Structure of Riemann solvers on networks (preliminary version 3)

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Simplicity is the ultimate improvement.

Science unveils simplicity in nature.

Abstract

In this work we consider scalar Riemann solvers on networks, associated to scalar conservation laws. A junction is a particular network which is a finite set of half lines glued together at the origin. Riemann solvers solve uniquely the Riemann problem on the junction. We also assume that Riemann solutions are stable by passage to the limit.

In part I, we only address fundamental questions concerning Riemann problems on junctions. We show a characterization of Riemann solvers either by their set of stationary solutions (the germ), or equivalently by their junction Godunov flux. Moreover, we show that the gluing of two junctions with Riemann solvers is well defined and leads to a new junction with a new Riemann solver. We also show that given a quasi Godunov flux (which does not satisfies all the properties of Godunov fluxes), there exists a nonlinear projection which allows to associate a unique Godunov flux. This projection is called relaxation. Our theory encompasses in particular Kružkov germs, (quasi) Hamilton-Jacobi germs, monotone germs, conservative and non-conservative germs.

In part II, we focus on the special case of importance in applications, where the fluxes of the network are bell-shaped. In this case, we show that the junction Godunov flux has a polar decomposition into a preflux composed with an explicit capacity. Hence all the information is encoded in the preflux. We then find explicit formulas in order to glue prefluxes, or to relax quasi-prefluxes. We also give some explicit characterizations of some conservative prefluxes where flux limiters appear naturally.

In part III, we give an existence and uniqueness theory for conservation laws on networks in the special case where Riemann solvers are associated to Kružkov germs.

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Keywords: network, junction, Riemann problem, Riemann solver, conservation laws, discontinuous flux.

Contents

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1 Introduction

In part I of this work, we consider scalar conservations laws on a junction. A junction of type $n : m$ consists in a set of $N := n + m$ half lines (branches) glued together at the origin, with n ingoing branches and m outgoing branches. We consider fluxes $f = (f^1, \ldots, f^N)$, with one flux on each branch. We assume that the solution at the origin (the junction point) is given by a Riemann solver. Recall that the Riemann problem consists to solve the problem with initial data which is constant on each branch. By definition, a Riemann solver allows to solve uniquely the Riemann problem on the junction. Moreover, by definition, the germ is the set $\mathcal{G} \subset \mathbb{R}^N$ of stationary solutions for this Riemann problem on the junction. We show that Riemann solvers are characterized by their germ $\mathcal{G}$. We assume moreover that the Riemann solver is stable, i.e. that the set of solutions to the $\mathcal{G}$-Riemann problem is closed by passage to the limit.

We show that there is a 1-1 correspondence between any germ $\mathcal{G}$ and its associated Godunov flux $\hat{f}_\mathcal{G}$ at the junction, such that $\mathcal{G} = \left\{\hat{f}_\mathcal{G} = f\right\}$. We also show that the Godunov flux $\hat{f}_\mathcal{G}$ enjoys a certain Riemann monotonicity property which implies for instance that $\hat{f}_\mathcal{G} + \varepsilon \cdot id$ is injective for all positive ε, where $id : \mathbb{R}^N \to \mathbb{R}^N$ is the identity. This monotonicity property allows to define the gluing $\mathcal{G}_1 \sharp \mathcal{G}_2$ of two Riemann germs $\mathcal{G}_\gamma$ for junctions of type $n_\gamma : m_\gamma$ for $\gamma = 1, 2$. We glue together two branches with the same flux, one outgoing branch from germ $\mathcal{G}_1$ with one ingoing branch of germ $\mathcal{G}_2$. Then the glued set $\mathcal{G}_1 \sharp \mathcal{G}_2$ is again a Riemann germ and is of type $(n_1 + n_2 - 1) : (m_1 + m_2 - 1)$.

This work is the result of a project that we started more than ten years ago, strongly inspired by the work of Andreianov, Karlsen, Risebro in [4]. There, the convenient notion of (what we call Kružkov) germ was introduced in order to describe the transmission condition between two domains. We were also motivated by problems coming from traffic on networks (see for instance the book of Garavello, Piccoli [24]). Notice that in the present work we do not address at all the further difficulties that arise when drivers' turning preferences are transported by the traffic itself (see Bressan, Fang Yu [9]). The notion of (Kružkov) germs has been recently generalized by Musch, Fjordholm, Risebro in [40] to the case of networks (see also the recent work Cardaliaguet, Forcadel, Monneau [11] for an application of [40] to Kružkov germs for traffic). Our reflection reached a certain maturity that allows us to deliver a quite general theory of germs/Riemann solvers, that we develop in the present work.

Let us call $\mathcal{G}$-entropy solution any entropy solution with traces in $\mathcal{G}$ at the origin. Then our work opens the door to the following natural question.

Open question: *for which Riemann germ $\mathcal{G}$, is there existence and/or uniqueness of $\mathcal{G}$-entropy solutions?*

There are potentially as many open problems as Riemann germs. This question is largely open, and we will try to provide partial answers in future works. For instance, a satisfactory theory can be developed for Hamilton-Jacobi germs (see Forcadel, Monneau [23] and Forcadel, Imbert, Monneau [22]).

In part II of this work, we provide an existence and uniqueness theory for PDE solutions associated to germs in the special class of Kružkov germs. A subclass of Kružkov germs is the one of monotone Kružkov germs. For this subclass, we show that a theory of subsolutions/supersolutions is available, with a natural L^1-comparison principle.

2 Framework and main results

2.1 Preliminaries for Part I

2.1.1 Riemann germ on a junction

Let $N \geq 1$. A branch is an oriented half line. We describe a junction of type $m : n$, i.e. with m ingoing branches $(-\infty, 0)$ and n outgoing branches $(0, +\infty)$, where $N = m + n$. Precisely, consider branches

$J^k \simeq (0,+\infty)$ or $(-\infty,0)$ for $k=1,\dots,N$, and the following junction set

$$J := \{0\} \bigcup_{k=1,\dots,N} J^k$$

with the topology of N branches glued together at the junction point 0. For later use, we also define the orientation of the branch

$$-\sigma^j = \begin{cases} -1 & \text{if} \quad J^j \simeq (-\infty,0) \\ +1 & \text{if} \quad J^j \simeq (0,+\infty) \end{cases} \tag{2.1}$$

For $a=(a^1,\dots,a^N), b=(b^1,\dots,b^N) \in \bar{\mathbb{R}}^N$ with $\bar{\mathbb{R}} := \mathbb{R}\cup\{-\infty,+\infty\}$, we write $a<b$ if $a^k<b^k$ for all k, and $a\le b$ if $a^k \le b^k$ for all k. Then we consider a box $[a,b]$ defined as follows and a vectorial flux function f satisfying

(2.2)

$$\begin{cases} [a,b] := \prod_{k=1,\dots,N} [a^k,b^k] \subset \mathbb{R}^N, \quad \text{with} \quad a<b \\ \text{the function } f=(f^k)_{k=1,\dots,N}, \text{ s.t.} \begin{cases} & f^k : [a^k,b^k] \to \mathbb{R} \text{ is locally Lipschitz continuous} \\ \text{and} & \\ & \text{there exists } \theta^k \in \{\pm 1\} \text{ s.t.} \\ & \theta^k f^k(p^k) \to +\infty \quad \text{if} \quad |p^k| \to +\infty \quad \text{and} \quad p^k \in [a^k,b^k] \quad \textbf{(coercivity)} \end{cases} \end{cases}$$

where we take the convention in the whole work that for all $c,d \in \bar{\mathbb{R}}$ with $c \le d$, we set

$$\begin{cases} [c,d] := \mathbb{R} \cap \{x \in \bar{\mathbb{R}}, \quad c \le x \le d\} \\ (c,d] := \mathbb{R} \cap \{x \in \bar{\mathbb{R}}, \quad c < x \le d\} \\ [c,d) := \mathbb{R} \cap \{x \in \bar{\mathbb{R}}, \quad c \le x < d\} \end{cases} \quad \textbf{(box convention)} \tag{2.3}$$

We mainly have in mind this convention for c,d finite, but allow more generally convention $c,d \in \{\pm\infty\}$. In particular, coercivity assumption in (2.2) is useful only when $a^k=-\infty$ and/or $b^k=+\infty$.

Even if the local Lipschitz continuity of f seems technical in assumption (2.2), it is very useful, because this insures that the velocity of propagation is always finite. For $p=(p^1,\dots,p^N)$, we will also denote

$$f^k(p) := f^k(p^k)$$

by abuse of notation.

We then consider functions $u^k : [0,+\infty) \times J^k \to [a^k,b^k]$, with $u^k(t,x)$ solution of scalar conservation laws on the branch J^k

$$\begin{cases} \partial_t u^k + \partial_x (f^k(u^k)) = 0 & \text{on} & (0,+\infty) \times J^k, \qquad k=1,\dots,N \\ u(t,0) \in \mathcal{G} & \text{for a.e. time} \quad t \in & (0,+\infty) \end{cases} \tag{2.4}$$

and where the junction condition satisfied by the trace of $u=(u^1,\dots,u^N)$ on $\{x=0\}$, is encoded by a given set $\mathcal{G} \subset [a,b]$. We will see later why the trace is well defined in the case of a Riemann problem.

The box $[a,b]$ where the solutions take their values presents the advantage to be preserved by the PDE, under suitable conditions. Moreover, even if the box is equal to $\mathbb{R}^N$, we will see that under our assumptions, if the initial data is bounded, then there is a general procedure that allows to replace $\mathbb{R}^N$ by a bounded box $[a,b]$ (see Proposition 6.1).

We now want to recall the definition of Kružkov entropy solutions (see Kružkov [35]). For $x \in \mathbb{R}$, we set $\text{sign}(x) := 1_{\{x>0\}} - 1_{\{x<0\}}$. We recall that the Kružkov pairs (entropy/ flux of entropy) for $u = (u^1,\dots,u^N), v=(v^1 \dots, v^N) \in \mathbb{R}^N$ are given by

$$\eta^k(u,v) := \eta^k(u^k,v^k) := |u^k - v^k| \quad \text{and} \quad \psi^k(u,v) := \psi^k(u^k,v^k) := \text{sign}(u^k-v^k)\cdot\{f^k(u^k) - f^k(v^k)\} \tag{2.5}$$

Then we recall the following standard notion.

Definition 2.1 (Kružkov entropy solution, [35])
We say that $u^k \in L^\infty([0,+\infty)\times J^k;[a^k,b^k])$ is a Kružkov entropy solution of the first line of (2.4), with initial data $u_0^k \in L^\infty(J^k;[a^k,b^k])$, if for any constant $c=(c^1,\dots,c^N) \in \mathbb{R}^N$, and for any (test) function $0 \le \varphi^k \in C^1_c([0,+\infty)\times J^k)$, we have

$$\int_{(0,+\infty)\times J^k} \{\eta^k(u,c)\varphi^k_t + \psi^k(u,c)\varphi^k_x\}\ dtdx + \int_{\{0\}\times J^k} \eta^k(u_0,c)\varphi^k\ dx \ge 0$$

Definition 2.2 **($\mathcal{G}$-entropy solution)**
Under assumption (2.2), we say that $u = (u^k)_{k=1,\dots,N}$ is a $\mathcal{G}$-entropy solution, if each component u^k is a Kružkov entropy solution of the first line of (2.4), and the trace $u(\cdot, 0)$ (when it is defined) satisfies the second line of (2.4).

We introduce the following definition.

Definition 2.3 **($\mathcal{G}$-Riemann problem)**
Assume (2.2). Given any $p = (p^1, \dots, p^N) \in [a, b]$, we say that $u = (u^k)_{k=1,\dots,N}$ is a $\mathcal{G}$-entropy solution to the $\mathcal{G}$-Riemann problem with initial data p, if $u^k : [0, +\infty) \times J^k \to [a^k, b^k]$ is a Kružkov entropy solution of (2.4) such that for all k, we have

(2.6)
$$\begin{cases} u^k(t,x) = u^k\left(1, \dfrac{x}{t}\right) & \textit{for all} \quad (t,x) \in (0,+\infty)\times J^k \quad \textbf{(0-homogeneity)} \\ u^k = p^k & \textit{on} \quad \{0\}\times J^k, \quad \textbf{(initial data)} \\ u^k(t,0) = \hat{p}^k & \textit{for a.e. time} \quad t \in (0,+\infty) \quad \textbf{(trace at the origin along } J^k\textbf{)} \end{cases}$$

for some $\hat{p} = (\hat{p}^1, \dots, \hat{p}^N) \in \mathcal{G}$. The solution is then denoted by $u = u^{\mathcal{G}}_{p,\hat{p}}$.

Notice that the meaning of the trace is the following

$$\text{ess} \lim_{J^j \ni x \to 0} \int_{(0,T)} |u^j(t,x) - \hat{p}^j| \, dt = 0 \quad \text{for all index } j \text{ and all } T > 0$$

Still, in part I of this work, we will never have to use such a delicate notion of trace; here the notion will be much more elementar. Indeed, for classical Riemann problem (and then also for our $\mathcal{G}$-Riemann problem), it is known that each component map $x \mapsto u^k(t,x)$ is monotone (see Lemma 10.1), and then the trace $u^k(t,0)$ is well-defined without requiring further assumptions on f which are usually required to get strong traces.

We have the following result.

Proposition 2.4 **(L^1 estimate for Riemann problem)**
Assume (2.2) and let $p \in [a, b]$ and some set $\mathcal{G} \subset [a, b]$ such that there exists $\hat{p} \in \mathcal{G}$ and a $\mathcal{G}$-entropy solution $u := u^{\mathcal{G}}_{p,\hat{p}}$. Then for all $t \geq 0$ and $j = 1, \dots, N$, we have

$$\text{(2.7)} \qquad \int_{\{t\}\times J^j} (u^j - p^j) = t\left\{f^j(\hat{p}^j) - f^j(p^j)\right\} \quad \textit{and} \quad \int_{\{t\}\times J^j} |u^j - p^j| = t\left|f^j(\hat{p}^j) - f^j(p^j)\right|$$

Definition 2.5 **(Generalized Riemann germ, generalized Godunov flux, Riemann germ)**
Assume (2.2).
i) (Generalized Riemann germ and the projection $\pi_{\mathcal{G}}$)
A set $\mathcal{G} \subset [a, b]$ is called a generalized Riemann germ (with respect to (J, f)) if for any initial data $p \in [a, b]$, there exists a unique $\mathcal{G}$-entropy solution $u^{\mathcal{G}}_{p,\hat{p}}$ to $\mathcal{G}$-Riemann problem (2.6) with some trace $\hat{p} \in \mathcal{G}$ at $x = 0$. Then it defines a map $\pi := \pi_{\mathcal{G}}$

$$\begin{array}{rccl} \pi_{\mathcal{G}} : & [a,b] & \to & \mathcal{G} \\ & p & \mapsto & \pi(p) := \hat{p} \end{array}$$

which is a nonlinear projection, i.e. satisfies $\pi \circ \pi = \pi$.
ii) (Generalized Godunov flux)
Given a generalized Riemann germ $\mathcal{G}$, we define the generalized Godunov flux at the junction $\hat{f} := \hat{f}_{\mathcal{G}}$ as

$$\begin{array}{rccl} \hat{f}_{\mathcal{G}} : & [a,b] & \to & \mathbb{R}^N \\ & p & \mapsto & \hat{f}_{\mathcal{G}}(p) := (f \circ \pi_{\mathcal{G}})(p) \end{array}$$

iii) (Stability and Riemann germ)
Given a generalized Riemann germ $\mathcal{G}$, we say that $\mathcal{G}$-Riemann problem (2.6) is stable if for any sequence $(p_n)_{n\in\mathbb{N}}$ with $[a, b] \ni p_n \to p_\infty \in [a, b]$, we have

$$u^{\mathcal{G}}_{p_n,\pi(p_n)} \to u^{\mathcal{G}}_{p_\infty,\pi(p_\infty)} \quad \textit{in} \quad L^1_{loc}([0,+\infty)\times J) \quad \textit{as} \quad n \to +\infty$$

By abuse of terminology, we will say that $\mathcal{G}$ is stable. In such a case, $\mathcal{G}$ is called a Riemann germ (with respect to (J, f)).

Remark 2.6 *The (generalized) Riemann solver is the map* $p \mapsto u^{\mathcal{G}}_{p,\pi(p)}$. *By abuse of terminology (in the literature), the map* π *itself is sometimes also called a (generalized) Riemann solver.*

The (generalized) Godunov flux at the junction $\hat{f}_{\mathcal{G}}$ has to be distinguished from the standard Godunov flux G^{f^j}.

Definition 2.7 (Standard Godunov flux)
Assume (2.2). The standard Godunov flux associated to the flux f^j *is given by*

$$G^j(p^j,q^j) := G^{f^j}(p^j,q^j) := \begin{cases} \inf_{[p^j,q^j]} f^j & \text{if} \quad p^j \le q^j \\ \sup_{[q^j,p^j]} f^j & \text{if} \quad p^j \ge q^j \end{cases}$$

with monotonicities (not in the strict sense) indicated by the arrows $G^j(\uparrow,\downarrow)$.

We introduce the following.

Definition 2.8 (Subclasses of germs)
Assume (2.2). Let $\mathcal{G} \subset [a,b]$ *be a generalized Riemann germ.*
i) (Kružkov germs = D**-germs)**
We say that $\mathcal{G}$ *is a Kružkov germ (also called a D-germ), if it satisfies*

$$D^f(p,q) \ge 0 \quad \text{for all} \quad p,q \in \mathcal{G}$$

where the dissipation is defined by

$$(2.8) \quad D^f(p,q) := \sum_{k=1,\dots,N} D^{f^k}(p,q) = IN - OUT \quad \text{with} \quad D^{f^k}(p,q) := \sigma^k \cdot sign(p^k - q^k) \cdot \left\{ f^k(p^k) - f^k(q^k) \right\}$$

(which is a Kružkov entropy production at the junction point).
i') (Monotone Kružkov germs = D_+**-germs)**
We say that $\mathcal{G}$ *is a* D_+*-germ, if it satisfies*

$$D^f_+(p,q) \ge 0 \quad \text{for all} \quad p,q \in \mathcal{G}$$

where the semi-dissipation is defined by

$$(2.9) \quad D^f_+(p,q) := \sum_{k=1,\dots,N} D^{f^k}_+(p,q) = IN - OUT \quad \text{with} \quad D^{f^k}_+(p,q) := \sigma^k \cdot sign^+(p^k - q^k) \cdot \left\{ f^k(p^k) - f^k(q^k) \right\}$$

(which is a Kružkov semi-entropy production at the junction point).
ii) (quasi HJ germs)
We say that $\mathcal{G}$ *is a quasi Hamilton-Jacobi germ (quasi HJ germ for short) if there exists a scalar function* $h : \mathcal{G} \to \mathbb{R}$ *and* $\omega = (\omega^1, \dots, \omega^N) \in (0,+\infty)^N$ *such that*

$$f = (\omega^1 h, \dots, \omega^N h) \quad \text{on} \quad \mathcal{G}$$

(with our convention $f^j(p) := f^j(p^j)$*).*
ii') (HJ germs)
We say that $\mathcal{G}$ *is a Hamilton-Jacobi germ (HJ germ for short) if its is a quasi HJ germ with* $\omega^j = 1$ *for all* $j = 1, \dots, N$.
iii) (Monotone germs)
We say that $\mathcal{G}$ *is a monotone germ if for all* $p,q \in [a,b]$ *we have*

$$(2.10) \qquad p \ge q \quad \text{implies} \quad \pi_{\mathcal{G}}(p) \ge \pi_{\mathcal{G}}(q)$$

iv) (Conservative germs)
We say that $\mathcal{G}$ *is a conservative germ if IN= OUT, i.e. if*

$$\sum_{J^j \simeq (-\infty,0)} f^j(p) = \sum_{J^j \simeq (0,+\infty)} f^j(p) \quad \text{for all} \quad p \in \mathcal{G}$$

For later use, we also define the Rankine-Hugoniot function

$$RH^f(p) := \sum_{j=1,\dots,N} \sigma^j f^j(p) = IN - OUT$$

which vanishes on conservative germs.

Remark 2.9 (Relation between quasi HJ gems and HJ germs)
Notice that up to a linear change of variables, quasi HJ gems and HJ gems do coincide. We still have to distinguish them, because for $N \geq 3$, genuine HJ gems are never conservative in general, while we may still consider conservative quasi HJ germs.
Conservative quasi HJ germs do appear naturally in examples. Moreover, after a linear change of variables, those conservative quasi HJ germs are transformed into HJ germs which are no longer conservative. The point is that for HJ germs, a Hamilton-Jacobi approach can be rigorously developed (see for instance [30]). Still after the inverse linear change of variables applied to HJ germs, we get back to the original problem and can recover the forgotten conservative property.

In what follows, most of the results are stated for HJ germs, but can also be stated for quasi HJ germs, just using a linear change of variables.

Notice that Kružkov, monotone Kružkov, (quasi) HJ, monotone and conservative germs $\mathcal{G}$ can equivalently (and conveniently) be defined by the properties of their Godunov flux $\hat{f}_{\mathcal{G}}$ (see Lemma 5.5). We will see that the terminology "monotone Kružkov" germs is justified for D_+-germs, because we have exactly

$$\{\text{Kružkov germs}\} \cap \{\text{monotone germs}\} = \{D_+\text{-germs}\}$$

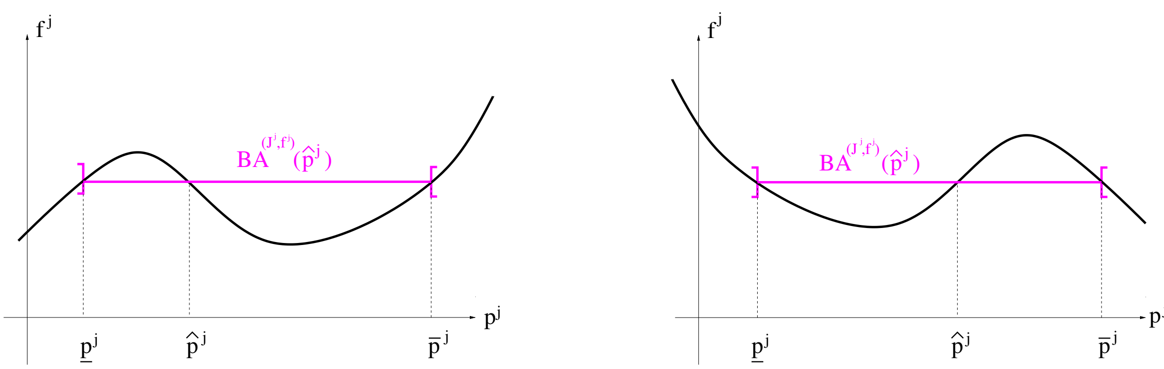
as shows Theorem 2.28. We also have

$$\{\text{conservative germs}\} \cap \{\text{Kružkov germs}\} = \{\text{conservative germs}\} \cap \{\text{monotone germs}\}$$

as shows Theorem 2.29.

2.1.2 Basin of Attraction

We will need the following key notion of basin of attraction of a point $\hat{p} \in [a, b]$. As we will see later, the basin of attraction of $\hat{p}$ is the set of initial data $p \in [a, b]$ such that Riemann problem (2.6) can reach the value $\hat{p}$ at the junction point $x = 0$. It turns out that this notion is independent on the germ $\mathcal{G}$, and indeed reduces to the question for each component j. Hence the basin of attraction of $\hat{p}$, is simply given by the product of the basins of attraction of each component $\hat{p}^j$. The basin of attraction of $\hat{p}^j$ depends on the orientation of the branch J^j. For a single branch J^j with flux f^j, the "Basin of Attraction" around $\hat{p}^j$ is called $BA^{(J^j,f^j)}(\hat{p}^j)$ and is pictured in the generic case on the associated figure.



(a) Basin of Attraction for an ingoing branch $J^j \simeq (-\infty, 0)$

(b) Basin of Attraction for an outgoing branch $J^j \simeq (0, +\infty)$

The limit cases correspond to $\underline{p}^j = \hat{p}^j, a^j$ or $\overline{p}^j = \hat{p}^j, b^j$. For $J^j \simeq (-\infty, 0)$, the Basin of Attraction of $\hat{p}^j$ is itself given by the largest interval $I^j := BA^{(J^j,f^j)}(\hat{p}^j)$ containing $\hat{p}^j$, such that for $\lambda^j := f^j(\hat{p}^j)$, we have $I^j \cap \{f^j = \lambda^j\} = \{\hat{p}^j\}$ with f^j strictly bigger than λ^j on the left of $\hat{p}^j$ and f^j strictly less than λ^j on the right of $\hat{p}^j$. In the case where f^j is increasing, then I^j is reduced to the singleton $\{\hat{p}^j\}$.

The basin of attraction of $\hat{p}$ is simply given by the product of the basins of attraction for each component $\hat{p}^j$. We now give the precise definition.

Definition 2.10 (Basin of attraction [1])
Assume (2.2) and let $\hat{p} \in [a,b]$.
Then the Basin of Attraction of the point $\hat{p}$ is the set

$$BA(\hat{p}) = \prod_{j=1,\dots,N} BA^{(J^j,f^j)}(\hat{p}^j)$$

i) (Case A: $J^j \simeq (-\infty,0)$)
Then

$$BA^{(J^j,f^j)}(\hat{p}^j) := BA^j_-(\hat{p}^j) \cup \left\{\hat{p}^j\right\} \cup BA^j_+(\hat{p}^j)$$

with for $\lambda^j := f^j(\hat{p}^j)$

$$\begin{cases} BA^j_-(\hat{p}^j) := \left\{q^j \in [a^j,p^j), \quad f^j > \lambda^j \quad on \quad [q^j,p^j)\right\} \\ BA^j_+(\hat{p}^j) := \left\{q^j \in (p^j,b^j], \quad f^j < \lambda^j \quad on \quad (p^j,q^j]\right\} \end{cases}$$

where the intervals $BA^j_+(\hat{p}^j) = BA^j(\hat{p}^j) \cap (\hat{p}^j,+\infty)$ and $BA^j_-(\hat{p}^j) = BA^j(\hat{p}^j) \cap (-\infty,\hat{p}^j)$ may be empty.
ii) (Case B: $J^j \simeq (0,+\infty)$)
Then

$$BA^{(J^j,f^j)}(\hat{p}^j) := BA^{(-J^j,-f^j)}(\hat{p}^j) \quad with \quad -J^j \simeq (-\infty,0)$$

In both Cases A or B, and when there will be no ambiguity, we will denote the basin of attraction simply by $BA^j(\hat{p}^j)$, or $BA^j(\hat{p})$.

2.1.3 Riemann monotonicity and local constancy

We now introduce certain properties of functions which will be satisfied by the the projection $\pi_{\mathcal{G}}$ and the Godunov flux $\hat{f}_{\mathcal{G}}$, under natural conditions. To this end, we first need to recall the following definition.

Definition 2.11 (Hadamard product of two vectors)
Assume $N \geq 1$. For $p,q \in \mathbb{R}^N$, we define $p \diamond q \in \mathbb{R}^N$ as the vector of components given by the product of components

$$(p \diamond q)^k = p^k q^k$$

Then we introduce the following.

Definition 2.12 (Riemann monotonicity)
Assume (2.2) and consider a function $h : [a,b] \to \mathbb{R}^N$ for $N \geq 1$. We set $[h]^p_q := h(p) - h(q)$.
We say that h is Riemann monotone if it satisfies for all $p,q \in [a,b]$

$$(p-q) \diamond [h]^p_q \leq 0 \quad \Longrightarrow \quad [h]^p_q = 0 \tag{2.11}$$

Notice that condition (2.11) means $\max_{j=1,\dots,N} (p^j - q^j) \cdot \left\{h^j(p) - h^j(q)\right\} \leq 0$ implies $h(p) = h(q)$.

Notice in particular that if h is Riemann monotone, then the map $p^j \mapsto h^j(p)$ is nondecreasing. Moreover, for every $\varepsilon > 0$, the map $h + \varepsilon Id : [a,b] \to \mathbb{R}^N$ is injective (see Lemma 5.3).

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[1] In case $J^j \simeq (-\infty,0)$, we have more precisely

$$\underline{p}^j := \begin{cases} \inf BA^j_-(\hat{p}^j) & \text{if} \quad BA^j_-(\hat{p}^j) \neq \emptyset \\ \hat{p}^j & \text{otherwise} \end{cases} \quad \text{and} \quad BA^j_-(\hat{p}^j) = \begin{cases} [a^j,\hat{p}^j) & \text{if} \quad \underline{p}^j = a^j \quad \text{and} \quad f^j(a^j) > \lambda^j \\ (\underline{p}^j,\hat{p}^j) & \text{otherwise} \end{cases}$$

and

$$\overline{p}^j := \begin{cases} \sup BA^j_+(\hat{p}^j) & \text{if} \quad BA^j_+(\hat{p}^j) \neq \emptyset \\ \hat{p}^j & \text{otherwise} \end{cases} \quad \text{and} \quad BA^j_+(\hat{p}^j) = \begin{cases} (\hat{p}^j,b^j] & \text{if} \quad \overline{p}^j = b^j \quad \text{and} \quad f^j(b^j) < \lambda^j \\ (\hat{p}^j,\overline{p}^j) & \text{otherwise} \end{cases}$$

Definition 2.13 (Local constancy[2])
Assume (2.2) and consider a map $h : [a,b] \to \mathbb{R}^N$. Let $(e_1, \dots, e_N)$ be the canonical basis of $\mathbb{R}^N$. Then we say that h is locally constant on $\{h \neq f\}$, if
(2.12)

$$\left\{\begin{array}{ll} \text{for all } p \in [a,b] \text{ and } K_p := \left\{j \in \{1,\dots,N\}, \quad h^j(p) \neq f^j(p)\right\}, & \text{there exists } \varepsilon > 0 \\ \text{such that for } Q_\varepsilon(p) := p + \displaystyle\sum_{j\in K_p} (-\varepsilon,\varepsilon)e_j, \text{ we have} & \\ h = const \quad \text{on} \quad [a,b] \cap Q_\varepsilon(p) & \textbf{(Constant on a local box)} \end{array}\right.$$

2.2 Main results of Part I

2.2.1 Fundamental results: properties and gluing of Riemann germs

Proposition 2.14 (Properties of generalized Riemann germs)
Assume (2.2). Let $\mathcal{G} \subset [a,b]$ be a generalized Riemann germ in the sense of Definition 2.5. Let $\pi := \pi_{\mathcal{G}}$ be the projection map and $\hat{f} := \hat{f}_{\mathcal{G}}$ be the generalized Godunov flux at the junction, as introduced in Definition 2.5). Then the following holds true.
i) (Inverse of π)
We have

$$\pi^{-1}(\hat{p}) = BA(\hat{p}) \quad \text{for all} \quad \hat{p} \in \mathcal{G}$$

where $BA(\hat{p})$ is the basin of attraction of $\hat{p}$ given in Definition 2.10. In particular π is locally constant on $\{\pi \neq Id\}$.
ii) (Dissipation)
We have

$$D^f(q,\hat{p}) < 0 \quad \text{for all} \quad q \in BA(\hat{p}) \backslash \{\hat{p}\}$$

where the dissipation D^f is defined in (2.8).
iii) (Riemann monotonicity of π)
The map π is Riemann monotone.
iv) (Local constancy of the generalized Godunov flux $\hat{f}$)
The function $\hat{f}$ is locally constant on $\left\{\hat{f} \neq f\right\}$.
v) (Germ as a level set)
The generalized Riemann germ $\mathcal{G}$ can be recovered as follows

$$(2.13) \qquad \mathcal{G} = \mathcal{G}_{\hat{f}} \quad \text{where} \quad \mathcal{G}_{\hat{f}} := \left\{p \in [a,b], \quad \hat{f}(p) = f(p)\right\} \quad \textbf{(Germ as a level set)}$$

vi) (Partial relaxation formula)
Moreover fix some $p \in [a,b]$ and some index j, and define the map

$$\hat{f}^j_p(q^j) := \hat{f}^j(\iota^j_p(q^j)) \quad \text{for all} \quad q^j \in [a^j,b^j], \quad \text{with} \quad \iota^j_p(q^j) := (p^1,\dots,p^{j-1},q^j,p^{j+1},\dots,p^N)$$

Then this map satisfies the following partial relaxation formula

$$\left\{\hat{f}^j_p(p^j)\right\} = \left\{\begin{array}{ll} \displaystyle\bigcup_{q^j\in[a^j,b^j]} \left\{\hat{f}^j_p(q^j)\right\} \cap \left\{G^j(q^j,p^j)\right\} & \text{if} \quad J^j \simeq (0,+\infty) \\ \displaystyle\bigcup_{q^j\in[a^j,b^j]} \left\{G^j(p^j,q^j)\right\} \cap \left\{\hat{f}^j_p(q^j)\right\} & \text{if} \quad J^j \simeq (-\infty,0) \end{array}\right| \quad \textbf{(partial relaxation formula)}$$

where G^j is the standard Godunov flux associated to the function f^j.
vii) (Partial Lipschitz estimate and basic monotonicity)
Moreover for any $p \in [a,b]$ and any index j, the map $\hat{f}^j_p : [a^j,b^j] \to \mathbb{R}$ is locally Lipschitz continuous and satisfies with $\sigma^j \in \{\pm 1\}$

$$\sigma^j (\partial_j \hat{f}^j) \circ \iota^j_p = \sigma^j (\hat{f}^j_p)' \in \left\{0, \ \max\left\{0, \sigma^j (f^j)'\right\}\right\} \quad \text{a.e. on} \quad [a^j,b^j] \quad \text{if} \quad J^j \simeq \sigma^j(-\infty,0)$$

[2] When the map h is continuous, then it is easy to see that local constancy of h on $\{h \neq f\}$ is equivalent to the fact that if for some index $j \in \{1,\dots,N\}$, the j-component of h satisfies $h^j(p) \neq f^j(p)$ for some $p \in [a,b]$, then there exists some $\varepsilon > 0$ such that the maps $(-\varepsilon,\varepsilon) \ni x \mapsto h^k(p + xe_j)$ are constant for all $k = 1,\dots,N$, while it is defined, i.e. while $p + xe_j \in [a,b]$.

Notice that π is not a continuous map in general. We have the following structure result.

Theorem 2.15 (Structure of generalized Riemann germs)
Assume (2.2) and consider a set $\mathcal{G} \subset [a,b]$.
i) (First characterization)
The set $\mathcal{G}$ is a generalized Riemann germ if and only if $(BA(\hat{p}))_{\hat{p}\in\mathcal{G}}$ is a partition of $[a,b]$.
ii) (Second characterization)
The set $\mathcal{G}$ is a generalized Riemann germ if and only if $\mathcal{G} = \mathcal{G}_{\hat{f}}$ with $\mathcal{G}_{\hat{f}}$ given in (2.13) for some function $\hat{f} : [a,b] \to \mathbb{R}^N$ which is locally constant on $\left\{\hat{f} \neq f\right\}$ and satisfying for all j the following additional conditions:

$$(2.14) \qquad \left\{ \begin{array}{lll} p^j \mapsto \sigma^j \hat{f}^j(p) \quad \text{is nondecreasing on} \quad [a,b] & & \textbf{(Basic monotonicities)} \\ f_- \leq \hat{f} \leq f_+ & & \textbf{(Monotone bounds)} \end{array} \right.$$

where σ^j is defined in (2.1) and
(2.15)

$$\left\{ \begin{array}{llll} f^j_-(p^j) := \inf\limits_{[a^j,p^j]} f^j = G^j(a^j,p^j) \quad \text{and} \quad f^j_+(p^j) := \sup\limits_{[p^j,b^j]} f^j = G^j(b^j,p^j) & \text{if} \quad J^j \simeq (0,+\infty), & \sigma^j = -1 \\ f^j_-(p^j) := \inf\limits_{[p^j,b^j]} f^j = G^j(p^j,b^j) \quad \text{and} \quad f^j_+(p^j) := \sup\limits_{[a^j,p^j]} f^j = G^j(p^j,a^j) & \text{if} \quad J^j \simeq (-\infty,0), & \sigma^j = +1 \end{array} \right.$$

When it is the case, then we necessarily have $\hat{f} = \hat{f}_{\mathcal{G}}$ where $\hat{f}_{\mathcal{G}}$ is the generalized Godunov flux defined in Proposition 2.14.

Remark 2.16 *Notice that the second line of (2.14) means simply*

$$\inf_{[p^j,b^j]} \sigma^j f^j \leq \sigma^j \hat{f}^j(p) \leq \sup_{[a^j,p^j]} \sigma^j f^j \quad \text{for all} \quad p \in [a,b]$$

For $N = 1$, it is a fact that $\mathcal{G}$ must be a closed set and the function $\hat{f}$ must be continuous (see Theorem 2.30 for $N = 1$). For $N \geq 2$, this is no longer the case (see counter-examples in Lemma 9.1).

We have the following

Theorem 2.17 (Riemann germs)
Assume (2.2) and let $\mathcal{G}$ be a generalized Riemann germ.
i) (Characterization of Riemann germs)
Then $\mathcal{G}$ is a Riemann germ if and only if the generalized Godunov flux $\hat{f} = \hat{f}_{\mathcal{G}}$ is continuous.
ii) (Characterization of $\hat{f}$ for Riemann germs)
If $\mathcal{G}$ is a Riemann germ, then the generalized Godunov flux $\hat{f}_{\mathcal{G}} : [a,b] \to \mathbb{R}^N$ is then fully characterized as a continuous function $\hat{f}$, locally constant on $\left\{\hat{f} \neq f\right\}$ and satisfying $\mathcal{G} = \left\{\hat{f} = f\right\}$.

Notice that the continuity of $\hat{f}_{\mathcal{G}}$ implies in particular the closedness of Riemann germs $\mathcal{G}$ in $[a,b]$.

