times \{r_0\}$, then $U \leq V$ in $\mathbb{R} \times [0, r_0]$.

Remark 6.2 (The Neumann boundary condition)**.** Notice that condition (6.1) can be seen as the evaluation on the boundary $r = 0$ of the inequalities in equation (1.3) associated to subsolutions U and supersolutions V.

Proof of Proposition 6.1. The proof proceeds in several steps.

Step 1: subsolution $W = U - V$. We set $W = U - V$. We write the difference of the two inequalities satisfied by U and V, which gives

$$rW_t \le G(rU_r) - G(rV_r) + U_r(K(rU_r) - K(rV_r)) \\ + (U_r - V_r)K(rV_r) + rU_{rr}(H(rU_r) - H(rV_r)) + rH(rV_r)W_{rr}$$

with

$$G(p) = \sqrt{1+p^2}, \quad K(p) = \frac{2+p^2}{1+p^2}, \quad H(p) = \frac{1}{1+p^2}. \tag{6.3}$$

This leads to

$$W_t \le AW_r + H(rV_r)W_{rr} \quad \text{on} \quad \mathbb{R} \times (0, r_0) \tag{6.4}$$

with

$$A = a + \frac{K(rV_r)}{r} + b + c \tag{6.5}$$

where

$$\begin{cases} a = \int_0^1 ds\ G'(r(U_r - sW_r)), \\ b = U_r \int_0^1 ds\ K'(r(U_r - sW_r)), \\ c = rU_{rr} \int_0^1 ds\ H'(r(U_r - sW_r)). \end{cases} \tag{6.6}$$

Using (6.2) and the fact that $|G'(p)| \le |p|$ and $|K'(p)| = |H'(p)| \le 2|p|$, this implies that

$$A \ge -r_0 C + \frac{2}{r_0} - 2r_0C^2 - 2r_0C^2.$$

Choosing then $r_0 = r_0(C) > 0$ small enough, we deduce that

$$A \ge 0 \quad \text{and} \quad H(rV_r) \ge \frac{1}{2}. \tag{6.7}$$

Step 2: supersolution Ψ. The goal is now to construct a non-negative supersolution (i.e. satisfying the reverse inequality in (6.4)) which explodes as $|t| \to +\infty$. We define for some $\mu > 0$

$$\Psi(r,t) = e^{-\mu t}\zeta(r) + f(t)$$

with

$$0 \le f \in C^\infty(\mathbb{R}) \quad \text{s.t.} \quad \begin{cases} f(t) = 0 \quad \text{if } t < 0 \\ f' \ge 0 \\ f(t) \to +\infty \quad \text{as } t \to +\infty \end{cases}$$

such that we have

$$\begin{cases} -\mu\zeta \ge \frac{1}{2}\zeta_{rr} & \text{in} \quad (0,r_0), \\ \zeta_r(0) = 0. \end{cases}$$

We can simply choose $\zeta(r) := \cos\left(\frac{\pi}{4}\frac{r}{r_0}\right)$ with $2\mu := \left(\frac{\pi}{4r_0}\right)^2$. Because $\zeta_r \le 0, \zeta_{rr} \le 0$ on $(0,r_0)$, we get, using (6.7), that

$$\Psi_t \ge \frac{1}{2}\Psi_{rr} \ge A\Psi_r + H(ru_r)\Psi_{rr} \quad \text{on} \quad \mathbb{R}\times(0,r_0).$$

Step 3: contact point. Notice that $\Psi \ge \delta > 0$ on $\mathbb{R}\times[0,r_0]$. Then for $\varepsilon > 0$ large enough, we have:

$$\varepsilon\Psi \ge W \quad \text{on} \quad \mathbb{R}\times[0,r_0].$$

We can then decrease ε untill we get a contact point,

$$\varepsilon^* = \inf\{\varepsilon \ge 0,\ \varepsilon\Psi \ge W \quad \text{on} \quad \mathbb{R}\times[0,r_0]\}.$$

We now want to show that $\varepsilon^* = 0$. By contradiction, assume that $\varepsilon^* > 0$. We have

$$\inf_{(t,r)\in\mathbb{R}\times[0,r_0]}\{\varepsilon^*\Psi - W\} = 0. \tag{6.8}$$

Because W is bounded and

$$\liminf_{|t|\to+\infty}\inf_{r\in[0,r_0]}\Psi(t,r) = +\infty$$

we deduce that the infimum in (6.8) is reached at some point $(t^*,r^*)\in\mathbb{R}\times[0,r_0]$. Because $\varepsilon^*\Psi \ge \varepsilon^*\delta > 0$ and $W \le 0$ for $r = r_0$ we deduce that $r^* \in [0,r_0)$. Recall that

$$\bar{W} = \varepsilon^*\Psi - W$$

solves

$$\begin{cases} \left.\begin{array}{l} \bar{W}_t \ge A\bar{W}_r + H(rV_r)\bar{W}_{rr}, \\ \bar{W} \ge 0 \end{array}\right| & \text{on} \quad \mathbb{R}\times(0,r_0), \\ \bar{W}(t^*,r^*) = 0, \\ \bar{W}_r(t,0) \le 0 \quad \text{for all} \quad t\in\mathbb{R}, \end{cases}$$

and as a consequence of our assumptions, the functions A and $H(rV_r)$ are continuous on $\mathbb{R}\times(0,r_0]$.

Case 1: $r^* > 0$. Then we can apply the strong maximum principle (see Theorem A.1) and deduce that

$$\varepsilon^*\Psi = W \quad \text{on} \quad (-\infty,t^*]\times[0,r_0], \tag{6.9}$$

which is absurd for $r = r_0$.

