A junction condition by specified homogenization

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Abstract

Given a coercive Hamiltonian which is quasi-convex with respect to the gradient variable and periodic with respect to time and space at least "far away from the origin", we consider the solution of the Cauchy problem of the corresponding Hamilton-Jacobi equation posed on the real line. Compact perturbations of coercive periodic quasi-convex Hamiltonians enter into this framework for example. We prove that the rescaled solution converges towards the solution of the expected effective Hamilton-Jacobi equation, but whose "flux" at the origin is limited in the sense of (Imbert, Monneau [9]). In other words, the homogenization of such a Hamilton-Jacobi equation yields to supplement the expected homogenized Hamilton-Jacobi equation with a junction condition at the single discontinuous point of the effective Hamiltonian. We also illustrate possible applications of such a result by deriving, for a traffic flow problem, the effective flux limiter generated by the presence of a finite number of traffic lights on an ideal road.

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1 Introduction

1.1 Setting of the problem

This article is concerned with the study of the limit of the solution $u^\varepsilon(t,x)$ of the following equation

$$u_t^\varepsilon + H\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, u_x^\varepsilon\right) = 0 \quad \text{for } (t,x) \in (0,T) \times \mathbb{R} \tag{1}$$

submitted to the initial condition

$$u^\varepsilon(0,x) = u_0(x) \quad \text{for } x \in \mathbb{R} \tag{2}$$

for a Hamiltonian H satisfying the following assumptions:

(A0) (Continuity) $H \in C(\mathbb{R}^3; \mathbb{R})$.

(A1) (Time periodicity) For all $k \in \mathbb{Z}$ and $(t,x,p) \in \mathbb{R}^3$,

$$H(t+k, x, p) = H(t,x,p).$$

(A2) (Uniform modulus of continuity in time) There exists a modulus of continuity ω such that for all $t,s,x,p \in \mathbb{R}$,

$$H(t,x,p) - H(s,x,p) \le \omega(|t-s|\,(1 + \max(H(s,x,p),0))).$$

(A3) (Uniform coercivity)

$$\lim_{|q|\to+\infty} H(t,x,q) = +\infty$$

uniformly with respect to (t,x).

(A4) (Quasi-convexity of H for large x's) There exists some $\rho_0 > 0$ such that for all $x \in \mathbb{R} \setminus (-\rho_0, \rho_0)$, there exists a continuous map $t \mapsto p^0(t,x)$ such that

$$\begin{cases} H(t,x,\cdot) & \text{is non-increasing in} \quad (-\infty, p^0(t,x)), \\ H(t,x,\cdot) & \text{is non-decreasing in} \quad (p^0(t,x), +\infty). \end{cases}$$

(A5) (Left and right Hamiltonians) There exist two Hamiltonians $H_\alpha(t,x,p)$, $\alpha = L, R$, such that

$$\begin{cases} H(t,x+k,p) - H_L(t,x,p) \to 0 & \text{as} \quad \mathbb{Z} \ni k \to -\infty \\ H(t,x+k,p) - H_R(t,x,p) \to 0 & \text{as} \quad \mathbb{Z} \ni k \to +\infty \end{cases}$$

uniformly with respect to $(t,x,p) \in [0,1]^2 \times \mathbb{R}$, and for all $k,j \in \mathbb{Z}$, $(t,x,p) \in \mathbb{R}^3$ and $\alpha \in \{L, R\}$,

$$H_\alpha(t+k, x+j, p) = H_\alpha(t,x,p).$$

Finally, in order to ensure appropriate properties of the effective Hamiltonians $\bar{H}_\alpha$, we assume one of the following properties.

(B-i) (Quasi-convexity of the left and right Hamiltonians) For each $\alpha = L, R$, H_α does not depend on time and there exists p_α^0 (independent on (t, x)) such that

$$\begin{cases} H_\alpha(x,\cdot) & \text{is non-increasing on} \quad (-\infty, p_\alpha^0) \\ H_\alpha(x,\cdot) & \text{is non-decreasing on} \quad (p_\alpha^0, +\infty) \end{cases}$$

(B-ii) (Convexity of the left and right Hamiltonians) For each $\alpha = L, R$, and for all $(t, x) \in \mathbb{R} \times \mathbb{R}$, the map $p \mapsto H_\alpha(t, x, p)$ is convex.

Example 1.1. A simple example of such a Hamiltonian is

$$H(t, x, p) = |p| - f(t, x)$$

with a continuous function f satisfying $f(t+1, x) = f(t, x)$ and $f(t, x) \to 0$ as $|x| \to +\infty$ uniformly with respect to $t \in \mathbb{R}$.

1.2 Main results

Our main result is concerned with the limit of the solution u^ε of (1)-(2). This limit satisfies an effective Hamilton-Jacobi equation posed on the real line whose Hamiltonian is discontinuous. More precisely, the effective Hamiltonian equals the one which is expected (see **(A5)**) in $(-\infty; 0)$ and $(0; +\infty)$; in particular, it is discontinuous in the space variable (piecewise constant in fact). In order to get a unique solution, a flux limiter should be identified [9].

Homogenized Hamiltonians and effective flux limiter

The homogenized left and right Hamiltonians are classically determined by the study of some "cell problems".

Proposition 1.2 (Homogenized left and right Hamiltonians). *Assume* **(A0)-(A5)**, *and either* **(B-i)** *or* **(B-ii)**. *Then for every $p \in \mathbb{R}$, and $\alpha = L, R$, there exists a unique $\lambda \in \mathbb{R}$ such that there exists a bounded solution v^α of*

$$\begin{cases} v_t^\alpha + H_\alpha(t, x, p + v_x^\alpha) = \lambda & \text{in} \quad \mathbb{R} \times \mathbb{R}, \\ v^\alpha \text{ is } \mathbb{Z}^2\text{-periodic.} \end{cases} \tag{3}$$

If $\bar{H}_\alpha(p)$ denotes such a λ, then the map $p \mapsto \bar{H}_\alpha(p)$ is continuous, coercive and quasi-convex.

Remark 1.3. We recall that a function $\bar{H}_\alpha$ is quasi-convex if the sets $\{\bar{H}_\alpha \leq \lambda\}$ are convex for all $\lambda \in \mathbb{R}$. If $\bar{H}_\alpha$ is also coercive, then $\bar{p}_\alpha^0$ denotes in proofs some $p \in \operatorname{argmin} \bar{H}_\alpha$.

The effective flux limiter is the smallest $\lambda \in \mathbb{R}$ for which there exists a solution w of the following global-in-time Hamilton-Jacobi equation

$$\begin{cases} w_t + H(t,x,w_x) = \lambda, & (t,x) \in \mathbb{R} \times \mathbb{R}, \\ w \text{ is 1-periodic w.r.t. } t. \end{cases} \tag{4}$$

Theorem 1.4 (Effective flux limiter)**.** *Assume* **(A0)-(A5)***, and either* **(B-i)** *or* **(B-ii)***. The set*

$$E = \{\lambda \in \mathbb{R} : \exists w \textit{ sub-solution of } (4)\}$$

is not empty and bounded from below. Moreover, if $\bar{A}$ denotes the infimum of E, then

$$\bar{A} \geq A_0 := \max_{\alpha=L,R} \left(\min \bar{H}_\alpha\right). \tag{5}$$

Remark 1.5. We will see below (Theorem 4.6) that the infimum is in fact a minimum: there exists a global corrector which, in particular, can be rescaled properly.

We can now define the effective junction condition.

Definition 1.6 (Effective junction condition)**.** The *effective junction function* $F_{\bar{A}}$ is defined by

$$F_{\bar{A}}(p_L, p_R) := \max(\bar{A}, \bar{H}_L^+(p_L), \bar{H}_R^-(p_R))$$

where

$$\bar{H}_\alpha^-(p) = \begin{cases} \bar{H}_\alpha(p) & \text{if } \; p < \bar{p}_\alpha^0, \\ \bar{H}_\alpha(\bar{p}_\alpha^0) & \text{if } \; p \geq \bar{p}_\alpha^0 \end{cases} \quad \text{and} \quad \bar{H}_\alpha^+(p) = \begin{cases} \bar{H}_\alpha(\bar{p}_\alpha^0) & \text{if } \; p \leq \bar{p}_\alpha^0, \\ \bar{H}_\alpha(p) & \text{if } \; p > \bar{p}_\alpha^0 \end{cases}$$

where $\bar{p}_\alpha^0 \in \operatorname{argmin} \bar{H}_\alpha$.

The convergence result

Our main result is the following theorem.

Theorem 1.7 (Junction condition by homogenization)**.** *Assume* **(A0)-(A5)** *and either* **(B-i)** *or* **(B-ii)***. Assume that the initial datum u_0 is Lipschitz continuous and for $\varepsilon > 0$, let u^ε be the solution of* (1)-(2)*. Then u^ε converges locally uniformly to the unique flux-limited solution u^0 of*

$$\begin{cases} u_t^0 + \bar{H}_L(u_x^0) = 0, & t > 0, x < 0, \\ u_t^0 + \bar{H}_R(u_x^0) = 0, & t > 0, x > 0, \\ u_t^0 + F_{\bar{A}}(u_x^0(t,0^-), u_x^0(t,0^+)) = 0, & t > 0, x = 0 \end{cases} \tag{6}$$

submitted to the initial condition (2)*.*

Remark 1.8. Recall that the notion of flux-limited solution for (6) is developed in [9].

This theorem asserts in particular that the slopes of the limit solution at the origin are characterized by the effective flux limiter $\bar{A}$. Its proof relies on the construction of a global "corrector", *i.e.* a solution of (4), which is close to an appropriate V-shape function after rescaling. This latter condition is necessary so that the slopes at infinity of the corrector fit the expected slopes of the solution of the limit problem at the origin. Here is a precise statement.

Theorem 1.9 (Existence of a global corrector for the junction)**.** *Assume* **(A0)-(A5)** *and either* **(B-i)** *or* **(B-ii)***. There exists a solution w of* (4) *with $\lambda = \bar{A}$ such that, the function*

$$w^\varepsilon(t,x) = \varepsilon w(\varepsilon^{-1}t, \varepsilon^{-1}x)$$

converges locally uniformly (along a subsequence $\varepsilon_n \to 0$) towards a function $W = W(x)$ which satisfies $W(0) = 0$ and

$$\hat{p}_R x 1_{\{x>0\}} + \hat{p}_L x 1_{\{x<0\}} \geq W(x) \geq \bar{p}_R x 1_{\{x>0\}} + \bar{p}_L x 1_{\{x<0\}}. \tag{7}$$

where

$$\begin{cases} \bar{p}_R = \min E_R \\ \hat{p}_R = \max E_R \end{cases} \quad \textit{with} \quad E_R := \left\{p \in \mathbb{R}, \quad \bar{H}_R^+(p) = \bar{H}_R(p) = \bar{A}\right\} \tag{8}$$

$$\begin{cases} \bar{p}_L = \max E_L \\ \hat{p}_L = \min E_L \end{cases} \quad \textit{with} \quad E_L := \left\{p \in \mathbb{R}, \quad \bar{H}_L^-(p) = \bar{H}_L(p) = \bar{A}\right\}. \tag{9}$$

The construction of this global corrector is the reason why homogenization is referred to as "specified". See also Section 1.3 about related results. As a matter of fact, we will prove a stronger result, see Theorem 4.6.

