Junction of elastic plates and beams

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Abstract - We consider the linearized elasticity system in a multidomain of $\mathbf{R}^3$. This multidomain is the union of a horizontal plate with fixed cross section and small thickness ε, and of a vertical beam with fixed height and small cross section of radius r^ε. The lateral boundary of the plate and the top of the beam are assumed to be clamped. When ε and r^ε tend to zero simultaneously, with $r^\varepsilon \gg \varepsilon^2$, we identify the limit problem. This limit problem involves six junction conditions.

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1 Introduction

Let ω^a and ω^b (a for "above", b for "below") be two bounded regular domains in $\mathbf{R}^2$. In the whole paper, the origin and axes are chosen so that

$$\int_{\omega^a} x_1\, dx_1\, dx_2 = \int_{\omega^a} x_2\, dx_1\, dx_2 = \int_{\omega^a} x_1 x_2\, dx_1\, dx_2 = 0 \text{ and } 0 \in \omega^b. \tag{1.1}$$

Let ε be a parameter taking values in a sequence of positive numbers converging to zero, and let r^ε be another positive parameter tending to zero with ε. We introduce the thin multidomain $\Omega^\varepsilon = \Omega^{a\varepsilon} \bigcup J^\varepsilon \bigcup \Omega^{b\varepsilon}$, where $\Omega^{a\varepsilon} = r^\varepsilon \omega^a \times (0, 1)$ represents a vertical beam with fixed height and small cross section, $\Omega^{b\varepsilon} = \omega^b \times (-\varepsilon, 0)$ represents a horizontal plate with small thickness and fixed cross section, and $J^\varepsilon = r^\varepsilon \omega^a \times \{0\}$ represents the interface at the junction between the beam and the plate.

In this thin multidomain, we consider the displacement $\overline{U}^\varepsilon$, solution of the three-dimensional linearized elasticity system:

$$\begin{aligned} \overline{U}^\varepsilon \in Y^\varepsilon \text{ and } \forall U \in Y^\varepsilon, \int_{\Omega^\varepsilon} [A^\varepsilon e(\overline{U}^\varepsilon), e(U)]\, dx \quad &= \quad \int_{\Omega^\varepsilon} F^\varepsilon . U\, dx + \int_{\Omega^\varepsilon} [G^\varepsilon, e(U)]\, dx + \\ &+ \quad \int_{\Sigma^{a\varepsilon} \bigcup T^{b\varepsilon} \bigcup B^{b\varepsilon}} H^\varepsilon . U\, d\sigma, \end{aligned} \tag{1.2}$$

where

- $Y^\varepsilon = \{U \in (H^1(\Omega^\varepsilon))^3,\, U = 0 \text{ on } T^{a\varepsilon} = r^\varepsilon \omega^a \times \{1\} \text{ and on } \Sigma^{b\varepsilon} = \partial\omega^b \times (-\varepsilon, 0)\}$;
- $A^\varepsilon = A^\varepsilon(x) = \begin{cases} A^a, \text{ if } x \in \Omega^{a\varepsilon}, \\ k^\varepsilon A^b, \text{ if } x \in \Omega^{b\varepsilon}, \end{cases}$

with k^ε a positive parameter depending on ε and A^a, A^b tensors with constant coefficients

A^a_{ijkl} and A^b_{ijkl}, $i,j,k,l \in \{1,2,3\}$, satisfying the usual symmetry and coercivity conditions:

$$A^a_{ijkl} = A^a_{jikl} = A^a_{ijlk}, \quad A^b_{ijkl} = A^b_{jikl} = A^b_{ijlk},$$

$$\exists C > 0,\ \forall \xi \in \mathbf{R}^{3\times 3}_s, [A^a\xi, \xi] \geq C|\xi|^2,\ [A^b\xi, \xi] \geq C|\xi|^2,$$

where $\mathbf{R}^{3\times 3}_s$ denotes the set of symmetric 3×3-matrices, $(A^a\xi)_{ij} = \sum_{kl} A^a_{ijkl}\xi_{kl}$ (e.g.), the scalar product $[.,.]$ in $\mathbf{R}^{3\times 3}$ is defined by $[\eta, \xi] = \sum_{ij} \eta_{ij}\xi_{ij}$ and $|.|$ is the associated norm;

- $e_{ij}(U) = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)$;

- $F^\varepsilon \in (L^2(\Omega^\varepsilon))^3$; the euclidian scalar product in $\mathbf{R}^3$ is denoted by a dot;

- $G^\varepsilon \in (L^2(\Omega^\varepsilon))^{3\times 3}$;

- $H^\varepsilon \in (L^2(\Sigma^{a\varepsilon} \bigcup T^{b\varepsilon} \bigcup B^{b\varepsilon}))^3$, where $\Sigma^{a\varepsilon}$ denotes the lateral boundary of the beam, $T^{b\varepsilon}$ and $B^{b\varepsilon}$ are respectively the top and the bottom of the plate:

$$\Sigma^{a\varepsilon} = r^\varepsilon \partial\omega^a \times (0,1), \quad T^{b\varepsilon} = (\omega^b \setminus r^\varepsilon\omega^a) \times \{0\}, \quad B^{b\varepsilon} = \omega^b \times \{-\varepsilon\}.$$

The constraint "$U = 0$" in the definition of Y^ε means that the multistructure is clamped on the top $T^{a\varepsilon}$ of the beam and on the lateral boundary $\Sigma^{b\varepsilon}$ of the plate. The case k^ε tending to zero or infinity corresponds to very different materials in $\Omega^{a\varepsilon}$ and $\Omega^{b\varepsilon}$. (Note that breaking the symmetry between $\Omega^{a\varepsilon}$ and $\Omega^{b\varepsilon}$ is not restrictive.) In the right hand side of (1.2), the second term is written in divergence form like in [21], [22] and [13]. It is well known that, by means of the Green formula, this second term can contribute to the other ones, giving possibly less regular (not necessarily L^2) volume and surface source terms. For convenience of the reader, we have chosen to write the three integrals: one recovers the classical formulation by setting $G^\varepsilon = 0$, but the simplest case corresponds to $F^\varepsilon = 0, H^\varepsilon = 0$ and $G^\varepsilon \neq 0$. This case was considered in the short preliminary version [13].

Problem (1.2) admits a unique solution $\overline{U}^\varepsilon$ (see e.g. [23]). The aim of this paper is to describe the limit behaviour of the displacement $\overline{U}^\varepsilon$, as ε tends to zero. We prove that this behaviour depends on the limit of the sequence

$$q^\varepsilon = k^\varepsilon \frac{\varepsilon^3}{(r^\varepsilon)^2}.$$

When $k^\varepsilon\varepsilon^3$ and $(r^\varepsilon)^2$ have same order (i.e. $q^\varepsilon \to q \in (0,\infty)$), the limit problem (after suitable rescaling) is coupled between a two-dimensional plate and a one-dimensional beam, with six junction conditions. If $k^\varepsilon\varepsilon^3 \gg (r^\varepsilon)^2$, the multistructure has the limit behaviour of a thin rigid plate and a thin elastic beam which are independent of each other, the beam being clamped at both ends; on the contrary, if $k^\varepsilon\varepsilon^3 \ll (r^\varepsilon)^2$, the structure behaves as a thin rigid beam and a thin elastic plate which are independent of each other, the plate being clamped on its contour and fixed vertically at the junction.

The reader is referred to [1], [3], [4], [6], [7], [9], [10], [17], [18], [19], [21], [22], [24], [25], for the asymptotic behaviour of plates and beams. Junction problems are considered in [5], [8], [11], [12], [14], [15], [16]. The present work is a natural follow up of [21], [22], which deal with reduction of dimension for elastic thin cylinders, and [11], [12], which deal with the diffusion equation in the thin multistructure considered in this paper. Our results were announced in a short note [13].

2 The result

2.1 The rescaled problem

In the sequel, the indexes α and β take values in the set $\{1,2\}$. Moreover, $x = (x', x_3)$ denotes the generic point in $\mathbf{R}^3$.

Let $\Omega^a = \omega^a \times (0,1)$, $\Omega^b = \omega^b \times (-1,0)$, $T^a = \omega^a \times \{1\}$, $\Sigma^a = \partial\omega^a \times (0,1)$ and $\Sigma^b = \partial\omega^b \times (-1,0)$. The asymptotic behaviour of $\overline{U}^\varepsilon$ can be described by using a convenient rescaling. (The reader is referred to Section 3.1 for details.) This rescaling maps the space Y^ε onto the space $\mathcal{Y}^\varepsilon$ defined by:

$$\begin{array}{l}\mathcal{Y}^\varepsilon = \{u = (u^a, u^b) \in (H^1(\Omega^a))^3 \times (H^1(\Omega^b))^3,\ u^a = 0 \text{ on } T^a, u^b = 0 \text{ on } \Sigma^b, \\ \qquad \text{for a.e. } x' \in \omega^a,\ u^a_\alpha(x', 0) = \varepsilon r^\varepsilon u^b_\alpha(r^\varepsilon x', 0) \text{ and } u^a_3(x', 0) = u^b_3(r^\varepsilon x', 0)\}.\end{array} \tag{2.3}$$

In particular, we denote by $\overline{u}^\varepsilon = (\overline{u}^{a\varepsilon}, \overline{u}^{b\varepsilon})$ the rescaling of the solution $\overline{U}^\varepsilon$ of Problem (1.2). We set

$$e^{a\varepsilon}(u^a) = \begin{pmatrix} \frac{1}{(r^\varepsilon)^2} e_{\alpha\beta}(u^a) & \frac{1}{r^\varepsilon} e_{\alpha 3}(u^a) \\ \\ \frac{1}{r^\varepsilon} e_{3\alpha}(u^a) & e_{33}(u^a) \end{pmatrix}, \quad e^{b\varepsilon}(u^b) = \begin{pmatrix} e_{\alpha\beta}(u^b) & \frac{1}{\varepsilon} e_{\alpha 3}(u^b) \\ \\ \frac{1}{\varepsilon} e_{3\alpha}(u^b) & \frac{1}{\varepsilon^2} e_{33}(u^b) \end{pmatrix}. \tag{2.4}$$

Then $\overline{u}^\varepsilon$ is the unique solution of the following problem:

$$\begin{array}{l} \overline{u}^\varepsilon \in \mathcal{Y}^\varepsilon \text{ and } \forall u \in \mathcal{Y}^\varepsilon, \\ \\ \int_{\Omega^a} [A^a e^{a\varepsilon}(\overline{u}^{a\varepsilon}), e^{a\varepsilon}(u^a)]\, dx + q^\varepsilon \int_{\Omega^b} [A^b e^{b\varepsilon}(\overline{u}^{b\varepsilon}), e^{b\varepsilon}(u^b)]\, dx = \\ \\ \quad = \int_{\Omega^a} f^{a\varepsilon}.u^a\, dx + \int_{\Omega^b} f^{b\varepsilon}.u^b\, dx + \int_{\Omega^a} [g^{a\varepsilon}, e^{a\varepsilon}(u^a)]\, dx + \int_{\Omega^b} [g^{b\varepsilon}, e^{b\varepsilon}(u^b)]\, dx + \\ \\ \qquad \int_{\Sigma^a} h^{a\varepsilon}.u^a\, d\sigma + \int_{\omega^b} \left(h^{b\varepsilon}_+ . u^b_{|x_3=0} + h^{b\varepsilon}_- . u^b_{|x_3=-1} \right) dx', \end{array} \tag{2.5}$$

where q^ε is defined by

$$q^\varepsilon = k^\varepsilon \frac{\varepsilon^3}{(r^\varepsilon)^2} \tag{2.6}$$

and where the source terms are suitable transforms of $(F^\varepsilon, G^\varepsilon, H^\varepsilon)$ (see Section 3.1).

2.2 The setting of the limit problem

For the definition of the limit problem, we introduce the following functional spaces:

$$\mathcal{U}^a = \{u^a \in (H_0^2(0,1))^2 \times H^1(\Omega^a), \exists \zeta^a \in H^1(0,1), \zeta^a(1) = 0, u_3^a = \zeta^a - x_1 \tfrac{d u_1^a}{d x_3} - x_2 \tfrac{d u_2^a}{d x_3}\},$$

$$\mathcal{V}^a = \{v^a \in (H^1(\Omega^a))^2 \times L^2(0,1; H^1(\omega^a)),\ \exists c \in H_0^1(0,1), v_1^a = -c\,x_2, v_2^a = c\,x_1,$$
$$\text{for a.e. } x_3 \in (0,1),\ \textstyle\int_{\omega^a} v_3^a(x', x_3)\, dx' = 0\},$$

$$\mathcal{W}^a = \{w^a \in (L^2(0,1; H^1(\omega^a)))^2 \times \{0\},$$
$$\text{for a.e. } x_3 \in (0,1),\ \textstyle\int_{\omega^a} w_\alpha^a\, dx' = \int_{\omega^a} (x_1 w_2^a - x_2 w_1^a)\, dx' = 0\},$$

$$\mathcal{U}^b = \{u^b \in (H^1(\Omega^b))^2 \times H_0^2(\omega^b),\ \exists \zeta_\alpha^b \in H_0^1(\omega^b), \quad u_\alpha^b = \zeta_\alpha^b - x_3 \tfrac{\partial u_3^b}{\partial x_\alpha}\},$$

$$\mathcal{V}^b = \{v^b \in (L^2(\omega^b; H^1(-1,0)))^2 \times \{0\}, \quad \text{for a.e. } x' \in \omega^b,\ \textstyle\int_{-1}^0 v_\alpha^b(x', x_3)\, dx_3 = 0\},$$

$$\mathcal{W}^b = \{w^b \in (\{0\})^2 \times L^2(\omega^b; H^1(-1,0)), \quad \text{for a.e. } x' \in \omega^b,\ \textstyle\int_{-1}^0 w_3^b(x', x_3)\, dx_3 = 0\},$$

$$\mathcal{Z}^a = \mathcal{U}^a \times \mathcal{V}^a \times \mathcal{W}^a, \quad \mathcal{Z}^b = \mathcal{U}^b \times \mathcal{V}^b \times \mathcal{W}^b.$$

Without loss of generality, we assume that q^ε defined by (2.6) satisfies

$$q^\varepsilon \to q \in [0, \infty]. \tag{2.7}$$

According to the value of q, the functional space for the limit problem is the following one:

$$\mathcal{Z} = \{z = (z^a, z^b) = ((u^a, v^a, w^a), (u^b, v^b, w^b)) \in \mathcal{Z}^a \times \mathcal{Z}^b,$$
$$\text{for a.e. } x' \in \omega^a,\ u_3^a(x', 0) = u_3^b(0)\}, \qquad \text{if } q \in (0, +\infty),$$

$$\mathcal{Z}_\infty = \{z^a = (u^a, v^a, w^a) \in \mathcal{Z}^a, \quad \text{for a.e. } x' \in \omega^a,\ u_3^a(x', 0) = 0\}, \qquad \text{if } q = +\infty,$$

$$\mathcal{Z}_0 = \{z^b = (u^b, v^b, w^b) \in \mathcal{Z}^b,\ u_3^b(0) = 0\}, \qquad \text{if } q = 0.$$

Let us remark that $\mathcal{U}^a$ (resp. $\mathcal{U}^b$) is a Bernouilli-Navier (resp. Kirchhoff-Love) space of displacements. Less classical spaces are $\mathcal{V}^a$, $\mathcal{W}^a$, $\mathcal{V}^b$, $\mathcal{W}^b$, which are introduced in a way similar to [21] and [22] (see also Appendix, Section 8.1). As for the boundary conditions, some of them are due to the clamping. These are more or less standard ones:

$$u_\alpha^a(1) = \frac{du_\alpha^a}{dx_3}(1) = c(1) = 0, \quad u_3^b = 0 \text{ and } \frac{\partial u_3^b}{d\nu} = 0 \text{ on } \partial\omega^b.$$

In contrast with the other requirements, the six conditions

$$u_\alpha^a(0) = \frac{d\,u_\alpha^a}{d\,x_3}(0) = c(0) = 0, \quad u_3^a(x', 0) = u_3^b(0) \text{ (respectively } u_3^a(x', 0) = 0 \text{ or } u_3^b(0) = 0),$$

which appear in the definition of the above spaces $\mathcal{U}^a$, $\mathcal{V}^a$ and $\mathcal{Z}$ (respectively $\mathcal{Z}_\infty$ or $\mathcal{Z}_0$), are specific to the junction between the beam and the plate. Note also that, in view of the definition of $\mathcal{U}^a$, the condition $u_3^a(x',0) = u_3^b(0)$ (respectively $u_3^a(x',0) = 0$) reduces to $\zeta^a(0) = u_3^b(0)$ (respectively $\zeta^a(0) = 0$).

We finally introduce, for $z^a = (u^a, v^a, w^a)$ in $\mathcal{Z}^a$ and $z^b = (u^b, v^b, w^b)$ in $\mathcal{Z}^b$:

$$e^a(z^a) = \begin{pmatrix} e_{\alpha\beta}(w^a) & e_{\alpha 3}(v^a) \\ \\ e_{3\alpha}(v^a) & e_{33}(u^a) \end{pmatrix}, \quad e^b(z^b) = \begin{pmatrix} e_{\alpha\beta}(u^b) & e_{\alpha 3}(v^b) \\ \\ e_{3\alpha}(v^b) & e_{33}(w^b) \end{pmatrix}. \tag{2.8}$$

2.3 The main result

We describe the limit behaviour of Problem (2.5), as ε tends to zero. In the sequel, we assume that

$$f^{a\varepsilon} \rightharpoonup f^a \text{ weakly in } (L^2(\Omega^a))^3, \tag{2.9}$$
$$f^{b\varepsilon} \rightharpoonup f^b \text{ weakly in } (L^2(\Omega^b))^3, \tag{2.10}$$
$$g^{a\varepsilon} \rightharpoonup g^a \text{ weakly in } (L^2(\Omega^a))^{3\times 3}, \tag{2.11}$$
$$g^{b\varepsilon} \rightharpoonup g^b \text{ weakly in } (L^2(\Omega^b))^{3\times 3}, \tag{2.12}$$
$$h^{a\varepsilon} \rightharpoonup h^a \text{ weakly in } (L^2(\Sigma^a))^3, \tag{2.13}$$
$$h_+^{b\varepsilon} \rightharpoonup h_+^b \text{ and } h_-^{b\varepsilon} \rightharpoonup h_-^b \text{ weakly in } (L^2(\omega^b))^3, \tag{2.14}$$

which is not restrictive, as proved in Remark 4 hereafter.

Our main result is the following one:

Theorem 1 *Assume that $\frac{r^\varepsilon}{\varepsilon^2} \to +\infty$ and that (2.7), (2.9) to (2.14) hold true. Then, with $e^{a\varepsilon}$, $e^{b\varepsilon}$ defined in (2.4) and e^a, e^b defined in (2.8),*

1) If $q \in (0, +\infty)$, there exists $\overline{z} = (\overline{z}^a, \overline{z}^b) = ((\overline{u}^a, \overline{v}^a, \overline{w}^a), (\overline{u}^b, \overline{v}^b, \overline{w}^b)) \in \mathcal{Z}$, such that

$$(\overline{u}^{a\varepsilon}, \overline{u}^{b\varepsilon}) \rightharpoonup (\overline{u}^a, \overline{u}^b) \text{ weakly in } (H^1(\Omega^a))^3 \times (H^1(\Omega^b))^3, \tag{2.15}$$

$$(e^{a\varepsilon}(\overline{u}^{a\varepsilon}), e^{b\varepsilon}(\overline{u}^{b\varepsilon})) \rightharpoonup (e^a(\overline{z}^a), e^b(\overline{z}^b)) \text{ weakly in } (L^2(\Omega^a))^{3\times 3} \times (L^2(\Omega^b))^{3\times 3}, \tag{2.16}$$

and $\overline{z}$ is the unique solution of the following problem:

$$\begin{aligned} &\overline{z} \in \mathcal{Z} \text{ and } \forall z \in \mathcal{Z}, \\ &\textstyle\int_{\Omega^a} [A^a e^a(\overline{z}^a), e^a(z^a)]\, dx + q \int_{\Omega^b} [A^b e^b(\overline{z}^b), e^b(z^b)]\, dx = \\ &\quad = \textstyle\int_{\Omega^a} f^a.u^a\, dx + \int_{\Omega^b} f^b.u^b\, dx + \int_{\Omega^a} [g^a, e^a(z^a)]\, dx + \int_{\Omega^b} [g^b, e^b(z^b)]\, dx + \\ &\qquad + \textstyle\int_{\Sigma^a} h^a.u^a\, d\sigma + \int_{\omega^b} \left(h_+^b.u^b_{|x_3=0} + h_-^b.u^b_{|x_3=-1} \right) dx'. \end{aligned} \tag{2.17}$$

Moreover, if the convergences in (2.11), (2.12) are strong, then the convergences in (2.15) and (2.16) are strong.