Case 2: $r^* = 0$**.** If the coefficient A would have been continuous up to $r = 0$, then we would have applied Hopf lemma (see Lemma A.2) to deduce again (6.9), in order to get the same contradiction.

The difficulty here is that the coefficient A blows-up as r goes to zero. We can easily circumvent this difficulty, if we replace Ψ with

$$\tilde{\Psi} := \Psi - \eta r$$

for some $\eta > 0$ small enough. Now at the point $(t^*, 0)$ of minimum of $\bar{W} = \varepsilon^* \tilde{\Psi} - W$, we get in particular that

$$0 \le \bar{W}_r(t^*, 0) = -\varepsilon^* \eta - W_r(t^*, 0).$$

On the other hand, we have by assumption

$$W_r(t^*, 0) = (U_r - V_r)(t^*, 0) \ge 0$$

which gives a contradiction. Therefore, in all cases, we conclude that $\varepsilon^* = 0$, which means that $W \le 0$. This ends the proof of the proposition. □

Proposition 6.3 (Comparison principle for large r's)**.** *Given some constants $\lambda > 0$, $\delta > 0$ and $L_0 \ge 1$, there exists $r_0^+ = r_0^+(\delta, L_0, \lambda) > 0$ such that the following holds for all $r_0 \in [r_0^+, +\infty)$. Let $U \in C^{2,1}(\mathbb{R} \times [r_0, +\infty))$ be a subsolution and $V \in C^{2,1}(\mathbb{R} \times [r_0, +\infty))$ be a supersolution of (1.3) on $\mathbb{R} \times (r_0, +\infty)$, satisfying in $\mathbb{R} \times [r_0, +\infty)$,*

$$\begin{cases} -L_0 \le U_r, V_r \le -\delta, \\ |U(t,r) - \lambda t - \Phi_0(r)| \le C, \\ |V(t,r) - \lambda t - \Phi_0(r)| \le C, \\ |(\Phi_0)_r(r)| \le L_0 \end{cases} \tag{6.10}$$

for some function Φ_0 and some constant $C > 0$.

If $U \le V$ on $\mathbb{R} \times \{r_0\}$, then $U \le V$ in $\mathbb{R} \times [r_0, +\infty)$.

Proof of Proposition 6.3. We have:

$$U_t \le \frac{1}{r} G(rU_r) + \frac{U_r}{r} K(rU_r) + U_{rr} \sigma^2(rU_r)$$

and

$$V_t \ge \frac{1}{r} G(rV_r) + \frac{V_r}{r} K(rV_r) + V_{rr} \sigma^2(rV_r)$$

with

$$G(a) = \sqrt{1 + a^2}, \quad K(a) = \frac{2 + a^2}{1 + a^2}, \quad \sigma(a) = \frac{1}{\sqrt{1 + a^2}}.$$

By contradiction, assume that

$$M = \sup_{(t,r) \in \mathbb{R} \times [r_0, +\infty)} \{U(t,r) - V(t,r)\} > 0.$$

For $\alpha, \eta > 0$, we set

$$M_{\alpha,\eta} = \sup_{t \in \mathbb{R},\ r, \rho \ge r_0} \left\{ U(t,r) - V(t,\rho) - \frac{|r - \rho|^2}{2\alpha} - \alpha \frac{r^2}{2} - \eta \frac{t^2}{2} \right\}$$

which satisfies

$$M_{\alpha,\eta} \geq \frac{M}{2} > 0 \quad \text{for } \alpha, \eta \text{ small enough.} \tag{6.11}$$

Since $U(t,r) - V(t,\rho) \leq 2C + \Phi_0(r) - \Phi_0(\rho) \leq 2C + L_0|r-\rho|$ (using the L_0-Lipschitz property of the profile Φ_0), we deduce that this supremum is reached at a point that we denote by (t,r,ρ). It satisfies

$$\eta\frac{t^2}{2} + \alpha\frac{r^2}{2} \leq 2C - \frac{M}{2} + L_0|r-\rho| - \frac{|r-\rho|^2}{2\alpha} \leq 2C - \frac{M}{2} + \frac{\alpha L_0^2}{2}$$

which in turn implies (for fixed $\alpha > 0$)

$$\lim_{\eta\to 0} \eta t = 0. \tag{6.12}$$

We next distinguish two cases.

Case 1: $r, \rho > r_0$**.** In that case, setting $\tilde{U}(t,r) = U(t,r) - \alpha\frac{r^2}{2}$, we get with $a = \tilde{U}_t(t,r), b = V_t(t,\rho), A = \tilde{U}_{rr}(t,r), B = V_{rr}(t,\rho)$ that

$$a \leq \frac{1}{r}G(rp + \alpha r^2) + \frac{p+\alpha r}{r}K(rp + \alpha r^2) + (A+\alpha)\sigma^2(rp + \alpha r^2) \tag{6.13}$$

$$b \geq \frac{1}{\rho}G(\rho p) + \frac{p}{\rho}K(\rho p) + B\sigma^2(\rho p) \tag{6.14}$$

$$a - b = \eta t$$

$$\begin{pmatrix} A & 0 \\ 0 & -B \end{pmatrix} \leq \frac{1}{\alpha}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \tag{6.15}$$

where $p := \dfrac{r-\rho}{\alpha}$ satisfies (using equation (6.10) with $p = V_r(t,\rho) = \tilde{U}_r(t,r)$)