Extension: periodic oscillations near the origin

The techniques developed to prove the Theorem 1.7 allow us to deal with a different situation inspired from traffic flow problems. As explained in [10], such problems are related to the study of some Hamilton-Jacobi equations. The problem that we address in Theorem 1.10 below is motivated by its meaningful application to traffic lights. We aim at figuring out how the fraffic flow on an ideal (infinite, straight) road is modified by the presence of a finite number of traffic lights.

We can consider a Hamilton-Jacobi equation whose Hamiltonian does not depend on (t,x) for x outside a (small) interval of the form $N_\varepsilon = (b_1\varepsilon, b_N\varepsilon)$ and is piecewise constant with respect to x in $(b_1\varepsilon, b_N\varepsilon)$. At space discontinuities, junction conditions are imposed with ε-time periodic flux limiters. The limit solution satifies the equation after the "neighbourhood" N_ε disappeared. We will see that the equation keeps memory of what happened there through a flux limiter at the origin $x = 0$.

Let us be more precise now. For $N \geq 1$, we consider that (a finite number of) junction points $-\infty = b_0 < b_1 < b_2 < \cdots < b_N < b_{N+1} = +\infty$ and (a finite number of) times $0 = \tau_0 < \tau_1 < \cdots < \tau_K < 1 = \tau_{K+1}$, $K \in \mathbb{N}$ are given. We then consider the solution u^ε of (1) with

(C1) the global Hamiltonian is given by the following formula

$$H(t,x,p) = \begin{cases} \bar{H}_\alpha(p) & \text{if } b_\alpha < x < b_{\alpha+1} \\ \max(\bar{H}^+_{\alpha-1}(p^-), \bar{H}^-_\alpha(p^+), a_\alpha(t)) = 0 & \text{if } x = b_\alpha, \alpha \neq 0. \end{cases}$$

(C2) For $\alpha = 0, \dots, N$, the Hamiltonians $\bar{H}_\alpha$ are continuous, coercive and quasi-convex.

(C3) the flux limiters a_α satisfy for each $\alpha = 1, \dots, N$ and $i = 0, \dots, K$,

$$a_\alpha(s+1) = a_\alpha(s) \quad \text{with} \quad a_\alpha(s) = A^i_\alpha \quad \text{for all} \quad s \in [\tau_i, \tau_{i+1})$$

with $(A^i_\alpha)^{i=0,\dots,K}_{\alpha=1,\dots,N}$ satisfying $A^i_\alpha \geq \max_{\beta=\alpha-1,\alpha}\left(\min \bar{H}_\beta\right)$.

The fact that Hamiltonians outside N_ε do not depend on time and space is emphasized by denoting them $\bar{H}_\alpha$ instead of H_α.

The equation is supplemented with the following initial condition

$$u^\varepsilon(0,x) = U^\varepsilon_0(x) \quad \text{for } x \in \mathbb{R} \tag{10}$$

with

$$U^\varepsilon_0 \text{ is equi-Lipschitz continuous and } U^\varepsilon_0 \to u_0 \text{ locally uniformly.} \tag{11}$$

Then the following convergence result holds true.

Theorem 1.10 (Homogenization in time of a junction condition)**.** *Assume* **(C1)-(C3)** *and* (11)*. Let u^ε be the solution of* (1)-(10) *for all $\varepsilon > 0$. Then:*

i) (Homogenization) *There exists some $\bar{A} \in \mathbb{R}$, such that u^ε converges locally uniformly as ε tends to zero towards the unique viscosity solution u^0 of* (6)-(2) *with*

$$\bar{H}_L := \bar{H}_0, \quad \bar{H}_R := \bar{H}_N.$$

ii) (Identification of $\bar{A}$) *Moreover we have*

$$\bar{A} \geq \max_{\alpha=1,\dots,N} \langle a_\alpha \rangle \quad \textit{with} \quad \langle a_\alpha \rangle := \int_0^1 a_\alpha(s)\, ds \tag{12}$$

and

$$\bar{A} = \langle a_1 \rangle \quad \textit{if} \quad N = 1. \tag{13}$$

Remark 1.11. Since the function $a(t)$ is piecewise constant, the way u^ε satisfies (1) has to be made precise. An L^1 theory in time (following the approach of [5, 6]) could probably be developed for such a problem, but we will use here a different, elementary approach. The Cauchy problem is understood as the solution of successive Cauchy problems. This is the reason why we will first prove a global Lipschitz bound on the solution so that there indeed exists such a solution.

Remark 1.12. It is also possible to adress the homogenization with any finite number of junctions (with limiter functions $a_\alpha(t)$ piecewise constants – or continuous – and 1-periodic), either separated with distance of order $O(1)$ or with distance of order $O(\varepsilon)$, or mixing both, and even on a complicated network. See also [11] for other examples of traffic light problems and [2] for green waves modelling.

Remark 1.13. Note that the result of Theorem 1.4 still holds for equation (1) under Assumptions **(C1)-(C3)**, with the set E defined for sub-solutions which are moreover assumed to be globally Lipschitz (without fixed bound on the Lipschitz constant). The reader can check that the proof is unchanged.

Remark 1.14. Inequality (12) has a natural traffic interpretation, saying that the average limitation on the traffic flow created by several traffic lights on a single road is higher or equal to the one created by the traffic light which creates the highest limitation. For the connection between our problem and traffic problems, see also the literature cited in Subsection 1.4.

Remark 1.15. Note that it is quite easy to see that (13) holds true when the Hamiltonians $\bar{H}_\alpha$ are convex, because of the optimal control interpretation of the problem. In the more general case of quasi-convex Hamiltonians, this is really non trivial, but we prove the result to be true.

Remark 1.16. We may have $\bar{A} > \max_{\alpha=1,\dots,N}\langle a_\alpha\rangle$. It is possible to see it for $N = 2$, using the traffic light interpretation of the problem. If we have two traffic lights very close to each other (let us say that the distance in between is at most the place for only one car), and if the equal period of the traffic lights are exactly in opposite phases (with for instance one minute for the green phase, and one minute for the red phase), then the effect of the two traffic lights together, gives a very low flux which is much lower than the effect of a single traffic light alone (i.e. here at most one car every two minutes will go through the two traffic lights).

1.3 Related results

Achdou and Tchou [1] studied a singular perturbation problem which has the same flavor as the one we are looking at in the present paper. More precisely, they consider the simplest network (a so-called junction) embedded in a star-shaped domain. They prove that the value function of an infinite horizon control problem converges, as the star-shaped domain "shrinks" to the junction to the value function of a control problem posed on the junction. We borrow from them the idea of studying the cell problem on truncated domains with state constraints. We provide a different approach, which is also in some sense more general because it can be applied to problems outside the framework of optimal control theory. Our approach is strongly based on [9] where a general theory for networks is developed in details.

The general theme of Lions's 2013-2014 lectures at Collège de France [12] is "Elliptic or parabolic equations and specified homogenization". As far as first order Hamilton-Jacobi equations are concerned, the term "specified homogenization" refers to the problem of constructing correctors to cell problems associated with Hamiltonians that are typically the sum of a periodic one H and a compactly supported function f depending only on x, say. Lions exhibits sufficient conditions on f such that the effective Hamilton-Jacobi equation is not perturbed. In terms of flux limiters [9], it corresponds to look for sufficient conditions such that the effective flux limiter $\bar{A}$ is (less than or) equal to $A_0 = \min H$.

Barles, Briani and Chasseigne [4, Theorem 6.1] considered the case

$$H(x,p) = \varphi\left(\frac{x}{\varepsilon}\right) H_R(p) + \left(1 - \varphi\left(\frac{x}{\varepsilon}\right)\right) H_L(p)$$

for some continuous increasing function $\varphi : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{s\to-\infty} \varphi(s) = 0 \quad \text{and} \quad \lim_{s\to+\infty} \varphi(s) = 1.$$

They prove that u^ε converges towards a value function denoted by U^-, that they characterize as the solution to a particular optimal control problem. It is proved in [9] that U^- is the solution of (6) with $\bar{H}_\alpha = H_\alpha$ and $\bar{A}$ replaced with $A_I^+ = \max(A_0, A^*)$ with

$$A_0 = \max(\min H_R, \min H_L) \quad \text{and} \quad A^* = \max_{q\in[\min(p_R^0,p_L^0),\max(p_R^0,p_L^0)]} (\min(H_R(q), H_L(q))).$$

1.4 Further extensions

Note that the method presented in this paper can be readily applied (without modifying proofs) to the study of homogeneization on a finite number of branches and not only two branches; the theory developed in [9] should also be used for the limit problem.

Similar questions in higher dimensions with point defects of other co-dimensions will be addressed in future works.

1.5 Organization of the article

Section 2 is devoted to the proof of the convergence result (Theorem 1.7). Section 3 is devoted to the construction of correctors far from the junction point (Proposition 1.2) while the junction case, i.e. the proof of Theorem 4.6 is addressed in Section 4. We recall that Theorem 1.9 is a straightforward corollary of this stronger result. The proof of Theorem 4.6 makes use of a comparison principle which is expected but not completely standard. This is the reason why a proof is sketched in Appendix, together with two other ones that are rather standard but included for the reader's convenience.

Notation. A ball centered at x of radius r is denoted by $B_r(x)$. If $\{u^\varepsilon\}_\varepsilon$ is locally bounded, the upper and lower relaxed limits are defined as

$$\begin{cases} \limsup_\varepsilon{}^* u^\varepsilon(X) = \limsup_{Y\to X,\varepsilon\to 0} u^\varepsilon(Y), \\ \liminf_\varepsilon{}_* u^\varepsilon(X) = \liminf_{Y\to X,\varepsilon\to 0} u^\varepsilon(Y). \end{cases}$$

In our proofs, constants may change from line to line.

2 Proof of convergence

This section is devoted to the proof of Theorem 1.7. We first construct barriers.

Lemma 2.1 (Barriers)**.** *There exists a nonnegative constant C such that for any $\varepsilon > 0$*

$$|u^\varepsilon(t,x) - u_0(x)| \le Ct \quad \textit{for} \quad (t,x) \in (0,T) \times \mathbb{R}\,. \tag{14}$$

Proof. Let L_0 be the Lipschitz constant of the initial datum $u_0(x)$. Taking

$$C = \sup_{\substack{(t,x)\in\mathbb{R}\times\mathbb{R} \\ |p|\le L_0}} |H(t,x,p)| < +\infty$$

owing to **(A0)** and **(A5)**, the function $u^\pm(t,x) = u_0(x) \pm Ct$ are super and sub-solution of (1)-(2) and (14) follows via comparison principle. $\square$

We can now prove the convergence theorem.

Proof of Theorem 1.7. We classically consider the upper and lower relaxed semi-limits

$$\begin{cases} \overline{u} = \limsup_{\varepsilon} {}^* u^\varepsilon \\ \underline{u} = \liminf_{\varepsilon} {}_* u^\varepsilon\,. \end{cases}$$

Notice that these functions are well defined because of Lemma 2.1. In order to prove convergence of u^ε towards u^0 it is sufficient to prove that $\overline{u}$ and $\underline{u}$ are respectively a sub- and a super-solution of (6)-(2). The initial condition immediately follows from (14). We focus our attention on the sub-solution case, the other one can be handle similarly.