2) If $q = +\infty$, there exists $\overline{z}^a = (\overline{u}^a, \overline{v}^a, \overline{w}^a) \in \mathcal{Z}_\infty$, such that

$$\overline{u}^{a\varepsilon} \rightharpoonup \overline{u}^a \text{ weakly in } (H^1(\Omega^a))^3, \ \ \overline{u}^{b\varepsilon} \to 0 \text{ strongly in } (H^1(\Omega^b))^3, \tag{2.18}$$

$$e^{a\varepsilon}(\overline{u}^{a\varepsilon}) \rightharpoonup e^a(\overline{z}^a) \text{ weakly in } (L^2(\Omega^a))^{3\times 3}, \ \ e^{b\varepsilon}(\overline{u}^{b\varepsilon}) \to 0 \text{ strongly in } (L^2(\Omega^b))^{3\times 3}, \tag{2.19}$$

and $\overline{z}^a$ is the unique solution of the following problem:

$$\begin{aligned} &\overline{z}^a \in \mathcal{Z}_\infty \text{ and } \forall z^a \in \mathcal{Z}_\infty, \\ &\textstyle\int_{\Omega^a} [A^a e^a(\overline{z}^a), e^a(z^a)]\, dx = \int_{\Omega^a} f^a.u^a\, dx + \int_{\Omega^a} [g^a, e^a(z^a)]\, dx + \int_{\Sigma^a} h^a.u^a\, d\sigma. \end{aligned} \tag{2.20}$$

Moreover, if the convergence in (2.11) is strong, then

$$\overline{u}^{a\varepsilon} \to \overline{u}^a \text{ strongly in } (H^1(\Omega^a))^3, \tag{2.21}$$

$$e^{a\varepsilon}(\overline{u}^{a\varepsilon}) \to e^a(\overline{z}^a) \text{ strongly in } (L^2(\Omega^a))^{3\times 3}, \ \ \sqrt{q^\varepsilon}\, e^{b\varepsilon}(\overline{u}^{b\varepsilon}) \to 0 \text{ strongly in } (L^2(\Omega^b))^{3\times 3}. \tag{2.22}$$

3) If $q = 0$, there exists $\overline{z}^b = (\overline{u}^b, \overline{v}^b, \overline{w}^b) \in \mathcal{Z}_0$, such that

$$q^\varepsilon\, \overline{u}^{a\varepsilon} \to 0 \text{ strongly in } (H^1(\Omega^a))^3, \ q^\varepsilon\, \overline{u}^{b\varepsilon} \rightharpoonup \overline{u}^b \text{ weakly in } (H^1(\Omega^b))^3, \tag{2.23}$$

$$q^\varepsilon e^{a\varepsilon}(\overline{u}^{a\varepsilon}) \to 0 \text{ strongly in } (L^2(\Omega^a))^{3\times 3}, \ q^\varepsilon e^{b\varepsilon}(\overline{u}^{b\varepsilon}) \rightharpoonup e^b(\overline{z}^b) \text{ weakly in } (L^2(\Omega^b))^{3\times 3}, \tag{2.24}$$

and $\overline{z}^b$ is the unique solution of the following problem:

$$\begin{aligned} &\overline{z}^b \in \mathcal{Z}_0 \text{ and } \forall z^b \in \mathcal{Z}_0, \\ &\textstyle\int_{\Omega^b} [A^b e^b(\overline{z}^b), e^b(z^b)]\, dx = \int_{\Omega^b} f^b.u^b\, dx + \int_{\Omega^b} [g^b, e^b(z^b)]\, dx + \\ &\qquad + \textstyle\int_{\omega^b} \left(h^b_+.u^b_{|x_3=0} + h^b_-.u^b_{|x_3=-1} \right) dx'. \end{aligned} \tag{2.25}$$

Moreover, if the convergence in (2.12) is strong, then

$$q^\varepsilon \overline{u}^{b\varepsilon} \to \overline{u}^b \text{ strongly in } (H^1(\Omega^b))^3, \tag{2.26}$$

$$\sqrt{q^\varepsilon}\, e^{a\varepsilon}(\overline{u}^{a\varepsilon}) \to 0 \text{ strongly in } (L^2(\Omega^b))^{3\times 3}, \ \ q^\varepsilon e^{b\varepsilon}(\overline{u}^{b\varepsilon}) \to e^b(\overline{z}^b) \text{ strongly in } (L^2(\Omega^b))^{3\times 3}. \tag{2.27}$$

Remark 1 *The condition $\frac{r^\varepsilon}{\varepsilon^2} \to +\infty$ is only used to prove that $\overline{u}^a_3(x', 0) \equiv \overline{u}^b_3(0)$ and $\overline{c}(0) = 0$, via a convenient Sobolev imbedding theorem, as regards the second equality. At this stage of our understanding, we do not know if it is just a technical condition or not.*

Remark 2 *In the Appendix, Section 8.1, we prove that the functions $\overline{v}^a$ and $\overline{w}^a$ (resp. $\overline{v}^b$ and $\overline{w}^b$) which appear in the limit problem are the limits of suitable expressions of $\overline{u}^{a\varepsilon}$ (resp. $\overline{u}^{b\varepsilon}$).*

2.4 Back to the problem in the thin multidomain

As far as the asymptotic behaviour of the "energy" of the solution of Problem (1.2) in the thin multidomain is concerned, we define the following renormalized energy by:

$$\mathcal{E}^\varepsilon := \left(\frac{\lambda^\varepsilon}{r^\varepsilon}\right)^2 \int_{\Omega^\varepsilon} [A^\varepsilon e(\overline{U}^\varepsilon), e(\overline{U}^\varepsilon)]\, dx, \tag{2.28}$$

where λ^ε can be made explicit in terms of $\varepsilon, r^\varepsilon, F^\varepsilon, G^\varepsilon, H^\varepsilon$ (see (3.37) in Section 3.1); we also have

$$\mathcal{E}^\varepsilon = \int_{\Omega^a} [A^a e^{a\varepsilon}(\overline{u}^{a\varepsilon}), e^{a\varepsilon}(\overline{u}^{a\varepsilon})]\, dx + q^\varepsilon \int_{\Omega^b} [A^b e^{b\varepsilon}(\overline{u}^{b\varepsilon}), e^{b\varepsilon}(\overline{u}^{b\varepsilon})]\, dx$$

and from Theorem 1 we deduce the following Corollary:

Corollary 1 *Assume that $\frac{r^\varepsilon}{\varepsilon^2} \to +\infty$ and that (2.7), (2.9) to (2.14) hold true.*

1) If $q \in (0, +\infty)$ and the convergences in (2.11), (2.12) are strong, then

$$\mathcal{E}^\varepsilon \to \mathcal{E} = \int_{\Omega^a} [A^a e^a(\overline{z}^a), e^a(\overline{z}^a)]\, dx + q \int_{\Omega^b} [A^b e^b(\overline{z}^b), e^b(\overline{z}^b)]\, dx.$$

2) If $q = +\infty$ and the convergence in (2.11) is strong, then

$$\mathcal{E}^\varepsilon \to \mathcal{E}_\infty = \int_{\Omega^a} [A^a e^a(\overline{z}^a), e^a(\overline{z}^a)]\, dx.$$

3) If $q = 0$ and the convergence in (2.12) is strong, then

$$q^\varepsilon \mathcal{E}^\varepsilon \to \mathcal{E}_0 = \int_{\Omega^b} [A^b e^b(\overline{z}^b), e^b(\overline{z}^b)]\, dx.$$

Actually, the proof of Corollary 1 is part of proof of Theorem 1, since the strong convergences of $\overline{u}^{a\varepsilon}$ to $\overline{u}^a$ (resp. $\overline{u}^{b\varepsilon}$ to $\overline{u}^b$) and $e^{a\varepsilon}(\overline{u}^{a\varepsilon})$ to $e^a(\overline{z}^a)$ (resp. $e^{b\varepsilon}(\overline{u}^{b\varepsilon})$ to $e^b(\overline{z}^b)$) follow from the convergence of the energy $\mathcal{E}^\varepsilon$. The following interpretation is a direct consequence of the strong convergences of $e^{a\varepsilon}(\overline{u}^{a\varepsilon})$ and $e^{b\varepsilon}(\overline{u}^{b\varepsilon})$.

Interpretation: For example, let us consider the particular case of Problem (1.2), for which $G^\varepsilon \equiv 0$, $H^\varepsilon \equiv 0$, $k^\varepsilon = 1$ and $A^a = A^b = A$:

$$\overline{U}^\varepsilon \in Y^\varepsilon \text{ and } \forall U \in Y^\varepsilon, \int_{\Omega^\varepsilon} [Ae(\overline{U}^\varepsilon), e(U)]\, dx = \int_{\Omega^\varepsilon} F^\varepsilon . U\, dx,$$

and let us assume that $r^\varepsilon = \varepsilon^{3/2}$ and that

$$\frac{1}{\varepsilon^9} \sum_\alpha \|F^\varepsilon_\alpha\|^2_{L^2(\Omega^{a\varepsilon})} + \frac{1}{\varepsilon^6} \sum_\alpha \|F^\varepsilon_3\|^2_{L^2(\Omega^{a\varepsilon})} + \frac{1}{\varepsilon^6} \sum_\alpha \|F^\varepsilon_\alpha\|^2_{L^2(\Omega^{b\varepsilon})} + \frac{1}{\varepsilon^8} \sum_\alpha \|F^\varepsilon_3\|^2_{L^2(\Omega^{b\varepsilon})} = 1. \tag{2.29}$$

(This last condition is not restrictive, since it is just a matter of normalization. One can remark that, in this case, the parameter λ^ε introduced in (3.37), in Section 3.1, has value $\varepsilon^{-3/2}$.) Defining the rescaled force and the rescaled solution by

$$f^{a\varepsilon}_\alpha(x) = \frac{1}{\varepsilon^3} F^\varepsilon_\alpha(\varepsilon^{\frac{3}{2}} x', x_3), \qquad f^{a\varepsilon}_3(x) = \frac{1}{\varepsilon^{\frac{3}{2}}} F^\varepsilon_3(\varepsilon^{\frac{3}{2}} x', x_3), \qquad \text{for } x \in \Omega^a,$$

$$f^{b\varepsilon}_\alpha(x) = \frac{1}{\varepsilon^{\frac{5}{2}}} F^\varepsilon_\alpha(x', \varepsilon x_3), \qquad f^{b\varepsilon}_3(x) = \frac{1}{\varepsilon^{\frac{7}{2}}} F^\varepsilon_3(x', \varepsilon x_3), \qquad \text{for } x \in \Omega^b,$$

$$\overline{u}^{a\varepsilon}_\alpha(x) = \overline{U}^\varepsilon_\alpha(\varepsilon^{\frac{3}{2}} x', x_3), \qquad \overline{u}^{a\varepsilon}_3(x) = \frac{1}{\varepsilon^{\frac{3}{2}}} \overline{U}^\varepsilon_3(\varepsilon^{\frac{3}{2}} x', x_3), \qquad \text{for } x \in \Omega^a,$$

$$\overline{u}^{b\varepsilon}_\alpha(x) = \frac{1}{\varepsilon^{\frac{5}{2}}} \overline{U}^\varepsilon_\alpha(x', \varepsilon x_3), \qquad \overline{u}^{b\varepsilon}_3(x) = \frac{1}{\varepsilon^{\frac{3}{2}}} \overline{U}^\varepsilon_3(x', \varepsilon x_3), \qquad \text{for } x \in \Omega^b,$$

one can check that $\overline{u}^\varepsilon$ solves the rescaled problem

$$\begin{cases} \overline{u}^\varepsilon \in \mathcal{Y}^\varepsilon \text{ and } \forall u \in \mathcal{Y}^\varepsilon, \\ \int_{\Omega^a} [A e^{a\varepsilon}(\overline{u}^{a\varepsilon}), e^{a\varepsilon}(u^a)]\, dx + \int_{\Omega^b} [A e^{b\varepsilon}(\overline{u}^{b\varepsilon}), e^{b\varepsilon}(u^b)]\, dx = \int_{\Omega^a} f^{a\varepsilon}.u^a\, dx + \int_{\Omega^b} f^{b\varepsilon}.u^b\, dx. \end{cases}$$

Since, thanks to (2.29),

$$\int_{\Omega^a} |f^{a\varepsilon}|^2\, dx + \int_{\Omega^b} |f^{b\varepsilon}|^2\, dx = 1,$$

it is not restrictive to assume that, for some subsequence of ε, still denoted by ε, and for some f^a in $L^2(\Omega^a)$ and f^b in $L^2(\Omega^b)$,

$$f^{a\varepsilon} \rightharpoonup f^a \text{ in } L^2(\Omega^a) \text{ and } f^{b\varepsilon} \rightharpoonup f^b \text{ in } L^2(\Omega^b).$$

Then, Theorem 1 asserts that

$$\overline{u}^{a\varepsilon} \to \overline{u}^a \text{ strongly in } (H^1(\Omega^a))^3 \text{ and } \overline{u}^{b\varepsilon} \to \overline{u}^b \text{ strongly in } (H^1(\Omega^b))^3, \tag{2.30}$$

$$e^{a\varepsilon}(\overline{u}^{a\varepsilon}) \to \overline{e}^a \text{ strongly in } (L^2(\Omega^a))^{3\times 3} \text{ and } e^{b\varepsilon}(\overline{u}^{b\varepsilon}) \to \overline{e}^b \text{ strongly in } (L^2(\Omega^b))^{3\times 3}, \tag{2.31}$$

where $\overline{e}^a = e^a(\overline{z}^a)$, $\overline{e}^b = e^b(\overline{z}^b)$ and $\overline{z} = (\overline{z}^a, \overline{z}^b)$ is the unique solution of the rescaled limit problem:

$$\begin{cases} \overline{z} \in \mathcal{Z} \text{ and } \forall z \in \mathcal{Z}, \\ \int_{\Omega^a} [A e^a(\overline{z}^a), e^a(z^a)]\, dx + \int_{\Omega^b} [A e^b(\overline{z}^b), e^b(z^b)]\, dx = \int_{\Omega^a} f^a.u^a\, dx + \int_{\Omega^b} f^b.u^b\, dx. \end{cases} \tag{2.32}$$

Coming back to the initial domain, we define $\overline{E}^{a\varepsilon}$ and $\overline{E}^{b\varepsilon}$ by

$$\overline{E}^{a\varepsilon} = \varepsilon^{\frac{3}{2}}\, \overline{e}^a(\frac{x'}{r^\varepsilon}, x_3), \text{ for } x \in \Omega^{a\varepsilon}, \text{ and } \overline{E}^{b\varepsilon} = \varepsilon^{\frac{5}{2}}\, \overline{e}^b(x', \frac{x_3}{\varepsilon}), \text{ for } x \in \Omega^{b\varepsilon},$$

and we define the relative errors $\Delta^{a\varepsilon}$, $\Delta^{b\varepsilon}$ and Δ^{ε} by

$$\Delta^{a\varepsilon} = \frac{\int_{\Omega^{a\varepsilon}} |e(\overline{U}^{\varepsilon}) - \overline{E}^{a\varepsilon}|^2\,dx}{\int_{\Omega^{a\varepsilon}} |\overline{E}^{a\varepsilon}|^2\,dx}, \quad \Delta^{b\varepsilon} = \frac{\int_{\Omega^{b\varepsilon}} |e(\overline{U}^{\varepsilon}) - \overline{E}^{b\varepsilon}|^2\,dx}{\int_{\Omega^{b\varepsilon}} |\overline{E}^{b\varepsilon}|^2\,dx},$$

$$\Delta^{\varepsilon} = \frac{\int_{\Omega^{a\varepsilon}} |e(\overline{U}^{\varepsilon}) - \overline{E}^{a\varepsilon}|^2\,dx + \int_{\Omega^{b\varepsilon}} |e(\overline{U}^{\varepsilon}) - \overline{E}^{b\varepsilon}|^2\,dx}{\int_{\Omega^{a\varepsilon}} |\overline{E}^{a\varepsilon}|^2\,dx + \int_{\Omega^{b\varepsilon}} |\overline{E}^{b\varepsilon}|^2\,dx}.$$

Assuming that $\overline{e}^a \neq 0$ and $\overline{e}^b \neq 0$, an easy computation gives that

$$\Delta^{a\varepsilon} = \frac{\int_{\Omega^a} |e^{a\varepsilon}(\overline{u}^{a\varepsilon}) - \overline{e}^a|^2\,dx}{\int_{\Omega^a} |\overline{e}^a|^2\,dx}, \quad \Delta^{b\varepsilon} = \frac{\int_{\Omega^b} |e^{b\varepsilon}(\overline{u}^{b\varepsilon}) - \overline{e}^b|^2\,dx}{\int_{\Omega^b} |\overline{e}^b|^2\,dx}.$$

Hence the strong convergences in (2.31) imply that $\Delta^{a\varepsilon}$, $\Delta^{b\varepsilon}$, and then Δ^{ε}, tend to zero with ε. These convergences of the relative errors mean that the deformation of the original displacement is well described by $\overline{E}^{a\varepsilon}$ and $\overline{E}^{b\varepsilon}$:

$$e(\overline{U}^{\varepsilon}) \simeq \overline{E}^{a\varepsilon} \text{ in } \Omega^{a\varepsilon}, \quad e(\overline{U}^{\varepsilon}) \simeq \overline{E}^{b\varepsilon} \text{ in } \Omega^{b\varepsilon}.$$

In the same spirit, from the solution $\overline{z} = (\overline{z}^a, \overline{z}^b) = ((\overline{u}^a, \overline{v}^a, \overline{w}^a), (\overline{u}^b, \overline{v}^b, \overline{w}^b))$ of Problem (2.32), we are going to define $\hat{U}^{a\varepsilon}$ and $\hat{U}^{b\varepsilon}$, which are good approximates of the restrictions of $\overline{U}^{\varepsilon}$ to $\Omega^{a\varepsilon}$ and $\Omega^{b\varepsilon}$, respectively. Actually, let us set:

$$\hat{u}^{a\varepsilon} = \overline{u}^a + r^{\varepsilon}\overline{v}^a + (r^{\varepsilon})^2\overline{w}^a = \overline{u}^a + \varepsilon^{\frac{3}{2}}\overline{v}^a + \varepsilon^3\overline{w}^a,$$

$$\hat{u}^{b\varepsilon} = \overline{u}^b + \varepsilon\overline{v}^b + \varepsilon^2\overline{w}^b,$$

$$\hat{U}^{a\varepsilon}_{\alpha}(x) = \hat{u}^{a\varepsilon}_{\alpha}(\tfrac{x'}{\varepsilon^{\frac{3}{2}}}, x_3), \qquad \hat{U}^{a\varepsilon}_3(x) = \varepsilon^{\frac{3}{2}}\hat{u}^{a\varepsilon}_3(\tfrac{x'}{\varepsilon^{\frac{3}{2}}}, x_3), \qquad \text{for } x \in \Omega^{a\varepsilon},$$

$$\hat{U}^{b\varepsilon}_{\alpha}(x) = \varepsilon^{\frac{5}{2}}\hat{u}^{b\varepsilon}_{\alpha}(x', \tfrac{x_3}{\varepsilon}), \qquad \hat{U}^{b\varepsilon}_3(x) = \varepsilon^{\frac{3}{2}}\hat{u}^{b\varepsilon}_3(x', \tfrac{x_3}{\varepsilon}), \qquad \text{for } x \in \Omega^{b\varepsilon}.$$

As

$$e^{a\varepsilon}(\hat{u}^{a\varepsilon}) = \overline{e}^a + \varepsilon^{\frac{3}{2}} \begin{pmatrix} 0 & e_{\alpha 3}(\overline{w}^a) \\ e_{3\beta}(\overline{w}^a) & e_{33}(\overline{v}^a) \end{pmatrix} \text{ and } e^{b\varepsilon}(\hat{u}^{b\varepsilon}) = \overline{e}^b + \varepsilon \begin{pmatrix} e_{\alpha\beta}(\overline{v}^b) & e_{\alpha 3}(\overline{w}^b) \\ e_{3\beta}(\overline{w}^b) & 0 \end{pmatrix},$$

it is clear that, as ε tends to zero, $e^{a\varepsilon}(\hat{u}^{a\varepsilon}) \to \overline{e}^a$ and $e^{b\varepsilon}(\hat{u}^{b\varepsilon}) \to \overline{e}^b$ in L^2-norm and then, from (2.31),

$$\int_{\Omega^a} |e^{a\varepsilon}(\overline{u}^{a\varepsilon} - \hat{u}^{a\varepsilon})|^2\,dx + \int_{\Omega^b} |e^{b\varepsilon}(\overline{u}^{b\varepsilon} - \hat{u}^{b\varepsilon})|^2\,dx \to 0.$$

As above, if $\overline{e}^a \neq 0$ and $\overline{e}^b \neq 0$, we get that

$$\hat{\Delta}^{a\varepsilon} := \frac{\int_{\Omega^{a\varepsilon}} |e(\overline{U}^{\varepsilon} - \hat{U}^{a\varepsilon})|^2\,dx}{\int_{\Omega^{a\varepsilon}} |e(\hat{U}^{a\varepsilon})|^2\,dx} = \frac{\int_{\Omega^a} |e^{a\varepsilon}(\overline{u}^{a\varepsilon} - \hat{u}^{a\varepsilon})|^2\,dx}{\int_{\Omega^a} |e^{a\varepsilon}(\hat{u}^{a\varepsilon})|^2\,dx} \to 0,$$

$$\hat{\Delta}^{b\varepsilon} := \frac{\int_{\Omega^{b\varepsilon}} |e(\overline{U}^{\varepsilon} - \hat{U}^{b\varepsilon})|^2\,dx}{\int_{\Omega^{a\varepsilon}} |e(\hat{U}^{b\varepsilon})|^2\,dx} = \frac{\int_{\Omega^{b}} |e^{b\varepsilon}(\overline{u}^{b\varepsilon} - \hat{u}^{b\varepsilon})|^2\,dx}{\int_{\Omega^{b}} |e^{b\varepsilon}(\hat{u}^{b\varepsilon})|^2\,dx} \to 0,$$

$$\hat{\Delta}^{\varepsilon} = \frac{\int_{\Omega^{a\varepsilon}} |e(\overline{U}^{\varepsilon} - \hat{U}^{a\varepsilon})|^2\,dx + \int_{\Omega^{b\varepsilon}} |e(\overline{U}^{\varepsilon} - \hat{U}^{b\varepsilon})|^2\,dx}{\int_{\Omega^{a\varepsilon}} |e(\hat{U}^{a\varepsilon})|^2\,dx + \int_{\Omega^{b\varepsilon}} |e(\hat{U}^{b\varepsilon})|^2\,dx} \to 0.$$

As least formally, this means that

$$\overline{U}^{\varepsilon} \simeq \hat{U}^{a\varepsilon} \text{ in } \Omega^{a\varepsilon}, \quad \overline{U}^{\varepsilon} \simeq \hat{U}^{b\varepsilon} \text{ in } \Omega^{b\varepsilon}. \tag{2.33}$$

Let us prove that, from the eqivalence (2.33), one can recover heuristically the conditions at the junction. As a matter of fact, suppose we just know that

$$\overline{u}^a_\alpha = \overline{u}^a_\alpha(x_3), \quad \overline{u}^a_3 = \overline{\zeta}^a(x_3) - x_1\frac{d\overline{u}^a_1}{dx_3}(x_3) - x_2\frac{d\overline{u}^a_2}{dx_3}(x_3),$$

$$\overline{v}^a_\alpha = \overline{c}(x_3)x'^R_\alpha, \text{ that is } \overline{v}^a_1 = -\overline{c}(x_3)x_2 \text{ and } \overline{v}^a_2 = \overline{c}(x_3)x_1,$$

$$\overline{w}^a_3 = 0,$$

$$\overline{u}^b_3 = \overline{u}^b_3(x'), \quad \overline{u}^b_\alpha = \overline{\zeta}^b_\alpha(x') - x_3\frac{\partial \overline{u}^b_3}{\partial x_\alpha}(x'),$$

$$\overline{v}^b_3 = 0, \quad \overline{w}^b_\alpha = 0.$$

¿From the above expressions of $\overline{u}^a, \overline{v}^a, \overline{w}^a, \overline{u}^b, \overline{v}^b, \overline{w}^b$, we deduce that

$$\begin{cases} \hat{U}^{a\varepsilon}_\alpha(\varepsilon^{\frac{3}{2}}x', x_3) = \overline{u}^a_\alpha(x_3) + \varepsilon^{\frac{3}{2}}\overline{c}(x_3)x'^R_\alpha + \varepsilon^3\overline{w}^a_\alpha(x), & \text{for } x \in \Omega^a, \\ \hat{U}^{b\varepsilon}_\alpha(x', \varepsilon x_3) = \varepsilon^{\frac{5}{2}}\left(\overline{\zeta}^b_\alpha(x') - x_3\frac{\partial \overline{u}^b_3}{\partial x_\alpha}(x') + \varepsilon\overline{v}^b_\alpha(x) + \varepsilon^2\cdot 0\right), & \text{for } x \in \Omega^b, \end{cases} \tag{2.34}$$

$$\begin{cases} \hat{U}^{a\varepsilon}_3(\varepsilon^{\frac{3}{2}}x', x_3) = \varepsilon^{\frac{3}{2}}\left(\overline{\zeta}^a(x_3) - x_1\frac{d\overline{u}^a_1}{dx_3}(x_3) - x_2\frac{d\overline{u}^a_2}{dx_3}(x_3) + \varepsilon^{\frac{3}{2}}\overline{v}^a_3(x) + \varepsilon^3\cdot 0\right), & \text{for } x \in \Omega^a, \\ \hat{U}^{b\varepsilon}_3(x', \varepsilon x_3) = \varepsilon^{\frac{3}{2}}\left(\overline{u}^b_3(x') + \varepsilon\cdot 0 + \varepsilon^2\overline{w}^b_3(x)\right), & \text{for } x \in \Omega^b. \end{cases} \tag{2.35}$$