We now introduce the following

Definition 2.18 (Godunov flux at the junction)
Assume (2.2) with $N \geq 1$. Recall that $\sigma \in \{\pm 1\}^N$ encodes the orientations of the junction J with fluxes f. A function $\hat{f} : [a,b] \to \mathbb{R}^N$ is said to be a Godunov flux with respect to (J,f) if it satisfies

$$(2.16) \qquad \left\{ \begin{array}{l} \hat{f} : [a,b] \to \mathbb{R}^N \quad \text{is continuous,} \\ \\ p^j \mapsto \sigma^j \hat{f}^j(p) \quad \text{is nondecreasing on} \quad [a,b] \\ \\ \hat{f} \ \text{is locally constant on} \ \left\{\hat{f} \neq f\right\} \ \text{in the sense of Definition 2.13,} \\ \\ \inf\limits_{[p^j,b^j]} \sigma^j f^j \leq \sigma^j \hat{f}^j(p) \leq \sup\limits_{[a^j,p^j]} \sigma^j f^j \qquad \text{for all} \quad p \in [a,b] \end{array} \right.$$

Then we have the following corollary of Theorems 2.15 and (2.17).

Corollary 2.19 (Reconstruction of a Riemann germ from a Godunov flux)
Assume (2.2) with $N \geq 1$. Let $\hat{f}$ be a Godunov flux in the sense of Definition 2.18. Then the set

$$\mathcal{G} := \left\{p \in [a,b], \quad \hat{f}(p) = f(p)\right\}$$

is a Riemann germ, and its generalized Godunov flux $\hat{f}_{\mathcal{G}}$ is equal to the Godunov flux $\hat{f}$, i.e. $\hat{f}_{\mathcal{G}} = \hat{f}$.

We have the following important result.

Theorem 2.20 (Riemann monotonicity of $\sigma \diamond \hat{f}$)
Assume (2.2) with $N \geq 1$ and recall that $\sigma \in \{\pm 1\}^N$ encodes the orientations of the branches. Assume moreover that f satisfies
(2.17)
$f^k : [a^k, b^k] \to \mathbb{R}$ *is not constant on any nondegenerate interval, for all $k = 1, \dots, N$* **(Nondegeneracy)**

i) (General result)
Then for any Riemann germ $\mathcal{G}$, the Godunov flux $\hat{f}_{\mathcal{G}} : [a,b] \to \mathbb{R}^N$ is such that $\sigma \diamond \hat{f}_{\mathcal{G}}$ is Riemann monotone in the sense of Definition 2.12.
ii) (Case of Kružkov germs)
If $\mathcal{G}$ is a Kružkov germ, then the conclusion of point i) still holds true, without assuming nondegeneracy condition (2.17).

Remark 2.21 *Notice that Riemann monotonicity of $\sigma \diamond \hat{f}_{\mathcal{G}}$ means that for two distinct points p, q of the germ, the entropy flux can not be negative for each components, at the junction point. In other words, at least one branch must dissipate a positive entropy flux at the junction point.*

Notice that without condition (2.17), there exist Riemann germs $\mathcal{G}$ such that $\sigma \diamond \hat{f}_{\mathcal{G}}$ is not Riemann monotone (see counter-example Lemma 9.3).

We now show that Riemann monotonicity is a key feature to build Riemann germs. We start with the following definition.

Definition 2.22 (Godunov quasi-flux at the junction)
Assume (2.2) with $N \geq 1$ and compact box $[a,b] \subset \mathbb{R}^N$. Recall that $\sigma \in \{\pm 1\}^N$ encodes the orientations of the branches for the junction J with fluxes f.
Then a function $g_0 : [a,b] \to \mathbb{R}^N$ is said to be a Godunov quasi-flux with respect to (J, f), if it satisfies the following conditions

$$\text{(2.18)} \qquad \begin{cases} g_0 : [a,b] \to \mathbb{R}^N \ \textit{is continuous} \\ \\ \sigma \diamond g_0 : [a,b] \to \mathbb{R}^N \ \textit{is Riemann monotone in the sense of Definition 2.11} \\ \\ \sigma^j f^j(b^j) \leq \sigma^j g_0^j(q)_{|q^j = b^j} \quad \textit{and} \quad \sigma^j g_0^j(q)_{|q^j = a^j} \leq \sigma^j f^j(a^j) \end{cases}$$

Under nondegeneracy assumption (2.17), notice that any generalized Godunov flux $\hat{f}_{\mathcal{G}}$ associated to a Riemann germ $\mathcal{G} \subset [a,b]$ (with $[a,b]$ compact) is a Godunov quasi-flux in the sense of Definition 2.22.

Theorem 2.23 (Riemann relaxation of Godunov quasi-fluxes)
Assume (2.2) with $N \geq 1$ and compact box $[a,b] \subset \mathbb{R}^N$. Let $g_0 : [a,b] \to \mathbb{R}^N$ be a Godunov quasi-flux in the sense of Definition 2.22. We define for $p, q \in [a,b]$ the vectorial standard Godunov flux with orientation σ

$$G^f_\sigma(p,q) := (G^{f^1}_{\sigma^1}(p^1,q^1), \dots, G^{f^N}_{\sigma^N}(p^N,q^N)) \quad \textit{with} \quad G^{f^j}_{\sigma^j}(p^j,q^j) = \begin{cases} G^{f^j}(p^j,q^j) & \textit{if} \quad \sigma^j = 1 \\ G^{f^j}(q^j,p^j) & \textit{if} \quad \sigma^j = -1 \end{cases}$$

Then the following formula defines uniquely $g_1(p) \in \mathbb{R}^N$ for all $p \in [a,b]$

$$\text{(2.19)} \qquad \{g_1(p)\} := \bigcup_{q \in [a,b]} \left(\{G^f_\sigma(p,q)\} \cap \{g_0(q)\}\right)$$

Then the set

$$\mathcal{G} := \{p \in [a,b], \quad g_1(p) = f(p)\} \tag{2.20}$$

is a Riemann germ with respect to (J,f).
We set in particular the Riemann relaxation operator

$$\mathfrak{R} g_0 := g_1 \tag{2.21}$$

Moreover g_1 satisfies (2.18) and is also a Godunov flux in the sense of Definition 2.18, and $\mathfrak{R}\circ\mathfrak{R} = \mathfrak{R}$ is a nonlinear projection, that we call Riemann relaxation.
Assume now that f satisfies furthermore nondegeneracy condition (2.17). Then, conversely, all Riemann germs $\mathcal{G}$ are obtained like in (2.20), and their Godunov flux satisfy $\hat{f}_{\mathcal{G}} = \mathfrak{R}(\hat{f}_{\mathcal{G}})$.

Moreover Riemann monotonicity is also required for gluing Riemann germs, as shows Lemma 9.4.

Theorem 2.24 (Gluing of Riemann germs along $J_\alpha^{j_\alpha}$ and $J_\beta^{j_\beta}$)
For $\gamma = \alpha, \beta$, consider Riemann germs $\mathcal{G}_\gamma \subset [a_\gamma, b_\gamma]$ for junctions of type $n_\gamma : m_\gamma$ with $N_\gamma := n_\gamma + m_\gamma$, and assume (2.2) with $J_\gamma^j = \sigma_\gamma^j \cdot (-\infty, 0)$.

For each $\gamma = \alpha$ or β, assume either 1) nondegeneracy condition (2.17) for the flux function $f_\gamma : [a_\gamma, b_\gamma] \to \mathbb{R}^{N_\gamma}$, or 2) that $\mathcal{G}_\gamma$ is a Kružkov germ. We allow mixing cases for α and β.

Define $[a,b]_\gamma^j := [a_\gamma^j, b_\gamma^j]$. Fix $j_\gamma \in \{1, \ldots, N_\gamma\}$ such that

$$\begin{cases} f_\alpha^{j_\alpha} = f_\beta^{j_\beta} =: f^0 \quad \textit{and} \quad \sigma_\alpha^{j_\alpha} = -\sigma_\beta^{j_\beta} \\ \\ \textit{with} \quad f^0 : [a^0, b^0] \to \mathbb{R} \quad \textit{where} \quad [a^0, b^0] := [a,b]_\alpha^{j_\alpha} = [a,b]_\beta^{j_\beta} \end{cases}$$

Define the gluing $\mathcal{G} := \mathcal{G}_\alpha \sharp \mathcal{G}_\beta$ of the germs $\mathcal{G}_\alpha$ and $\mathcal{G}_\beta$ along $J_\alpha^{j_\alpha}$ and $J_\beta^{j_\beta}$ as

$$\mathcal{G} := \mathcal{G}_\alpha \underset{j_\alpha : j_\beta}{\sharp} \mathcal{G}_\beta := \left\{ (p'_\alpha, p'_\beta) \in [a,b]'_\alpha \times [a,b]'_\beta, \quad \textit{there exists } r_\alpha, r_\beta \in [a^0, b^0] \textit{ s.t. } \left| \begin{array}{l} (r_\alpha, p'_\alpha) \in \mathcal{G}_\alpha, \\ (r_\beta, p'_\beta) \in \mathcal{G}_\beta, \\ f^0(r_\alpha) = G^{f^0}(r_\alpha, r_\beta) = f^0(r_\beta) \end{array} \right. \right\}$$

where G^{f^0} is the standard Godunov flux associated to the function f^0, and where we use the abuse of notation $(r_\gamma, p'_\gamma) := (p_\gamma^1, \ldots, p_\gamma^{j_\gamma - 1}, r_\gamma, p_\gamma^{j_\gamma + 1}, \ldots, p_\gamma^{N_\gamma})$, and

$$[a,b]'_\gamma := \prod_{j \in \{1, \ldots, N_\gamma\} \setminus \{j_\gamma\}} [a,b]_\gamma^j$$

i) (Glued Riemann germ)
Then $\mathcal{G}$ is a Riemann germ for a junction $(n_\alpha + n_\beta - 1) : (m_\alpha + m_\beta - 1)$.
ii) (Associativity of gluing)
The gluing of germs is associative.
iii) (Identity element of gluing)
For general gluing, the identity element is the standard Godunov germ for $1:1$ junction

$$\mathcal{G}^{f^0} := \left\{ (r,s) \in [a^0, b^0]^2, \quad f^0(r) = G^{f^0}(r,s) = f^0(s) \right\}$$

i.e.

$$\mathcal{G}_\alpha \sharp \mathcal{G}^{f^0} = \mathcal{G}_\alpha \quad \textit{and} \quad \mathcal{G}^{f^0} \sharp \mathcal{G}_\beta = \mathcal{G}_\beta$$

iv) (Nature of the glued germs)
Moreover if $\mathcal{G}_\gamma$ are Kružkov germs (resp. HJ germs, resp. monotone germs, resp. conservative germs), then $\mathcal{G}$ is a Kružkov germ (resp. HJ germs, resp. monotone germ, resp. conservative germ).

Notice that the identity element $\mathcal{G}^{f^0}$ is a conservative germ for $1:1$ junctions and then by Theorem 2.35, it is also a Kružkov, quasi HJ and monotone germ. Moreover, from i) of Definition 2.5, there exists the

Godunov projection map $\pi := \pi_{\mathcal{G}^{f^0}} : [a^0, b^0]^2 \to \mathcal{G}^{f^0}$ with $\pi = (\pi^\alpha, \pi^\beta) =: (\pi^L, \pi^R)$, such that the standard Godunov flux satisfies

$$G^{f^0}(p^L, p^R) = (f^0 \circ \pi^L)(p^L, p^R) = (f^0 \circ \pi^R)(p^L, p^R) \quad \text{for all} \quad (p^L, p^R) \in [a^0, b^0]^2$$

and π is monotone in the sense of (2.10). Already for the standard Godunov flux G^{f^0}, this result seems new.

Remark 2.25 *In Theorem 2.24, with abuse of notation, we have formally the gluing along the axis of flux f^0 as follows*

$$\mathcal{G}_\alpha \sharp \mathcal{G}_\beta \ni (p'_\alpha, p'_\beta) = \overbrace{(p'_\alpha, \underbrace{r_\alpha) \sharp (r_\beta}_{\in \mathcal{G}^{f^0}}}^{\in \mathcal{G}_\alpha}\overbrace{, p'_\beta)}^{\in \mathcal{G}_\beta}$$

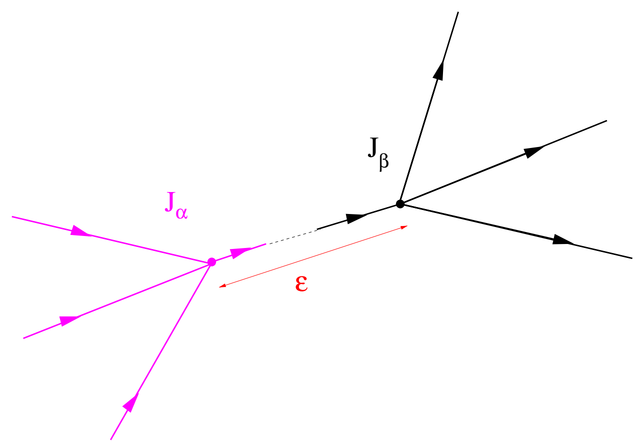

Figure 1: Illustration of a gluing of two particular junctions, formally as $\varepsilon \to 0$

2.2.2 Applications to Kružkov, Hamilton-Jacobi, monotone or conservative germs

We have the following results.

Theorem 2.26 (Properties of Kružkov germs)
Assume (2.2) with $N \geq 1$. Let $\mathcal{G} \subset [a, b]$ be a generalized Riemann germ with respect to (J, f).
i) (Characterization of Kružkov germs)
Then $\mathcal{G}$ is a Kružkov germ if and only if $\mathcal{G} = \left\{p \in [a, b], \ \hat{f}(p) = f(p)\right\}$ for some locally Lipschitz continuous function $\hat{f} : [a, b] \to \mathbb{R}^N$ whose Jacobian matrix satisfies the following column diagonally dominant inequality

$$\sigma^i \partial_i \hat{f}^i \geq \sum_{j \in \{1,\dots,N\} \setminus \{i\}} |\partial_i \hat{f}^j| \quad a.e. \ on \quad [a, b], \quad for\ all \quad i = 1, \dots, N$$

When it is the case, then we have $\hat{f} = \hat{f}_{\mathcal{G}}$.
ii) (D-maximality)
If $\mathcal{G}$ is a Kružkov germ, then it satisfies the following D-maximality property: for all $p \in [a, b]$

$$\left(D^f(p, q) \geq 0 \quad for\ all \quad q \in \mathcal{G}\right) \quad \Longrightarrow \quad p \in \mathcal{G}$$

Theorem 2.27 (Characterization of monotone germs)
Assume (2.2) with $N \geq 1$. Let $\mathcal{G} \subset [a, b]$ be a generalized Riemann germ with respect to (J, f), and let $\hat{f} := \hat{f}_{\mathcal{G}}$ be its associated Godunov flux. Then $\mathcal{G}$ is monotone if and only if

$$(2.22) \qquad p \mapsto \sigma^j \hat{f}^j(p) \quad is\ nonincreasing\ in\ p^k\ for\ all\ k \neq j$$

Theorem 2.28 (Characterization of D_+-germs)
Assume (2.2) with $N \geq 1$. Let $\mathcal{G} \subset [a,b]$ be a generalized Riemann germ with respect to (J,f). Then we have as a germ

$$\mathcal{G} \text{ is a } D_+\text{-germ} \quad \Leftrightarrow \quad (\mathcal{G} \text{ monotone and } \quad \mathcal{G} \text{ Kružkov})$$

Theorem 2.29 (Properties of conservative Riemann germs)
Assume (2.2) with $N \geq 1$. Let $\mathcal{G} \subset [a,b]$ be a Riemann germ with respect to (J,f). Assume that $\mathcal{G}$ is conservative. Then we have

$$\mathcal{G} \text{ monotone} \quad \Leftrightarrow \quad \mathcal{G} \text{ Kružkov} \quad \Leftrightarrow \quad \mathcal{G} \quad D_+\text{-germ}$$

Notice that even for $N = 2$, there exist monotone and nonmonotone nonconservative Kružkov germs (see ii) of Lemma 9.8).

Notice also that for conservative germs $\mathcal{G}$, the fact that the monotonicity of $\mathcal{G}$ is equivalent to the fact that $\mathcal{G}$ is Kružkov (which can itself be seen as equivalent to the fact that $\mathcal{G}$ generates a L^1-contraction semi-group), can be interpreted as a version of Crandall-Tartar's Lemma (see Proposition 1 in [15]).

Notice also that there exist monotone Kružkov germs which are not necessarily conservative, as it is already the case for all non-conservative germs for $N = 1$ branch.

Theorem 2.30 (Properties of quasi HJ germs)
Assume (2.2) with $N \geq 1$.
i) (Regularity of $\hat{f}$)
Every quasi HJ germ $\mathcal{G}$ is a Riemann germ, i.e. $\hat{f} = \hat{f}_{\mathcal{G}}$ is continuous. Moreover there exists a function $\hat{h}$ and $\omega = (\omega^1, \ldots, \omega^N) \in (0,+\infty)^N$ such that

$$(2.23) \qquad \left\{ \begin{array}{l} \hat{h} : [a,b] \to \mathbb{R} \quad \text{is continuous} \\ p \mapsto \sigma^j \hat{h}(p) \quad \text{is nondecreasing in } p^j \text{ for all } j = 1, \ldots, N, \\ \hat{f} := (\omega^1 \hat{h}, \ldots, \omega^N \hat{h}) \text{ satisfies the monotone bounds given in the second line of (2.14).} \end{array} \right.$$

ii) (HJ-relaxation formula)
Assume furthermore that $\mathcal{G}$ is a HJ germ (i.e. with $\omega^j = 1$ for all $j = 1, \ldots, N$). Then for each $p \in [a,b]$, we have
(2.24)

$$\left\{\hat{h}(p)\right\} = \bigcup_{q \in [a,b]} \left(\left\{\hat{h}(q)\right\} \cap \bigcap_{J^j \simeq (-\infty,0)} \{G^j(p^j, q^j)\} \cap \bigcap_{J^j \simeq (0,+\infty)} \{G^j(q^j, p^j)\} \right) \quad \textbf{(HJ-relaxation formula)}$$

Remark 2.31 *Here HJ-relaxation can be seen as a special case of Riemann relaxation given in Theorem 2.23, in the particular case where all the components of the junction Godunov flux are the same (equal to $\hat{h}$ here).*

Remark 2.32 *Notice that HJ-relaxation formula (2.24) can be used to define some HJ-relaxation operator (see Lemma 6.7) which computes the analogue for $N \geq 1$ of the effective boundary condition that was obtained for $N = 1$ by* Andreianov, Sbihi *in [5].*

Definition 2.33 (Characteristic subset of a HJ germ)
Assume (2.2) with $N \geq 1$. Let $\mathcal{G}$ be a generalized Riemann germ which is a HJ germ. We introduce the characteristic subset of the HJ germ $\mathcal{G}$ as the set $\chi\mathcal{G}$ defined by

$$\chi\mathcal{G} := \underline{\chi}\mathcal{G} \cup \overline{\chi}\mathcal{G} \quad \text{with} \quad \left\{ \begin{array}{l} \underline{\chi}\mathcal{G} := \left\{\hat{p} \in \mathcal{G}, \quad BA^j_{\sigma^j} \neq \emptyset \quad \text{for all} \quad j = 1, \ldots, N\right\} \\ \\ \overline{\chi}\mathcal{G} := \left\{\hat{p} \in \mathcal{G}, \quad BA^j_{-\sigma^j}(\hat{p}^j) \neq \emptyset \quad \text{for all} \quad j = 1, \ldots, N\right\} \end{array} \right.$$

with $BA^j_+(\hat{p}^j) := BA^j(\hat{p}^j) \cap (\hat{p}^j, +\infty)$ and $BA^j_-(\hat{p}^j) := BA^j(\hat{p}^j) \cap (-\infty, \hat{p}^j)$.

Theorem 2.34 (HJ germ determined by its characteristic subset)
Assume (2.2) with $N \geq 1$. Let $\mathcal{G}, \mathcal{G}_0$ be two generalized Riemann germs which are both HJ germs. Then we have

$$\chi\mathcal{G} \subset \mathcal{G}_0 \quad \text{implies} \quad \mathcal{G}_0 = \mathcal{G}$$

Theorem 2.34 is very important, because in many practical situations, the characteristic subset $\chi\mathcal{G}$ is usually a finite set (see for instance example given in Lemma 9.12, where the characteristic subset is a set of four points for a HJ germ on a $1:1$ junction).

Notice that monotone germs have less good properties than Kružkov or quasi HJ germs. For instance there exist monotone generalized germs which are not Riemann germs (see Lemma 9.1 for $N=2$).
Notice also that for any Kružkov or quasi HJ germs, the Godunov flux at the junction is indeed always locally Lipschitz (see Theorem 2.26 for Kružkov germs, and see Proposition 6.9 for HJ germs). On the contrary, there exists monotone germs $\mathcal{G}$ such that $\hat{f}_{\mathcal{G}}$ is continuous but not locally Lipschitz (see Lemma 9.2).

Notice also that for $N=1$, all generalized Riemann germs are indeed Kružkov germs, HJ germs and monotone germs (but not conservative germs in general). This result extends to conservative germs for $N=2$ in the case of $1:1$ junctions.

Theorem 2.35 (Properties of conservative germs on $1:1$ junctions)
Assume (2.2) for $N=2$ and $1:1$ junctions with $J^1 \simeq (-\infty,0)$ and $J^2 \simeq (0,+\infty)$. Let $\mathcal{G} \subset [a,b]$ be a generalized Riemann germ.
i) (Nature of the germ)
Then $\mathcal{G}$ is conservative if and only if it is a HJ germ. Moreover in that case, $\mathcal{G}$ is determined by its characteristic subset $\chi\mathcal{G}$ given in Definition 2.33, and $\mathcal{G}$ is a D_+-germ. Moreover we have

$$\mathcal{G} = \left\{ p \in [a,b], \quad RH^f(p) = 0, \quad D^f(p,q) \geq 0 \quad \text{for all} \quad q \in \chi\mathcal{G} \right\} \tag{2.25}$$

ii) (HJ-relaxation formula)
When $\mathcal{G}$ is conservative, then the Godunov flux satisfies $(\hat{f}^1,\hat{f}^2) = (\hat{h},\hat{h})$ with $\hat{h}:[a,b]\to\mathbb{R}$ with monotonicities $\hat{h}(\uparrow,\downarrow)$. And for any $p\in[a,b]$, there exists some $q\in[a,b]$ (possibly non unique) such that

$$\hat{h}(p) = G^{f^1}(p^1,q^1) = \hat{h}(q) = G^{f^2}(q^2,p^2) \tag{2.26}$$

Moreover we have

$$\max\left\{ \inf_{[p^1,b^1]} f^1, \ \inf_{[a^2,p^2]} f^2 \right\} \leq \hat{h}(p) \leq \min\left\{ \sup_{[a^1,p^1]} f^1, \ \sup_{[p^2,b^2]} f^2 \right\} \tag{2.27}$$

Notice that for $1:1$ junctions, there are examples of Kružkov germs which are not conservative (see for instance ii) of Lemma 9.8 for $2:0$ junctions). Notice also that HJ-relaxation formula (2.26) allows to construct all conservative germs for $1:1$ junctions (see for instance the Relaxation operator in Lemma 6.7).
As a straightforward corollary of Theorem 2.24, we get the following result.

Corollary 2.36 (Semigroup of germs for the gluing of $1:1$ junctions with the same flux)
Assume (2.2) and nondegeneracy assumption (2.17) for $N=2$ and $1:1$ junctions $J=\{0\}\cup\bigcup_{j=1,2} J^j$ with $J^1 \simeq (-\infty,0)$ and $J^2 \simeq (0,+\infty)$ and the same flux function

$$f^1 = f^2 =: f^0, \quad [a^1,b^1] = [a^2,b^2] =: [a^0,b^0], \quad [a,b] := [a^0,b^0]^2$$

Let $\mathfrak{G}^{(J,f)}$ be the set of Riemann germs $\mathcal{G}\subset[a,b]$ with respect to (J,f). Let

$$\mathcal{G}^{f^0} := \left\{ (r,s) \in [a^0,b^0], \quad f^0(r) = G^{f^0}(r,s) = f^0(s) \right\}$$

where G^{f^0} is the standard Godunov flux associated to the flux $f^0:[a^0,b^0]\to\mathbb{R}$.
Then $(\mathfrak{G}^{(J,f)},\sharp)$ is a semigroup, with identity element equal to $\mathcal{G}^{f^0}$. We have

$$\begin{array}{cccc} \sharp: & \mathfrak{G}^{(J,f)} \times \mathfrak{G}^{(J,f)} & \to & \mathfrak{G}^{(J,f)} \\ & (\mathcal{G}_2,\mathcal{G}_1) & \mapsto & \mathcal{G}_2\sharp\mathcal{G}_1 \end{array}$$

where we recall that $\mathcal{G}_2\sharp\mathcal{G}_1$ is the germ obtained by gluing of the outgoing branch $J^2\simeq(0,+\infty)$ of $\mathcal{G}_2$ with the ingoing branch $J^1\simeq(-\infty,0)$ of $\mathcal{G}_1$.

Notice that this semigroup can be not commutative (see Lemma 9.13 for an example).

2.3 Introduction to Part II

Almost all applications do concern only bell-shaped fluxes. Part II is devoted to this study where many simplifications and practical computations arise.

2.3.1 Bell-shaped fluxes, prefluxes and polar decomposition

We introduce the following

Definition 2.37 (Bell-shaped fluxes)
Assume (2.2) for $N \geq 1$. We say that $f = (f^1, \dots, f^N)$ is bell-shaped, if each scalar function $f^k : [a^k, b^k] \to [0, +\infty)$ is continuous, satisfies $f^k(a^k) = 0 = f^k(b^k)$, has a maximum value $f^k_{\max} := f^k(c^k)$ at $c^k \in (a^k, b^k)$ and is increasing on (a^k, c^k) and decreasing on (c^k, b^k) for $k = 1, \dots, N$. We set the monotone functions

$$(2.28) \qquad f^{k,+}(q) = \begin{cases} f^k(q) & \text{for} \quad q \in [a^k, c^k] \\ f^k(c^j) & \text{for} \quad q \in [c^k, b^k] \end{cases} \quad \text{and} \quad f^{k,-}(q) = \begin{cases} f^k(c^k) & \text{for} \quad q \in [a^k, c^k] \\ f^k(q) & \text{for} \quad q \in [c^k, b^k] \end{cases}$$

We now define below the notion of preflux.

Definition 2.38 (Preflux)
*Let $N \geq 1$. We say that $\hat{\gamma}$ is a **preflux** if it satisfies the following set of conditions*

$$(2.29) \qquad \begin{cases} \hat{\gamma} : [0, +\infty)^N \to [0, +\infty)^N \quad \text{is continuous} & \textbf{(Continuity)} \\ \hat{\gamma} \text{ is locally constant on } \left\{\hat{\gamma} \neq id_{[0,+\infty)^N}\right\} \text{ in the sense of Definition 2.13} & \textbf{(Local constancy)} \\ 0 \leq \hat{\gamma} \leq id_{[0,+\infty)^N} & \textbf{(Bounds)} \end{cases}$$

Moreover, for a function $\hat{\gamma}$ satisfying the first and third line of (2.29), then its local constancy in the second line of (2.29) is equivalent to

$$(2.30) \qquad \partial_j \hat{\gamma} = 0 \quad \text{on the open set} \quad \left\{p \in [0, +\infty)^N, \quad \hat{\gamma}^j(p) < p^j\right\}, \quad \text{for all} \quad j = 1, \dots, N$$

Remark 2.39 *Notice that (2.30) is a rigid condition which strongly mixes the behaviours of different components of $\hat{\gamma} = (\hat{\gamma}^1, \dots, \hat{\gamma}^N)$.*

For bell-shaped fluxes, a preflux is a key notion, because of the following result for junction Godunov fluxes.

Theorem 2.40 (Polar decomposition of Godunov fluxes for bell-shaped fluxes)
Assume (2.2) for $N \geq 1$ for a junction (J, f) with $\sigma \in \{\pm 1\}^N$. Assume that f is bell-shaped in the sense of Definition 2.37, and call $\bar{\gamma} : [a, b] \to [0, +\infty)^N$ the capacity defined by

$$(2.31) \qquad \bar{\gamma}^j(p) := f^{j,\sigma_j}(p^j), \quad j = 1, \dots, N.$$

i) (Polar decomposition)
Let $\mathcal{G} \subset [a, b]$ be a Riemann germ with respect to (J, f). Then the Godunov flux $\hat{f}_{\mathcal{G}}$ associated to $\mathcal{G}$ has the following polar decomposition

$$(2.32) \qquad \hat{f}_{\mathcal{G}} = \hat{\gamma} \circ \bar{\gamma} \qquad \textbf{(preflux} \circ \textbf{capacity)}$$

where $\hat{\gamma} : [0, +\infty)^N \to [0, +\infty)^N$ is a preflux (as in Definition 2.38) and $\bar{\gamma}$ is the capacity defined in (2.31). Moreover the preflux $\hat{\gamma}$ is unique on the image of the capacity

$$\hat{K} := Im(\bar{\gamma}) = \prod_{k=1,\dots,N} [0, f^k_{\max}]$$

ii) (Riemann germ construction)
Given any preflux $\hat{\gamma} : [0, +\infty)^N \to [0, +\infty)^N$, we define

$$(2.33) \qquad \mathcal{G} := \mathcal{G}_{\hat{f}} := \left\{p \in [a, b], \quad \hat{f}(p) = f(p)\right\} \quad \text{with} \quad \hat{f} := \hat{\gamma} \circ \bar{\gamma} : [a, b] \to [0, +\infty)^N.$$

Then $\mathcal{G}$ is a Riemann germ.

Remark 2.41 (Supply and Demand law of Lebacque)
In the Polar decomposition, the capacity $\bar\gamma^j = f^{j,\sigma_j}$ gives a solid and mathematically rigorous fundation to the heuristic law of Lebacque (see [37]). Recall that the Supply and Demand law of Lebacque suggests to look for junction Godunov fluxes which involve on Demand functions $f^{j,\sigma_j=1}$ for ingoing branches and only Supply functions $f^{j,\sigma_j=-1}$ for outgoing branches.

Hence we see that for bell-shaped fluxes, all the information about a Riemann germ is contained in its preflux, which is a quite simple notion. It is also possible to translate all usual properties of Godunov fluxes (like for instance σ-monotonicity, conservativity, Kružkov, HJ) as immediate properties on the prefluxes.

Even if prefluxes seem to be simple objects, it is not so easy to construct them. A nice way to do it, consists to consider the weaker notion of quasi-preflux and then to relax it into a genuine preflux.

Definition 2.42 (Quasi-preflux)
Let $N \geq 1$. Let $\bar\lambda = (\bar\lambda^1, \dots, \bar\lambda^N) \in (0,+\infty)^N \cup \{+\infty\}^N$. A function $\hat\gamma_0 : [0,+\infty)^N \to \mathbb{R}^N$ is said to be a **$\bar\lambda$-quasi-preflux** *(or simply quasi-preflux) if it satisfies the following conditions*
(2.34)
$$\left\{\begin{array}{l}
\hat\gamma_0 : [0,+\infty)^N \to \mathbb{R}^N \quad \text{is continuous,} \qquad \text{with moreover } \hat\gamma_0 : [0,+\infty)^N \to [0,+\infty)^N \text{ if } \bar\lambda = (+\infty, \dots, +\infty),\\[2ex]
\hat\gamma_0 \quad \text{is Riemann monotone in the sense of Definition 2.12,}\\[2ex]
(\hat\gamma_0^j(q))_{|\{q^j=0\}} \leq 0 \quad \text{and} \quad (\hat\gamma_0^j(q))_{|\left\{q^j \geq \bar\lambda^j\right\}} \geq 0 \quad \text{for all} \quad j = 1, \dots, N
\end{array}\right.$$

Because every preflux is known to be Riemann monotone (see i) of Lemma 11.11), we see that any preflux is in particular a quasi-preflux. As a very natural conservative case is the example (see [37]) of Lebacque quasi-preflux $\hat\gamma_0 = (\hat\gamma_0^0, \hat\gamma_0^1, \hat\gamma_0^2) : [0,+\infty)^3 \to [0,+\infty)^3$ for 1:2 junctions given for $\bar\gamma = (\bar\gamma^0, \bar\gamma^1, \bar\gamma^2) \in [0,+\infty)^3$ by
$$\left\{\begin{array}{l}
\hat\gamma_0^1(\bar\gamma) := \min\left\{\bar\gamma^1, \theta^1\bar\gamma^0\right\},\\
\hat\gamma_0^2(\bar\gamma) := \min\left\{\bar\gamma^2, \theta^2\bar\gamma^0\right\}, \qquad \text{with} \quad 0 < \theta^1, \theta^2 < 1, \quad \theta^1 + \theta^2 = 1\\
\hat\gamma_0^0 = \hat\gamma_0^1 + \hat\gamma_0^2
\end{array}\right.$$
Here $\hat\gamma_0$ is a not a preflux, but only a (conservative) quasi-preflux.

Theorem 2.43 (Riemann relaxation $\mathfrak{R}_{\bar\lambda}$ of $\bar\lambda$-quasi-prefluxes)
Let $N \geq 1$ and $\bar\lambda \in (0,+\infty)^N$ be a flux limiter, and consider a junction whose orientations of the branches are described by $\sigma_0 \in \{\pm 1\}^N$. Let us consider a $\bar\lambda$-quasi-preflux $\hat\gamma_0 : [0,+\infty)^N \to \mathbb{R}^N$ in the sense of Definition 13.1.
i) (Practical relaxation calculus using the doubling set)
We define the doubling set

$$\mathbb{D}_{\bar\lambda} := \left\{(\lambda^L, \lambda^R) \in [0,\bar\lambda]^2, \quad \max\left\{\lambda^{L,j}, \lambda^{R,j}\right\} = \bar\lambda^j \quad \text{for all} \quad j = 1, \dots, N\right\} \tag{2.35}$$

For any $\bar\gamma \in [0,\bar\lambda]$, we consider $(\lambda^L, \lambda^R) \in \mathbb{D}_{\bar\lambda}$ solutions of

$$\min\left\{\bar\gamma^j, (\lambda^L)^j\right\} = \hat\gamma_0^j(\lambda^R), \quad j = 1, \dots, N. \tag{2.36}$$

Then the set
$$\mathcal{R}_{\bar\gamma} := \left\{(\lambda^L, \lambda^R) \in \mathbb{D}_{\bar\lambda} \quad \text{solution of (13.6)}\right\}$$
is non-empty, and the set
$$\Lambda_{\bar\gamma} := \left\{\hat\gamma_0(\lambda^R), \quad (\lambda^L, \lambda^R) \in \mathcal{R}_{\bar\gamma}\right\}$$
is reduced to a singleton
$$\Lambda_{\bar\gamma} = \{\hat\gamma_{\bar\lambda}(\bar\gamma)\}.$$
This defines a map $\hat\gamma_{\bar\lambda} : [0,\bar\lambda] \to [0,\bar\lambda]$, that we extend to $[0,+\infty)^N$, setting
$$\begin{array}{llll}
\hat\gamma_{\bar\lambda} : & [0,+\infty)^N & \to & [0,+\infty)^N\\
 & \bar\gamma & \mapsto & (\hat\gamma_{\bar\lambda} \circ T_{\bar\lambda})(\bar\gamma)
\end{array}$$
where $T_{\bar\lambda}$ is the truncation operator

$$T_{\bar\lambda}(\bar\gamma) := (\min\left\{\bar\gamma^1, \bar\lambda^1\right\}, \dots, \min\left\{\bar\gamma^N, \bar\lambda^N\right\}). \tag{2.37}$$

ii) (Riemann relaxation operator $\mathfrak{R}_{\bar\lambda}$ "on the box $[0,\bar\lambda]$")
Then $\hat\gamma_{\bar\lambda} : [0,+\infty)^N \to [0,+\infty)^N$ is a preflux. Moreover we set

$$\mathfrak{R}_{\bar\lambda}\hat\gamma_0 := \hat\gamma_{\bar\lambda}. \tag{2.38}$$

Under certain additional conditions, it is also possible to define the operator

$$\mathfrak{R}_\infty := \lim_{\bar\lambda\to(\infty,\dots,\infty)} \mathfrak{R}_{\bar\lambda}$$

where the flux limiter $\bar\lambda$ disappears. In that case, it is possible to show the following "compatibility formula"

$$\mathfrak{R}(\hat\gamma_0 \circ \bar\gamma) = (\mathfrak{R}_\infty\hat\gamma_0)\circ\bar\gamma$$

where $\mathfrak{R}$ is defined in (2.21) and where $\bar\gamma^j = f^{j,\sigma_j}$ is the capacity. In other words, to relax a junction Godunov flux, it is sufficient to relax its associated quasi-preflux. The analogue of this formula also works with $\mathfrak{R}_\infty$ replaced by $\mathfrak{R}_{\bar\lambda}$ if furthermore the quasi-preflux $\hat\gamma_0$ satisfies $\hat\gamma_0 = \hat\gamma_0 \circ T_{\bar\gamma}$.

As an illustration, the relaxation of Lebacque quasi-preflux is explicitely given by the following conservative preflux

$$\mathfrak{R}_\infty\hat\gamma_0 =: \hat\gamma \quad \text{with} \quad \left\{\begin{array}{l} \hat\gamma^1(\bar\gamma) := \min\left\{\bar\gamma^1, \max\left\{\theta^1\bar\gamma^0, \bar\gamma^0-\bar\gamma^2\right\}\right\},\\ \hat\gamma^2(\bar\gamma) := \min\left\{\bar\gamma^2, \max\left\{\theta^2\bar\gamma^0, \bar\gamma^0-\bar\gamma^1\right\}\right\},\\ \hat\gamma^0(\bar\gamma) = (\hat\gamma^1+\hat\gamma^2)(\bar\gamma) = \min\left\{\bar\gamma^0, \bar\gamma^1+\bar\gamma^2\right\}.\end{array}\right. \tag{2.39}$$

We also have the following result about gluing of prefluxes, where a doubling set also plays a central role.