Let φ be a test function such that

$$(\overline{u} - \varphi)(t,x) < (\overline{u} - \varphi)(\bar{t},\overline{x}) = 0 \quad \forall (t,x) \in B_{\overline{r}}(\bar{t},\overline{x}) \setminus \big\{(\bar{t},\overline{x})\big\}\,. \tag{15}$$

We argue by contradiction by assuming that

$$\varphi_t(\bar{t},\overline{x}) + \bar{H}\left(\bar{x}, \varphi_x(\bar{t},\overline{x})\right) = \theta > 0, \tag{16}$$

where

$$\bar{H}\left(\bar{x}, \varphi_x(\bar{t},\overline{x})\right) := \begin{cases} \bar{H}_R(\varphi_x(\bar{t},\overline{x})) & \text{if} \quad \overline{x} > 0, \\ \bar{H}_L(\varphi_x(\bar{t},\overline{x})) & \text{if} \quad \overline{x} < 0, \\ F_A(\varphi_x(\bar{t},0^-), \varphi_x(\bar{t},0^+)) & \text{if} \quad \overline{x} = 0. \end{cases}$$

We only treat the case where $\overline{x} = 0$ since the case $x \neq 0$ is somewhat classical. This latter case is detailed in Appendix for the reader's convenience. Using [9, Proposition 2.5], we may suppose

$$\varphi(t,x) = \phi(t) + \bar{p}_L x 1_{\{x<0\}} + \bar{p}_R x 1_{\{x>0\}} \tag{17}$$

where ϕ is a C^1 function defined in $(0,+\infty)$. In this case, Eq. (16) becomes

$$\phi'(\bar{t}) + F_{\bar{A}}\left(\bar{p}_L, \bar{p}_R\right) = \phi'(\bar{t}) + \bar{A} = \theta > 0. \tag{18}$$

Let us consider a solution w of the equation

$$w_t + H(t, x, w_x) = \bar{A} \tag{19}$$

provided by Theorem 1.9, which is in particular 1-periodic with respect to time. We recall that the function W is the limit of w after rescaling. We claim that, if $\varepsilon > 0$ is small enough, the perturbed test function $\varphi^\varepsilon(t, x) = \phi(t) + w^\varepsilon(t, x)$ is a viscosity super-solution of

$$\varphi_t^\varepsilon + H\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \varphi_x^\varepsilon\right) = \frac{\theta}{2} \quad \text{in} \quad B_r(\bar{t}, 0)$$

for some sufficiently small $r > 0$. In order to justify this fact, let $\psi(t, x)$ be a test function touching φ^ε from below at $(t_1, x_1) \in B_r(\bar{t}, 0)$. In this way

$$w\left(\frac{t_1}{\varepsilon}, \frac{x_1}{\varepsilon}\right) = \frac{1}{\varepsilon}\left(\psi(t_1, x_1) - \phi(t_1)\right)$$

and

$$w\left(s, y\right) \geq \frac{1}{\varepsilon}\left(\psi(\varepsilon s, \varepsilon y) - \phi(\varepsilon s)\right)$$

for (s, y) in a neighborhood of $\left(\frac{t_1}{\varepsilon}, \frac{x_1}{\varepsilon}\right)$. Hence from (18)-(19)

$$\psi_t(t_1, x_1) - \phi'(t_1) + H\left(\frac{t_1}{\varepsilon}, \frac{x_1}{\varepsilon}, \psi_x(t_1, x_1)\right) \geq \bar{A}$$

which implies

$$\begin{aligned}\psi_t(t_1, x_1) + H\left(\frac{t_1}{\varepsilon}, \frac{x_1}{\varepsilon}, \psi_x(t_1, x_1)\right) &\geq \bar{A} + \phi'(t_1) \\ &\geq \bar{A} + \phi'(\bar{t}) - \frac{\theta}{2} \geq \frac{\theta}{2}\end{aligned}$$

provided r is small enough. Hence, the claim is proved.

Combining (7) from Theorem 1.9 with (15) and (17), we can fix $\kappa_r > 0$ and $\varepsilon > 0$ small enough so that

$$u^\varepsilon + \kappa_r \leq \varphi^\varepsilon \quad \text{on} \quad \partial B_r(\bar{t}, 0).$$

By comparison principle the previous inequality holds in $B_r(\bar{t}, 0)$. Passing to the limit as $\varepsilon \to 0$ and $(t, x) \to (\bar{t}, \bar{x})$, we get the following contradiction

$$\overline{u}(\bar{t}, 0) + \kappa_r \leq \varphi(\bar{t}, 0) = \overline{u}(\bar{t}, 0).$$

The proof of convergence is now complete. □

Remark 2.2. For the super-solution property, φ in (17) should be replaced with

$$\varphi(t, x) = \phi(t) + \hat{p}_L x 1_{\{x<0\}} + \hat{p}_R x 1_{\{x>0\}}.$$

3 Homogenized Hamiltonians

In order to prove Proposition 1.2, we first prove the following lemma. Even if the proof is standard, we give it in full details since we will adapt it when constructing global correctors for the junction.

Lemma 3.1 (Existence of a corrector)**.** *There exists $\lambda \in \mathbb{R}$ and a bounded (discontinuous) viscosity solution of* (3)*.*

Remark 3.2. If H_α does not depend on t, then it is possible to construct a corrector which does not depend on time either. We leave details to the reader.

Proof. For any $\delta > 0$, it is possible to construct a (possibly discontinuous) viscosity solution v^δ of

$$\begin{cases} \delta v^\delta + v^\delta_t + H_\alpha(t,x,p+v^\delta_x) = 0 & \text{in } \mathbb{R}\times\mathbb{R}, \\ v^\delta \text{ is } \mathbb{Z}^2\text{-periodic.} \end{cases}$$

First, the comparison principle implies

$$|\delta v^\delta| \le C_\alpha \tag{20}$$

where

$$C_\alpha = \sup_{(t,x)\in[0,1]^2} |H_\alpha(t,x,p)|.$$

Second, the function

$$m^\delta(x) = \sup_t v^\delta(t,x)$$

is a sub-solution of

$$H_\alpha(t(x),x,p+m^\delta_x) \le C_\alpha$$

(for some function $t(x)$). Assumptions **(A3)** and **(A5)** imply in particular that there exists $C > 0$ independent of δ such that

$$|m^\delta_x| \le C$$

and

$$v^\delta_t \le C.$$

In particular, the comparison principle implies that for all $t \in \mathbb{R}$ and $x \in \mathbb{R}$ and $h \ge 0$,

$$v^\delta(t+h,x) \le v^\delta(t,x) + Ch.$$

Combining this inequality with the time-periodicity of v^δ yields

$$|v^\delta(t,x) - m^\delta(x)| \le C;$$

in particular,

$$|v^\delta(t,x) - v^\delta(0,0)| \le C. \tag{21}$$

Hence, the half relaxed limits

$$\bar{v} = \limsup_{\delta \to 0}{}^*(v^\delta - v^\delta(0,0)) \quad \text{and} \quad \underline{v} = \liminf_{\delta \to 0}{}_*(v^\delta - v^\delta(0,0))$$

are finite. Moreover, (20) implies that $\delta v^\delta(0,0) \to -\lambda$ (at least along a subsequence). Hence, discontinuous stability of viscosity solutions implies that $\bar{v}$ is a $\mathbb{Z}^2$-periodic sub-solution of (3) and $\underline{v}$ is a $\mathbb{Z}^2$-periodic super-solution of the same equation. Perron's method then allows us to construct a corrector between $\bar{v}$ and $\underline{v} + C$ with $C = \sup(\bar{v} - \underline{v})$. The proof of the lemma is now complete. □

The following lemma is completely standard; the proof is given in Appendix for the reader's convenience.

Lemma 3.3 (Uniqueness of λ)**.** *The real number λ given by Lemma 3.1 is unique. If $\bar{H}_\alpha(p)$ denotes such a real number, the function $\bar{H}_\alpha$ is continuous.*

Lemma 3.4 (Coercivity of $\bar{H}_\alpha$)**.** *The continuous function $\bar{H}_\alpha$ is coercive,*

$$\lim_{|p| \to +\infty} \bar{H}_\alpha(p) = +\infty.$$

Proof. In view of the uniform coercivity in p of H_α with respect to (t,x) (see **(A3)**), for any $R > 0$ there exists a positive constant C_R such that

$$|p| \geq C_R \quad \Rightarrow \quad H_\alpha(t,x,p) \geq R \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}. \tag{22}$$

Let v^α be the discontinuous corrector given by Lemma 3.1 and $(\bar{t}, \bar{x})$ be point of supremum of its upper semi-continuous envelope $(v^\alpha)^*$. Then we have

$$H_\alpha(\bar{t}, \bar{x}, p) \leq \bar{H}_\alpha(p)$$

which implies

$$\bar{H}_\alpha(p) \geq R \quad \text{for} \quad |p| \geq C_R. \tag{23}$$

The proof of the lemma is now complete. □

We first prove the quasi-convexity of $\bar{H}_\alpha$ under assumption **(B-ii)**. We prove in fact more: the effective Hamiltonian is convex in this case.

Lemma 3.5 (Convexity of $\bar{H}_\alpha$ under **(B-ii)**)**.** *Assume* **(A0)-(A5)** *and* **(B-ii)**. *Then the function $\bar{H}_\alpha$ is convex.*

Proof. For p, $q \in \mathbb{R}$, let v_p, v_q be respectively solutions of (3) with $\lambda = \bar{H}_\alpha(p)$, $\bar{H}_\alpha(q)$ respectively. We also set

$$u_p(t,x) = v_p(t,x) + px - t\bar{H}_\alpha(p)$$

and define similarly u_q.

Step 1: u_p and u_q are locally Lipschitz continuous. In this case, we have almost everywhere:

$$\begin{cases} (u_p)_t + H_\alpha(t,x,(u_p)_x) = 0, \\ (u_q)_t + H_\alpha(t,x,(u_q)_x) = 0. \end{cases}$$

For $\mu \in [0,1]$, let

$$\bar{u} = \mu u_p + (1-\mu) u_q.$$

By convexity, we get almost everywhere

$$\bar{u}_t + H_\alpha(t,x,\bar{u}_x) \leq 0. \tag{24}$$

We claim that the convexity of H_α (in the gradient variable) implies that $\bar{u}$ is a viscosity sub-solution. To see it, we use an argument of [3, Proposition 5.1]. For $P = (t,x)$, we define a mollifier $\rho_\delta(P) = \delta^{-2}\rho(\delta^{-1}P)$ and set

$$\bar{u}_\delta = \bar{u} \star \rho_\delta$$

Then by convexity, we get with $Q = (s,y)$:

$$(\bar{u}_\delta)_t + H_\alpha(P,(\bar{u}_\delta)_x) \leq \int dQ \ \left\{ H_\alpha(P,\bar{u}_x(Q)) - H_\alpha(Q,\bar{u}_x(Q) \right\} \rho_\delta(P-Q).$$

The fact that $\bar{u}_x$ is locally bounded and the fact that H_α is continuous imply that the right hand side goes to zero as $\delta \to 0$. We deduce (by stability of viscosity sub-solutions) that (24) holds true in the viscosity sense. Then the comparison principle implies that

$$\mu \bar{H}_\alpha(p) + (1-\mu)\bar{H}_\alpha(q) \geq \bar{H}_\alpha(\mu p + (1-\mu)q). \tag{25}$$

Step 2: u_p and u_q are continuous. We proceed in two (sub)steps.
Step 2.1: the case of a single function u. We first want to show that if $u = u_p$ is continuous and satisfies (24) almost everywhere, then u is a viscosity sub-solution. To this end, we will use the structural assumptions satisfied by the Hamiltonian. The ones that were useful to prove the comparison principle, will be also useful to prove the result we want. Indeed, we will revisit the proof of the comparison principle. We also use the fact that

$$u(t,x) - px + t\bar{H}_\alpha(p) \quad \text{is bounded.} \tag{26}$$

For $\nu > 0$, we set

$$u^\nu(t,x) = \sup_{s \in \mathbb{R}} \left(u(s,x) - \frac{(t-s)^2}{2\nu} \right) = u(s_\nu, x) - \frac{(t-s_\nu)^2}{2\nu}$$

