EXISTENCE AND NON-EXISTENCE OF SEMI-DISCRETE SHOCKS FOR A CAR-FOLLOWING MODEL IN TRAFFIC FLOW

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Abstract. We consider a (microscopic) car-following model in traffic flow that can be seen as a semi-discrete scheme (discretization in space only) of a (macroscopic) Hamilton-Jacobi equation. For this discrete model, and for general velocity laws satisfying a strict chord inequality, we construct traveling solutions that are naturally associated to "traveling shocks" for the conservation law derived from the Hamilton-Jacobi equation. These shocks can be interpreted as a phase transition between two states of different car densities. There is no smallness condition on the size of these shocks. This existence and uniqueness of the solution is done at the level of the Hamilton-Jacobi equation using the notion of viscosity solution. A surprising non-existence result of semi-discrete shocks for this microscopic model is also presented in the case where a shock exists for the associated macroscopic model, but the velocity law V satisfies a non strict chord inequality.

Key words. car-following model, Hamilton-Jacobi equation, semi-discrete shock, traffic flow, viscosity solutions

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1. Introduction.

In this paper we are interested in a discrete car-following model for traffic flow. In this model, the vehicles of positions $(X_i)_{i\in\mathbb{Z}}$, satisfying $X_i < X_{i+1}$, move with the velocity

$$\dot{X}_i = V(X_{i+1} - X_i) \tag{1.1}$$

where V is a given function describing the behaviour of the drivers. Usually V is assumed to be a non-decreasing function, *i.e.* the velocity of the driver is higher if its distance to the vehicle in front of him, is higher. We look for particular shock solutions of (1.1) of the form

$$X_i(t) = h(i + \frac{t}{T}) + ct \tag{1.2}$$

where h solves

$$c + \frac{1}{T}h'(y) = V(h(y+1) - h(y)) \tag{1.3}$$

and

$$\begin{cases} h(y+1) - h(y) \longrightarrow b & \text{as} \quad y \to -\infty \\ h(y+1) - h(y) \longrightarrow a & \text{as} \quad y \to +\infty. \end{cases} \tag{1.4}$$

Notice that (1.2), (1.3) and (1.4) provide a solution of (1.1) which satisfies

$$\begin{cases} X_i(t+T) = X_{i+1}(t) + c\,T \\ \text{and} \\ X_{i+1}(t) - X_i(t) \longrightarrow b \quad \text{as} \quad i \to -\infty \\ X_{i+1}(t) - X_i(t) \longrightarrow a \quad \text{as} \quad i \to +\infty. \end{cases} \tag{1.5}$$

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The first equation of (1.5) means that after a period of time T, each vehicle replaces its neighboor in front of it, up to a shift of a distance cT. This means that if we have an air plane view of the traffic and the cars are assumed to be not distinguishable, then we realize that the discrete shock also moves with velocity c. Here b is the limit interdistance between vehicles far before the shock, and a is the corresponding one far after the shock. Therefore the interdistance function $h(y+1)-h(y)$ can be interpreted as a phase transition between two states b and a and is then a sort of discrete shock. Such solutions are also related to Hamilton-Jacobi equations (see Section 2). Moreover our methods of proof are heavily based on the notion of viscosity solutions for (1.3) (see Definition 3.1).

1.1. Main results. We assume that V satisfies the following properties

Assumption (A)
(A_1) **(Regularity)**
$V\in C^1(\mathbb{R})$, $V'\in L^\infty(\mathbb{R})$, $V_{|\mathbb{R}^+}\in L^\infty(\mathbb{R}^+)$,
(A_2) **(Monotonicity)**

$$V'>0 \quad \text{on} \quad \mathbb{R}\,,$$

(A_3) **(Strict chord inequality)**
There exists $T>0$ and $c\in\mathbb{R}$ such that

$$\begin{cases} \dfrac{p}{T}+c\le V(p) \quad \text{for} \quad p\in\mathbb{R} \quad \text{if and only if} \quad p\in[a,b], \\ \\ \text{with equality if and only if } p\in\{a,b\}, \end{cases}$$

(A_4) **(Non-degeneracy)**

$$V'(b)<\frac{1}{T}<V'(a).$$

The regularity assumption (A_1) is natural in order to apply Cauchy-Lipschitz theorem for ODEs. For simplification in the proofs, we assume that V is defined on $\mathbb{R}$. The monotonicity assumption on V plays a crucial role in maximum principle arguments, and the strict monotonicity (A_2) is essential for strong maximum principle arguments. Assumption (A_3) means that the graph of V has only two intersection points with the straight line $z=\dfrac{p}{T}+c$, and is above this straight line on the interval $[a,b]$ (see (2.4) and Figure 1.1). Assumption (A_4) is a kind of non-degeneracy condition at the points a and b, which allows us to get exponential asymptotics of the solution at infinity, and then simplifies the analysis and the construction of solutions.
Our first main result is

Theorem 1.1. (Existence and uniqueness of a semi-discrete shock) *Assume that (A) holds for some $a<b$ and $T>0$.*
i) (existence)
Then there exists a concave solution $h\in C^1(\mathbb{R})$ of (1.3) satisfying for some constants $\gamma>0$, $C>0$:

$$\left|h(y)-\bar{h}(y)\right|\le Ce^{-\gamma|y|} \quad \text{with} \quad \bar{h}(y)=by\mathbb{1}_{\{y<0\}}+ay\mathbb{1}_{\{y\ge0\}} \tag{1.6}$$

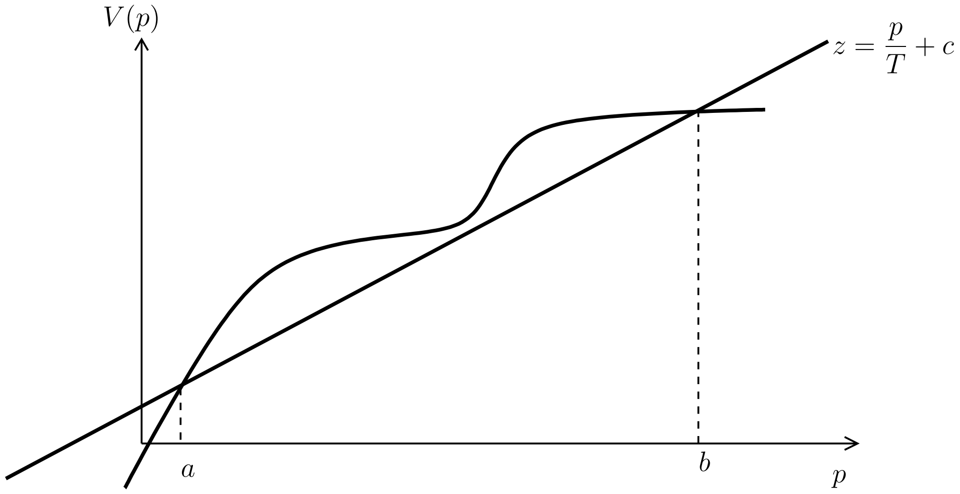

FIG. 1.1. *Example of V*

and

$$h'(+\infty) = a \leq h(y+1) - h(y) \leq b = h'(-\infty)\,. \tag{1.7}$$

ii) (uniqueness)
Moreover h is unique (up to translations and addition of constants) among the solutions $g \in C^1(\mathbb{R})$ of (1.3) satisfying $\left|g - \bar{h}\right| \leq C$ for some constant $C > 0$.

Notice that (1.7) implies (1.4) and then the function h given by Theorem 1.1 corresponds to the one we were looking for.
We emphasize the fact that Theorem 1.1 stays true if we only assume that V is defined on $[a, b]$ and satisfies (A) on this interval. This is related to the fact that, on the one hand such a function V can always be extended as a function satisfying (A) on $\mathbb{R}$, and on the other hand the solution satisfies (1.7), and then does not see the part where the function V has been extended. In this spirit, existence and uniqueness results can be obtained under weaker assumptions (see Theorem 8.1).
We underline the fact that assumption (A_3) is crucial for the existence. We may think to relax assumption (A_3) to the following condition:

(A'_3) **(Non strict chord inequality)**
There exists $T > 0$ and $c \in \mathbb{R}$ such that

$$\begin{cases} \dfrac{p}{T} + c \leq V(p) \quad \text{for} \quad p \in \mathbb{R} \quad \text{if and only if} \quad p \in [a, b], \\ \\ \text{with equality at least for } p = a, b, p_0 \text{ with } p_0 \in (a, b). \end{cases}$$

Then we have the following surprising non-existence result:

Theorem 1.2. (Non-existence of semi-discrete shocks)
Assume that (A_1), (A_2), (A'_3) hold for some $a < b$ and $T > 0$. Then there is no solution $h \in C^1(\mathbb{R})$ of (1.3) satisfying

$$h'(+\infty) = a \quad \textit{and} \quad h'(-\infty) = b. \tag{1.8}$$

A consequence of Theorem 1.2, is that if we consider a continuous family of functions V_ε for $\varepsilon \in [0,1]$ such that V_ε satisfies (A) for $\varepsilon > 0$ and only (A_1), (A_2), (A_3') for $\varepsilon = 0$, then the solution h^ε will split as ε goes to zero, into (at least) two solutions h_1 and h_2 as above. This illustrates how condition (A_3) is delicate. It would be very interesting to identify the long time dynamics describing the separation of microscopic shocks in the case of condition (A_3'), in a spirit similar to [10, 2].