At least formally, from (2.33), (2.34) and (2.35), it follows that

$$\overline{U}^{\varepsilon}_\alpha = O(1) \text{ in } \Omega^{a\varepsilon}, \quad \overline{U}^{\varepsilon}_\alpha = O(\varepsilon^{\frac{5}{2}}) \text{ in } \Omega^{b\varepsilon},$$

$$\overline{U}^{\varepsilon}_3 = O(\varepsilon^{\frac{3}{2}}) \text{ in } \Omega^{a\varepsilon}, \quad \overline{U}^{\varepsilon}_3 = O(\varepsilon^{\frac{3}{2}}) \text{ in } \Omega^{b\varepsilon}.$$

We deduce from these estimates that the main observable displacement is the transversal displacement of the beam. At the junction, the continuity of $\overline{U}^{\varepsilon}$ formally implies that, for $x' \in \omega^a$,

$$\overline{u}^a_\alpha(0) + \varepsilon^{\frac{3}{2}}\overline{c}(0)x'^R_\alpha + \varepsilon^3\overline{w}^a_\alpha(x', 0) \;\simeq\; \varepsilon^{\frac{5}{2}}\left(\overline{\zeta}^b_\alpha(\varepsilon^{\frac{3}{2}}x') - 0\cdot\frac{\partial \overline{u}^b_3}{\partial x_\alpha}(\varepsilon^{\frac{3}{2}}x', 0) + \varepsilon\overline{v}^b_\alpha(\varepsilon^{\frac{3}{2}}x', 0) + \varepsilon^2\cdot 0\right),$$

$$\overline{\zeta}^a(0) - x_1\frac{d\overline{u}_1^a}{dx_3}(0) - x_2\frac{d\overline{u}_2^a}{dx_3}(0) + \varepsilon^{\frac{3}{2}}\overline{v}_3^a(x',0) + \varepsilon^3\cdot 0 \;\simeq\; \overline{u}_3^b(\varepsilon^{\frac{3}{2}}x') + \varepsilon\cdot 0 + \varepsilon^2\overline{w}_3^b(\varepsilon^{\frac{3}{2}}x',0).$$

This gives formally

$$\overline{u}_\alpha^a(0) = 0, \quad \overline{c}(0) = 0,$$

$$\overline{\zeta}^a(0) = \overline{u}_3^b(0), \quad \frac{d\overline{u}_\alpha^a}{dx_3}(0) = 0,$$

which are the six conditions on the junction, a rigourous proof of which is given in Section 5. Moveover we get at the junction the following estimates

$$\overline{U}_\alpha^\varepsilon = O(\varepsilon^{\frac{5}{2}}), \quad \overline{U}_3^\varepsilon = O(\varepsilon^{\frac{3}{2}}) \text{ on } J^\varepsilon.$$

Remark 3 *We could go further and formally deduce for instance that*

$$\overline{\zeta}_\alpha^b(0) = 0, \quad \overline{v}_3^a(x',0) = x_1\frac{\partial\overline{u}_3^b}{\partial x_1}(0) + x_2\frac{\partial\overline{u}_3^b}{\partial x_2}(0).$$

But these relations have no sense, from a functionnal analysis point of view, because of the too low regularity of the solutions. For instance $\overline{\zeta}_\alpha^b$ only belongs to $H^1(\omega^b)$, and then its value $\overline{\zeta}_\alpha^b(0)$ is not well defined. On the contrary, the functions involved in the conditions at the junction have a value in zero, if they belong to the functional space given by the limit problem.

The remaining part of the paper is devoted to the proofs of Theorem 1 and Corollary 1.

3 The derivation of the rescaled problem

Let us emphasize that we perform different scalings for the respective restrictions of $U \in Y^\varepsilon$ to the respective subdomains $\Omega^{a\varepsilon}$ and $\Omega^{b\varepsilon}$, in order to get convenient transmission conditions for their transforms u^a and u^b. We mean that, with the transmission conditions appearing in the definition (2.3) of $\mathcal{Y}^\varepsilon$, namely

$$\text{for a.e. } x' \in \omega^a,\; u_\alpha^a(x',0) = \varepsilon r^\varepsilon u_\alpha^b(r^\varepsilon x',0) \text{ and } u_3^a(x',0) = u_3^b(r^\varepsilon x',0), \tag{3.36}$$

we are abble to derive the junction conditions, for the limit problem. (The derivation of the limit junction conditions seems to be delicate otherwise.) Moreover this is the scaling for which the coupling is maximum at the limit, at least for the third component of the displacement.

3.1 The result of the scaling

In this subsection, we give the explicit expressions of the source terms and the solution of the rescaled problem (2.5), as functions of the corresponding terms of the initial problem (1.2). An explanation is given in Subsection 3.2.

On the first hand, assuming that $(F^\varepsilon, G^\varepsilon, H^\varepsilon) \neq (0,0,0)$ (otherwise the problem is trivial), we define λ^ε by:

$$\begin{aligned}
&\tfrac{1}{(r^\varepsilon)^2} \textstyle\sum_{\alpha=1}^2 \|F_\alpha^\varepsilon\|^2_{L^2(\Omega^{a\varepsilon})} + \|F_3^\varepsilon\|^2_{L^2(\Omega^{a\varepsilon})} + \tfrac{\varepsilon^3}{(r^\varepsilon)^2} \textstyle\sum_{\alpha=1}^2 \|F_\alpha^\varepsilon\|^2_{L^2(\Omega^{b\varepsilon})} + \tfrac{\varepsilon}{(r^\varepsilon)^2} \|F_3^\varepsilon\|^2_{L^2(\Omega^{b\varepsilon})} + \\
&+ \|G^\varepsilon\|^2_{(L^2(\Omega^{a\varepsilon}))^{3\times 3}} + \tfrac{\varepsilon^3}{(r^\varepsilon)^2} \|G^\varepsilon\|^2_{(L^2(\Omega^{b\varepsilon}))^{3\times 3}} + \tfrac{1}{(r^\varepsilon)^3} \textstyle\sum_{\alpha=1}^2 \|H_\alpha^\varepsilon\|^2_{L^2(\Sigma^{a\varepsilon})} + \tfrac{1}{r^\varepsilon} \|H_3^\varepsilon\|^2_{L^2(\Sigma^{a\varepsilon})} + \\
&+ \tfrac{\varepsilon^2}{(r^\varepsilon)^2} \textstyle\sum_{\alpha=1}^2 \|H_\alpha^\varepsilon\|^2_{L^2(T^{b\varepsilon} \bigcup B^{b\varepsilon})} + \tfrac{1}{(r^\varepsilon)^2} \|H_3^\varepsilon\|^2_{L^2(T^{b\varepsilon} \bigcup B^{b\varepsilon})} = \left(\tfrac{r^\varepsilon}{\lambda^\varepsilon}\right)^2;
\end{aligned} \tag{3.37}$$

then we set

$$\begin{aligned}
&f_\alpha^{a\varepsilon}(x) = \tfrac{\lambda^\varepsilon}{r^\varepsilon} F_\alpha^\varepsilon(r^\varepsilon x', x_3), && f_3^{a\varepsilon}(x) = \lambda^\varepsilon F_3^\varepsilon(r^\varepsilon x', x_3), && \text{for } x \in \Omega^a; \\
&f_\alpha^{b\varepsilon}(x) = \lambda^\varepsilon \tfrac{\varepsilon^2}{(r^\varepsilon)^2} F_\alpha^\varepsilon(x', \varepsilon x_3), && f_3^{b\varepsilon}(x) = \lambda^\varepsilon \tfrac{\varepsilon}{(r^\varepsilon)^2} F_3^\varepsilon(x', \varepsilon x_3), && \text{for } x \in \Omega^b;
\end{aligned} \tag{3.38}$$

$$g^{a\varepsilon}(x) = \lambda^\varepsilon G^\varepsilon(r^\varepsilon x', x_3), \ \text{for } x \in \Omega^a; \quad g^{b\varepsilon}(x) = \lambda^\varepsilon \frac{\varepsilon^2}{(r^\varepsilon)^2} G^\varepsilon(x', \varepsilon x_3), \ \text{for } x \in \Omega^b; \tag{3.39}$$

$$\begin{aligned}
&h_\alpha^{a\varepsilon}(x) = \lambda^\varepsilon \tfrac{1}{(r^\varepsilon)^2} H_\alpha^\varepsilon(r^\varepsilon x', x_3), && h_3^{a\varepsilon}(x) = \lambda^\varepsilon \tfrac{1}{r^\varepsilon} H_3^\varepsilon(r^\varepsilon x', x_3), && \text{for } x \in \Sigma^a; \\
&h^{b\varepsilon}_{+\,\alpha}(x') = h^{b\varepsilon}_{+\,3}(x') = 0, && && \text{for } x' \in r^\varepsilon \omega^a, \\
&h^{b\varepsilon}_{+\,\alpha}(x') = \lambda^\varepsilon \tfrac{\varepsilon}{(r^\varepsilon)^2} H_\alpha^\varepsilon(x', 0), && h^{b\varepsilon}_{+\,3}(x') = \lambda^\varepsilon \tfrac{1}{(r^\varepsilon)^2} H_3^\varepsilon(x', 0), && \text{for } x' \in \omega^b \setminus r^\varepsilon \omega^a, \\
&h^{b\varepsilon}_{-\,\alpha}(x') = \lambda^\varepsilon \tfrac{\varepsilon}{(r^\varepsilon)^2} H_\alpha^\varepsilon(x', -\varepsilon), && h^{b\varepsilon}_{-\,3}(x') = \lambda^\varepsilon \tfrac{1}{(r^\varepsilon)^2} H_3^\varepsilon(x', -\varepsilon), && \text{for } x' \in \omega^b.
\end{aligned} \tag{3.40}$$

(Note that $h_+^{b\varepsilon} = 0$ on $r^\varepsilon \omega^a$, since there is no contribution of H^ε on J^ε.)

On the other hand, for any $U \in Y^\varepsilon$, we define the rescaled function $u = (u^a, u^b)$ by:

$$u_\alpha^a(x) = \lambda^\varepsilon r^\varepsilon \, U_\alpha(r^\varepsilon x', x_3), \ \ u_3^a(x) = \lambda^\varepsilon \, U_3(r^\varepsilon x', x_3), \ \text{for } x \in \Omega^a, \tag{3.41}$$

$$u_\alpha^b(x) = \lambda^\varepsilon \frac{1}{\varepsilon} \, U_\alpha(x', \varepsilon x_3), \ \ u_3^b(x) = \lambda^\varepsilon \, U_3(x', \varepsilon x_3), \ \text{for } x \in \Omega^b. \tag{3.42}$$

Remark 4 *Let us remark that the rescaled souce terms are bounded, but not strongly converging to zero, since, by definition of λ^ε (see (3.37)) and by (3.38) to (3.40),*

$$\begin{aligned}
\|f^{a\varepsilon}\|^2_{(L^2(\Omega^a))^3} + \|f^{b\varepsilon}\|^2_{(L^2(\Omega^b))^3} + \|g^{a\varepsilon}\|^2_{(L^2(\Omega^a))^{3\times 3}} + \|g^{b\varepsilon}\|^2_{(L^2(\Omega^b))^{3\times 3}} + \\
+ \|h^{a\varepsilon}\|^2_{(L^2(\Sigma^a))^3} + \|h_+^{b\varepsilon}\|^2_{(L^2(\omega^b))^3} + \|h_-^{b\varepsilon}\|^2_{(L^2(\omega^b))^3} = 1.
\end{aligned}$$

3.2 The derivation of the scaling

Let us consider the possible scalings for the solution $\overline{U}^\varepsilon$ and test function U. If, instead of a multidomain, one considers a single thin cylinder, the natural scaling is (see [17], [21], [22]):

$$u_\alpha(x) = r^\varepsilon\, U_\alpha(r^\varepsilon x', x_3), \ \ u_3(x) = U_3(r^\varepsilon x', x_3), \text{ for } x \in \Omega^a,$$

and for a single plate, it is (see [5], [17])

$$u_\alpha(x) = U_\alpha(x', \varepsilon x_3), \ \ u_3(x) = \varepsilon\, U_3(x', \varepsilon x_3), \text{ for } x \in \Omega^b.$$

For the multidomain made of the union of the beam and the plate, the idea is to consider different coefficients of normalization, $\lambda^{a\varepsilon}$ and $\lambda^{b\varepsilon}$, for $\Omega^{a\varepsilon}$ and $\Omega^{b\varepsilon}$ respectively, that is we set

$$\begin{array}{lll} u^a_\alpha(x) = \lambda^{a\varepsilon} r^\varepsilon\, U_\alpha(r^\varepsilon x', x_3), & u^a_3(x) = \lambda^{a\varepsilon}\, U_3(r^\varepsilon x', x_3), & \text{for } x \in \Omega^a, \\ u^b_\alpha(x) = \lambda^{b\varepsilon}\, U_\alpha(x', \varepsilon x_3), & u^b_3(x) = \lambda^{b\varepsilon} \varepsilon\, U_3(x', \varepsilon x_3), & \text{for } x \in \Omega^b. \end{array}$$

Then one has, with $e^{a\varepsilon}$, $e^{b\varepsilon}$ defined in (2.4),

$$e(U)(r^\varepsilon x', x_3) = \frac{1}{\lambda^{a\varepsilon}} e^{a\varepsilon}(u^{a\varepsilon})(x) \text{ for } x \in \Omega^a \text{ and } e(U)(x', \varepsilon x_3) = \frac{1}{\lambda^{b\varepsilon}} e^{b\varepsilon}(u^{b\varepsilon})(x) \text{ for } x \in \Omega^b$$

and it is easy to check that the variational equality in (1.2) reads, once each integral is written on the corresponding fixed domain,

$$\begin{aligned}
&\tfrac{(r^\varepsilon)^2}{(\lambda^{a\varepsilon})^2} \int_{\Omega^a} [A^a e^{a\varepsilon}(\overline{u}^{a\varepsilon}), e^{a\varepsilon}(u^a)]\, dx + k^\varepsilon \tfrac{\varepsilon}{(\lambda^{b\varepsilon})^2} \int_{\Omega^b} [A^b e^{b\varepsilon}(\overline{u}^{b\varepsilon}), e^{b\varepsilon}(u^b)]\, dx = \\
&= \tfrac{1}{\lambda^{a\varepsilon}} \left(\textstyle\sum_{\alpha=1}^2 \int_{\Omega^a} r^\varepsilon F^\varepsilon_\alpha(r^\varepsilon x', x_3) u^a_\alpha(x)\, dx + \int_{\Omega^a} (r^\varepsilon)^2 F^\varepsilon_3(r^\varepsilon x', x_3) u^a_3(x)\, dx \right) + \\
&+ \tfrac{1}{\lambda^{b\varepsilon}} \left(\textstyle\sum_{\alpha=1}^2 \int_{\Omega^b} \varepsilon F^\varepsilon_\alpha(x', \varepsilon x_3) u^b_\alpha(x)\, dx + \int_{\Omega^b} F^\varepsilon_3(x', \varepsilon x_3) u^b_3(x)\, dx \right) + \\
&+ \tfrac{(r^\varepsilon)^2}{\lambda^{a\varepsilon}} \int_{\Omega^a} [G^\varepsilon(r^\varepsilon x', x_3), e^{a\varepsilon}(u^a)]\, dx + \tfrac{\varepsilon}{\lambda^{b\varepsilon}} \int_{\Omega^b} [G^\varepsilon(x', \varepsilon x_3), e^{b\varepsilon}(u^b)]\, dx + \\
&+ \tfrac{1}{\lambda^{a\varepsilon}} \left(\textstyle\sum_{\alpha=1}^2 \int_{\Sigma^a} H^\varepsilon_\alpha(r^\varepsilon x', x_3) u^a_\alpha(x)\, d\sigma + \int_{\Sigma^a} r^\varepsilon H^\varepsilon_3(r^\varepsilon x', x_3) u^a_3(x)\, d\sigma \right) + \\
&+ \tfrac{1}{\lambda^{b\varepsilon}} \left(\textstyle\sum_{\alpha=1}^2 \int_{\omega^b \setminus r^\varepsilon \omega^a} H^\varepsilon_\alpha(x', 0) u^b_\alpha(x', 0)\, dx' + \int_{\omega^b \setminus r^\varepsilon \omega^a} \tfrac{1}{\varepsilon} H^\varepsilon_3(x', 0) u^b_3(x', 0)\, dx' \right) + \\
&+ \tfrac{1}{\lambda^{b\varepsilon}} \left(\textstyle\sum_{\alpha=1}^2 \int_{\omega^b} H^\varepsilon_\alpha(x', -\varepsilon) u^b_\alpha(x', -1)\, dx' + \int_{\omega^b} \tfrac{1}{\varepsilon} H^\varepsilon_3(x', -\varepsilon) u^b_3(x', -1)\, dx' \right).
\end{aligned} \tag{3.43}$$

We decide to choose $\lambda^{a\varepsilon} = \varepsilon \lambda^{b\varepsilon}$, so that the junction condition written for (u^a, u^b) reads: for almost every x' in ω^a,

$$u^a_\alpha(x', 0) = \frac{\lambda^{a\varepsilon}}{\lambda^{b\varepsilon}} r^\varepsilon u^b_\alpha(r^\varepsilon x', 0) = \varepsilon r^\varepsilon u^b_\alpha(r^\varepsilon x', 0) \text{ and } u^a_3(x', 0) = \frac{\lambda^{a\varepsilon}}{\lambda^{b\varepsilon}} \frac{1}{\varepsilon} u^b_3(r^\varepsilon x', 0) = u^b_3(r^\varepsilon x', 0)$$

(see also the begining of Section 3). Then, after dividing by $(r^\varepsilon)^2/(\lambda^{a\varepsilon})^2$, writing λ^ε instead of $\lambda^{a\varepsilon}$, for simplicity, and defining the rescaled source terms by (3.38), (3.39), (3.40), the equality (3.43) is exactly the variational equality in (2.5). Finally, we recall that the particular choice of λ^ε given in (3.37) makes the source terms bounded, but not strongly converging to zero (see also Remark 4).

Remark 5 *Since the first member of (3.43) is another way of writing $\int_{\Omega^\varepsilon}[Ae(\overline{U}^\varepsilon), e(U)]\,dx$, it follows that*

$$\left(\frac{r^\varepsilon}{\lambda^\varepsilon}\right)^2 \left(\int_{\Omega^a}[A^a e^{a\varepsilon}(\overline{u}^{a\varepsilon}), e^{a\varepsilon}(\overline{u}^{a\varepsilon})]\,dx + q^\varepsilon \int_{\Omega^b}[A^b e^{b\varepsilon}(\overline{u}^{b\varepsilon}), e^{b\varepsilon}(\overline{u}^{b\varepsilon})]\,dx\right) = \\ = \int_{\Omega^\varepsilon}[Ae(\overline{U}^\varepsilon), e(\overline{U}^\varepsilon)]\,dx, \tag{3.44}$$

which gives the definition of the renormalized energy in (2.28). In [13], we took $\lambda^\varepsilon = r^\varepsilon$, since the initial problem (1.2) was supposed to be suitably normalized, a priori.

4 The *a priori* estimates and the compactness arguments

4.1 *A priori* estimates:

In the following, we denote by C any positive constant which does not depend on ε and **we write** $\overline{e}^{a\varepsilon}$ **(resp.** $\overline{e}^{b\varepsilon}$**) for** $e^{a\varepsilon}(\overline{u}^{a\varepsilon})$ **(resp.** $e^{b\varepsilon}(\overline{u}^{b\varepsilon})$**).** Taking $u = \overline{u}^\varepsilon = (\overline{u}^{a\varepsilon}, \overline{u}^{b\varepsilon})$ as test function in (2.5), we get

$$\int_{\Omega^a}[A^a\overline{e}^{a\varepsilon}, \overline{e}^{a\varepsilon}]\,dx + q^\varepsilon \int_{\Omega^b}[A^b\overline{e}^{b\varepsilon}, \overline{e}^{b\varepsilon}]\,dx = \\ = \int_{\Omega^a} f^{a\varepsilon}.\overline{u}^{a\varepsilon}\,dx + \int_{\Omega^b} f^{b\varepsilon}.\overline{u}^{b\varepsilon}\,dx + \int_{\Omega^a}[g^{a\varepsilon}, \overline{e}^{a\varepsilon}]\,dx + \int_{\Omega^b}[g^{b\varepsilon}, \overline{e}^{b\varepsilon}]\,dx + \\ \int_{\Sigma^a} h^{a\varepsilon}.\overline{u}^{a\varepsilon}\,d\sigma + \int_{\omega^b}\left(h_+^{b\varepsilon}.\overline{u}^{b\varepsilon}_{|x_3=0} + h_-^{b\varepsilon}.\overline{u}^{b\varepsilon}_{|x_3=-1}\right)\,dx'. \tag{4.45}$$

¿From the Korn inequality, since $\overline{u}^{a\varepsilon}$ vanishes on T^a and $\overline{u}^{b\varepsilon}$ vanishes on Σ^b, we get for $\varepsilon \leq 1$ and $r^\varepsilon \leq 1$,

$$\|\overline{u}^{a\varepsilon}\|_{(H^1(\Omega^a))^3} \leq C\|e(\overline{u}^{a\varepsilon})\|_{(L^2(\Omega^a))^{3\times 3}} \leq C\|\overline{e}^{a\varepsilon}\|_{(L^2(\Omega^a))^{3\times 3}},$$

$$\|\overline{u}^{b\varepsilon}\|_{(H^1(\Omega^b))^3} \leq C\|e(\overline{u}^{b\varepsilon})\|_{(L^2(\Omega^b))^{3\times 3}} \leq C\|\overline{e}^{b\varepsilon}\|_{(L^2(\Omega^b))^{3\times 3}},$$

and, by continuity of the trace mapping,

$$\|\overline{u}^{a\varepsilon}\|_{(L^2(\Sigma^a))^3} \leq C\|\overline{u}^{a\varepsilon}\|_{(H^1(\Omega^a))^3},$$

$$\|\overline{u}^{b\varepsilon}_{|x_3=0}\|_{(L^2(\omega^b))^3} \text{ and } \|\overline{u}^{b\varepsilon}_{|x_3=-1}\|_{(L^2(\omega^b))^3} \leq C\|\overline{u}^{a\varepsilon}\|_{(H^1(\Omega^b))^3}.$$

By using the above inequalities, the coercivity of A^a and A^b and the boundedness of the source terms (see (2.9) to (2.14) and Remark 4), it follows from (4.45) that

$$C\|\overline{e}^{a\varepsilon}\|^2_{(L^2(\Omega^a))^{3\times3}} + Cq^\varepsilon\|\overline{e}^{b\varepsilon}\|^2_{(L^2(\Omega^b))^{3\times3}} \le$$

$$\le \left(\|f^{a\varepsilon}\|_{(L^2(\Omega^a))^3} + \|g^{a\varepsilon}\|_{(L^2(\Omega^a))^{3\times3}} + \|h^{a\varepsilon}\|_{(L^2(\Sigma^a))^3}\right)\|\overline{e}^{a\varepsilon}\|_{(L^2(\Omega^a))^{3\times3}} +$$

$$+ \left(\|f^{b\varepsilon}\|_{(L^2(\Omega^b))^3} + \|g^{b\varepsilon}\|_{(L^2(\Omega^a))^{3\times3}} + \|h^{b\varepsilon}_+\|_{(L^2(\omega^b))^3} + \|h^{b\varepsilon}_-\|_{(L^2(\omega^b))^3}\right)\|\overline{e}^{b\varepsilon}\|_{(L^2(\Omega^b))^{3\times3}} \le$$

$$\le C\left(\|\overline{e}^{a\varepsilon}\|_{(L^2(\Omega^a))^{3\times3}} + \|\overline{e}^{b\varepsilon}\|_{(L^2(\Omega^b))^{3\times3}}\right).$$

• If q^ε is bounded below by some positive constant, that is if q defined in (2.7) equals some positive number or $+\infty$, it follows that $\overline{e}^{a\varepsilon}$ is bounded in $(L^2(\Omega^a))^{3\times3}$ and $\overline{e}^{b\varepsilon}$ is bounded in $(L^2(\Omega^b))^{3\times3}$. Then, from Korn's inequality, it results that $\overline{u}^{a\varepsilon}$ is bounded in $(H^1(\Omega^a))^3$ and $\overline{u}^{b\varepsilon}$ is bounded in $(H^1(\Omega^b))^3$. Moreover, in the particular case $q = +\infty$, $\overline{e}^{b\varepsilon}$ tends to zero (strongly) in $(L^2(\Omega^b))^{3\times3}$ and $\overline{u}^{b\varepsilon}$tends to zero (strongly) in $(H^1(\Omega^b))^3$.