$$-L_0 \leq p, p + \alpha r \leq -\delta.$$

Subtracting (6.14) to (6.13), we get that

$$\eta t \leq I_1 + I_2 + I_3 \tag{6.16}$$

where

$$I_1 := \frac{1}{r}G(rp + \alpha r^2) - \frac{1}{\rho}G(\rho p), \quad I_2 := \frac{p+\alpha r}{r}K(rp + \alpha r^2) - \frac{p}{\rho}K(\rho p)$$

and

$$I_3 := (A+\alpha)\sigma^2(rp + \alpha r^2) - B\sigma^2(\rho p).$$

Estimate on I_1. We have

$$\begin{aligned} I_1 =& \frac{1}{r}G(rp+\alpha r^2) - \frac{1}{r}G(\rho p) + \frac{1}{r}G(\rho p) - \frac{1}{\rho}G(\rho p) \\ \leq & G'(-r_0\delta)\left(\frac{(r-\rho)p}{r} + \alpha r\right) + \left(\frac{\rho - r}{r\rho}\right) G(\rho p) \\ \leq & G'(-r_0\delta)\left(\frac{\alpha p^2}{r} + \alpha r\right) + \left(\frac{\rho - r}{r\rho}\right)(1+\rho|p|) \\ \leq & G'(-r_0\delta)\left(\frac{\alpha p^2}{r} + \alpha r\right) + \frac{\alpha|p|}{r\rho} + \frac{\alpha p^2}{r} \end{aligned}$$

where, for the second line, we have used that $rp+\alpha r^2, \rho p \leq -r_0\delta$ and G' is non-decreasing on $(-\infty, -\delta r_0)$ and for the third line, we have used that $G(a) \leq 1+|a|$. Choosing r_0 such that $G'(-r_0\delta) \leq -\frac{1}{2}$, and such that $r_0 \geq \frac{1}{\delta} \geq 1$ we get that

$$I_1 \leq -\frac{1}{2}\alpha r + 2\alpha L_0^2. \tag{6.17}$$

where we have used that $|p| \leq L_0$ and $L_0 \geq 1$.

Estimate on I_2. Using that K is bounded by 2, we have

$$\begin{aligned} I_2 &\leq \frac{p}{r}K(rp+\alpha r^2) - \frac{p}{\rho}K(\rho p) + 2\alpha \\ &\leq \frac{p}{r}\left(K(rp+\alpha r^2) - K(\rho p)\right) + 2\alpha \end{aligned}$$

where we have used the fact that $p \leq 0$, $\rho \geq r$. Using now the fact that K is non-decreasing on $(-\infty, 0)$ and that $0 \geq rp + \alpha r^2 \geq \rho p$, we get that

$$I_2 \leq 2\alpha. \tag{6.18}$$

Estimate on I_3. Using the matrix inequality (6.15), we have that for all $\xi, \zeta \in \mathbb{R}$

$$A\xi^2 \leq B\zeta^2 + \frac{(\xi-\zeta)^2}{\alpha}.$$

Using also that σ is bounded by 1, we get

$$I_3 \leq \alpha + \frac{1}{\alpha}\left(\sigma(rp+\alpha r^2) - \sigma(\rho p)\right)^2 \leq \alpha + \frac{1}{\alpha}\left(\|\sigma'\|_{L^\infty(\rho p, rp+\alpha r^2)}((r-\rho)p + \alpha r^2)\right)^2.$$

Since $|\sigma'(a)| \leq \frac{1}{a^2}$, we have $\|\sigma'\|_{L^\infty(\rho p, rp+\alpha r^2)} \leq \frac{1}{(r(p+\alpha r))^2} \leq \frac{1}{(r\delta)^2}$. Hence we get

$$I_3 \leq \alpha + \frac{1}{\alpha}\left(\frac{(r-\rho)p + \alpha r^2}{(r\delta)^2}\right)^2 = \alpha + \frac{1}{\delta^4\alpha}\left(\frac{\alpha p^2}{r^2} + \alpha\right)^2 \leq \alpha + \frac{4\alpha L_0^4}{\delta^4} \tag{6.19}$$

where for the last inequality, we have used that $r \geq r_0 \geq 1$ and $|p| \leq L_0$ with $L_0 \geq 1$.

Combining (6.16), (6.17), (6.18) and (6.19), we finally get

$$\eta t \leq -\frac{1}{2}\alpha r + 5\alpha L_0^2 + \frac{4\alpha L_0^4}{\delta^4}.$$

Taking the limit $\eta \to 0$ and using (6.12), we get (using $L_0 \geq 1$)

$$0 \leq -\frac{1}{2}\alpha r + 5\alpha L_0^2 + \frac{4\alpha L_0^4}{\delta^4}$$

which is absurd for $r \geq r_0 > 10L_0^2 + \dfrac{8L_0^4}{\delta^4}$.

Case 2: $r = r_0$ or $\rho = r_0$. Assume for instance that $r = r_0$ (the case $\rho = r_0$ being similar). Using that $M_{\alpha,\eta} > 0$ for α and η small enough, we get that

$$\frac{|r_0 - \rho|^2}{2\alpha} + \frac{\alpha}{2}r_0^2 \leq U(t, r_0) - V(t, \rho) \leq V(t, r_0) - V(t, \rho) \leq L_0|r_0 - \rho|.$$

This implies in particular that $|r_0 - \rho| \leq 2\alpha L_0$. Injecting this in the previous inequality, we obtain that

$$\frac{\alpha}{2}r_0^2 \leq 2\alpha L_0^2$$

which is absurd for $r_0 > 2L_0$. This ends the proof of the proposition. □

Before proving Liouville Theorem 1.8, we first prove Theorem 4.7 that has been used in Subsection 4.3.