As usual, we get from (26) that

$$|t - s_\nu| \leq C\sqrt{\nu} \quad \text{with} \quad C = C(p,T) \tag{27}$$

for $t \in (-T,T)$. In particular $s_\nu \to t$ locally uniformly. If a test function φ touches u^ν from above at some point (t,x), then we have $\varphi_t(t,x) = -\dfrac{t-s_\nu}{\nu}$ and

$$\begin{aligned}
\varphi_t(t,x) + H_\alpha(t,x,\varphi_x(t,x)) &\le H_\alpha(t,x,\varphi_x(t,x)) - H_\alpha(s_\nu,x,\varphi_x(t,x)) \\
&\le \omega(|t-s_\nu|\,(1+\max(0,H_\alpha(s_\nu,x,\varphi_x(t,x))))) \\
&\le \omega\left(\frac{(t-s_\nu)^2}{\nu} + |t-s_\nu|\right)
\end{aligned} \tag{28}$$

where we have used **(A2)** in the third line. The right hand side goes to zero as ν goes to zero, because

$$\frac{(t-s_\nu)^2}{\nu} \to 0 \quad \text{locally uniformly w.r.t. } (t,x)$$

because u is continuous. Indeed, this can be checked for (t,x) replaced by (t_ν,x_ν) because for any sequence $(t_\nu,s_\nu,x_\nu) \to (t,t,x)$, we have

$$u(t_\nu,x_\nu) \le u^\nu(t_\nu,x_\nu) = u(s_\nu,x_\nu) - \frac{(t_\nu-s_\nu)^2}{2\nu}$$

where the continuity of u implies the result. For a given $\nu>0$, we see that (27) and (28) imply that

$$|\varphi_t|\,,|\varphi_x| \le C_{\nu,p}.$$

This implies in particular that u^ν is Lipschitz continuous, and then

$$u^\nu_t + H(t,x,u^\nu_x) \le o_\nu(1) \quad \text{a.e.}$$

where $o_\nu(1)$ is locally uniform with respect to (t,x).

Step 2.2: application. Applying Step 2.1, we get for $z=p,q$

$$(u^\nu_z)_t + H(t,x,(u^\nu_z)_x) \le o_\nu(1) \quad \text{a.e.}$$

where $o_\nu(1)$ is locally uniform with respect to (t,x). Step 1 implies that

$$\bar u^\nu := \mu u^\nu_p + (1-\mu)u^\nu_q$$

is a viscosity sub-solution of

$$(\bar u^\nu)_t + H_\alpha(t,x,(\bar u^\nu)_x) \le o_\nu(1)$$

where $o_\nu(1)$ is locally uniform with respect to (t,x). In the limit $\nu \to 0$, we recover (by stability of sub-solutions) that $\bar u$ is a viscosity sub-solution, i.e. satisfies (24) in the viscosity sense. This gives then the same conclusion as in Step 1.

Step 3: the general case. To cover the general case, we simply replace u_p by $\tilde{u}_p$ which is the solution to the Cauchy problem

$$\begin{cases} (\tilde{u}_p)_t + H_\alpha(t,x,(\tilde{u}_p)_x) = 0, & \text{for} \quad (t,x) \in (0,+\infty)\times\mathbb{R} \\ \tilde{u}_p(0,x) = px, \end{cases}$$

Then $\tilde{u}_p$ is continuous and satisfies $|\tilde{u}_p - u_p| \le C$. Proceeding similarly with $\tilde{u}_q$ and using Step 2, we deduce the result (25). □

We finally prove the quasi-convexity of $\bar{H}_\alpha$ under assumption **(B-i)**.

Lemma 3.6 (Quasi-convexity of $\bar{H}_\alpha$ under **(B-i)**)**.** *Assume* **(A0)-(A5)** *and* **(B-i)**. *Then the function $\bar{H}_\alpha$ is quasi-convex.*

Proof. We reduce quasi-convexity to convexity by composing with an increasing function γ.

We first assume that H_α satisfies

$$\begin{cases} H_\alpha \in C^2, \\ D^2_{pp}H_\alpha(x,p^0_\alpha) > 0, \\ D_pH_\alpha(x,p) < 0 \quad \text{for} \quad p \in (-\infty, p^0_\alpha), \\ D_pH_\alpha(x,p) > 0 \quad \text{for} \quad p \in (p^0_\alpha, +\infty), \\ H_\alpha(x,p) \to +\infty \quad \text{as} \quad |p| \to +\infty \quad \text{uniformly w.r.t. } x \in \mathbb{R}. \end{cases} \tag{29}$$

For a function γ such that

$$\gamma \quad \text{is convex}, \quad \gamma \in C^2(\mathbb{R}) \quad \text{and} \quad \gamma' \ge \delta_0 > 0$$

we have

$$D^2_{pp}(\gamma(H_\alpha(x,p))) > 0$$

if and only if

$$(\ln \gamma')'(\lambda) > -\frac{D^2_{pp}H_\alpha(x,p)}{(D_pH_\alpha(x,p))^2} \quad \text{for} \quad p = \pi^\pm_\alpha(x,\lambda) \quad \text{and} \quad \lambda \ge H_\alpha(x,p) \tag{30}$$

where $\pi^\pm_\alpha(x,\lambda)$ is the only real number r such that $\pm r \ge 0$ and $H_\alpha(x,r) = \lambda$. Because $D^2_{pp}H_\alpha(x,p^0_\alpha) > 0$, we see that the right hand side is negative for λ close enough to $H_\alpha(x,p^0_\alpha)$ and it is indeed possible to construct such a function γ.

In view of Remark 3.2, we can construct a solution of $\delta v^\delta + \gamma \circ H_\alpha(x, p + v^\delta_x) = 0$ with $-\delta v^\delta \to \overline{\gamma \circ H_\alpha}(p)$ as $\delta \to 0$, and a solution of

$$\gamma \circ H_\alpha(x, p + v_x) = \overline{\gamma \circ H_\alpha}(p)$$

This shows that

$$\bar{H}_\alpha = \gamma^{-1} \circ \overline{\gamma \circ H_\alpha}.$$

Thanks to Lemmas 3.4 and 3.5, we know that $\overline{\gamma \circ H_\alpha}$ is coercive and convex. Hence $\bar{H}_\alpha$ is quasi-convex.

If now H_α does not satisfies (29), then for all $\varepsilon > 0$, there exists $H_\alpha^\varepsilon \in C^2$ such that

$$\begin{cases} (D^2_{pp} H^\varepsilon_\alpha)(x, p^0_\alpha) > 0 \\ D_p H^\varepsilon_\alpha(x,p) < 0 \quad \text{for} \quad p \in (-\infty, p^0_\alpha), \\ D_p H^\varepsilon_\alpha(x,p) > 0 \quad \text{for} \quad p \in (p^0_\alpha, +\infty), \\ |H^\varepsilon_\alpha - H_\alpha| < \varepsilon. \end{cases}$$

Then we can argue as in the proof of continuity of $\bar{H}_\alpha$ and deduce that

$$\bar{H}_\alpha(p) = \lim_{\varepsilon \to 0} \bar{H}^\varepsilon_\alpha(p).$$

Moreover, the previous case implies that $\bar{H}^\varepsilon_\alpha$ is quasi-convex. Hence, so is $\bar{H}_\alpha$. The proof of the lemma is now complete. □

Proof of Proposition 1.2. Combine Lemmas 3.1, 3.3, 3.4, 3.5 and 3.6. □

4 Truncated cell problems

We consider the following problem: find $\lambda_\rho \in \mathbb{R}$ and w such that

$$\begin{cases} w_t + H(t,x,w_x) = \lambda_\rho, & (t,x) \in \mathbb{R} \times (-\rho, \rho), \\ w_t + H^-(t,x,w_x) = \lambda_\rho, & (t,x) \in \mathbb{R} \times \{-\rho\}, \\ w_t + H^+(t,x,w_x) = \lambda_\rho, & (t,x) \in \mathbb{R} \times \{\rho\}, \\ w \text{ is 1-periodic w.r.t. } t. \end{cases} \tag{31}$$

Even if our approach is different, we borrow here an idea from [1] by truncating the domain and by considering correctors in $[-\rho, \rho]$ with $\rho \to +\infty$.

4.1 A comparison principle

Proposition 4.1 (Comparison principle for a mixed boundary value problem)**.** *Let us consider the following problem for $\rho_2 > \rho_1 > \rho_0$ and $\lambda \in \mathbb{R}$*

$$\begin{cases} v_t + H(t,x,v_x) \geq \lambda & \textit{for } (t,x) \in \mathbb{R} \times (\rho_1, \rho_2), \\ v_t + H^+(t,x,v_x) \geq \lambda & \textit{for } (t,x) \in \mathbb{R} \times \{\rho_2\}, \\ v(t,x) \geq U_0(t) & \textit{for } (t,x) \in \mathbb{R} \times \{\rho_1\}, \\ v \textit{ is 1-periodic w.r.t. } t \end{cases} \tag{32}$$

where U_0 is continuous and for $\varepsilon_0 > 0$

$$\begin{cases} u_t + H(t,x,u_x) \leq \lambda - \varepsilon_0 & \textit{for } (t,x) \in \mathbb{R} \times (\rho_1, \rho_2), \\ u_t + H^+(t,x,u_x) \leq \lambda - \varepsilon_0 & \textit{for } (t,x) \in \mathbb{R} \times \{\rho_2\}, \\ u(t,x) \leq U_0(t) & \textit{for } (t,x) \in \mathbb{R} \times \{\rho_1\}, \\ u \textit{ is 1-periodic w.r.t. } t. \end{cases} \tag{33}$$

Then $u \leq v$ in $\mathbb{R} \times [\rho_1, \rho_2]$.

Remark 4.2. A similar result holds true if the Dirichlet condition is imposed at $x = \rho_2$ and junction conditions

$$\begin{aligned} v_t + H^-(t, x, v_x) &\geq \lambda && \text{at } x = \rho_1 \\ u_t + H^-(t, x, u_x) &\leq \lambda - \varepsilon_0 && \text{at } x = \rho_1 \end{aligned}$$

are imposed at $x = \rho_1$.

The proof of Proposition 4.1 is very similar to (in fact simpler than) the proof of the comparison principle for Hamilton-Jacobi equations on networks contained in [9]. The main difference lies in the fact that in our case, u and v are global in time and the space domain is bounded. A sketched proof is provided in Appendix shedding some light on the main differences. Here the $\varepsilon_0 > 0$ in (33) is used in place of the standard correction term $-\eta/(T - t)$ for a Cauchy problem.