• Otherwise, i.e. if q^ε tends to zero, we define $\tilde{u}^\varepsilon$ by

$$\tilde{u}^\varepsilon = (\tilde{u}^{a\varepsilon}, \tilde{u}^{b\varepsilon}) = q^\varepsilon\overline{u}^\varepsilon = q^\varepsilon(\overline{u}^{a\varepsilon}, \overline{u}^{b\varepsilon}). \tag{4.46}$$

It is clear that $\tilde{u}^\varepsilon$ solves

$$\begin{aligned}
&\tilde{u}^\varepsilon \in \mathcal{Y}^\varepsilon \text{ and } \forall u \in \mathcal{Y}^\varepsilon,\\
&\quad \frac{1}{q^\varepsilon}\int_{\Omega^a}[A^a e^{a\varepsilon}(\tilde{u}^{a\varepsilon}), e^{a\varepsilon}(u^a)]\,dx + \int_{\Omega^b}[A^b e^{b\varepsilon}(\tilde{u}^{b\varepsilon}), e^{b\varepsilon}(u^b)]\,dx =\\
&\quad = \int_{\Omega^a} f^{a\varepsilon}.u^a\,dx + \int_{\Omega^b} f^{b\varepsilon}.u^b\,dx + \int_{\Omega^a}[g^{a\varepsilon}, e^{a\varepsilon}(u^a)]\,dx + \int_{\Omega^b}[g^{b\varepsilon}, e^{b\varepsilon}(u^b)]\,dx +\\
&\quad\quad \int_{\Sigma^a} h^{a\varepsilon}.u^a\,d\sigma + \int_{\omega^b}\left(h^{b\varepsilon}_+.u^b_{|x_3=0} + h^{b\varepsilon}_-.u^b_{|x_3=-1}\right)dx'.
\end{aligned} \tag{4.47}$$

Taking $u = \tilde{u}^\varepsilon$ as test function in (4.47), it is easy to prove (as we have done in the case $q^\varepsilon \ge C > 0$) that $\tilde{e}^{a\varepsilon} := e^{a\varepsilon}(\tilde{u}^{a\varepsilon}) = q^\varepsilon\overline{e}^{a\varepsilon}$ tends to zero in $(L^2(\Omega^a))^{3\times3}$, $\tilde{e}^{b\varepsilon} := e^{b\varepsilon}(\tilde{u}^{b\varepsilon}) = q^\varepsilon\overline{e}^{b\varepsilon}$ is bounded in $(L^2(\Omega^b))^{3\times3}$, $\tilde{u}^{a\varepsilon} = q^\varepsilon\overline{u}^{a\varepsilon}$ tends to zero in $(H^1(\Omega^a))^3$ and $\tilde{u}^{b\varepsilon} = q^\varepsilon\overline{u}^{b\varepsilon}$ is bounded in $(H^1(\Omega^b))^3$.

4.2 Compactness arguments:

• If $q^\varepsilon \to q \in (0, +\infty]$, it results from the a priori estimates that there exist $\overline{u} = (\overline{u}^a, \overline{u}^b)$ in $(H^1(\Omega^a))^3 \times (H^1(\Omega^b))^3$ and $\overline{e} = (\overline{e}^a, \overline{e}^b)$ in $(L^2(\Omega^a))^{3\times3} \times (L^2(\Omega^b))^{3\times3}$, such that

$$\overline{u}^\varepsilon = (\overline{u}^{a\varepsilon}, \overline{u}^{b\varepsilon}) \rightharpoonup \overline{u} = (\overline{u}^a, \overline{u}^b) \text{ weakly in } (H^1(\Omega^a))^3 \times (H^1(\Omega^b))^3, \tag{4.48}$$

$$\overline{e}^\varepsilon = (\overline{e}^{a\varepsilon}, \overline{e}^{b\varepsilon}) \rightharpoonup \overline{e} = (\overline{e}^a, \overline{e}^b) \text{ weakly in } (L^2(\Omega^a))^{3\times3} \times (L^2(\Omega^b))^{3\times3}. \tag{4.49}$$

Clearly $\overline{u}^a = 0$ on T^a, $\overline{u}^b = 0$ on Σ^b and $\overline{e}^a$, $\overline{e}^b$ are symmetric matrices. Moreover, from the boundedness of $\overline{e}^\varepsilon = (\overline{e}^{a\varepsilon}, \overline{e}^{b\varepsilon})$ and a classical semicontinuity argument, we get that $\overline{u}^a$ is a Bernouilli-Navier displacement and $\overline{u}^b$ is a Kirchhoff-Love displacement:

$$e_{\alpha\beta}(\overline{u}^a) = 0 \text{ and } e_{\alpha 3}(\overline{u}^a) = 0, \quad e_{\alpha 3}(\overline{u}^b) = 0 \text{ and } e_{33}(\overline{u}^b) = 0,$$

which, combined with the constraints $\overline{u}^a = 0$ on T^a, $\overline{u}^b = 0$ on Σ^b, is equivalent to (see [16]):

$$\begin{gathered}\overline{u}^a \in (H^2(0,1))^2 \times H^1(\Omega^a), \overline{u}^a_\alpha(1) = \frac{d\overline{u}^a_\alpha}{dx_3}(1) = 0,\\ \exists \overline{\zeta}^a \in H^1(0,1), \overline{\zeta}^a(1) = 0, \overline{u}^a_3 = \overline{\zeta}^a - x_1 \frac{d\overline{u}^a_1}{dx_3} - x_2 \frac{d\overline{u}^a_2}{dx_3},\\ \overline{u}^b \in \mathcal{U}^b.\end{gathered}$$

Moreover one can prove as in [22] that there exist $(\overline{v}^a, \overline{w}^a)$ and $(\overline{v}^b, \overline{w}^b)$ such that $\overline{e}^a = e^a(\overline{u}^a, \overline{v}^a, \overline{w}^a)$ and $\overline{e}^b = e^b(\overline{u}^b, \overline{v}^b, \overline{w}^b)$ (see the definitions of e^a and e^b in (2.8)) and such that

$$\begin{aligned}\overline{v}^a \in\ & (H^1(\Omega^a))^2 \times L^2(0,1; H^1(\omega^a)),\ \exists \overline{c} \in H^1(0,1), \overline{c}(1) = 0, \overline{v}^a_1 = -\overline{c}\, x_2, \overline{v}^a_2 = \overline{c}\, x_1,\\ & \text{for a.e. } x_3 \in (0,1),\ \textstyle\int_{\omega^a} \overline{v}^a_3(x', x_3)\, dx' = 0,\end{aligned}$$

$$\overline{w}^a \in \mathcal{W}^a, \qquad \overline{v}^b \in \mathcal{V}^b, \qquad \overline{w}^b \in \mathcal{W}^b$$

and suitable expressions of $\overline{u}^{a\varepsilon}$ (resp. $\overline{u}^{b\varepsilon}$) tend to $(\overline{v}^a, \overline{w}^a)$ (resp. $(\overline{v}^b, \overline{w}^b)$). For convenience of the reader, the proof of this fact is given in the Appendix (see Section 8.1). In particular, $\overline{v}^a$ defines some $\overline{c} \in H^1(0,1)$ with $\overline{c}(1) = 0$, which is actually the limit in $L^2(0,1)$ of $\overline{c}^\varepsilon$ given by

$$\overline{c}^\varepsilon(x_3) = \frac{\int_{\omega^a} (x_1 \overline{u}^{a\varepsilon}_2(x', x_3) - x_2 \overline{u}^{a\varepsilon}_1(x', x_3))\ dx'}{r^\varepsilon \int_{\omega^a} (x_1^2 + x_2^2)\ dx'}. \tag{4.50}$$

To summarize, we have proved (2.15), (2.16).

In the particular case $q = +\infty$, we have already noticed (see the *a priori* estimates) that

$$\overline{u}^{b\varepsilon} \to \overline{u}^b = 0 \text{ strongly in } (H^1(\Omega^b))^3 \text{ and } \overline{e}^{b\varepsilon} \to \overline{e}^b = 0 \text{ strongly in } (L^2(\Omega^b))^{3\times 3}, \tag{4.51}$$

that is we have proved (2.18), (2.19).

- If $q^\varepsilon \to 0$, it results from the *a priori* estimates that

$$\begin{gathered} q^\varepsilon\, \overline{u}^{a\varepsilon} \to 0 \text{ strongly in } (H^1(\Omega^a))^3,\ q^\varepsilon\, \overline{u}^{b\varepsilon} \rightharpoonup \overline{u}^b \text{ weakly in } (H^1(\Omega^b))^3,\\ q^\varepsilon \overline{e}^{a\varepsilon} \to 0 \text{ strongly in } (L^2(\Omega^a))^{3\times 3},\ q^\varepsilon \overline{e}^{b\varepsilon} \rightharpoonup \overline{e}^b \text{ weakly in } (L^2(\Omega^b))^{3\times 3},\end{gathered} \tag{4.52}$$

for some $\overline{u}^b \in \mathcal{U}^b$ and some symmetric matrix $\overline{e}^b \in (L^2(\Omega^b))^{3\times 3}$. Again (see the Appendix, Section 8.1), there exists $(\overline{v}^b, \overline{w}^b)$ in $\mathcal{V}^b \times \mathcal{W}^b$, which are limits of suitable expressions of $\overline{u}^{b\varepsilon}$ and such that $\overline{e}^b = e^b(\overline{u}^b, \overline{v}^b, \overline{w}^b) = e^b(\overline{z}^b)$. In other words, we have proved (2.23), (2.24).

5 The limit constraints that are due to the junction

As for the limit constraints, it remains to prove that

1) $\overline{u}^a_\alpha(0) = 0$,
2) $\overline{u}^a_3(x', 0) \equiv \overline{u}^b_3(0)$, which is equivalent to $\overline{\zeta}^a(0) = \overline{u}^b_3(0)$ and $\frac{d\overline{u}^a_\alpha}{dx_3}(0) = 0$,
3) $\overline{c}(0) = 0$,

since the above three conditions give $\overline{u}^a_\alpha \in (H^2_0(0,1))^2$ and $\overline{c} \in H^1_0(0,1)$, so that $\overline{u}^a \in \mathcal{U}^a$ and $\overline{v}^a \in \mathcal{V}^a$. These limit constraints are derived below.

5.1 Proof of $\overline{u}^a_\alpha(0) = 0$

The fact that $\overline{u}^a_\alpha(0) = 0$ results from the following easy Lemma.

Lemma 1 *Assume that $\{u^{b\varepsilon}\}_\varepsilon$ is bounded in $L^2(\omega^b)$. Then $\{r^\varepsilon u^{b\varepsilon}(r^\varepsilon .)\}_\varepsilon$ is bounded in $L^2(\omega^a)$, for every ω^a such that $r^\varepsilon \omega^a \subset \omega^b$, for any ε.*

Proof: We have

$$\int_{\omega^a} |r^\varepsilon u^{b\varepsilon}(r^\varepsilon x')|^2 \, dx' = \int_{r^\varepsilon \omega^a} |u^{b\varepsilon}(x')|^2 \, dx' \leq \int_{\omega^b} |u^{b\varepsilon}(x')|^2 \, dx' \leq C.$$

Application: If $q \neq 0$, we write the junction condition for $\overline{u}^\varepsilon_\alpha$:

$$\text{for a.e. } x' \in \omega^a, \ \overline{u}^{a\varepsilon}_\alpha(x', 0) = \varepsilon r^\varepsilon \overline{u}^{b\varepsilon}_\alpha(r^\varepsilon x', 0).$$

The first member tends to $\overline{u}^a_\alpha(x', 0) = \overline{u}^a_\alpha(0)$ in $L^2(\omega^a)$. The second member tends to zero in this space, by Lemma 1, since $\overline{u}^{b\varepsilon}_\alpha(., 0)$ is bounded in $L^2(\omega^b)$, so that $\overline{u}^a_\alpha(0) = 0$. If $q = 0$, the same proof applies to $\tilde{u}^\varepsilon = q^\varepsilon \overline{u}^\varepsilon$.

5.2 Proof of $\overline{u}^a_3(x', 0) \equiv \overline{u}^b_3(0)$

This is a crucial part of this paper. It is derived from the following general Lemma.

Lemma 2 *Assume that ε and r^ε tend to zero, with $0 < \varepsilon^2 \ll r^\varepsilon$. Let $u^{b\varepsilon} \in (H^1(\Omega^b))^3$ be such that*

$$u^{b\varepsilon} = 0 \text{ on } \Sigma^b,$$

$$\{e^{b\varepsilon}(u^{b\varepsilon})\}_\varepsilon \text{ is bounded in } (L^2(\Omega^b))^{3\times 3}, \tag{5.53}$$

with $e^{b\varepsilon}$ defined in (2.4). Then, up to a subsequence,

$$u^{b\varepsilon} \rightharpoonup u^b \text{ weakly in } (H^1(\Omega^b))^3, \tag{5.54}$$

for some $u^b \in \mathcal{U}^b$ (in particular $u^b_3 \in H^2_0(\omega^b)$). Moreover $u^{b\varepsilon}_3(r^\varepsilon., 0)$ tends to $u^b_3(0)$ strongly in $L^2(\omega^a)$, for every ω^a such that $r^\varepsilon \omega^a \subset \omega^b$, for any ε.

Proof: The first part of the lemma is classical (see [5]). Let us prove the convergence of $u_3^{b\varepsilon}(r^\varepsilon.,0)$. We define $U^\varepsilon : \omega^b \to \mathbf{R}$ by

$$\begin{aligned} U^\varepsilon(x') &= k \int_{-1}^0 \int_{-1}^0 \int_{t<x_3<t'} u_3^{b\varepsilon}(x',x_3)\, dx_3\, dt\, dt' \\ &= k \int_{-1}^0 \int_{-1}^0 \rho(t,t') \int_t^{t'} u_3^{b\varepsilon}(x',x_3)\, dx_3\, dt\, dt', \end{aligned} \tag{5.55}$$

with $\rho(t,t') = 1$ if $t < t'$, 0 otherwise, and with k chosen so that

$$k \int_{-1}^0 \int_{-1}^0 \rho(t,t')(t'-t)\, dt\, dt' = 1. \tag{5.56}$$

Moreover we define $e_\alpha^\varepsilon : \Omega^b \to \mathbf{R}$, E_α^ε and $D_\alpha^\varepsilon : \omega^b \to \mathbf{R}$ by

$$e_\alpha^\varepsilon = 2\, e_{\alpha 3}(u^{b\varepsilon}) = \frac{\partial u_\alpha^{b\varepsilon}}{\partial x_3} + \frac{\partial u_3^{b\varepsilon}}{\partial x_\alpha}, \tag{5.57}$$

$$E_\alpha^\varepsilon(x') = k \int_{-1}^0 \int_{-1}^0 \rho(t,t') \int_t^{t'} e_\alpha^\varepsilon(x',x_3)\, dx_3\, dt\, dt', \tag{5.58}$$

$$\begin{aligned} D_\alpha^\varepsilon(x') &= k \int_{-1}^0 \int_{-1}^0 \rho(t,t') \int_t^{t'} \frac{\partial u_\alpha^{b\varepsilon}}{\partial x_3}(x',x_3)\, dx_3\, dt\, dt' \\ &= k \int_{-1}^0 \int_{-1}^0 \rho(t,t') \left(u_\alpha^{b\varepsilon}(x',t') - u_\alpha^{b\varepsilon}(x',t) \right) dt\, dt'. \end{aligned} \tag{5.59}$$

It is clear that

$$\nabla U^\varepsilon = E^\varepsilon - D^\varepsilon. \tag{5.60}$$

Still denoting by C various constants that do not depend on ε, we have from Cauchy-Schwarz inequality:

$$|E_\alpha^\varepsilon(x')|^2 \le C \int_{-1}^0 |e_\alpha^\varepsilon(x',x_3)|^2\, dx_3,$$

which gives, by definition of e_α^ε and by (5.53),

$$\|E_\alpha^\varepsilon\|_{L^2(\omega^b)} \le C \|e_\alpha^\varepsilon\|_{L^2(\Omega^b)} = C \|e_{\alpha 3}(u^{b\varepsilon})\|_{L^2(\Omega^b)} \le C\varepsilon. \tag{5.61}$$

¿From (5.59), Cauchy-Schwarz inequality and the boundedness of $u_\alpha^{b\varepsilon}$ in $H_0^1(\Omega^b)$, we have

$$\|D_\alpha^\varepsilon\|_{H_0^1(\omega^b)} \le C \|u_\alpha^{b\varepsilon}\|_{H_0^1(\Omega^b)} \le C. \tag{5.62}$$

¿From (5.60), we get the following decomposition:

$$U^\varepsilon = \hat{U}^\varepsilon + \tilde{U}^\varepsilon,$$

with $\hat{U}^\varepsilon$, $\tilde{U}^\varepsilon$ the respective solutions in $H_0^1(\omega^b)$ of

$$-\Delta \hat{U}^\varepsilon = -div E^\varepsilon \quad \text{and} \quad -\Delta \tilde{U}^\varepsilon = div D^\varepsilon \quad \text{in } \omega^b,$$

and from (5.61), (5.62),

$$\|\nabla\hat{U}^\varepsilon\|_{(L^2(\omega^b))^2} \leq \|E^\varepsilon\|_{(L^2(\omega^b))^2} \leq C\varepsilon, \tag{5.63}$$

$$\hat{U}^\varepsilon \to 0 \text{ in } H^1_0(\omega^b), \tag{5.64}$$

$$\|\tilde{U}^\varepsilon\|_{H^2(\omega^b)} \leq C\|divD^\varepsilon\|_{L^2(\omega^b)} \leq C. \tag{5.65}$$

But, using (5.54) and (5.56), it is easy to prove that

$$U^\varepsilon \rightharpoonup u_3^b = u_3^b(x') \text{ weakly in } L^2(\omega^b),$$

which gives, by virtue of (5.64), (5.65),

$$\tilde{U}^\varepsilon = U^\varepsilon - \hat{U}^\varepsilon \rightharpoonup u_3^b \text{ weakly in } H^2(\omega^b).$$

Then, as the embedding $H^2(\omega^b) \subset \mathcal{C}^0(\overline{\omega}^b)$ is compact, for ω^b bidimensional, we get that

$$\tilde{U}^\varepsilon \to u_3^b \text{ in } \mathcal{C}^0(\overline{\omega}^b). \tag{5.66}$$

This is enough to prove that $u_3^{b\varepsilon}(r^\varepsilon.,0)$ tends to $u_3^b(0)$ strongly in $L^2(\omega^a)$.

Actually we have, for a.e. x' in ω^a,

$$\begin{aligned} u_3^{b\varepsilon}(r^\varepsilon x',0) - u_3^b(0) &= \left[u_3^{b\varepsilon}(r^\varepsilon x',0) - U^\varepsilon(r^\varepsilon x')\right] + \\ &+ \left[U^\varepsilon(r^\varepsilon x') - \tilde{U}^\varepsilon(r^\varepsilon x')\right] + \\ &+ \left[\tilde{U}^\varepsilon(r^\varepsilon x') - u_3^b(r^\varepsilon x')\right] + \\ &+ \left[u_3^b(r^\varepsilon x') - u_3^b(0)\right]. \end{aligned} \tag{5.67}$$

We are going to show that each of the above brackets tends to zero, strongly in $L^2(\omega^a)$.

As for the first bracket, we have:

$$\int_{\omega^a} |u_3^{b\varepsilon}(r^\varepsilon x',0) - U^\varepsilon(r^\varepsilon x')|^2\, dx' = \frac{1}{(r^\varepsilon)^2}\int_{r^\varepsilon\omega^a} |u_3^{b\varepsilon}(x',0) - U^\varepsilon(x')|^2\, dx'. \tag{5.68}$$

But, by using (5.56),

$$\begin{aligned} U^\varepsilon(x') &= k\int_{-1}^0\int_{-1}^0 \rho(t,t')\int_t^{t'}\left(u_3^{b\varepsilon}(x',0) + \int_0^{x_3}\frac{\partial u_3^{b\varepsilon}}{\partial x_3}(x',y_3)\,dy_3\right)\,dx_3\,dt\,dt' = \\ &= u_3^{b\varepsilon}(x',0) + k\int_{-1}^0\int_{-1}^0 \rho(t,t')\int_t^{t'}\int_0^{x_3}\frac{\partial u_3^{b\varepsilon}}{\partial x_3}(x',y_3)\,dy_3\,dx_3\,dt\,dt', \end{aligned}$$

so that

$$|U^\varepsilon(x') - u_3^{b\varepsilon}(x',0)| \leq C\int_{-1}^0\left|\frac{\partial u_3^{b\varepsilon}}{\partial x_3}(x',y_3)\right|\,dy_3,$$

and this gives with (5.53)

$$\int_{r^\varepsilon\omega^a} |u_3^{b\varepsilon}(x',0) - U^\varepsilon(x')|^2\, dx' \leq C\int_{\Omega^b}\left|\frac{\partial u_3^{b\varepsilon}}{\partial x_3}\right|^2\,dx \leq C\varepsilon^4.$$

Coming back to (5.68), it results that

$$\int_{\omega^a} |u_3^{b\varepsilon}(r^\varepsilon x',0) - U^\varepsilon(r^\varepsilon x')|^2\,dx' \le C\frac{\varepsilon^4}{(r^\varepsilon)^2},$$

which tends to zero, since we have assumed that $\varepsilon^2 \ll r^\varepsilon$. Now we consider the second bracket in (5.67), that is $\hat U^\varepsilon(r^\varepsilon x')$, and we are going to prove that its L^2-norm tends to zero, again if $\varepsilon^2 \ll r^\varepsilon$. In fact, from Cauchy-Schwarz inequality, the continuity of the imbedding $H_0^1(\omega^b) \subset L^4(\omega^b)$ (actually $L^q(\omega^b)$, for every finite q, in dimension 2) and from (5.63),

$$\begin{aligned}
\int_{\omega^a} |\hat U^\varepsilon(r^\varepsilon x')|^2\,dx' &= \tfrac{1}{(r^\varepsilon)^2}\int_{r^\varepsilon\omega^a} |\hat U^\varepsilon(x')|^2\,dx' \\
&\le \tfrac{1}{(r^\varepsilon)^2}\left(\int_{r^\varepsilon\omega^a} |\hat U^\varepsilon(x')|^4\,dx'\right)^{\frac12} |r^\varepsilon\omega^a|^{\frac12} \\
&\le C(r^\varepsilon)^{-1}\|\hat U^\varepsilon\|^2_{L^4(\omega^b)} \\
&\le C(r^\varepsilon)^{-1}\|\hat U^\varepsilon\|^2_{H_0^1(\omega^b)} \\
&\le C\tfrac{(\varepsilon)^2}{r^\varepsilon}.
\end{aligned}$$

The third bracket in (5.67) tends to zero with ε, in $L^\infty-$norm, and also the fourth one tends to zero, by virtue of (5.66). This concludes the proof of Lemma 2.