Indeed, Theorem 1.2 is a straightforward consequence of the following classification result:

Theorem 1.3. (Classification of solutions)
Assume (A_1) and (A_2). Let $h \in C^1(\mathbb{R})$ be a solution of (1.3) for some $T > 0$ and $c \in \mathbb{R}$, satisfying for some constant $C > 0$

$$|h(y+1) - h(y)| \le C. \tag{1.9}$$

Then there exists $\tilde{a}, \tilde{b} \in \mathbb{R}$ such that

$$h'(-\infty) = \tilde{b} \quad \text{and} \quad h'(+\infty) = \tilde{a}, \tag{1.10}$$

we have $h \in C^2(\mathbb{R})$ and one of the following three cases holds.
Case 1: $\tilde{a} < \tilde{b}$
Then $h'' < 0$ and

$$\begin{cases} c + \dfrac{p}{T} < V(p) \quad \text{for all} \quad p \in (\tilde{a}, \tilde{b}), \\ \\ \text{with equality for} \quad p \in \left\{\tilde{a}, \tilde{b}\right\}. \end{cases} \tag{1.11}$$

Case 2: $\tilde{a} > \tilde{b}$
Then $h'' > 0$ and

$$\begin{cases} c + \dfrac{p}{T} > V(p) \quad \text{for all} \quad p \in (\tilde{b}, \tilde{a}), \\ \\ \text{with equality for} \quad p \in \left\{\tilde{a}, \tilde{b}\right\}. \end{cases} \tag{1.12}$$

Case 3: $\tilde{a} = \tilde{b}$
Then $h'' = 0$.
As an interesting application to traffic, it seems that we never observe traffic flow going to the right, with a shock where the traffic jam is on the left and the “fluid” traffic is on the right, i.e. a case where $a > b$. Therefore condition (1.12) of Theorem 1.3 suggests strongly that the velocity function V has to be concave in traffic applications in the range where V is increasing. Nevertheless we can find non concave velocity function V and Fundamental diagrams in the literature on traffic (see Li *et al.*[15]).

1.2. Brief review of the literature. In the case of Newell's model (see [17]) the velocity function is given by

$$V(p) = V_0(1 - e^{-\gamma(p-L)}) \tag{1.13}$$

for positive constants V_0, γ, L. For this model, exact solutions are known (see formula (11) in [21]) such that

$$\dot{X}_i(t) = V(X_{i+1}(t) - X_i(t)) = \frac{V(a)+V(b)}{2} + \left(\frac{V(a)-V(b)}{2}\right)\tanh\left(\beta\left(i + \frac{t}{T}\right)\right)$$

with

$$\begin{cases} \dfrac{1}{T} = \dfrac{V(a) - V(b)}{a - b}, \\ \beta = \gamma \left(\dfrac{b-a}{2} \right) > 0. \end{cases}$$

We mention that the exact solutions exist for two models with delay: Newell original model (1.13) and a version of Newell's model where the exponential is replaced by a hyperbolic tangent. The reader can consult [21, 11, 12, 20] for explicit solutions. Up to our knowledge, it seems that no exact solutions are known for other models like Bando *et al.* model [3]. Car-following models are related to the following Lighthill, Whitham and Richards model (see [16],[19])

$$\rho_t + f(\rho)_x = 0 \quad \text{with} \quad f(\rho) = \rho V(\frac{1}{\rho}) .$$

In this framework of conservation laws, shocks arise naturally. Discrete shocks have been constructed for fully discrete monotone schemes (with discretization in space and time), by Jennings [14] using "maximum principle" arguments and a fixed point approach (see also Serre [18] for systems). Semi-discrete shocks for semi-discrete schemes (with discretization in space only), have been constructed for systems of conservation laws in the case of small shocks, by a center manifold approach (see [6, 5] and also [7] for a study of the stability). In the case of Theorem 1.1, we construct large semi-discrete shocks (associated to $X_{i+1} - X_i$) using maximum principle arguments and Perron's method applied at the level of Hamilton-Jacobi equation (associated to X_i) instead of the scalar conservation law (associated to $X_{i+1} - X_i$).

1.3. Organization of the paper. In Section 2, we present the link of our problem with Hamilton-Jacobi equations. Section 2 is not necessary for the proof of our main results and can be skipped by the reader. In Section 3, we define a viscosity solution and recall that any C^1 solution is a viscosity solution and vice versa. In Section 4, we give preliminary results that will be used later in the next sections. In particular, we explain how equation (1.3) can be seen as an ODE with delay and then can be solved towards the left (see Lemma 4.1). We also provide a powerful self-contained proof of the exponential behaviour of the solution at $-\infty$ which is true for instance under assumption (A_4) (see Proposition 4.2). This exponential asymptotics will be used later to show the existence of a solution. In Section 5, we prove qualitative concavity/convexity properties of the solutions which provide a proof of the classification result (Theorem 1.3). In Section 6, we show the uniqueness (and concavity) of solutions (see Proposition 6.2). Section 7 is devoted to the proof of existence of a solution by Perron's method (in the framework of viscosity solutions). The difficult part is to construct the subsolution. After the presentation of the method in a first subsection, we provide qualitative properties of the subsolution in the second subsection. In the third and last subsection, we present the existence result and give the proof of Theorem 1.1. In Section 8, we extend our existence, uniqueness result and non-existence result to cases under weaker assumptions on the velocity function V (respectively Theorems 8.1 and 8.3). Finally Section 9 is an appendix where we recall the strong maximum principle (Lemma 9.1) used in the proofs.

1.4. Normalization. Notice that in Theorem 1.3, we can always come back from case 2 to case 1 by a simple change of unknowns. Indeed, if h is solution of (1.3) satisfying (1.10) with $\tilde{a} > \tilde{b}$ then $\check{h}(y) := -h(y)$ is solution of

$$\begin{cases} \check{c} + \frac{1}{T}\check{h}'(y) = \check{V}(\check{h}(y+1) - \check{h}(y)), \\ \\ \check{h}'(-\infty) = \check{b} \quad \text{and} \quad \check{h}'(+\infty) = \check{a}, \end{cases} \quad \text{with} \quad \begin{cases} \check{V}(p) := -V(-p), \\ \\ \check{c} = -c, \quad \check{a} = -\tilde{a}, \quad \check{b} = -\tilde{b}, \end{cases}$$

where we have now the condition:

$$\check{a} < \check{b}.$$

Still having equation (1.3) in mind, up to consider the new velocity function

$$\tilde{V}(p) = T(V(p) - c)$$

and replace V by $\tilde{V}$, we can assume that

$$T = 1 \quad \text{and} \quad c = 0\,. \tag{1.14}$$

From now on, except in Section 2 and part of Section 8, and up to the end of the paper, we will use normalization (1.14) and then (1.3) can be rewritten as

$$h'(y) = V(h(y+1) - h(y)). \tag{1.15}$$

We will also assume that $a < b$.

2. Link with Hamilton-Jacobi equations. It is known (see [9]) that this microscopic model is related to the following macroscopic model

$$\mathcal{X}_t = V(\mathcal{X}_y) \tag{2.1}$$

which is a Hamilton-Jacobi equation, where $t > 0$ is the time and $y \in \mathbb{R}$ is a continuous index of the vehicles and where $\mathcal{X}_t = \frac{\partial \mathcal{X}}{\partial t}$, $\mathcal{X}_y = \frac{\partial \mathcal{X}}{\partial y}$. We now define the viscosity subsolution, supersolution and solution of (2.1).

Definition 2.1. (Viscosity solution for Hamilton-Jacobi equations)
Let $T_1 > 0$ and $\mathcal{X} : (0, T_1) \times \mathbb{R} \to \mathbb{R}$ be a locally bounded function. We denote $\Omega = (0, T_1) \times \mathbb{R}$.

- *A function $\mathcal{X}$ is a subsolution (resp. a supersolution) of (2.1) if $\mathcal{X}$ is upper semi-continuous (resp. lower semi-continuous) and if for all test function $\psi \in C^1(\Omega)$ such that $\mathcal{X} - \psi$ attains a local maximum (resp. a local minimum) at (t^*, y^*), we have*

$$\psi_t \le V(\psi_y) \quad (\textit{resp.} \quad \psi_t \ge V(\psi_y)) \quad \textit{at} \quad (t^*, y^*).$$

- *A function $\mathcal{X} \in C(\Omega)$ is a viscosity solution of (2.1) if it is both a viscosity subsolution and a viscosity supersolution of (2.1).*

Lemma 2.2. (A particular viscosity solution)
The function

$$\mathcal{X}(t, y) = \min(ay + tV(a), by + tV(b)) \tag{2.2}$$

is a viscosity solution of (2.1).

Proof of Lemma 2.2
We can see that for $a < b$, the function $\mathcal{X}$ is C^1 except on the line

$$y = -\frac{t}{T} \quad \text{with} \quad \frac{1}{T} = \frac{V(a) - V(b)}{a - b}$$

with

$$\mathcal{X}(t, y) = \mathcal{X}(0, y + \frac{t}{T}) + ct \quad \text{with} \quad c = \frac{aV(b) - bV(a)}{a - b}\ . \tag{2.3}$$

We can check that $\mathcal{X}$ is a viscosity solution of (2.1) if and only if the following condition is satisfied

$$c + \frac{p}{T} \leq V(p) \quad \text{for any} \quad p \in [a, b]\ . \tag{2.4}$$

Indeed, $\mathcal{X}$ is a viscosity solution of (2.1) if and only if $\mathcal{X}$ is a subsolution on the line $y = -\frac{t}{T}$ for any test function $ct + \phi(y + \frac{t}{T})$ with $a \leq \phi'(0) \leq b$.

□

Our goal is to construct a discrete analogue of $\mathcal{X}$ for equation (1.1), which corresponds to a shock in traffic flow. Theorem 1.2 shows that even if $\mathcal{X}$ given in (2.2) is a solution of the macroscopic equation (2.1), there is no corresponding solution of (1.15) at the microscopic level. Indeed, when equality in assumption (A_3') only arises at the three points $p = a, b, p_0$, then there are two solutions h_1, h_2 at the microscopic level, such that the interdistance $h_1(y+1) - h_1(y)$ provides a transition between b and p_0 and the interdistance $h_2(y+1) - h_2(y)$ is a transition between p_0 and a. Theorem 1.2 exhibits an important difference between the macroscopic model (2.1) and the microscopic model (1.15).