Proof of Theorem 4.7. For all $\nu \in \mathbb{R}$, we define

$$w^\nu = \Phi^1 - \Phi^2 + \nu.$$

In view of (4.15), we can choose $\nu \geq 0$ big enough so that $w^\nu \geq 0$. We then define

$$\nu^* = \inf\{\bar{\nu} \geq 0 : \ w^\nu \geq 0 \text{ in } [0, +\infty), \quad \text{for all} \quad \nu \geq \bar{\nu}\}.$$

We want to show that $\nu^* = 0$. By contradiction, let us assume that $\nu^* > 0$. Using (4.15), we then have

$$\begin{cases} w^{\nu^*} \geq 0 \\ w^{\nu^*}(r) > 0 \quad \text{for } r \text{ large enough} \\ \inf\limits_{r \in [0,+\infty)} w^{\nu^*}(r) = 0. \end{cases}$$

From Propositions 6.1 and 6.3, we deduce that we have

$$\inf_{r \in [r_0^-, r_0^+]} w^{\nu^*}(r) = \inf_{r \in [0,+\infty)} w^{\nu^*}(r) = 0$$

with $0 < r_0^- < r_0^+$. Using again the Strong Maximum Principle (Theorem A.1), we deduce that $w^{\nu^*} \equiv 0$. For $r = +\infty$, this implies that $\nu^* = 0$. Contradition. Therefore $\nu^* = 0$ and $\Phi^1 \geq \Phi^2$. Exchanging Φ^1 and Φ^2, we get the reverse inequality. This shows that $\Phi^1 = \Phi^2$ and ends the proof. □

We now prove Theorem 1.8.

Proof of Theorem 1.8. The proof proceeds in several steps.

Step 0: regularity and condition at $r = 0$. Because U is globally Lipschitz continuous (in space and time), we can apply Proposition 2.2 to conclude that $U \in C^{1,2}(\mathbb{R} \times [0, +\infty))$. By continuity in equation (1.3) up to $r = 0$, we deduce that U satisfies

$$U_r(t, 0) = -\frac{1}{2} \quad \text{for all} \quad t \in \mathbb{R}.$$

Finally, from Lemma 2.3, we have

$$|U_{rr}(t, r)| \leq C(1 + r^2) \quad \text{for all} \quad (t, r) \in \mathbb{R} \times [0, +\infty).$$

Step 1: preliminaries for the sliding method. We apply the sliding method (see [4]). For any $h \in \mathbb{R}$, we set

$$U^h(t, r) = U(t + h, r).$$

Since U satisfies (1.10), one can choose $b \geq 0$ large enough so that $U^h + b \geq U$ on $\mathbb{R} \times [0, +\infty)$. We now consider

$$b^* = \inf\{b \in \mathbb{R} : U^h + b \geq U\}$$

and we set

$$V := U^h + b^* \geq U.$$

Notice that, using in particular Step 0, we can check that the assumptions of Propositions 6.1 and 6.3 are fulfilled with $0 < r_0^- < r_0^+ < +\infty$ (decreasing r_0^- and increasing r_0^+ if necessary).

We claim that this implies

$$m := \inf_{(t,r) \in \mathbb{R} \times [r_0^-, r_0^+]} (V - U) = 0. \tag{6.20}$$

Indeed, if $m > 0$, applying Propositions 6.1 and 6.3, we deduce that

$$V - U \geq m > 0 \quad \text{on} \quad \mathbb{R} \times [0, +\infty)$$

which contradicts the definition of b^*. Therefore (6.20) holds true.

Step 2: consequence. We distinguish two cases.

Case 1: the infimum in (6.20) is reached at (t_0, r_0). We have

$$\begin{cases} V \geq U & \text{on} \quad \mathbb{R} \times [0, +\infty), \\ V = U & \text{at} \quad (t_0, r_0) \in \mathbb{R} \times [r_0^-, r_0^+]. \end{cases}$$

Notice that $W = V - U$ satisfies

$$W_t = AW_r + H(rV_r)W_{rr}$$

with A and H defined in (6.5) and (6.3). Moreover A and $H(rV_r)$ are continuous functions because $U, V \in C^{2,1}(\mathbb{R} \times [0, +\infty))$.

From the strong maximum principle (Theorem A.1) applied to W, we deduce that

$$V \equiv U$$

which gives for all $k \in \mathbb{Z}$

$$U(t, r) = U(t + h, r) + b^* = U(t + kh, x) + kb^*.$$

In view of (1.10), this implies that $b^* = -\lambda h$, i.e.

$$U(t + h, r) = U(t, r) + \lambda h.$$

Case 2: the infimum in (6.20) is reached at infinity. We now assume that there exists sequences $(t_n)_n$ and $r_n \in [r_0^-, r_0^+]$ such that $|t_n| \to +\infty$, $r_n \to r_\infty \in [r_0^-, r_0^+]$ and $(V-U)(t_n, r_n) \to m$. We define the functions

$$U_n(t,r) := U(t+t_n, r) - \lambda t_n, \quad V_n(t,r) = V(t+t_n, r) - \lambda t_n$$

which have the same Lipschitz constant (in space and time) as the one of U. We can then apply Ascoli-Arzelà Theorem, to deduce that, up to a subsequence, we have

$$U_n \to U_\infty, \quad V_n \to V_\infty, \quad \text{and} \quad V_\infty(t,r) = U_\infty(t+h, r) + b^*$$

where U_∞, V_∞ are two globally Lipschitz solutions of (1.3) on $\mathbb{R} \times (0, +\infty)$ satisfying again

$$\begin{cases} V_\infty \geq U_\infty & \text{on} \quad \mathbb{R} \times [0, +\infty), \\ V_\infty = U_\infty & \text{at} \quad (0, r_\infty) \in \mathbb{R} \times [r_0^-, r_0^+]. \end{cases}$$