Application: If $q \neq 0$, we write the junction condition for $\overline{u}_3^\varepsilon$:

$$\text{for a.e. } x' \in \omega^a,\ \overline{u}_3^{a\varepsilon}(x',0) = \overline{u}_3^{b\varepsilon}(r^\varepsilon x',0).$$

The first member tends to $\overline{u}_3^a(x',0)$ in $L^2(\omega^a)$. The second member tends to $\overline{u}_3^b(0)$ in this space, by Lemma 2. It follows that, for a.e. x' in ω^a, $\overline{u}_3^a(x',0) = \overline{u}_3^b(0)$. If $q = 0$, the same proof applies to $\tilde u^\varepsilon = q^\varepsilon\overline{u}^\varepsilon$.

5.3 Proof of $\overline{c}(0) = 0$

This part also is crucial.

Lemma 3 *Assume that ε and r^ε tend to zero, with $0 < \varepsilon^2 \ll r^\varepsilon$. Let $(u^{a\varepsilon}, u^{b\varepsilon}) \in (H^1(\Omega^a))^3 \times (H^1(\Omega^b))^3$ be such that*

$$u^{a\varepsilon}_{|x_3=1} = 0, \tag{5.69}$$

$$a.e.\ x' \in \omega^a,\ u_\alpha^{a\varepsilon}(x',0) = \varepsilon r^\varepsilon u_\alpha^{b\varepsilon}(r^\varepsilon x',0), \tag{5.70}$$

$$\{e^{a\varepsilon}(u^{a\varepsilon})\}_\varepsilon \text{ is bounded in } (L^2(\Omega^a))^{3\times3}, \tag{5.71}$$

$$\{u_\alpha^{b\varepsilon}\}_\varepsilon \text{ is bounded in } H^1(\Omega^b). \tag{5.72}$$

Let c^ε be defined by

$$c^\varepsilon(x_3) = \frac{\int_{\omega^a} (x_1 u_2^{a\varepsilon}(x',x_3) - x_2 u_1^{a\varepsilon}(x',x_3))\,dx'}{r^\varepsilon \int_{\omega^a}(x_1^2 + x_2^2)\,dx'}. \tag{5.73}$$

Then $c^\varepsilon \to c$ in $L^2(0,1)$, for some c in $H_0^1(0,1)$.

Proof: For $\alpha = 1, 2$, we define x_α^R by $x_1^R = -x_2$, $x_2^R = x_1$ and we set

$$v^{a\varepsilon} = \frac{u^{a\varepsilon}}{r^\varepsilon},$$

$$e_\alpha^\varepsilon = 2e_{\alpha 3}(v^{a\varepsilon}) = 2e_{\alpha 3}^{a\varepsilon}(u^{a\varepsilon})$$

(without confusion with e_α^ε appearing in (5.57)),

$$m_\alpha^\varepsilon = \frac{1}{|\omega^a|}\int_{\omega^a} v_\alpha^{a\varepsilon}\,dx',$$

$$\rho_\alpha^\varepsilon = \frac{1}{r^\varepsilon}\left[v_\alpha^{a\varepsilon} - c^\varepsilon x_\alpha^R - m_\alpha^\varepsilon\right],$$

with c^ε given by (8.104).

We begin by giving **two** *a priori* **estimates**. Due to (5.71), we have

$$\|e^\varepsilon\|_{(L^2(\Omega^a))^2} \leq C. \tag{5.74}$$

As for ρ^ε, it follows from (1.1) that $\rho_\alpha^\varepsilon(., x_3)$ has mean value zero on ω^a, for every x_3 and, as $e_{\alpha\beta}(\rho^\varepsilon) = (1/r^\varepsilon)e_{\alpha\beta}(v^{a\varepsilon})$, we get from the Poincaré-Wirtinger inequality for elasticity

$$\begin{aligned} \|\rho^\varepsilon\|^2_{(L^2(\Omega^a))^2} &\leq C\textstyle\sum_{\alpha\beta}\|e_{\alpha\beta}(\rho^\varepsilon)\|^2_{L^2(\Omega^a)} = \frac{C}{(r^\varepsilon)^2}\sum_{\alpha\beta}\|e_{\alpha\beta}(v^{a\varepsilon})\|^2_{L^2(\Omega^a)} \\ &= C\textstyle\sum_{\alpha\beta}\left\|e_{\alpha\beta}^{a\varepsilon}(u^{a\varepsilon})\right\|^2_{L^2(\Omega^a)}, \end{aligned}$$

which gives, with (5.71),

$$\|\rho^\varepsilon\|_{(L^2(\Omega^a))^2} \leq C. \tag{5.75}$$

Now we prove that **one can derive a single equation**, of the form $c^\varepsilon = K^\varepsilon - r^\varepsilon R^\varepsilon$, **from the system of two equations** $c^\varepsilon x_\alpha^R + m_\alpha^\varepsilon = v_\alpha^{a\varepsilon} - r^\varepsilon\rho_\alpha^\varepsilon$. (This is a very tricky argument appearing in [22], see also Section 8.1.) Indeed we get by differentiating the previous system with respect to x_3,

$$\frac{dc^\varepsilon}{dx_3}x_\alpha^R + \frac{dm_\alpha^\varepsilon}{dx_3} + \frac{\partial v_3^{a\varepsilon}}{\partial x_\alpha} = e_\alpha^\varepsilon - r^\varepsilon\frac{\partial\rho_\alpha^\varepsilon}{\partial x_3}, \quad \forall\alpha = 1, 2. \tag{5.76}$$

After multiplying (5.76) by a test function $\varphi_\alpha \in \mathcal{D}(\omega^a)$, summing over α and integrating over ω^a, we have

$$\begin{aligned} &\tfrac{dc^\varepsilon}{dx_3}\textstyle\int_{\omega^a}\sum_\alpha \varphi_\alpha x_\alpha^R\,dx' + \sum_\alpha \frac{dm_\alpha^\varepsilon}{dx_3}\int_{\omega^a}\varphi_\alpha\,dx' - \int_{\omega^a} v_3^{a\varepsilon} div\varphi\,dx' = \\ &\qquad = \textstyle\int_{\omega^a}\sum_\alpha e_\alpha^\varepsilon\varphi_\alpha\,dx' - r^\varepsilon\frac{d}{dx_3}\int_{\omega^a}\sum_\alpha \rho_\alpha^\varepsilon\varphi_\alpha\,dx'. \end{aligned} \tag{5.77}$$

We choose the test function φ_α so that

$$\int_{\omega^a}\sum_\alpha \varphi_\alpha x_\alpha^R\,dx' = 1 \tag{5.78}$$

$$\int_{\omega^a} \varphi_\alpha \, dx' = 0, \quad \forall \alpha = 1, 2, \tag{5.79}$$

$$div\varphi = 0. \tag{5.80}$$

It is easy to check that such test function does exist: take e.g.

$$\varphi_1 = \frac{\partial \phi}{\partial x_2}, \varphi_2 = -\frac{\partial \phi}{\partial x_1}, \text{ with } \phi \in \mathcal{D}(\omega^a), \int_{\omega^a} \phi \, dx' = \frac{1}{2}.$$

Now we set (without confusion with E^ε appearing in (5.58))

$$E^\varepsilon = \int_{\omega^a} e^\varepsilon . \varphi \, dx', \qquad K^\varepsilon = -\int_{x_3}^1 E^\varepsilon(y_3) \, dy_3, \qquad R^\varepsilon = \int_{\omega^a} \rho^\varepsilon . \varphi \, dx',$$

where "." denotes the scalar product in $\mathbf{R}^2$. Then (5.77) reads:

$$\frac{dc^\varepsilon}{dx_3} = \frac{dK^\varepsilon}{dx_3} - r^\varepsilon \frac{dR^\varepsilon}{dx_3},$$

which gives by integration

$$c^\varepsilon = K^\varepsilon - r^\varepsilon R^\varepsilon, \tag{5.81}$$

since $c^\varepsilon(1) = K^\varepsilon(1) = 0$ and since also $R^\varepsilon(1) = 0$, because $\rho^\varepsilon(1) = 0$.

Now **we pass to the limit** in (5.81). Due to (5.74) and (5.75), E^ε and R^ε are bounded in $L^2(0,1)$. Moreover it follows that K^ε is bounded in $H^1(0,1)$, since by Poincaré inequality,

$$\|K^\varepsilon\|^2_{H^1(0,1)} \leq C \int_0^1 \left| \frac{dK^\varepsilon}{dx_3} \right|^2 dx_3 = C \int_0^1 |E^\varepsilon|^2 \, dx_3 \leq C.$$

Then we deduce that there exists c in $H^1(0,1)$, with $c(1) = 0$, such that

$$K^\varepsilon \rightharpoonup c \text{ weakly in } H^1(0,1), \text{ hence } K^\varepsilon \to c \text{ strongly in } \mathcal{C}^0(0,1).$$

As $r^\varepsilon R^\varepsilon$ tends to zero strongly in $L^2(0,1)$, it follows from (5.81) and the above that c^ε tends to c strongly in $L^2(0,1)$.

It remains to **prove that c vanishes at the origin**. For this, we notice that $K^\varepsilon(0) \to c(0)$. But one has also another expression for K^ε. Actually, from (5.81), (5.78), (5.79) and by definition of ρ^ε_α,

$$\begin{aligned} K^\varepsilon = r^\varepsilon R^\varepsilon + c^\varepsilon &= r^\varepsilon \textstyle\int_{\omega^a} \rho^\varepsilon . \varphi \, dx' + c^\varepsilon \int_{\omega^a} \sum_\alpha \varphi_\alpha x^R_\alpha \, dx' + \sum_\alpha m^\varepsilon_\alpha \int_{\omega^a} \sum_\alpha \varphi_\alpha \, dx' \\ &= \textstyle\sum_\alpha \int_{\omega^a} \left(r^\varepsilon \rho^\varepsilon_\alpha + c^\varepsilon x^R_\alpha + m^\varepsilon_\alpha \right) \varphi_\alpha \, dx' = \sum_\alpha \int_{\omega^a} v^{a\varepsilon}_\alpha \varphi_\alpha \, dx', \end{aligned}$$

that is

$$K^\varepsilon = \sum_\alpha \int_{\omega^a} v^{a\varepsilon}_\alpha \varphi_\alpha \, dx' = \sum_\alpha \int_{\omega^a} \frac{1}{r^\varepsilon} u^{a\varepsilon}_\alpha \varphi_\alpha \, dx'$$

and in particular, due to the boundary condition (5.70),

$$K^\varepsilon(0) = \varepsilon \sum_\alpha \int_{\omega^a} u^{b\varepsilon}_\alpha(r^\varepsilon x', 0)\varphi_\alpha \, dx'.$$

Hence, by using Hölder inequality, the continuity of the imbedding of $H^{\frac{1}{2}}(\omega^b)$ in $L^4(\omega^b)$ (in dimension 2), the continuity of the trace mapping from $H^1(\Omega^b)$ to $H^{\frac{1}{2}}(\omega^b)$ and (5.72), we get

$$\begin{aligned}
|K^\varepsilon(0)| &\le C\varepsilon \textstyle\sum_\alpha \int_{\omega^a} |u^{b\varepsilon}_\alpha(r^\varepsilon x', 0)| \, dx' = C\frac{\varepsilon}{(r^\varepsilon)^2} \sum_\alpha \int_{r^\varepsilon\omega^a} |u^{b\varepsilon}_\alpha(x', 0)| \, dx' \\
&\le C\frac{\varepsilon}{(r^\varepsilon)^2} \textstyle\sum_\alpha \left[\int_{r^\varepsilon\omega^a} |u^{b\varepsilon}_\alpha(x', 0)|^4 \, dx'\right]^{\frac{1}{4}} |r^\varepsilon\omega^a|^{\frac{3}{4}} \\
&= C\varepsilon(r^\varepsilon)^{-\frac{1}{2}} \textstyle\sum_\alpha \left[\int_{r^\varepsilon\omega^a} |u^{b\varepsilon}_\alpha(x', 0)|^4 \, dx'\right]^{\frac{1}{4}} \\
&\le C\varepsilon(r^\varepsilon)^{-\frac{1}{2}} \textstyle\sum_\alpha \left\| u^{b\varepsilon}_\alpha(.,0) \right\|_{L^4(\omega^b)} \le C\varepsilon(r^\varepsilon)^{-\frac{1}{2}} \sum_\alpha \left\| u^{b\varepsilon}_\alpha(.,0) \right\|_{H^{\frac{1}{2}}(\Omega^b)} \le C\varepsilon(r^\varepsilon)^{-\frac{1}{2}},
\end{aligned}$$

which tends to zero, since $0 < \varepsilon^2 \ll r^\varepsilon$. As we have proved that $K^\varepsilon(0) \to c(0)$, we conclude that $c(0) = 0$.

6 The use of convenient test functions

This is the third crucial part of the paper, at least in the case $q \in (0, +\infty)$.

6.1 The case $q = +\infty$

Remark that $\mathcal{Z}_\infty = \{z^a \in \mathcal{Z}^a, \zeta^a(:= \zeta^a(u^a)) \in H^1_0(0,1)\}$. Let $u^a \in \mathcal{U}^a$, with $\zeta^a \in H^1_0(0,1)$ and let (v^a, w^a) be such that

$$\begin{aligned}
&v^a_1 = -cx_2 \text{ and } v^a_2 = cx_1, \text{ with } c \in H^1_0(0,1); \quad v^a_3 \in \mathcal{C}^1(\overline{\Omega^a}), v^a_{3\,|x_3=0} = 0; \\
&w^a_\alpha \in \mathcal{C}^1(\overline{\Omega^a}), w^a_{\alpha\,|x_3=0} = 0; \quad w^a_3 = 0.
\end{aligned} \tag{6.82}$$

In other words, v^a and w^a satisfy all the conditions given in the definitions of $\mathcal{V}^a$ and $\mathcal{W}^a$, but the integral ones; moreover v^a_3 and w^a_α belong to

$$\mathcal{R} = \{v \in \mathcal{C}^1(\overline{\Omega^a}), \, v_{|x_3=0} = 0\}. \tag{6.83}$$

Let $u^{a\varepsilon} = u^a + r^\varepsilon v^a + (r^\varepsilon)^2 w^a$. Then it is easy to check that $u = (u^{a\varepsilon}, 0)$ is in $\mathcal{Y}^\varepsilon$. Taking it as test function in the variational equation of Problem (1.2), we get

$$\int_{\Omega^a} [A^a \overline{e}^{a\varepsilon}, e^{a\varepsilon}(u^{a\varepsilon})] \, dx = \int_{\Omega^a} f^{a\varepsilon}.u^{a\varepsilon} \, dx + \int_{\Omega^a} [g^{a\varepsilon}, e^{a\varepsilon}(u^{a\varepsilon})] \, dx + \int_{\Sigma^a} h^{a\varepsilon}.u^{a\varepsilon} \, d\sigma. \tag{6.84}$$

But we have, since $e_{\alpha\beta}(u^a) = e_{\alpha 3}(u^a) = e_{\alpha\beta}(v^a) = e_{33}(w^a) = 0$,

$$e^{a\varepsilon}(u^{a\varepsilon}) = \begin{pmatrix} e_{\alpha\beta}(w^a) & e_{\alpha 3}(v^a) \\ \\ e_{3\alpha}(v^a) & e_{33}(u^a) \end{pmatrix} + r^\varepsilon \begin{pmatrix} 0 & e_{\alpha 3}(w^a) \\ \\ e_{3\alpha}(w^a) & e_{33}(v^a) \end{pmatrix},$$

so that $e^{a\varepsilon}(u^{a\varepsilon})$ tends to $e^a(z^a) = e^a(u^a, v^a, w^a)$ (strongly) in $(L^2(\Omega^a))^{3\times 3}$. Moreover $u^{a\varepsilon}$ tends to u^a (strongly) in $(H^1(\Omega^a))^3$ and we have seen in (4.49) that $\overline{e}^{a\varepsilon}$ tends to $e^a(\overline{z}^a)$ weakly in $(L^2(\Omega^a))^{3\times 3}$. By passing to the limit in (6.84), using (2.9), (2.11), (2.13), it follows that

$$\int_{\Omega^a} [A^a e^a(\overline{z}^a), e^a(z^a)]\, dx = \int_{\Omega^a} f^a.u^a\, dx + \int_{\Omega^a} [g^a, e^a(z^a)]\, dx + \int_{\Sigma^a} h^a.u^a\, d\sigma, \tag{6.85}$$

which is the variational equation of (2.20). It holds also true, by density and continuity, for every (v^a, w^a) such that

$$v_1^a = -cx_2 \text{ and } v_2^a = cx_1, \text{ with } c \in H_0^1(0,1); \quad v_3^a \in L^2(0,1; H^1(\omega^a));$$

$$w_\alpha^a \in L^2(0,1; H^1(\omega^a)); \quad w_3^a = 0,$$

i.e. for every (v^a, w^a) satisfying the conditions given in the definitions of $\mathcal{V}^a$ and $\mathcal{W}^a$, but the integral ones. (Note that $\mathcal{R}$ defined in (6.83) is dense in $L^2(0,1; H^1(\omega^a))$.) In particular (6.85) is also true for any $z^a \in \mathcal{Z}_\infty$. This means that $\overline{z}^a$ solves the variational problem (2.20).

6.2 The case $q = 0$

Then we have seen that $\tilde{u}^\varepsilon = q^\varepsilon \overline{u}^\varepsilon = q^\varepsilon(\overline{u}^{a\varepsilon}, \overline{u}^{b\varepsilon})$ solves (4.47) and that (see (4.52)) $\tilde{u}^{a\varepsilon} = q^\varepsilon \overline{u}^{a\varepsilon}$ tends to $\overline{u}^a = 0$ in $(H^1(\Omega^a))^3$, $\tilde{e}^{b\varepsilon} = q^\varepsilon \overline{e}^{b\varepsilon}$ tends to $\overline{e}^b = e^b(\overline{z}^b)$ weakly in $(L^2(\Omega^b))^{3\times 3}$, for some $\overline{z}^b$ in $\mathcal{Z}_0$. (In particuler $\overline{u}_3^b(0) = 0$.)

Let B be some given small ball, with center zero, in ω^b. Let $z^b = (u^b, v^b, w^b)$ be such that

$$\begin{aligned} & u^b \in \mathcal{U}^b, \quad \zeta_\alpha^b (:= \zeta_\alpha^b(u^b)) \equiv u_3^b \equiv 0 \text{ in } B; \\ & v_\alpha^b \in \mathcal{C}^1(\overline{\Omega^b}), v_\alpha^b \equiv 0 \text{ in } B \times \{0\}; \quad v_3^b = 0; \\ & w_\alpha^b = 0; \quad w_3^b \in \mathcal{C}^1(\overline{\Omega^b}),\ w_3^b \equiv 0 \text{ in } B \times \{0\}. \end{aligned} \tag{6.86}$$

In other words, z^b satisfies all the conditions given in the definition of $\mathcal{Z}_0$, except the integral ones; moreover ζ_α^b and u_3^b vanish in B, v_α^b and w_3^b belong to $\mathcal{C}^1(\overline{\Omega^b})$ and vanish in $B \times \{0\}$. Let $u^{b\varepsilon} = u^b + \varepsilon v^b + (\varepsilon)^2 w^b$. Then it is easy to check that $u = (0, u^{b\varepsilon})$ is in $\mathcal{Y}^\varepsilon$, for ε small enough. Taking it as test function in the variational equation of Problem (1.2), we get

$$\begin{aligned} \textstyle\int_{\Omega^b} [A^b \tilde{e}^{b\varepsilon}, e^{b\varepsilon}(u^{b\varepsilon})]\, dx \quad &= \textstyle\int_{\Omega^b} f^{b\varepsilon}.u^{b\varepsilon}\, dx + \int_{\Omega^b} [g^{b\varepsilon}, e^{b\varepsilon}(u^{b\varepsilon})]\, dx \\ &\quad + \textstyle\int_{\omega^b} \left(h_+^{b\varepsilon}.u^{b\varepsilon}_{|x_3=0} + h_-^{b\varepsilon}.u^{b\varepsilon}_{|x_3=-1} \right) dx'. \end{aligned} \tag{6.87}$$

But we have, since $e_{\alpha 3}(u^b)=e_{33}(u^b)=e_{33}(v^b)=e_{\alpha\beta}(w^b)=0$,

$$e^{b\varepsilon}(u^{b\varepsilon})=\begin{pmatrix} e_{\alpha\beta}(u^b) & e_{\alpha 3}(v^b) \\ \\ e_{3\alpha}(v^b) & e_{33}(w^b)\end{pmatrix}+\varepsilon\begin{pmatrix} e_{\alpha\beta}(v^b) & e_{\alpha 3}(w^b) \\ \\ e_{3\alpha}(w^b) & 0\end{pmatrix},$$

so that $e^{b\varepsilon}(u^{b\varepsilon})$ tends to $e^b(z^b)$ (strongly) in $(L^2(\Omega^b))^{3\times 3}$. Moreover $u^{b\varepsilon}$ tends to u^b (strongly) in $(H^1(\Omega^b))^3$ and we have seen that $\tilde{e}^{b\varepsilon}$ tends to $e^b(\overline{z}^b)$ weakly in $(L^2(\Omega^b))^{3\times 3}$. By passing to the limit in (6.87), using (2.10), (2.12), (2.14), it follows that

$$\begin{aligned}\int_{\Omega^b}[A^b e^b(\overline{z}^b), e^b(z^b)]\,dx &= \int_{\Omega^b} f^b.u^b\,dx+\int_{\Omega^b}[g^b,e^b(z^b)]\,dx \\ &\quad+\int_{\omega^b}\left(h_+^b.u^b_{|x_3=0}+h_-^b.u^b_{|x_3=-1}\right)dx',\end{aligned}\tag{6.88}$$

for every $z^b=(u^b,v^b,w^b)$ having the regularity of (6.86).