Theorem 2.44 (Gluing of prefluxes)
Let $\gamma := \alpha,\beta$, and $N_\gamma \geq 2$, and some prefluxes $\hat\lambda_\gamma : [0,+\infty)^{N_\gamma} \to \mathbb{R}^{N_\gamma}$, with notation $\hat\lambda_\gamma = (\hat\lambda^0_\gamma, \dots, \hat\lambda^{N_\gamma-1}_\gamma)$. Let $j_\gamma \in \{0,\dots,N_\gamma-1\}$. In order to simplify the presentation, let us assume that $j_\alpha = 0 = j_\beta$.
i) (Practical calculus for gluing prefluxes)
Given some limiter $\bar\lambda^0 \in [0,+\infty)$, let us consider the doubling set

$$\mathbb{D}_{\bar\lambda^0} := \left\{(\lambda^{L,0},\lambda^{R,0}) \in [0,\bar\lambda^0]^2, \quad \max\left\{\lambda^{L,0},\lambda^{R,0}\right\} = \bar\lambda^0\right\}.$$

Then for any $\bar\gamma = (\bar\gamma_\alpha,\bar\gamma_\beta) \in [0,+\infty)^{N_\alpha-1}\times[0,+\infty)^{N_\beta-1}$, consider the equation

$$\hat\lambda^0_\alpha(\lambda^{L,0},\bar\gamma_\alpha) = \hat\lambda^0_\beta(\lambda^{R,0},\bar\gamma_\beta) \tag{2.40}$$

Then the set

$$\mathcal{R} := \left\{(\lambda^{L,0},\lambda^{R,0}) \in \mathbb{D}_{\bar\lambda^0}, \quad \textit{with } (\lambda^{L,0},\lambda^{R,0}) \textit{ solution of (2.40)}\right\}$$

is non empty and let us consider the set

$$\Lambda := \left\{\tilde\lambda(\lambda^{L,0},\lambda^{R,0},\bar\gamma) \quad \textit{with } (\lambda^{L,0},\lambda^{R,0}) \in \mathcal{R}\right\}$$

with

$$\tilde\lambda(\lambda^{L,0},\lambda^{R,0},\bar\gamma) = ((\hat\lambda^1_\alpha,\dots,\hat\lambda^{N_\alpha-1}_\alpha)(\lambda^{L,0},\bar\gamma_\alpha);(\hat\lambda^1_\beta,\dots,\hat\lambda^{N_\beta-1}_\beta)(\lambda^{R,0},\bar\gamma_\beta))$$

Then Λ is reduced to a singleton $\Lambda = \{\lambda\}$ and this defines the following map

$$\begin{array}{rccl} \underline{\hat\lambda}: & [0,+\infty)^{N_\alpha-1}\times[0,+\infty)^{N_\beta-1} & \to & \mathbb{R}^{N_\alpha+N_\beta-2}\\ & (\bar\gamma_\alpha,\bar\gamma_\beta) & \mapsto & \underline{\hat\lambda}(\bar\gamma_\alpha,\bar\gamma_\beta) := \lambda. \end{array}$$

ii) (The $\bar\lambda^0$-glued preflux)
Then $\underline{\hat\lambda}$ is a preflux. We use the following notation for the glued preflux

$$\underline{\hat\lambda} =: \hat\lambda_\alpha \underset{(j_\alpha:j_\beta,\bar\lambda^0)}{\sharp} \hat\lambda_\beta \tag{2.41}$$

which is defined here only for $j_\alpha = 0 = j_\beta$ (but can easily be generalized for indices $j_\alpha \in \{0,\dots,N_\alpha-1\}$ and $j_\beta \in \{0,\dots,N_\beta-1\}$). We also may use notation $\hat\lambda_\alpha \underset{(j_\alpha:j_\beta)}{\sharp} \hat\lambda_\beta$ where we drop the index $\bar\lambda^0$ when it is clear from the context.

Remark 2.45 (Independence for large enough limiter $\bar{\lambda}^0$)
Notice that if $\hat{\lambda}_\gamma \circ T_{\bar{\lambda}_\gamma} = \hat{\lambda}_\gamma$ with $\bar{\lambda}^0_\alpha, \bar{\lambda}^0_\beta \le \bar{\lambda}^0, \bar{\lambda}^{0\prime}$ and where the truncation operator is defined in (2.37), then the gluing is independent on the limiter:

$$\hat{\lambda}_\alpha \underset{(j_\alpha:j_\beta,\bar{\lambda}^0)}{\sharp} \hat{\lambda}_\beta = \hat{\lambda}_\alpha \underset{(j_\alpha:j_\beta,\bar{\lambda}^{0\prime})}{\sharp} \hat{\lambda}_\beta.$$

Remark 2.46 (Gluing quasi-prefluxes)
Notice also that the gluing extends to the the gluing of quasi-prefluxes, under a certain additional assumption on the glued branches j_α and j_β, which is automatically satisfied for prefluxes. Moreover, we have commutation of operations: gluing of relaxation of quasi-prefluxes is the relaxation of the gluing of quasi-prefluxes.

2.3.2 Explicit prefluxes

We start with an application of our theory.

Proposition 2.47 (A Kružkov germ product property for several conservative lines)
Assume (2.2) for $N = 2n$ with $n : n$ junctions, and call f^{jL} and f^{jR} respectively the j-th ingoing and j-the outgoing fluxes for $j = 1, \dots, n$. Assume that each $f^{j\alpha}$ is bell-shaped in the sense of Definition 2.37 with maximum at $c^{j\alpha} \in (a^{j\alpha}, b^{j\alpha})$ and $f^{j\alpha}(a^{j\alpha}) = 0 = f^{j\alpha}(b^{j\alpha})$. Let $\mathcal{G} \subset [a,b]$ be a Kružkov germ satisfying for $p = (p^{1L}, p^{1R}, \dots, p^{nL}, p^{nR})$

$$\mathcal{G} \subset \bigcap_{j=1,\dots,n} \Sigma^j, \quad \text{with} \quad \Sigma^j := \left\{p \in [a,b], \quad f^{jL}(p^{jL}) = f^{jR}(p^{jR})\right\}$$

which means that each j-th line $J^{jL} \cup \{0\} \cup J^{jR} \simeq \mathbb{R}$ is conservative. Then

$$\mathcal{G} = \prod_{j=1,\dots,n} \mathcal{G}^j \quad \text{with} \quad \mathcal{G}^j \subset \left\{(p^{jL}, p^{jR}) \in [a^{jL}, b^{jL}] \times [a^{jR}, b^{jR}], \quad f^{jL}(p^{jL}) = f^{jR}(p^{jR})\right\} \subset \mathbb{R}^2$$

where each $\mathcal{G}^j$ is a Kružkov conservative germ with respect to (f^{jL}, f^{jR}) for a $1:1$ junction with $J^{jL} \simeq (-\infty, 0)$ and $J^{jR} \simeq (0, +\infty)$.

This result has implications in the theory of traffic on networks. In particular for a Kružkov germ, if two conservative lines L_i for $i = 1, 2$ cross each other, then the flux limiter on line L_1 only depends on the structure of the junction (between the two lines), but not on the state of the traffic on line L_2.
On the contrary if the germ is not assumed to be Kružkov[3], then the flux limiter on line L_1 may depend on the density of cars on line L_2 at the junction point, and there exist such germs which are no longer products of germs as above (see for instance Lemma 12.12).

We now present the following characterizations.

Proposition 2.48 (Characterization of prefluxes for a single branch)
Let $\hat{\gamma} = \hat{\gamma}^1 : [0, +\infty) \to [0, +\infty)$ be a preflux. Then there exists a constant $\bar{\lambda}^0 \in [0, +\infty]$ (the flux limiter) such that

$$\hat{\gamma}^1(\bar{\gamma}^1) = \min\left\{\bar{\gamma}^1, \bar{\lambda}^0\right\} \tag{2.42}$$

and every function $\hat{\gamma}^1 : [0, +\infty) \to [0, +\infty)$ satisfying (2.42) is a preflux for $N = 1$.

Proposition 2.49 (Characterization of 1:1 conservative prefluxes)
Let $\hat{\gamma} = (\hat{\gamma}^1, \gamma^2) : [0, +\infty)^2 \to [0, +\infty)^2$ be a 1:1 conservative preflux, i.e. a preflux satisfying $\hat{\gamma}^1 = \hat{\gamma}^2$. Then there exists a constant $\bar{\lambda}^0 \in [0, +\infty]$ (the flux limiter) such that

$$\hat{\gamma}^1(\bar{\gamma}) = \hat{\gamma}^2(\bar{\gamma}) = \min\left\{\bar{\gamma}^1, \bar{\gamma}^2, \bar{\lambda}^0\right\} \tag{2.43}$$

and every function $\hat{\gamma}$ satisfying (2.43) is a 1:1 conservative preflux.

We also have the following characterization of certain prefluxes with bounded images.

[3] And contrarily to what the author thought and stated erroneously in a preliminary version of this work.

Theorem 2.50 (Characterization of 1:2 conservative prefluxes)
Let us consider a preflux

$$\hat{\gamma} := (\hat{\gamma}^0, \hat{\gamma}^1, \hat{\gamma}^2) : [0,+\infty)^3 \to K \subset [0,+\infty)^3, \tag{2.44}$$

which is assumed to be 1:2 conservative, i.e. which satisfies

$$\hat{\gamma}^0 = \hat{\gamma}^1 + \hat{\gamma}^2. \tag{2.45}$$

Assume moreover that K is compact.
i) (Necessary condition)
Then we have

$$\left\{ \begin{array}{l} \hat{\gamma}^1(\bar{\gamma}) = \min\left\{\bar{\gamma}^1, \bar{\lambda}^1(\bar{\gamma}^2), \max\left\{\hat{\theta}^1(\bar{\gamma}^0), \bar{\gamma}^0 - \bar{\gamma}^2\right\}\right\}, \\ \hat{\gamma}^2(\bar{\gamma}) = \min\left\{\bar{\gamma}^2, \bar{\lambda}^2(\bar{\gamma}^1), \max\left\{\hat{\theta}^2(\bar{\gamma}^0), \bar{\gamma}^0 - \bar{\gamma}^1\right\}\right\}, \end{array}\right. \tag{2.46}$$

for four continuous functions $\bar{\lambda}^1, \bar{\lambda}^2, \hat{\theta}^1, \hat{\theta}^2 : [0,+\infty) \to [0,+\infty)$ and a constant $A_* = (A_*^0, A_*^1, A_*^2) \in [0,+\infty)^3$ satisfying

$$\left\{ \begin{array}{l} A_*^0 = A_*^1 + A_*^2, \\ \bar{\lambda}^1 = const = \bar{\lambda}^1(A_*^2) \quad on \quad [A_*^2, +\infty), \\ \bar{\lambda}^2 = const = \bar{\lambda}^2(A_*^1) \quad on \quad [A_*^1, +\infty), \\ \hat{\theta}^j = const = \hat{\theta}^j(A_*^0) \quad on \quad [A_*^0, +\infty) \quad for \quad j = 1,2 \\ \\ \hat{\theta}^1(\bar{\gamma}^0) + \hat{\theta}^2(\bar{\gamma}^0) = \bar{\gamma}^0 \quad for\ all \quad \bar{\gamma}^0 \in [0, A_*^0], \\ \bar{\lambda}^2 \circ \hat{\theta}^1 \geq \hat{\theta}^2, \\ \bar{\lambda}^1 \circ \hat{\theta}^2 \geq \hat{\theta}^1, \end{array}\right. \tag{2.47}$$

where the two last lines can be interpreted as a further compatibility condition between the four functions $\bar{\lambda}^j, \hat{\theta}^j$ for $j = 1, 2$.
ii) (Sufficient condition)
Conversely, for every continuous functions $\bar{\lambda}^1, \bar{\lambda}^2, \hat{\theta}^1, \hat{\theta}^2$ and constant A_* as above satisfying (2.47), the function $\hat{\gamma}$ defined by (2.46) and (2.45) is a 1:2 conservative preflux satisfying (2.44) with K compact.

Remark 2.51 (Limiters)
In expression (2.46), the functions $\bar{\lambda}^1, \hat{\lambda}^2$ can be seen naturally as limiters.

Remark 2.52 (Case of unbounded prefluxes)
The condition K compact can be removed, and a characterization of 1:2 conservative prefluxes is still available, but more technical. Essentially, limit cases have to be added.

Remark 2.53 (Kružkov 1:2 conservative prefluxes)
Notice that the functions $\hat{\theta}^j$ for $j = 1, 2$ are not necessarily monotone here. On the contrary, there exists a notion of Kružkov 1:2 conservative prefluxes (see Definition 11.1) which indeed forces the function $\hat{\theta}^j$ to be monotone. Among prefluxes given by Theorem 2.50, those which are Kružkov are exactly those satisfying moreover the following condition

$$\left\{ \begin{array}{r} (\hat{\theta}^j)' \geq 0, \\ -1 \leq (\bar{\lambda}^j)' \leq 0, \end{array}\right. \quad for \quad j = 1, 2. \tag{2.48}$$

Theorem 2.54 (A large family of 1:n conservative prefluxes)
i) (Assumptions)
Let $N := 1 + n$ with $n \geq 2$. We consider functions $\hat{\theta}^j$ for $j = 0, \ldots, n$, satisfying

$$\left\{ \begin{array}{l} \hat{\theta}^j : [0,+\infty) \to [0,+\infty) \quad continuous\ increasing\ bijective, \quad for \quad j = 0, 1, \ldots, n \\ \\ \hat{\theta}^0 = id_{[0,+\infty)} = \displaystyle\sum_{j=1,\ldots,n} \hat{\theta}^j \end{array}\right. \tag{2.49}$$

Let for $\bar\gamma = (\bar\gamma^0, \bar\gamma')$ *with* $\bar\gamma' := (\bar\gamma^1, \ldots, \bar\gamma^n)$ *and for* $j = 1, \ldots, n$

$$B^j(\bar\gamma) := \hat\theta^j \left(\max_{I \subset \{1,\ldots,n\}\setminus\{j\}} Y_I \right) \quad \textit{with} \quad \left\{ \begin{array}{l} Y_I := \left(id_{[0,+\infty)} - \sum_{k\in I} \hat\theta^k \right)^{-1} (\bar\gamma^0 - \sum_{k\in I} \bar\gamma^k)_+, \\ \textit{(including } Y_\emptyset := \bar\gamma^0 \textit{ for } I := \emptyset\textit{)}, \end{array} \right.$$

Then $B^j : [0,+\infty)^N \to [0,+\infty)$ *is a continuous function, independent of the variable* $\bar\gamma^j$*.*
ii) (The result)
Then the function $\hat\lambda_* : [0,+\infty)^N \to [0,+\infty)^N$ *defined by*

$$\text{(2.50)} \qquad \left\{ \begin{array}{l} \hat\lambda_*^j(\bar\gamma) := \min\left\{\bar\gamma^j, B^j(\bar\gamma)\right\}, \quad j = 1, \ldots, n, \\ \hat\lambda_*^0 := \hat\lambda_*^1 + \cdots + \hat\lambda_*^n \end{array} \right.$$

is a 1:n conservative preflux, which satisfies moreover

$$\text{(2.51)} \qquad \hat\lambda_*^0(\bar\gamma) = \min\left\{\bar\gamma^0, \bar\gamma^1 + \cdots + \bar\gamma^n\right\}.$$

Remark 2.55 (Comments)
i) (Preflux seen as a generalization)
The preflux given by Theorem 2.54 generalizes preflux (2.39) to the case of $n \geq 2$ *branches, and with variable coefficients* $\hat\theta^j$ *instead of constant coefficients* θ^j*.*
ii) (No limiters)
Notice that the preflux $\hat\lambda_*$ *given by Theorem 2.54 does not contain any limiter. In order to introduce fixed limiters, it is sufficient to consider the preflux* $\hat\lambda_* \circ T_{\bar\lambda}$ *for some* $\bar\lambda \in [0,+\infty)^N$ *where the truncation operator* $T_{\bar\lambda}$ *is defined in (2.37). It is also possible to introduce variable limiters, but the construction is much more subtle, see Corollary 14.17.*

Remark 2.56 (A large family of n:m conservative prefluxes)
Gluing two conservative prefluxes, one of type n:1 with one of type 1:m, we get easily explicit n:m conservative prefluxes, see for instance expression (14.88).

2.4 Introduction to Part III

2.4.1 Preliminaries: $\mathcal{G}$-entropy solutions, semi-solutions, strong traces

In part I of the work we were focusing on self-similar solutions of Riemann problems on a junction. On the contrary, in Part III, we consider general Kružkov entropy solutions/subsolutions/supersolutions, for which we will need to be able to define the trace at the junction point. This will be done using the work of Panov [43].

For functions $u^k : [0,+\infty) \times J^k \to [a^k, b^k]$, with $u^k(t,x)$, we consider the equation of the first line of (2.4), namely the scalar conservation laws on the branch J^k

$$\text{(2.52)} \qquad \partial_t u^k + \partial_x(f^k(u^k)) = 0 \quad \text{on} \quad (0,+\infty) \times J^k, \quad k = 1, \ldots, N.$$

We want to recall the definition of Kružkov entropy solutions, subsolutions and supersolutions (see Kružkov [35]). For $x \in \mathbb{R}$, we set $\text{sign}(x) = 1_{\{x>0\}} - 1_{\{x<0\}}$. We recall that the Kružkov pairs (entropy/ flux of entropy) for $u = (u^1, \ldots, u^N), v = (v^1 \ldots, v^N) \in \mathbb{R}^N$ are given by (2.5), i.e.

$$\text{(2.53)} \qquad \eta^k(u^k, v^k) := |u^k - v^k| \quad \text{and} \quad \psi^{f^k}(u^k, v^k) = \psi^k(u^k, v^k) := \text{sign}(u^k - v^k) \cdot \left\{f^k(u^k) - f^k(v^k)\right\}$$

Similarly, for $x \in \mathbb{R}$, we set $|x|_\pm := \max(0, \pm x)$ and $\text{sign}^\pm(x) = 1_{\{\pm x>0\}}$. We recall that the Kružkov pairs (semi-entropy/ flux of semi-entropy) given by

$$\text{(2.54)} \qquad \eta^k_\pm(u^k, v^k) := |u^k - v^k|_\pm \quad \text{and} \quad \psi^{f^k}_\pm(u^k, v^k) = \psi^k_\pm(u^k, v^k) := \text{sign}^\pm(u^k - v^k) \cdot \left\{f^k(u^k) - f^k(v^k)\right\}$$

We will also use shorthands notations $\eta^k(u,v) := \eta^k(u^k, v^k)$ and $\psi^{f^k}(u,v) = \psi^k(u,v) := \psi^k(u^k, v^k)$ and similarly for $\eta^k_\pm$ and $\psi^k_\pm$, $\psi^{f^k}_\pm$. Then we recall the following standard notion (which recalls and contains Definition 2.1 for Kružkov entropy solutions).

Definition 2.57 (Kružkov entropy solution, subsolution and supersolution)
We say that u^k is a Kružkov entropy solution (resp. subsolution, resp. supersolution) of (2.52), with initial data $u_0^k \in L^\infty(J^k;[a^k,b^k])$, if $u^k \in L^\infty([0,+\infty)\times J^k;[a^k,b^k])$ and for any constant $c=(c^1,\dots,c^N)\in\mathbb{R}^N$, and for any (test) function $0\le\varphi^k\in C_c^1([0,+\infty)\times J^k)$, we have

$$\int_{(0,+\infty)\times J^k}\left\{\eta^k(u,c)\varphi_t^k+\psi^k(u,c)\varphi_x^k\right\}\ dtdx+\int_{\{0\}\times J^k}\eta^k(u_0,c)\varphi^k\ dx\ge 0$$

$$\left(\text{resp.}\quad \int_{(0,+\infty)\times J^k}\left\{\eta_\pm^k(u,c)\varphi_t^k+\psi_\pm^k(u,c)\varphi_x^k\right\}\ dtdx+\int_{\{0\}\times J^k}\eta_\pm^k(u_0,c)\varphi^k\ dx\ge 0\right)$$

with $+$ for subsolutions and $-$ for supersolutions.
For subsolutions u^k, we write

$$\partial_t u^k+\partial_x(f^k(u^k))\underset{\text{Kružkov}}{\le}0\quad\text{on}\quad(0,+\infty)\times J^k$$

and for supersolutions u^k, we write

$$\partial_t u^k+\partial_x(f^k(u^k))\underset{\text{Kružkov}}{\ge}0\quad\text{on}\quad(0,+\infty)\times J^k$$

Notice that the box $[a,b]$ is where all the values of the function u stay confined.

The standard Kružkov theory on the real line (hence without junctions) shows that BV norm of the initial data is preserved by the evolution. As it has been shown in an important counter-example by Adimurthi, Ghoshal, Dutta, Veerappa Gowda [1], already for $1:1$ junctions with convex fluxes on each branch, the BV norm (in space) of the solution may blow-up in finite time. For this reason, the notion of trace of the solution at the junction point can not be based on BV bounds which do not exist in general.

Fortunately, under suitable conditions on the fluxes, the notion of strong trace of the solution has first been shown to exist by Vasseur [47]. Then it has been generalized by Panov [43], in a way which is convenient for our work. We now recall this result, which plays a fundamental role in our analysis in Part II.

Theorem 2.58 (Existence of a strong trace; Theorem 1.1 in Panov [43])
Assume (2.2) for $N\ge 1$ and that f satisfies the nondegeneracy condition (2.17). Let u be a Kružkov entropy solution (resp. subsolution, supersolution) of (2.52) in the sense of Definition 2.57. Then for each index $j=1,\dots,N$, there exists a function $u^j(\cdot,0)\in L^1_{loc}(0,+\infty)$ satisfying

$$\text{ess}\lim_{J^j\ni x\to 0}\int_{(0,T)}|u^j(t,0)-u^j(t,x)|\ dt=0\quad\text{for all index } j \text{ and all } T>0 \tag{2.55}$$

Such function is called the strong trace of u^j on $(0,+\infty)_t\times\{0\}_x$.

Recall that without the nondegeneracy condition (2.17), the strong trace does not exist in general, but only a notion of quasi-trace is defined.

Definition 2.59 (Notion of $\mathcal{G}^{SUB}$ and $\mathcal{G}^{SUP}$)
Assume (2.2) for $N\ge 1$. Let $\mathcal{G}\subset[a,b]$ be a generalized Riemann germ, and let $\hat f:=\hat f_{\mathcal{G}}$ be its associated Godunov flux. Then we define

$$\mathcal{G}^{SUB}:=\Big\{p\in[a,b],\quad \sigma\diamond(\hat f-f)(p)\le 0\Big\}\quad\text{and}\quad \mathcal{G}^{SUP}:=\Big\{p\in[a,b],\quad \sigma\diamond(\hat f-f)(p)\ge 0\Big\}$$

Then we give the following definition (which recalls and contains Definition 2.2 for $\mathcal{G}$-entropy solutions).

Definition 2.60 ($\mathcal{G}$-entropy solution/subsolution/supersolution)
Assume (2.2) for $N\ge 1$ and that f satisfies the nondegeneracy condition (2.17). Let us consider some initial data $u_0=(u_0^1,\dots,u_0^N)$ with $u_0^k\in L^\infty(J^k;[a^k,b^k])$. We say that $u=(u^1,\dots,u^N)$ is a $\mathcal{G}$-entropy solution (resp. subsolution, resp. supersolution) of (2.52) with initial data u_0, if each u^k is a Kružkov entropy solution (resp. subsolution, resp. supersolution) of (2.52) with initial data u_0^k in the sense of Definition 2.57, and if the strong trace $u(t,0)=(u^1(t,0),\dots,u^N(t,0))$ of u given by Theorem 2.58 satisfies

$$u(t,0)\in\mathcal{G}\quad\text{for a.e. time}\quad t\in(0,+\infty)$$

$$\big(\text{resp. } u(t,0)\in\mathcal{G}^{SUB}\quad\text{for a.e. time}\quad t\in(0,+\infty)\big)$$

$$\big(\text{resp. } u(t,0)\in\mathcal{G}^{SUP}\quad\text{for a.e. time}\quad t\in(0,+\infty)\big)$$

If $\mathcal{G}$ is a generalized Riemann germ, then a function u is naturally a $\mathcal{G}$-entropy solution if and only if it is both a $\mathcal{G}$-entropy subsolution and supersolution (see Lemma 17.1). The point with $\mathcal{G}$-entropy subsolutions/supersolutions is that we do not expect them to be interesting, except for certain subclasses of germs, like the subclass of monotone Kružkov germs, as we will see below.

2.4.2 Main results of Part III: existence, uniqueness, contraction, comparison

Notice that we have a natural isomorphism $L^1(J) := L^1(J;\mathbb{R}) \simeq \prod_{k:=1,\dots,N} L^1(J^k;\mathbb{R})$ with the norm

$$\int_J |u_0| dx := \sum_{k=1,\dots,N} \int_{J^k} |u_0^k| dx \quad \text{for} \quad u_0 = (u_0^1, \dots, u_0^N)$$

that we use constantly. We recall the BV semi-norm

$$[u_0^k]_{BV(J^k)} := TV(u_0^k, J^k) := \sup_{|\varphi|\le 1,\ \varphi\in C_c^1(J^k)} \int_{J^k} -u_0^k \varphi_x$$

where $C_c^1(J^k)$ is the set of C^1 functions with compact support in the branch J^k, and where BV stands for bounded variations, and TV stands for Total Variation. Similarly, we set

$$[u_0]_{BV(J^*)} := \sum_{k=1,\dots,N} [u_0^k]_{BV(J^k)} = \sum_{k=1,\dots,N} TV(u_0^k; J^k) = TV(u_0; J^*)$$

For a measure $w = (w^1, \dots, w^N)$, we also set

$$|w|_{\mathcal{M}(J^*)} := \sum_{k=1,\dots,N} |w^j|_{\mathcal{M}(J^k)}$$

where $\mathcal{M}(J^*)$ is the set of measures on $J^* := J\backslash\{0\}$, and where $\mathcal{M}(J^k)$ is the set of (real valued) measures on J^k and $|\cdot|_{\mathcal{M}(J^k)}$ is the total variation of the measure. This space appears naturally when we consider $u_t = -f(u)_x$ as a measure.

Theorem 2.61 (Theory for Kružkov germs)
Assume (2.2) for $N \ge 1$, nondegeneracy condition (2.17), and that $\mathcal{G} \subset [a,b] \ni 0_{\mathbb{R}^N}$ is a Kružkov germ in the sense of i) of Definition 2.8. Let u_0 be an initial data satisfying

$$u_0^k \in (BV \cap L^1)(J^k; [a^k, b^k]) \quad \textit{for all index} \quad k = 1, \dots, N \tag{2.56}$$

where here BV denotes the space of functions with bounded variations.
i) (Existence and uniqueness)
Then there exists a unique $\mathcal{G}$-entropy solution u of (2.52) with initial data u_0. Moreover we have

$$u \in C^0([0,+\infty); L^1(J^*)) \cap Lip([0,+\infty); \mathcal{M}(J^*)) \quad \textit{with} \quad J^* := J\backslash\{0\} \tag{2.57}$$

ii) (L^1-contraction)
Moreover if u_0, v_0 are two initial data satisfying (2.56), and if u, v are respectively their associated $\mathcal{G}$-entropy solutions, then we have the following L^1-contraction property

$$\int_{\{t\}\times J} |u - v|(t,\cdot)\ dx \le \int_J |u_0 - v_0|\ dx \quad \textit{for all time } t > 0$$

iii) (A priori bounds)
Either the box $[a,b]$ is bounded and we set $[\bar a, \bar b] := [a,b]$, or the box $[a,b]$ is not bounded and there exists a bounded box $[\bar a, \bar b] \subset [a,b]$ such that

$$u_0(J^*) \subset [\bar a, \bar b] \quad \textit{and} \quad \mathcal{G} \cap [\bar a, \bar b] \quad \textit{is a Riemann germ on the bounded box } [\bar a, \bar b] \tag{2.58}$$

Then we have the following bounds for all $t > 0$

$$TV(f(u)(t,\cdot); J^*) \le K_0 \quad \textit{with} \quad K_0 := \sum_{j=1,\dots,N} \left\{ TV(f^j(u_0^j); J^j) + |(f^j - \hat f^j)(u_0(0))| \right\} \tag{2.59}$$

where $\hat{f} := \hat{f}_{\mathcal{G}}$ is Godunov flux associated to the germ $\mathcal{G}$, and

$$TV(u(t,\cdot); J\backslash \bar{B}_{2\delta})) \le TV(u_0; J\backslash \bar{B}_\delta) + \delta^{-1} t K_0 \quad \text{for all} \quad \delta \in (0,1) \tag{2.60}$$

iii') (Additional a priori bounds for $N = 1$)
For $N = 1$, we have for all $t > 0$

$$TV(f(u)(t,\cdot); J^*) + TV(f(u); (0,t) \times \{0\}) \le K_0 \tag{2.61}$$

with K_0 defined in (2.59) and where the second term of (2.61) denotes the Total Variation in time of the trace of the function $f(u)$ at $x = 0$ and on the time interval $(0,t)$.

Remark 2.62 *Notice that our BV assumption (2.56) on the initial data is technical, and indeed simplifies our proof of existence, and allows regularity (2.57) of the solutions, and gives nice a priori bounds. Moreover the condition $0_{\mathbb{R}^N} \in [a,b]$ is only here to allow the initial data to belong to $L^1(J)$. Obviously any shift from $0_{\mathbb{R}^N}$ can also be considered.*

Remark 2.63 (Reduction to a bounded box)
Notice that assumption (2.58) can always be be satisfied (for some suitable bounded box $[\bar{a}, \bar{b}]$) when $u_0(J^*)$ is compact. This non obvious result is due to key Proposition 6.1. This reduction to a bounded box here has nothing to do with the Kružkov property of the germ, but is a general property of germs (using our coercivity assumption on the fluxes).

Notice that prior to Theorem 2.61, only a few existence and uniqueness results were available in several important and pionnering works. Existence results were available for complete and conservative D-maximal L^1-dissipative sets $\mathcal{G}$ (see [4], [2], [40] and [20]). Nevertheless completeness was not fully understood, and it was not understood that D-maximality is an automatic consequence of completeness and L^1-dissipative properties.

Most of the time, existence was proved only for some particular Riemann solvers. Some nice uniqueness results were also obtained for Riemann solver $\mathcal{RS}_2$ in [25]. We indeed show (see Corollary 15.12) that this is due to the fact that Riemann solver $\mathcal{RS}_2$ is associated to a Kružkov germ. As a consequence of Theorem 2.61, we also get existence of a solution in a systematic way. From this perspective point of view, it seems that Theorem 2.61 provides a new progress in the understanding of scalar conservation laws on junctions.

We also have the following result.

Theorem 2.64 (Properties of semisolutions for monotone Kružkov germs)
Assume (2.2) for $N \ge 1$, nondegeneracy condition (2.17), and that $\mathcal{G} \subset [a,b] \ni 0_{\mathbb{R}^N}$ is a monotone Kružkov germ in the sense of i') of Definition 2.8. We consider $\mathcal{G}$-entropy subsolutions/supersolutions of (2.52).
i) (Stability of sub/supersolutions)
Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of $\mathcal{G}$-entropy subsolutions (resp. supersolutions) such that

$$u_n \to u \quad in \quad L^1_{loc}([0,+\infty) \times J)$$

Then u is a $\mathcal{G}$-entropy subsolution (resp. supersolution).
ii) (Max/Min for sub/supersolutions)
Let u, w be two $\mathcal{G}$-entropy subsolutions (resp. supersolutions). Then $\max\{u,w\}$ (resp. $\min\{u,w\}$) is a $\mathcal{G}$-entropy subsolution (resp. supersolution).
iii) (L^1-comparison principle)
Let u (resp. v) be a $\mathcal{G}$-entropy subsolution (resp. supersolution). Then for all $0 < s < t$, we have

$$\sum_{j=1,\dots,N} \int_{J^k} |u^k - v^k|_+(t,x)\, dx \le \sum_{j=1,\dots,N} \int_{J^k} |u^k - v^k|_+(s,x)\, dx$$

We end this presentation with a known, but key result. This is classically the following result which garantees the stability of $\mathcal{G}$-entropy solutions for Kružkov germs (see for instance [40]).

Proposition 2.65 (Equivalent characterization of $\mathcal{G}$-entropy solutions for Kružkov germs)
Assume (2.2) with $N \ge 1$ and let $\mathcal{G} \subset [a,b] \ni 0_{\mathbb{R}^N}$ be a Kružkov germ. Then $u = (u^1, \dots, u^N)$ is $\mathcal{G}$-entropy solution of (2.4) with initial data u_0 with $u_0 \in BV(J)$, if and only if $u^j : [0,+\infty) \times J^j \to [a^j, b^j]$ is a Kružkov

entropy solution of the first line of (2.4), and the trace condition in the second line of (2.4) is replaced by the following condition. For all test functions $0 \leq \varphi^j \in C_c^1([0,+\infty) \times \bar{J}^j)$ *with* $\bar{J}^j = \{0\} \cup J^j \simeq [0,+\infty)$ *or* $(-\infty, 0]$ *with*

$$\varphi^j(t,0) = \varphi^k(t,0) =: \varphi(t,0) \quad \text{for all } t \in [0,+\infty) \text{ and all indices } j,k$$

and for all elements $c = (c^1, \dots, c^N) \in \mathcal{G}$, *we have*

$$\sum_k \left\{ \int_{(0,+\infty)\times J^k} \{\eta^k(u,c)\varphi_t^k + \psi^k(u,c)\varphi_x^k\} \ dtdx + \int_{\{0\}\times J^k} \eta^k(u_0,c)\varphi \ dx \right\} \geq 0 \tag{2.62}$$

for (η^k, ψ^k) *given in (2.5).*

2.5 Organization of the work

The work is organized in three parts and several sections. The main results of Parts I, II and III presented in Section 2 are proved as indicated in the following tables.

Main results of Part I	topics	proof for $N = 1$	proof for $N \geq 1$
Proposition 2.4	L^1 estimate for Riemann problem		Subsection 4.2
Proposition 2.14	properties of gen. Riemann germs		
	i) inverse of π	ii) of Proposition 3.6	ii) of Proposition 4.10
	ii) dissipation	Lemma 3.4	Lemma 3.4
	iii) Riemann monotonicity of π	vi) of Proposition 3.6	Proposition 5.1
	iv) local constancy of $\hat{f}$	i) of Proposition 3.6	i) of Proposition 4.10
	v) germ as a level set	iii) of Proposition 3.6	iii) of Proposition 4.10
	vi) partial relaxation formula	Proposition 3.10	Proposition 4.15
	vii) { partial Lipschitz estimate and basic monotonicity	Proposition 3.8	Proposition 4.13
Theorem 2.15	character. of gen. Riemann germs	Subsection 3.4	Subsection 4.5
	i) 1rst characterization	ii) of Lemma 3.5	ii) of Lemma 4.9
	ii) 2nd characterization	Proposition 3.9	Proposition 4.14
Theorem 2.17	Riemann germs		Subsection 4.7
Theorem 2.20	Riemann monotonicity of $\sigma \diamond \hat{f}$	Proposition 3.8	Subsection 5.4
	i) general result	Proposition 3.8	Proposition 5.8
	ii) Kružkov case		Corollary 5.10
Theorem 2.23	Riemann relaxation		Subsection 7.1
Theorem 2.24	gluing of Riemann germs		Subsection 5.6
	i) gluing and iv) nature of glued germs		{ Proposition 5.13; i) of Corollary 5.14
	ii) associativity		Lemma 5.15
	iii) identity element		ii) of Corollary 5.14
Theorem 2.26	properties of Kružkov germs		Subsection 6.2
	i) characterization		Proposition 4.20
	ii) D-maximality		Lemma 6.2
Theorem 2.27	characterization of monotone germs		iii) of Lemma 5.5
Theorem 2.28	characterization of D_+-germs		{ Proposition 6.5; i') and iii) of Lemma 5.5
Theorem 2.29	conservative Riemann germs		Subsection 6.4
Theorem 2.30	properties of (quasi) HJ germs		Subsection 6.5
	i) regularity of $\hat{f}$		{ i) of Proposition 6.9; ii) of Theorem 2.15
	ii) relaxation formula		ii) of Proposition 6.9
Theorem 2.34	HJ germ $\mathcal{G}$ determined by $\chi\mathcal{G}$		Subsection 6.6
Theorem 2.35	properties of conservative $1:1$ germs		Subsection 6.7
	i) nature of the germs		{ Lemma 6.11; Theorem 2.34
	ii) relaxation formula		ii) of Proposition 6.9
	i) relation (2.25)		Lemma 8.17

Main results of Part II	topics	position
Theorem 2.40	Polar decomposition	Theorem 11.8
Theorem 2.43	Relaxation of quasi-prefluxes	Theorem 13.4
Theorem 2.44	Gluing of prefluxes	{ Theorem 11.17 Lemma 13.12
Proposition 2.47	Kružkov germ product property	Subsection 12.6
Proposition 2.48	Charact. of prefluxes for $N = 1$	Proposition 12.1
Proposition 2.49	Charact. of 1:1 conservative prefluxes	Proposition 12.1
Theorem 2.50	Charact. of 1:2 conservative prefluxes	Theorem 12.7
Theorem 2.54	Family of 1:n conservative prefluxes	Corollary 14.14
Remark 2.56	Family of n:m conservative prefluxes	Lemma 14.18

Main results of Part III	topics	position
Theorem 2.64	properties of semisolutions i) Stability ii) Max/Min iii) L^1-comparison	Subsection 17 Lemma 17.2 Lemma 17.3 Lemma 17.4
Theorem 2.61	theory for Kružkov germs i) existence and uniqueness ii) L^1-contraction	 Proposition 18.8 Lemma 17.4

For the precise material of the subsections, we refer the reader to the content given at the beginning of the manuscript. We insist below on the general structure/spirit of the work.