3. Viscosity solutions. We define here viscosity solutions for advanced differential equation which includes the microscopic model (1.15). Let $F : \mathbb{R}^2 \to \mathbb{R}$ be globally Lipschitz function such that $F(x_1, x_2)$ is increasing in x_1.

Definition 3.1. (Viscosity solution for advanced differential equations)

– *A function $h \in L^\infty_{loc}(\mathbb{R})$ is a subsolution (resp. supersolution) of*

$$h'(y) = F(h(y + 1), h(y)) \tag{3.1}$$

if h is upper semi-continuous (resp. lower semi-continuous) and if for all test function $\psi \in C^1(\mathbb{R})$ such that $h - \psi$ attains a local maximum (resp. a local minimum) at y^*, we have

$$\psi'(y^*) \leq F(\psi(y^* + 1), \psi(y^*)) \quad (\textit{resp.} \quad \psi'(y^*) \geq F(\psi(y^* + 1), \psi(y^*))\ ).$$

– *A function $h \in C(\mathbb{R})$ is a viscosity solution of (3.1) if it is both a subsolution and a supersolution of (3.1).*

We will indeed work with viscosity solutions, and we refer the reader to [4, 8] for an introduction to this notion using the definition of test functions.
We now give a regularity property of viscosity solution.

Proposition 3.2. (Regularity of viscosity solutions for (1.15))
Assume (A_1). Let h be a viscosity solution of (1.15). Then $h \in C^2(\mathbb{R})$.

Proof of Proposition 3.2
Step 1: h is locally Lipschitz.
Recall that by definition of viscosity solution $h \in L^\infty_{\rm loc}(\mathbb{R})$. Using equation (1.15) and (A_3), we see that for any $R > 0$, there exists a constant $L_R > 0$ such that

$$|V(h(y+1)-h(y))| \le L_R \quad \text{for} \quad |y| < R\,.$$

Therefore, we deduce that h satisfies in the sense of viscosity solutions for Hamilton-Jacobi equation (see Definition 3.1):

$$\begin{cases} h_y \le L_R & \text{on} \quad (-R,R) \\ h_y \ge L_R & \text{on} \quad (-R,R) \end{cases}$$

i.e. $\bar{g}(y) = h(y) - L_R$ and $\underline{g}(y) = h(y) + L_R$ solve

$$\begin{cases} \bar{g}'_y \le 0 & \text{on} \quad (-R,R) \\ \underline{g}'_y \ge 0 & \text{on} \quad (-R,R) \end{cases}$$

From the comparison principle for Hamilton-Jacobi equation (see [4, 8]) (here $g'_y = 0$ is a very particular Hamilton-Jacobi equation) we deduce that

$$\begin{cases} \bar{g}(y+z) \ge \bar{g}(y) & \text{for} \quad z \ge 0 \quad \text{and} \quad y, y+z \in (-R,R) \\ \underline{g}(y+z) \le \underline{g}(y) & \text{for} \quad z \ge 0 \quad \text{and} \quad y, y+z \in (-R,R) \end{cases}$$

This implies that

$$|h(y+z) - h(y)| \le L_R z \quad \text{for} \quad z \ge 0 \quad \text{and} \quad y, y+z \in (-R,R) \tag{3.2}$$

Step 2: higher regularity of h.
Let $g(y) = h(y) - \int_0^y (V(h(z+1)-h(z)))\,dz$. Because $g - h$ is $C^1(\mathbb{R})$, we deduce that in the viscosity sense we have

$$g' = 0 \quad \text{on} \quad \mathbb{R}\,.$$

The comparison principle for Hamilton-Jacobi (see [4, 8]) implies that $g(y) = \text{constant} = g(0)$ for $y \in \mathbb{R}$. This implies that $h \in C^1(\mathbb{R})$ and (1.15) holds for $C^1(\mathbb{R})$ functions. Moreover $V \in C^1(\mathbb{R})$ implies that $h \in C^2(\mathbb{R})$.

□

4. Preliminaries: Cauchy problem and asymptotics.

In a first subsection we show how to propagate the solution to the left (Lemma 4.1), and in a second subsection we provide exponential asymptotics of the solution (Proposition 4.2), which can be seen as the main result of this section.

4.1. Propagation of the solution to the left.

The following result shows that we can solve the Cauchy problem (1.15) towards the left.

Lemma 4.1. (Existence and uniqueness of the construction on the left) *Assume (A_1). Let us consider an "initial data" $h_0 \in C([0,1])$. Then there exists a unique function h on $(-\infty,1]$, with $h \in C^1(-\infty,0) \cap C((-\infty,1])$ solution of*

$$\begin{cases} h'(y) = V(h(y+1)-h(y)) & \text{for} \quad -\infty < y < 0, \\ h(y) = h_0(y) & \text{for} \quad 0 \le y \le 1. \end{cases}$$

Proof of Lemma 4.1

The proof seems very classical, but we give it for the convenience of the reader. In order to come back to a more familiar situation solving the equations in the direction of positive coordinates (as in a standard Cauchy problem), we set $\tilde{h}(y) := h(-y)$, $\tilde{h}_0(y) = h_0(-y)$. Then $\tilde{h}$ satisfies

$$\left\{\begin{array}{lll} \tilde{h}'(y) = -V(\tilde{h}(y-1) - \tilde{h}(y)) & \text{for} & 0 \le y < +\infty, \\ \tilde{h}(y) = \tilde{h}_0(y) & \text{for} & -1 \le y \le 0. \end{array}\right. \tag{4.1}$$

For $\delta > 0$, we set

$$\mathcal{A}(\tilde{h})(y) := \left\{\begin{array}{lll} \tilde{h}_0(y) - \displaystyle\int_0^y V(\tilde{h}(z-1) - \tilde{h}(z))\, dz & \text{if} & 0 < y \le \delta, \\ \tilde{h}_0(y) & \text{if} & -1 \le y \le 0. \end{array}\right.$$

The operator $\mathcal{A}$ is more generally defined on the following set

$$X = \left\{ \tilde{h} \in C([-1,\delta]) \quad \text{with} \quad \tilde{h} = \tilde{h}_0 \quad \text{on} \quad [-1,0] \right\}$$

which is a closed subset of the Banach space $C([-1,\delta])$. We easily have

$$\left| \mathcal{A}(\tilde{h}) - \mathcal{A}(\tilde{g}) \right|_{L^\infty(-1,\delta)} \le 2\delta |V'|_{L^\infty(\mathbb{R})}\, |\tilde{h} - \tilde{g}|_{L^\infty(-1,\delta)}$$

which shows that $\mathcal{A}$ is a contraction on X for δ small enough. This shows the existence and uniqueness of a fixed point $\tilde{h} \in X$ of $\mathcal{A}$ and provides a solution on the interval $[0,\delta)$ of the delayed equation (4.1) . By a classical iteration argument where we replace successively the interval $[0,\delta)$ (for instance by intervals $[k\frac{\delta}{2}, (k+2)\frac{\delta}{2})$ with $k \in \mathbb{N}$), we then extend uniquely the solution on $[-1,+\infty)$. □

4.2. Asymptotics. We have

Proposition 4.2. (Asymptotics close to $-\infty$)
Assume (A_1), $V'(b) \neq 1$ and consider a solution $g \in C^1(-\infty,0) \cap C((-\infty,1])$ of

$$\left\{\begin{array}{lll} g'(y) = V(g(y+1) - g(y)) & \textit{for} & -\infty < y < 0, \\ g & \textit{given on} & [0,1], \end{array}\right. \tag{4.2}$$

with

$$g'(y) \longrightarrow b \quad \textit{as} \quad y \longrightarrow -\infty\,. \tag{4.3}$$

Then there exist $K, \gamma > 0$ and $c_1 \in \mathbb{R}$ such that

$$|g(y) - by - c_1| \le K e^{\gamma y} \quad \textit{for} \quad y \le 0\,. \tag{4.4}$$

This result is different, but related to Lemma 1 in [7]. We provide here a self-contained and elementar proof of Proposition 4.2, which has an interest in itself and can be adapted to other frameworks. Our proof is in the same spirit as the proof of Proposition 2.1 in [13].

In order to prove Proposition 4.2, we will use Lemma 4.3 below. For the convenience of the authors and the reader, we prefer to work with positive coordinates. To this end, we make a change of function, setting

$$u(y) = g(-y) + by. \tag{4.5}$$

From (4.3) and (4.2), we deduce that

$$b = V(b).$$

As a consequence, a simple computation shows that we have

$$\begin{cases} u'(y) = V(b) - V\left(b + u(y-1) - u(y)\right) & \text{for} \quad 0 < y < +\infty, \\ u & \text{given on} \quad [-1, 0], \end{cases} \tag{4.6}$$

and

$$u'(y) \longrightarrow 0 \quad \text{as} \quad y \to +\infty\,. \tag{4.7}$$

We define

$$N(u,y) := \inf_{\alpha\in\mathbb{R}} \left(\int_{y-1}^{y+1} (u(z)-\alpha)^2\, dz \right)^{1/2} \tag{4.8}$$

and

$$M(u,y) := \sup_{z\geq y} N(u,z). \tag{4.9}$$

Lemma 4.3. (Basic estimate)
Assume (A_1) and $V'(b) \neq 1$. Then there exists $M_0 > 0$, $L > 1$, $\mu \in (0,1)$, such that if $u \in C^1(0,+\infty) \cap C([-1,+\infty))$ solves (4.6), then we have

$$M(u,0) \leq M_0 \quad \Longrightarrow \quad M(u, y+L) \leq \mu M(u,y) \quad \textit{for all} \quad y \geq 0\,.$$

The proof of Lemma 4.3 is postponed in this subsection. We now prove Proposition 4.2.