We can then repeat Step 0 and then case 1 for (U, V) replaced by (U_∞, V_∞) and get that $b^* = -\lambda h$, and then $V \geq U$ means

$$U(t+h, r) \geq U(t,r) + \lambda h. \tag{6.21}$$

Step 3: conclusion. Notice that (6.21) means that $t \mapsto U(t,r) - \lambda t$ is both nondecreasing (using $h > 0$) and nonincreasing (using $h < 0$). This implies that

$$U(t,r) - \lambda t = U(0,r).$$

From (1.11), we have in particular

$$U_r(0,r) \leq 0$$

and by our assumptions $U(0,r)$ is globally Lipschitz in the variable r. Then Theorem 1.4 i) implies that there exists a constant $a \in \mathbb{R}$ such that

$$U(0,r) = \Phi(r) + a.$$

This ends the proof of the theorem. □

7 Long time convergence

In order to prove Theorem 1.6 we need the following proposition, whose proof is postponed.

Proposition 7.1 (Gradient estimate from above). *Let $T > 0$ and let U be a solution of (1.3)-(1.4) in $(0,T) \times (0, +\infty)$, such that U is globally Lipschitz continuous with respect to time. Assume that there exists a constant C such that for all $(t,r) \in (0,T) \times (0,+\infty)$,*

$$|U(t,r) - \lambda t - \Phi(r)| \leq C. \tag{7.1}$$

If the initial datum U_0 satisfies

$$(U_0)_r \leq \Phi_r \quad \text{in} \quad (0, +\infty) \tag{7.2}$$

then we have

$$U_r \leq \Phi_r \quad \text{in} \quad (0,T) \times (0,+\infty).$$

Proof of Theorem 1.6. By Theorem 2.1, there exists a unique solution U to (1.3), (1.4) which is globally Lipschitz continuous (in space and time). Notice that $\lambda t + \Phi(r)$ is a global solution. Therefore, using (1.8) and applying the comparison principe (see [10, Theorem 1.3]), we deduce the following estimate for all times,

$$|U(t,r) - \lambda t - \Phi(r)| \le C \tag{7.3}$$

Finally using (1.9) and applying Proposition 7.1, we deduce that

$$U_r \le \Phi_r \le \delta < 0. \tag{7.4}$$

Then for any sequence $t_n \to +\infty$, by Ascoli-Arzelà theorem, we get the convergence (for a subsequence still denoted by $(t_n)_n$),

$$U(t+t_n, r) - U(t_n, 0) \to U_\infty(t,r) \quad \text{locally uniformly on compact sets}$$

where U_∞ is still globally Lipschitz continuous and still satisfies (7.3) and (7.4). Therefore the Liouville result (Theorem 1.8) implies that there exists a number $a \in \mathbb{R}$ such that

$$U_\infty(t,r) = \lambda t + \Phi(r) + a.$$

This ends the proof of the theorem. □

Proof of Proposition 7.1. We have to prove that for $r > \rho > 0$

$$U(t,r) - U(t,\rho) \le \Phi(r) - \Phi(\rho).$$

Using log coordinates and setting $u(t,x) = U(t,e^x)$ and $\phi(x) = \Phi(e^x)$, this is equivalent to prove that for $x > y > -\infty$

$$u(t,x) - u(t,y) \le \phi(x) - \phi(y).$$

Recall that u and $\lambda t + \phi(x)$ are both solutions of the following equation

$$u_t = F(x, u_x, u_{xx}) = e^{-x}\sqrt{1+u_x^2} + e^{-2x} u_x + e^{-2x} \frac{u_{xx}}{1+u_x^2}.$$

By contradiction, assume that

$$M = \sup_{x>y,\ t\in[0,T)} \{u(t,x) - u(t,y) - \phi(x) + \phi(y)\} > 0.$$

For $\varepsilon, \alpha, \eta > 0$, we consider the following approximate supremum,

$$M_{\varepsilon,\alpha,\eta} = \sup_{x>y,\ t,s\in[0,T)} \left\{ u(t,x) - u(s,y) - \phi(x) + \phi(y) - \frac{|t-s|^2}{2\varepsilon} - \frac{\alpha}{2}x^2 - \frac{\eta}{T-t} \right\}. \tag{7.5}$$

Remark that when the penalization parameters α and η are small enough, we have

$$M_{\varepsilon,\alpha,\eta} \ge M/2 > 0.$$

From (7.1), we deduce that $u(t,x) - u(s,y) - \phi(x) + \phi(y)$ is bounded by $2C + \lambda T$, and then the supremum in (7.5) is reached at a point that we denote by (t,x,s,y) which satisfies

$$\frac{|t-s|^2}{2\varepsilon} + \frac{\alpha}{2}x^2 + \frac{\eta}{T-t} \le 2C + \lambda T.$$

We deduce in particular that

$$\lim_{\alpha\to 0} \alpha x = 0. \tag{7.6}$$

The proof is divided into two cases.

Case 1: there exists $\varepsilon_n \to 0$ such that $t = 0$ or $s = 0$. Assume for example that $t = 0$ (if $s = 0$, a similar reasoning provides the same contradiction). Then we have

$$\begin{aligned}\frac{\eta}{T} &< u(0,x) - u(s,y) - \phi(x) + \phi(y) - \frac{s^2}{2\varepsilon} \\ &\le u(0,y) - u(s,y) - \frac{s^2}{2\varepsilon} \le Ls - \frac{s^2}{2\varepsilon} \le \frac{\varepsilon L^2}{2}\end{aligned}$$

where in the second line we have used (7.2) and then used L, which denotes the Lipschitz constant in time of U. This is absurd for ε small enough.