In Section 3, we develop the theory for a single branch ($N = 1$) in the case of $1 : 0$ junctions. The fundamental concepts (like basins of attraction $BA(\hat{p})$, the nonlinear projection $\pi_{\mathcal{G}}$, the Godunov flux $\hat{f}_{\mathcal{G}}$, generalized Riemann germs, level set formulation of the germ $\mathcal{G}_{\hat{g}}$, relaxation formula, Riemann monotonicity, local constancy) introduced in this work can already easily be understood in this section. Only the gluing requires the understanding of the case $N > 1$. For $N = 1$, the flux is denoted by g, while for $N \geq 1$, the flux vector is denoted by $f = (f^1, \dots, f^N)$, in order to avoid any confusion between these cases.

In Section 4, we extend the theory to the case of $N \geq 1$ branches, and focus on the case of N:0 junctions. We start the section, showing that the general case of n:m junctions can always be reduced to the case of N:0 junctions with $N := n + m$, and furthermore with coercive fluxes satisfying $f^j(p^j) \to +\infty$ as $|p^j| \to +\infty$. This reduction can be done using certain commutative transforms which are very convenient.

It turns out that there are two types of transforms that are interesting: the reversions (changing a branch $(-\infty, 0)$ into $(0, +\infty)$ and vice versa, and changing the sign of the flux) and the inversions (keeping the orientation of the branches unchanged, but changing the sign of the flux and of its arguments).

In the case $N \geq 1$, the key tool appears to be the slicing Lemma 4.12, which allows to reduce a germ from N branches to a germ with $N - k$ branches. Obviously any germ $\mathcal{G}$ can not be sliced in a naive way. But using its Godunov flux $p \mapsto \hat{f}_{\mathcal{G}}(p)$, we can freeze the last k components of p, and then define a frozen flux which is associated to a new germ for $N - k$ branches. The slicing lemma is used in many ways to analyse generalized Riemann germs, and then to deduce their fundamental properties, similar to the case $N = 1$.

In Section 5, we excavate a fundamental property of germs: their Riemann monotonicity. Except for pathological fluxes (for instance fluxes which are constant on some intervals), this property is satisfied by any Riemann germ. Riemann monotonicity may be seen as a property coming from nowhere, but this is not the case. This property is very natural and necessary, once we are interested in the gluing of germs. This is the careful study of the gluing of germs that made this property to appear as a natural property. We show that the natural projection $\pi_{\mathcal{G}}$ is already Riemann monotone, and in some sense, this implies that $\sigma \diamond \hat{f}_{\mathcal{G}}$ is also Riemann monotone, where σ encodes the orientations of the branches. With this key property in hands, it becomes then easy to glue Riemann germs together (when their fluxes and branch orientations agree in a suitable way). Again, we first perform the gluing on the Godunov fluxes, and then deduce from it and justify the natural gluing of the germs.

Still in Section 5, we show that gluing preserves certain classes of germs (Kružkov, HJ, monotone, conservative). Again, we first show it at the level of the associated Godunov fluxes, and then deduce these properties at the level of germs.

In Section 6, we give some applications of the theory mainly to the cases of Kružkov germs and of HJ germs. For both, we show that all generalized Riemann germs have necessarily continuous Godunov fluxes, and then are Riemann germs, just by definition. A key result for HJ germs is that they are completely characterized by a characteristic subset $\chi\mathcal{G}$, which is a finite set in many applications. A culminating application is the case of $1:1$ junctions with conservative (generalized) Riemann germs. Such germs have all the best properties that we can expect: they are at the same time Kružkov, HJ, monotone and conservative germs. They are characterized by their characteristic subset $\chi\mathcal{G}$ which is finite in many applications. And their Godunov flux $\hat{f}_{\mathcal{G}} =: (\hat{h}, \hat{h})$ is such that the function $\hat{h} : [a,b] \to \mathbb{R}$ satisfies the best relaxation formula that we can expect.

Section 7 deals with an important fact: the possibility of relaxing quasi Godunov fluxes to Godunov fluxes. This provides in particular a practical way to construct Riemann germs. We also show that gluing and relaxation do commute.

Section 8 provides complementary results. The main contribution concerns duality for monotone Kružkov germs.

Section 9 gives various exemples and counter-examples to illustrate the general theory.

Section 10 is an appendix. In the first subsection, we mainly recall the theory for solving the standard scalar Riemann problem on the real line, with initial data p involving two constant values (p^L, p^R), one on the left, and one on the right. The explicit expression of the solution involving concave/convex envelopes of the flux, is here a key ingredient. This ingredient allows us, say for a solution defined on $(0,+\infty)_t \times (0,+\infty)_x$, to decide which trace $\hat{p}$ the solution may be reached on the axis $(0,+\infty)_t \times \{0\}_x$, given its constant initial data $p := p^R$ on the set $\{0\}_t \times (0,+\infty)_x$. This allows us to study the set $\hat{\mathcal{P}}_p$ of such values $\hat{p}$. Indeed we can then see that the basin of attraction $BA(\hat{p})$ for $J^R = (0,+\infty)$, is nothing else than the inverse, i.e. the set of p's such that $\hat{p} \in \hat{\mathcal{P}}_p$.

In Section 10, the second subsection gives some important (independent) results about reduction of test functions for Hamilton-Jacobi equations. This is a key result which allows us to show that any HJ germ $\mathcal{G}$ is determined completely by its characteristic subset $\chi\mathcal{G}$.

Part II of the work focuses on the case of bell-shaped fluxes on the network. This case is of special importance for most of the applications. For bell-shaped fluxes, we show that the junction Godunov flux enjoys a Polar decomposition, i.e. it can be written as a preflux composed with an explicit capacity. This polar decomposition enlights previous results on the topic. On the one hand, the capacity contains and justifies rigorously Demand and Supply law of Lebacque. On the other hand, all the information of the junction is encoded in the preflux, which is a normalized object. Prefluxes are simple to define, but conservative prefluxes are very rigid objects. For this reason they are quite delicate to construct. And Part II gives many different methods to build prefluxes. Two general methods are available. First we can glue together given prefluxes to obtain a third preflux (in general associated to more branches). The second method consists to consider a quasi-preflux (i.e. a function which does not satisfy all the properties of prefluxes) that we can (nonlinearly) project to a single associated preflux. This method is called Riemann relaxation of quasi-prefluxes.

Independently of these general methods, we also provide an explicit characterization of general 1:2 conservative prefluxes. This allows us to see how all the known examples proposed in the literature for 1:2 or 2:1 junctions do fit in our classification. More generally, we discuss the construction of 1:n conservative prefluxes with limiters, ans also construct by gluing a large class of n:m conservative prefluxes with limiters. Moreover, on several explicit examples taken from the literature, we construct explicitly the germ associated to a given preflux. Those examples are illustrated with pictures of the threedimensional germ set.

Part III of the work focuses on Kružkov germs for general fluxes. In this framework, we show that the theory of Kružkov extends completely, with existence, uniqueness, L^1-contraction estimates for solutions. Both the vanishing method and numerical schemes (discrete and semi-discrete) are studied. In the particular case of conservative Kružkov germs, we show that there is a notion of subsolutions and supersolutions and also a comparison principle.

Finally the bibliography is quite reduced for part I. This is in particular due to the novelty of the notion of germ, and to the new point of view that we develop here. The bibliography of Part II provides additional materials which are mainly useful illustrations for some examples. The bibliography of Part III is quite reduced on some works of fundamental interest, or useful results that we need for technical proofs.

2.6 Main notations

$(e_1,\dots,e_N)$	= canonical basis of $\mathbb{R}^N$
$J^j \simeq (0,+\infty)$ or $(-\infty,0)$	= j-th (oriented) branch
$\sigma^j = \pm 1$	= "orientation" of j-th branch $J^j \simeq \sigma^j \cdot (-\infty,0)$
$f^j : [a^j,b^j] \to \mathbb{R}$	= j-th flux
$[a,b] := \prod_{j=1,\dots,N} [a^j,b^j]$	= the box (with box convention (2.3))
$f = (f^j)_{j=1,\dots,N} : [a,b] \to \mathbb{R}^N$	= the flux function
$f^j(p) := f^j(p^j)$	= abuse of notation, for $p = (p^1,\dots,p^N)$
$p = (p^1,\dots,p^N) \le 0$	= means $p^j \le 0$ for all j
$p = (p^1,\dots,p^N) < 0$	= means $p^j < 0$ for all j
$\mathcal{G} \subset [a,b]$	= set or germ
$\pi = \pi_{\mathcal{G}} : [a,b] \to \mathcal{G}$	= natural projection on $\mathcal{G}$
$\hat f = \hat f_{\mathcal{G}} = f \circ \pi_{\mathcal{G}}$	= generalized Godunov flux at the junction
$\mathcal{G}_{\hat f} := \left\{ \hat f = f \right\}$	= level set formulation of the germ
$f_\pm = (f_\pm^1,\dots,f_\pm^N)$	= monotone bounds
$BA^j(\hat p^j) = BA^{(J^j,f^j)}(\hat p^j)$	= Basin of Attraction of the point $\hat p^j \in [a^j,b^j]$
$BA^j_-(\hat p^j), BA^j_+(\hat p^j)$	= lower, upper Basins of Attraction
$BA(\hat p) = \prod_{j=1,\dots,N} BA^j(\hat p^j)$	= Basin of Attraction of the point $\hat p \in \mathcal{G}$
$G^j = G^{f^j} : [a^j,b^j]^2 \to \mathbb{R}$	= standard Godunov flux associated to f^j
$D^j = D^{f^j} = \sigma^j \psi^{f^j} : [a^j,b^j]^2 \to \mathbb{R}$	= j-th dissipation (see (2.8))
$D^f = \sum_{j=1,\dots,N} D^{f^j}$	= dissipation
$D^j_+ = D^{f^j}_+ = \sigma^j \psi^{f^j}_+ : [a^j,b^j]^2 \to \mathbb{R}$	= j-th semi-dissipation (see (2.9))
$D^f_+ = \sum_{j=1,\dots,N} D^{f^j}_+$	= semi-dissipation
$RH^{\hat f}(p) = \sum_{j=1,\dots,N} \sigma^j \hat f^j(p)$	= Rankine-Hugoniot function
$u_{p,\hat p} = (u^1_{p^1,\hat p^1},\dots,u^N_{p^N,\hat p^N})$	= solution with initial data p and trace $\hat p$
$u^{\mathcal{G}}_{p,\hat p}$	= $\mathcal{G}$-entropy solution $u_{p,\hat p}$, i.e. with $\hat p \in \mathcal{G}$
$\hat{\mathcal{P}}^j_{p^j}$	= set of $\left\{ \hat p^j = \hat p^j_{\lambda,p^j} \right\}$ for solutions $u_{p^j,\hat p^j}$ (see (4.7))
$\hat{\mathcal{P}}_p = \prod_{j=1,\dots,N} \hat{\mathcal{P}}^j_{p^j}$	= set of $\hat p$ for solutions $u_{p,\hat p}$
$p \diamond q$	= Hadamard product of two vectors
$\left[\hat f\right]_q^p := \hat f(p) - \hat f(q)$	= bracket of $\hat f$
$\mathcal{G}_{p_0''}$	= slicing of germ $\mathcal{G}$ w.r.t. p_0'' (see (4.12))
$\iota^j_p(q^j)$	= injection of j-th coordinate
$\mathcal{G}_1 \sharp \mathcal{G}_2$	= gluing of germs
$\underline{\chi}\mathcal{G}, \overline{\chi}\mathcal{G}, \chi\mathcal{G}$	= sub/super/characteristic subset of HJ germ $\mathcal{G}$

Recall that Section 3 focuses on the case $N = 1$. Then the index j is dropped everywhere, and f is replaced by g in order to avoid any confusion.

Part I
Structure of germs and of Godunov fluxes

3 Riemann problem on a single branch $N = 1$

3.1 Characterization of trace values on a junction $1:0$

In this section, we work with a single branch with $N = 1$ which is a junction $1 : 0$. Then we drop the index $k = 1$ everywhere and use the notation $g := f^1$, $a := a^1$, $b := b^1$ and $J^1 := (-\infty, 0)$ and work on the junction $J := (-\infty, 0]$. We assume

(3.1)
$$\left\{\begin{array}{lll} & g : \mathbb{R} \supset [a,b] \to \mathbb{R} \text{ is Lipschitz continuous with } -\infty \le a < b \le +\infty, & \text{with box convention (2.3)}\\ \text{and} & & \\ & g(p) \to +\infty \quad \text{if} \quad |p| \to +\infty \quad \text{and} \quad p \in [a,b] & \textbf{(coercivity)}\end{array}\right.$$

which at this stage are conditions slightly less general than in (2.2), because we impose the orientation of the junction, and also the direction of the coercivity.

We want to understand the $\mathcal{G}$-Riemann problem for $\mathcal{G} := \{\hat{p}\}$ with $\hat{p} \in [a, b]$ and initial data $p \in [a, b]$, namely we look for Kružkov entropy solutions $v(t, x) = v$ of the following Riemann problem

$$(3.2)\qquad \left\{\begin{array}{rlllll} & & v: & [0,+\infty) & \times(-\infty,0] \to [a,b] \\ v(t,x) & = v\left(1,\dfrac{x}{t}\right) & \text{on} & (0,+\infty) & \times(-\infty,0) \\ & & & & \\ v_t + (g(v))_x & = 0 & \text{on} & (0,+\infty) & \times(-\infty,0) \\ v(0,\cdot) & = p & \text{on} & \{0\} & \times(-\infty,0) \\ v(\cdot,0^-) & = \hat{p} & \text{a.e. on} & (0,+\infty) & \times\{0\} \end{array}\right.$$

where the last condition arises in the sense of traces. We want to determine the set of p such that such a solution does exist.

To this end, we recall that the Godunov flux $G(\uparrow, \downarrow) : [a, b]^2 \to \mathbb{R}$ associated to g is given by

$$(3.3)\qquad G(q,r) := \left\{\begin{array}{ll} \inf_{[q,r]} g & \text{if} \quad q \le r \\ \sup_{[r,q]} g & \text{if} \quad q \ge r \end{array}\right.$$

We set the following nondecreasing functions of p

$$(3.4)\qquad g_-(p) := \inf_{[p,b]} g = G(p,b) \quad \le \quad g_+(p) := \sup_{[a,p]} g = G(p,a)$$

and for $\lambda \in [g_-(p), g_+(p)]$, we define the following element of $[a, b]$

$$(3.5)\qquad \hat{p}_{\lambda,p} := \left\{\begin{array}{lll} p & & \text{if} \quad g(p) = \lambda \\ \sup\{q \in (p,b], \quad g > \lambda \quad \text{on} \quad (p,q)\} & & \text{if} \quad g(p) > \lambda \\ \inf\{q \in [a,p), \quad g < \lambda \quad \text{on} \quad (q,p)\} & & \text{if} \quad g(p) < \lambda \end{array}\right.$$

which is nonincreasing in λ. We then define the following set

$$(3.6)\qquad \hat{\mathcal{P}}_p := \{\hat{p}_{\lambda,p} \in [a,b], \quad \lambda \in [g_-(p), g_+(p)]\}$$

Precisely we have the following result

Lemma 3.1 (Set of $\hat{p}$'s for which $\{\hat{p}\}$-Riemann solutions exist with initial data p)
Assume (3.1). For any $p, \hat{p} \in [a, b]$, there exists an entropy solution $v = v_{p,\hat{p}}$ of (3.2) if and only if $\hat{p} \in \hat{\mathcal{P}}_p$, with $\hat{\mathcal{P}}_p$ given in (3.6).

Proof of Lemma 3.1
Step 1: Equivalence
Let v be an entropy solution to (3.2). We set $\bar{p} := \hat{p}$ and extend v as $\tilde{v}(t,x) := \begin{cases} v(t,x) & \text{for} \quad x < 0 \\ \bar{p} & \text{for} \quad x \geq 0 \end{cases}$.
Notice that for $J = (-\infty, 0]$, $J^1 = (-\infty, 0)$ and $\varphi \in C_c^1(([0,+\infty) \times J^1)$ with $\varphi \geq 0$, we have for all $k \in \mathbb{R}$

$$\int_{(0,+\infty)\times J} |\tilde{v} - k|\varphi_t + \text{sign}(\tilde{v} - k) \cdot \{g(\tilde{v}) - g(k)\} \varphi_x + \int_{\{0\}\times J} |\tilde{v}_0 - k|\varphi \geq 0.$$

In particular, we have for all $k \in \mathbb{R}$

$$|\tilde{v} - k|_t + \partial_x (\text{sign}(\tilde{v} - k) \cdot \{g(\tilde{v}) - g(k)\}) \leq 0 \quad \text{in} \quad \mathcal{D}'((0,+\infty) \times (\mathbb{R}\setminus \{0\}))$$

Using the fact that $\tilde{v}(t,x)$ is bounded in L^∞ and has trace $\bar{p}$ at $x = 0$, it is then easy to check that $\tilde{v}$ is an entropy solution to the following standard Riemann problem

$$\text{(3.7)} \qquad \begin{cases} \tilde{v}_t + (g(\tilde{v}))_x = 0 \quad \text{on} \quad (0,+\infty) \times \mathbb{R} \\ \tilde{v}(0,x) = \tilde{v}_0(x) := \begin{cases} \bar{p} & \text{if} \quad x > 0 \\ p & \text{if} \quad x < 0 \end{cases} \end{cases}$$

i.e. for all $\varphi \in C_c^1([0,+\infty) \times \mathbb{R})$ with $\varphi \geq 0$, we have for all $k \in [a,b]$

$$\int_{(0,+\infty)\times \mathbb{R}} |\tilde{v} - k|\varphi_t + \text{sign}(\tilde{v} - k) \cdot \{g(\tilde{v}) - g(k)\} \varphi_x + \int_{\{0\}\times \mathbb{R}} |\tilde{v}_0 - k|\varphi \geq 0.$$

Then we see that v solves (3.2), if and only if $\tilde{v}$ solves (3.7) and satisfies $\tilde{v}(t,0^-) = \bar{p}$ for a.e. $t > 0$.
Step 2: Characterization
From Lemma 10.1, we know that the solution $\tilde{v}$ to Riemann problem (3.7) is unique and has to satisfy $\tilde{v}(t,x) = U(x/t)$ with U given in (10.2) and $(p_L, p_R) := (p, \bar{p})$. In particular, when it makes sense, we have $\bar{p} = \tilde{v}(t,0^-) = ((\tilde{g}_{|I})')^{-1}(0^-)$, where $I := [\min(p_L,p_R), \max(p_L,p_R)]$ and $\tilde{g}$ is the convex (resp. the concave) envelope of g on the interval I if $p_L - p_R < 0$ (resp. $p_L - p_R > 0$).
Case 1: $p < \bar{p}$
This means $p_L < p_R$. Then $(\xi_L, \xi_R) = (\tilde{g}'(p_L^+), \tilde{g}'(p_R^-))$ with $\xi_L \leq \xi_R$, and we can not have $\xi_R > 0$. Hence $\tilde{g}'(p_R^-) = \xi_R \leq 0$. Either $\tilde{v}(t,0^-) = p_R$ if $\xi_R < 0$. Or $\xi_R = 0$ and (using the fact that $\tilde{g}$ is the convex envelop of g on some interval) we get

$$g(q) \geq g(p_R^-) + \xi_R(q - p_R) = g(p_R^-) \quad \text{for all} \quad q \in [p_L, p_R]$$

Then either there exists some $\varepsilon > 0$ such that $\tilde{g}' = \xi_R = 0$ on $[p_R - \varepsilon, p_R]$, and then $p_R = \bar{p}$ is not the trace of $\tilde{v}$ from the left side $\{x < 0\}$, because there is a jump just at the place $x = 0$. Or $\tilde{g}' < 0 = \xi_R = \tilde{g}'(p_R^-)$ on $[p_L, p_R)$, and the fact that $\tilde{g}'$ is nondecreasing implies that $\tilde{v}(t,0^-) = ((\tilde{g}_{|I})')^{-1}(0^-) = p_R$. In this case, we conclude that $\bar{p} = \tilde{v}(t,0^-)$ if and only if $\tilde{g}'(\bar{p}^-) < 0$ or $\tilde{g}'(\bar{p}^-) = 0 > \tilde{g}'$ on $[p,\bar{p})$. This means that

$$\bar{p} = \tilde{v}(t,0^-) \quad \text{if and only if} \quad \tilde{g}' < 0 \quad \text{on} \quad [p,\bar{p})$$

Case 2: $p > \bar{p}$
This case is very similar to Case 1 with $\tilde{g}$ concave instead of convex. Again, we conclude that

$$\bar{p} = \tilde{v}(t,0^-) \quad \text{if and only if} \quad \tilde{g}' < 0 \quad \text{on} \quad (\bar{p},p]$$

Case 3: $\bar{p} = p$
Then the unique solution is $\tilde{v} \equiv p$, and then the trace condition $\bar{p} = \tilde{v}(t,0^+)$ is obviously satisfied.
Conclusion
We conclude that $\bar{p} = \tilde{v}(t,0^-)$ iff $\begin{cases} \bar{p} = p \\ \bar{p} < p \quad \text{and} \quad \tilde{g}' < 0 \quad \text{on} \quad (\bar{p},p] \qquad (\text{with } \tilde{g} \text{ concave}) \\ \bar{p} > p \quad \text{and} \quad \tilde{g}' < 0 \quad \text{on} \quad [p,\bar{p}) \qquad (\text{with } \tilde{g} \text{ convex}) \end{cases}$, i.e.

$$\text{(3.8)} \qquad \bar{p} = \tilde{v}(t,0^-) \quad \text{iff} \quad \begin{cases} \bar{p} = p \\ \bar{p} < p \qquad \text{with } \tilde{g} \text{ concave decreasing on } [\bar{p},p] \\ \bar{p} > p \qquad \text{with } \tilde{g} \text{ convex decreasing on } [p,\bar{p}] \end{cases}$$

Let us call $\bar{\mathcal{P}}_p$ the set of $\bar{p} \in [a,b]$ characterized by the right hand side of (3.8). Still because of Lemma 10.1, we see that such a solution $\tilde{v}$ does exist in each case covered in (3.8).
Step 3: Equivalent characterization
For $\bar{p} \in [a,b]$ and $\lambda := g(\bar{p})$, it is easy to see from (3.8) that $\bar{p} = \hat{p}_{\lambda,p}$ and $\lambda \in [g_-(p), g_+(p)]$. This shows that $\bar{p} \in \hat{\mathcal{P}}_p$ and then $\bar{\mathcal{P}}_p \subset \hat{\mathcal{P}}_p$. The reverse inclusion is also straightforward to check. This shows $\bar{\mathcal{P}}_p = \hat{\mathcal{P}}_p$ and ends the proof of the lemma.

3.2 Basin of attraction: the map $\tilde{p} \mapsto BA(\tilde{p})$

The main result of this subsection is the following inverse characterization of the map $p \mapsto \hat{\mathcal{P}}_p$.

Lemma 3.2 (Inverse characterization of the map $p \mapsto \hat{\mathcal{P}}_p$)
Assume (3.1) with $J = (-\infty, 0]$. Let $p \in [a,b]$ and some $\tilde{p} \in \hat{\mathcal{P}}_p$.
Then for all $p' \in [a,b]$, we have

$$\tilde{p} \in \hat{\mathcal{P}}_{p'} \qquad \text{if and only if} \qquad p' \in BA(\tilde{p})$$

where

$$BA(\tilde{p}) := BA_-(\tilde{p}) \cup \{\tilde{p}\} \cup BA_+(\tilde{p}) \tag{3.9}$$

and $BA_\pm = BA_\pm(\tilde{p})$ are relative open sets of $[a,b]$ given for $\lambda := g(\tilde{p})$ by $\left\{ \begin{array}{ll} BA_+ := \{q \in (\tilde{p}, b], & g < \lambda \quad on \quad (\tilde{p}, q]\} \\ BA_- := \{q \in [a, \tilde{p}), & g > \lambda \quad on \quad [q, \tilde{p})\} \end{array} \right.$.

Proof of Lemma 3.2
Let $p \in [a,b]$ and $\tilde{p} \in \hat{\mathcal{P}}_p$. Notice that it is much more simple to make the reasoning on a picture (see Figure just before Subsubsection 2.1.2).
Step 1: proof that $p' \in BA(\tilde{p})$ implies $\tilde{p} \in \hat{\mathcal{P}}_{p'}$
Consider some $p' \in BA(\tilde{p})$. Then whatever is the position of p' with respect to $\tilde{p}$, we get

$$g_-(p') = \inf_{[p',b]} g \le g(\tilde{p}) = \lambda \le \sup_{[a,p']} g = g_+(p')$$

Therefore, we can consider the quantity $\hat{p}_{\lambda,p'}$, and whatever is the position of $g(p')$ with respect to $\lambda = g(\tilde{p})$, we easily get from its definition (3.5) that $\hat{p}_{\lambda,p'} = \tilde{p}$. Therefore $\tilde{p} = \hat{p}_{\lambda,p'} \in \hat{\mathcal{P}}_{p'}$.
Step 2: proof that $p' \in BA(\tilde{p})$ is implied by $\tilde{p} \in \hat{\mathcal{P}}_{p'}$
Now consider some $p' \in [a,b]$ such that $\tilde{p} \in \hat{\mathcal{P}}_{p'}$. Hence $\tilde{p} = \hat{p}_{\lambda',p'}$ for some $\lambda' \in [g_-(p'), g_+(p')]$. Now whatever is the position of $\lambda' = g(\tilde{p})$ with respect to $g(p')$, we easily get from (3.9) that $p' \in BA(\tilde{p})$.
This ends the proof of the lemma.

Then we get immediately

Corollary 3.3 (Set of p's for which $\{\hat{p}\}$-Riemann solutions exist with initial data p)
Assume (3.1). For any $p, \hat{p} \in [a,b]$, there exists an entropy solution $v = v_{p,\hat{p}}$ of (3.2) if and only if $p \in BA(\hat{p})$, with $BA(\hat{p})$ given in (3.9).

3.3 Dissipation property of basins of attraction

We have the following result.

Lemma 3.4 (Dissipation property of basins of attraction, $N = 1$, junction $1:0$)
Assume (3.1). Let $\hat{p}, \hat{q} \in [a,b]$ be such that $BA(\hat{p}) \cap BA(\hat{q}) \neq \emptyset$. Then we have

$$D^g(\hat{q}, \hat{p}) \le 0 \quad \text{with} \quad D^g(\hat{q}, \hat{p}) := sign(\hat{q} - \hat{p}) \cdot \{g(\hat{q}) - g(\hat{p})\} \tag{3.10}$$

Moreover

$$\text{either} \quad D^g(\hat{q}, \hat{p}) < 0, \quad \text{or} \quad \hat{p} = \hat{q} \tag{3.11}$$

Proof of Lemma 3.4
Notice that $D^g(\hat{p},\hat{q}) = D^g(\hat{q},\hat{p})$. Assume by symmetry that $\hat{p} \le \hat{q}$, and set $\underline{q} := \inf BA(\hat{q})$, $\overline{p} := \sup BA(\hat{p})$, and let us show that

$$g(\hat{q}) < g(\hat{p}) \quad \text{or} \quad \hat{p} = \hat{q} \tag{3.12}$$

We first notice that for all $p \in BA(\hat{p})$, we have $D^g(p,\hat{p}) \le 0$, with moreover $D^g(p,\hat{p}) = 0$ if and only if $p = \hat{p}$. Therefore if $\hat{p} \in BA(\hat{q})$ or $\hat{q} \in BA(\hat{p})$, this implies (3.12).
If $\underline{q} = \hat{q}$, then $\hat{q} \in BA(\hat{p})$. Similarly, if $\overline{p} = \hat{p}$, then $\hat{p} \in BA(\hat{q})$.

Assume now that $\hat{p} < \overline{p}$ and $\underline{q} < \hat{q}$. If $\overline{p} = b$, and $g(b) < g(\hat{p})$, then $BA(\hat{p}) \cap [\hat{p}, b] = [\hat{p}, b]$ and then $\hat{q} \in BA(\hat{p})$. Otherwise, we have $BA(\hat{p}) \cap [\hat{p}, b] = [\hat{p}, \overline{p})$ and $g(\overline{p}) = g(\hat{p})$.

Similarly, if $\underline{q} = a$ and $g(a) > g(\hat{q})$, then $BA(\hat{q}) \cap [a, \hat{q}] = [a, \hat{q}]$ and then $\hat{p} \in BA(\hat{q})$. Otherwise, we have $BA(\hat{q}) \cap [a, \hat{q}] = (\underline{q}, \hat{q}]$ and $g(\underline{q}) = g(\hat{q})$.

Now $BA(\hat{p}) \cap BA(\hat{q}) \neq \emptyset$ implies $\underline{q} < \overline{p}$. Now if $\underline{q} < \hat{p}$, then $\hat{p} \in BA(\hat{q})$. Similarly, if $\hat{q} < \overline{p}$, then $\hat{q} \in BA(\hat{p})$. Hence we can now assume that $\hat{p} \le \underline{q} < \overline{p} \le \hat{q}$ with $g(\hat{p}) = g(\overline{p})$ and $g(\underline{q}) = g(\hat{q})$. Now recall that we have

$$g < g(\hat{p}) = g(\overline{p}) \quad \text{on} \quad (\hat{p}, \overline{p}) \supset (\underline{q}, \overline{p}) \quad \text{which implies} \quad g(\underline{q}) < g(\overline{p})$$

Hence $g(\hat{q}) < g(\hat{p})$ which implies (3.12).

Finally, we conclude that in all cases, we have (3.12) which implies both (3.10) and (3.11). This ends the proof of the lemma.

3.4 Characterizations of generalized Riemann germs

The following result follows immediately from the definitions and Lemma 3.2.

Lemma 3.5 (First characterizations of generalized Riemann germs, $N = 1$)
Assume (3.1) and let $\mathcal{G} \subset [a, b]$ be a set.
i) (First characterization)
The set $\mathcal{G}$ is a generalized Riemann germ if and only if for all $p \in [a, b]$, we have the singleton property $\mathcal{G} \cap \hat{\mathcal{P}}_p = \{\hat{p}\}$ where $\hat{\mathcal{P}}_p$ is defined in (3.6).
ii) (Equivalent characterization)
The set $\mathcal{G}$ is a generalized Riemann germ if and only if $(BA(\hat{p}))_{\hat{p}\in\mathcal{G}}$ is a partition of $[a, b]$.

Proposition 3.6 (First properties of generalized Riemann germs, $N = 1$)
Assume (3.1) and let $\mathcal{G} \subset [a, b]$ be a generalized Riemann germ for a junction $1 : 0$ with $J = (-\infty, 0]$ and $f := g$. Given $p \in [a, b]$, the unique $\mathcal{G}$-entropy solution of (2.6) writes $u^{\mathcal{G}}_{p,\hat{p}}$ with $\pi(p) := \pi_{\mathcal{G}}(p) := \hat{p} \in \mathcal{G}$ and $\pi : [a, b] \to \mathcal{G}$ which satisfies $\pi \circ \pi = \pi$. We set $\hat{g} := \hat{g}_{\mathcal{G}} := g \circ \pi$.
i) (Local constancy): *The map $\hat{g}$ is locally constant on $\{\hat{g} \neq g\}$.*
ii) (Inverse of π): *For all $\hat{p} \in \mathcal{G}$, we have $\pi^{-1}(\hat{p}) = BA(\hat{p})$.*
iii) (Level set formulation of the germ): *We have $\mathcal{G} = \mathcal{G}_{\hat{g}} := \{p \in [a, b], \quad \hat{g}(p) = g(p)\}$.*
iv) (Characterization of $\hat{g}$): *The function $\hat{g} : [a, b] \to \mathbb{R}$ is fully characterized as a continuous function which is locally constant on $\{\hat{g} \neq g\}$ such that $\mathcal{G} = \mathcal{G}_{\hat{g}}$.*
v) (Monotone bounds): *We have $g_- \le \hat{g} \le g_+$ with $g_\pm$ defined in (3.4).*
vi) (Monotonicity of π): *The map π is nondecreasing on $[a, b]$.*
vii) (Continuity): *The map $\hat{g}$ is continuous.*

Remark 3.7 *Notice that the monotonicity of $\hat{g}$ is not proved in Proposition 3.6. This seems to be a delicate property. It will be proved later, using the locally Lipschitz properties of $\hat{g}$ in order to clean the possible accumulation of basins of attraction.*

Proof of Proposition 3.6
Step 1: proof of vi)
Let $p \le q$. Assume by contradiction that $\pi(p) = \hat{p} > \hat{q} = \pi(q)$. If $BA(\hat{p}) \cap BA(\hat{q}) = \emptyset$, then

$$BA(\hat{q}) < BA(\hat{p})$$

where for sets $A, B \subset \mathbb{R}$, for $A < B$, we mean $a < b$ for all $(a, b) \in A \times B$. This implies that $q < p$. Contradiction. Hence $BA(\hat{p}) \cap BA(\hat{q}) \neq \emptyset$, and the partition property in Lemma 3.5 shows that $\hat{p} = \hat{q}$. Again contradiction. Therefore, we conclude that $\hat{p} \le \hat{q}$, and the map π is nondecreasing.

Step 2: proof of ii)
From i) of Lemma 3.5 and Lemma 3.2, we deduce that $\pi^{-1}(\hat{p}) = BA(\hat{p})$ for any $\hat{p} \in \mathcal{G}$.
Step 3: proof of iii)
Recall that $\hat{g} = g \circ \pi$. Because $\pi^{-1}(\hat{p}) = BA(\hat{p})$ for any $\hat{p} \in \mathcal{G}$, we see that $\hat{g}$ is constant on each $BA(\hat{p})$ with value $\hat{g}(p) = g(\hat{p})$ for every $p \in BA(\hat{p})$. This implies $\mathcal{G} \subset \mathcal{G}_{\hat{g}}$. Conversely, let $q \in \mathcal{G}_{\hat{g}}$. Then $g(q) = \hat{g}(q) = g(\hat{q})$ for the unique $\hat{q} \in \mathcal{G}$ such that $q \in BA(\hat{q})$. Because $g \neq g(\hat{q})$ on $BA(\hat{q})\backslash\{\hat{q}\}$, we deduce that $q \in \{\hat{g} = g\} \cap BA(\hat{q}) = \{\hat{q}\} \subset \mathcal{G}$, which shows that $\mathcal{G}_{\hat{g}} \subset \mathcal{G}$. Therefore we have equality $\mathcal{G}_{\hat{g}} = \mathcal{G}$.
Step 4: proof of i)
Let $p \in [a,b] \cap \{\hat{g} \neq g\}$ and let $\hat{p} \in \mathcal{G}$ such that $p \in BA(\hat{p})$. Then p belongs to $BA(\hat{p})\backslash\{\hat{p}\}$ which is a relative open set of $[a,b]$. We deduce the existence of some $\varepsilon > 0$ such that

$$\omega := Q_p \cap [a,b] \quad \subset BA(\hat{p}) \quad \text{with} \quad Q_p := (p-\varepsilon, p+\varepsilon)$$

Because $\hat{g} = const = g(\hat{p})$ on ω, this shows the local constancy of the map $\hat{g}$.
Step 5: proof of v)
Consider $p \in [a,b]$ and let $\hat{p} \in \mathcal{G}$ such that $p \in BA(\hat{p})$. From Lemma 3.2, we deduce that $\hat{p} \in \hat{\mathcal{P}}_p$, and then there exists $\lambda = g(\hat{p}) \in [g_-(p), g_+(p)]$ such that $\hat{p} = \hat{p}_{\lambda,p}$. Because $\hat{g}(\hat{p}) = g(\hat{p})$, this shows point v).
Step 6: proof that $\mathcal{G}$ is closed
Consider a sequence $(\hat{p}_n)_{n\in\mathbb{N}}$ with $\hat{p}_n \in \mathcal{G}$ such that $\hat{p}_n \to \hat{p}_\infty \in [a,b]$. We set $\tilde{p}_\infty := \pi(\hat{p}_\infty) \in \mathcal{G}$. Either $\hat{p}_\infty = \tilde{p}_\infty$ and the proof is done, or

$$\hat{p}_\infty \in BA(\tilde{p}_\infty)\backslash\{\tilde{p}_\infty\} \tag{3.13}$$

Then Step 3 shows that

$$(\hat{p}_\infty - \varepsilon, \hat{p}_\infty + \varepsilon) \cap [a,b] \subset BA(\tilde{p}_\infty) \tag{3.14}$$

Therefore $\hat{p}_n \in BA(\tilde{p}_\infty)$ for n large enough, i.e. $\hat{p}_n = \pi(\hat{p}_n) = \tilde{p}_\infty$. This implies that $\hat{p}_\infty = \tilde{p}_\infty \in \mathcal{G}$. Contradiction. Hence we always have $\hat{p}_\infty = \tilde{p}_\infty \in \mathcal{G}$, and then $\mathcal{G}$ is closed.
Step 7: proof of vii)
The continuity of $\hat{g}$ is not straightforward because we may have accumulation of basins of attraction.
Consider a sequence $(p_n)_{n\in\mathbb{N}}$ with $p_n \in [a,b]$ such that we assume by contradiction that

$$p_n \to p_\infty, \quad \hat{g}(p_n) \not\to \hat{g}(p_\infty) \tag{3.15}$$

We set $\hat{p}_n = \pi(p_n)$ and $\tilde{p}_\infty = \pi(p_\infty)$, and then have $g(\hat{p}_n) = \hat{g}(p_n) \not\to \hat{g}(p_\infty) = g(\tilde{p}_\infty)$. Using the continuity of g, and up to extract a subsequence, we can assume furthermore (from the closedness of $\mathcal{G}$) that there exists $\hat{p}_\infty \in \mathcal{G}$ such that

$$p_n \to p_\infty, \quad \hat{p}_n \to \hat{p}_\infty \neq \tilde{p}_\infty \tag{3.16}$$

We exhaust the different possible cases.
Case A: $p_\infty \neq \tilde{p}_\infty$
Then $p_\infty \in BA(\tilde{p}_\infty)\backslash\{\tilde{p}_\infty\}$, and from the definition of the basin of attraction, we deduce that $p_n \in BA(\tilde{p}_\infty)$, and then $\pi(p_n) = \hat{p}_n = \tilde{p}_\infty$. Contradiction with (3.16).
Case B: $p_\infty = \tilde{p}_\infty$
From (3.16), we can assume that $\hat{p}_\infty < \tilde{p}_\infty$ (the case $\hat{p}_\infty > \tilde{p}_\infty$ can be dealt in a similar way). Hence we have $\hat{p}_n \to \hat{p}_\infty < \tilde{p}_\infty = \lim\limits_{n\to+\infty} p_n$. Therefore $[\hat{p}_n, p_n) \subset BA(\hat{p}_n)$ with $(\hat{p}_n, p_n) \to (\hat{p}_\infty, \tilde{p}_\infty)$. The fact that the family $(BA(\hat{p})_{\hat{p}\in\mathcal{G}}$ forms a partition of $[a,b]$ implies that the sequence $(\hat{p}_n)_n$ is stationary for large n, i,e. that $\hat{p}_n = \hat{p}_\infty$. Then

$$[\hat{p}_\infty, \tilde{p}_\infty) \subset BA(\hat{p}_\infty) \not\ni \tilde{p}_\infty \tag{3.17}$$

By definition of the basin of attraction, we deduce that

$$g(\tilde{p}_\infty) = g(\hat{p}_\infty) \quad \text{or} \quad (\tilde{p}_\infty = b \quad \text{and} \quad g(\tilde{p}_\infty) < g(\hat{p}_\infty))$$

Case B.1: $g(\tilde{p}_\infty) = g(\hat{p}_\infty)$
Then $\hat{g}(p_n) = g(\hat{p}_n) = g(\hat{p}_\infty) = g(\tilde{p}_\infty) = \hat{g}(p_\infty)$. Contradiction with (3.15).
Case B.2: $\tilde{p}_\infty = b \quad$ **and** $\quad g(\tilde{p}_\infty) < g(\hat{p}_\infty)$

The strict inequality and the construction of $BA(\hat{p}_\infty)$, imply that $\tilde{p}_\infty \in BA(\hat{p}_\infty)$, which gives a contradiction with (3.17). We conclude that (3.15) is impossible, and then $\hat{g}$ is continuous.