Proof of Proposition 4.2
Step 1: Normalization
We use definition (4.5). From the definitions of M and N (see (4.8) and (4.9)), we have as $y \to +\infty$, $N(u,y) \longrightarrow 0$ and $M(u,y) \longrightarrow 0$, because of (4.7). In particular there exists $y_1 > 0$ such that

$$M(u, y_1) \leq M_0\,.$$

Step 2: Decay estimate on M
Using Lemma 4.3, for $L > 1$, we have with $M(y) := M(u,y)$ and $l \in \mathbb{N}$

$$M(y_1 + lL) \leq \mu^l M(y_1).$$

If $lL \leq y - y_1 < (l+1)L$ then

$$\begin{aligned} M(y) &\leq M(y_1 + lL) \\ &\leq \mu^l M(y_1) \\ &\leq M(y_1) e^{(\ln\mu)(\frac{y-y_1}{L} - 1)} \\ &\leq (\mu^{\frac{1}{L}})^y M(y_1) e^{(\ln\mu)(-\frac{y_1}{L} - 1)} \\ &\leq e^{-\gamma y} K_1 \end{aligned}$$

where $e^{-\gamma} := \mu^{\frac{1}{L}}$, $\gamma > 0$ and $K_1 := M(y_1)e^{\ln\mu(-\frac{y_1}{L}-1)}$. Up to increase K_1, we can assume that

$$M(y) \le e^{-\gamma y} K_1 \quad \text{for all} \quad y \ge 0. \tag{4.10}$$

Step 3: Control of $u(y+1) - u(y)$
On the one hand, we first deduce from (4.6) that

$$|u'(y)| \le |V'|_{L^\infty(\mathbb{R})}\, |u(y-1) - u(y)|. \tag{4.11}$$

On the other hand, we have by Cauchy-Schwarz inequality

$$|u(y+1) - u(y)| \le \left| \int_y^{y+1} u'(s)\, ds \right| \le \sqrt{\int_y^{y+1} |u'(s)|^2\, ds} \tag{4.12}$$

and using (4.11), we get

$$\begin{aligned}
&\sqrt{\int_y^{y+1} |u'(s)|^2\, ds} \\
&\le |V'|_{L^\infty(\mathbb{R})} \sqrt{\int_y^{y+1} |(u(s-1)-\alpha) - (u(s)-\alpha)|^2\, ds} \\
&\le |V'|_{L^\infty(\mathbb{R})} \left(\sqrt{\int_y^{y+1} |u(s-1)-\alpha|^2\, ds} + \sqrt{\int_y^{y+1} |u(s)-\alpha|^2\, ds} \right) \\
&\le 2|V'|_{L^\infty(\mathbb{R})} N(u,y)
\end{aligned}$$

for a value α which reaches the infimum in the definition of $N(u,y)$, which implies

$$\sqrt{\int_y^{y+1} |u'(s)|^2\, ds} \le 2|V'|_{L^\infty(\mathbb{R})} M(y). \tag{4.13}$$

From (4.12) and (4.13), we get using (4.10)

$$|u(y+1) - u(y)| \le e^{-\gamma y} K_2 \quad \text{for all} \quad y \ge 0, \quad \text{with} \quad K_2 := 2|V'|_{L^\infty(\mathbb{R})} K_1. \tag{4.14}$$

We conclude this step with the following inequality of independent interest (as a consequence of (4.11), (4.12) and (4.13))

$$|u'(y+1)| \le 2(|V'|_{L^\infty(\mathbb{R})})^2 M(y). \tag{4.15}$$

Step 4: Conclusion
If $u(+\infty)$ exists, then we have

$$\begin{aligned}
|u(+\infty) - u(y)| &\le \sum_{k\ge 0} |u(y+k+1) - u(y+k)| \\
&\le \sum_{k\ge 0} K_2 e^{-\gamma(y+k)} \\
&\le \frac{K_2}{1-e^{-\gamma}} e^{-\gamma y}.
\end{aligned}$$

Indeed from the absolute convergence of the serie in the previous computation, we can also deduce that $u(+\infty)$ does exist and also this shows the exponential convergence of $u(y)$ to its limit in $y = +\infty$. This implies (4.4) through (4.5). □

We now give the proof of Lemma 4.3.

Proof of Lemma 4.3

Step 1: Construction of sequences

By contradiction, we suppose that there exist a sequence $(u_n)_n$ and some sequences

$$\left\{\begin{array}{l} M_n \longrightarrow 0 \\ L_n \longrightarrow +\infty \\ (0,1) \ni \mu_n \longrightarrow 1 \\ y^n \geq 0 \end{array}\right.$$

such that

$$M(u_n, 0) \leq M_n \quad \text{and} \quad M(u_n, y_n + L_n) > \mu_n M(u_n, y_n). \tag{4.16}$$

We set

$$\varepsilon_n := M(u_n, y_n + L_n) = \sup_{z \geq y_n + L_n} N(u_n, z). \tag{4.17}$$

Then there exists $z_n \geq y_n + L_n$ such that

$$\frac{\varepsilon_n}{1 + \frac{1}{n}} \leq N(u_n, z_n) = \sqrt{\int_{z_n - 1}^{z_n + 1} |u_n(z) - \alpha_n|^2 \, dz} \leq \varepsilon_n \tag{4.18}$$

for some α_n. Moreover from (4.16), we get that $\varepsilon_n \leq M_n \to 0$. Let us consider a rescaling of the functions u_n, that we call

$$v_n(y) = \frac{u_n(y + z_n) - \alpha_n}{\varepsilon_n}.$$

From the definition of ε_n, we deduce that

$$\left\{\begin{array}{l} 1 \geq N(v_n, 0) \geq \dfrac{1}{1 + \frac{1}{n}} \\ \\ M(v_n, -L_n) < \dfrac{1}{\mu_n} \end{array}\right. \tag{4.19}$$

where in the first line we have used (4.18), and in the second line we have used (4.16) and the fact that $y_n \leq z_n - L_n$.

Step 2: ODE satisfied by v_n and a priori bounds

We have

$$v_n'(y) = \frac{1}{\varepsilon_n} \left\{ V(b) - V\left(b + \varepsilon_n (v_n(y-1) - v_n(y))\right) \right\} \tag{4.20}$$

which can be written as

$$v_n'(y) = -\frac{1}{\varepsilon_n} \left\{ \int_0^1 ds \; V'(b + s\varepsilon_n (v_n(y-1) - v_n(y))) \right\} \varepsilon_n (v_n(y-1) - v_n(y)).$$

Then

$$|v_n'(y)| \leq |V'|_{L^\infty(\mathbb{R})}\ |v_n(y-1) - v_n(y)|. \tag{4.21}$$

A computation similar to Step 3 of the proof of Proposition 4.2, implies that we get an inequality analogous to (4.15), i.e. for $y \geq -L_n$

$$|v_n'(y+1)| \leq 2(|V'|_{L^\infty(\mathbb{R})})^2 M(v_n, y).$$

Using (4.19), we deduce for $y \geq -L_n$

$$|v_n'(y+1)| \leq \frac{2}{\mu_n}(|V'|_{L^\infty(\mathbb{R})})^2 \tag{4.22}$$

and we recall that

$$1 \geq \sqrt{\int_{-1}^{1} |v_n(y)|^2\ dy} = N(v_n, 0) \geq \frac{1}{1+\frac{1}{n}}\ . \tag{4.23}$$

Step 3: Passage to the limit
From (4.22) and (4.23), we deduce that v_n is bounded in $C^1_{\text{loc}}(\mathbb{R})$ uniformly as $n \to +\infty$. So $v_n \longrightarrow v$ locally uniformly, and passing to the limit in (4.20), we get

$$v'(y) = -\beta\,(v(y-1) - v(y)) \quad \text{in} \quad \mathcal{D}'(\mathbb{R}), \quad \text{with} \quad \beta := V'(b) \neq 1. \tag{4.24}$$

And from (4.19) and (4.22) we obtain respectively as $n \to \infty$ and for almost every $y \in \mathbb{R}$

$$1 \leq N(v, 0) \leq M(v, -\infty) \leq 1 \quad \text{and} \quad |v'(y)| \leq 2(|V'|_{L^\infty(\mathbb{R})})^2 \tag{4.25}$$

because $\mu_n \to 1$. Note that $M(v, -\infty)$ does exist because of the monotonicity of $M(v, \cdot)$.
Step 4: Getting a contradiction
Because of (4.25), we have $v \in \mathcal{S}'(\mathbb{R})$ and we can apply Fourier transform to equation (4.24), and get

$$i\xi \hat{v}(\xi) = -\beta(e^{-i\xi} - 1)\hat{v}(\xi)$$

i.e.

$$A(\xi)\hat{v}(\xi) = 0 \quad \text{with} \quad A(\xi) := \beta(\cos\xi - 1) + i(\xi - \beta \sin\xi).$$

It is easy to see that $A(\xi) = 0$ if and only if $\xi = 0$, which implies $\operatorname{supp} \hat{v} = \{0\}$. Therefore $\hat{v} = \sum_{\text{finite}} a_\gamma \partial^\gamma \delta_0$ and coming back to the real space, we deduce that v is a polynomial. Because v' is bounded, we deduce that $v(y) = c_1 y + c_2$. Plugging this expression in (4.24), and using the fact that $\beta \neq 1$, we deduce that $c_1 = 0$. Therefore $v(y) = c_2$ =constant. But $N(v, 0) = 1$. Contradiction. □

5. Qualitative properties of solutions. In a first subsection, we prove a very nice monotonicity property of the interdistance function $h(y+1) - h(y)$ (see Proposition 5.1), that can be seen as the main result of this section. As a consequence, we prove the classification result (Theorem 1.3) in a second subsection.