4.2 Correctors on truncated domains

Proposition 4.3 (Existence and properties of a corrector on a truncated domain)**.** *There exists a unique $\lambda_\rho \in \mathbb{R}$ such that there exists a solution $w^\rho = w$ of* (31)*. Moreover, there exists a constant $C > 0$ independent of $\rho \in (\rho_0, +\infty)$ such that*

$$\begin{cases} |\lambda_\rho| \leq C, & \\ |m^\rho(x) - m^\rho(y)| \leq C\,|x - y| & \text{for } x, y \in [-\rho, \rho], \\ |w^\rho(t, x) - m^\rho(x)| \leq C & \text{for } (t, x) \in \mathbb{R} \times [-\rho, \rho], \end{cases} \tag{34}$$

where

$$m^\rho(x) = \sup_{t \in \mathbb{R}} w^\rho(t, x).$$

Proof. In order to construct a corrector on the truncated domain, we proceed classically by considering

$$\begin{cases} \delta w^\delta + w_t^\delta + H(t, x, w_x^\delta) = 0, & (t, x) \in \mathbb{R} \times (-\rho, \rho)\,, \\ \delta w^\delta + w_t^\delta + H^-(t, x, w_x^\delta) = 0, & (t, x) \in \mathbb{R} \times \{-\rho\}\,, \\ \delta w^\delta + w_t^\delta + H^+(t, x, w_x^\delta) = 0, & (t, x) \in \mathbb{R} \times \{\rho\}\,, \\ w^\delta \text{ is 1-periodic w.r.t. } t. & \end{cases} \tag{35}$$

A discontinuous viscosity solution of (35) is constructed by Perron's method (in the class of 1-periodic functions with respect to time) since $\pm\delta^{-1}C$ are trivial super-/sub-solutions if C is chosen as follows

$$C = \sup_{t \in \mathbb{R},\ x \in \mathbb{R}} |H(t, x, 0)|.$$

In particular, the solution w^δ satisfies by construction

$$|w^\delta| \leq \frac{C}{\delta}. \tag{36}$$

We next consider
$$m^\delta(x) = \sup_{t\in\mathbb{R}} (w^\delta)^*(t,x).$$
We remark that the supremum is reached since w^δ is periodic with respect to time; we also remark that m^δ is a viscosity sub-solution of
$$H(t(x), x, m^\delta_x) \le C, \quad x \in (-\rho,\rho)$$
(for some function $t(x)$). In view of **(A3)**, we conclude that m^δ is globally Lipschitz continuous and
$$|m^\delta_x| \le C \tag{37}$$
for some constant C which still only depends on H. Assumption **(A3)** also implies that,
$$w^\delta_t \le C$$
(with C only depending on H). In particular, the comparison principle implies that for all $t \in \mathbb{R}$, $x \in (-\rho,\rho)$ and $h \ge 0$,
$$w^\delta(t+h,x) \le w^\delta(t,x) + Ch.$$
Combining this information with the periodicity of w^δ with respect to t, we conclude that for $t \in \mathbb{R}$ and $x \in (-\rho,\rho)$,
$$|w^\delta(t,x) - m^\delta(x)| \le C.$$
In particular,
$$|w^\delta(t,x) - w^\delta(0,0)| \le C.$$
We then consider
$$\overline{w} = \limsup_\delta{}^*(w^\delta - w^\delta(0,0)) \quad \text{and} \quad \underline{w} = \liminf_\delta{}_*(w^\delta - w^\delta(0,0)).$$
We next remark that (36) and (37) imply that there exists $\delta_n \to 0$ such that
$$\begin{aligned} m^{\delta_n} - m^{\delta_n}(0) &\to m^\rho \quad &&\text{as } n \to +\infty \\ \delta_n w^{\delta_n}(0,0) &\to -\lambda_\rho \quad &&\text{as } n \to +\infty \end{aligned}$$
(the first convergence being locally uniform). In particular, λ, $\overline{w}$, $\underline{w}$ and m^ρ satisfies
$$\begin{aligned} |\lambda_\rho| &\le C \\ |\overline{w} - m^\rho| &\le C \\ |\underline{w} - m^\rho| &\le C \\ |m^\rho_x| &\le C. \end{aligned}$$
Discontinuous stability of viscosity solutions of Hamilton-Jacobi equations imply that $\overline{w} - 2C$ and $\underline{w}$ are respectively a sub-solution and a super-solution of (31) and
$$\overline{w} - 2C \le \underline{w}.$$

Perron's method is used once again in order to construct a solution w^ρ of (31) which is 1-periodic with respect to time. In view of the previous estimates, λ_ρ, m^ρ and w^ρ satisfy (34). Proving the uniqueness of λ_ρ is classical so we skip it. The proof of the proposition is now complete. □

Proposition 4.4 (First definition of the effective flux limiter)**.** *The map $\rho \mapsto \lambda_\rho$ is non-decreasing and bounded in $(0,+\infty)$. In particular,*

$$\bar{A} = \lim_{\rho \to +\infty} \lambda_\rho$$

exists and $\bar{A} \geq \lambda_\rho$ for all $\rho > 0$.

Proof. For $\rho' > \rho > 0$, we see that the restriction of $w^{\rho'}$ to $[-\rho, \rho]$ is a sub-solution, as a consequence of [9, Proposition 2.15]. The boundedness of the map follows from Proposition 4.3. The proof is thus complete. □

We next prove that we can control w^ρ from below under appropriate assumptions on $\bar{A}$.

Proposition 4.5 (Control of slopes on a truncated domain)**.** *Assume first that $\bar{A} > \min \bar{H}_R$. Then for all $\delta > 0$, there exists $\rho_\delta > 0$ and $C_\delta > 0$ (independent on ρ) such that for $x \geq \rho_\delta$ and $h \geq 0$,*

$$w^\rho(t, x+h) - w^\rho(t,x) \geq (\bar{p}_R - \delta)h - C_\delta. \tag{38}$$

If now we assume that $\bar{A} > \min \bar{H}_L$, then for $x \leq -\rho_\delta$ and $h \geq 0$,

$$w^\rho(t, x-h) - w^\rho(t,x) \geq (-\bar{p}_L - \delta)h - C_\delta \tag{39}$$

for some $\rho_\delta > 0$ and $C_\delta > 0$ as above.

Proof. We only prove (38) since the proof of (39) follows along the same lines. Let $\delta > 0$. In view of **(A5)**, we know that there exists ρ_δ such that

$$|H(t,x,p) - H_R(t,x,p)| \leq \delta \quad \text{ for } \quad x \geq \rho_\delta. \tag{40}$$

Assume that $\bar{A} > \min \bar{H}_R$. Then Proposition 1.2 implies that we can pick p_R^δ such that

$$\bar{H}_R(p_R^\delta) = \bar{H}_R^+(p_R^\delta) = \lambda_\rho - 2\delta$$

for $\rho \geq \rho_0$ and $\delta \leq \delta_0$, by choosing ρ_0 large enough and δ_0 small enough.

We now fix $\rho \geq \rho_\delta$ and $x_0 \in [\rho_\delta, \rho]$. In view of Proposition 1.2 applied to $p = p_R^\delta$, we know that there exists a corrector v_R solving (3) with $\alpha = R$. Since it is $\mathbb{Z}^2$-periodic, it is bounded and $w_R = p_R^\delta x + v_R(t,x)$ solves

$$(w_R)_t + H_R(t, x, (w_R)_x) = \lambda_\rho - 2\delta, \quad (t,x) \in \mathbb{R} \times \mathbb{R}.$$

In particular, the restriction of w_R to $[\rho_\delta, \rho]$ satisfies (see [9, Proposition 2.15]),

$$\begin{cases} (w_R)_t + H_R(t, x, (w_R)_x) \le \lambda_\rho - 2\delta & \text{for } (t, x) \in \mathbb{R} \times (\rho_\delta, \rho), \\ (w_R)_t + H_R^+(t, x, (w_R)_x) \le \lambda_\rho - 2\delta & \text{for } (t, x) \in \mathbb{R} \times \{\rho\}\,. \end{cases}$$

In view of (40), this implies

$$\begin{cases} (w_R)_t + H(t, x, (w_R)_x) \le \lambda_\rho - \delta & \text{for } (t, x) \in \mathbb{R} \times (\rho_\delta, \rho), \\ (w_R)_t + H^+(t, x, (w_R)_x) \le \lambda_\rho - \delta & \text{for } (t, x) \in \mathbb{R} \times \{\rho\}\,. \end{cases}$$

Now we remark that $v = w^\rho - w^\rho(0, x_0)$ and $u = w_R - w_R(0, x_0) - 2C - 2\|v_R\|_\infty$ satisfies

$$v(t, x_0) \ge -2C \ge u(t, x_0)$$

where C is given by (34). Thanks to the comparison principle from Proposition 4.1, we thus get for $x \in [\rho_\delta, \rho]$,

$$w^\rho(t, x) - w^\rho(t, x_0) \ge p_R^\delta(x - x_0) - C_\delta$$

where C_δ is a large constant which does not depend on ρ. In particular, we get (38), reducing δ if necessary. □

4.3 Construction of global correctors

We now state and prove a result which implies Theorem 1.9 stated in the introduction.

Theorem 4.6 (Existence of a global corrector for the junction)**.** *Assume* **(A0)-(A5)** *and either* **(B-i)** *or* **(B-ii)***.*

i) (General properties) *There exists a solution w of (4) with $\lambda = \bar{A}$ such that for all $(t, x) \in \mathbb{R}^2$,*

$$|w(t, x) - m(x)| \le C \tag{41}$$

for some globally Lipschitz continuous function m, and

$$\bar{A} \ge A_0.$$

ii) (Bound from below at infinity) *If $\bar{A} > \max_{\alpha=L,R}\left(\min \bar{H}_\alpha\right)$, then there exists $\delta_0 > 0$ such that for every $\delta \in (0, \delta_0)$, there exists $\rho_\delta > \rho_0$ such that w satisfies*

$$\begin{cases} w(t, x+h) - w(t, x) \ge (\bar{p}_R - \delta)h - C_\delta & \text{for } x \ge \rho_\delta \quad \text{and} \quad h \ge 0, \\ w(t, x-h) - w(t, x) \ge (-\bar{p}_L - \delta)h - C_\delta & \text{for } x \le -\rho_\delta \quad \text{and} \quad h \ge 0. \end{cases} \tag{42}$$

The first line of (42) also holds only if $\bar{A} = \min \bar{H}_R$, while the second line of (42) also holds if $\bar{A} = \min \bar{H}_L$.

iii) (Rescaling w) *For $\varepsilon > 0$, we set*

$$w^\varepsilon(t,x) = \varepsilon w(\varepsilon^{-1}t, \varepsilon^{-1}x).$$

Then (along a subsequence $\varepsilon_n \to 0$), we have that w^ε converges locally uniformly towards a function $W = W(x)$ which satisfies

$$\begin{cases} |W(x) - W(y)| \le C\,|x-y| & \text{for all} \quad x,y \in \mathbb{R}, \\ \bar{H}_R(W_x) = \bar{A} \quad \text{and} \quad \hat{p}_R \ge W_x \ge \bar{p}_R & \text{for } x \in (0,+\infty), \\ \bar{H}_L(W_x) = \bar{A} \quad \text{and} \quad \hat{p}_L \le W_x \le \bar{p}_L & \text{for } x \in (-\infty,0). \end{cases} \tag{43}$$

In particular, we have $W(0) = 0$ and

$$\hat{p}_R x 1_{\{x>0\}} + \hat{p}_L x 1_{\{x<0\}} \ge W(x) \ge \bar{p}_R x 1_{\{x>0\}} + \bar{p}_L x 1_{\{x<0\}}. \tag{44}$$

Proof. We consider (up to some subsequence)

$$\overline{w} = \limsup_{\rho\to+\infty}{}^*(w^\rho - w^\rho(0,0)), \quad \underline{w} = \liminf_{\rho\to+\infty}{}_*(w^\rho - w^\rho(0,0)) \quad \text{and} \quad m = \lim_{\rho\to+\infty}(m^\rho - m^\rho(0)).$$

We derive from (34) that $\underline{w}$ and $\overline{w}$ are finite and

$$m - C \le \underline{w} \le \overline{w} \le m + C.$$

Moreover, discontinuous stability of viscosity solutions imply that $\overline{w} - 2C$ and $\underline{w}$ are respectively a sub-solution and a super-solution of (4) with $\lambda = \bar{A}$ (recall Proposition 4.4). Hence, a discontinuous viscosity solution w of (4) can be constructed by Perron's method (in the class of functions that are 1-periodic with respect to time).