Case 2: for all ε small enough we have $t, s > 0$. In that case, using that the function

$$(t',x') \mapsto u(t',x') - u(s,y) - \phi(x') + \phi(y) - \frac{|t'-s|^2}{2\varepsilon} - \frac{\alpha}{2}(x')^2 - \frac{\eta}{T-t'}$$

reaches a maximum at (t,x), we deduce that

$$\frac{t-s}{\varepsilon} + \frac{\eta}{(T-t)^2} \le F(x, \phi_x(x) + \alpha x, \phi_{xx}(x) + \alpha).$$

Similarly, we have that

$$\frac{t-s}{\varepsilon} \ge F(y, \phi_x(y), \phi_{xx}(y)).$$

Subtracting these two inequalities, we get

$$\begin{aligned}\frac{\eta}{T^2} \le & F(x, \phi_x(x) + \alpha x, \phi_{xx}(x) + \alpha) - F(y, \phi_x(y), \phi_{xx}(y)) \\ \le & F(x, \phi_x(x) + \alpha x, \phi_{xx}(x) + \alpha) - F(x, \phi_x(x), \phi_{xx}(x)) \\ & + F(x, \phi_x(x), \phi_{xx}(x)) - F(y, \phi_x(y), \phi_{xx}(y)) \\ \le & F(x, \phi_x(x) + \alpha x, \phi_{xx}(x) + \alpha) - F(x, \phi_x(x), \phi_{xx}(x)) + \lambda - \lambda\end{aligned}$$

which gives

$$\frac{\eta}{T^2} \le e^{-x}\alpha|x| + e^{-2x}\alpha x + e^{-2x}\alpha + I \tag{7.7}$$

with

$$I := e^{-2x}\phi_{xx}(x)\left(\frac{1}{1 + (\phi_x(x) + \alpha x)^2} - \frac{1}{1 + (\phi_x(x))^2}\right).$$

We write

$$I := e^{-2x}\phi_{xx}(x)\ J, \quad J := H(\phi_x(x) + \alpha x) - H(\phi_x(x)), \quad H(p) := \frac{1}{1+p^2}.$$

Estimate on J. We observe that the function H is concave in $\left[-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]$ and convex outside. Recalling (1.6), we also see that

$$e^{-x}\phi_x(x) = \Phi_r(e^x) \in [-1, -\lambda]. \tag{7.8}$$

We now define some $b > 0$ such that

$$\begin{cases} -\frac{1}{2\sqrt{3}} \le \phi_x(x) \le 0 & \text{for all} \quad x \le -b < 0, \\ \phi_x(x) \le -\frac{2}{\sqrt{3}} & \text{for all} \quad x \ge b > 0, \end{cases}$$

We call L_1 the Lipschitz constant of H. Using (7.6), we can assume αx small enough. For instance, for $|\alpha x| \le \frac{1}{2\sqrt{3}}$, we deduce from the convexity/concavity property of H that

(7.9)
$$\begin{cases} \alpha x \dfrac{C}{|\phi_x(x)|^3} \ge \alpha x H'(\phi_x(x) + \alpha x) \ge J \ge \alpha x H'(\phi_x(x)) \ge 0 & \text{for all} \quad x \ge b > 0, \\ \alpha x L_1 \le J \le \alpha x H'(\phi_x(x)) \le 0 & \text{for all} \quad x \le -b < 0, \end{cases}$$

where now $C > 0$ is generic constant that can change from line to line.

Estimate on I. Notice that $\lambda t + \Phi(r)$ is a globally Lipschitz continuous solution of (1.3), and then Lemma 2.3 implies the bound (2.2), namely

$$|\Phi_{rr}(r)| \le C(1 + r^2).$$

Because $\Phi_{rr}(e^x) = e^{-2x}\phi_{xx} - e^{-2x}\phi_x$, we deduce that

$$I \le e^{-2x}\phi_x J + C(1 + e^{2x})|J|.$$

We deduce, using (7.8) and (7.9), that

$$I \le \begin{cases} Ce^{-x}\alpha x & \text{for all} \quad x \ge b > 0, \\ Ce^{-x}\alpha|x| + C\alpha|x| & \text{for all} \quad x \le -b < 0, \end{cases}$$

which can be rewritten as

$$I \le Ce^{-x}\alpha|x| \quad \text{for all} \quad |x| \ge b > 0.$$

Using (7.7), this leads to

$$\frac{\eta}{T^2} \le Ce^{-x}\alpha|x| + e^{-2x}\alpha x \quad \text{for all} \quad |x| \ge b > 0 \tag{7.10}$$

We now distinguish several cases.

Assume first that there exists $\alpha \to 0$ such that $x \le -b$. Increasing $b > 0$ if necessary, we can assume that $Ce^{-x}\alpha|x| + e^{-2x}\alpha x \le 0$ for all $x \le -b$, which gives a contradiction.

Second, assume that there exists $\alpha \to 0$ such that $x \ge b$. For $x \ge b > 0$, sending $\alpha \to 0$ in (7.10), we get a contradiction.