5.1. Monotonicity properties of the interdistance function G. We first notice that given any solution h of (1.15), the function $G(y) := h(y+1) - h(y)$ solves the following equation:

$$G'(y) = V(G(y+1)) - V(G(y)) \quad \text{for} \quad y \in \mathbb{R}. \tag{5.1}$$

We now present a monotonicity result for the solutions of this equation.

Proposition 5.1. (Monotonicity of G)
Assume (A_1) and (A_2). Let $G \in C^1(\mathbb{R})$ be a bounded solution of (5.1). Then we have the following three cases:
either

$$G' > 0 \quad on \quad \mathbb{R},$$

or

$$G' < 0 \quad on \quad \mathbb{R},$$

or

$$G' = 0 \quad on \quad \mathbb{R}.$$

Proof of Proposition 5.1
Step 1: if G has a global maximum, then G is constant
Assume that G has a global maximum at y_0. Comparing G to the constant function equal to $G(y_0)$, and using the strict monotonicity of V (see (A_2)), we deduce from Lemma 9.1 that

$$G(y) = G(y_0) \quad \text{for all} \quad y \in \mathbb{R}.$$

Step 2: case where G has a local maximum at y_0
Assume by contradiction that G is not non decreasing on $[y_0, +\infty)$.
Then either we have

$$\left\{ \begin{array}{l} G'(y_0) = 0, \\ \\ \text{and for every} \quad \varepsilon > 0, \quad \text{there exists} \quad y_\varepsilon \in (y_0, y_0+\varepsilon) \quad \text{such that} \quad G(y_\varepsilon) < G(y_0), \end{array} \right. \tag{5.2}$$

or G is constant on some interval $[y_0, y_0 + \varepsilon_0]$ for some $\varepsilon_0 > 0$. In this last case, let us define

$$\bar{y}_0 = \sup \{x_0 \geq y_0, \quad G \quad \text{is non decreasing on} \quad [y_0, x_0]\}.$$

Then we have $y_0 < y_0 + \varepsilon_0 \leq \bar{y}_0 < +\infty$. This implies that $\bar{y}_0$ satisfies property (5.2), and up to replace y_0 by $\bar{y}_0$, we can now assume (5.2).

Step 2.1: definition of a sequence
By (5.2), we have

$$0 = G'(y_0) = V(G(y_0+1)) - V(G(y_0)).$$

The strict monotonicity of V (see (A_2)) implies that

$$G(y_0 + 1) = G(y_0).$$

Up to redefine z_0, let us call z_0 a point of minimum of G on $[y_0, y_0+1]$ which satisfies

$$G(z_0) < G(y_0) \quad \text{and} \quad G'(z_0) = 0.$$

Therefore $G(z_0+1) = G(z_0)$ and $y_0 + 1 \in (z_0, z_0+1)$. We deduce that a maximum y_1 of G on $[z_0, z_0+1]$ satisfies

$$G(y_1) \geq G(y_0+1) = G(y_0) \quad \text{and} \quad G'(y_1) = 0.$$

Similarly, we can consider a minimum z_1 of G on $[y_1, y_1+1] \ni z_0 + 1$ which satisfies

$$G(z_1) \leq G(z_0+1) = G(z_0) \quad \text{and} \quad G'(z_1) = 0.$$

More generally, we define for $n \geq 1$

$$y_{n+1} \in \text{Argmax}_{[z_n, 1+z_n]} G \quad \text{and then} \quad z_{n+1} \in \text{Argmin}_{[y_{n+1}, 1+y_{n+1}]} G$$

which satisfy

$$\begin{cases} G(y_{n+1}) \geq G(y_n) \geq G(y_0) > G(z_0) \geq G(z_n) \geq G(z_{n+1}), \\ \\ y_0 < z_0 < y_1 < z_n < y_{n+1} < z_{n+1}. \end{cases}$$

Notice that G is Lipschitz (let us say of constant L), because G is bounded and solves (5.1). Therefore

$$1 \geq y_{n+1} - z_n \geq d \quad \text{and} \quad 1 \geq z_{n+1} - y_{n+1} \geq d \quad \text{with} \quad d := \frac{G(y_0) - G(z_0)}{L} > 0.$$

This shows that $G(y)$ oscillates as $y \to +\infty$. Moreover the sequence $G(y_n)$ is non decreasing and bounded, and then converges.

Step 2.2: further properties (5.3) and (5.4)

We now need a further property. Let us call

$$y'_n \in \text{Argmax}_{[y_n, 1+y_n)} G \quad \text{and} \quad z'_n \in \text{Argmax}_{[z_n, 1+z_n)} G.$$

Then $G(y'_n) = G(1 + y'_n)$ and then

$$\begin{cases} \text{if} \quad y'_n \geq z_n, \qquad \text{then} \quad G(y'_n) \leq G(y_{n+1}), \\ \\ \text{if} \quad y'_n < z_n, \qquad \text{then} \quad G(y'_n) = G(1+y'_n) \leq G(y_{n+1}), \quad \text{because} \quad 1 + y'_n < 1 + z_n. \end{cases}$$

Therefore

$$\sup_{[y_n, 1+y_n]} G \leq G(y_{n+1}). \tag{5.3}$$

Similarly, we show that

$$\inf_{[z_n, 1+z_n]} G \geq G(z_{n+1}). \tag{5.4}$$

Step 2.3: contradiction, passing to the limit

This implies that (up to pass to the limit on a subsequence)

$$G_n(y) := G(y + y_n) \to G_\infty(y) \quad \text{and} \quad z_n - y_n \to d_\infty \in [d, 1]$$

where G_∞ solves (5.1) and we deduce from (5.3) that

$$G_\infty(y) \le G_\infty(0) \tag{5.5}$$

and

$$G_\infty(d_\infty) \le G(z_0) < G(y_0) \le G_\infty(0). \tag{5.6}$$

Then Step 1 applied to (5.5) implies that G_∞ is constant which is in contradiction with (5.6). This implies that G has to be non decreasing on $[y_0, +\infty)$.

Step 3: case where G has a local minimum at z_0

As in Step 2, we conclude to a contradiction.

Step 4: monotonicity of G

Steps 2 and 3 imply that G is monotone.

Step 5: $\pm G' > 0$ or $G' = 0$

Assume by contradiction that there exists $y_0 \in \mathbb{R}$ such that

$$G'(y_0) = 0\,.$$

As above, we deduce that

$$G(y_0 + 1) = G(y_0)\,.$$

Because G is monotone, this implies that

$$G(y) = G(y_0) \quad \text{on} \quad [y_0, y_0 + 1]$$

and therefore

$$G'(y_0 + 1) = 0\,.$$

Iterating the argument, we deduce that

$$G(y) = G(y_0) \quad \text{for} \quad y \ge y_0\,.$$

Applying a Cauchy-Lipschitz type argument (like in the proof of Lemma 4.1), we deduce that

$$G(y) = G(y_0) \quad \text{for all} \quad y \in \mathbb{R}.$$

□

5.2. Qualitative properties of h: proof of Theorem 1.3. We now prove Theorem 1.3.

Proof of Theorem 1.3

Step 1: sign of G' and h''

We define $G(y) := h(y+1) - h(y)$. Recall that $h' \in C^1(\mathbb{R})$ and solves (1.15). Therefore $h \in C^2(\mathbb{R})$. Using (1.9), we get from Proposition 5.1 that $G' > 0$, $G' < 0$ or $G' = 0$. Deriving (1.15), we get

$$h''(y) = V'(G(y))G'(y)$$

and then $h'' > 0$, $h'' < 0$ or $h'' = 0$.
Step 2: coming back to a Hamilton-Jacobi equation
Using again (1.9), we deduce the existence of $\tilde{a}, \tilde{b} \in \mathbb{R}$ such that

$$h'(-\infty) = \tilde{b} \quad \text{and} \quad h'(+\infty) = \tilde{a}.$$

Defining

$$u(t,y) := h(y + \frac{t}{T}) + ct$$

we see that u is a C^1 solution (and then a viscosity solution) of

$$u_t = V(u_y). \tag{5.7}$$

We now define

$$u^\varepsilon(t,y) = \varepsilon u\left(\frac{t}{\varepsilon}, \frac{y}{\varepsilon}\right).$$

We have as $\varepsilon \to 0$

$$u^\varepsilon(t,y) \to u^0(t,y) = \tilde{h}(y + \frac{t}{T}) + ct \quad \text{with} \quad \tilde{h}(y) = \tilde{b}y\mathbb{1}_{\{y<0\}} + \tilde{a}y\mathbb{1}_{\{y\geq 0\}}.$$

By stability of viscosity solutions (see [4, 8]), we deduce that u^0 is still a viscosity solution of (5.7).
Step 3: necessary conditions
Case 1: $\tilde{a} < \tilde{b}$
Then testing the viscosity solution u^0 from above with any test function of the form

$$\varphi(y + \frac{t}{T}) + ct \quad \text{where} \quad \tilde{a} \leq \varphi'(0) \leq \tilde{b}$$

we deduce that

$$c + \frac{p}{T} \leq V(p) \quad \text{for all} \quad p \in [\tilde{a}, \tilde{b}]$$

with equality for $p = \tilde{a}, \tilde{b}$.
Because we have $h'' < 0$, we deduce that

$$\tilde{a} < p := h(y+1) - h(y) < h'(y) < \tilde{b}.$$

Therefore, we deduce from (1.15) that

$$c + \frac{p}{T} < V(p) \quad \text{for all} \quad p \in (\tilde{a}, \tilde{b}).$$

Case 2: $\tilde{a} > \tilde{b}$
Similarly, we get (1.12). □

6. Uniqueness. The main result of this section is the uniqueness result, namely Proposition 6.2. We start with the following result:

Lemma 6.1. (Asymptotics of concave functions)
Let h be a concave function satisfying $|h-\bar{h}| \le C$ for $\bar{h}(y) = \min(ay, by)$ with $a < b$. Then there exist constants α, β such that

$$\lim_{|y|\to+\infty} (\tilde{h}(y) - \bar{h}(y)) = 0 \quad \textit{with} \quad \tilde{h}(y) = \alpha + h(y+\beta) \tag{6.1}$$

and

$$h'(+\infty) = a \le h(y+1) - h(y) \le b = h'(-\infty). \tag{6.2}$$

where $h'(\pm\infty)$ are the limits of $h'(y)$ at the points y where h is derivable.