Using again (34), w and m satisfy (41). We also get (42) from Proposition 4.5 (use (34) and pass to the limit with m instead of w if necessary).

We now study $w^\varepsilon(t,x) = \varepsilon w(\varepsilon^{-1}t, \varepsilon^{-1}x)$. Remark that (34) implies in particular that

$$w^\varepsilon(t,x) = \varepsilon m(\varepsilon^{-1}x) + O(\varepsilon).$$

In particular, we can find a sequence $\varepsilon_n \to 0$ such that

$$w^{\varepsilon_n}(t,x) \to W(x) \quad \text{locally uniformly as } n \to +\infty,$$

with $W(0) = 0$. Arguing as in the proof of convergence away from the junction point (see the case $\bar{x} \neq 0$ in Appendix), we deduce that W satisfies

$$\begin{aligned} \bar{H}_R(W_x) &= \bar{A} \text{ for } x > 0, \\ \bar{H}_L(W_x) &= \bar{A} \text{ for } x < 0. \end{aligned}$$

We also deduce from (42) that for all $\delta > 0$ and $x > 0$,

$$W_x \ge \bar{p}_R - \delta$$

in the case where $\bar{A} > \min \bar{H}_R$. Assume now that $\bar{A} = \min \bar{H}_R$. This implies that

$$\bar{p}_R \leq W_x \leq \hat{p}_R$$

and, in all cases, we thus get (44) for $x > 0$.

Similarly, we can prove for $x < 0$ that

$$\hat{p}_L \leq W_x \leq \bar{p}_L$$

and the proof of (43) of is achieved. This implies (44). The proof of Theorem 4.6 is now complete. □

4.4 Proof of Theorem 1.4

Proof of Theorem 1.4. Let $\bar{A}$ denote the limit of A_ρ (see Proposition 4.4). We want to prove that $\bar{A} = \inf E$ where we recall that

$$E = \{\lambda \in \mathbb{R} : \exists w \text{ sub-solution of } (4)\}.$$

We argue by contradiction by assuming that there exists $\lambda < \bar{A}$ and w_λ sub-solution of (4).

The function

$$m_\lambda(x) = \sup_{t \in \mathbb{R}} w_\lambda(t, x)$$

satisfies

$$H(t(x), x, (m_\lambda)_x) \leq C$$

(for some function $t(x)$). Assumption **(A3)** implies that m_λ is globally Lipschitz continuous. Moreover, since w_λ is 1-periodic w.r.t. time and $(w_\lambda)_t \leq C$, then

$$|w_\lambda(t, x) - m_\lambda(x)| \leq C.$$

Hence

$$w_\lambda^\varepsilon(t, x) = \varepsilon w_\lambda(\varepsilon^{-1} t, \varepsilon^{-1} x)$$

has a limit W^λ which satisfies

$$\bar{H}_R(W_x^\lambda) \leq \lambda \quad \text{for } x > 0.$$

In particular, for $x > 0$,

$$W_x^\lambda \leq \hat{p}_R^\lambda := \max\{p \in \mathbb{R} : \bar{H}_R(p) = \lambda\} < \bar{p}_R$$

where $\bar{p}_R$ is defined in (8). Similarly,

$$W_x^\lambda \geq \hat{p}_L^\lambda := \min\{p \in \mathbb{R} : \bar{H}_L(p) = \lambda\} > \bar{p}_L$$

with $\bar{p}_L$ defined in (9). Those two inequalities imply in particular that for all $\delta > 0$, there exists $\tilde{C}_\delta$ such that

$$w_\lambda(t,x) \le \begin{cases} (\hat{p}_R^\lambda + \delta)x + \tilde{C}_\delta & \text{for } x > 0, \\ (\hat{p}_L^\lambda + \delta)x + \tilde{C}_\delta & \text{for } x < 0. \end{cases}$$

In particular,

$$w_\lambda < w \text{ for } |x| \ge R$$

if δ is small enough and R is large enough. In particular,

$$w_\lambda < w + C_R \text{ for } x \in \mathbb{R}.$$

Remark finally that $u(t,x) = w(t,x) + C_R - \bar{A}t$ is a solution and $u_\lambda(t,x) = w_\lambda(t,x) - \lambda t$ is a sub-solution of (1) with $\varepsilon = 1$ and $u_\lambda(0,x) \le u(0,x)$. Hence the comparison principle implies that

$$w_\lambda(t,x) - \lambda t \le w(t,x) - \bar{A}t + C_R.$$

Dividing by t and letting t go to $+\infty$, we get the following contradiction

$$\bar{A} \le \lambda.$$

The proof is now complete. □

5 Proof of Theorem 1.10

This section is devoted to the proof of Theorem 1.10. As previously pointed out in Remark 1.11, the notion of solutions for (1) has to be first made precise because the Hamiltonian is discontinuous with respect to time.

Notion of solutions for (1)**.** For $\varepsilon = 1$, a function u is a *solution* of (1) if it is globally Lipschitz continuous (in space and time) and if it solves successively the Cauchy problems on time intervals $[\tau_i + k, \tau_{i+1} + k)$ for $i = 0, \dots, K$ and $k \in \mathbb{N}$.

Because of this definition (approach), we have to show that if the initial data u_0 is globally Lipschitz continuous, then the solution to the successive Cauchy problems is also globally Lipschitz continuous (which of course insures its uniqueness from the classical comparison principle). The statement of these Lipschitz bounds is the goal of the next step.

Proof of Theorem 1.10 i). In view of the proof of Theorem 1.7, the reader can check that it is enough to get a global Lipschitz bound on the solution u^ε and to construct a global corrector in this new framework. The proof of these two facts is postponed, see Lemmas 5.1 and 5.2 following this proof. Notice that half-relaxed limits are not necessary anymore and that the reasoning can be completed by considering locally converging subsequences of $\{u^\varepsilon\}_\varepsilon$. □

Lemma 5.1 (Global Lipschitz bound)**.** *The function u^ε is equi-Lipschitz continuous with respect to time and space.*

Proof. Remark that it is enough to get the result for $\varepsilon = 1$ since $u(t,x) = \varepsilon^{-1}u^\varepsilon(\varepsilon t, \varepsilon x)$ satisfies the equation with $\varepsilon = 1$ and the initial condition

$$u_0^\varepsilon(x) = \frac{1}{\varepsilon} U_0^\varepsilon(\varepsilon x)$$

is equi-Lipschitz continuous. For the sake of clarity, we drop the ε superscript in u_0^ε and simply write u_0. We next proceed in several steps.

Step 1. Bounds on the time interval $[\tau_0, \tau_1) = [0, \tau_1)$**.** We assume that the initial data satisfies $|(u_0)_x| \le L$. Then as usual, there is a constant $C > 0$ such that

$$u^\pm(t,x) = u_0(x) \pm Ct$$

are super-/sub-solutions of (6) with H given by **(C1)** with for instance

$$C := \max\left(\max_{\alpha=1,\dots,N} |a_\alpha|_\infty,\ \max_{\alpha=0,\dots,N} \left(\max_{|p|\le L} |\bar{H}_\alpha(p)| \right)\right). \tag{45}$$

Let u be the standard (continuous) viscosity solution of (1) on the time interval $(0, \tau_1)$ with initial data given by u_0 (recall that $\varepsilon = 1$). Then for any $h > 0$ small enough, we have $-Ch \le u(h,x) - u(0,x) \le Ch$. The comparison principle implies for $t \in (0, \tau_1 - h)$

$$-Ch \le u(t+h, x) - u(t,x) \le Ch$$

which shows the Lipschitz bound in time, on the time interval $[0, \tau_1)$:

$$|u_t| \le C. \tag{46}$$

From the Hamilton-Jacobi equation, we now deduce the following Lipschitz bound in space on the time interval $(0, \tau_1)$:

$$|\bar{H}_\alpha(u_x(t,\cdot))|_{L^\infty(b_\alpha, b_{\alpha+1})} \quad \le C, \quad \text{for} \quad \alpha = 0, \dots, N. \tag{47}$$

Step 2. Bounds on the time interval $[\tau_1, \tau_2)$**.** We deduce in particular that (47) still holds true at time $t = \tau_1$. Combined with our definition (45) of the constant C, we also deduce that

$$v^\pm(t,x) = u(\tau_1, x) \pm C(t - \tau_1)$$

are sub/super-solutions of (6) for $t \in (\tau_1, \tau_2)$ where H is given by **(C1)**. Applying the reasoning of Step 2.1, we get bounds (46) and (47) on the time interval $[\tau_1, \tau_2)$.

Step 3. Conclusion and global bounds. Repeating the previous resoning in Steps 2.1 and 2.2, we deduce that the Lipschitz bounds (46) and (47) hold true for all $t \in [0, +\infty)$. $\square$

Lemma 5.2. *The conclusion of Theorem 4.6 still holds true in this new framework.*

Proof. The proof proceeds in several steps.

Step 1. Construction of a time periodic corrector w^ρ on $[-\rho, \rho]$. We first construct a Lipschitz corrector on a truncated domain. In order to do so, we proceed in several steps.

Step 1.1. First Cauchy problem on $(0, +\infty)$. The method presented in the proof of Proposition 4.3, using a term δw^δ has the inconvenient that it would not clearly provide a Lipschitz solution, which will be a difficulty to use a comparison principle. In order to stay with our notion of globally Lipschitz solutions, we simply solve the Cauchy problem for $\rho > \rho_0 := \max_{\alpha=1,\dots,N} |b_\alpha|$:

$$\begin{cases} w_t^\rho + H(t, x, w_x^\rho) = 0 & \text{on} \quad (0, +\infty) \times (-\rho, \rho)\,, \\ w_t^\rho + \bar{H}_L^-(w_x^\rho) = 0 & \text{on} \quad (0, +\infty) \times \{-\rho\}\,, \\ w_t^\rho + \bar{H}_R^+(w_x^\rho) = 0 & \text{on} \quad (0, +\infty) \times \{\rho\}\,, \\ w^\rho(0, x) = 0 & \text{for} \quad x \in [-\rho, \rho]\,. \end{cases} \tag{48}$$

As in the proof of the previous lemma, we get global Lipschitz bounds with a constant C (independent on $\rho > 0$):

$$|w_t^\rho|, \quad |\bar{H}_\alpha(w_x^\rho(t, \cdot))|_{L^\infty((b_\alpha, b_{\alpha+1})\cap(-\rho,\rho))} \quad \le C, \quad \text{for} \quad \alpha = 0, \dots, N. \tag{49}$$