Step 8: proof of iv)

Assume that $\tilde{g} : [a,b] \to \mathbb{R}$ is continuous which is locally constant on $\{\tilde{g} \neq g\}$ such that $\mathcal{G} = \mathcal{G}_{\tilde{g}}$. Then we have $\tilde{g} = g = \hat{g}$ on $\mathcal{G}$, where $\mathcal{G} = \mathcal{G}_{\hat{g}}$ is a closed set, because $\hat{g}$ is continuous. Moreover, we have $\tilde{g}' = 0$ on $[a,b]\backslash\mathcal{G}$. By continuity of g, $\tilde{g}$ and the structure of each basin on attraction, we deduce that $\tilde{g} = g(\hat{p}) = \hat{g}$ on $BA(\hat{p})$ for all $\hat{p} \in \mathcal{G}$. Because the family of basins of attraction forms a partition of $[a,b]$, we deduce that $\tilde{g} = \hat{g}$. This ends the proof of the proposition.

Proposition 3.8 (Monotonicity of $\hat{g}$, $N = 1$)

We work under assumptions of Proposition 3.6. Then $\hat{g}$ is nondecreasing and locally Lipschitz continuous, satisfying

$$\max(0, g') \geq \hat{g}' \geq 0 \quad a.e. \ on \quad [a,b]. \tag{3.18}$$

and

$$(\hat{g}' \in \{0, \max(0,g')\} \quad and\ \hat{g}' = g' > 0\ implies\ \hat{g} = g) \quad holds\ a.e.\ on \quad [a,b] \tag{3.19}$$

Proof of Proposition 3.8

Step 1: Lipschitz continuity of $\hat{g}$

Consider two points $p, q \in (a,b)$ such that $p < q$, and set $\hat{p} = \pi(p), \hat{q} = \pi(q)$. From the monotonicity of π, we deduce that $\hat{p} \leq \hat{q}$. Assume moreover that

$$\hat{p} < \hat{q}$$

With notation of Lemma 3.2, we can then write $BA_+(\hat{p}) = (\hat{p}, \overline{p})$, $BA_-(\hat{q}) = (\underline{q}, \hat{q})$, where we recall that

$$\left\{ \begin{array}{l} BA_+(\hat{p}) := \{r \in (\hat{p}, b], \quad g < g(\hat{p}) \quad \text{on} \quad (\hat{p}, r]\} \\ BA_-(\hat{q}) := \{r \in [a, \hat{q}), \quad g > g(\hat{q}) \quad \text{on} \quad [r, \hat{q})\} \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} \overline{p} := \left\{ \begin{array}{ll} \hat{p} & \text{if} \quad BA_+(\hat{p}) = \emptyset \\ \sup BA_+(\hat{p}) & \text{if} \quad BA_+(\hat{p}) \neq \emptyset \end{array} \right. \\ \\ \underline{q} := \left\{ \begin{array}{ll} \hat{q} & \text{if} \quad BA_-(\hat{q}) = \emptyset \\ \inf BA_-(\hat{q}) & \text{if} \quad BA_-(\hat{q}) \neq \emptyset \end{array} \right. \end{array} \right.$$

Because $BA(\hat{p}) \cap BA(\hat{q}) = \emptyset$, we deduce that $\hat{p} \leq \overline{p} < \underline{q} \leq \hat{q}$. Moreover $p \in BA(\hat{p})$, $q \in BA(\hat{q})$ imply that

$$a < p \leq \overline{p} < \underline{q} \leq q < b \quad \text{with} \quad g(\overline{p}) = g(\hat{p}), \quad g(\underline{q}) = g(\hat{q})$$

We deduce that $\hat{g}(p) - \hat{g}(q) = g(\hat{p}) - g(\hat{q}) = g(\overline{p}) - g(\underline{q})$. Hence

$$\frac{|\hat{g}(p) - \hat{g}(q)|}{|p-q|} = \frac{|g(\overline{p}) - g(\underline{q})|}{|p-q|} \leq \frac{|g(\overline{p}) - g(\underline{q})|}{|\overline{p} - \underline{q}|} \leq \text{Lip}(g; K)$$

for any compact interval $K \subset [a,b]$ containing p and q. This implies that $\text{Lip}(\hat{g}; K) \leq \text{Lip}(g; K)$ and then $\hat{g}$ is locally Lipschitz continuous on $[a,b]$. Moreover, consider a point $p_0 \in (a,b)$ where both $\hat{g}$ and g have a derivative. Then choosing $p := p_0 < q$ or $p < q := p_0$, we see in the limit $|p-q| \to 0$ that $|\hat{g}'(p_0)| \leq |g'(p_0)|$. In particular, we get

$$|\hat{g}'| \leq |g'| \quad \text{a.e. on} \quad [a,b] \tag{3.20}$$

Step 2: Monotonicity of $\hat{g}$

Assume by contradiction that $\hat{g}$ is not nondecreasing on $[a,b]$. Then, because $\hat{g}$ and g are Lipschitz continuous, by Rademacher's theorem, there exists at least a point $p_0 \in (a,b)$ where both $\hat{g}$ and g are differentiable, such that

$$\hat{g}'(p_0) < 0 \tag{3.21}$$

Because $\hat{g}' = 0$ on the open set $[a,b]\backslash\{\hat{g} = g\}$, we deduce that $\hat{g}(p_0) = g(p_0)$, i.e. $p_0 \in \mathcal{G}$, and then $p_0 = \hat{p}_0 := \pi(p_0)$.

Case A: p_0 is an accumulation point of $\mathcal{G}$

Then along the accumulation sequence, we get $g = \hat{g}$ and then $g'(p_0) = \hat{g}'(p_0) < 0$. Then the structure of each basin of attraction implies that $p_0 = \hat{p}_0$ satisfies for some small $\varepsilon > 0$

$$Q := (\hat{p}_0 - \varepsilon, \hat{p}_0 + \varepsilon) \subset BA(\hat{p}_0) \tag{3.22}$$

Case B: p_0 is an isolated point of $\mathcal{G}$
Again, the structure of each basin of attraction implies that $p_0 = \hat{p}_0$ satisfies (3.22).
Conclusion in both cases
Then (3.22) implies that $\hat{g} = const$ on Q. Therefore $\hat{g}'(p_0) = \hat{g}'(\hat{p}_0) = 0$. Contradiction with (3.21). This finally shows that $\hat{g}$ is nondecreasing on $[a, b]$. This implies with (3.20) that we have $|g'| \geq \hat{g}' \geq 0$ a.e. on $[a, b]$. Now recall that $\mathcal{G}$ is a closed set, and $\hat{g}' = 0$ on $(a, b)\backslash\mathcal{G}$, with $\hat{g} = g$ on $\mathcal{G}$. Therefore $\hat{g}' = g'$ a.e. on $\mathcal{G}$. We deduce that $\max(0, g') \geq \hat{g}' \geq 0$ a.e. on $[a, b]$, which shows (3.18). Moreover we also get (3.19).
This ends the proof of the proposition.

Proposition 3.9 (Second characterization of (generalized) Riemann germs, $N = 1$)
Assume (3.1). Let $\mathcal{G} \subset [a, b]$ be a set. Then $\mathcal{G}$ is a generalized Riemann germ with respect to $(J, f) = ((-\infty, 0], g)$ if and only we have

$$\mathcal{G} = \mathcal{G}_{\tilde{g}} \quad \text{with} \quad \mathcal{G}_{\tilde{g}} := \{p \in [a, b], \quad \tilde{g}(p) = g(p)\} \tag{3.23}$$

for some function $\tilde{g}$ satisfying

$$\tilde{g} : [a, b] \to \mathbb{R} \quad \text{nondecreasing, locally constant on } \{\tilde{g} \neq g\}, \text{ and satisfying } g_- \leq \tilde{g} \leq g_+ \tag{3.24}$$

Moreover, when this is the case, the function $\tilde{g}$ is continuous and $\mathcal{G}$ is a Riemann germ.

Proof of Proposition 3.9
Part I: proof that $\mathcal{G} = \mathcal{G}_{\tilde{g}}$ with $\tilde{g} := \hat{g}$
Assume that $\mathcal{G}$ is a generalized Riemann germ. Then from iii), iv) and v) of Proposition 3.6 and from monotonicity property of $\hat{g}$ given in Proposition 3.8, we deduce that $\hat{g}$ satisfies (3.23) with (3.24). Moreover vii) of Proposition shows that $\hat{g}$ is continuous.
Part II: proof that (3.23)-(3.24) implies that $\mathcal{G}$ is a generalized Riemann germ
We want to check that $\mathcal{G} := \mathcal{G}_{\tilde{g}}$ is a generalized Riemann germ, i.e. satisfies the following singleton property

$$\mathcal{G} \cap \hat{\mathcal{P}}_p = \{\hat{p}\} \quad \text{for all} \quad p \in [a, b]$$

Step 1: nonemptyness of $\mathcal{G} \cap \hat{\mathcal{P}}_p$
Recall that $g_- \leq \tilde{g} \leq g_+$. We first define $\hat{p} = \pi(p)$ for $p \in [a, b]$. Setting $\lambda := \tilde{g}(p)$, we define

$$\hat{p} := \pi(p) := \begin{cases} p & \text{if} \quad g(p) = \lambda \\ \sup\{q \in [p, b], \quad g > \lambda \quad \text{on} \quad (p, q]\} & \text{if} \quad g(p) > \lambda \\ \inf\{q \in [a, p], \quad g < \lambda \quad \text{on} \quad [q, p)\} & \text{if} \quad g(p) < \lambda \end{cases}$$

Because $\tilde{g}$ is locally constant where it differs from g, we deduce the following.
If $g(p) > \lambda$, then either $g(\hat{p}) = \lambda = \tilde{g}(\hat{p})$ or $g(\hat{p}) > \lambda = \tilde{g}(\hat{p})$ and $\hat{p} = a$. In the second case, we conclude that $\tilde{g}(\hat{p}) = \tilde{g}(a) \leq g_+(a) = g(a) = g(\hat{p})$. Contradiction. Therefore only the first case arises, i.e. $g(\hat{p}) = \lambda = \tilde{g}(\hat{p})$ and $\hat{p} \in \mathcal{G}$. Moreover with $\hat{p}_{\lambda,p}$ defined in (3.5), we have

$$\hat{p} = \hat{p}_{\lambda,p} \quad \text{with} \quad g_-(p) \leq g(p) < \lambda = \tilde{g}(\hat{p}) \leq \tilde{g}(p) \leq g_+(p)$$

Similarly, if $g(p) < \lambda$, then either $g(\hat{p}) = \lambda = \tilde{g}(\hat{p})$ or $g(\hat{p}) < \lambda$ and $\hat{p} = b$, and we exclude the second case. Hence in all cases, this defines $\mathcal{G} \ni \hat{p} = \hat{p}_{\lambda,p}$ with $\lambda \in [g_-(p), g_+(p)]$. Hence $\mathcal{G} \cap \hat{\mathcal{P}}_p \supset \{\hat{p}\}$.
Step 2: $\mathcal{G} \cap \hat{\mathcal{P}}_p$ is reduced to a singleton
Consider any $\hat{p}_\alpha \in \mathcal{G} \cap \hat{\mathcal{P}}_p$ for $\alpha = 1, 2$. Then there exists $\lambda_\alpha \in [g_-(p), g_+(p)]$ such that $\lambda_\alpha = g(\hat{p}_\alpha)$ and moreover $\hat{p}_\alpha = \hat{p}_{\lambda_\alpha,p}$. Hence we have $\tilde{g}(\hat{p}_\alpha) = g(\hat{p}_\alpha) = \lambda_\alpha$. Notice that the definition of $\mathcal{G} := \mathcal{G}_{\tilde{g}}$ and the fact that $\tilde{g}$ is locally constant on $\{\tilde{g} \neq g\}$ imply that

$$\tilde{g} = const = \tilde{g}(\hat{p}_\alpha) = g(\hat{p}_\alpha) = \lambda_\alpha \quad \text{on} \quad BA(\hat{p}_\alpha)$$

Because $p \in BA(\hat{p}_1) \cap BA(\hat{p}_2)$, this implies that $\lambda_1 = \tilde{g}(p) = \lambda_2$ and then $\hat{p}_1 = \hat{p}_2$, which shows that $\mathcal{G} \cap \hat{\mathcal{P}}_p$ is reduced to a singleton. We conclude that $\mathcal{G}_{\tilde{f}}$ is a generalized Riemann germ.
Part III: getting a Riemann germ
Moreover, from Proposition 3.6, we have $\hat{g}_{\mathcal{G}_{\tilde{g}}} = \tilde{g}$ which is continuous. Hence $\mathcal{G}$ is a Riemann germ.
This ends the proof of the proposition.

3.5 Basic relaxation formula

Proposition 3.10 (Basic relaxation formula, $N = 1$, junction $1:0$)
Assume (3.1). Let g_0 be a function satisfying

$$(3.25)\qquad \begin{cases} g_0 : [a,b] \to \mathbb{R} \textit{ is continuous nondecreasing} & \textbf{(Monotonicity)} \\ g_- \le g_0 \le g_+ \quad \textit{with } g_\pm \textit{ defined in (3.4)} & \textbf{(Bounds)} \end{cases}$$

i) (Basic relaxation formula)
Then the following formula defines uniquely a map $g_1 : [a,b] \to \mathbb{R}$

$$(3.26)\qquad \{g_1(p)\} = \bigcup_{q\in[a,b]} \{G(p,q)\} \cap \{g_0(q)\} \quad \textit{for all} \quad p \in [a,b]$$

where G is the standard Godunov flux associated to the function g, defined in (3.3).
ii) (Application to generalized Riemann germs)
Moreover the map g_1 satisfies condition (3.25), and is such that $\mathcal{G}_{g_1} = \{g_1 = g\}$ is a generalized Riemann germ, and g_1 is the generalized Godunov flux associated to $\mathcal{G}_{g_1}$ on a junction $1:0$, i.e. $g_1 = \hat{g}_{\mathcal{G}_{g_1}}$.
Conversely, if $\mathcal{G}$ is a generalized Riemann germ, then the associated function $\hat{g} := \hat{g}_{\mathcal{G}}$ satisfies (3.25) and (3.26) with $g_1 = g_0 = \hat{g}$.

Proof of Proposition 3.10
Step 1: g_1 is well-defined, nondecreasing, continuous and satisfies (3.25)
Let $g_0 : [a,b] \to \mathbb{R}$ satisfying (3.25). Given $p, q \in [a,b]$, we define $\Phi_p(q) := G(p,q) - g_0(q)$ where we recall the monotonicities $g_0(\uparrow)$ and $G(\uparrow,\downarrow)$. Recall that we have $g_-(p) = G(p,b) \le g_0(p) \le g_+(p) = G(p,a)$.
Step 1.1: discussion for a, b finite or not
Case A: finite a and b
Hence $\Phi_p(a) \ge 0 \ge \Phi_p(b)$ for finite a, b.
Case B: infinite a or b
If $a = -\infty$, then

$$\lim_{q\to a} \Phi_p(q) = \lim_{q\to a} \{G(p,q) - g_0(q)\} = \lim_{q\to a} \left\{ \sup_{[q,p]} g - g_0(q) \right\} = +\infty$$

where we have used the coercivity of g (as assumed in (3.1)) and the monotonicity of g_0.
If $b = +\infty$, then

$$\lim_{q\to b} \Phi_p(q) = \lim_{q\to b} \{G(p,q) - g_0(q)\} \le \lim_{q\to b} \{G(p,q) - g_-(q)\} = \lim_{q\to b} \left\{ \inf_{[p,q]} g - \inf_{[q,b]} g \right\} = -\infty$$

where we have used again the coercivity of g.
Step 1.2: remaining part of the argument
Because Φ_p is continuous nonincreasing, we deduce that there exists at least some $c \in [a,b]$ such that $\Phi_p(c) = 0$. Now even if c is non unique, the common value $g_0(c) = G(c,p)$ is unique, because of the monotonicities of g_0 and $G(p,\cdot)$. This defines uniquely the value $g_1(p)$ in (3.26). Consider $p' \ge p$. Then there exists $c, c' \in [a,b]$ such that

$$g_1(p) = g_0(c) = G(p,c) \le G(p',c), \quad g_1(p') = g_0(c') = G(p',c')$$

Hence $\Phi_{p'}(c) \ge 0 = \Phi_p(c)$. This shows that we can always choose $c' \ge c$, and then $g_1(p') = g_0(c') \ge g_0(c) = g_1(p)$, which shows that g_1 is nonincreasing, and moreover, we can always choose a map $p \mapsto c = c(p)$ which is nondecreasing such that $\Phi_p(c(p)) = 0$. Moreover, the continuity of g_0, G and the uniqueness of the value $g_1(p)$ imply the continuity of g_1.
Finally, by construction, for $p \in [a,b]$ we have $g_1(p) = g_0(c) = G(p,c) \in [G(p,b), G(p,a)] = [g_-(p), g_+(p)]$, which shows (3.25).
Step 2: local constancy of g_1

Assume that $g_1(p) \ne g(p)$ and recall that $g(p) \ne g_1(p) = g_0(c) = G(p,c) = \begin{cases} \sup_{[c,p]} g = \sup_{[c,p)} g & \text{if} \quad c < p \\ \inf_{[p,c]} g = \inf_{(p,c]} g & \text{if} \quad c > p \end{cases}$,

where we have used the fact that the inf / sup can not be reached at p, because $g(p) \ne g_1(p)$. Notice also that we can not have $c = p$, otherwise we would get $g_1(p) = g_0(p) = G(p,p) = g(p)$, which is impossible by assumption. This shows now that for p_ε close to p, we also have (by continuity of g, g_1, g_0)

$$g(p_\varepsilon) \neq g_1(p_\varepsilon) = g_0(c) = G(p_\varepsilon, c) = \left\{ \begin{array}{lll} \sup\limits_{[c,p_\varepsilon]} g = \sup\limits_{[c,p)} g & \text{if} & c < p_\varepsilon \\ \inf\limits_{[p_\varepsilon,c]} g = \inf\limits_{(p,c]} g & \text{if} & c > p_\varepsilon \end{array} \right\} = g_1(p)$$

This justifies a posteriori that we can choose $c_\varepsilon := c$ in $g_1(p_\varepsilon) = g_0(c_\varepsilon) = G(p_\varepsilon, c_\varepsilon)$. Hence g_1 is locally constant on $\{g_1 \neq g\}$.

Step 3: conclusion for g_1

From Proposition 3.9, we deduce that $\mathcal{G}_{g_1}$ is a Riemann germ on a junction $1:0$ and $g_1 = \hat{g}_{\mathcal{G}_{g_1}}$.

Step 4: conclusion for $\hat{g}$

Consider a generalized Riemann germ $\mathcal{G} \subset [a,b]$. Let $\hat{g} := \hat{g}_{\mathcal{G}}$ be the generalized Godunov flux associated to $\mathcal{G}$ on a junction $1:0$. Then we have $\mathcal{G} = \mathcal{G}_{\hat{g}} := \{\hat{g} = g\}$. From Propositions 3.6, 3.8, the function $\hat{g}$ satisfies (3.25). Consider g_1 given by formula (3.26) for $g_0 := \hat{g}$. Now if $\hat{p} \in \mathcal{G}$, then we check immediately that $g_1(\hat{p}) = \hat{g}(\hat{p}) = g(\hat{p})$. Therefore $\mathcal{G} \subset \mathcal{G}_{g_1} := \{g_1 = g\}$. Because the family $(BA(\hat{p}))_{\hat{p}\in\mathcal{G}}$ is already a partition of $[a,b]$, where $\hat{g}$ and g_1 coincide, we deduce that $g_1 = \hat{g}$, and then $\mathcal{G}_{g_1} = \mathcal{G}_{\hat{g}} = \mathcal{G}$. Moreover, this shows that (3.26) holds true with $g_1 = g_0 = \hat{g}$. This ends the proof of the lemma.

3.6 Proposition 2.14, Theorem 2.15 and their proofs on a junction $1:0$

Proof of Proposition 2.14 for $1:0$ **junction**

For the proof we refer to the table of Subsection 2.5. Notice that for $1:0$ junctions, Riemann monotonicities of π and $\hat{f}$ reduces to the fact that those functions are nondecreasing. Then the result of Proposition 2.14 for $1:0$ junction follows from Propositions 3.6, 3.8, 3.10 (respectively first properties of gen. Riemann germs, monotonicity, relaxation formula) and Lemma 3.4 (dissipation property).

Proof of Theorem 2.15 for $1:0$ **junction**

For the proof we refer to the table of Subsection 2.5. Point i) of Theorem 2.15 follows from ii) of Lemma 3.5 (first character. of gen. Riemann germs). Point ii) of Theorem 2.15 follows from Propositions 3.9 (second character. of (gen.) Riemann germs).

3.7 Weak stability of the Riemann problem

Lemma 3.11 (Weak stability of Riemann problem on a single branch $1:0$**)**

Assume (3.1). Let $p_n \in [a,b]$ and $p_n \in BA(\hat{p}_n)$. We call $v_n = v_{p_n,\hat{p}_n}$ the unique entropy solution of (3.2) with $(p,\hat{p}) := (p_n,\hat{p}_n)$. Assume that $(p_n,\hat{p}_n) \to (p_\infty,\hat{p}_\infty) \in [a,b]^2$ as $n \to \infty$. Then, up to extraction of a subsequence (still denoted by v_n), there exists $\tilde{p}_\infty \in [a,b]$ such that $p_\infty \in BA(\tilde{p}_\infty)$ and

$$v_{p_n,\hat{p}_n} = v_n \to v_\infty := v_{p_\infty,\tilde{p}_\infty} \quad \textit{in} \quad L^1_{loc}([0,+\infty) \times (-\infty,0])$$

Here $\tilde{p}_\infty$ is the trace of the limit, while $\hat{p}_\infty$ is the limit of the trace. Moreover, we have

$$g(\tilde{p}_\infty) = g(\hat{p}_\infty) \quad \textit{with} \quad \tilde{p}_\infty \in \{p_\infty, \hat{p}_\infty\} \tag{3.27}$$

(even if $\hat{p}_n$ may not converge towards $\tilde{p}_\infty$, and p_∞ may not belong to $BA(\hat{p}_\infty)$).
In other words, for self-similar solutions on a half line, if the limit is not constant in space and time, then the trace of the limit is equal to the limit of the trace.

Proof of Lemma 3.11

Step 1: first results

Recall that $v_n(t,x) = V_n(x/t)$ with V_n monotone and bounded in $L^\infty((-\infty,0];[a,b])$. By classical Helly's theorem for monotone functions, we know that, up to extract a subsequence, we have $V_n \to V_\infty$ a.e. on $(-\infty,0]$, and then $v_n \to v_\infty$ in $L^1_{loc}([0,+\infty) \times (-\infty,0])$ with $v_\infty(t,x) := V_\infty(x/t)$. By stability of entropy solutions and of their initial data, we have that v_∞ is a solution of (3.2) with initial data $p := p_\infty$. Because V_∞ is monotone, it also has a trace $\tilde{p}_\infty \in [a,b]$ at $x = 0^-$. Hence $v_\infty = v_{p_\infty,\tilde{p}_\infty}$ with $p_\infty \in BA(\tilde{p}_\infty)$.

Moreover, we have $\left\{ \begin{array}{lll} \hat{p}_n := \hat{p}_{\hat{\lambda}_n,p_n}, & \text{with } \hat{\lambda}_n := g(\hat{p}_n) & \text{and } \hat{\lambda}_\infty := g(\hat{p}_\infty) \\ \tilde{p}_\infty := \hat{p}_{\tilde{\lambda}_\infty,p_\infty} & \text{with } \tilde{\lambda}_\infty := g(\tilde{p}_\infty) & \end{array} \right.$.

Step 2: proof that $\tilde{\lambda}_\infty = \hat{\lambda}_\infty$
Case A: $p_\infty \neq \tilde{p}_\infty$
Because $BA(\tilde{p}_\infty)\backslash\{\tilde{p}_\infty\}$ is a relative open set of $[a,b]$, we deduce that $p_n \in BA(\tilde{p}_\infty)\backslash\{\tilde{p}_\infty\}$ for n large enough. Hence $\hat{p}_n = \tilde{p}_\infty$, and then $\tilde{p}_\infty = \hat{p}_\infty$ and $\tilde{\lambda}_\infty = \hat{\lambda}_\infty$.
Case B: $p_\infty = \tilde{p}_\infty$
If $\hat{p}_\infty = p_\infty$, then we get immediately that $\hat{\lambda}_\infty = \tilde{\lambda}_\infty$. Assume therefore that $\hat{p}_\infty \neq p_\infty$. Precisely assume that $\hat{p}_\infty < p_\infty$ (the case $\hat{p}_\infty > p_\infty$ is similar).

Because $(\hat{p}_n, p_n) \to (\hat{p}_\infty, p_\infty)$, this implies for n large enough that

$$g < \hat{\lambda}_n = g(\hat{p}_n) \quad \text{on} \quad (\hat{p}_n, p_n] \tag{3.28}$$

Let us call the interval $I_n := [\hat{p}_n, p_n]$. Then from the proof of Lemma 3.1, Step 2, Case 2, we have

$$\tilde{g}_n := \text{concave envelope of } g^{I_n} \quad \text{with} \quad g^{I_n}(a) := \begin{cases} g(a) & \text{if} \quad a \in I_n \\ -\infty & \text{if} \quad a \in \mathbb{R}\backslash I_n \end{cases}$$

Hence $\tilde{g}_n(\hat{p}_n) = g(\hat{p}_n) = \hat{\lambda}_n$ and $\tilde{g}_n(p_n) < \hat{\lambda}_n$, and from (3.28), we deduce that the concave function $\tilde{g}_n$ satisfies

$$\tilde{g}_n < \hat{\lambda}_n \quad \text{on} \quad (\hat{p}_n, p_n] \tag{3.29}$$

We deduce (see Lemma 10.1) that

$$(\xi_L^n, \xi_R^n) := (\tilde{g}_n'(p_n^-), \tilde{g}_n'(\hat{p}_n^+)) \quad \text{satisfies} \quad \xi_L^n \leq \xi_R^n \leq 0 \tag{3.30}$$

and get

$$V_n(\xi) = \begin{cases} p_n & \text{if} \quad \xi < \xi_L^n \\ \hat{p}_n & \text{if} \quad \xi_R^n < \xi \leq 0 \\ ((\tilde{g}_{n|I_n})')^{-1}(\xi) & \text{if} \quad \xi \in [\xi_L^n, \xi_R^n] \end{cases} \tag{3.31}$$

which is well defined for almost every $\xi > 0$. Moreover V_n is nonincreasing.
We deduce that $\hat{\lambda}_n > g(p_n) \to g(p_\infty)$ and then

$$\hat{\lambda}_\infty \geq g(p_\infty) \tag{3.32}$$

Assume by contradiction that

$$\hat{\lambda}_\infty > g(p_\infty) \tag{3.33}$$

Setting $I_\infty := [\hat{p}_\infty, p_\infty]$, we see that $\tilde{g}_n \to \tilde{g}_\infty$ locally uniformly on $[\hat{p}_\infty, p_\infty]$, where $\tilde{g}_\infty$ is the concave envelope of g^{I_∞}. Then passing to the limit in (3.29), we get $\begin{cases} \tilde{g}_\infty \leq \hat{\lambda}_\infty = g(\hat{p}_\infty) & \text{on} \quad [\hat{p}_\infty, p_\infty] \\ \tilde{g}_\infty = g & \text{on} \quad \partial[\hat{p}_\infty, p_\infty] \end{cases}$. Because of (3.33) and the concavity of $\tilde{g}_\infty$, we deduce that $\tilde{g}_\infty$ is above its chord on $[\hat{p}_\infty, p_\infty]$ and then

$$\tilde{g}_\infty > g(p_\infty) \quad \text{on} \quad [\hat{p}_\infty, p_\infty) \tag{3.34}$$

Then is is easy to check that V_n converges almost everywhere (by Helly's theorem for sequences of monotone functions) towards V_∞ which is given by (3.30)-(3.31) for $n = \infty$. Moreover (3.34) insures that $V_\infty \neq const$ a.e. on $(-\infty, 0)$. On the other hand, notice that $v_\infty = v_{p_\infty, \tilde{p}_\infty} = v_{p_\infty, p_\infty}$ and then this function is uniquely given by $v_\infty \equiv p_\infty$. Therefore $V_\infty \equiv p_\infty = const$. Contradiction. Hence (3.33) is false, and we conclude from (3.32) that $\hat{\lambda}_\infty = g(p_\infty) = g(\tilde{p}_\infty) = \tilde{\lambda}_\infty$, where we have used our assumption $p_\infty = \tilde{p}_\infty$. This shows (3.27). This ends the proof of the lemma.

4 Riemann problem on a junction with $N \geq 1$ branches

4.1 Two families of transformations and reduction to $N:0$ junctions

Given assumption (2.2), there are naturally two families of transformations with actions on $(J, f, \mathcal{G})$ where J is a junction with $N \geq 1$ branches, $f^j : [a^j, b^j] \to \mathbb{R}$ are maps for $j = 1, \ldots, N$ and $\mathcal{G} \subset [a,b]$ is a set. Those are I-inversions and I'-reversions where $I, I' \subset \{1, \ldots, N\}$ are subsets of indices. The first family of I-inversions does not modify the junction J, but only modify $(f, \mathcal{G})$. On the contrary the second family of I-reversions modifies (J, f) but does not modify the set $\mathcal{G}$.

4.1.1 I-inversions

We will need the following partial inversion transform defined for a subset I of indices.

Definition 4.1 **(I-inversion)**
Let (J,f) satisfying (2.2) with $N \ge 1$, and let $\mathcal{G} \subset [a,b]$ be a set. Given a subset $I \subset \{1,\dots,N\}$, we define the I-inversion with respect to I as the map $\bar{(\cdot)} : (J,f,\mathcal{G}) \mapsto (\bar{J},\bar{f},\bar{\mathcal{G}})$ defined for $\bar{\varepsilon}^j := \begin{cases} -1 & \text{if } \; j \in I \\ +1 & \text{otherwise} \end{cases}$ as

$$\tag{4.1} \begin{cases} \bar{J} := J \\ \\ \bar{f}^j(\bar{p}^j) := \bar{\varepsilon}^j f^j(\bar{\varepsilon}^j \bar{p}^j) \quad \text{for} \quad \bar{p}^j \in \left[\bar{a}^j, \bar{b}^j\right] = \bar{\varepsilon}^j \left[a^j, b^j\right] \\ \\ \bar{\mathcal{G}} := \left\{\bar{p} \in [\bar{a},\bar{b}], \quad p \in \mathcal{G}\right\} \quad \text{with} \quad \bar{p}^j := \bar{\varepsilon}^j p^j \end{cases}$$

If $\hat{f} : [a,b] \to \mathbb{R}^N$ is any map, we also define the action of the I-inversion $\bar{(\cdot)}$ on $\hat{f}$ as

$$\tag{4.2} \bar{\hat{f}}^j(\bar{p}) := \bar{\varepsilon}^j \hat{f}^j(p) \quad \text{with } \bar{p} \text{ defined in (4.1)}$$

Remark 4.2 *Notice that the notation for $\bar{a},\bar{b}$ is not consistent with the general definition of $\bar{p}$ as a function of p. But this inconsistency will not bring any confusion.*

Then we have the following result

Lemma 4.3 **(I-inversion of a germ)**
Let (J,f) satisfying (2.2) with $N \ge 1$, and let $\mathcal{G} \subset [a,b]$ be a set. Given a subset $I \subset \{1,\dots,N\}$, consider the I-inversion which maps $(J,f,\mathcal{G})$ to $(\bar{J},\bar{f},\bar{\mathcal{G}})$ with $(\bar{J},\bar{f})$ satisfying (2.2).
i) (Case of a generalized Riemann germ)
Then $\mathcal{G}$ is a generalized Riemann germ with respect to (J,f), if and only if $\bar{\mathcal{G}}$ is also a generalized Riemann germ with respect to $(\bar{J},\bar{f})$. Moreover for $\bar{p}$ defined in (4.1), we have

$$\tag{4.3} \begin{cases} BA^{(\bar{J}^j,\bar{f}^j)}(\bar{p}^j) & = \bar{\varepsilon}^j BA^{(J^j,f^j)}(p^j) \\ \pi^j_{\bar{\mathcal{G}}}(\bar{p}) & = \bar{\varepsilon}^j \pi^j_{\mathcal{G}}(p) \end{cases}$$

ii) (Case of a set with special expression)
Assume that there exists some function $\hat{f} : [a,b] \to \mathbb{R}^N$ such that the set $\mathcal{G}$ satisfies $\mathcal{G} = \left\{p \in [a,b], \; \hat{f}(p) = f(p)\right\}$. Given $\bar{\hat{f}}$ in (4.2), we have $\bar{\mathcal{G}} = \left\{\bar{p} \in [\bar{a},\bar{b}], \; \bar{\hat{f}}(\bar{p}) = \bar{f}(\bar{p})\right\}$.

Proof of Lemma 4.3
It is straightforward to check point ii). Let us now prove point i). Consider an entropy solution u of $u^j_t + \partial_x(f^j(u^j)) = 0$ on $(0,+\infty) \times J^j$. We set

$$\bar{u}^j(t,x) := \bar{\varepsilon}^j u^j(t,x)$$

Then $\bar{u}$ is an entropy solution u of $\bar{u}^j_t + \partial_x(\bar{f}^j(\bar{u}^j)) = 0$ on $(0,+\infty) \times \bar{J}^j$. Moreover $\bar{u}$ is self-similar if u is, and

$$u(t,0) \in \mathcal{G} \quad \text{is equivalent to} \quad \bar{u}(t,0) \in \bar{\mathcal{G}}$$

This implies that $\bar{\mathcal{G}}$ is generalized Riemann germ with respect to $(\bar{J},\bar{f})$, if $\mathcal{G}$ is with respect to (J,f). It is is indeed an equivalence. Moreover, under I-inversion, the properties (4.3) of the basins of attraction and of the natural projection follow from the definitions. This ends the proof of the lemma.

4.1.2 I-reversions

We will need the following partial reversion transform defined for a subset I of indices.

Definition 4.4 **(I-reversion)**
Let (J,f) satisfying (2.2) with $N \geq 1$, and let $\mathcal{G} \subset [a,b]$ be a set. Given a subset $I \subset \{1,\dots,N\}$, we define the I-reversion with respect to I as the map $\tilde{(\cdot)} : (J,f,\mathcal{G}) \mapsto (\tilde{J},\tilde{f},\tilde{\mathcal{G}})$ defined for $\tilde{\varepsilon}^j := \begin{cases} -1 & \text{if } j \in I \\ +1 & \text{otherwise} \end{cases}$ *as*

$$\begin{cases} \tilde{J} := \{0\} \cup \bigcup_{j=1,\dots,N} \tilde{J}^j \quad \text{with} \quad \tilde{J}^j := \tilde{\varepsilon}^j J^j \\ \\ \tilde{f}^j(p^j) := \tilde{\varepsilon}^j f^j(p^j) \\ \\ \tilde{\mathcal{G}} := \mathcal{G} \end{cases}$$

If $\hat{f} : [a,b] \to \mathbb{R}^N$ is any map, we also define the action of the I-reversion $\tilde{(\cdot)}$ on $\hat{f}$ as

$$\tilde{\hat{f}}^j(p) := \tilde{\varepsilon}^j \hat{f}^j(p) \tag{4.4}$$

We see that for indices in I, the I*-reversion transform* changes outgoing branches into ingoing branches, and vice versa.

Then we have the straightforward result.