Proof of Lemma 6.1
Step 1: limits at infinity
Up to a change the variables, we can reduce the problem to a function ϕ (associated to h on $\mathbb{R}_+$ or $\mathbb{R}_-$), such that

$$\left\{ \begin{array}{l} \phi'' \ge 0 \\ 0 \le \phi \le C \end{array} \right| \quad \text{on} \quad \mathbb{R}_+ .$$

Under those conditions, we deduce that $\phi' \le 0$ on $\mathbb{R}_+$ (otherwise we would get a contradiction with the boundedness of ϕ, using the convexity of ϕ). Therefore $\phi(+\infty)$ exists. Using this argument, we deduce that

$$(h-\bar{h})(y) \longrightarrow c^{\pm}, \text{ as } y \to \pm\infty\,. \tag{6.3}$$

Step 2: normalization
Assuming (6.3), we set

$$\tilde{h}(y) = \alpha + h(y+\beta) \quad \text{“}= \text{''} \left\{ \begin{array}{lll} \alpha + c^- + b(y+\beta) = by & \text{as} & y \to -\infty \\ \alpha + c^+ + a(y+\beta) = ay & \text{as} & y \to +\infty \end{array} \right. \text{''}$$

which implies (6.1) for a good choice of the constants α, β.
Step 3: proof of (6.2)
From (6.1) and the concavity of h, we deduce easily that

$$h'(+\infty) = a \le h'(y) \le b = h'(-\infty)$$

which implies (6.2). □

We have the following result.

Proposition 6.2. (Uniqueness and concavity)
Assume (A_1) and (A_2). Let $a < b$. If $h \in C^1(\mathbb{R})$ is a solution of (1.15) satisfying

$$\left|h - \bar{h}\right| \le C \quad \textit{with} \quad \bar{h}(y) = \min(ay, by) \tag{6.4}$$

then h is unique up to translation and addition of constants and satisfies on $\mathbb{R}$

$$h'' < 0 \quad \textit{and} \quad h'(+\infty) = a \le h(y+1) - h(y) \le b = h'(-\infty).$$

Proof of Proposition 6.2
Step 1: concavity of h
From Theorem 1.3 and (6.4), we deduce that $h'' < 0$.
Step 2: uniqueness
Let h^1 and h^2 be two solutions of (1.15) satisfying (6.4). Apply Lemma 6.1 and up to replace h^i by $\tilde{h}^i$, we can assume that $|\tilde{h}^i - \bar{h}| \longrightarrow 0$ as $|y| \to +\infty$ for $i = 1, 2$, which implies

$$|\tilde{h}^1 - \tilde{h}^2| \longrightarrow 0 \quad \text{as} \quad |y| \to +\infty. \tag{6.5}$$

Assume by contradiction that we do not have $\tilde{h}^2 \leq \tilde{h}^1$.
Let y_0 be such that

$$M = \sup_y (\tilde{h}^2 - \tilde{h}^1)(y) = (\tilde{h}^2 - \tilde{h}^1)(y_0) > 0\,. \tag{6.6}$$

From assumptions (A_1) and (A_2), using Lemma 9.1 applied to $\tilde{h}^2 - M \leq \tilde{h}^1$ with equality at y_0, we conclude that

$$\tilde{h}^2 - M = \tilde{h}^1$$

which is a contradiction with (6.5) and (6.6). Therefore we have

$$\tilde{h}^2 \leq \tilde{h}^1\ .$$

Similarly we show that $\tilde{h}^1 \leq \tilde{h}^2$, which implies $\tilde{h}^1 = \tilde{h}^2$. □

7. Existence of a solution. The goal of this section is to show the existence of a solution of (1.15) by Perron's method. In a first subsection, we propose a natural supersolution and a general construction of subsolutions. The second subsection is devoted to prove further properties of the subsolution (in particular of its extension towards $-\infty$) which will be crucial in the third subsection to show that we can set this subsolution below the supersolution. The solution is then constructed in the third subsection where we also give the proof of Theorem 1.1.

7.1. Sub and supersolutions. Our goal is to construct a solution of (1.15) in between a sub and a supersolution, using Perron's method. Indeed, the supersolution is easily given by the following result.

Lemma 7.1. (supersolution)
Assume (A_1) and (A_3) with $a < b$. Let

$$\bar{h}(y) = by\,\mathbb{1}_{\{y<0\}} + ay\,\mathbb{1}_{\{y\geq 0\}}\,. \tag{7.1}$$

Then $\bar{h} := \min(ay, by)$ is a viscosity supersolution of (1.15) in the sense of Definition 3.1.

Proof of Lemma 7.1
Because of (A_3), we know that the functions $y \mapsto ay$ and $y \mapsto by$ are two solutions of (1.15). Then the result follows from the fact that the minimum of two solutions is a viscosity supersolution (see [4]). □

The delicate part is the construction of a subsolution (such that it is below our supersolution). We indicate below a way to do it, and will need further developments on the subsolution in the next subsections in order to construct the solution.

For some $y_0 \in \mathbb{R}$, let us now consider a function g satisfying

$$\left\{\begin{array}{ll} g \in C^1([y_0,+\infty)), & \\ \dfrac{d}{dy}(g(y+1)-g(y)) \le 0, & \\ g'(y) < V(g(y+1)-g(y)), & \text{for} \quad y > y_0. \\ ay-\delta \le g(y) < ay \quad \text{for} \quad \delta > 0, & \end{array}\right. \tag{7.2}$$

Using Lemma 4.1, we extend by continuity g on $y \le y_0$ as the solution of

$$g'(y) = V(g(y+1)-g(y)) \quad \text{for} \quad y \le y_0. \tag{7.3}$$

Then we have the following lemma.

Lemma 7.2. (A subsolution)
Assume (A_1) and (A_2). If g satisfies (7.2) and is extended on $\{y \le y_0\}$ by (7.3), then g is a viscosity subsolution of (1.15) on $\mathbb{R}$ in the sense of Definition 3.1.

Proof of Lemma 7.2
This is clear that g is a subsolution on $\mathbb{R} \setminus \{y_0\}$. Let us assume that $\varphi \in C^1$ is a test function such that

$$\left\{\begin{array}{l} g \le \varphi \quad \text{on} \quad \mathbb{R}, \\ g(y_0) = \varphi(y_0). \end{array}\right. \tag{7.4}$$

Then we have

$$\varphi'(y_0) \le g'(y_0^-) = V(g(y_0+1)-g(y_0)) \le V(\varphi(y_0+1)-\varphi(y_0))$$

where the last inequality follows from (7.4) and the monotonicity of V (see (A_2)). This shows that g is a viscosity subsolution at $y = y_0$ and finally g is a viscosity subsolution on $\mathbb{R}$. □

7.2. Qualitative properties of our subsolution. We have the following result which is analogous to the monotonicity for solutions (see Proposition 5.1).

Lemma 7.3. (Monotonicity property for G associated to our subsolution)
Let us assume (A_1), (A_2) and (7.2). Let us define

$$G(y) := g(y+1) - g(y) \tag{7.5}$$

where g is the subsolution given by Lemma 7.2. Then G is nonincreasing on $\mathbb{R}$.

Proof of Lemma 7.3
Recall that by construction, g and G are continuous on $\mathbb{R}$. Let us define

$$y^* = \inf\{z_0 \in (-\infty, y_0] : \ G \text{ is non increasing on } (z_0,+\infty)\} \le y_0.$$

Assume by contradiction that $y^* > -\infty$.
Case 1: $\boldsymbol{G(y^*+1) < G(y^*)}$ or $\boldsymbol{y^* \in (y_0-1, y_0]}$
By (7.2), we have $G'(y) \le 0$ if $y > y_0$. In both cases $y^*+1 = y_0$ or $y^*+1 \neq y_0$, there

exists $\eta > 0$ small enough such that $y+1 \neq y_0$ if $y \in (y^* - \eta, y^*)$. Therefore for such y, we have

$$G'(y) = g'(y+1) - g'(y) \leq V(G(y+1)) - V(G(y)) =: F(y) \tag{7.6}$$

and (using the third line of (7.2))

$$\begin{cases} G'(y^*) < F(y^*) & \text{if} \quad y^* \in (y_0 - 1, y_0], \\ F(y^*) \leq 0 & \text{because} \quad G(y^*+1) \leq G(y^*). \end{cases} \tag{7.7}$$