Then following the proof of Proposition 4.1 in [7], we deduce that there exists a real λ_ρ with

$$|\lambda_\rho| \le C$$

and a constant C_0 (that depends on ρ) such that we have

$$|w^\rho(t, x) - \lambda_\rho t| \le C_0. \tag{50}$$

Step 1.2. Getting global sub and super-solutions. Let us now define the following function (up to some subsequence $k_n \to +\infty$):

$$w_\infty^\rho(t, x) = \lim_{k_n \to +\infty} (w^\rho(t + k_n, x) - \lambda_\rho k_n)$$

which still satisfies (49) and (50). Then we also define the two functions

$$\overline{w}_\infty^\rho(t, x) = \inf_{k\in\mathbb{Z}} (w_\infty^\rho(t + k, x) - k\lambda_\rho)\,, \quad \underline{w}_\infty^\rho(t, x) = \sup_{k\in\mathbb{Z}} (w_\infty^\rho(t + k, x) - k\lambda_\rho)\,,$$

which still satisfy (49) and (50), and are respectively super and sub-solutions of the problem in $\mathbb{R}\times[-\rho,\rho]$. They satisfy moreover that $\overline{w}^\rho_\infty(t,x)-\lambda_\rho t$ and $\underline{w}^\rho_\infty(t,x)-\lambda_\rho t$ are 1-periodic in time, which implies the following bounds

$$|\overline{w}^\rho_\infty(t,x)-\overline{w}^\rho_\infty(0,x)-\lambda_\rho t|\le C,\quad |\underline{w}^\rho_\infty(t,x)-\overline{w}^\rho_\infty(0,x)-\lambda_\rho t|\le C.$$

Step 1.3: A new Cauchy problem on $(0,+\infty)$ and construction of a time periodic solution. We note that $\overline{w}^\rho_\infty+2C_0\ge \underline{w}^\rho_\infty$, and we now solve the Cauchy problem with new initial data $\underline{w}^\rho_\infty(0,x)$ instead of the zero initial data and call $\tilde{w}^\rho$ the solution of this new Cauchy problem. From the comparison principle, we get

$$\underline{w}^\rho_\infty\le\tilde{w}^\rho\le\overline{w}^\rho_\infty+2C_0.$$

In particular, we have

$$\tilde{w}^\rho(1,x)\ge\underline{w}^\rho_\infty(1,x)=\tilde{w}^\rho(0,x)+\lambda_\rho.$$

This implies, by comparison, that

$$\tilde{w}^\rho(k+1,x)\ge\tilde{w}^\rho(k,x)+\lambda_\rho. \tag{51}$$

Moreover $\tilde{w}^\rho$ still satisfies (49) (indeed with the same constant because, by construction, this is also the case for $\underline{w}^\rho_\infty$). We now define (up to some subsequence $k_n\to+\infty$):

$$\tilde{w}^\rho_\infty(t,x)=\lim_{k_n\to+\infty}\left(\tilde{w}^\rho(t+k_n,x)-\lambda_\rho k_n\right)$$

which, because of (51) and the fact that $\tilde{w}^\rho(t,x)-\lambda_\rho t$ is bounded, satisfies

$$\tilde{w}^\rho_\infty(k+1,x)=\tilde{w}^\rho_\infty(k,x)$$

and then is 1-periodic in time. Moreover $\tilde{w}^\rho_\infty$ is still a solution and satisfies (49). We define

$$w^\rho:=\tilde{w}^\rho_\infty$$

which satisfies (34) and then provides the analogue of the function given in Proposition 4.3.

Step 2. Contruction of w on $\mathbb{R}$. The result of Theorem 4.6 still holds true for

$$w=\lim_{\rho\to+\infty}\left(w^\rho-w^\rho(0,0)\right)$$

and

$$\bar{A}=\lim_{\rho\to+\infty}\lambda_\rho.\quad\square$$

Proof of Theorem 1.10 ii). We first prove the result in the case $N=1$. We recall that $\bar{H}_L=\bar{H}_0$ and $\bar{H}_R=\bar{H}_1$ and set $a=a_1$ and (up to translation) $b_1=0$.

Step 1: The convex case: identification of $\bar{A}$.

STEP 1.1: A CONVEX SUBCASE. We first work in the particular case where both $\bar{H}_\alpha$ for $\alpha = L, R$ are convex and given by the Legendre-Fenchel transform of convex Lagrangians L_α which satisfy for some compact interval I_α:

$$L_\alpha(p) = \begin{cases} \text{finite} & \text{if} \quad q \in I_\alpha, \\ +\infty & \text{if} \quad q \notin I_\alpha. \end{cases} \tag{52}$$

Then it is known (see for instance the section on optimal control in [9]) that the solution of (1) on the time interval $[0, \varepsilon\tau_1)$, is given by

$$u^\varepsilon(t,x) = \inf_{y\in\mathbb{R}} \left(\inf_{X \in S_{0,y;t,x}} \left\{ u^\varepsilon(0, X(0)) + \int_x^t L_\varepsilon(s, X(s), \dot{X}(s)) \right\} ds \right) \tag{53}$$

with

$$L_\varepsilon(s,x,p) = \begin{cases} \bar{H}_L^*(p) & \text{if} \quad x < 0, \\ \bar{H}_R^*(p) & \text{if} \quad x > 0, \\ \min(-a(\varepsilon^{-1}s), \min\limits_{\alpha=L,R} L_\alpha(0)) & \text{if} \quad x = 0, \end{cases}$$

and for $s < t$, the following set of trajectories:

$$S_{s,y;t,x} = \{X \in \text{Lip}((s,t);\mathbb{R}), \quad X(s) = y, \quad X(t) = x\}.$$

Combining this formula with the other one on the time interval $[\varepsilon\tau_1, \varepsilon\tau_2)$, and iterating on all necessary intervals, we get that (53) is a representation formula of the solution u^ε of (1) for all $t > 0$. We also know (see the section on optimal control in [9]), that the optimal trajectories from $(0, y)$ to (t_0, x_0) intersect the axis $x = 0$ at most on a time interval $[t_1^\varepsilon, t_2^\varepsilon]$ with $0 \le t_1^\varepsilon \le t_2^\varepsilon \le t_0$. If this interval is not empty, then we have $t_i^\varepsilon \to t_i^0$ for $i = 1, 2$ and we can easily pass to the limit in (53). In general, it is easy to see that u^ε converges to u^0 given by the formula

$$u^0(t,x) = \inf_{y\in\mathbb{R}} \left(\inf_{X \in S_{0,y;t,x}} \left\{ u^0(0, X(0)) + \int_x^t L_0(s, X(s), \dot{X}(s))\, ds \right\} \right)$$

with

$$L_0(s,x,p) = \begin{cases} \bar{H}_L^*(p) & \text{if} \quad x < 0, \\ \bar{H}_R^*(p) & \text{if} \quad x > 0, \\ \min(-\langle a \rangle, \min\limits_{\alpha=L,R} L_\alpha(0)) & \text{if} \quad x = 0, \end{cases}$$

and from [9] we see that u^0 is the unique solution of (6)-(2). with $\bar{A} = \langle a \rangle$.
STEP 1.2: THE GENERAL CONVEX CASE. The general case of convex Hamiltonians is recovered, because for Lipschitz initial data u_0, we know that the solution is globally Lipschitz. Therefore, we can always modify the Hamiltonians $\bar{H}_\alpha$ outside some compact interval such that the modified Hamiltonians satisfy (52).

Step 2: General quasi-convex Hamiltonians: identification of $\bar{A}$.

STEP 2.1: SUB-SOLUTION INEQUALITY. From Proposition 2.9 in [9], we know that $w(t,0)$, as a function of time only, satisfies in the viscosity sense

$$w_t(t,0) + a(t) \le \bar{A} \quad \text{for all} \quad t \in \mathbb{R}\backslash(\bigcup_{i=1,\dots,K+1} \tau_i\mathbb{Z}).$$

Using the 1-periodicity in time of w, we see that the integration in time on one period implies:

$$\langle a \rangle \le \bar{A}. \tag{54}$$

STEP 2.2: SUPER-SOLUTION INEQUALITY. Recall that $\bar{A} \ge \ \langle a \rangle \ \ge A_0 := \max_{\alpha=L,R} \min(\bar{H}_\alpha)$. If $\bar{A} = A_0$, then obviously, we get $\bar{A} = \langle a \rangle$. Hence, it remains to treat the case $\bar{A} > A_0$.

STEP 2.3: CONSTRUCTION OF A SUPER-SOLUTION FOR $x \neq 0$. Then there exists some $\delta > 0$ such that

$$\bar{p}_L + 2\delta < \bar{p}_L^0 \qquad \text{and} \quad \bar{p}_R^0 < \bar{p}_R - 2\delta. \tag{55}$$

If w denotes a global corrector given by Theorem 4.6, let us define

$$\underline{w}_R(t,x) = \inf_{h\ge 0} \big(w(t,x+h) - \bar{p}_R^0 h\big) \quad \text{for} \quad x \ge 0,$$

and similarly

$$\underline{w}_L(t,x) = \inf_{h\ge 0} \big(w(t,x-h) + \bar{p}_L^0 h\big) \quad \text{for} \quad x \le 0.$$

From (42), we deduce that we have for some $\bar{h} \ge 0$

$$w(t,x) \ge \underline{w}_R(t,x) = w(t,x+\bar{h}) - \bar{p}_R^0\bar{h} \ge w(t,x) + (\bar{p}_R - \delta - \bar{p}_R^0)\bar{h} - C_\delta.$$

From (55), this implies

$$0 \le \bar{h} \le C_\delta/\delta$$

and using the fact that w is globally Lipschitz, we deduce that for $\alpha = R$:

$$w \ge \underline{w}_\alpha \ge w - C_1. \tag{56}$$

Moreover, by constrution (as an infimum of (globally Lipschitz) super-solutions), $\underline{w}_R$ is a (globally Lipschitz) super-solution of the problem in $\mathbb{R} \times (0,+\infty)$. We also have for $x = y + z$ with $z \ge 0$:

$$\begin{aligned}\underline{w}_R(t,x) - \underline{w}_R(t,y) &= w(t,x+\bar{h}) - \bar{p}_R^0\bar{h} - \underline{w}_R(t,y)\\ &\ge w(t,x+\bar{h}) - \bar{p}_R^0\bar{h} - \big(w(t,y+\bar{h}+z) - \bar{p}_R^0(\bar{h}+z)\big)\\ &\ge \bar{p}_R^0 z = \bar{p}_R^0(x-y)\end{aligned}$$

which shows that

$$(\underline{w}_R)_x \ge \bar{p}_R^0. \tag{57}$$

Similarly (and we can also use a symmetry argument to see it), we get that $\underline{w}_L$ is a (globally Lipschitz) super-solution in $\mathbb{R} \times (-\infty, 0)$, it satisfies (56) with $\alpha = L$ and

$$(\underline{w}_L)_x \le \bar{p}_L^0. \tag{58}$$

We now define

$$\underline{w}(t,x) = \begin{cases} \underline{w}_R(t,x) & \text{if} \quad x > 0, \\ \underline{w}_L(t,x) & \text{if} \quad x < 0, \\ \min(\underline{w}_L(t,0), \underline{w}_R(t,0)) & \text{if} \quad x = 0 \end{cases} \tag{59}$$

which by constrution is lower semicontinuous and satisfies (56), and is a super-solution for $x \neq 0$.

STEP 2.4: CHECKING THE SUPER-SOLUTION PROPERTY AT $x = 0$. Let φ be a test function touching $\underline{w}$ from below at $(t_0, 0)$ with $t_0 \in \mathbb{R} \setminus \left(\bigcup_{i=1,\dots,K+1} \tau_i \mathbb{Z} \right)$. We want to check that

$$\varphi_t(t_0, 0) + F_{a(t_0)}(\varphi_x(t_0, 0^-), \varphi_x(t_0, 0^+)) \ge \bar{A}. \tag{60}$$

We may assume that

$$\underline{w}(t_0, 0) = \underline{w}_R(t_0, 0) = w(t_0, 0 + \bar{h}) - \bar{p}_R^0 \bar{h}$$

since the case $\underline{w}(t_0, 0) = \underline{w}_L(t_0, 0)$ is completely similar.