But the following density results are proved in the Appendix (Section 8.2). First, from Lemma 5, any $\zeta_\alpha^b\in H_0^1(\omega^b)$ can be approached (in $H_0^1(\omega^b)$-norm) by a sequence ζ_α^{bn}, with $\zeta_\alpha^{bn}\equiv 0$ in a ball B^n of radius r^n, tending to zero. Moreover, from Lemma 6, any $u_3^b\in H_0^2(\omega^b)$, with $u_3^b(0)=0$ can be approached (in the weak topology of $H_0^2(\omega^b)$) by a sequence u_3^{bn} that vanishes in B^n. Finally, from Lemma 7, any v_α^b (or w_3^b) in $L^2(\omega^b;H^1(-1,0))$ can be approached (in $L^2(\omega^b;H^1(-1,0))$-norm) by a sequence of functions v_α^{nb} (or w_3^{nb}) in $\mathcal{C}^1(\overline{\Omega^b})$ that vanish in $B^n\times\{0\}$.

By continuity, it results that (6.88) holds true for any

$$\begin{aligned}&u^b\in\mathcal{U}^b;\quad u_3^b(0)=0;\\ &v_\alpha^b\in L^2(\omega^b;H^1(-1,0));\quad v_3^b=0;\\ &w_\alpha^b=0;\quad w_3^b\in L^2(\omega^b;H^1(-1,0)),\end{aligned}$$

i.e. for every z^b satisfying the conditions given in the definitions of $\mathcal{Z}_0$, but the integral ones. In particular (6.88) is also true for any $z^b\in\mathcal{Z}_0$. This means that $\overline{z}^b$ solves the variational problem (2.25).

6.3 The case $q\in(0,+\infty)$

Let $z=(z^a,z^b)=((u^a,v^a,w^a),(u^b,v^b,w^b))\in(\mathcal{C}^1(\overline{\Omega^a}))^3\times(\mathcal{C}^1(\overline{\Omega^b}))^3$. We assume that z satisfies all the conditions required in the definition of Z, except the integral ones, and we assume that it is "more regular". In particular $u_3^a(x',0)\equiv u_3^b(0)$, that is $\zeta^a(0)=u_3^b(0)$.

The precise requirements are given below:

$$
\begin{aligned}
&u^a \in \mathcal{U}^a,\ u^a_\alpha \in \mathcal{C}^2[0,1],\ \zeta^a \in \mathcal{C}^1[0,1];\\
&v^a_1 = -cx_2 \text{ and } v^a_2 = cx_1 \text{ with } c \in \mathcal{C}^1[0,1], c(0)=c(1)=0;\\
&v^a_3 \in \mathcal{C}^1(\overline{\Omega^a}), v^a_{3\,|x_3=0} = 0;\\
&w^a_\alpha \in \mathcal{C}^1(\overline{\Omega^a}), w^a_{\alpha\,|x_3=0} = 0; w^a_3 = 0;\\
&u^b \in \mathcal{U}^b, \zeta^b_\alpha \in \mathcal{C}^1(\overline{\omega^b}) \cap H^1_0(\omega^b);\ u^b_3 \in \mathcal{C}^1(\overline{\omega^b}) \cap H^2_0(\omega^b);\\
&v^b \in (\mathcal{C}^1(\overline{\Omega^b}))^2 \times \{0\};\ w^b \in (\{0\})^2 \times \mathcal{C}^1(\overline{\Omega^b});\ v^b_\alpha \text{ and } w^b_3 = 0 \text{ on } \Sigma^b;\\
&u^b_3(0) = \zeta^a(0).
\end{aligned}
\tag{6.89}
$$

We are going to define a convenient test function $u^\varepsilon = (u^{a\varepsilon}, u^{b\varepsilon})$ in $\mathcal{Y}^\varepsilon$.
• In Ω^b, we choose

$$
u^{b\varepsilon} = u^b + \varepsilon v^b + \varepsilon^2 w^b. \tag{6.90}
$$

As the couple $u^\varepsilon = (u^{a\varepsilon}, u^{b\varepsilon})$ needs to satisfy the transmission conditions (3.36), i.e.

$$
\text{for a.e. } x' \in \omega^a,\ u^a_\alpha(x',0) = \varepsilon r^\varepsilon u^b_\alpha(r^\varepsilon x',0) \text{ and } u^a_3(x',0) = u^b_3(r^\varepsilon x',0)\},
$$

then, necessarily, $u^{a\varepsilon}(x',0)$ is given by:

$$
u^{a\varepsilon}_\alpha(x',0) = \varepsilon r^\varepsilon \left(\zeta^b_\alpha(r^\varepsilon x') + \varepsilon v^b_\alpha(r^\varepsilon x',0)\right),
$$

$$
u^{a\varepsilon}_3(x',0) = u^b_3(r^\varepsilon x') + \varepsilon^2 w^b_3(r^\varepsilon x',0).
$$

• In $\Omega^a \cap \{x_3 > r^\varepsilon\}$, we choose

$$
u^{a\varepsilon} = u^a + r^\varepsilon v^a + (r^\varepsilon)^2 w^a. \tag{6.91}
$$

• In $\Omega^a \cap \{0 < x_3 < r^\varepsilon\}$, $u^{a\varepsilon}$ is obtained by linear interpolation between $u^{a\varepsilon}(x',0)$ and $u^{a\varepsilon}(x', r^\varepsilon)$:

$$
u^{a\varepsilon}(x',x_3) = \frac{x_3}{r^\varepsilon}\left(u^a(x',r^\varepsilon) + r^\varepsilon v^a(x',r^\varepsilon) + (r^\varepsilon)^2 w^a(x',r^\varepsilon)\right) + \left(1 - \frac{x_3}{r^\varepsilon}\right) u^{a\varepsilon}(x',0),
$$

that is (see above)

$$
\begin{aligned}
\text{for } 0 < x_3 < r^\varepsilon, \quad u^{a\varepsilon}_\alpha(x',x_3) &= \frac{x_3}{r^\varepsilon}\left(u^a_\alpha(r^\varepsilon) + r^\varepsilon v^a_\alpha(x',r^\varepsilon) + (r^\varepsilon)^2 w^a_\alpha(x',r^\varepsilon)\right) + \\
&\quad + \left(1 - \tfrac{x_3}{r^\varepsilon}\right) \varepsilon r^\varepsilon \left(\zeta^b_\alpha(r^\varepsilon x') + \varepsilon v^b_\alpha(r^\varepsilon x',0)\right),
\end{aligned}
\tag{6.92}
$$

$$\text{for } 0 < x_3 < r^\varepsilon, \quad u_3^{a\varepsilon}(x', x_3) = \frac{x_3}{r^\varepsilon}\left(u_3^a(x', r^\varepsilon) + r^\varepsilon v_3^a(x', r^\varepsilon)\right) + \\ + \left(1 - \tfrac{x_3}{r^\varepsilon}\right)\left(u_3^b(r^\varepsilon x') + \varepsilon^2 w_3^b(r^\varepsilon x', 0)\right). \tag{6.93}$$

Taking $u^\varepsilon = (u^{a\varepsilon}, u^{b\varepsilon})$ as test function in the variational equation of Problem (2.5), we get

$$\int_{\Omega^a} [A^a \overline{e}^{a\varepsilon}, e^{a\varepsilon}(u^{a\varepsilon})]\, dx + q^\varepsilon \int_{\Omega^b} [A^b \overline{e}^{b\varepsilon}, e^{b\varepsilon}(u^{b\varepsilon})]\, dx = \\ = \int_{\Omega^a} f^{a\varepsilon}.u^{a\varepsilon}\, dx + \int_{\Omega^b} f^{b\varepsilon}.u^{b\varepsilon}\, dx + \int_{\Omega^a} [g^{a\varepsilon}, e^{a\varepsilon}(u^{a\varepsilon})]\, dx + \int_{\Omega^b} [g^{b\varepsilon}, e^{b\varepsilon}(u^{b\varepsilon})]\, dx + \\ \int_{\Sigma^a} h^{a\varepsilon}.u^{a\varepsilon}\, d\sigma + \int_{\omega^b} \left(h_+^{b\varepsilon}.u^{b\varepsilon}_{|x_3=0} + h_-^{b\varepsilon}.u^{b\varepsilon}_{|x_3=-1}\right) dx'. \tag{6.94}$$

Passing to the limit in the integral terms in Ω^b is easy. As for the terms in $\Omega^a \cap \{x_3 > r^\varepsilon\}$ and $\Sigma^a \cap \{x_3 > r^\varepsilon\}$, we have from Lebesgue's theorem, with $u^{a\varepsilon} = u^a + r^\varepsilon v^a + (r^\varepsilon)^2 w^a$ and χ^ε the characteristic function of $\{x_3 > r^\varepsilon\}$,

$$\begin{array}{l} \chi^\varepsilon e^{a\varepsilon}(u^{a\varepsilon}) \to e^a(z^a) \text{ strongly in } (L^2(\Omega^a))^{3\times 3}, \\ \chi^\varepsilon u^{a\varepsilon} \to u^a \text{ strongly in } (L^2(\Omega^a))^3, \\ \chi^\varepsilon u^{a\varepsilon}_{|\Sigma^a} \to u^a_{|\Sigma^a} \text{ strongly in } (L^2(\Sigma^a))^3, \end{array}$$

so that, by virtue of (2.9), (2.11), (2.13), (4.49),

$$\begin{array}{l} \int_{\Omega^a \cap \{x_3 > r^\varepsilon\}} [A^a \overline{e}^{a\varepsilon} - g^{a\varepsilon}, e^{a\varepsilon}(u^{a\varepsilon})]\, dx - \int_{\Omega^a \cap \{x_3 > r^\varepsilon\}} f^{a\varepsilon}.u^{a\varepsilon}\, dx - \int_{\Sigma^a \cap \{x_3 > r^\varepsilon\}} h^{a\varepsilon}.u^{a\varepsilon}\, d\sigma \\ \to \int_{\Omega^a} [A^a \overline{e}^a - g^a, e^a(z^a)]\, dx - \int_{\Omega^a} f^a.u^a\, dx - \int_\Sigma h^a.u^a\, d\sigma. \end{array}$$

For the terms in $\Omega^a \cap \{0 < x_3 < r^\varepsilon\}$ and $\Sigma^a \cap \{0 < x_3 < r^\varepsilon\}$, it is clear, from (2.9), (2.13) and from the uniform boundedness of $u^{a\varepsilon}$, that

$$\int_{\Omega^a \cap \{0 < x_3 < r^\varepsilon\}} f^{a\varepsilon}.u^{a\varepsilon}\, dx - \int_{\Sigma^a \cap \{0 < x_3 < r^\varepsilon\}} h^{a\varepsilon}.u^{a\varepsilon}\, d\sigma \to 0.$$

Hence it remains to show that

$$\int_{\Omega^a \cap \{0 < x_3 < r^\varepsilon\}} [A^a \overline{e}^{a\varepsilon} - g^{a\varepsilon}, e^{a\varepsilon}(u^{a\varepsilon})]\, dx \to 0.$$

But we have, from Cauchy-Schwarz inequality, (2.11) and (4.49),

$$\int_{\Omega^a \cap \{0 < x_3 < r^\varepsilon\}} [A^a \overline{e}^{a\varepsilon} - g^{a\varepsilon}, e^{a\varepsilon}(u^{a\varepsilon})]\, dx \le C\, \|e^{a\varepsilon}(u^{a\varepsilon})\|_{(L^2(\Omega^a \cap \{0 < x_3 < r^\varepsilon\}))^{3\times 3}}$$

and it is enough to prove that

$$\|e^{a\varepsilon}(u^{a\varepsilon})\|_{(L^2(\Omega^a \cap \{0 < x_3 < r^\varepsilon\}))^{3\times 3}} \to 0. \tag{6.95}$$

Then, by passing to the limit in (6.94), we will get

$$\int_{\Omega^a}[A^a e^a(\overline{z}^a), e^a(z^a)]\,dx + q\int_{\Omega^b}[A^b e^b(\overline{z}^b), e^b(z^b)]\,dx =$$

$$\int_{\Omega^a} f^a.u^a\,dx + \int_{\Omega^b} f^b.u^b\,dx + \int_{\Omega^a}[g^a, e^a(z^a)]\,dx + \int_{\Omega^b}[g^b, e^b(z^b)]\,dx + \tag{6.96}$$

$$\int_{\Sigma^a} h^a.u^a\,d\sigma + \int_{\omega^b}\left(h^b_+.u^b_{|x_3=0} + h^b_-.u^b_{|x_3=-1}\right)\,dx',$$

for any z having the regularity given in (6.89), and then, by a density argument given in Lemma 8 of the Appendix (Section 8.2), for any z satisfying all the requirements of $\mathcal{Z}$, except the integral conditions. *A fortiori*, the same variational equality holds true for any z in $\mathcal{Z}$, that is $(\overline{z}^a, \overline{z}^b)$ solves (2.17).

Proof of (6.95): We are going to prove that the terms $e_{33}(u^{a\varepsilon})$, $\frac{1}{(r^\varepsilon)^2}e_{\alpha\beta}(u^{a\varepsilon})$ and $\frac{1}{r^\varepsilon}e_{\alpha 3}(u^{a\varepsilon})$ tend to zero (strongly) in $L^2(\{0 < x_3 < r^\varepsilon\})$.

- **Term $e_{33}(u^{a\varepsilon})$:** We easily get from (6.93)

$$e_{33}(u^{a\varepsilon}) = \frac{\partial u_3^{a\varepsilon}}{\partial x_3} = \frac{1}{r^\varepsilon}\left(u_3^a(x', r^\varepsilon) - u_3^b(r^\varepsilon x')\right) + v_3^a(x', r^\varepsilon) - \frac{(\varepsilon)^2}{r^\varepsilon}w_3^b(r^\varepsilon x', 0). \tag{6.97}$$

Clearly the last two terms in (6.97) tend to zero in $L^2(\{0 < x_3 < r^\varepsilon\})$, since $v_3^a(x', r^\varepsilon)$ and $w_3^b(r^\varepsilon x', 0)$ are uniformly bounded:

$$\int_{0<x_3<r^\varepsilon} |v_3^a(x', r^\varepsilon)|^2\,dx \le C r^\varepsilon \to 0,$$

$$\int_{0<x_3<r^\varepsilon}\left(\frac{(\varepsilon)^2}{r^\varepsilon}\right)^2 |w_3^b(r^\varepsilon x', 0)|^2\,dx \le C\frac{\varepsilon^4}{r^\varepsilon} \ll C\frac{\varepsilon^2}{r^\varepsilon} \to 0,$$

since, by assumption, $\varepsilon^2 \ll r^\varepsilon$. As for the first term in (6.97), it is uniformly bounded, because of the junction condition:

$$\frac{1}{r^\varepsilon}\left(u_3^a(x', r^\varepsilon) - u_3^b(r^\varepsilon x')\right) = \frac{1}{r^\varepsilon}\left(u_3^a(x', 0) + \int_0^{r^\varepsilon}\frac{\partial u_3^a}{\partial x_3}(x', t)\,dt - u_3^b(0) - \int_0^{r^\varepsilon}\nabla u_3^b(tx').x'\,dt\right)$$

$$= \frac{1}{r^\varepsilon}\int_0^{r^\varepsilon}\frac{\partial u_3^a}{\partial x_3}(x', t)\,dt - \frac{1}{r^\varepsilon}\int_0^{r^\varepsilon}\nabla u_3^b(tx').x'\,dt \le C$$

and hence, its norm in $L^2(\{0 < x_3 < r^\varepsilon\})$ tends to zero.

- **Term $\frac{1}{(r^\varepsilon)^2}e_{\alpha\beta}(u^{a\varepsilon})$:** ¿From (6.92),

$$\frac{\partial u_\alpha^{a\varepsilon}}{\partial x_\beta} = \frac{x_3}{r^\varepsilon}\left(r^\varepsilon\frac{\partial v_\alpha^a}{\partial x_\beta}(x', r^\varepsilon) + (r^\varepsilon)^2\,\frac{\partial w_\alpha^a}{\partial x_\beta}(x', r^\varepsilon)\right)$$

$$+ \left(1 - \frac{x_3}{r^\varepsilon}\right)\varepsilon(r^\varepsilon)^2\left(\frac{\partial \zeta_\alpha^b}{\partial x_\beta}(r^\varepsilon x') + \varepsilon\frac{\partial v_\alpha^b}{\partial x_\beta}(r^\varepsilon x', 0)\right)$$

and hence, since $e_{\alpha\beta}(v^a) = 0$,

$$\frac{1}{(r^\varepsilon)^2} e_{\alpha\beta}(u^{a\varepsilon}) = \frac{x_3}{r^\varepsilon} e_{\alpha\beta}(w^a)(x', r^\varepsilon) + \left(1 - \frac{x_3}{r^\varepsilon}\right) \varepsilon \left(e_{\alpha\beta}(\zeta^b)(r^\varepsilon x') + \varepsilon e_{\alpha\beta}(v^b)(r^\varepsilon x', 0)\right),$$

which gives, from the regularity of w^a, ζ^b and v^b (see (6.89))

$$\left|\frac{1}{(r^\varepsilon)^2} e_{\alpha\beta}(u^{a\varepsilon})\right| \leq C + C\varepsilon(1+\varepsilon) \leq C,$$

and hence, the norm of this term, in $L^2(\{0 < x_3 < r^\varepsilon\})$, tends to zero.

- **Term $\frac{1}{r^\varepsilon} e_{\alpha 3}(u^{a\varepsilon})$:** ¿From (6.92),

$$\frac{\partial u_\alpha^{a\varepsilon}}{\partial x_3} = \frac{1}{r^\varepsilon}\left(u_\alpha^a(r^\varepsilon) + r^\varepsilon v_\alpha^a(x', r^\varepsilon) + (r^\varepsilon)^2 w_\alpha^a(x', r^\varepsilon)\right) - \varepsilon\left(\zeta_\alpha^b(r^\varepsilon x') + \varepsilon v_\alpha^b(r^\varepsilon x', 0)\right)$$

and, from (6.93),

$$\frac{\partial u_3^{a\varepsilon}}{\partial x_\alpha} = \frac{x_3}{r^\varepsilon}\left(\frac{\partial u_3^a}{\partial x_\alpha}(x', r^\varepsilon) + r^\varepsilon \frac{\partial v_3^a}{\partial x_\alpha}(x', r^\varepsilon)\right) + \left(1 - \frac{x_3}{r^\varepsilon}\right) r^\varepsilon \left(\frac{\partial u_3^b}{\partial x_\alpha}(r^\varepsilon x') + \varepsilon^2 \frac{\partial w_3^b}{\partial x_\alpha}(r^\varepsilon x', 0)\right),$$

so that

$$\frac{2}{r^\varepsilon} e_{\alpha 3}(u^{a\varepsilon}) = \frac{1}{r^\varepsilon}\left(\frac{\partial u_\alpha^{a\varepsilon}}{\partial x_3} + \frac{\partial u_3^{a\varepsilon}}{\partial x_\alpha}\right) = T_1 + T_2 + T_3 + T_4, \tag{6.98}$$

with

$$T_1 = \tfrac{1}{(r^\varepsilon)^2}\left(u_\alpha^a(r^\varepsilon) + x_3 \tfrac{\partial u_3^a}{\partial x_\alpha}(x', r^\varepsilon)\right),$$

$$T_2 = \tfrac{1}{r^\varepsilon}\left(v_\alpha^a(x', r^\varepsilon) - \varepsilon\left(\zeta_\alpha^b(r^\varepsilon x') + \varepsilon v_\alpha^b(r^\varepsilon x', 0)\right)\right),$$

$$T_3 = w_\alpha^a(x', r^\varepsilon),$$

$$T_4 = \tfrac{x_3}{r^\varepsilon}\tfrac{\partial u_3^a}{\partial x_\alpha}(x', r^\varepsilon) + (1 - \tfrac{x_3}{r^\varepsilon})\left(\tfrac{\partial u_3^b}{\partial x_\alpha}(r^\varepsilon x') + \varepsilon^2 \tfrac{\partial w_3^b}{\partial x_\alpha}(r^\varepsilon x', 0)\right).$$

We are going to show that each term tends to zero, in $L^2(\{0 < x_3 < r^\varepsilon\})$.

- **Term T_1:** As $u_\alpha^a(0) = 0$,

$$u_\alpha^a(r^\varepsilon) = \int_0^{r^\varepsilon} \frac{du_\alpha^a}{dx_3}(t)\, dt$$

and, as $e_{\alpha 3}(u^a) = 0$,

$$x_3 \frac{\partial u_3^a}{\partial x_\alpha}(x', r^\varepsilon) = -x_3 \frac{du_\alpha^a}{dx_3}(r^\varepsilon) = -\frac{x_3}{r^\varepsilon}\int_0^{r^\varepsilon} \frac{du_\alpha^a}{dx_3}(t)\, dt + \frac{x_3}{r^\varepsilon}\int_0^{r^\varepsilon}\left(\frac{du_\alpha^a}{dx_3}(t) - \frac{du_\alpha^a}{dx_3}(r^\varepsilon)\right) dt,$$

so that, since $\frac{du^a_\alpha}{dx_3}(0)=0$,

$$u^a_\alpha(r^\varepsilon)+x_3\tfrac{\partial u^a_3}{\partial x_\alpha}(x',r^\varepsilon) \;=\left(1-\tfrac{x_3}{r^\varepsilon}\right)\int_0^{r^\varepsilon}\frac{du^a_\alpha}{dx_3}(t)\,dt+\frac{x_3}{r^\varepsilon}\int_0^{r^\varepsilon}\left(\frac{du^a_\alpha}{dx_3}(t)-\frac{du^a_\alpha}{dx_3}(r^\varepsilon)\right)dt$$

$$=\int_0^{r^\varepsilon}\int_0^t\frac{d^2u^a_\alpha}{dx_3^2}(\tau)\,d\tau\,dt$$

and, from the regularity of u^a_α,

$$|T_1|=\frac{1}{(r^\varepsilon)^2}\left|u^a_\alpha(r^\varepsilon)+x_3\frac{\partial u^a_3}{\partial x_\alpha}(x',r^\varepsilon)\right|\le\frac{1}{(r^\varepsilon)^2}\int_0^{r^\varepsilon}\int_0^{r^\varepsilon}\left|\frac{d^2u^a_\alpha}{dx_3^2}(\tau)\right|d\tau\,dt\le C,$$

so that T_1 tends to zero, in $L^2(\{0<x_3<r^\varepsilon\})$.