Because V is increasing (assumption (A_2)), we deduce that

$$F(y^*) < 0 \quad \text{if} \quad G(y^*+1) < G(y^*). \tag{7.8}$$

From the continuity of F and using either (7.7) or (7.8), we deduce that we have in all cases (up to reduce $\eta > 0$)

$$G'(y) \leq F(y) < 0 \quad \text{for} \quad y \in (y^* - \eta, y^*).$$

Contradiction with the definition of y^*.
Case 2: $G(y^*+1) = G(y^*)$ and $y^* \leq y_0 - 1$
Step A: We show that $G(y) = G(y^*)$ for all $y \in [y^*, y_0+1]$
Because G is nonincreasing on $(y^*, +\infty)$ and $G(y^*+1) = G(y^*)$, we deduce that

$$G = G(y^*) \quad \text{on} \quad [y^*, y^*+1].$$

Therefore, we have for any $y_1 \in (y^*, y^*+1) \setminus (\{y_0 - 1\} \cup [y_0, +\infty))$

$$0 = G'(y_1) \leq V(G(y_1+1)) - V(G(y_1))\,.$$

This implies

$$V(G(y_1+1)) \geq V(G(y_1)) \quad \text{for} \quad y_1 \in (y^*, y^*+1) \setminus (\{y_0-1\} \cup [y_0, +\infty))\ .$$

Because V is increasing (assumption (A_2)), we deduce that

$$G(y_1+1) \geq G(y_1) \quad \text{for all} \quad y_1 \in [y^*, y^*+1] \setminus [y_0, +\infty)$$

where we have used the continuity of G to add (when it is useful) the point $y_0 - 1$ and the endpoints $\{y^*, y^*+1\}$. Since G is nonincreasing on $[y^*, +\infty)$, we get in particular that

$$G = G(y_1) = G(y^*) \quad \text{on} \quad [y_1, y_1+1].$$

If $y_1 < y_0$, we can repeat the argument with y_1 replaced by some $y_2 \in (y_1, y_1+1) \setminus (\{y_0 - 1\} \cup [y_0, +\infty))$, and so on, and get that

$$G(y) = G(y^*) =: C_1 \quad \text{for all} \quad y \in [y^*, y_0+1]. \tag{7.9}$$

Step B: Consequences
In particular, the equation on g and the fact that $y^* \leq y_0 - 1$ imply that

$$g(y) = g(y_0) + V(C_1)(y - y_0) \quad \text{for} \quad y \in [y_0 - 1, y_0]$$

and by uniqueness of the extention on $(-\infty, y_0 - 1]$ (see Lemma 4.1), we deduce that

$$g(y) = g(y_0) + V(C_1)(y - y_0) \quad \text{for} \quad y \le y_0$$

and then G is constant on $(-\infty, y_0 - 1]$, i.e.

$$G(y) = G(y_0 - 1) = G(y^*) \quad \text{on} \quad (-\infty, y_0 - 1)$$

where we have used again the fact that $y^* \le y_0 - 1$. Joint to (7.9), we deduce that G is constant on $(-\infty, y^*]$ and then G is globally nonincreasing on $\mathbb{R}$. Contradiction with the definition of y^*.

Conclusion

In cases 1 and 2, we get a contradiction. This implies that $y^* = -\infty$. □

Corollary 7.4. (Bound and limit of G)
Assume (A_1), (A_2), (A_3) and (7.2). For G defined in Lemma 7.3 by (7.5), we have

$$|G(y)| \le M_1 := \max(|\sup_{\mathbb{R}^+} V|, |V(G(y_0))|) \quad \textit{for} \quad y \le y_0 - 1 \tag{7.10}$$

and

$$\begin{cases} G(y) \longrightarrow b & \textit{as} \quad y \to -\infty, \\ G(y) \longrightarrow a & \textit{as} \quad y \to +\infty\,. \end{cases}$$

Proof of Corollary 7.4

Step 1: limit of G

We recall that $g'(y) = V(G(y))$ for $y < y_0$. Because G is nonincreasing and $V' \ge 0$, we deduce that

$$\sup_{\mathbb{R}^+} V \ge g'(y) = V(G(y)) \ge V(G(y_0)) \quad \text{for} \quad y < y_0\,.$$

Then we have

$$|G(y)| = |g(y+1) - g(y)| \le \text{Lip}(g) \le M_1 \quad \text{for} \quad y \le y_0 - 1$$

with M_1 defined in (7.10). But G is nonincreasing, which implies that the following limit exists

$$\lim_{y \to -\infty} G(y) = A \tag{7.11}$$

Step 2: $A \in \{a, b\}$

Using the equation satisfied by g for $y < y_0$, we get

$$G(y) = g(y+1) - g(y) = \int_0^1 g'(y+s)\, ds = \int_0^1 V(G(y+s))\, ds \longrightarrow V(A) \quad \text{as} \quad y \to -\infty.$$

where we have used (7.11) for the passage to the limit. This shows that $G(-\infty) = A = V(A)$, and then $A \in \{a, b\}$ by (A_3).

Step 3: $G(+\infty) = a$

From the last line of (7.2), we deduce that

$$a - \delta \le G(y) \le a + \delta \quad \text{for} \quad y > y_0$$

and as in Step 1, we deduce that G has a limit in $+\infty$.
We define for $k \in \mathbb{N} \setminus \{0\}$

$$I_k(y) = \frac{1}{k}\sum_{l=0}^{k-1} G(y+l).$$

From the definition of G, we get

$$I_k(y) = \frac{1}{k}(g(y+k) - g(y)).$$

Then (7.2) implies for $y \geq y_0$

$$\frac{1}{k}(ak-\delta) \leq I_k(y) \leq \frac{1}{k}(ak+\delta)$$

i.e.

$$|I_k(y) - a| \leq \frac{\delta}{k}.$$

On the other hand, we have

$$I_k(y) \longrightarrow G(+\infty) \quad \text{as} \quad y \to +\infty\,.$$

This shows that

$$|G(+\infty) - a| \leq \frac{\delta}{k}\,.$$

Taking the limit $k \to +\infty$, we get

$$G(+\infty) = a\,.$$

Step 4: $A = b$
Assume by contradiction that $A = a$. Then $G(-\infty) = a = G(+\infty)$ and because G is nonincreasing , we have

$$G(y) = a \quad \text{on} \quad \mathbb{R}\,.$$

This means that the function

$$k(y) = g(y) - ay$$

is 1-periodic. On the other hand, by the third line of (7.2), we get

$$a + k'(y) = g'(y) < V(g(y+1) - g(y)) = V(G(y)) = V(a) = a \quad \text{for} \quad y > y_0$$

i.e.

$$k'(y) < 0 \quad \text{for all} \quad y > y_0$$

which is impossible for a periodic function k. Contradiction, and then $A = b$. □

7.3. Construction of a solution. We start with:

Lemma 7.5. (Candidate for g)
Assume (A_1), (A_4). Then, there exists a constant $\gamma > 0$ such that the function

$$g(y) := ay - \delta e^{-\gamma y}$$

satisfies (7.2) for $y_0 = 0$ and $\delta > 0$ small enough.

Proof of Lemma 7.5
We have to check the following properties for $y \geq 0$.

$(H1)$ $\quad \dfrac{d}{dy}(g(y+1) - g(y)) < 0\,,$
$(H2)$ $\quad g'(y) < V(g(y+1) - g(y))\,,$
$(H3)$ $\quad ay - \delta \leq g(y) < ay \quad \text{with} \quad \delta > 0,$
where (H3) is obvious.

i) Checking (H1)
We have

$$G(y) = g(y+1) - g(y) = a + \delta e^{-\gamma y}(1 - e^{-\gamma}).$$

Therefore $G'(y) < 0$.

ii) Checking (H2)
On the one hand, we have

$$g'(y) = a + \gamma \delta e^{-\gamma y} = V(a) + \gamma \delta e^{-\gamma y}.$$

On the other hand, we have

$$\begin{aligned} V(G(y)) &= V(a + \delta e^{-\gamma y}(1 - e^{-\gamma})) \\ &= V(a) + V'(\xi)\delta e^{-\gamma y}(1 - e^{-\gamma}) \end{aligned}$$

for some $\xi \in [a, G(y)]$. To check (H2), it is enough to check

$$\gamma \delta e^{-\gamma y} < V'(\xi)\delta e^{-\gamma y}(1 - e^{-\gamma})\,. \tag{7.12}$$

Then it is enough to check

$$F(\gamma) := \frac{\gamma}{1 - e^{-\gamma}} < V'(a) \tag{7.13}$$

which will implies (7.12) for δ small enough (by continuity of V'). We have

$$F' > 0 \quad \text{and} \quad F(0) = 1.$$

Therefore to check (7.13), it is sufficient to have

$$F(0) = 1 < V'(a)$$

which is true by (A_4). $\square$

We have the following

Corollary 7.6. (Existence)
Assume (A). Let g be given by Lemmata 7.5 and 7.2 and let $\bar{h}$ given by (7.1). Then

there exist two constants $\alpha, \beta \in \mathbb{R}$ and a constant $C > 0$ and a viscosity solution h of (1.15) in the sense of Definition 3.1 such that

$$\underline{h}(y) \le h(y) \le \bar{h}(y) \quad \textit{for all} \quad y \in \mathbb{R}$$

with $\underline{h}(y) := \alpha + g(y + \beta)$ and

$$|h - \bar{h}| \le \bar{h}(y) - \underline{h}(y) \le Ce^{-\gamma|y|}. \tag{7.14}$$

Proof of Corollary 7.6
Step 1: preliminaries on g
By Lemma 7.3 and Corollary 7.4, we have for $G(y) := g(y + 1) - g(y)$

$$G \quad \text{is nonincreasing} \quad \text{and} \quad a = G(+\infty) \le G(y) \le G(-\infty) = b.$$