Finally, assume that for all α small enough, we have $-b \le x \le b$. In that case, we have from (7.7)

$$\frac{\eta}{T^2} \le e^{-x}\alpha|x| + e^{-2x}\alpha x + e^{-2x}\alpha + e^{-2x}|\phi_{xx}(x)|L_1\alpha|x|.$$

Again, sending $\alpha \to 0$, we get a contradiction. This ends the proof of the proposition. □

A Appendix

1.1 Strong maximum principle and Hopf lemma

In this subsection, we recall the classical strong maximum principle and Hopf lemma.
For $-\infty \leq t_1 < t_2 \leq +\infty$ and $0 < R \leq +\infty$, let us consider the following general linear parabolic equation

$$w_t = a(t,r)w_{rr} + b(t,r)w_r + c(t,r)w \quad \text{on} \quad Q := (t_1,t_2) \times (0,R) \tag{A.1}$$

with the following assumptions on the coefficients

$$\begin{cases} a,b,c \in C(\overline{Q}), \\ a \geq \delta > 0 \quad \text{on} \quad \overline{Q} \end{cases} \tag{A.2}$$

For $A = Q$ or $\overline{Q}$, we recall that we say that a function $w \in C^{2,1}(A)$ if and only if $w, w_r, w_{rr}, w_t \in C(A)$. Then we have the following classical result.

Theorem A.1 (Strong maximum principle [24]). *Consider a function $w \in C^{2,1}(Q)$ which is a supersolution of* (A.1). *If*

$$\begin{cases} w \geq 0 & \text{on} \quad Q, \\ w = 0 & \text{at} \quad (t_0, r_0) \in Q \end{cases}$$

then $w \equiv 0$ on $Q \cap \{t \leq t_0\}$.

We also have (see [19, Lemma 2.8]).

Lemma A.2 (Hopf lemma). *Consider a function $w \in C^{2,1}(\overline{Q})$ which is a supersolution of* (A.1). *If*

$$\begin{cases} w \geq 0 & \text{on} \quad \overline{Q}, \\ w = 0 & \text{at} \quad (t_0, 0) \in \partial Q \quad \text{with} \quad t_0 \in (t_1, t_2) \end{cases}$$

then either $w \equiv 0$ on $\overline{Q} \cap \{t \leq t_0\}$ or $w_r(t_0, 0) > 0$.

1.2 Interior Schauder estimate

The following result can be found in Krylov [17] (see also [18, 19]).

Proposition A.3 (Interior Schauder estimates). *Let $T > 0$, $\delta > 0$, $R > 0$ and $\alpha \in (0,1)$ and $N \geq 1$. Assume that w solves (in the sense of distributions)*

$$w_t = a\Delta w + b \quad \text{on} \quad (T-\delta, T+\delta) \times B_R$$

with B_R the ball of radius R in $\mathbb{R}^N$. Assume that $a, b \in C^{\frac{\alpha}{2},\alpha}_{t,x}((T-\delta,T+\delta)\times B_R)$ with for some $\eta > 0$:

$$0 < \eta \leq a \leq 1/\eta \quad \text{on} \quad (T-\delta, T+\delta) \times B_R$$

and

$$\|a\|_{C^{\frac{\alpha}{2},\alpha}_{t,x}((T-\delta,T+\delta)\times B_R)} \leq C_0.$$

Then there exists a constant $C = (\delta, R, \alpha, N, \eta, C_0) > 0$ such that

$$\|w\|_{C^{\frac{\alpha}{2},\alpha}_{t,x}([T,T+\delta)\times B_{R/2})} \leq C \left\{ \|b\|_{C^{\frac{\alpha}{2},\alpha}_{t,x}((T-\delta,T+\delta)\times B_R)} + |w|_{L^\infty((T-\delta,T+\delta)\times B_R)} \right\}.$$

1.3 A technical lemma

Lemma A.4 (A Hölder estimate)**.** *Let* $\alpha \in (0,1)$ *and* $N \geq 1$*. For* $X \in \mathbb{R}^N$*, let us define the function*

$$\zeta(X) := \begin{cases} |X|^\alpha \dfrac{X}{|X|} & \text{if} \quad X \neq 0, \\ 0 & \text{if} \quad X = 0 \end{cases}$$

Then there exists a constant $C = C(\alpha) > 0$ *such that for all* $X', X \in \mathbb{R}^N$*, we have*

$$|\zeta(X') - \zeta(X)| \leq C|X' - X|^\alpha.$$

Proof of Lemma A.4. Let us assume that $|X'| \geq |X| > 0$. We write

$$\zeta(X') - \zeta(X) = T_1 + T_2$$

with

$$T_1 = (|X'|^\alpha - |X|^\alpha)\,\frac{X'}{|X'|} \quad \text{and} \quad T_2 = |X|^\alpha \left(\frac{X'}{|X'|} - \frac{X}{|X|}\right).$$

Step 1: estimate on T_1. We have

$$||X'|^\alpha - |X|^\alpha| = |X|^\alpha\,|r^\alpha - 1| \quad \text{with} \quad r = \frac{|X'|}{|X|} \geq 1. \tag{A.3}$$

Case A: $1 \leq r \leq 2$. We write $r = 1 + \delta$ with $0 \leq \delta \leq 1$. Then we have

$$\begin{aligned} |r^\alpha - 1| &= \alpha\delta + O(\delta^2) \\ &\leq C\delta^\alpha = C|r - 1|^\alpha. \end{aligned}$$

Case B: $r \geq 2$. Then we have

$$|r^\alpha - 1| \leq C|r - 1|^\alpha. \tag{A.4}$$

Putting together cases A and B, we see that (A.4) holds true for any $r \geq 1$. Using (A.3), we get for some $C \geq 1$:

$$|T_1| = ||X'|^\alpha - |X|^\alpha| \leq C||X'| - |X||^\alpha \leq C|X' - X|^\alpha. \tag{A.5}$$