∘ **Term T_2:** We have

$$T_2=\frac{1}{r^\varepsilon}v^a_\alpha(x',r^\varepsilon)-\frac{\varepsilon}{r^\varepsilon}\left(\zeta^b_\alpha(r^\varepsilon x')+\varepsilon v^b_\alpha(r^\varepsilon x',0)\right).$$

But, as $c(0)=0$,

$$\left|\frac{1}{r^\varepsilon}v^a_\alpha(x',r^\varepsilon)\right|\le\frac{C}{r^\varepsilon}\left|c(r^\varepsilon)\right|\le C$$

and this term tends to zero, in $L^2(\{0<x_3<r^\varepsilon\})$. Moreover, as $\varepsilon^2\ll r^\varepsilon$ and as ζ^b_α and v^b_α are uniformly bounded, due to the regularity conditions (6.89),

$$\left|\frac{\varepsilon}{r^\varepsilon}\left(\zeta^b_\alpha(r^\varepsilon x')+\varepsilon v^b_\alpha(r^\varepsilon x',0)\right)\right|\le C\frac{\varepsilon}{r^\varepsilon},$$

so that the $L^2(\{0<x_3<r^\varepsilon\})$-norm of this term is bounded by $C\varepsilon/\sqrt{r^\varepsilon}$, which tends to zero by assumption.

∘ **Terms T_3 and T_4:** Clearly these terms are bounded, due to the regularity conditions (6.89).

7 Proof of stronger convergences and proof of Corollary 1

Actually, the stronger convergences in Theorem 1 are deduced from Corollary 1. The proof preceeds as follows.

Taking $\overline{u}^\varepsilon=(\overline{u}^{a\varepsilon},\overline{u}^{b\varepsilon})$ as test function in the variational equation of Problem (2.5), we get

$$\begin{aligned}\mathcal{E}^\varepsilon&=\textstyle\int_{\Omega^a}[A^a\overline{e}^{a\varepsilon},\overline{e}^{a\varepsilon}]\,dx+q^\varepsilon\int_{\Omega^b}[A^b\overline{e}^{b\varepsilon},\overline{e}^{b\varepsilon}]\,dx=\\&=\textstyle\int_{\Omega^a}f^{a\varepsilon}.\overline{u}^{a\varepsilon}\,dx+\int_{\Omega^b}f^{b\varepsilon}.\overline{u}^{b\varepsilon}\,dx+\int_{\Omega^a}[g^{a\varepsilon},\overline{e}^{a\varepsilon}]\,dx+\int_{\Omega^b}[g^{b\varepsilon},\overline{e}^{b\varepsilon}]\,dx+\\&\quad\textstyle\int_{\Sigma^a}h^{a\varepsilon}.\overline{u}^{a\varepsilon}\,d\sigma+\int_{\omega^b}\left(h^{b\varepsilon}_+.\overline{u}^{b\varepsilon}_{|x_3=0}+h^{b\varepsilon}_-.\overline{u}^{b\varepsilon}_{|x_3=-1}\right)dx'.\end{aligned}\tag{7.99}$$

We are going to pass to the limit in the last member of the above equality.

- If $q \in (0, +\infty)$, we have, from the convergences already proved in Theorem 1 and from classical compactness arguments,

$$(\overline{e}^{a\varepsilon}, \overline{e}^{b\varepsilon}) \rightharpoonup (\overline{e}^{a}, \overline{e}^{b}) \quad \text{weakly in } (L^2(\Omega^a))^{3\times 3} \times (L^2(\Omega^b))^{3\times 3},$$

$$(\overline{u}^{a\varepsilon}, \overline{u}^{b\varepsilon}) \to (\overline{u}^{a}, \overline{u}^{b}) \quad \text{strongly in } (L^2(\Omega^a))^{3} \times (L^2(\Omega^b))^{3},$$

$$\overline{u}^{a\varepsilon}_{|\Sigma^a} \to \overline{u}^{a}_{|\Sigma^a} \quad \text{strongly in } (L^2(\Sigma^a))^3,$$

$$\overline{u}^{b\varepsilon}_{|x_3=0} \quad (\text{ resp } \overline{u}^{b\varepsilon}_{|x_3=-1}) \to \overline{u}^{b}_{|x_3=0} \quad (\text{ resp } \overline{u}^{b}_{|x_3=-1}) \quad \text{strongly in } (L^2(\omega^b))^3.$$

If $(g^{a\varepsilon}, g^{b\varepsilon})$ tends to (g^a, g^b) strongly in $(L^2(\Omega^a))^{3\times 3} \times (L^2(\Omega^b))^{3\times 3}$, it follows that

$$\begin{aligned}
\mathcal{E}^\varepsilon = {} & \textstyle\int_{\Omega^a} f^{a\varepsilon}.\overline{u}^{a\varepsilon}\,dx + \int_{\Omega^b} f^{b\varepsilon}.\overline{u}^{b\varepsilon}\,dx + \int_{\Omega^a}[g^{a\varepsilon}, \overline{e}^{a\varepsilon}]\,dx + \int_{\Omega^b}[g^{b\varepsilon}, \overline{e}^{b\varepsilon}]\,dx + \int_{\Sigma^a} h^{a\varepsilon}.\overline{u}^{a\varepsilon}\,d\sigma + \\
& + \textstyle\int_{\omega^b} \left(h^{b\varepsilon}_{+}.\overline{u}^{b\varepsilon}_{|x_3=0} + h^{b\varepsilon}_{-}.\overline{u}^{b\varepsilon}_{|x_3=-1} \right) dx' \longrightarrow \\
& \textstyle\int_{\Omega^a} f^{a}.\overline{u}^{a}\,dx + \int_{\Omega^b} f^{b}.\overline{u}^{b}\,dx + \int_{\Omega^a}[g^{a}, \overline{e}^{a}]\,dx + \int_{\Omega^b}[g^{b}, \overline{e}^{b}]\,dx + \int_{\Sigma^a} h^{a}.\overline{u}^{a}\,d\sigma + \\
& + \textstyle\int_{\omega^b} \left(h^{b}_{+}.\overline{u}^{b}_{+|x_3=0} + h^{b}_{-}.\overline{u}^{b}_{|x_3=-1} \right) dx' = \\
& = \textstyle\int_{\Omega^a}[A^a\overline{e}^a, \overline{e}^a]\,dx + q\int_{\Omega^b}[A^b\overline{e}^b, \overline{e}^b]\,dx = \mathcal{E},
\end{aligned}$$

which proves the first part of Corollary 1. Moreover, we get, from the convergence of $\mathcal{E}^\varepsilon$ to $\mathcal{E}$ and from a classical lower semicontinuity argument:

$$\begin{aligned}
0 \; & = \liminf \left(\textstyle\int_{\Omega^a}[A^a\overline{e}^{a\varepsilon}, \overline{e}^{a\varepsilon}]\,dx - \int_{\Omega^a}[A^a\overline{e}^{a}, \overline{e}^{a}]\,dx + q^\varepsilon \int_{\Omega^b}[A^b\overline{e}^{b\varepsilon}, \overline{e}^{b\varepsilon}]\,dx - q\int_{\Omega^b}[A^b\overline{e}^{b}, \overline{e}^{b}]\,dx \right) \\
& \geq \liminf \left(\textstyle\int_{\Omega^a}[A^a\overline{e}^{a\varepsilon}, \overline{e}^{a\varepsilon}]\,dx - \int_{\Omega^a}[A^a\overline{e}^{a}, \overline{e}^{a}]\,dx \right) + \\
& \qquad \liminf \left(q^\varepsilon \textstyle\int_{\Omega^b}[A^b\overline{e}^{b\varepsilon}, \overline{e}^{b\varepsilon}]\,dx - q\int_{\Omega^b}[A^b\overline{e}^{b}, \overline{e}^{b}]\,dx \right) \\
& = \liminf \left(\textstyle\int_{\Omega^a}[A^a\overline{e}^{a\varepsilon}, \overline{e}^{a\varepsilon}]\,dx - \int_{\Omega^a}[A^a\overline{e}^{a}, \overline{e}^{a}]\,dx \right) + \\
& \qquad \liminf q \left(\textstyle\int_{\Omega^b}[A^b\overline{e}^{b\varepsilon}, \overline{e}^{b\varepsilon}]\,dx - \int_{\Omega^b}[A^b\overline{e}^{b}, \overline{e}^{b}]\,dx \right) \geq 0,
\end{aligned}$$

which gives, up to extraction of a new subsequence,

$$\textstyle\int_{\Omega^a}[A^a\overline{e}^{a\varepsilon}, \overline{e}^{a\varepsilon}]\,dx \longrightarrow \int_{\Omega^a}[A^a\overline{e}^{a}, \overline{e}^{a}]\,dx,$$

$$\textstyle\int_{\Omega^b}[A^b\overline{e}^{b\varepsilon}, \overline{e}^{b\varepsilon}]\,dx \longrightarrow \int_{\Omega^b}[A^b\overline{e}^{b}, \overline{e}^{b}]\,dx.$$

It follows that (e.g.)

$$C \, \|\overline{e}^{a\varepsilon} - \overline{e}^{a}\|^2_{(L^2(\Omega^a))^{3\times 3}} \leq \int_{\Omega^a} [A^a(\overline{e}^{a\varepsilon} - \overline{e}^{a}), (\overline{e}^{a\varepsilon} - \overline{e}^{a})] \, dx =$$

$$= \int_{\Omega^a} [A^a \overline{e}^{a\varepsilon}, \overline{e}^{a\varepsilon}] \, dx + \int_{\Omega^a} [A^a \overline{e}^{a}, \overline{e}^{a}] \, dx - \int_{\Omega^a} [A^a \overline{e}^{a\varepsilon}, \overline{e}^{a}] \, dx - \int_{\Omega^a} [A^a \overline{e}^{a}, \overline{e}^{a\varepsilon}] \, dx \longrightarrow 0,$$

and hence $\overline{e}^{a\varepsilon}$ tends to $\overline{e}^{a}$ strongly in $(L^2(\Omega^a))^{3\times 3}$. It follows that $e(\overline{u}^{a\varepsilon}) \to e(\overline{u}^{a})$ strongly in $(L^2(\Omega^a))^{3\times 3}$ and then, from Korn's inequality, $\overline{u}^{a\varepsilon} \to \overline{u}^{a}$ strongly in $H^1(\Omega^a)^3$. By the same way, $\overline{e}^{b\varepsilon}$ tends to $\overline{e}^{b}$ strongly in $(L^2(\Omega^b))^{3\times 3}$ and $\overline{u}^{b\varepsilon} \to \overline{u}^{b}$ strongly in $H^1(\Omega^b)^3$. The conclusion is that we get the stronger convergences mentionned in Theorem 1, if $q \in (0, +\infty)$.

- If $q = +\infty$, we have seen that

$$\overline{u}^{b\varepsilon} \to \overline{u}^{b} = 0 \text{ strongly in } (H^1(\Omega^b))^3,$$
$$\overline{e}^{b\varepsilon} \to \overline{e}^{b} = 0 \text{ strongly in } (L^2(\Omega^b))^{3\times 3},$$

and, with appropriate changes in the above proof, we have, if $g^{a\varepsilon}$ tend to g^a strongly in $(L^2(\Omega^a))^{3\times 3}$,

$$\mathcal{E}^\varepsilon \longrightarrow \int_{\Omega^a} f^a.\overline{u}^a \, dx + \int_{\Omega^a} [g^a, \overline{e}^a] \, dx + \int_{\Sigma^a} h^a.\overline{u}^a \, d\sigma = \int_{\Omega^a} [A^a \overline{e}^a, \overline{e}^a] \, dx = \mathcal{E}_\infty,$$

$$0 = \liminf \left(\int_{\Omega^a} [A^a \overline{e}^{a\varepsilon}, \overline{e}^{a\varepsilon}] \, dx - \int_{\Omega^a} [A^a \overline{e}^{a}, \overline{e}^{a}] \, dx + q^\varepsilon \int_{\Omega^b} [A^b \overline{e}^{b\varepsilon}, \overline{e}^{b\varepsilon}] \, dx \right)$$

$$\geq \liminf \left(\int_{\Omega^a} [A^a \overline{e}^{a\varepsilon}, \overline{e}^{a\varepsilon}] \, dx - \int_{\Omega^a} [A^a \overline{e}^{a}, \overline{e}^{a}] \, dx \right) + \liminf \left(q^\varepsilon \int_{\Omega^b} [A^b \overline{e}^{b\varepsilon}, \overline{e}^{b\varepsilon}] \, dx \right) \geq 0,$$

$$\int_{\Omega^a} [A^a \overline{e}^{a\varepsilon}, \overline{e}^{a\varepsilon}] \, dx \longrightarrow \int_{\Omega^a} [A^a \overline{e}^{a}, \overline{e}^{a}] \, dx,$$

$$q^\varepsilon \int_{\Omega^b} [A^b \overline{e}^{b\varepsilon}, \overline{e}^{b\varepsilon}] \, dx \longrightarrow 0,$$

$$\overline{e}^{a\varepsilon} \to \overline{e}^{a} \text{ strongly in } (L^2(\Omega^a))^{3\times 3},$$

$$\sqrt{q^\varepsilon} \overline{e}^{b\varepsilon} \to 0 \text{ strongly in } (L^2(\Omega^a))^{3\times 3},$$

$$\overline{u}^{a\varepsilon} \to \overline{u}^{a} \text{ strongly in } H^1(\Omega^a)^3.$$

- If $q = 0$, we have, with $\tilde{u}^\varepsilon = q^\varepsilon \overline{u}^\varepsilon$, $\tilde{e}^\varepsilon = q^\varepsilon \overline{e}^\varepsilon$,

$$\mathcal{E}^\varepsilon = \frac{1}{q^\varepsilon} \int_{\Omega^a} [A^a \tilde{e}^{a\varepsilon}, \tilde{e}^{a\varepsilon}] \, dx + \int_{\Omega^b} [A^b \tilde{e}^{b\varepsilon}, \tilde{e}^{b\varepsilon}] \, dx =$$

$$= \int_{\Omega^a} f^{a\varepsilon}.\tilde{u}^{a\varepsilon} \, dx + \int_{\Omega^b} f^{b\varepsilon}.\tilde{u}^{b\varepsilon} \, dx + \int_{\Omega^a} [g^{a\varepsilon}, \tilde{e}^{a\varepsilon}] \, dx + \int_{\Omega^b} [g^{b\varepsilon}, \tilde{e}^{b\varepsilon}] \, dx + \tag{7.100}$$

$$\int_{\Sigma^a} h^{a\varepsilon}.\tilde{u}^{a\varepsilon} \, d\sigma + \int_{\omega^b} \left(h^{b\varepsilon}_+.\tilde{u}^{b\varepsilon}_{|x_3=0} + h^{b\varepsilon}_-.\tilde{u}^{b\varepsilon}_{|x_3=-1} \right) dx',$$

$$\tilde{u}^{a\varepsilon} \to \overline{u}^a = 0 \text{ strongly in } (H^1(\Omega^a))^3,$$
$$\tilde{e}^{a\varepsilon} \to \overline{e}^a = 0 \text{ strongly in } (L^2(\Omega^a))^{3\times 3},$$

and we have, if $g^{b\varepsilon}$ tend to g^b strongly in $(L^2(\Omega^b))^{3\times 3}$,

$$\mathcal{E}^\varepsilon \longrightarrow \int_{\Omega^b} f^b.\overline{u}^b\,dx + \int_{\Omega^b}[g^b,\overline{e}^b]\,dx + \int_{\omega^b}\Big(h^b_+.\overline{u}^b_{|x_3=0} + h^b_-.\overline{u}^b_{|x_3=-1}\Big)\,dx' = \int_{\Omega^b}[A^b\overline{e}^b,\overline{e}^b]\,dx = \mathcal{E}_0,$$

$$\begin{aligned} 0 &= \liminf\Big(\tfrac{1}{q^\varepsilon}\textstyle\int_{\Omega^a}[A^a\tilde{e}^{a\varepsilon},\tilde{e}^{a\varepsilon}]\,dx + \int_{\Omega^b}[A^b\tilde{e}^{b\varepsilon},\tilde{e}^{b\varepsilon}]\,dx - \int_{\Omega^b}[A^b\overline{e}^b,\overline{e}^b]\,dx\Big) \\ &\geq \liminf\Big(\tfrac{1}{q^\varepsilon}\textstyle\int_{\Omega^a}[A^a\tilde{e}^{a\varepsilon},\tilde{e}^{a\varepsilon}]\,dx\Big) + \liminf\Big(\int_{\Omega^b}[A^b\tilde{e}^{b\varepsilon},\tilde{e}^{b\varepsilon}]\,dx - \int_{\Omega^b}[A^b\overline{e}^b,\overline{e}^b]\,dx\Big) \geq 0, \end{aligned}$$

$$\textstyle\int_{\Omega^b}[A^b\tilde{e}^{b\varepsilon},\tilde{e}^{b\varepsilon}]\,dx \longrightarrow \int_{\Omega^b}[A^a\overline{e}^b,\overline{e}^b]\,dx,$$

$$\tfrac{1}{q^\varepsilon}\textstyle\int_{\Omega^a}[A^a\tilde{e}^{a\varepsilon},\tilde{e}^{a\varepsilon}]\,dx \longrightarrow 0,$$

$$q^\varepsilon\overline{e}^{b\varepsilon} = \tilde{e}^{b\varepsilon} \to \overline{e}^b \text{ strongly in } (L^2(\Omega^b))^{3\times 3},$$

$$\sqrt{q^\varepsilon}\;\overline{e}^{a\varepsilon} = \tfrac{1}{\sqrt{q^\varepsilon}}\tilde{e}^{a\varepsilon} \to 0 \text{ strongly in } (L^2(\Omega^b))^{3\times 3},$$

$$q^\varepsilon\overline{u}^{b\varepsilon} \to \overline{u}^b \text{ strongly in } H^1(\Omega^b)^3.$$

8 Appendix

8.1 The definitions of (v^a,w^a) and (v^b,w^b) as suitable limits

For the convenience of the reader, we give in this Appendix a sketch of proof of the following result, mentionned in Section 4.2. (For thin cylinders, a complete proof can be find in [22]. The case of plates is analogous and simpler, but it is not published, as far as we know.)

Lemma 4 *(i) Let $\{u^\varepsilon\}_\varepsilon$ be a sequence in $(H^1(\Omega^a))^3$, such that $u^\varepsilon = 0$ on $T^a = \omega^a \times \{1\}$ and*

$$\{e^{a\varepsilon}(u^\varepsilon)\}_\varepsilon \text{ is bounded in } (L^2(\Omega^a))^{3\times 3}. \tag{8.101}$$

Let $\mathcal{W}^a$ be the space defined in Section 2.2 and let

$$\begin{aligned}\mathcal{V}^a_- = \{v^a \in (H^1(\Omega^a))^2 \times L^2(0,1;H^1(\omega^a)),\ \exists c \in H^1(0,1), c(1)=0, v^a_1 = -c\,x_2, v^a_2 = c\,x_1,\\ \text{for a.e. } x_3 \in (0,1),\ \textstyle\int_{\omega^a} v^a_3(x',x_3)\,dx' = 0\}.\end{aligned}$$

(Note that $\mathcal{V}^a_-$ satisfies the same requirements as $\mathcal{V}^a$, in Section 2.2, except $c(0)=0$.) Then there exists a pair $(v^a,w^a) \in \mathcal{V}^a_- \times \mathcal{W}^a$, such that, for all $\alpha,\beta = 1,2$:

$$\frac{1}{r^\varepsilon}e_{\alpha 3}(u^\varepsilon) \rightharpoonup e_{\alpha 3}(v^a) \text{ weakly in } L^2(\Omega^a), \tag{8.102}$$

$$\frac{1}{(r^\varepsilon)^2} e_{\alpha\beta}(u^\varepsilon) \rightharpoonup e_{\alpha\beta}(w^a) \text{ weakly in } L^2(\Omega^a). \tag{8.103}$$

Morever, denoting by (c, v_3^a) *the couple defining* v^a *and setting*

$$c^\varepsilon(x_3) = \frac{\int_{\omega^a} \left(x_1 u_2^\varepsilon(x', x_3) - x_2 u_1^\varepsilon(x', x_3)\right) \, dx'}{r^\varepsilon \int_{\omega^a} \left(x_1^2 + x_2^2\right) \, dx'}, \tag{8.104}$$

$$v_3^\varepsilon = \frac{u_3^\varepsilon}{r^\varepsilon} - \frac{1}{|\omega^a|} \int_{\omega^a} \frac{u_3^\varepsilon}{r^\varepsilon} \, dx' + \frac{1}{|\omega^a|} \sum_\alpha x_\alpha \frac{d}{dx_3} \int_{\omega^a} u_\alpha^\varepsilon \, dx', \tag{8.105}$$

we have

$$c^\varepsilon \to c \text{ strongly in } L^2(0,1), \tag{8.106}$$

$$v_3^\varepsilon \rightharpoonup v_3^a \text{ weakly in } H^{-1}(0,1; H^1(\omega^a)). \tag{8.107}$$

Finally, setting

$$d_\alpha^\varepsilon(x_3) = \frac{1}{|\omega^a|} \int_{\omega^a} \frac{u_\alpha^\varepsilon(x', x_3)}{r^\varepsilon} \, dx' \tag{8.108}$$

and $x_1^R = -x_2$, $x_2^R = x_1$, *we have*

$$\frac{u_\alpha^\varepsilon}{(r^\varepsilon)^2} - \frac{1}{r^\varepsilon} \left(c^\varepsilon x_\alpha^R + d_\alpha^\varepsilon\right) \rightharpoonup w_\alpha^a \text{ weakly in } L^2(0,1; H^1(\omega^a)). \tag{8.109}$$

(ii) If $\{u^\varepsilon\}_\varepsilon$ *is a sequence in* $(H^1(\Omega^b))^3$, *such that* $u^\varepsilon = 0$ *on* $\Sigma^b = \partial\omega^b \times (-1, 0)$ *and*

$$\{e^{b\varepsilon}(u^\varepsilon)\}_\varepsilon \text{ is bounded in } (L^2(\Omega^b))^{3\times 3}, \tag{8.110}$$

then there exists a pair $(v^b, w^b) \in \mathcal{V}^b \times \mathcal{W}^b$, *such that, for all* $\alpha = 1, 2$*:*

$$\frac{1}{\varepsilon} e_{\alpha 3}(u^\varepsilon) \rightharpoonup e_{\alpha 3}(v^b) \text{ weakly in } L^2(\Omega^b), \tag{8.111}$$

$$\frac{1}{\varepsilon^2} e_{33}(u^\varepsilon) \rightharpoonup e_{33}(w^b) \text{ weakly in } L^2(\Omega^b). \tag{8.112}$$

In addition, we have

$$\frac{u_\alpha^\varepsilon}{\varepsilon} - \tilde{u}_\alpha^\varepsilon - \int_{-1}^0 \left(\frac{u_\alpha^\varepsilon}{\varepsilon} - \tilde{u}_\alpha^\varepsilon\right) dx_3 \rightharpoonup v_\alpha^b \text{ weakly in } L^2(\omega^b; H^1(-1,0)), \quad \text{for } \alpha = 1, 2, \tag{8.113}$$

with $\tilde{u}^\varepsilon$ *defined by*

$$\tilde{u}_\alpha^\varepsilon = -\int_0^{x_3} \frac{1}{\varepsilon} \frac{\partial u_3^\varepsilon}{\partial x_\alpha}(x', s) ds.$$

Moreover,

$$\frac{u_3^\varepsilon}{\varepsilon^2} - \int_{-1}^0 \frac{u_3^\varepsilon}{\varepsilon^2} \, dx_3 \rightharpoonup w_3^b \text{ weakly in } L^2(\omega^b; H^1(-1,0)). \tag{8.114}$$

Proof of (i): We use the following decomposition and estimate, whose proof may be found for instance in [16] or in [17]: there exists a positive constant C such that, for every u in $L^2(0,1;H^1(\omega^a))^2$, there exist $\overline{u}$ and $\hat{u}$ satisfying:

$$\begin{cases} u = \overline{u} + \hat{u}, \\ \int_{\omega^a} \overline{u}_\alpha(x', x_3)\, dx' \equiv 0,\ \int_{\omega^a} (x_1 \overline{u}_2(x', x_3) - x_2 \overline{u}_1(x', x_3))\, dx' \equiv 0, \\ e_{\alpha\beta}(\hat{u}) = 0, \qquad \forall \alpha, \beta = 1, 2, \end{cases} \tag{8.115}$$

$$\|\overline{u}\|_{(L^2(0,1;H^1(\omega^a)))^2} \leq C \sum_{\alpha,\beta} \|e_{\alpha\beta}(u)\|_{L^2(\Omega^a)}. \tag{8.116}$$

The function $\hat{u}$ is a rigid displacement:

$$\hat{u}_\alpha(x', x_3) = c(x_3) x^R_\alpha + d_\alpha(x_3), \tag{8.117}$$

with $x^R_1 = -x_2$, $x^R_2 = x_1$ (R for "rotation"). Applying (8.115) and (8.117) to $u = \frac{1}{r^\varepsilon}(u^\varepsilon_1, u^\varepsilon_2)$, we get:

$$\frac{1}{r^\varepsilon} u^\varepsilon_\alpha = \overline{u}^\varepsilon_\alpha + \hat{u}^\varepsilon_\alpha, \text{ with } \hat{u}^\varepsilon_\alpha = c^\varepsilon(x_3) x^R_\alpha + d^\varepsilon_\alpha(x_3). \tag{8.118}$$

One can check easily that the functions c^ε and d^ε_α are given in terms of u^ε by the formulae (8.104) and (8.108). ¿From (8.116), we obtain:

$$\|\overline{u}^\varepsilon_\alpha\|_{L^2(0,1;H^1(\omega^a))} \leq C \sum_{\alpha,\beta} \|e_{\alpha\beta}(\frac{1}{r^\varepsilon} u^\varepsilon)\|_{L^2(\Omega^a)}.$$

Setting $w^\varepsilon_\alpha = \overline{u}^\varepsilon_\alpha / r^\varepsilon$ and using (8.101), it follows that

$$\|w^\varepsilon_\alpha\|_{L^2(0,1;H^1(\omega^a))} \leq C.$$

So, taking a subsequence of ε, still denoted by the same letter, we may assume the existence of w^a_α such that

$$w^\varepsilon_\alpha \rightharpoonup w^a_\alpha \text{ weakly in } L^2(0,1;H^1(\omega^a)), \forall \alpha = 1, 2,$$

that is (8.109). Moreover it is clear that $(w^\varepsilon_1, w^\varepsilon_2, 0)$ and $w^a = (w^a_1, w^a_2, 0)$ belong to $\mathcal{W}^a$. Since (8.118) implies that

$$\frac{1}{(r^\varepsilon)^2} e_{\alpha\beta}(u^\varepsilon) = e_{\alpha\beta}(w^\varepsilon),$$

we see that (8.103) is proved.