From the equation $g' = V(G)$ for $y \le y_0 = 0$, we deduce that g is concave on $(-\infty, 0]$. On the other hand g is concave on $[0, +\infty)$. Passing to the limit $y \to 0 = y_0$ with $y > 0$ in the last line of (7.2), and using equation (7.3) to evaluate $g'(0^-)$, we get

$$g'(0^+) \le g'(0^-).$$

This implies that g is globally concave. Moreover

$$g'(-\infty) = V(G(-\infty)) = V(b) = b.$$

Then we can apply Proposition 4.2 to conclude that g is asymptotic to the straight line $z = by + c_1$ as $y \to -\infty$, for some suitable constant $c_1 \in \mathbb{R}$. On the other hand the expression of g is explicit on $[0, +\infty)$. We conclude that

$$|g - \bar{h}| \le C'.$$

Step 2: consequences
From Lemma 6.1, we deduce that we can find $\alpha, \beta \in \mathbb{R}$ such that $\underline{h}(y) := \alpha + g(y + \beta)$ satisfies

$$\lim_{|y| \to +\infty} (\bar{h} - \underline{h})(y) = 0.$$

Applying the reasoning of Step 2 of the proof of Proposition 6.2, we also conclude that

$$\underline{h} \le \bar{h}$$

and then the right inequality of (7.14) holds true.
Step 3: Perron's method
We are now ready to apply Perron's method in the framework of viscosity solutions (see for instance [4, 8]) and to conclude to the existence of a solution h of (1.15) as in the statement of the corollary. □

Proof of Theorem 1.1
i) Uniqueness
This follows from Proposition 6.2.
ii) Existence
This follows from Corollary 7.6. □

8. Results under weaker assumptions. In this section, we give generalizations of Theorems 1.1 and 1.2, under weaker assumptions. In particular, we show in Theorem 8.1 below, that the existence of solutions is very robust (under the strict chord inequality (A_3)). In general, we can always relax C^1 regularity of V to Lipschitz, and remove condition (A_2), assuming that V is increasing. For some results, we can even have weaker assumptions, as it is shown below in Theorems 8.1 and 8.3.

Theorem 8.1. (Existence and uniqueness under weak assumptions) *Let $a < b$ and assume that $V \in C([a,b])$ in a nondecreasing function on $[a,b]$, satisfying for some $T > 0$ and $c \in \mathbb{R}$*

$$\begin{cases} \frac{p}{T} + c < V(p) \quad \text{for} \quad p \in (a,b), \\ \\ \text{with equality for } p \in \{a,b\}. \end{cases} \tag{8.1}$$

i) (existence)
Then there exists a concave function $h \in C^1(\mathbb{R})$ solution of (1.15) satisfying

$$h'(+\infty) = a \le h'(y) \le b = h'(-\infty)\,. \tag{8.2}$$

ii) (uniqueness)
Moreover, if $V \in Lip([a,b])$ and V is increasing on $[a,b]$, then such a function h is unique (up to translations and to addition of constants).

Remark 8.2. (Logarithmic branches without (A_4))
For V smooth such that $V'(b) = 1/T$ and $V''(b) > 0$, we expect to loose the exponential asymptotics. More precisely we expect that h will have a logarithmic branch (and then will no longer be asymptotic to a straight line as $y \to -\infty$). Similarly if $V'(a) = 1/T$ and $V''(a) > 0$, we also expect a logarithmic branch of h as $y \to +\infty$.

Proof of Theorem 8.1
We do the proof with the normalization $T = 1$ and $c = 0$.
i) (existence)
We approximate V by a function V_ε that satisfies assumption (A), and get by Theorem 1.1 a solution h_ε of

$$h'_\varepsilon(y) = V_\varepsilon(h_\varepsilon(y+1) - h_\varepsilon(y))$$

which a concave function satisfying (8.2). Up to redefine h_ε, we can fix the origin such that

$$h_\varepsilon(0) = 0 \quad \text{and} \quad h'_\varepsilon(0) = (a+b)/2.$$

Let us call h the limit of h_ε as ε goes to zero. By construction h is concave, satisfies

$$a \le a' := h'(+\infty) \le h'(y) \le h'(-\infty) =: b' \le b$$

and solves

$$h'(y) = V(h(y+1) - h(y)).$$

Thefere we have

$$a' = V(a'), \quad b' = V(b') \quad \text{and} \quad a \le a' \le (a+b)/2 \le b' \le b.$$

Condition (8.1) implies that $a' = a$ and $b' = b$, which shows (8.2).

ii) (uniqueness)

We notice that Proposition 5.1 is still true if we only assume that $V \in \mathrm{Lip}([a,b])$ and V is increasing on $[a,b]$, because those are the conditions used in the strong maximum principle (see Lemma 9.1). Then the proof of uniqueness given in Step 2 of the proof of Proposition 6.2 still applies. □

When we are not able to apply the true strong maximum principle, we can still get a non-existence result as shows the following

Theorem 8.3. (Non-existence under weak assumptions)
Let $a < b$ and assume that $V \in C([a,b])$ is an increasing function on $[a,b]$, satisfying for some $T > 0$ and $c \in \mathbb{R}$

$$\left\{ \begin{array}{l} \dfrac{p}{T} + c \le V(p) \quad \textit{for} \quad p \in (a,b), \\ \\ \textit{with equality at least for } p = a, b, p_0, \textit{ with } p_0 \in (a,b). \end{array} \right. \tag{8.3}$$

Then there is no solution $h \in C^1(\mathbb{R})$ of (1.15) satisfying $a \le h(y+1) - h(y) \le b$ and

$$|h - \bar{h}| \le C \quad \textit{with} \quad \bar{h}(y) := \min(ay, by). \tag{8.4}$$

Proof of Theorem 8.3

We do the proof with the normalization $T = 1$ and $c = 0$.

Step 1: exclusion of the case V linear

Let us assume that $V(p) = p$. Then the equation is

$$h'(y) = h(y+1) - h(y)$$

and we can apply the Fourier transform argument of Step 4 of the proof of Lemma 4.3 (because the function h is globally Lipschitz) to show that

$$h(y) = c_1 y + c_2.$$

Therefore there is no solution satisfying (8.4).

Step 2: existence of another candidate

Up to shift h, we can deduce from (8.4) that for some $C_0 > 0$

$$\bar{h} \ge h \ge \bar{h} - C_0. \tag{8.5}$$

From (8.3) and Step 1, we know that we can assume that there exists an interval $[a', b'] \subset [a,b]$ with $a' < b'$ and $(a', b') \ne (a,b)$ such that

$$\left\{ \begin{array}{l} p < V(p) \quad \text{for} \quad p \in (a', b'), \\ \\ \text{with equality if and only if } p \in \{a', b'\}. \end{array} \right.$$

Then Theorem 8.1 shows the existence of a concave solution g of

$$g'(y) = V(g(y+1) - g(y))$$

satisfying

$$g'(+\infty) = a' \le g'(y) \le b' = g'(-\infty).$$

As a consequence, up to shift g, we can assume that

$$g \geq 1 + \underline{h} \quad \text{with} \quad \underline{h}(y) := \min(a'y, b'y) \geq \bar{h}(y). \tag{8.6}$$

Moreover, up to add the same linear function to g and h, we can assume that $a > 0$.
Step 3: getting a contradiction
The idea of the proof is to shift the graph of h below the graph of g in order to get a contact point, and then to conclude to a contradiction using a sort of weak version of the strong maximum principle. We will avoid to get a contact point at infinity, using in a suitable way the behaviour of the functions at infinity.
We start with the following inequalities

$$g \geq 1 + \underline{h} \geq 1 + \bar{h} \geq 1 + h. \tag{8.7}$$

Case 1: $a < a'$
Let us define the function h^λ whose the graph is the translation of the graph of h of vector $\lambda(1, b)$, i.e.

$$h^\lambda(y) := \lambda b + h(y - \lambda).$$

We set

$$\lambda_0 := \sup \left\{ \lambda_1 \in [0, +\infty), \quad g \geq h^{\lambda_1} \right\} \quad \geq 0$$

where the bound from below of λ_0 also follows from (8.7). If $\lambda_0 = +\infty$, then we would deduce from (8.5) that $g(y) \geq by - C_0$, which is impossible because $g'(+\infty) = a' < b' \leq b$. Therefore $\lambda_0 < +\infty$. We also have (with a similar definition of $\bar{h}^{\lambda_0}$)

$$\begin{cases} g - h^{\lambda_0} \geq g - \bar{h}^{\lambda_0} = g - \bar{h} \geq 1 & \text{on} \quad (-\infty, 0], \\ \\ (g - h^{\lambda_0})(y) \geq (\underline{h} - \bar{h}^{\lambda_0})(y) \geq -C_1 + (a' - a)y & \text{for} \quad y \in [0, +\infty) \end{cases} \tag{8.8}$$

for some constant $C_1 > 0$. From the definition of λ_0, and from (8.8) with $a' - a > 0$, we then deduce that there exists $y_0 \in \mathbb{R}$ such that

$$g - h^{\lambda_0} \geq 0 = (g - h^{\lambda_0})(y_0).$$

Because V is increasing, from the equation satisfied by g and h^{λ_0}, we deduce that

$$(g - h^{\lambda_0})(y_0) = (g - h^{\lambda_0})(y_0 + 1) = (g - h^{\lambda_0})(y_0 + k) \quad \text{for all} \quad k \in \mathbb{N}.$$

Contradiction with (8.8).
Case 2: $b' < b$
We get a contradiction similarly as in case 1.
Step 4: conclusion
There is no solution h as stated in the theorem. □

9. Appendix. We now give the following result which is a special case of Lemma 6.2 b) given in [1].

Lemma 9.1. (Strong maximum principle)
Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a globally Lipschitz function such that $F(x_1, x_2)$ is increasing in x_1. We consider the following equation

$$h'(y) = F(h(y+1), h(y)) \quad \textit{for} \quad y \in \mathbb{R} \tag{9.1}$$

Let h^1, h^2 be respectively a viscosity sub and supersolution of (9.1) in the sense of Definition 3.1. Assume that

$$\begin{cases} h^1 \le h^2 \quad on \quad \mathbb{R} \\ h^2(y_0) = h^1(y_0)\,. \end{cases}$$

Then we have

$$h^1 = h^2 \quad for\ all \quad y \in \mathbb{R}.$$

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