Lemma 4.5 **(I-reversion of germ)**
Let (J,f) satisfying (2.2) with $N \geq 1$, and let $\mathcal{G} \subset [a,b]$ be a set. Given a subset $I \subset \{1,\dots,N\}$, consider the I-inversion which maps $(J,f,\mathcal{G})$ to $(\tilde{J},\tilde{f},\tilde{\mathcal{G}})$ with $(\tilde{J},\tilde{f})$ satisfying (2.2).
i) (Case of a generalized Riemann germ)
Then $\mathcal{G}$ is a generalized Riemann germ with respect to (J,f), if and only if $\tilde{\mathcal{G}}$ is also a generalized Riemann germ with respect to $(\tilde{J},\tilde{f})$. Moreover, for all $q \in [a,b]$ we have

$$\begin{cases} BA^{(\tilde{J},\tilde{f})}(q) &= BA^{(J,f)}(q) \\ \pi_{\tilde{\mathcal{G}}} &= \pi_{\mathcal{G}} \end{cases} \tag{4.5}$$

ii) (Case of a set with special expression)
Assume that there exists some function $\hat{f} : [a,b] \to \mathbb{R}^N$ such that the set $\mathcal{G}$ satisfies $\mathcal{G} = \left\{p \in [a,b],\ \hat{f}(p) = f(p)\right\}$. Given $\tilde{\hat{f}}$ in (4.4), we have $\tilde{\mathcal{G}} = \left\{p \in [a,b],\ \tilde{\hat{f}}(p) = \tilde{f}(p)\right\}$.

Proof of Lemma 4.5
It is straightforward to check point ii). Let us now prove point i). Consider an entropy solution u of $u_t^j + \partial_x(f^j(u^j)) = 0$ on $(0,+\infty) \times J^j$. We set

$$\tilde{u}^j(t,x) := u^j(t,\tilde{\varepsilon}^j x)$$

Then $\tilde{u}$ is an entropy solution u of $\tilde{u}_t^j + \partial_x(\tilde{f}^j(\tilde{u}^j)) = 0$ on $(0,+\infty) \times \tilde{J}^j$. Moreover $\tilde{u}$ is self-similar if u is, and

$$u(t,0) \in \mathcal{G} \quad \text{is equivalent to} \quad \tilde{u}(t,0) \in \tilde{\mathcal{G}}$$

This implies that $\tilde{\mathcal{G}}$ is generalized Riemann germ with respect to $(\tilde{J},\tilde{f})$, if $\mathcal{G}$ is with respect to (J,f). Moreover it is an equivalence. Moreover, under I-reversion, the properties (4.5) of the basins of attraction and of the natural projection follow from the definitions. This ends the proof of the lemma.

It is straightforward to check the following result.

Lemma 4.6 **(Commutativity of inversions and reversions)**
Let (J,f) satisfying (2.2) with $N \geq 1$, and let $\mathcal{G} \subset [a,b]$ be a set. Given two subsets $\bar{I},\tilde{I} \subset \{1,\dots,N\}$, consider the $\bar{I}$-inversion $\bar{(\cdot)}^{\bar{I}}$ and the $\tilde{I}$-reversion $\tilde{(\cdot)}^{\tilde{I}}$. Then we have the commutativity $\bar{(\cdot)}^{\bar{I}} \circ \tilde{(\cdot)}^{\tilde{I}} = \tilde{(\cdot)}^{\tilde{I}} \circ \bar{(\cdot)}^{\bar{I}}$.

4.1.3 Reduction to $N:0$ junctions

The following result is a straightforward corollary of Lemma 4.5.

Corollary 4.7 (Reduction to $N:0$ junctions)
Let (J,f) satisfying (2.2) with $N\ge 1$, and let $\mathcal{G}\subset[a,b]$ be a generalized Riemann germ with respect to (J,f). Set

$$I:=\left\{j\in\{1,\dots,N\},\quad \sigma^j=-1\right\}$$

The action of a I-reversion on $(J,f,\mathcal{G})$ defines $(\tilde J,\tilde f,\tilde{\mathcal{G}})$. Then $\tilde{\mathcal{G}}$ is a generalized Riemann germ with respect to $(\tilde J,\tilde f)$ with $\tilde J^j\simeq(-\infty,0)=\tilde\sigma^j\cdot(-\infty,0)$ for all $j=1,\dots,N$, i.e. with $\tilde\sigma^j=1$ for all j. This means that $\tilde J$ is a junction of type $N:0$.

This way, we see that we reduce the problem of $n:m$ junctions with $N:=n+m$, to the case $n=N$ and $m=0$. Such junctions are then called $N:0$ junctions. In the remaining part of the section, we will work which such junctions which have more symmetries. The result for the original problem can then be obtained easily by the inverse of the reverse transform.

4.2 Proof of Proposition 2.4: L^1 estimate for Riemann problem

We can do the proof of Proposition 2.4 in two steps.
Step 1: reduction
We first notice that the problem reduces to a single branch $N=1$ and using a reversion, we can assume that we work on a $1:0$ junction.
Step 2: the computation
We now compute formally for $j=1$ with $J^j=(-\infty,0)$

$$\begin{aligned}\frac{d}{dt}\int_{J^j}(u^j(t,\cdot)-p^j)\ &=\int_{J^j}u^j_t(t,\cdot)\\&=-\int_{J^j}\partial_x\left\{f^j(u^j(t,\cdot))\right\}\ dx\\&=-[f^j]^{p^j}_{\hat p^j}\\&=f^j(\hat p^j)-f^j(p^j)\end{aligned}$$

This computation can easily be justified (in the sense of distributions), and this implies the first equality of (2.7), because $u^j(0,\cdot)=p^j$. The second equality of (2.7) follows from the fact that the sign of u^j-p^j only depends on j, for self-similar solutions (see Lemma 10.1). This ends the proof of the proposition.

4.3 First results on generalized Riemann germs

Assume (2.2) with $N\ge 1$ branches $\simeq(-\infty,0)$. We recall briefly some definitions/notations. For $j\in\{1,\dots,N\}$, we recall that the standard Godunov flux $G^j:[a^j,b^j]^2\to\mathbb{R}$ associated to f^j, is $G^j(q,r):=\begin{cases}\inf\limits_{[q,r]}f^j & \text{if}\quad q\le r\\ \sup\limits_{[r,q]}f^j & \text{if}\quad q\ge r\end{cases}$. We also define the following nondecreasing functions of $p^j\in[a^j,b^j]$

$$f^j_-(p^j):=\inf_{[p^j,b^j]}f^j=G^j(p^j,b^j)\quad\le\quad f^j_+(p):=\sup_{[a^j,p^j]}f^j=G^j(p^j,a^j)\tag{4.6}$$

and for $\lambda\in[f^j_-(p^j),f^j_+(p^j)]$, we define the following element of $[a^j,b^j]$

$$\hat p^j_{\lambda,p^j}:=\begin{cases}p^j & \text{if}\quad f^j(p^j)=\lambda\\ \sup\left\{q^j\in(p^j,b^j],\quad f^j>\lambda\quad\text{on}\quad(p^j,q^j)\right\} & \text{if}\quad f^j(p^j)>\lambda\\ \inf\left\{q^j\in[a^j,p^j),\quad f^j<\lambda\quad\text{on}\quad(q^j,p^j)\right\} & \text{if}\quad f^j(p^j)<\lambda\end{cases}\tag{4.7}$$

which is nonincreasing in λ. For $p=(p^1,\dots,p^N)\in[a,b]$, we also define the following subset of $[a,b]$

$$\hat{\mathcal{P}}_p:=\prod_{j=1,\dots,N}\hat{\mathcal{P}}^j_{p^j}\quad\text{with}\quad\hat{\mathcal{P}}^j_{p^j}:=\left\{\hat p^j_{\lambda,p^j}\in[a^j,b^j],\quad\lambda\in[f^j_-(p^j),f^j_+(p^j)]\right\}\tag{4.8}$$

We also recall the basins of attraction defined for all $p\in[a,b]$ by

$$BA(p):=\prod_{j=1,\dots,N}BA^j(p^j)\quad\text{with}\quad BA^j(p^j):=BA^j_-(p^j)\cup\left\{p^j\right\}\cup BA^j_+(p^j)\tag{4.9}$$

where $BA^j_\pm = BA^j_\pm(p^j)$ are given for $\lambda^j := f^j(p^j)$ by $\left\{\begin{array}{l} BA^j_+ := \left\{q^j \in (p^j, b^j], \quad f^j < \lambda^j \quad \text{on} \quad (p^j, q^j]\right\} \\ BA^j_- := \left\{q^j \in [a^j, p^j), \quad f^j > \lambda^j \quad \text{on} \quad [q^j, p^j)\right\} \end{array}\right.$.

Then from Lemma 3.2 and the definitions, we get immediately

Lemma 4.8 (Inverse characterization of the map $p \mapsto \hat{\mathcal{P}}_p$)
Assume (2.2) with $N \geq 1$ and a $N:0$ junction. Then for two arbitrary vectors $p, \hat{p} \in [a,b]$, we have

$$\hat{p} \in \hat{\mathcal{P}}_p \qquad \textit{if and only if} \qquad p \in BA(\hat{p})$$

The following result follow immediately from the definitions and Lemma 3.1.

Lemma 4.9 (Generalized Riemann germ characterization, $N \geq 1$)
Assume (2.2) with $N \geq 1$ and let $\mathcal{G} \subset [a,b]$ be a set.
i) (First characterization)
The set $\mathcal{G}$ is a generalized Riemann germ if and only if for all $p \in [a,b]$, we have the singleton property $\mathcal{G} \cap \hat{\mathcal{P}}_p = \{\hat{p}\}$, where $\hat{\mathcal{P}}_p$ is defined in (4.8).
ii) (Equivalent characterization)
The set $\mathcal{G}$ is a generalized Riemann germ if and only if $(BA(\hat{p}))_{\hat{p}\in\mathcal{G}}$ is a partition of $[a,b]$.

Proposition 4.10 (First properties of generalized Riemann germs, $N \geq 1$)
Assume (2.2) with $N \geq 1$ and let $\mathcal{G} \subset [a,b]$ be a generalized Riemann germ for a junction $N:0$. Given $p \in [a,b]$, the unique $\mathcal{G}$-entropy solution of (2.6) writes $u^{\mathcal{G}}_{p,\hat{p}}$ with $\pi(p) := \pi_{\mathcal{G}}(p) := \hat{p} \in \mathcal{G}$ and $\pi : [a,b] \to \mathcal{G}$ which satisfies $\pi \circ \pi = \pi$. We set $\hat{f} := \hat{f}_{\mathcal{G}} := f \circ \pi$.
i) (Local constancy): *The map $\hat{f}$ is locally constant on $\left\{\hat{f} \neq f\right\}$.*
ii) (Inverse of π): *For all $\hat{p} \in \mathcal{G}$, we have $\pi^{-1}(\hat{p}) = BA(\hat{p})$.*
iii) (Level set formulation of the germ): *We have $\mathcal{G} = \mathcal{G}_{\hat{f}} := \left\{p \in [a,b], \quad \hat{f}(p) = f(p)\right\}$.*
iv) (Characterization of $\hat{f}$): *Assume that $\hat{f} : [a,b] \to \mathbb{R}^N$ is continuous. Then $\hat{f}$ is fully characterized as the continuous function which is locally constant on $\left\{\hat{f} \neq f\right\}$ such that $\mathcal{G} = \mathcal{G}_{\hat{f}}$.*
v) (Monotone bounds): *We have $f_- \leq \hat{f} \leq f_+$ with $f_\pm$ defined in (4.6).*

Remark 4.11 *Notice that the monotonicity properties of $\hat{f}$ are not proved in Proposition 4.10. Again it seems to be a delicate result. We will prove it later using as a key step a slicing lemma which reduces to germs for a single branch, for which we already have proved some monotonicity.*

Proof of Proposition 4.10
Step 1: proof of ii)
From i) of Lemma 4.9 and Lemma 4.8, we have $\pi^{-1}(\hat{p}) = BA(\hat{p})$ for all $\hat{p} \in \mathcal{G}$.
Step 2: proof of iii)
From b) in hands, then Step 3 of the proof of Proposition 3.6 applies word by word.
Step 3: proof of i)
Let $p \in [a,b] \cap \left\{\hat{f} \neq f\right\}$ and let $\hat{p} = \pi(p) \in \mathcal{G}$ such that $p \in BA(\hat{p})$. We set

$$I_p(\hat{f}) := \left\{j \in \{1,\dots,N\}, \quad \hat{f}^j(p) \neq f^j(p^j)\right\} \tag{4.10}$$

Because $\hat{f} = f \circ \pi$, we deduce that for all $j \in I_p(\hat{f})$ that $p^j \in BA(\hat{p}^j)\backslash\left\{\hat{p}^j\right\}$, which is a relative open set of $[a^j, b^j]$. We deduce the existence of some $\varepsilon > 0$ small enough such that $[a^j, b^j] \cap Q^j_{p^j} \subset BA^j(\hat{p}^j)$ with $Q^j_{p^j} := (p^j - \varepsilon, p^j + \varepsilon)$. This means that

$$[a,b] \cap Q^{\hat{f}}_\varepsilon(p) \subset BA(\hat{p}) \quad \text{with} \quad Q^{\hat{f}}_\varepsilon(p) := p + \sum_{j \in I_p(\hat{f})} (-\varepsilon, \varepsilon) e_j \tag{4.11}$$

with the convention that $Q^{\hat{f}}_\varepsilon(p) = \{p\}$ when $I_p(\hat{f}) = \emptyset$. Then $\hat{f} = const = \hat{f}(p) = f(\hat{p})$ on $[a,b] \cap Q^{\hat{f}}_\varepsilon(p)$, which means exactly that $\hat{f}$ is locally constant on $\left\{\hat{f} \neq f\right\}$.
Step 4: proof of v)

Consider $p \in [a,b]$. We know that $p \in BA(\hat{p})$ with $\hat{p} \in \mathcal{G}$. Hence for each index j, we have $\hat{p}^j = \hat{p}^j_{\lambda^j,p^j}$ with $\hat{f}^j(p) = f^j(\hat{p}^j) = \lambda^j \in [f^j_-(p^j), f^j_+(p^j)]$ (by assumption). This gives the monotone bounds.

Step 5: proof of iv)

Assume that $\hat{f}$ is continuous. Now consider some continuous function $\tilde{f} : [a,b] \to \mathbb{R}$ which is locally constant on $\left\{\tilde{f} \neq f\right\}$ such that $\mathcal{G} = \mathcal{G}_{\tilde{f}}$. We want to show that $\tilde{f} = \hat{f}$.

We already know that $\tilde{f} = f = \hat{f}$ on $\mathcal{G}$, where $\mathcal{G} = \mathcal{G}_{\hat{f}}$ is a closed set, because $\hat{f}$ is continuous. Moreover, we have for all $p \in [a,b]\backslash\mathcal{G}$, there exists $\varepsilon > 0$ such that

$$\frac{\partial \tilde{f}}{\partial p^j} = 0 \quad \text{in} \quad [a,b] \cap Q^{\tilde{f}}_\varepsilon(p) \quad \text{for all} \quad j \in I_p(\tilde{f})$$

where $I_p(\tilde{f})$ and $Q^{\tilde{f}}_\varepsilon(p)$ are defined respectively in (4.10) and (4.11). By continuity of f, $\tilde{f}$ and the structure of each basin on attraction, we deduce that $\tilde{f} = f(\hat{p}) = \hat{f}$ on $BA(\hat{p})$ for all $\hat{p} \in \mathcal{G}$. Because the family of basins of attraction forms a partition of $[a,b]$, we deduce that $\tilde{f} = \hat{f}$.

This ends the proof of the proposition.

4.4 Slicing lemma and basic monotonicities

The following simple lemma is a key tool.

Lemma 4.12 (Slicing lemma)

Assume (2.2) with $N \geq 2$, and let $\mathcal{G} \subset [a,b]$ be a generalized Riemann germ for a $N:0$ junction. Let $1 \leq n < N$. Then for $p \in [a,b]$, we write

$$p = (p', p'') \quad \textit{with} \quad p' = (p^1, \dots, p^n) \in [a',b'] \quad \textit{and} \quad p'' = (p^{n+1}, \dots, p^N) \in [a'',b''].$$

Given some $p''_0 \in [a'',b'']$, we set $\Pi := [a',b'] \times \{p''_0\}$ and define the slicing of the germ $\mathcal{G}$ with respect to p''_0:

$$\mathcal{G}_{p''_0} := \{\hat{p}' \in [a',b'], \quad \textit{s.t. there exists } \hat{p} = (\hat{p}', \hat{p}'') \in \mathcal{G} \textit{ with } BA(\hat{p}) \cap \Pi \neq \emptyset\}$$

Then $\mathcal{G}_{p''_0} \subset [a',b']$ is a generalized Riemann germ for a $n:0$ junction. Moreover

$$\mathcal{G}_{p''_0} = \left\{p' = (p^1, \dots, p^n) \in [a',b'], \quad \hat{f}^j(p', p''_0) = f^j(p^j) \quad \textit{for} \quad j = 1, \dots, n\right\} \tag{4.12}$$

and the effective flux function associated to $\mathcal{G}_{p''_0}$ is

$$\hat{f}_{\mathcal{G}_{p''_0}} = (\hat{f}^j(\cdot, p''_0))_{j=1,\dots,n} \tag{4.13}$$

Proof of Lemma 4.12

Recall that $BA(\hat{p})$ is given in (4.9) and that in the present case all branches satisfy $J^j \simeq (-\infty, 0)$.
With $\mathcal{G}_\Pi := \{\hat{p} \in \mathcal{G},\ BA(\hat{p}) \cap \Pi \neq \emptyset\}$, we can write

$$\mathcal{G}_{p''_0} := \{\hat{p}' \in [a',b'], \quad \text{s.t. there exists } \hat{p} = (\hat{p}', \hat{p}'') \in \mathcal{G}_\Pi\}$$

Because $(BA(\hat{p}))_{\hat{p} \in \mathcal{G}}$ is a partition of $[a,b]$, we see that Π is covered exactly once by the family $(BA(\hat{p}))_{\hat{p} \in \mathcal{G}_\Pi}$. Hence $(\Pi \cap BA(\hat{p}))_{\hat{p} \in \mathcal{G}_\Pi}$ is a partition of Π. This implies that $\mathcal{G}_{p''_0} \subset [a',b']$ is a generalized Riemann germ.
Now define

$$\tilde{\mathcal{G}}_{p''_0} := \left\{p' = (p^1, \dots, p^n) \in [a',b'], \quad \hat{f}^j(p', p''_0) = f^j(p^j) \quad \text{for} \quad j = 1, \dots, n\right\}$$

For any $\hat{p}' \in \mathcal{G}_{p''_0}$, we set $\hat{p} := \pi(\hat{p}', p''_0) \in \mathcal{G}_\Pi$, where $\pi = \pi_{\mathcal{G}} : [a,b] \to \mathcal{G}$ is the natural projection associated to $\mathcal{G}$. Then $\hat{p}$ writes $\hat{p} = (\hat{p}', \hat{p}'')$. Moreover $\hat{f} = \textit{const} = f(\hat{p})$ on $BA(\hat{p})$, and $BA(\hat{p}) \cap \Pi \ni (\hat{p}', p''_0)$. This shows in particular that $\mathcal{G}_{p''_0} \subset \tilde{\mathcal{G}}_{p''_0}$.

Conversely, consider $p' \in \tilde{\mathcal{G}}_{p''_0}$ and set $\pi(p', p''_0) =: \hat{p} = (\hat{p}', \hat{p}'') \in \mathcal{G}_\Pi$ with $p' \in BA'(\hat{p}')$ (with obvious notation). Hence $f^j(p^j) = \hat{f}^j(p', p''_0) = f^j(\hat{p}) = f^j(\hat{p}^j)$ for $j = 1, \dots, n$. Because we have $BA^j(\hat{p}^j) \cap \left\{f^j = f^j(\hat{p}^j)\right\} = \left\{\hat{p}^j\right\}$ for all $j = 1, \dots, N$, we deduce that $p' = \hat{p}'$ with $(\hat{p}', \hat{p}'') \in \mathcal{G}_\Pi$. This implies the reverse inclusion $\mathcal{G}_{p''_0} \supset \tilde{\mathcal{G}}_{p''_0}$. This shows (4.12). Moreover, we get that $\hat{f}_{\mathcal{G}_{p''_0}} = (\hat{f}^j(\cdot, p''_0))_{j=1,\dots,n}$, i.e. (4.13).
This ends the proof of the lemma.

We now state the following result.

Proposition 4.13 (Basic monotonicities of $\hat{f}$, $N \geq 1$)
We work under assumptions of Proposition 4.10. Then we have on $[a,b]$

$$\left.\begin{array}{l} p \mapsto \hat{f}^j(p) \\ p \mapsto \pi^j(p) \end{array}\right\} \text{ are nondecreasing in } p^j \text{, for all } j \quad \textbf{(Basic monotonicities)}$$

Moreover we have

$$(4.14) \quad \left\{\begin{array}{l} \text{for all index } j \text{, the function } \hat{f}^j \text{ is locally Lipschitz continuous in the variable } p^j, \\ \text{and for all } q_0 \in [a,b] \text{ and for all } j \text{, we have with } \iota^j_{q_0}(p^j) := (q_0^1, \ldots, q_0^{j-1}, p^j, q_0^{j+1}, \ldots, q_0^N) \\ \\ \max(0, (f^j)') \geq (\hat{f}^j_{q_0})' \geq 0 \quad \text{a.e. on } [a^j, b^j] \text{ with } \hat{f}^j_{q_0} := \hat{f}^j \circ \iota^j_{q_0} \end{array}\right.$$

and for all $q_0 \in [a,b]$ and all index j we have

$$\left((\hat{f}^j_{q_0})' \in \left\{0, \min(0,(f^j)')\right\} \quad \text{and } (\hat{f}^j_{q_0})' = (f^j)' > 0 \text{ implies } \hat{f}^j_{q_0} = f^j\right) \quad \text{holds a.e. on } [a^j, b^j]$$

Proof of Proposition 4.13
Applying the Slicing Lemma 4.12 with $n = 1$, we get for $j = 1$, that $\hat{f}^j_{q_0}$ is the flux function at the junction $1:0$ associated to a generalized Riemann germ. Applying vi) of Proposition 3.6 and Proposition 3.8, we deduce the result for $j = 1$. Up to relabel the indices, we get the result for all indices $j = 1, \ldots, N$. This ends the proof of the lemma.

4.5 Theorem 2.15 and its proof: characterization of generalized Riemann germs

We first start with the following result and then give the proof of Theorem 2.15.

Proposition 4.14 (Generating generalized Riemann germs, $N \geq 1$)
Assume (2.2) for a $N:0$ junction with $N \geq 1$ and let $\mathcal{G} \subset [a,b]$ be a set.
Then $\mathcal{G}$ is a generalized Riemann germs if and only if $\mathcal{G} = \mathcal{G}_{\hat{f}} := \left\{\hat{f} = f\right\}$ for some function $\hat{f} : [a,b] \to \mathbb{R}^N$ which is locally constant on $\left\{\hat{f} \neq f\right\}$ and satisfying for all j

$$(4.15) \quad \left\{\begin{array}{lll} p^j \mapsto \hat{f}^j(p) \quad \text{is nondecreasing on} & [a,b] & \textbf{(Basic monotonicities)} \\ f_- \leq \hat{f} \leq f_+ & & \textbf{(Monotone bounds)} \end{array}\right.$$

with $f_\pm$ defined in (4.6).

Proof of Proposition 4.14
Part I: properties of generalized Riemann germs
Notice that if $\mathcal{G}$ is a generalized Riemann germ, then Proposition 4.10 and Proposition 4.13 shows that $\hat{f} := \hat{f}_\mathcal{G}$ satisfies all the required conditions.
Part II: constructing a generalized Riemann germ
Conversely, assume that $\tilde{f} : [a,b] \to \mathbb{R}^N$ is locally constant on $\left\{\tilde{f} \neq f\right\}$ and satisfies (4.15) for all index j. Then let $\mathcal{G} := \mathcal{G}_{\tilde{f}} := \left\{\tilde{f} = f\right\}$. Let us check that $\mathcal{G}$ is a generalized Riemann germ, i.e. that $\mathcal{G}$ satisfies the following singleton property $\mathcal{G} \cap \hat{\mathcal{P}}_p = \{\hat{p}\}$ for all $p \in [a,b]$.
Step 1: nonemptyness of $\mathcal{G} \cap \hat{\mathcal{P}}_p$
Recall that $f_- \leq \tilde{f} \leq f_+$. We first define $\pi(p) := \hat{p}$ for $p \in [a,b]$. For each index j, setting $\lambda := \tilde{f}^j(p)$, we define

$$\hat{p}^j := \pi^j(p) := \left\{\begin{array}{ll} p^j & \text{if } \quad f^j(p^j) = \lambda \\ \sup\left\{q^j \in [p^j, b^j], \quad f^j > \lambda \quad \text{on} \quad (p^j, q^j]\right\} & \text{if } \quad f^j(p^j) > \lambda \\ \inf\left\{q^j \in [a^j, p^j], \quad f^j < \lambda \quad \text{on} \quad [q^j, p^j)\right\} & \text{if } \quad f^j(p^j) < \lambda \end{array}\right.$$

Exactly as in Step 1 of the proof of Proposition 3.9, we conclude that $\mathcal{G} \cap \hat{\mathcal{P}}_p \supset \{\hat{p}\}$.
Step 2: $\mathcal{G} \cap \hat{\mathcal{P}}_p$ is reduced to a singleton
Exactly as in Step 2 of the proof of Proposition 3.9, we conclude that $\mathcal{G} \cap \hat{\mathcal{P}}_p = \{\hat{p}\}$. which shows that $\mathcal{G}$ is a generalized Riemann germ. This ends the proof of the proposition.

Proof of Theorem 2.15
For the proof we refer to the table of Subsection 2.5. Point i) of Theorem 2.15 follows from ii) of Lemma 4.9, while point ii) of Theorem 2.15 follows from follows from Proposition 4.14. This ends the proof of the theorem.

4.6 Partial relaxation formula

Proposition 4.15 (Partial relaxation formula, $N \geq 1$)
Assume (2.2) for a $N:0$ junction with $N \geq 1$. Let $\mathcal{G} \subset [a,b]$ be a generalized Riemann germ and $\hat{f} := \hat{f}_{\mathcal{G}}$. For any $p \in [a,b]$ and index j, we define for all $q^j \in [a^j, b^j]$

$$\hat{f}^j_p := \hat{f}^j \circ \iota^j_p \quad \textit{with} \quad \iota^j_p(q^j) := \hat{f}^j(p^1, \ldots, p^{j-1}, q^j, p^{j+1}, \ldots, p^N)$$

Then for all index j, the function $\hat{f}^j_p$ satisfies

$$(4.16) \qquad \begin{cases} \hat{f}^j_p : [a^j, b^j] \to \mathbb{R} \textit{ is continuous nondecreasing} & \textbf{(Monotonicity)} \\ \hat{f}^j_{p,-} \leq \hat{f}^j_p \leq \hat{f}^j_{p,+} & \textbf{(Bounds)} \\ \hat{f}^j_{p,-}(q^j) = \inf\limits_{[q^j, b^j]} \hat{f}^j_p, & \\ \hat{f}^j_{p,+}(q^j) = \sup\limits_{[a^j, q^j]} \hat{f}^j_p & \end{cases}$$

and we have the following partial relaxation formula

$$(4.17) \qquad \left\{ \hat{f}^j_p(p^j) \right\} = \bigcup_{q^j \in [a^j, b^j]} \left\{ G^j(p^j, q^j) \right\} \cap \left\{ \hat{f}^j_p(q^j) \right\}$$

Proof of Proposition 4.15
The result follows from the slicing lemma 4.12, considering $\mathcal{G}^j_p := \left\{ q^j \in [a^j, b^j],\ \hat{f}^j_p(q^j) = f^j(q^j) \right\}$, which is a generalized Riemann germ in $[a^j, b^j]$. Moreover the associated effective flux function is $\hat{f}_{\mathcal{G}^j_p} = \hat{f}^j_p$. Then (4.16) and (4.17) follow from Proposition 3.10. This ends the proof of the lemma.

4.7 Theorem 2.17 and its proof: characterization of Riemann germs

Before to start the proof of Theorem 2.17, recall that we consider Kružkov entropy solution

$$u^{\mathcal{G}}_{p,\hat{p}} := (u^1_{p^1,\hat{p}^1}, \ldots, u^N_{p^N,\hat{p}^N}) \in L^1_{loc}\left([0,+\infty) \times (-\infty, 0)\right)^N \simeq L^1_{loc}\left([0,+\infty) \times J\right)$$

to the $\mathcal{G}$-Riemann problem (2.6).

Proof of Theorem 2.17
Part I: proof of i)
We only do the proof for $N:0$ junctions. The general case of junctions $n:m$ is recovered from $N:0$ with $N := n+m$, using suitable I-reversions (see Lemma 4.5).
Step 1: necessary condition
Assume that $\mathcal{G} \subset [a,b]$ is a Riemann germ. For $p_n \to p_\infty$, we set $\tilde{p}_\infty :=: \pi(p_\infty)$, and up to extract a subsequence (still denoted by p_n), we have $\hat{p}_n := \pi(p_n) \to \hat{p}_\infty \in [a,b]$. By Definition 2.5 of Riemann germs, we have

$$u^{\mathcal{G}}_{p_n,\hat{p}_n} \to u^{\mathcal{G}}_{p_\infty,\tilde{p}_\infty} \quad \text{in} \quad L^1_{loc}([0,+\infty) \times J), \quad \text{as} \quad n \to +\infty$$

Hence for $p_n \in BA(\hat{p}_n)$ and from the weak stability Lemma 3.11, we conclude that $f(\hat{p}_\infty) = f(\tilde{p}_\infty)$. By definition, this shows that $\hat{f}(p_n) = f(\hat{p}_n) \to f(\hat{p}_\infty) = f(\tilde{p}_\infty) = f(\pi(p_\infty)) = \hat{f}(p_\infty)$, which shows the continuity of $\hat{f}$.
Step 2: sufficient condition
Let $\mathcal{G}$ be a generalized Riemann germ and assume that $\hat{f} : [a,b] \to \mathbb{R}^N$ is continuous. We want to show that $\mathcal{G}$ is a Riemann germ. Because $\mathcal{G} = \left\{ \hat{f} = f \right\}$, we deduce that $\mathcal{G}$ is a closed set. Now if $p_n \in [a,b]$ satisfies $p_n \to p_\infty \in [a,b]$, then, up to extract a subsequence, we have $\mathcal{G} \ni \pi(p_n) =: \hat{p}_n \to \hat{p}_\infty \in \mathcal{G}$. From the weak

stability Lemma 3.11, there exists $\tilde{p}_\infty \in [a,b]$ with $p_\infty \in BA(\tilde{p}_\infty)$ such that, up to extract a subsequence, we have

$$u_n := u^{\mathcal{G}}_{p_n,\hat{p}_n} \to u_\infty := u_{p_\infty,\tilde{p}_\infty} \quad \text{in} \quad L^1_{loc}([0,+\infty)\times J), \quad \text{as} \quad n \to +\infty$$

and we want to show that $u_{p_\infty,\tilde{p}_\infty} = u^{\mathcal{G}}_{p_\infty,\pi(p_\infty)}$, i.e. that $\tilde{p}_\infty = \pi(p_\infty)$. From Lemma 3.11, we also know that

$$f(\hat{p}_\infty) = f(\tilde{p}_\infty) \quad \text{with} \quad \tilde{p}_\infty \in \{p_\infty, \hat{p}_\infty\} \tag{4.18}$$

Up to extract a subsequence (still denoted by p_n), we can assume that there exists $s = (s^j)_{j=1,\dots,N} \in \{\pm 1\}^N$

such that for all index j, we have

$$s^j(\hat{p}^j_n - p^j_n) \le 0 \quad \text{for all } n \tag{4.19}$$

We set $I := \left\{j \in \{1,\dots,N\}, \quad s^j = -1\right\}$. Then, up to use the I-inversion and (4.19), we can assume that $\hat{p}_n \le p_n$. Hence

$$\hat{f} = \hat{\lambda}_n := f(\hat{p}_n) \quad \text{on} \quad BA(\hat{p}_n) \supset [\hat{p}_n, p_n)$$

Passing to the limit, we get from the continuity of $\hat{f}$ that

$$\hat{f} = \hat{\lambda}_\infty = f(\hat{p}_\infty) \quad \text{on} \quad [\hat{p}_\infty, p_\infty] \supset \{\tilde{p}_\infty\}$$

where we have used (4.18) for the last inclusion. Using again (4.18), we get $f(\tilde{p}_\infty) = f(\hat{p}_\infty) = \hat{\lambda}_\infty = \hat{f}(\tilde{p}_\infty)$. This shows that $\tilde{p}_\infty \in \mathcal{G}$. Hence $p_\infty \in BA(\tilde{p}_\infty)$ implies $\tilde{p}_\infty \in \mathcal{G} \cap \hat{\mathcal{P}}_{p_\infty} = \{\pi(p_\infty)\}$, and then $\tilde{p}_\infty = \pi(p_\infty)$ which shows $u_\infty = u_{p_\infty,\tilde{p}_\infty} = u^{\mathcal{G}}_{p_\infty,\pi(p_\infty)}$. This shows that the limit u_∞ is unique, and independent of the extracted subsequence. Therefore, the full sequence u_n converges towards u_∞. This establishes that $\mathcal{G}$ is a Riemann germ.

Part II: proof of ii)

The result follows from iv) of Proposition 4.10. This ends the proof of the proposition.

4.8 Dissipation properties for $N:0$ junctions

Lemma 4.16 (Dissipative points are in the germ)

Assume (2.2) for $N:0$ junction with $N \ge 1$. Assume that $\mathcal{G} \subset [a,b]$ is a generalized Riemann germ, and let $\hat{f}$ be its associated effective flux function. Let $\hat{p},\hat{q} \in \mathcal{G}$ and $p \in BA(\hat{p})$, $q \in BA(\hat{q})$. Recall that $\sigma^j = +1$,

$$D^{f^j}(p,q) := sign(p^j - q^j) \cdot \left\{f^j(p) - f^j(q)\right\} \quad \text{and} \quad D^{f^j}_+(p,q) := sign^+(p^j - q^j) \cdot \left\{f^j(p) - f^j(q)\right\}$$

i) (Dissipative points are in the germ)

We have for all index j

$$D^{f^j}(p,\hat{p}) \le 0 \tag{4.20}$$

and

$$\sum_{j=1,\dots,N} D^{f^j}(p,\hat{p}) =: D^f(p,\hat{p}) \ge 0 \quad \Longrightarrow \quad p = \hat{p} \in \mathcal{G}$$

ii) (Properties of $D^{\hat{f}^j}$ and $D^{\hat{f}^j}_+$)

We have for all index j

$$D^{\hat{f}^j}(p,q) \ge D^{f^j}(\hat{p},\hat{q}) \quad \text{and} \quad D^{\hat{f}^j}_+(p,q) \ge D^{f^j}_+(\hat{p},\hat{q}) \tag{4.21}$$

Proof of Lemma 4.16

Step 1: proof of i)

This is a corollary of Lemma 3.4.

Step 2: proof of ii)

We only do the proof for D_+, because $D(p,q) = D_+(p,q) + D_+(q,p)$ then implies the result for D.

We set $\lambda_p := \hat{f}(p)$, $\lambda_q := \hat{f}(q)$, and $\hat{p}^j = \hat{p}^j_{\lambda^j_p,p^j}$, $\hat{q}^j = \hat{p}_{\lambda^j_q,q^j}$. Recall that (here with $\sigma^j = +1$)

$$D^j_+ := D^{\hat{f}^j}_+(p,q) = \left\{\text{sign}^+(p^j - q^j)\right\} \cdot \left\{\hat{f}^j(p) - \hat{f}^j(q)\right\}$$

and let us set

$$\hat{D}^j_+ := D^{f^j}_+(\hat{p},\hat{q}) = \left\{\text{sign}^+(\hat{p}^j - \hat{q}^j)\right\} \cdot \left\{f^j(\hat{p}) - f^j(\hat{q})\right\}$$

Because $\hat{f}(p) = \hat{f}(\hat{p})$ and the same for q, we deduce that only the change of value of $\text{sign}^+(p^j - q^j) \neq \text{sign}^+(\hat{p}^j - \hat{q}^j)$ can affect the difference $D^j_+ - \hat{D}^j_+$. We then distinguish the only two cases where this happens.

Case A: $p^j > q^j$ and $\hat{p}^j \le \hat{q}^j$

This means that $\hat{p}^j = \hat{p}_{\lambda^j_p, p^j} \le \hat{p}_{\lambda^j_q, q^j} = \hat{q}^j$. Then either $\hat{p}^j < \hat{q}^j$ or $\hat{p}^j = \hat{q}^j$, and from the monotonicities of the map $(\lambda^j, p^j) \mapsto \hat{p}^j_{\lambda^j, p^j}$ which are $(\downarrow, \uparrow)$, we deduce that

$$\lambda^j_p > \lambda^j_q \quad \text{or} \quad \left(\hat{p}^j = \hat{q}^j \quad \text{and} \quad \lambda^j_p = \lambda^j_q\right)$$

i.e.

$$D^j_+ > 0 = \hat{D}^j_+ \quad \text{or} \quad D^j_+ = 0 = \hat{D}^j_+$$

Case B: $p^j \le q^j$ and $\hat{p}^j > \hat{q}^j$

From the monotonicities, we deduce that $\lambda^j_p < \lambda^j_q$. Hence $D^j_+ = 0 > \hat{D}^j_+$.

Conclusion

In all cases, we deduce that $D^j_+ \ge \hat{D}^j_+$ which shows (4.21).

This ends the proof of the lemma.

4.9 Characterization of Kružkov functions and technical approximations

Notice that while D^{f^j} is a continuous function, $D^{\hat{f}^j}$ is not continuous in general. For later use, we will need the following technical result.