We distinguish two cases.

Case a). $\bar{h} > 0$. Then we have for all $h \ge 0$

$$\varphi(t, 0) \le w(t, 0 + h) - \bar{p}_R^0 h$$

with equality for $(t, h) = (t_0, \bar{h})$. This implies the viscosity inequality

$$\varphi_t(t_0, 0) + \bar{H}_R(\bar{p}_R^0) \ge \bar{A}$$

which implies (60), because $F_{a(t_0)}(\varphi_x(t_0, 0^-), \varphi_x(t_0, 0^+)) \ge a(t_0) \ge A_0 \ge \min \bar{H}_R = \bar{H}_R(\bar{p}_R^0)$.

Case b). $\bar{h} = 0$. Then we have $\varphi \le \underline{w} \le w$ with equality at $(t_0, 0)$. This implies immediately (60).

STEP 2.5: CONCLUSION. We deduce that $\underline{w}$ is a super-solution on $\mathbb{R} \times \mathbb{R}$. Now let us consider a C^1 function $\psi(t)$ such that

$$\psi(t) \le \underline{w}(t, 0)$$

with equality at $t = t_0$. Because of (57) and (58), we see that

$$\varphi(t, x) = \psi(t) + \bar{p}_L^0 x 1_{\{x<0\}} + \bar{p}_R^0 x 1_{\{x>0\}}$$

satisfies

$$\varphi \le \underline{w}$$

with equality at $(t_0, 0)$. This implies (60), and at almost every point t_0 where the Lipschitz continuous function $\underline{w}(t, 0)$ is differentiable, we have

$$\underline{w}_t(t_0, 0) + a(t_0) \ge \bar{A}.$$

Because w is 1-periodic in time, we get after an integration on one period:

$$\langle a \rangle \ge \bar{A}.$$

Together with (54), we deduce that $\langle a \rangle = \bar{A}$, which is the desired result, for $N = 1$.

It remains to deal with the case $N \ge 2$. We simply remark, using the sub-solution viscosity inequality at each junction condition, that

$$\bar{A} \ge \langle a_\alpha \rangle \quad \text{for all} \quad \alpha = 1, \dots, N.$$

The proof of Theorem 1.10-ii) is now complete. □

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A Proofs of some technical results

A.1 The case $x \neq 0$ in the proof of convergence

The case $x \neq 0$ in the proof of Theorem 1.7. We only deal with the subcase $x > 0$ since the subcase $x < 0$ is treated in the same way. Reducing $\overline{r}$ if necessary, we may assume that $B_{\overline{r}}(\overline{t}, \overline{x})$ is compactly embedded in the set $\{(t, x) \in (0, +\infty) \times (0, +\infty) : \ x > 0\}$: there exists a positive constant $c_{\overline{r}}$ such that

$$(t, x) \in B_{\overline{r}}(\overline{t}, \overline{x}) \quad \Rightarrow \quad x > c_{\overline{r}}\,. \tag{61}$$

Let $p = \varphi_x(\overline{t}, \overline{x})$ and let $v^R = v^R(t, x)$ be a solution of the cell problem

$$v^R_t + H_R\left(t, x, p + v^R_x\right) = \bar{H}_R(p) \quad \text{in} \quad \mathbb{R} \times \mathbb{R}\,. \tag{62}$$

We claim that if $\varepsilon > 0$ is small enough, the perturbed test function

$$\varphi^\varepsilon(t, x) = \varphi(t, x) + \varepsilon v^R\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right)$$

satisfies, in the viscosity sense, the inequality

$$\varphi_t^\varepsilon + H\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \varphi_x^\varepsilon\right) \geq \frac{\theta}{2} \quad \text{in} \quad B_r(\overline{t}, \overline{x}) \tag{63}$$

for sufficiently small $r > 0$. To see this let ψ be a test function touching φ^ε from below at $(t_1, x_1) \in B_r(\overline{t}, \overline{x}) \subseteq B_{\overline{r}}(\overline{t}, \overline{x})$. In this way the function

$$\eta(s, y) = \frac{1}{\varepsilon}\left(\psi(\varepsilon s, \varepsilon y) - \varphi(\varepsilon s, \varepsilon y)\right)$$

touches v^R from below at $(s_1, y_1) = \left(\frac{t_1}{\varepsilon}, \frac{x_1}{\varepsilon}\right)$ and (62) yields

$$\psi_t(t_1, x_1) - \varphi_t(t_1, x_1) + H_R\left(\frac{t_1}{\varepsilon}, \frac{x_1}{\varepsilon}, p + \psi_x(t_1, x_1) - \varphi_x(t_1, x_1)\right) \geq \bar{H}_R(p). \tag{64}$$

Since (61) implies that $\frac{x}{\varepsilon} \to +\infty$, as $\varepsilon \to 0$, uniformly with respect to $(t, x) \in B_{\overline{r}}(\overline{t}, \overline{x})$, we can find, owing to **(A5)**, an $\varepsilon_0 > 0$ independent of ψ and (t_1, x_1) such that the inequality

$$H\left(\frac{t_1}{\varepsilon}, \frac{x_1}{\varepsilon}, \psi_x(t_1, x_1)\right) \geq H_R\left(\frac{t_1}{\varepsilon}, \frac{x_1}{\varepsilon}, \psi_x(t_1, x_1)\right) - \frac{\theta}{4} \tag{65}$$

holds true for $\varepsilon < \varepsilon_0$. Combining (16)-(64)-(65) and using the continuity of φ_x and φ_t we have

$$\begin{aligned}
&\psi_t(t_1, x_1) + H\left(\frac{t_1}{\varepsilon}, \frac{x_1}{\varepsilon}, \psi_x(t_1, x_1)\right) \\
&\geq \psi_t(t_1, x_1) + H_R\left(\frac{t_1}{\varepsilon}, \frac{x_1}{\varepsilon}, p + \psi_x(t_1, x_1) - \varphi_x(t_1, x_1)\right) + \\
&\quad H_R\left(\frac{t_1}{\varepsilon}, \frac{x_1}{\varepsilon}, \psi_x(t_1, x_1)\right) - H_R\left(\frac{t_1}{\varepsilon}, \frac{x_1}{\varepsilon}, \varphi_x(\bar{t}, \bar{x}) + \psi_x(t_1, x_1) - \varphi_x(t_1, x_1)\right) - \frac{\theta}{4} \\
&\geq \frac{\theta}{2}
\end{aligned}$$

if r is sufficiently close to 0. The claim (63) is proved.

Since φ is strictly above $\overline{u}$, if ε and r are small enough

$$u^\varepsilon + \kappa_r \leq \varphi^\varepsilon \quad \text{on} \quad \partial B_r(\overline{t}, \overline{x})$$

for a suitable positive constant κ_r. By comparison principle we deduce

$$u^\varepsilon + \kappa_r \leq \varphi^\varepsilon \quad \text{in} \quad B_r(\overline{t}, \overline{x})$$

and passing to the limit as $\varepsilon \to 0$ and $(t, x) \to (\bar{t}, \bar{x})$ on both sides of the previous inequality, we produce the contradiction

$$\overline{u}(\overline{t}, \overline{x}) < \overline{u}(\overline{t}, \overline{x}) + \kappa_r \leq \varphi(\overline{t}, \overline{x}) = \overline{u}(\overline{t}, \overline{x})\,. \qquad \square$$

A.2 Proof of Lemma 3.3

Proof of Lemma 3.3. We first adress uniqueness. Let us assume that we have two solutions (v^i, λ^i) for $i = 1, 2$ of (3). Let

$$u^i(t,x) = v^i(t,x) + px - \lambda^i t$$

Then u^i solves

$$u^i_t + H_\alpha(t, x, u^i_x) = 0$$

with

$$u^1(0,x) \le u^2(0,x) + C$$

The comparison principle implies

$$u^1 \le u^2 + C \quad \text{for all} \quad t > 0$$

and then $\lambda^1 \ge \lambda^2$. Similarly we get the reverse inequality and then $\lambda^1 = \lambda^2$.

We now turn to the continuity of the map $p \mapsto \bar{H}_\alpha(p)$. It follows from the stability of viscosity sub- and super-solutions, from the fact that the constant C in (21) is bounded for bounded p's and from the comparison principle. This achieves the proof of the lemma. $\square$

A.3 Sketch of the proof of Proposition 4.1

Sketch of the proof of Proposition 4.1. Consider

$$M_\nu = \sup_{x \in [\rho_1, \rho_2], s,t \in \mathbb{R}} \left\{ u(t,x) - v(s,x) - \frac{(t-s)^2}{2\nu} \right\}.$$

We want to prove that

$$M = \lim_{\nu \to 0} M_\nu \le 0.$$

We argue by contradiction by assuming that $M > 0$. The supremum defining M_ν is reached, let s_ν, t_ν, x_ν denote a maximizer. Choose ν small enough so that $M_\nu \ge \frac{M}{2} > 0$. We classically get,

$$|t_\nu - s_\nu| \le C\sqrt{\nu}.$$

If there exists $\nu_n \to 0$ such that $x_{\nu_n} = \rho_1$ for all $n \in \mathbb{N}$, then

$$\frac{M}{2} \le M_{\nu_n} \le U_0(t_{\nu_n}) - U_0(s_{\nu_n}) \le \omega_0(t_{\nu_n} - s_{\nu_n}) \le \omega_0(C\sqrt{\nu_n})$$

where ω_0 denotes the modulus of continuity of U_0. The contradiction $M \le 0$ is obtained by letting n go to $+\infty$.

Hence, we can assume that for ν small enough, $x_\nu > \rho_1$. Reasoning as in [9, Theorem 7.8], we can easily reduce to the case where $H(t_\nu, x_\nu, \cdot)$ reaches its minimum for $p = p_0 = 0$. We can also consider the vertex test function G^γ associated with the single

Hamiltonian H (using notation of [9], it corresponds to the case $N = 1$) and the free parameter γ. If $x_\nu < \rho_2$, then $G^\gamma(x, y)$ reduces to the standard test function $\frac{(x-y)^2}{2}$.

We next consider

$$M_{\nu,\varepsilon} = \sup_{\substack{x,y\in[\rho_1,\rho_2]\cap\overline{B_r(x_\nu)} \\ s,t\in\mathbb{R}}} \left\{ u(t,x) - v(s,y) - \frac{(t-s)^2}{2\nu} - \varepsilon G^\gamma(\varepsilon^{-1}x, \varepsilon^{-1}y) - \varphi^\nu(t,s,x) \right\}$$

where $r = r_\nu$ is chosen so that $\rho_1 \notin \overline{B_r(x_\nu)}$ and the localization function

$$\varphi^\nu(t,s,x) = \frac{1}{2}((t-t_\nu)^2 + (s-s_\nu)^2 + (x-x_\nu)^2).$$

The supremum defining $M_{\nu,\varepsilon}$ is reached and if (t, s, x, y) denotes a maximizer, then

$$(t, s, x, y) \to (t_\nu, s_\nu, x_\nu, x_\nu) \quad \text{as } (\varepsilon, \gamma) \to 0.$$

In particular, $x, y \in B_r(x_\nu)$ for ε and γ small enough. The remaining of the proof is completely analogous (in fact much simpler). $\square$

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