It remains to prove the convergences involving v^a. In Section 5.3, it is proved that there exists c in $H^1(0,1)$, $c(1) = 0$, such that, for a subsequence of ε, (8.106) holds true. As for the other convergences involving v^a, we use again the decomposition (8.118), from which we deduce the following equality:

$$\frac{2}{r^\varepsilon} e_{\alpha 3}(u^\varepsilon) = \frac{\partial \overline{u}^\varepsilon_\alpha}{\partial x_3} + \frac{dc^\varepsilon}{dx_3} x^R_\alpha + \frac{dd^\varepsilon_\alpha}{dx_3} + \frac{1}{r^\varepsilon} \frac{\partial u^\varepsilon_3}{\partial x_\alpha}, \forall \alpha = 1, 2. \tag{8.119}$$

Now, setting

$$v_3^\varepsilon = \frac{u_3^\varepsilon}{r^\varepsilon} - \frac{1}{|\omega^a|}\int_{\omega^a} \frac{u_3^\varepsilon}{r^\varepsilon}\, dx' + x_\beta \frac{d}{dx_3} d_\beta^\varepsilon(x_3)$$

(the summation convention is used, concerning the index β), equality (8.119) can be written as:

$$\frac{2}{r^\varepsilon} e_{\alpha 3}(u^\varepsilon) = \frac{dc^\varepsilon}{dx_3} x_\alpha^R + \frac{\partial v_3^\varepsilon}{\partial x_\alpha} + \frac{\partial \overline{u}_\alpha^\varepsilon}{\partial x_3}. \tag{8.120}$$

The following estimate is proved in [20]:

$$\|v_3^\varepsilon\|_{H^{-1}(0,1;H^1(\omega^a))} \le C \left(\sum_{\alpha\beta} \|e_{\alpha\beta}(\frac{u^\varepsilon}{r^\varepsilon})\|_{L^2(\Omega^a)} + \sum_\alpha \|e_{\alpha 3}(\frac{u^\varepsilon}{r^\varepsilon})\|_{L^2(\Omega^a)} \right),$$

so that, from (8.101), the sequence $\{v_3^\varepsilon\}_\varepsilon$ is bounded in $H^{-1}(0,1;H^1(\omega^a))$. Hence there exists v_3^a in $H^{-1}(0,1;H^1(\omega^a))$, having zero mean-value on ω^a, such that (8.107) holds true (for a subsequence). It follows also from (8.101) that

$$\frac{1}{r^\varepsilon} e_{\alpha 3}(u^\varepsilon) \rightharpoonup \tau_{\alpha 3} \text{ weakly in } L^2(\Omega^a) \tag{8.121}$$

(again for some subsequence and some $\tau_{\alpha 3}$ in $L^2(\Omega^a)$). Moreover, since w_α^ε is bounded in $L^2(0,1;H^1(\omega^a))$,

$$\frac{\partial \overline{u}_\alpha^\varepsilon}{\partial x_3} = r^\varepsilon \frac{\partial w_\alpha^\varepsilon}{\partial x_3} \text{ tends to 0 in the sense of distributions .} \tag{8.122}$$

By passing to the limit in (8.120), using (8.106), (8.107), (8.121) and (8.122), we get:

$$2\tau_{\alpha 3} = \frac{dc}{dx_3} x_\alpha^R + \frac{\partial}{\partial x_\alpha} v_3^a, \tag{8.123}$$

which implies that

$$\frac{\partial}{\partial x_\alpha} v_3^a \in L^2(\Omega^a). \tag{8.124}$$

¿From (8.124) and from the fact that v_3^a is in $H^{-1}(0,1;H^1(\omega^a))$, it is then an exercise to get that v_3^a is in $L^2(0,1;H^1(\omega^a))$, so that $v^a := (c(x_3)x_\alpha^R, v_3^a) \in \mathcal{V}_-^a$ and satisfies (8.102).

Proof of (ii): Now we prove the analogous of the previous properties in the framework of 3d-2d reduction of dimension. This is much easier. Indeed, in order to prove (8.111) and (8.113), we consider the sequence $\{v_\alpha^\varepsilon\}_\varepsilon$ defined by:

$$v_\alpha^\varepsilon = \frac{u_\alpha^\varepsilon}{\varepsilon} - \tilde{u}_\alpha^\varepsilon - \int_{-1}^0 \left(\frac{u_\alpha^\varepsilon}{\varepsilon} - \tilde{u}_\alpha^\varepsilon \right)$$

and

$$\tilde{u}^\varepsilon = \left(-\int_0^{x_3} \frac{1}{\varepsilon}\frac{\partial u_3^\varepsilon}{\partial x_1}(x',s)ds, -\int_0^{x_3} \frac{1}{\varepsilon}\frac{\partial u_3^\varepsilon}{\partial x_2}(x',s)ds, \frac{u_3^\varepsilon}{\varepsilon} \right),$$

Then we have as above that

$$\frac{\partial v_\alpha^\varepsilon}{\partial x_3} = \frac{2}{\varepsilon} e_{\alpha 3}(u^\varepsilon) \tag{8.125}$$

is bounded in $L^2(\Omega^b)$, as a consequence of (8.110), and, as v_α^ε has meanvalue zero with respect to x_3, it results that it is bounded in $L^2(\omega^b; H^1(-1,0))$, so that (8.113) holds true, i.e.

$$v_\alpha^\varepsilon \rightharpoonup v_\alpha^b \text{ weakly in } L^2(\omega^b; H^1(-1,0)), \tag{8.126}$$

for some subsequence of ε and for some v_α^b in $L^2(\omega^b; H^1(-1,0))$. Setting $v^b = (v_1^b, v_2^b, 0)$, we get

$$e_{\alpha 3}(v^b) = \frac{1}{2}\frac{\partial v_\alpha^b}{\partial x_3},$$

so that we derive (8.111) from (8.125) and (8.126).

Finally we prove (8.112) and (8.114), by introducing the sequence of functions

$$w^\varepsilon = \frac{1}{\varepsilon^2} u_3^\varepsilon - \int_{-1}^0 \frac{1}{\varepsilon^2} u_3^\varepsilon \, dx_3,$$

which is bounded in $L^2(\omega^b; H^1(-1,0))$, since

$$\frac{\partial w^\varepsilon}{\partial x_3} = \frac{1}{\varepsilon^2} e_{33}(u^\varepsilon)$$

is bounded in $L^2(\Omega^b)$, due to (8.110). So, extracting a subsequence, we can find w_3^b in $L^2(\omega^b; H^1(-1,0))$, having meanvalue zero in x_3, such that (8.112) and (8.114) hold true, which ends the proof of Lemma 8.

8.2 The density arguments

In Section 6, we have mentionned four density arguments. These are stated in the following lemmata and proved hereafter. This is done for sake of completeness, since Lemmata 7 and 8 are very classical, Lemma 5 is classical and very similar to the density result proved in [11], while Lemma 6, though less classical, results from Theorem 9.1.3 of [2].

Lemma 5 *Let $v \in H_0^1(\omega^b)$, $0 \in \omega^b \subset \mathbb{R}^2$. There exist a sequence of positive numbers r^n, tending to zero, and a sequence of functions $v^n \in H_0^1(\omega^b)$, such that*

$$v^n \equiv 0 \textit{ in the ball } B^n \textit{ of center } 0 \textit{ and radius } r^n \ ,$$

$$v^n \to v \textit{ in } H_0^1(\omega^b).$$

Proof: Let $\tilde{V} = \{v \in \mathcal{C}^1(\overline{\omega^b}),\ v = 0 \text{ on } \partial\omega^b\}$. As $\tilde{V}$ is dense in $H_0^1(\omega^b)$, we may restrict to v in $\tilde{V}$. Then the proof goes as follows. For any integer n, we consider two balls B^n and B'^n in $\omega^b \subset \mathbf{R}^2$, with center 0 and respective radii r^n and R^n, to be determined later on, and such that $0 < r^n < R^n$, R^n tends to zero as n tends to infinity. We define $v^n \in H_0^1(\omega^b)$ by:

$$v^n = 0 \text{ in } B^n,\ v^n = v \text{ in } \omega^b \setminus B'^n,\ v^n = (1-\phi^n)v \text{ in } B'^n \setminus B^n,$$

where ϕ^n is the solution of the capacity problem in $B'^n \setminus B^n$:

$$\Delta\phi^n = 0 \text{ in } B'^n \setminus B^n,\ \phi^n = 1 \text{ on } \partial B^n,\ \phi^n = 0 \text{ on } \partial B'^n.$$

It is clear that $v^n \in W^{1,\infty}(\omega^b) \cap H_0^1(\omega^b)$ and $v^n \equiv 0$ in B^n. We are going to prove that, for convenient r^n and R^n, $v^n \to v$ in $H_0^1(\omega^b)$. Actually, as $0 \leq \phi^n \leq 1$,

$$\begin{aligned}
\|v^n - v\|^2_{H_0^1(\omega^b)} &= \textstyle\int_{B'^n} |\nabla(v^n - v)|^2\, dx' \\
&= \textstyle\int_{B^n} |\nabla v|^2\, dx' + \int_{B'^n \setminus B^n} |\phi^n \nabla v + v\nabla\phi^n|^2\, dx' \\
&\leq \textstyle\int_{B^n} |\nabla v|^2\, dx' + 2\int_{B'^n \setminus B^n} |\phi^n \nabla v|^2\, dx' + 2\int_{B'^n \setminus B^n} |v\nabla\phi^n|^2\, dx' \\
&\leq 3\textstyle\int_{B'^n} |\nabla v|^2\, dx' + 2\int_{B'^n \setminus B^n} |v\nabla\phi^n|^2\, dx' \\
&\leq 3\pi R^n \|\nabla v\|^2_{L^\infty(\omega^b)} + 2\|v\|^2_{L^\infty(\omega^b)} \textstyle\int_{B'^n \setminus B^n} |\nabla\phi^n|^2\, dx' \\
&= 3\pi R^n \|\nabla v\|^2_{L^\infty(\omega^b)} + 4\pi\|v\|^2_{L^\infty(\omega^b)} \left(\log \frac{R^n}{r^n}\right)^{-1}.
\end{aligned}$$

It is enough to take (e.g.) $r^n = 1/n^2$ and $R^n = 1/n$, in order to get $v^n \to v$ in $H_0^1(\omega^b)$.

Lemma 6 *Let $v \in H_0^2(\omega^b)$, $0 \in \omega^b \subset \mathbf{R}^2$, $v(0) = 0$. There exist a sequence of positive numbers r^n, tending to zero, and a sequence of functions $v^n \in H_0^2(\omega^b)$, such that*

$$v^n \equiv 0 \textit{ in the ball } B^n \textit{ of center } 0 \textit{ and radius } r^n\ ,$$

$$v^n \rightharpoonup v \textit{ weakly in } H_0^2(\omega^b).$$

Proof: 1) For any $v \in H_0^2(\omega^b)$, with $v(0) = 0$, there exists $\overline{v}^n \in \mathcal{C}^2(\overline{\omega^b}) \cap H_0^2(\omega^b)$, such that $\overline{v}^n \to v$ in $H^2(\omega^b)$ and hence in $\mathcal{C}^0(\overline{\omega^b})$. In particular, $\overline{v}^n(0) \to v(0) = 0$. Setting $v^n = \overline{v}^n - \overline{v}^n(0)\phi$, with $\phi \in \mathcal{D}(\omega^b)$ and $\phi(0) = 1$, it is clear that $v^n \in \mathcal{C}^2(\overline{\omega^b}) \cap H_0^2(\omega^b)$, $v^n(0) = 0$ and $v^n \to v$ in $H^2(\omega^b)$.

2) From step 1), we may restrict to v in $\mathcal{C}^2(\overline{\omega^b}) \cap H_0^2(\omega^b)$, $v(0) = 0$. Let $v^n = v\phi^n$, with $\phi^n(x') = \phi(n|x'|)$ and $\phi \in \mathcal{C}^\infty(\mathbf{R})$, $0 \leq \phi \leq 1$, $\phi \equiv 0$ on $(-\infty, 1]$, $\phi \equiv 1$ on $[2, +\infty)$. Clearly $v^n \in H_0^2(\omega^b)$, $v^n \equiv 0$ in the ball of center 0 and radius $1/n$ and

$$\int_{\omega^b} |v^n - v|^2\, dx' \leq \int_{|x'| < \frac{2}{n}} |v|^2\, dx' \to 0,$$

that is $v^n \to v$ in $L^2(\omega^b)$. Hence the Lemma is proved, as soon as we have proved that v^n is bounded uniformly in $H_0^2(\omega^b)$, i.e.

$$\frac{\partial^2 v^n}{\partial x_\alpha \partial x_\beta} \text{ is bounded in } L^2(\omega^b). \tag{8.127}$$

But

$$\frac{\partial^2 v^n}{\partial x_\alpha \partial x_\beta} = v\frac{\partial^2 \phi^n}{\partial x_\alpha \partial x_\beta} + \phi^n \frac{\partial^2 v}{\partial x_\alpha \partial x_\beta} + \frac{\partial v}{\partial x_\alpha}\frac{\partial \phi^n}{\partial x_\beta} + \frac{\partial v}{\partial x_\beta}\frac{\partial \phi^n}{\partial x_\alpha}.$$

The second term is obviously bounded in $L^\infty(\omega^b)$. Moreover, since

$$\frac{\partial \phi^n}{\partial x_\alpha} = n\phi'(n|x'|)\frac{x_\alpha}{|x'|} \text{ and } \frac{\partial^2 \phi^n}{\partial x_\alpha \partial x_\beta} = n^2\phi"(n|x'|)\frac{x_\alpha x_\beta}{|x'|^2} + n\phi'(n|x'|)\left(\frac{\delta_{\alpha\beta}}{|x'|} - \frac{x_\alpha x_\beta}{|x'|^3}\right),$$

it follows that

$$\left|\frac{\partial \phi^n}{\partial x_\alpha}\right| \leq Cn \text{ and } \left|\frac{\partial^2 \phi^n}{\partial x_\alpha \partial x_\beta}\right| \leq Cn^2,$$

$$\int_{\omega^b}\left|\frac{\partial v}{\partial x_\beta}\frac{\partial \phi^n}{\partial x_\alpha}\right|^2 dx' \leq C\left\|\frac{\partial v}{\partial x_\beta}\right\|_\infty^2 \int_{\frac{1}{n}<|x'|<\frac{2}{n}} n^2\, dx' = C\left\|\frac{\partial v}{\partial x_\beta}\right\|_\infty^2 \int_{1<|x'|<2} dx' = C,$$

$$\int_{\omega^b}|v\frac{\partial^2 \phi^n}{\partial x_\alpha \partial x_\beta}|^2\, dx' \leq \|v\|^2_{L^\infty(\frac{1}{n}<|x'|<\frac{2}{n})}\int_{\frac{1}{n}<|x'|<\frac{2}{n}} Cn^4\, dx' = Cn^2\|v\|^2_{L^\infty(\frac{1}{n}<|x'|<\frac{2}{n})}.$$

But, for $1/n < |x'| < 2/n$, $|v(x')| \leq C|x'| \leq C/n$, since v is regular and $v(0) = 0$. It follows that

$$\int_{\omega^b}|v\frac{\partial^2 \phi^n}{\partial x_\alpha \partial x_\beta}|^2\, dx' \leq C$$

and finally, (8.127) holds true, ending the proof of Lemma 5.

Lemma 7 *Let $v \in L^2(\omega^b; H^1(-1,0))$, $0 \in \omega^b \subset \mathbb{R}^2$. There exist a sequence of positive numbers r^n, tending to zero, and a sequence of functions v^n, such that*

$$v^n \in \mathcal{C}^1(\overline{\Omega^b}),$$

$$v^n \equiv 0 \textit{ in } B^n \times \{0\}, B^n \textit{ denoting the ball of center } 0 \textit{ and radius } r^n\ ,$$

$$v^n \to v \textit{ in } L^2(\omega^b; H^1(-1,0)).$$

Proof: By density of $\mathcal{C}^1(\overline{\Omega^b})$ in $L^2(\omega^b; H^1(-1,0))$, we may restrict to $v \in \mathcal{C}^1(\overline{\Omega^b})$. We consider a sequence r^n of positive numbers, converging to zero, and a sequence of functions $\phi^n : \omega^b \to \mathbf{R}$, of class $\mathcal{C}^\infty$, with $\phi^n \equiv 0$ in the ball B^n of center 0 and radius r^n, $\phi^n \equiv 1$

outside the ball B'^n of center 0 and radius $2r^n$, $0 \leq \phi^n \leq 1$ in $B'^n \setminus B^n$. We set $v^n = \phi^n v$. Then clearly $v^n \in \mathcal{C}^1(\overline{\Omega^b})$ and

$$\begin{aligned}
\|v^n - v\|^2_{L^2(\omega^b;H^1(-1,0))} &= \int_{\Omega^b} |v^n - v|^2\, dx + \int_{\Omega^b} |\tfrac{\partial}{dx_3}(v^n - v)|^2\, dx \\
&= \int_{B^n\times(-1,0)} |v|^2\, dx + \int_{(B'^n\setminus B^n)\times(-1,0)} |(1-\phi^n)v|^2\, dx + \\
&+ \int_{B^n\times(-1,0)} |\tfrac{\partial v}{\partial x_3}|^2\, dx + \int_{(B'^n\setminus B^n)\times(-1,0)} |(1-\phi^n)\tfrac{\partial v}{\partial x_3}|^2\, dx \\
&\leq \int_{B'^n\times(-1,0)} \left(|v|^2 + |\tfrac{\partial v}{\partial x_3}|^2\right) dx,
\end{aligned}$$

which tends to zero, as soon as r^n tends to zero.

Lemma 8 *Let* $U = H^1_T(0,1) = \{u \in H^1(0,1),\ u(1) = 0\}$, $\tilde{U} = \{u \in \mathcal{C}^1[0,1],\ u(1) = 0\}$, $V = H^2_0(\omega^b)$, $0 \in \omega^b \subset \mathbb{R}^2$, $\tilde{V} = \mathcal{C}^1(\overline{\omega^b}) \cap H^2_0(\omega^b)$. *Now let* $W = \{(u,v) \in U \times V,\ u(0) = v(0)\}$, $\tilde{W} = \{(u,v) \in \tilde{U} \times \tilde{V},\ u(0) = v(0)\}$. *Then* $\tilde{W}$ *is dense in* W.

Proof: It is clear that $\tilde{U}$ is dense in U and that $\tilde{V}$ is dense in V. Therefore, for any $(u,v) \in W$, there exists $(\overline{u}^n, \overline{v}^n) \in \tilde{U} \times \tilde{V}$ such that

$$\begin{aligned}
&\overline{u}^n \to u \text{ in } H^1(0,1) \text{ and hence in } \mathcal{C}^0[0,1], \\
&\overline{v}^n \to v \text{ in } H^2(\omega^b) \text{ and hence in } \mathcal{C}^0(\overline{\omega^b}).
\end{aligned}$$

Let $\phi^1 \in \mathcal{C}^\infty[0,1]$ with $\phi^1(0) = 1$, $\phi^1(1) = 0$, $\phi^2 \in \mathcal{D}(\omega^b)$ with $\phi^2(0) = 1$ and let

$$u^n = \overline{u}^n - (\overline{u}^n(0) - u(0))\phi^1,$$

$$v^n = \overline{v}^n - (\overline{v}^n(0) - v(0))\phi^2.$$

It is clear that $u^n \in \tilde{U}$, $v^n \in \tilde{V}$ and $u^n(0) = u(0) = v(0) = v^n(0)$, so that $(u^n, v^n) \in \tilde{W}$. Moreover

$$\|u^n - \overline{u}^n\|_{H^1(0,1)} = |\overline{u}^n(0) - u(0)| \|\phi^1\|_{H^1(0,1)} \to 0$$

and hence $u^n \to u$ in $H^1(0,1)$. Similarly $v^n \to v$ in $H^2(\omega^b)$.

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