Lemma 4.17 (Approximation of $D^{\hat{f}}$ and $D^{\hat{f}}_+$)

Assume (2.2) with $N \ge 1$. Consider the relative open set Ω of $[a,b]^2$ defined by

$$\Omega := \left\{(p,q) \in [a,b]^2, \quad p^j \neq q^j \quad \textit{for all} \quad j = 1,\dots,N\right\} \tag{4.22}$$

and any continuous function $\hat{f} : [a,b] \to \mathbb{R}^N$. Then for all $(p,q) \in [a,b]^2$, there exists a sequence $(p_\delta, q_\delta) \in \Omega$ such that $(p_\delta, q_\delta) \to (p,q)$ as $\delta \to 0^+$ and

$$D^{\hat{f}}(p,q) \ge \liminf_{\delta \to 0^+} D^{\hat{f}}(p_\delta, q_\delta) \tag{4.23}$$

and

$$D^{\hat{f}}_+(p,q) \ge \liminf_{\delta \to 0^+} D^{\hat{f}}_+(p_\delta, q_\delta) \tag{4.24}$$

Proof of Lemma 4.17

We only do the proof for $0 : N$ junctions (because the signs of the present computation are then more natural than for $N : 0$ junctions). The general case can be recovered using reversion transforms. We first notice that $D^{\hat{f}}$ is continuous on Ω. Assume now that $(p,q) \in [a,b]^2 \backslash \Omega$, and define the sets

$$I_\pm := \left\{j \in \{1,\dots,N\}, \quad \pm(p^j - q^j) > 0\right\}, \quad I_0 := \left\{j \in \{1,\dots,N\}, \quad (p^j - q^j) = 0\right\}$$

and

$$I_{0,\pm} := \left\{j \in I_0, \quad \pm\left\{\hat{f}^j(p) - \hat{f}^j(q)\right\} > 0\right\}, \quad I_{0,0} := \left\{j \in I_0, \quad \left\{\hat{f}^j(p) - \hat{f}^j(q)\right\} = 0\right\}$$

We set

$$h := p - q, \quad \bar{h} := \sum_{j \in I_{0,+} \cup I_{0,0}} e_j - \sum_{j \in I_{0,-}} e_j$$

Notice that for $\delta > 0$, we have

$$p - q + \delta \bar{h} = h + \delta \bar{h} \in E_K := E_{K_+} + E_{K_-} \subset \mathbb{R}^N \quad \text{with} \quad E_{K_\pm} := \pm \sum_{j \in K_\pm} (0, +\infty) e_j \tag{4.25}$$

and

$$K_+ \cap K_- = \emptyset, \quad K_+ \cup K_- = \{1, \dots, N\} \quad \text{with} \quad K_+ := I_+ \cup I_{0,+} \cup I_{0,0}, \quad K_- := I_- \cup I_{0,-}$$

If p, q belong to the interior of the box $[a, b]$, then we can consider the couple $(p + \delta \bar{h}, q) \in \Omega$ for $\delta > 0$ small enough. In general, we may have $a, b \in \partial[a, b]$. Nevertheless in all cases, we can find $\bar{p}, \bar{q} \in \mathbb{R}^N$ such that $\bar{p} - \bar{q} = \bar{h}$, and for $\delta > 0$ small enough, we have

$$p_\delta := p + \delta \bar{p}, \quad q_\delta := q + \delta \bar{q}, \quad \text{satisfy} \quad p_\delta, q_\delta \in [a, b]$$

Because $p_\delta - q_\delta = h + \delta \bar{h} \in E_K$, we deduce that $(p_\delta, q_\delta) \in \Omega$, and then

$$\begin{aligned} \lim_{\delta \to 0^+} D^{\hat{f}}(p_\delta, q_\delta) &= - \left\{ \sum_{j \in K_+} \left\{ \hat{f}^j(p) - \hat{f}^j(q) \right\} - \sum_{j \in K_-} \left\{ \hat{f}^j(p) - \hat{f}^j(q) \right\} \right\} \\ &= D^{\hat{f}}(p, q) - \sum_{j \in I_{0,+} \cup I_{0,0}} [\hat{f}^j]_q^p + \sum_{j \in I_{0,-}} [\hat{f}^j]_q^p \\ &\le D^{\hat{f}}(p, q) \end{aligned}$$

where we have used the definition of $I_{0,\pm}$ and $I_{0,0}$ in the last line. This shows (4.23).
Similarly, we have

$$\begin{aligned} \lim_{\delta \to 0^+} D_+^{\hat{f}}(p_\delta, q_\delta) &= - \sum_{j \in K_+} \left\{ \hat{f}^j(p) - \hat{f}^j(q) \right\} \\ &= D_+^{\hat{f}}(p, q) - \sum_{j \in I_{0,+} \cup I_{0,0}} [\hat{f}^j]_q^p \\ &\le D_+^{\hat{f}}(p, q) \end{aligned}$$

which shows (4.24). This ends the proof of the lemma.

Lemma 4.18 (A property of Kružkov functions for $0 : N$ junctions)
Assume (2.2) with $N \ge 1$ for some $0 : N$ junction. Consider a continuous function $\hat{f} : [a, b] \to \mathbb{R}^N$.
i) (Kružkov functions)
Then $\hat{f}$ satisfies

$$D^{\hat{f}} \ge 0 \quad \text{on} \quad [a, b]^2 \tag{4.26}$$

if and only if in the sense of distributions in $\mathcal{D}'(a, b)$ with $(a, b) := \prod_{j=1,\dots,N} (a^j, b^j)$

$$\left\{ \begin{array}{lll} \bar{S}_K := \sum_{j \in K} \hat{f}^j - \sum_{j \notin K} \hat{f}^j & & \\ \partial_k \bar{S}_K \le 0 & \text{for all} & k \in K \\ \partial_k \bar{S}_K \ge 0 & \text{for all} & k \notin K \end{array} \right| , \quad \text{for all} \quad K \subset \{1, \dots, N\} \tag{4.27}$$

ii) (Semi Kružkov functions)
Then $\hat{f}$ satisfies

$$D_+^{\hat{f}} \ge 0 \quad \text{on} \quad [a, b]^2 \tag{4.28}$$

if and only if in $\mathcal{D}'(a, b)$

$$\left\{ \begin{array}{lll} S_K := \sum_{j \in K} \hat{f}^j & & \\ \partial_k S_K \le 0 & \text{for all} & k \in K \\ \partial_k S_K \ge 0 & \text{for all} & k \notin K \end{array} \right| , \quad \text{for all} \quad K \subset \{1, \dots, N\} \tag{4.29}$$

Proof of Lemma 4.18
We first do the proof in part I, assuming more regularity on $\hat{f}$, and then in Part II for $\hat{f}$ only continuous.
Part I: assuming $\hat{f}$ locally Lipschitz continuous
Step 1: proof of i)
Step 1.1: (4.26) implies (4.27)

Let us consider two subsets $K_\pm \subset \{1,\dots,N\}$ such that $K_+ \cap K_- = \emptyset$, a point $p \in (a,b) := \prod_{j=1,\dots,N} (a^j, b^j)$ and $h \in E := E_{K_+} + E_{K_-} \subset \mathbb{R}^N$ with $E_{K_\pm} := \pm \sum_{j\in K_\pm} (0,+\infty) e_j$ and $\varepsilon > 0$ small enough such that we have $p_\varepsilon := p + \varepsilon h \in (a,b)$. Then we have $0 \le D^{\hat f}(p_\varepsilon, p) = -\left\{ \sum_{j\in K_+} \left\{ \hat f^j(p_\varepsilon) - \hat f^j(p) \right\} - \sum_{j\in K_-} \left\{ \hat f^j(p_\varepsilon) - \hat f^j(p) \right\} \right\}$.

Setting $S_{K_+,K_-} := \sum_{j\in K_+} \hat f^j - \sum_{j\in K_-} \hat f^j$, we get

$$h \cdot DS_{K_+,K_-} \le 0 \quad \text{a.e. on} \quad (a,b), \quad \text{for all} \quad h \in E$$

Now choosing

$$h := \pm e_k + \delta \left\{ \sum_{j\in K_+} e_j - \sum_{j\in K_-} e_j \right\} \quad \text{if} \quad k \in K_\pm$$

then in the limit $\delta \to 0^+$, we get $\pm \partial_k S_{K_+,K_-} \le 0$ for all $k \in K_\pm$. If moreover we choose $K_\pm$ such that $K := K_+ = \{1,\dots,N\} \backslash K_-$, then we get relation (4.27).

Step 1.2: (4.27) implies (4.26)

Conversely, assume that (4.27) holds true with $K := K_+ := \{1,\dots,N\} \backslash K_-$. With the same notation as in Step 1 with $\varepsilon := 1$ and $q := p_\varepsilon = p + h$, and $h \in E := E_{K_+} + E_{K_-}$, we get

$$D^{\hat f}(q,p) = -\left\{ \bar S_K(q) - \bar S_K(p) \right\} = -\int_0^1 h \cdot D\bar S_K(p+th)\, dt \ge 0 \quad \text{with} \quad h := q - p \in E = E_{K_+} + E_{K_-}$$

For Ω defined in (4.22), this implies $D^{\hat f} \ge 0$ on Ω. From Lemma 4.17, we deduce that $D^{\hat f} \ge 0$ on $\overline{\Omega} = [a,b]^2$, which is (4.26).

Step 2: proof of ii)

The proof follows the same lines as the one of point i).

Step 2.1: (4.28) implies (4.29)

As in Step 1.1, we consider a point $p \in (a,b)$, $h \in E := E_{K_+} + E_{K_-} \subset \mathbb{R}^N$ and $p_\varepsilon := p + \varepsilon h \in (a,b)$. Then we get

$$\left\{ \begin{array}{l} 0 \le D_+^{\hat f}(p_\varepsilon, p) = -\sum_{j\in K_+} \left\{ \hat f^j(p_\varepsilon) - \hat f^j(p) \right\} \\ 0 \le D_+^{\hat f}(p, p_\varepsilon) = -\sum_{j\in K_-} \left\{ \hat f^j(p) - \hat f^j(p_\varepsilon) \right\} \end{array} \right.$$

Hence for $S_{K_\pm} := \sum_{j\in K_\pm} \hat f^j$, we get

$$\pm h \cdot DS_{K_\pm} \le 0 \quad \text{a.e. on} \quad (a,b), \quad \text{for all} \quad h \in E$$

Now choosing h as in Step 1.1, in the limit $\delta \to 0^+$, we get (4.29) with $K := K_+ = \{1,\dots,N\} \backslash K_-$.

Step 2.2: (4.29) implies (4.28)

Conversely, assume that (4.29) holds true with $K := K_+ = \{1,\dots,N\} \backslash K_-$. With the same notation as in Step 1.2 with $\varepsilon := 1$ and $q := p_\varepsilon = p + h$, we get

$$D_+^{\hat f}(q,p) = -\sum_{j\in K_+} \left\{ \hat f^j(q) - \hat f^j(p) \right\} = -\int_0^1 h \cdot D \left\{ \sum_{j\in K_+} \hat f^j(p+th) \right\} dt \ge 0 \quad \text{with} \quad h := q - p \in E = E_{K_+} + E_{K_-}$$

For Ω defined in (4.22), this implies $D_+^{\hat f} \ge 0$ on Ω. From Lemma 4.17, we deduce that $D_+^{\hat f} \ge 0$ on $\overline{\Omega} = [a,b]^2$, which is (4.28).

Part II: general continuous $\hat f$

We now only assume $\hat f$ continuous. Let $0 \le \rho_\varepsilon := \varepsilon^{-N} \rho(\varepsilon^{-1} \cdot)$ be a mollifier with $\text{supp}(\rho) \subset [-1,1]^N$. Extending $\hat f$ (for instance by zero outside the box $[a,b]$), and for p,q in the interior of the box $[a,b]$, and integrating $D^{\hat f}(p+\xi, q+\xi) \ge 0$ (resp. $D_+^{\hat f}(p+\xi, q+\xi) \ge 0$) over the measure $\rho_\varepsilon(\xi) d\xi$ and setting $\hat f^\varepsilon := \hat f \star \rho_\varepsilon$, we get $D^{\hat f^\varepsilon} \ge 0$ (resp. $D_+^{\hat f^\varepsilon} \ge 0$) on $[a,b]_\varepsilon := [a + \varepsilon(1,\dots,1), b - \varepsilon(1,\dots,1)]$.

Defining $\bar{S}_K^\varepsilon$ (resp. S_K^ε) as $\bar{S}_K$ (resp. S_K) with $\hat{f}^\varepsilon$ instead of $\hat{f}$, we see that Steps 1 and 2 do apply, and in the limit $\varepsilon \to 0$, we recover the desired result, using the continuity of $\hat{f}$.
This ends the proof of the lemma.

As a straightforward application of a reversion transform, we get

Corollary 4.19 (A property of Kružkov functions for $N : 0$ junctions)
Assume (2.2) with $N \geq 1$ for some $N : 0$ junction. Consider a continuous function $\hat{f} : [a, b] \to \mathbb{R}^N$.
i) (Kružkov functions)
Then $\hat{f}$ satisfies

$$D^{\hat{f}} \geq 0 \quad on \quad [a,b]^2 \tag{4.30}$$

if and only if in $\mathcal{D}'(a,b)$ with $(a,b) := \prod_{j=1,\ldots,N}(a^j, b^j)$

$$\left\{\begin{array}{lll} \bar{S}_K := \sum_{j\in K} \hat{f}^j - \sum_{j\notin K} \hat{f}^j & & \\ \partial_k \bar{S}_K \geq 0 & for\ all & k \in K \\ \partial_k \bar{S}_K \leq 0 & for\ all & k \notin K \end{array}\right| , \quad for\ all \quad K \subset \{1, \ldots, N\} \tag{4.31}$$

ii) (Semi Kružkov functions)
Then $\hat{f}$ satisfies

$$D_+^{\hat{f}} \geq 0 \quad on \quad [a,b]^2 \tag{4.32}$$

if and only if in $\mathcal{D}'(a,b)$

$$\left\{\begin{array}{lll} S_K := \sum_{j\in K} \hat{f}^j & & \\ \partial_k S_K \geq 0 & for\ all & k \in K \\ \partial_k S_K \leq 0 & for\ all & k \notin K \end{array}\right| , \quad for\ all \quad K \subset \{1, \ldots, N\} \tag{4.33}$$

4.10 First application to Kružkov germs

Proposition 4.20 (Characterization of Kružkov germs among generalized Riemann germs)
Assume (2.2) with $N \geq 1$.
i) (Kružkov germs)
Let $\mathcal{G} \subset [a, b]$ be a generalized Riemann germ, and let $\hat{f} := \hat{f}_\mathcal{G}$ be its associated generalized Godunov flux. Then $\mathcal{G}$ is a Kružkov germ if and only if $\hat{f}$ is locally Lipschitz continuous on $[a, b]$ and its Jacobian matrix is **diagonally dominant***, i.e. precisely*

$$\sigma^i \partial_i \hat{f}^i \geq \sum_{j\in\{1,\ldots,N\}\setminus\{i\}} |\partial_i \hat{f}^j| \quad a.e.\ on \quad [a,b], \quad i = 1, \ldots, N \tag{4.34}$$

ii) (General continuous Kružkov functions, which are partially Lipschitz)
Assume that $h : [a, b] \to \mathbb{R}^N$ is a continuous map. Assume also that the map $p \mapsto h^j(p)$ is locally Lipschitz continuous in p^j uniformly in the other coordinates p^k for $k \neq j$, and for all $j = 1, \ldots, N$.

Then $D^h \geq 0$ on $[a, b]^2$ if and only if h is locally Lipschitz continuous and satisfies

$$\sigma^i \partial_i h^i \geq \sum_{j\in\{1,\ldots,N\}\setminus\{i\}} |\partial_i h^j| \quad a.e.\ on \quad [a,b], \quad i = 1, \ldots, N \tag{4.35}$$

Proof of Proposition 4.20
Part I: proof of i)
We first recall that from Lemma 5.5, the generalized Riemann germ $\mathcal{G}$ is a Kružkov germ if and only if it satisfies $D^{\hat{f}} \geq 0$ on $[a, b]^2$ for $\hat{f} := \hat{f}_\mathcal{G}$. We only do the proof for $N : 0$ junctions. The general case then follows by reversion transforms.
Step 1: necessary condition
Step 1.1: interior regularization

Consider a Kružkov germ $\mathcal{G}$. Then recall that $D^{\hat f} \geq 0$ implies for all points p, q in the interior of the box $[a, b]$ with $\sigma^j = 1$ for all j

$$D^{\hat f}(p+\xi, q+\xi) = \sum_{j=1,\dots,N} \text{sign}(p^j - q^j) \cdot \left\{ \hat f^j(p+\xi) - \hat f^j(q+\xi) \right\} \geq 0 \tag{4.36}$$

Let $0 \leq \rho_\varepsilon := \varepsilon^{-N} \rho(\varepsilon^{-1} \cdot)$ be a mollifier with $\text{supp}(\rho) \subset [-1,1]^N$. Extending $\hat f$ (for instance by zero outside the box $[a, b]$), and integrating (4.36) over the measure $\rho_\varepsilon(\xi) d\xi$ and setting $\hat f^\varepsilon := \hat f \star \rho_\varepsilon$, we get $D^{\hat f^\varepsilon} \geq 0$ on $[a, b]_\varepsilon := [a + \varepsilon(1, \dots, 1), b - \varepsilon(1, \dots, 1)]$.

Step 1.2: application

Recall that Corollary 4.19 holds for $N : 0$ juntions, and (4.31) means for $\hat f^\varepsilon$ that a.e. on $[a, b]_\varepsilon$

$$\left\{ \begin{array}{ll} \bar S_K := \sum_{j \in K} (\hat f^\varepsilon)^j - \sum_{j \notin K} (\hat f^\varepsilon)^j & \\ \partial_k \bar S_K \geq 0 & \text{for all} \quad k \in K \\ \partial_k \bar S_K \leq 0 & \text{for all} \quad k \notin K \end{array} \right| , \quad \text{for all} \quad K \subset \{1, \dots, N\} \tag{4.37}$$

Fix a point $p \in [a, b]$ where $\hat f^\varepsilon$ has a derivative, and fix the index 1. Let $K_-(p) := \left\{ j \in \{2, \dots, N\}, \quad \partial_1 (\hat f^\varepsilon)^j(p) \leq 0 \right\}$. Then for $K := K(p) := K_-(p) \cup \{1\}$, we get from (4.37)

$$\partial_1 (\hat f^\varepsilon)^1(p) + \sum_{j \in K_-(p)} \partial_1 (\hat f^\varepsilon)^j(p) - \sum_{j \in \{2,\dots,N\} \setminus K_-(p)} \partial_1 (\hat f^\varepsilon)^j(p) \geq 0$$

i.e.

$$\partial_1 (\hat f^\varepsilon)^1 \geq \sum_{j \in \{2,\dots,N\}} |\partial_1 (\hat f^\varepsilon)^j| \quad \text{a.e. on} \quad [a, b]_\varepsilon \tag{4.38}$$

The same result holds true for all other index than 1, which shows that $\hat f^\varepsilon$ satisfies (4.34) on $[a, b]_\varepsilon$.

Step 1.3: the limit $\varepsilon \to 0$

We also know that $0 \leq \partial_j \hat f^j \leq |(f^j)'|$ a.e. for $N : 0$ junctions. Joint to (4.38), this shows that $\hat f^\varepsilon$ is locally Lipschitz continuous, uniformly in ε in the interior of the box $[a, b]$. From Lemma 6.3, we know that $\hat f$ is continuous on $[a, b]$. Hence $\hat f^\varepsilon \to \hat f$ locally uniformly in the interior of the box $[a, b]$, and we conclude that $\hat f$ is locally Lipschitz continuous in the interior of the box $[a, b]$. Then, inside the interior of the box $[a, b]$, we can apply Corollary 4.19 to $\hat f$ as in Step 1.2. This shows (4.34) for $\sigma^j = 1$ for all j. Finally, the continuity of $\hat f$ on $[a, b]$ implies that $\hat f$ is locally Lipschitz continuous on the whole box $[a, b]$.

Step 2: sufficient condition

Assume now that $\hat f$ is locally Lipschitz continuous on $[a, b]$ and satisfies (4.34) for $\sigma^j = 1$ for all j. Then this implies immediately (4.37), which from Corollary 4.19 is a characterization of $D^{\hat f} \geq 0$ for a $N : 0$ junctions. Finally recall that Lemma 5.5 shows that $\mathcal{G}$ is a Kružkov germ.

Part II: proof of ii)

The same argument as in Part I applies and give the result. This ends the proof of the proposition.

Lemma 4.21 (A contraction property for Kružkov functions)

Assume that $h : [a, b] \to \mathbb{R}^N$ is a continuous map. Assume also that the map $p \mapsto h^j(p)$ is locally Lipschitz continuous in p^j uniformly in the other coordinates p^k for $k \neq j$, and for all $j = 1, \dots, N$. Assume also that h is Kružkov, i.e. satisfies $D^h \geq 0$ on $[a, b]^2$. Let $\varepsilon > 0$ and assume also that for $p, p' \in [a, b]$ satisfy

$$p + \varepsilon h(p) = q \quad \textit{and} \quad p' + \varepsilon h(p') = q'$$

Then

$$|p' - p|_1 \leq |q' - q|_1 \quad \textit{with} \quad |z|_1 := \sum_{j=1,\dots,N} |z^j|$$

In particular the map $(Id + \varepsilon h)^{-1}$ is a 1-Lipschitz map with respect to the norm $|\cdot|_1$.

Proof of Lemma 4.21

We have

$$(p' - p) + \varepsilon \{h(p') - h(p)\} = q' - q$$

We set $\bar{p} := p' - p$ and $\bar{q} := q' - q$. Taking the scalar product with the vector $\bar{s} := (\text{sign}(\bar{p}^1), \dots, \text{sign}(\bar{p}^N))^T$, we get

$$\begin{aligned} |\bar{p}|_1 &= \bar{s} \cdot \bar{p} \\ &\leq \bar{s} \cdot \bar{p} + \varepsilon D^h(p', p) \\ &= \bar{s} \cdot \bar{p} + \varepsilon \bar{s} \cdot \{h(p') - h(p)\} \\ &= \bar{s} \cdot \bar{q} \\ &\leq |\bar{q}|_1 \end{aligned}$$

where in the second line we have used $D^h \geq 0$. This implies the result and ends the proof of the lemma.

5 Gluing and Riemann monotonicity

5.1 Riemann monotonicity

Proposition 5.1 (The map π is Riemann monotone on $N:0$ junctions)
Assume (2.2) for a $N:0$ junction with $N \geq 1$. Let $\mathcal{G}$ be a generalized Riemann germ, and let $\pi := \pi_{\mathcal{G}}$ its associated projection. Then the map π is Riemann monotone in the sense of Definition 2.12.

Proof of Proposition 5.1
For all $p, q \in [a, b]$, we set $\hat{p} := \pi(p)$, $\hat{q} := \pi(q)$ and $[\pi]_q^p := \pi(p) - \pi(q) = \hat{p} - \hat{q}$. Assume by contradiction that π is not Riemann monotone, i.e. that we have

$$(5.1) \qquad (p - q) \diamond [\pi]_q^p \leq 0 \quad \text{and} \quad [\pi]_q^p \neq 0$$

Recall that

$$BA(\hat{p}) \cap BA(\hat{q}) \neq \emptyset \quad \Longrightarrow \quad \hat{p} = \hat{q} \quad \Longleftrightarrow \quad [\pi]_q^p = 0$$

From (5.1), we then deduce

$$(5.2) \qquad BA(\hat{p}) \cap BA(\hat{q}) = \emptyset$$

We set

$$I := \left\{k \in \{1, \dots, N, \quad p^k \geq q^k\}\right\}, \quad \bar{I} := \left\{k \in \{1, \dots, N, \quad p^k < q^k\}\right\}$$

and then $I \cup \bar{I} = \{1, \dots, N\}$. Up to apply a $\bar{I}$-inversion (see Definition 4.1), we can assume that

$$(5.3) \qquad p \geq q, \quad \hat{p} \leq \hat{q}, \quad \hat{p} \neq \hat{q}$$

We now do the proof by recurrence on $N \geq 1$.
Step 1: Case $N = 1$
Then we get $p \geq q$, $\pi(p) \leq \pi(q)$. Because π is nondecreasing (see vi) of Proposition 3.6), we get $\hat{p} = \pi(p) = \pi(q) = \hat{q}$. Contradiction.
Step 2: Case $N \geq 2$, and Proposition 5.8 holds true for $N' := N - 1$
Step 2.1: Case of the hyperplane intersection
Step 2.1.1: setting of the problem
Assume by contradiction that there exists an index $k_0 \in \{1, \dots, N\}$ and some $p_0^{k_0} \in [a^{k_0}, b^{k_0}]$ such that the "hyperplane" intersection with the box $[a, b]$ defined as $\Pi_{k_0} := \left\{r \in [a, b],\ r^{k_0} = p_0^{k_0}\right\}$ satisfies

$$(5.4) \qquad \begin{cases} \Pi_{k_0} \cap BA(\hat{p}) \neq \emptyset \\ \Pi_{k_0} \cap BA(\hat{q}) \neq \emptyset \end{cases}$$

Up to relabel the indices, we can assume that $k_0 = N$, and then (5.4) means

$$(5.5) \qquad BA(\hat{p}^N) \cap BA(\hat{q}^N) \neq \emptyset$$

From the slicing Lemma 4.12, we know that the flux function $\hat{f}_{\mathcal{G}}$ restricted to $\Pi_N \cap [a, b]$ is associated to a generalized Riemann germ $\mathcal{G}_{p_0^N} \subset \tilde{Q} := \prod_{k=1,\dots,N-1} [a_k, b_k]$, and we set the associated projection map $\tilde{\pi} : \tilde{Q} \to \mathcal{G}_{p_0^N}$ defined as $\tilde{\pi} := \pi_{\mathcal{G}_{p_0^N}} := \pi'(\cdot, p_0^N)$ for $\pi = (\pi', \pi^N)$. Moreover, writing

$$\hat{p} = (\hat{p}', \hat{p}^N), \quad \hat{q} = (\hat{q}', \hat{q}^N), \quad p = (p', p^N), \quad q = (q', q^N)$$

ans using $(p', p_0^N) \in BA(\hat{p})$ and $(q', p_0^N) \in BA(\hat{q})$, we get that $\hat{p}' = \tilde{\pi}(p')$, $\hat{q}' = \tilde{\pi}'(q') \in \mathcal{G}_{p_0^N}$. Then (5.3) implies $(p' - q') \diamond [\tilde{\pi}]_{q'}^{p'} \leq 0$. By recurrence assumption, notice that $\tilde{\pi}$ is Riemann monotone, and then $[\tilde{\pi}]_{q'}^{p'} = 0$, i.e. $\hat{p}' = \hat{q}'$.

Step 2.1.2: consequences

Hence we get $BA(\hat{p}) = Q' \times BA^N(\hat{p}^N)$, $BA(\hat{q}) = Q' \times BA^N(\hat{q}^N)$ with $Q' := BA(\hat{p}') = BA(\hat{q}')$. Hence (5.2) means $BA^N(\hat{p}^N) \cap BA^N(\hat{q}^N) = \emptyset$. Contradiction with (5.5). This implies there is no index k_0 such that (5.4) holds true, which is the next case.

Step 2.2: Case of no hyperplane intersection

We now assume that there is no index k_0 such that (5.4) holds true. Then this implies that $BA(\hat{p})$ and $BA(\hat{q})$ are well separated in all directions, and then in particular we have $p \geq q$, $\hat{p} < \hat{q}$. Again, because $BA(\hat{p})$ and $BA(\hat{q})$ are well separated in all directions, we deduce that

$$p^k \in BA(\hat{p}^k) < BA(\hat{q}^k) \ni q^k$$

which implies $p^k < q^k$. Contradiction.

Step 3: conclusion

Therefore (5.1) is false and we conclude that π is Riemann monotone. This ends the proof of the proposition.

Notice that if $\hat{f}$ is Riemann monotone on $[a, b]$, then f is in particular Riemann monotone on $\mathcal{G}$. Conversely, we also have

Lemma 5.2 (Transfer of Riemann monotonicity, $N : 0$ junctions)
Assume (2.2) for $N \geq 1$. Let $\mathcal{G} \subset [a, b]$ be a generalized Riemann germ.
Assume that f is Riemann monotone on $\mathcal{G}$. Then $\hat{f} := \hat{f}_{\mathcal{G}}$ is Riemann monotone on $[a, b]$.

Proof of Lemma 5.2

Assume that f is Riemann monotone on $\mathcal{G}$. This means that for all $\hat{p}, \hat{q} \in \mathcal{G}$, we have

$$(\hat{p} - \hat{q}) \diamond [f]_{\hat{q}}^{\hat{p}} \leq 0 \quad \Longrightarrow \quad [f]_{\hat{q}}^{\hat{p}} = 0 \tag{5.6}$$

Now consider any $p \in BA(\hat{p})$ and $q \in BA(\hat{q})$, and assume that $(p - q) \diamond [\hat{f}]_q^p \leq 0$. From ii) of Lemma 4.16 on dissipation properties, we deduce that $(\hat{p} - \hat{q}) \diamond [f]_{\hat{q}}^{\hat{p}} \leq 0$ and then (5.6) shows that $0 = [f]_{\hat{q}}^{\hat{p}} = [\hat{f}]_q^p$, i.e. that $\hat{f}$ is Riemann monotone. This ends the proof of the lemma.

We finish this subsection with the following results.

Lemma 5.3 (Injectivity of perturbed Riemann monotone functions for $N : 0$ junctions)
Assume (2.2) for $N \geq 1$. Assume that $h : [a, b] \to \mathbb{R}^N$ is Riemann monotone.
Then for any $\varepsilon > 0$, the function $h_\varepsilon := h + \varepsilon Id : [a, b] \to \mathbb{R}^N$ is injective.

Proof of Lemma 5.3

First, up to a reversion transform (see Definition 4.4), we can assume that $\sigma^j = 1$ for all indices j. Let $\varepsilon > 0$ and consider $p, q \in [a, b]$ such that $h_\varepsilon(p) = h_\varepsilon(q)$. Set

$$I := \left\{ j \in \{1, \ldots, N\}, \quad p^j < q^j \right\}$$

Then up to a I-inversion (see Definition 4.1), we can assume that $p \geq q$ and $h_\varepsilon(p) = h_\varepsilon(q)$. Hence

$$(p^j - q^j) \cdot [h^j]_q^p = (p^j - q^j) \cdot [h_\varepsilon^j]_q^p - \varepsilon (p^j - q^j)^2 = -\varepsilon (p^j - q^j)^2 \leq 0$$

Therefore $(p - q) \diamond [h]_q^p \leq 0$ and the Riemann monotonicity of h implies that $[h]_q^p = 0$. Therefore

$$\varepsilon [Id]_q^p = [h_\varepsilon]_q^p - [h]_q^p = 0$$

i.e. $p = q$, which shows the injectivity of h_ε. This ends the proof of the lemma.

Remark 5.4 *All results of Subsection 5.1 generalize to junctions of type $n : m$. This follows from the use of suitable I-reversions (see Definition 4.4 and Lemma 4.5).*

5.2 Proposition 2.14 and its proof: properties of generalized Riemann germs

We are now ready to give the proof of Proposition 2.14.

Proof of Proposition 2.14
The result follows from Propositions 4.10, 4.13, 5.1, 4.10 and Lemma 3.4. This ends the proof of the proposition.

5.3 Subclasses of germs and effect of slicing

Lemma 5.5 (Flux properties of subclasses of germs)
Assume (2.2) for $N \geq 1$. Let $\mathcal{G} \subset [a,b]$ be a generalized germ. Let $\hat{f} := \hat{f}_{\mathcal{G}}$. Then the following holds.
i) (Kružkov germs)
Then $\mathcal{G}$ is a Kružkov germ if and only if

$$D^{\hat{f}} \geq 0 \quad on \quad [a,b]^2 \tag{5.7}$$

with

$$D^{\hat{f}}(p,q) = \sum_{j=1,\dots,N} D^{\hat{f}^j}(p,q) \quad with \quad D^{\hat{f}^j}(p,q) = \sigma^j \cdot sign(p^j - q^j) \cdot \left\{\hat{f}^j(p) - \hat{f}^j(q)\right\}$$

i') (D_+-germs)
Then $\mathcal{G}$ is a D_+-germ if and only if

$$D_+^{\hat{f}} \geq 0 \quad on \quad [a,b]^2 \tag{5.8}$$

with

$$D_+^{\hat{f}}(p,q) = \sum_{j=1,\dots,N} D_+^{\hat{f}^j}(p,q) \quad with \quad D_+^{\hat{f}^j}(p,q) = \sigma^j \cdot sign^+(p^j - q^j) \cdot \left\{\hat{f}^j(p) - \hat{f}^j(q)\right\}$$

ii) (HJ germs)
Then $\mathcal{G}$ is a HJ germ if and only if there exists some function $\hat{h} : [a,b] \to \mathbb{R}$ such that

$$\hat{f}^j = \hat{h} \quad for\ all \quad j = 1,\dots,N \tag{5.9}$$

iii) (Monotone germs)
Then $\mathcal{G}$ is a monotone germ if and only if

$$p \mapsto \sigma^j \hat{f}^j(p) \quad is\ nonincreasing\ in\ p^k\ for\ all\ k \neq j \tag{5.10}$$

iv) (Conservative germs)
Then $\mathcal{G}$ is a conservative germ if and only if

$$\sum_{J^j \simeq (-\infty,0)} \hat{f}^j = \sum_{J^j \simeq (0,+\infty)} \hat{f}^j \quad on \quad [a,b]. \tag{5.11}$$

This means

$$RH^{\hat{f}} = 0 \quad on \quad [a,b], \quad with \quad RH^{\hat{f}}(p) := \sum_{j=1,\dots,N} \sigma^j \hat{f}^j(p) = IN - OUT.$$

v) (Effect of reversion tranform)
Moreover Kružkov germs, monotone germs and conservative germs are preserved by reversion transform of Definition 4.4, while HJ germs are not preserved in general by reversion transforms.
vi) (Effect of inversion transforms)
Moreover Kružkov germs are preserved by any inversion of Definition 4.1, while HJ germs, monotone germs and conservative are not preserved in general by inversion transforms.

Proof of Lemma 5.5
We mainly to the proof in the case of $N : 0$ junctions, the general case following by reversion transforms.
Step 1: proof of i)

Recall that $\mathcal{G}$ is a Kružkov germ, i.e.

$$D^f \geq 0 \quad \text{on} \quad \mathcal{G}^2 \tag{5.12}$$

and $\hat{f} = f \circ \pi$. Hence for all $p, q \in [a, b]$, and $\hat{p} := \pi(p)$ and $\hat{q} := \pi(q)$, and $f^j(\hat{p}) = f^j(\hat{p}^j)$, we have from (4.21) that $D^{\hat{f}}(p, q) \geq D^f(\hat{p}, \hat{q}) \geq 0$, which shows (5.7). Conversely it is straightforward that (5.7) implies (5.12). This shows point i).

Step 1': proof of i')

Recall that $\mathcal{G}$ is a D_+-germ, i.e.

$$D^f_+ \geq 0 \quad \text{on} \quad \mathcal{G}^2 \tag{5.13}$$

and $\hat{f} = f \circ \pi$. Hence for all $p, q \in [a, b]$, and $\hat{p} := \pi(p)$ and $\hat{q} := \pi(q)$, and $f^j(\hat{p}) = f^j(\hat{p}^j)$, we have from (4.21) that $D^{\hat{f}}_+(p, q) \geq D^f_+(\hat{p}, \hat{q}) \geq 0$, which shows (5.8). Conversely it is straightforward that (5.8) implies (5.13). This shows point i').

Step 2: proof of ii)

Recall that $\mathcal{G}$ is a HJ germ, i.e. there exists some function $h : \mathcal{G} \to \mathbb{R}$ such that

$$f^j = h \quad \text{on} \quad \mathcal{G}, \quad \text{for all} \quad j = 1, \dots, N \tag{5.14}$$

Setting $\hat{h} := h \circ \pi$, we see that (5.14) is equivalent to (5.9) which shows point ii).

Step 3: proof of iii)

Recall that a generalized Riemann germ $\mathcal{G}$ is monotone if and only if

$$\text{for all} \quad p, q \in [a, b], \quad (p \geq q \quad \text{implies} \quad \pi(p) \geq \pi(q)) \tag{5.15}$$

and we want to show that it is equivalent to (5.10). Fix some $k \in \{1, \dots, N\}$, consider $p, q \in [a, b]$ such that $p - q = \varepsilon e_k$ for some $\varepsilon \geq 0$, and call $\hat{p} := \pi(p)$, $\hat{q} := \pi(q)$.

On the one hand, recall from Proposition 5.1 that π is Riemann monotone, and then that $\pi^k(p) \geq \pi^k(q)$, i.e. that $\hat{p}^k \geq \hat{q}^k$ is always satisfied. On the other hand, (even if it is not used in this proof), notice that the monotonicity in (2.14) shows that $\sigma^k \hat{f}^k(p) \geq \sigma^k \hat{f}^k(q)$ is also always satisfied.

Now in order to show equivalence between (5.15) and (5.10), we see that it is sufficient to show that for all $j \in \{1, \dots, N\} \setminus \{k\}$, inequality $\hat{p}^j \geq \hat{q}^j$ is equivalent to $\sigma^j \hat{f}^j(p) \leq \sigma^j \hat{f}^j(q)$. We now focus on the case $\sigma^j = 1$ (the reasoning is similar for $\sigma^j = -1$). Because $\hat{f}^j(p) = f^j \circ \pi(p) = f^j(\hat{p}^j)$, it remains to show that

$$\hat{p}^j \geq \hat{q}^j \quad \text{is equivalent to} \quad f^j(\hat{p}^j) \leq f^j(\hat{q}^j), \quad \text{for} \quad J^j \simeq (-\infty, 0) \tag{5.16}$$

Recall that $BA^j = BA^{(J^j, f^j)}$. Because $p^j = q^j$, we deduce that $BA^j(\hat{p}^j) \cap BA^j(\hat{q}^j) \neq \emptyset$. Then Definition 2.10 of BA^j for $J^j \simeq (-\infty, 0)$ (see also the associated figure) implies (5.16). This ends the proof of the equivalence of (5.15) with (5.10).