Step 2: estimate on T_2. Writing $e = \dfrac{X}{|X|}$, $Y = \dfrac{X'}{|X|}$ with $|Y| \geq 1$, and using the fact that the map $Z \mapsto \dfrac{Z}{|Z|}$ is 1-Lipschitz (for the euclidean norm) on $\mathbb{R}^N \backslash B(0,1)$, we get that

$$\left|\frac{X'}{|X'|} - \frac{X}{|X|}\right| = \left|\frac{Y}{|Y|} - \frac{e}{|e|}\right| \leq |Y - e| = \frac{|X' - X|}{|X|}. \tag{A.6}$$

Case A: $|X| \leq |X'| \leq 2|X|$. Using (A.6), we deduce that

$$|T_2| \leq |X|^\alpha \frac{|X' - X|}{|X|} = \frac{|X' - X|^{1-\alpha}}{|X|^{1-\alpha}}|X' - X|^\alpha \leq 2^{1-\alpha}\left(\frac{|X' - X|}{|X'|}\right)^{1-\alpha}|X' - X|^\alpha$$

which implies

$$|T_2| \leq 4^{1-\alpha}|X' - X|^\alpha. \tag{A.7}$$

Case B: $|X'| \geq 2|X|$. We have

$$|T_2| \leq |X|^\alpha \leq ||X'| - |X||^\alpha \leq |X' - X|^\alpha$$

Putting together cases A and B, we see that (A.7) holds true for any $|X'| \geq |X| > 0$.

Step 3: conclusion. From Steps 1 and 2, we deduce that there exists a constant $C > 0$ such that

$$|\zeta(X') - \zeta(X)| \leq C|X' - X|^\alpha$$

This last estimate is also true if $X = 0$. By symmetry between X' and X, we see that it is finally true for any $X, X' \in \mathbb{R}^N$. This ends the proof of the lemma. $\square$

1.4 Equation satisfied by the curvature

The following result is not used in the rest of the paper. We give it as an interesting result of independent interest.

Lemma A.5 (Equation satisfied by the curvature)**.** *Let Φ be the profile given by Theorem 1.4. The curvature $\kappa(x) = \kappa_\Phi(e^x)$ solves the following equation*

$$\kappa_t = \frac{e^{-2x}\kappa_{xx}}{1+u_x^2} + \kappa^2(1+\kappa) + e^{-2x}\kappa_x \left\{ -1 + \frac{2u_x^2}{1+u_x^2} + \frac{e^x u_x}{\sqrt{1+u_x^2}} \right\}. \tag{A.8}$$

Proof of Lemma A.5. We start from

$$u_t = e^{-x}\sqrt{1+u_x^2}(1+\kappa)$$

with

$$\kappa = e^{-x}\left(\frac{u_x}{\sqrt{1+u_x^2}} + \frac{u_{xx}}{(1+u_x^2)^{\frac{3}{2}}} \right).$$

Let us define

$$M(a) = \frac{a}{\sqrt{1+a^2}} \quad \text{with} \quad M'(a) = \frac{1}{(1+a^2)^{\frac{3}{2}}}.$$

Then we can write

$$\kappa = e^{-2x}(e^x M(u_x))_x$$

and

$$\kappa_t = e^{-2x}(e^x M'(u_x)u_{xt})_x.$$

We now compute

$$\begin{aligned}
u_{xt} &= e^{-x}\sqrt{1+u_x^2}(\kappa_x - (1+\kappa)) + \frac{e^{-x}}{\sqrt{1+u_x^2}}(1+\kappa)u_x u_{xx} \\
&= e^{-x}\sqrt{1+u_x^2}(\kappa_x - (1+\kappa)) + e^{-x}\sqrt{1+u_x^2}(1+\kappa)u_x \left(\frac{u_{xx}}{1+u_x^2} + u_x - u_x \right) \\
&= e^{-x}\sqrt{1+u_x^2}\kappa_x - e^{-x}\sqrt{1+u_x^2}(1+\kappa)(1+u_x^2) + (1+u_x^2)(1+\kappa)\kappa u_x \\
&= (1+\kappa)\left\{ (1+u_x^2)\kappa u_x - e^{-x}(1+u_x^2)^{\frac{3}{2}} \right\} + e^{-x}\sqrt{1+u_x^2}\kappa_x.
\end{aligned}$$

This gives

$$
\begin{aligned}
e^{2x}\kappa_t &= \partial_x\left((1+\kappa)\left\{\kappa\frac{e^x u_x}{\sqrt{1+u_x^2}}-1\right\}+\frac{\kappa_x}{1+u_x^2}\right)\\
&= \partial_x\left((1+\kappa)\left\{\kappa e^x M(u_x)-1\right\}+\frac{\kappa_x}{1+u_x^2}\right)\\
&= \kappa_x\left\{\kappa e^x M(u_x)-1+(1+\kappa)e^x M(u_x)-\frac{2u_x u_{xx}}{(1+u_x^2)^2}\right\}+\frac{\kappa_{xx}}{1+u_x^2}+(1+\kappa)\kappa e^{2x}\kappa.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\kappa_t &= \frac{e^{-2x}\kappa_{xx}}{1+u_x^2}+(1+\kappa)\kappa^2\\
&+e^{-2x}\kappa_x\left\{-1+(1+2\kappa)e^x M(u_x)-2e^x M(u_x)(\kappa-e^{-x}M(u_x))\right\}\\
&= \frac{e^{-2x}\kappa_{xx}}{1+u_x^2}+(1+\kappa)\kappa^2+e^{-2x}\kappa_x\left\{-1+e^x M(u_x)+2(M(u_x))^2\right\}
\end{aligned}
$$

which shows the result. This ends the proof of the lemma. □

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