Step 4: proof of iv)

Recall that $\mathcal{G}$ is a conservative germ, i.e. $\displaystyle\sum_{J^j \simeq (-\infty, 0)} f^j = \sum_{J^j \simeq (0, +\infty)} f^j$ on $\mathcal{G}$. For $\hat{f}^j = f^j \circ \pi$, this shows that this is equivalent to (5.11), and this proves iv).

Step 5: proof of v) and vi)

The result is straightforward. This ends the proof of the lemma.

Lemma 5.6 (Nature of germs after slicing)

Assume (2.2) for $N \geq 2$. Let $\mathcal{G} \subset [a, b]$ be a generalized germ. Let $1 \leq n < N$. Then for $p \in [a, b]$, we write

$$p = (p', p'') \quad \textit{with} \quad p' = (p^1, \dots, p^n) \in [a', b'] \quad \textit{and} \quad p'' = (p^{n+1}, \dots, p^N) \in [a'', b''].$$

Given some $p''_0 \in [a'', b'']$, let $\mathcal{G}_{p''_0}$ be the sliced germ defined in Lemma 4.12.

If $\mathcal{G}$ is a Kružkov (resp. HJ, resp. monotone) germ, then the sliced germ $\mathcal{G}_{p''_0}$ is also a Kružkov (resp. HJ, resp. monotone) germ.

Proof of Lemma 5.6
From the definitions and Lemma 5.5, the result follows from the following property

$$\mathcal{G}_{p_0''} = \left\{ p' = (p^1, \dots, p^n) \in [a', b'], \quad \hat{f}^j(p', p_0'') = f^j(p^j) \quad \text{for} \quad j = 1, \dots, n \right\}$$

which is provided by the slicing Lemma 4.12. This ends the proof of the lemma.

Remark 5.7 *In Lemma 5.6, if $\mathcal{G}$ is a conservative germ, then $\mathcal{G}_{p_0''}$ satisfies the following relation*

$$\sum_{J^j \simeq (-\infty,0)} f^j(p) = \sum_{J^j \simeq (0,+\infty)} f^j(p) \quad \textit{for all} \quad p = (p', p_0'') \in \mathcal{G}_{p_0''} \times \{p_0''\}$$

which is the memory at the level of the sliced germ $\mathcal{G}_{p_0''}$ of the Rankine-Hugoniot relation satisfied by $\mathcal{G}$.

5.4 Theorem 2.20 and its proof: Riemann monotonicity of $\sigma \diamond \hat{f}$

The proof of Theorem 2.20 is done at the end of the subsection as a corollary of the following more general proposition.

Proposition 5.8 (The map $h \circ \pi$ is Riemann monotone on $N:0$ junctions)
Assume (2.2) for a $N:0$ junction with $N \geq 1$. Let $\mathcal{G}$ be a generalized Riemann germ, and let $\pi := \pi_{\mathcal{G}}$ its associated projection. Let $h = (h^1, \dots, h^N) : [a,b] \to \mathbb{R}^N$ be a continuous map with for $k = 1, \dots, N$

(5.17) *each component $h^k : [a^k, b^k] \to \mathbb{R}$ is non constant on any nondegenerate interval*

Assume that the map $\hat{h} := h \circ \pi : [a,b] \to \mathbb{R}^N$ is continuous and satisfies

$$(5.18) \qquad \left\{ \begin{array}{l} p \mapsto \hat{h}^k(p) \quad \textit{is nondecreasing in } p^k\textit{, for each } k = 1, \dots, N \\ \hat{h} \textit{ is locally constant on } \left\{ \hat{h} \neq h \right\} \end{array} \right.$$

Then $\hat{h}$ satisfies

$$(5.19) \qquad (p-q) \diamond [\hat{h}]_q^p \leq 0 \quad \Longrightarrow \quad \hat{h} = const \quad on \quad co\,(BA(\hat{p}) \cup BA(\hat{q}))$$

where $co(A)$ is the convex hull of a set $A \subset \mathbb{R}^N$. In particular $\hat{h}$ is Riemann monotone.

Proof of Proposition 5.8
The proof follows the one of Proposition 5.1 with $\hat{h}$ instead of π, and with some adaptations.
For all $p, q \in [a,b]$, we set $\hat{p} := \pi(p)$, $\hat{q} := \pi(q)$ and $[\hat{h}]_q^p := \hat{h}(p) - \hat{h}(q)$. Assume by contradiction that $\hat{h}$ does not satisfy (5.19), i.e. that we have

$$(5.20) \qquad (p-q) \diamond [\hat{h}]_q^p \leq 0 \quad \text{and} \quad \hat{h} \neq const \quad \text{on} \quad \text{co}\,(BA(\hat{p}) \cup BA(\hat{q}))$$

Recall that

$$BA(\hat{p}) \cap BA(\hat{q}) \neq \emptyset \quad \Longrightarrow \quad \hat{p} = \hat{q} \quad \Longrightarrow \quad \hat{h} = const \quad \text{on} \quad \text{co}\,(BA(\hat{p}) \cup BA(\hat{q})) = BA(\hat{p}) = BA(\hat{q})$$

From (5.20), we then deduce $BA(\hat{p}) \cap BA(\hat{q}) = \emptyset$ and up to apply an inversion, we can assume that $p \geq q$, $\hat{p} \leq \hat{q}$, $\hat{p} \neq \hat{q}$. We now do the proof by recurrence on $N \geq 1$.
Step 1: Case $N = 1$
Then (5.20) means

$$\hat{h}(p) \leq \hat{h}(q) \quad \text{and} \quad \hat{h} \neq const \quad \text{on} \quad \text{co}\,(BA(\hat{p}) \cup BA(\hat{q}))$$

Because $\hat{h}$ is nondecreasing, we get $\hat{h}(p) = \hat{h}(q)$, and then $\hat{h} = const$ on $\text{co}\,(BA(\hat{p}) \cup BA(\hat{q}))$. Contradiction.
Step 2: Case $N \geq 2$, and Proposition 5.8 holds true for $N' := N - 1$
Step 2.1: Case of the hyperplane intersection
Step 2.1.1: setting of the problem

Assume by contradiction that there exists an index $k_0 \in \{1,\dots,N\}$ and some $p_0^{k_0} \in [a^{k_0}, b^{k_0}]$ such that the "hyperplane" intersection with the box $[a,b]$, defined as $\Pi_{k_0} := \left\{ r \in [a,b], \quad r^{k_0} = p_0^{k_0} \right\}$, satisfies

$$\left\{ \begin{array}{l} \Pi_{k_0} \cap BA(\hat p) \neq \emptyset \\ \Pi_{k_0} \cap BA(\hat q) \neq \emptyset \end{array} \right. \tag{5.21}$$

Up to relabel the indices, we can assume that $k_0 = N$, and then (5.21) means

$$BA(\hat p^N) \cap BA(\hat q^N) \neq \emptyset \tag{5.22}$$

From the slicing Lemma 4.12, we know that the flux function $\hat f_{\mathcal G}$ restricted to $\Pi_N \cap [a,b]$ is associated to a generalized Riemann germ $\mathcal G_{p_0^N} \subset \tilde Q := \prod_{k=1,\dots,N-1} [a_k, b_k]$, and we set the associated projection map $\tilde\pi : \tilde Q \to \mathcal G_{p_0^N}$ defined as $\tilde\pi := \pi_{\mathcal G_{p_0^N}} := \pi'(\cdot, p_0^N)$ for $\pi = (\pi', \pi^N)$. Moreover, we write

$$\hat p = (\hat p', \hat p^N), \quad \hat q = (\hat q', \hat q^N), \quad p = (p', p^N), \quad q = (q', q^N), \quad h = (h', h^N), \quad \hat h = (\hat h', \hat h^N)$$

and (on the model of $\hat f_{\mathcal G_{p_0^N}} := \hat f_{\mathcal G}(\cdot, p_0^N)$) we set $\hat{\hat h} := \hat h'(\cdot, p_0^N) = h' \circ (\tilde\pi, p_0^N) = (h^k \circ \tilde\pi^k)_{k=1,\dots,N-1} = h' \circ \tilde\pi$, where we use our usual abuse of notation for the argument of h or h', and where $r' \mapsto (h' \circ \tilde\pi)^k(r')$ is nondecreasing in r'^k for every $k = 1,\dots,N-1$.
Using $(p', p_0^N) \in BA(\hat p)$ and $(q', p_0^N) \in BA(\hat q)$, we get that $\hat p' = \tilde\pi(p')$, $\hat q' = \tilde\pi'(q') \in \mathcal G_{p_0^N}$ and

$$\hat h'(p) = \hat h'(p', p^N) = h' \circ \pi(p', p^N) = h' \circ \pi(p', p_0^N) = \hat{\hat h}(p') \quad \text{and} \quad \hat h'(q) = \hat{\hat h}(q')$$

Hence (5.20) implies $(p' - q') \diamond [\hat{\hat h}]_{q'}^{p'} \le 0$. Notice that by construction, $\hat{\hat h}$ is locally constant on $\left\{ \hat{\hat h} \neq h' \right\}$ and such that $r' \mapsto \hat{\hat h}^k(r')$ is nondecreasing in r'^k for $k = 1,\dots,N-1$. This shows that $\hat{\hat h}$ satisfies (5.18) for $N' := N-1$. Then by recurrence assumption, we know that $\hat{\hat h} = h' \circ \tilde\pi$ satisfies (5.19) for $N' := N-1$. We deduce that

$$h' \circ \tilde\pi = \hat{\hat h} = const = \lambda' := h'(\hat p') = h'(\hat q') \quad \text{on} \quad Q' := \prod_{k=1,\dots,N-1} Q^k \quad \text{with} \quad Q^k := \mathrm{co}\left(BA(\hat p^k) \cup BA(\hat q^k) \right)$$

where we recall that $\mathrm{co}(A)$ is the convex hull of A.
Step 2.1.2: consequences
Let $\Omega' := \prod_{k=1,\dots,N-1} \Omega^k$ with $\Omega^k := \left\{ r^k \in [a^k, b^k],\ h^k(r^k) \neq \lambda'^k \right\}$. From (5.17), h^k is non constant on any nondegenerate interval. Hence Ω^k is a relative open set of $[a^k, b^k]$, which is also dense in Q^k. By assumption $\hat h$ is locally constant on $\left\{ \hat h \neq h \right\}$. This implies that $\hat h^N(\cdot, p_0^N)$ is locally constant on $\Omega' \subset \left\{ \lambda' = \hat h' \neq h' \right\}$. Notice that $\Omega' \cap Q'$ is a relative open set of Q'. Now $\Omega' \cap Q'$ is dense in Q', and $\hat h^N$ is continuous and locally constant on $\Omega' \cap Q'$. Therefore we deduce that $\hat h^N(\cdot, p_0^N) = const$ on Q'. Therefore

$$h^N(\hat p^N) = h^N(\pi^N(p', p_0^N)) = \hat h^N(p', p_0^N) = const = \hat h^N(q', p_0^N) = h^N(\hat q^N)$$

From (5.22), we deduce that $BA(\hat p^N) \cap BA(\hat q^N) \neq \emptyset$ with $h^N(\hat p^N) = h^N(\hat q^N)$, which from the structure of the Basins of Attraction, implies that $\hat p^N = \hat q^N$, and then we can choose any $p_0^N \in BA(\hat p^N) = BA(\hat q^N)$, which shows that

$$\hat h = const = h(\hat p) = h(\hat q) \quad \text{on} \quad \mathrm{co}(BA(\hat p) \cup BA(\hat q)) = Q' \times BA(\hat p^N)$$

Contradiction with (5.20). This implies there is no index k_0 such that (5.21) holds true, which is the next case.
Step 2.2: Case of no hyperplane intersection
We now assume that there is no index k_0 such that (5.21) holds true. Then this implies that $BA(\hat p)$ and $BA(\hat q)$ are well separated in all directions, and we get $\hat p < \hat q$ and then $p > q$, and also $p \in BA(\hat p) < BA(\hat q) \ni q$, which implies $p < q$. Contradiction.
Step 3: conclusion
Therefore (5.20) is false and this implies (5.19). This ends the proof of the proposition.

We will need the following proposition

Proposition 5.9 (Riemann monotonicity of Lipschitz Kružkov functions)
Assume (2.2) with $N \geq 1$, and recall that $\sigma \in \{\pm 1\}^N$ encodes the orientations of the branches. Assume that $h : [a,b] \to \mathbb{R}^N$ is a continuous map. Assume also that the map $p \mapsto h^j(p)$ is locally Lipschitz continuous in p^j uniformly in the other coordinates p^k for $k \neq j$, and for all $j = 1, \dots, N$.

If h satisfies

$$D^h(p,q) = \sum_{j=1,\dots,N} \sigma^j \cdot sign(p^j - q^j) \cdot \left\{h^j(p) - h^j(q)\right\} \geq 0 \quad \text{for all} \quad p,q \in [a,b] \tag{5.23}$$

then $\sigma \diamond h$ is Riemann monotone on $[a,b]$.

and its straightforward corollary.

Corollary 5.10 (Riemann monotonicity for Kružkov germs)
Assume (2.2) with $N \geq 1$, and recall that $\sigma \in \{\pm 1\}^N$ encodes the orientations of the branches. Let $\mathcal{G} \subset [a,b]$ be a Kružkov germ, and $\hat{f}_{\mathcal{G}}$ be its associated (generalized) Godunov flux. Then the map $\sigma \diamond \hat{f}_{\mathcal{G}}$ is Riemann monotone.

Proof of Proposition 5.9
Up to apply some suitable reversions, we can assume that we work with $\sigma^j = 1$ for all indices j, i.e. for $N:0$ junctions. By assumption, we consider a continuous Kružkov function $h : [a,b] \to \mathbb{R}^N$, i.e. satisfying

$$D^h(p,q) = \sum_{j=1,\dots,N} \operatorname{sign}(p^j - q^j) \cdot \left[h^j\right]_q^p \geq 0 \quad \text{for all} \quad p,q \in [a,b]\,. \tag{5.24}$$

and such that the map $p \mapsto h^j(p)$ is locally Lipschitz continuous in p^j uniformly in the other coordinates p^k for $k \neq j$, and for all $j = 1, \dots, N$.

Now assume that $p, q \in [a,b]$ satisfy $(p-q) \diamond [h]_q^p \leq 0$. Then (5.24) implies $(p-q) \diamond [h]_q^p = 0$. Up to apply suitable inversions, we can assume that $(p-q) \geq 0$. We then set $I := \left\{i \in \{1,\dots,N\}\,,\ (p-q)^i > 0\right\}$, $\bar{I} := \{1,\dots,N\} \setminus I$. In particular, we get

$$\begin{cases} \left[h^i\right]_q^p = 0 & \text{for all} \quad i \in I \\ (p-q)^i = 0 & \text{for all} \quad i \in \bar{I} \end{cases} \tag{5.25}$$

Moreover, consider the set $\Omega := \left\{\xi \in \mathbb{R}^N, \quad p+\xi, q+\xi \in [a,b]\right\}$ and consider the quantity

$$T^j(\xi,t) := \partial_j h^j(q + \xi + t(p-q)) - \sum_{i \in \{1,\dots,N\} \setminus \{j\}} |\partial_j h^i(q + \xi + t(p-q))| \geq 0$$

which is defined for a.e. ξ and is nonnegative from ii) of Proposition 4.20. Then we get for a.e. $\xi \in \Omega$

$$\begin{aligned}
&\sum_{i\in I} \left[h^i\right]_{q+\xi}^{p+\xi} \\
= &\sum_{i,j\in I} \int_0^1 dt\ \partial_j h^i(q+\xi+t(p-q))\cdot(p-q)^j \\
= &\sum_{j\in I}(p-q)^j \cdot \int_0^1 dt\ \left\{\sum_{i\in I}\partial_j h^i(q+\xi+t(p-q))\right\} \\
\ge &\sum_{j\in I}(p-q)^j \cdot \int_0^1 dt\ \left\{\partial_j h^i(q+\xi+t(p-q)) - \sum_{i\in I\setminus\{j\}}|\partial_j h^i(q+\xi+t(p-q))|\right\} \\
= &\sum_{j\in I}(p-q)^j \cdot \int_0^1 dt\ \left\{T^j(\xi,t) + \sum_{i\in \bar I}|\partial_j h^i(q+\xi+t(p-q))|\right\} \\
\ge &\sum_{j\in I}(p-q)^j \cdot \int_0^1 dt\ \left\{\sum_{i\in \bar I}|\partial_j h^i(q+\xi+t(p-q))|\right\} \\
= &\sum_{i\in \bar I}\int_0^1 dt\ \left\{\sum_{j\in I}(p-q)^j\cdot|\partial_j h^i(q+\xi+t(p-q))|\right\} \\
\ge &\sum_{i\in \bar I}\left|\int_0^1 dt\ \left\{\sum_{j\in I}(p-q)^j\cdot\partial_j h^i(q+\xi+t(p-q))\right\}\right| \\
= &\sum_{i\in \bar I}\left|\int_0^1 dt\ \left\{\sum_{j\in\{1,\dots,N\}}(p-q)^j\cdot\partial_j h^i(q+\xi+t(p-q))\right\}\right| \\
= &\sum_{i\in \bar I}\left|\left[h^i\right]_{q+\xi}^{p+\xi}\right|
\end{aligned}$$

i.e.

$$\sum_{i\in I}\left[h^i\right]_{q+\xi}^{p+\xi} \quad\ge\quad \sum_{i\in\bar I}\left|\left[h^i\right]_{q+\xi}^{p+\xi}\right| \quad \text{for a.e.}\quad \xi\in\Omega$$

By continuity of both sides, we deduce for $\xi = 0$ that $\sum_{i\in I}\left[h^i\right]_q^p \ \ge\ \sum_{i\in\bar I}\left|\left[h^i\right]_q^p\right|$. From (5.25), we deduce $[h]_q^p = 0$. Hence we have shown that $(p-q)\diamond[h]_q^p \le 0$ implies $[h]_q^p = 0$, i.e. that h is Riemann monotone. This ends the proof of the proposition.

Proof of Theorem 2.20
We want to show that $\sigma \diamond \hat f_{\mathcal G}$ is Riemann monotone. We only do it for $N:0$ junctions, and then show that $\hat f := \hat f_{\mathcal G}$ is Riemann monotone. For the general case of junctions $n:m$ with $N := n+m$, we use suitable I-reversions (see Lemma 4.5).
Step 1: proof of i)
For junctions $N:0$ and under nondegeneracy assumption (2.17), it follows from Proposition 5.8 with $h := f$ and $\hat h := \hat f$ which is continuous.
Step 2: proof of ii)
For Kružkov germs $\mathcal G$, we know from Lemma 5.5 that $h := \hat f_{\mathcal G}$ satisfies (5.23). We know moreover that $|\partial_j h^j(p)| \le |(f^j)'(p^j)|$ for all j, as do all generalized Godunov fluxes. Then the result follows from Corollary 5.10. This ends the proof of the theorem.

5.5 Local quasi-constancy criterion

We start with the following useful notion which will be useful in the next subsection.

Definition 5.11 **(j-local quasi-constancy)**

Given a box $[a,b] \subset \mathbb{R}^N$, let us consider a map $\hat{f} = (\hat{f}^1, \dots, \hat{f}^N) : [a,b] \to \mathbb{R}^N$. Let $j \in \{1,\dots,N\}$. Then we say that $\hat{f}$ is. j-locally quasi-constant if the following holds true. For any $p \in [a,b]$, let

$$\Phi(x) := \hat{f}(p + x \cdot e_j) \quad \text{defined for} \quad p_j + x \in [a^j, b^j].$$

Then

$$\Phi = const = \Phi(0) \quad on \quad \left\{\Phi^j = \Phi^j(0)\right\}$$

Then we have the following result.

Lemma 5.12 (Criterion for j-local quasi-constancy)
Assume (2.2) for $N \geq 1$ and orientations $\sigma \in \{\pm 1\}^N$. Fix $j \in \{1,\dots,N\}$.
1) (Godunov flux, with nondegeneracy condition)
Assume that $\hat{f} : [a,b] \to \mathbb{R}^N$ is a Godunov flux in the sense of Definition 2.18. If the component f^j is nondegenerate in the sense of (2.17), then $\hat{f}$ is j-locally quasi-constant.
2) (Godunov quasi-flux, which is Kružkov)
Assume that the box $[a,b]$ is compact (inside $\mathbb{R}^N$) and that $\hat{f} : [a,b] \to \mathbb{R}^N$ is a Godunov quasi-flux in the sense of Definition 2.22. Assume that the quasi-flux $\hat{f}$ is Kružkov, i.e. satisfies the following property

$$0 \leq D^{\hat{f}}(p,q) = \sum_{k=1,\dots,N} \sigma^j \cdot sign(p^k - q^k) \cdot \left\{\hat{f}^k(p) - \hat{f}^k(q)\right\} \quad for\ all \quad p,q \in [a,b]$$

Then $\hat{f}$ is j-locally quasi-constant.

Proof of Lemma 5.12
First, up to a permutation on the indices, we can assume that $j = 1$. Fix some $p \in [a,b]$ and denote $p' := (p^2, \dots, p^N)$. Let

$$\Phi(x) := \hat{f}(p + xe_1)$$

and consider the set

$$R := \left\{r = p^1 + x \in [a^1, b^1], \quad \Phi^1(x) = \Phi^1(0)\right\}$$

From the monotonicity of the map $x \mapsto \sigma^1 \Phi^1(x)$, we deduce that R is an interval.
Step 1: proof of i)
From nondegeneracy condition (2.17) for $k = j = 1$, we deduce that $\Omega^1 := \left\{\hat{f}^1(\cdot, p') \neq f^1\right\}$ is (relatively) open and dense in the interval R. Because $\hat{f}$ is locally constant on $\left\{\hat{f} \neq f\right\}$ and is continuous on $[a,b]$, we deduce that not only the 0-component of $\hat{f}(\cdot, p')$ is constant, but the full function satisfies $\hat{f}(\cdot, p') = const$ on R. This shows that $\hat{f}$ is j-locally quasi-constant for $j = 1$.
Step 2: proof of ii)
If interval R is a singleton, there is nothing to prove. Assume now that R is not reduced to a singleton. Then there exists q^1 in the interior of R.
Step 2.1: case with p in the interior of $[a,b]$
Then for any $\bar{\sigma}' = (\bar{\sigma}^2, \dots, \bar{\sigma}^N) \in \{\pm 1\}^{N-1}$, and $\varepsilon > 0$, we have

$$0 \leq D^{\hat{f}}(q_\varepsilon, p) \quad \text{with} \quad q_\varepsilon := (q^1, p' + \varepsilon \bar{\sigma}')$$

In the limit $\varepsilon \to 0$, this gives for $q_0 := (q^1, p')$

$$0 \leq \sigma^1 \text{sign}(q^1 - p^1) \cdot [\hat{f}^1]_p^{q_0} + \sum_{j=2,\dots,N} \sigma^j \bar{\sigma}^j [\hat{f}^j]_p^{q_0}$$

Because the choice of $\bar{\sigma}'$ is arbitrary, we get

$$\sum_{j=2,\dots,N} |[\hat{f}^j]_p^{q_0}| \leq |[\hat{f}^1]_p^{q_0}| \tag{5.26}$$

But the fact that $p^1, q^1 \in R$ implies $[\hat{f}^1]_p^{q_0} = 0$. Therefore $\hat{f}$ is constant on R, and then $\hat{f}$ is j-locally quasi-constant on the interior of $[a,b]$, for $j = 1$.

Step 2.2: case $p \in \partial[a,b]$
We first approximate p by some interior point p_δ of $[a,b]$. Then (5.26) implies

$$\sum_{j=2,\dots,N} |[\hat{f}^j]^{q_0}_{p_\delta}| \le |[\hat{f}^1]^{q_0}_{p_\delta}|$$

Finally in the limit $p_\delta \to p$, we recover (5.26), and we conclude again as in Step 2.1 that $\hat{f}$ is j-locally quasi-constant on the interior of $[a,b]$, for $j=1$.
This ends the proof of the lemma.

5.6 Theorem 2.24 and its proof: gluing Riemann germs

The proof of Theorem 2.24 is mainly based on Proposition 5.13, where we first glue the fluxes. Then in Corollary 5.14, we show that this is equivalent to glue the germs. Finally in Lemma 5.15, we check the associativity of the gluing. The proof of Theorem 2.24 is done at the end of this subsection, as a consequence of those three results.

Proposition 5.13 (Gluing of flux functions $\hat{f}_\gamma$ for $n_\gamma : m_\gamma$ junctions)
For $\gamma = \alpha, \beta$, assume that f_γ satisfies (2.2) for $N_\gamma = n_\gamma + m_\gamma$ and $n_\gamma : m_\gamma$ junctions J_γ with $J^j_\gamma \simeq \sigma^j_\gamma \cdot (-\infty, 0)$ and $\sigma_\gamma \in \{\pm 1\}^{N_\gamma}$, and consider Riemann germs $\mathcal{G}_\gamma$ with respect to (J_γ, f_γ).

We set $[a,b]^j_\gamma := [a^j_\gamma, b^j_\gamma]$. We assume that for each $\gamma = \alpha, \beta$, there exists one index $j_\gamma \in \{1, \dots, N_\gamma\}$ such that

$$f^{j_\alpha}_\alpha = f^{j_\beta}_\beta =: f^0 \quad on \quad [a,b]^{j_\alpha}_\alpha = [a,b]^{j_\beta}_\beta =: [a^0, b^0] \quad with \quad J^{j_\alpha}_\alpha \simeq (0,+\infty) \quad and \quad J^{j_\beta}_\beta \simeq (-\infty, 0) \tag{5.27}$$

*and we glue those two branches. Assume that both $\hat{f}_\gamma$ are j_γ-**locally quasi-constant**, and $\sigma_\gamma \diamond \hat{f}_\gamma$ **are Riemann monotone**.*

To simplify the notation, up to relabel the indices, we now assume that $j_\alpha = 0 = j_\beta$, and the indices now go through the values $\{0, \dots, N_\gamma - 1\}$. Hence we now have

$$\begin{cases} f_\gamma = (f^0_\gamma, \dots, f^{N_\gamma - 1}_\gamma) \\ \mathcal{G}_\gamma \subset [a,b]_\gamma := \prod_{i=0,\dots,N_\gamma-1} [a,b]^i_\gamma \\ [a,b]'_\gamma := \prod_{i=1,\dots,N_\gamma-1} [a,b]^i_\gamma \\ J^0_\alpha \simeq (0,+\infty) \quad and \quad J^0_\beta \simeq (-\infty, 0) \end{cases}$$

and consider $\hat{f}_\gamma = (\hat{f}^0_\gamma, \dots, \hat{f}^{N_\gamma-1}_\gamma)$ the associated flux. Let the new junction after gluing be defined by

$$\underline{J} := \{0\} \cup \left(\bigcup_{j=1,\dots,N_\alpha - 1} J^j_\alpha \right) \cup \left(\bigcup_{k=1,\dots,N_\beta-1} J^k_\beta \right)$$

and

$$\underline{\sigma}^I := \begin{cases} \sigma^k_\alpha & if \quad I = (k, \alpha) \\ \sigma^k_\beta & if \quad I = (k,\beta) \end{cases}$$

For $p_\gamma = (p^1_\gamma, \dots, p^{N_\gamma - 1}_\gamma) \in [a,b]'_\gamma$ (avoiding notation p'_γ to keep light notations), let us consider the set

$$R := \left\{ r \in [a^0, b^0], \quad \hat{f}^0_\alpha(r, p_\alpha) = \hat{f}^0_\beta(r, p_\beta) \right\} \quad with \quad \hat{f}^0_\alpha(\downarrow, p_\alpha), \quad \hat{f}^0_\beta(\uparrow, p_\beta) \tag{5.28}$$

Then R is non empty, and define the set

$$\Lambda := \left\{ \lambda = \tilde{f}(r, p_\alpha, p_\beta) \in \mathbb{R}^{N_\alpha + N_\beta - 2}, \quad r \in R \right\}$$

with

$$\tilde{f}(r,p_\alpha,p_\beta) := (\hat{f}^1_\alpha(r,p_\alpha),\dots,\hat{f}^{N_\alpha-1}_\alpha(r,p_\alpha);\hat{f}^1_\beta(r,p_\beta),\dots,\hat{f}^{N_\beta-1}_\beta(r,p_\beta)) \in \mathbb{R}^{N_\alpha+N_\beta-2}$$

Then Λ is reduced to a singleton $\Lambda = \{\lambda\}$, and this defines the following map

$$\begin{array}{rccl} \underline{\hat{f}}: & [a,b]'_\alpha \times [a,b]'_\beta & \to & \mathbb{R}^{N_\alpha+N_\beta-2} \\ & (p_\alpha,p_\beta) & \mapsto & \underline{\hat{f}}(p_\alpha,p_\beta) := \lambda \end{array}$$

and we set the map

$$\underline{f} := (f^1_\alpha,\dots,f^{N_\alpha-1}_\alpha; f^1_\beta,\dots,f^{N_\beta-1}_\beta) : [a,b]'_\alpha \times [a,b]'_\beta \to \mathbb{R}^{N_\alpha+N_\beta-2}$$

0) (Gluing Riemann germs)
Then $\underline{\hat{f}}$ is continuous, the map $\underline{\sigma} \diamond \underline{\hat{f}}$ is Riemann monotone, and the set

$$\mathcal{G} := \left\{ P \in [a,b]'_\alpha \times [a,b]'_\beta, \quad \underline{\hat{f}}(P) = \underline{f}(P) \right\}$$

is a Riemann germ with respect to $(\underline{J},\underline{f})$ and $\underline{\hat{f}}$ is the associated Godonuv flux at the junction $(n_\alpha + n_\beta - 1)$: $(m_\alpha + m_\beta - 1)$, i.e.

$$\underline{\hat{f}} = \underline{\hat{f}}_{\mathcal{G}}$$

i) (Gluing Kružkov germs)
Assume that $\mathcal{G}_\gamma$ are Kružkov germs for $\gamma = \alpha,\beta$. Then $\mathcal{G}$ is also a Kružkov germ.
ii) (Gluing HJ germs)
Assume that $\mathcal{G}_\gamma$ are HJ germs for $\gamma = \alpha,\beta$. Then $\mathcal{G}$ is also a HJ germ.
iii) (Gluing monotone germs)
Assume that $\mathcal{G}_\gamma$ are monotone germs for $\gamma = \alpha,\beta$. Then $\mathcal{G}$ is also a monotone germ.
iv) (Gluing conservative germs)
Assume that $\mathcal{G}_\gamma$ are conservative germs for $\gamma = \alpha,\beta$. Then $\mathcal{G}$ is also a conservative germ.

Proof of Proposition 5.13
Step 1: non emptyness of R
We set $p := p_\alpha$, $q := p_\beta$ and for $P := (p,q)$, and $r \in [a^0,b^0]$, we set

(5.29) $$g : [a^0,b^0] \to \mathbb{R} \quad \text{with} \quad g(r) := \tilde{g}(r,p,q) := \hat{f}^0_\alpha(r,p) - \hat{f}^0_\beta(r,q) \quad \text{with} \quad \hat{f}^0_\alpha(\downarrow,p), \quad \hat{f}^0_\beta(\uparrow,q)$$

Recall that using (5.27) and (2.15), we get

$$g(a^0) = \hat{f}^0_\alpha(a^0,p) - \hat{f}^0_\beta(a^0,q) \geq (f^0_\alpha)_-(a^0) - (f^0_\beta)_+(a^0) = G^{f^0}(a^0,a^0) - G^{f^0}(a^0,a^0) = 0$$

$$g(b^0) = \hat{f}^0_\alpha(b^0,p) - \hat{f}^0_\beta(b^0,q) \leq (f^0_\alpha)_+(b^0) - (f^0_\beta)_-(b^0) = G^{f^0}(b^0,b^0) - G^{f^0}(b^0,b^0) = 0$$

Therefore $g(a^0) \geq 0 \geq g(b^0)$, and by continuity of g, we get $R \neq \emptyset$.
Step 2: Λ is a singleton
From the monotonicities of $\hat{f}^0_\gamma(\cdot,p_\gamma)$ given in (5.29), we deduce that g is nonincreasing. Assume that there exists $r,r' \in R$ with $r < r'$. Then this implies that g is constant (and vanishes) on $[r,r']$, and we deduce that R is a closed interval. Moreover, from the monotonicities of $\hat{f}^0_\gamma$ in the 0-component, we deduce that both maps $\hat{f}^0_\gamma(\cdot,p_\gamma)$ for $\gamma = \alpha,\beta$ are also constant on $[r,r'] \subset R$, and then on the whole R.
We then use the fact that both $\hat{f}^0_\gamma$ are 0-locally quasi-constant, to deduce that $\hat{f}_\gamma(\cdot,p_\gamma)$ are both constant on R. Therefore Λ is reduced to a singleton.
Step 3: Continuity of $\underline{\hat{f}}$
Consider a sequence $(r_n,p_{\alpha,n},p_{\beta,n}) \to (r,p_\alpha,p_\beta)$ as $n \to +\infty$, such that

$$\begin{cases} \lambda_n := \underline{\hat{f}}(p_{\alpha,n},p_{\beta,n}) = \tilde{f}(r_n,p_{\alpha,n},p_{\beta,n}) \to \tilde{f}(r,p_\alpha,p_\beta) =: \lambda_0, \\ 0 = g(r_n) = \hat{f}^0_\alpha(r_n,p_{\alpha,n}) - \hat{f}^0_\beta(r,p_{\beta,n}) \to 0 = g(r) = \hat{f}^0_\alpha(r,p_\alpha) - \hat{f}^0_\beta(r,p_\beta) \end{cases}$$

The second line shows that $r \in R$, and then $\lambda_0 = \tilde{f}(r,p_\alpha,p_\beta) = \underline{\hat{f}}(p_\alpha,p_\beta)$. This shows the continuity of $\underline{\hat{f}}$.
Step 4: bounds on $\underline{\hat{f}}$

For $I \in \mathcal{I} := \{(1,\alpha),\dots,(n_\alpha-1,\alpha);(1,\beta),\dots,(n_\beta-1,\beta)\}$, we set

$$\underline{f}^I = \left\{ \begin{array}{ll} f_\alpha^k & \text{if} \quad I=(k,\alpha) \\ f_\beta^k & \text{if} \quad I=(k,\beta) \end{array} \right. \quad \text{and} \quad c^I = \left\{ \begin{array}{ll} c_\alpha^k & \text{if} \quad I=(k,\alpha) \\ c_\beta^k & \text{if} \quad I=(k,\beta) \end{array} \right. \quad \text{for} \quad c=a,b$$

By definition of $\underline{\hat{f}}$, we still get that $\underline{f}_-^I \le \underline{\hat{f}}^I \le \underline{f}_+^I$ for all $I \in \mathcal{I}$, i.e. $\underline{f}_- \le \underline{\hat{f}} \le \underline{f}_+$.

Step 5: local constancy of $\underline{\hat{f}}$

Consider $P=(p_\alpha,p_\beta) \in [a,b]'_\alpha \times [a,b]'_\beta$ be such that $\underline{f}(P) \neq \underline{\hat{f}}(P) = \tilde{f}(r,P)$, and let us set $K_P := \left\{ I \in \mathcal{I},\ \underline{\hat{f}}^I(P) \neq \underline{f}^I(P) \right\}$. Recall that $\tilde{f}(\cdot,P) = \underline{\hat{f}}(P)$ on R. In particular, for $Q_\varepsilon := P + \sum_{I \in K_P} (-\varepsilon,\varepsilon)e_I$, we get from the continuity of $\tilde{f}$ that for $\varepsilon > 0$ small enough, we have $\tilde{f} = const = \underline{\hat{f}}(P)$ on $R \times \left(\left([a,b]'_\alpha \times [a,b]'_\beta \right) \cap Q_\varepsilon \right)$. Therefore $\underline{\hat{f}} = const = \underline{\hat{f}}(P)$ on $\left([a,b]'_\alpha \times [a,b]'_\beta \right) \cap Q_\varepsilon$, which shows the local constancy of $\underline{\hat{f}}$.

Step 6: Riemann monotonicity of $\underline{\sigma} \diamond \underline{\hat{f}}$

Step 6.1: preliminary

By assumption, we know that both $\sigma_\gamma \diamond \hat{f}_\gamma$ are Riemann monotone. Precisely, we know that for $P=(p_\alpha,p_\beta)$ and $p := p_\alpha$, $q := p_\beta$, we have

$$\left\{ \begin{array}{lll} ((\bar{r},\bar{p})-(r,p)) \diamond \sigma_\alpha \diamond [\hat{f}_\alpha]_{(r,p)}^{(\bar{r},\bar{p})} \le 0 & \Longrightarrow & [\hat{f}_\alpha]_{(r,p)}^{(\bar{r},\bar{p})} = 0 \\ ((\bar{r},\bar{q})-(r,q)) \diamond \sigma_\beta \diamond [\hat{f}_\beta]_{(r,q)}^{(\bar{r},\bar{q})} \le 0 & \Longrightarrow & [\hat{f}_\beta]_{(r,q)}^{(\bar{r},\bar{q})} = 0 \end{array} \right. \tag{5.30}$$

and consider $(\bar{p},\bar{q})$ and (p,q) such that $((\bar{p},\bar{q})-(p,q)) \diamond \underline{\sigma} \diamond [\underline{\hat{f}}]_{(p,q)}^{(\bar{p},\bar{q})} \le 0$, and we want to show that $[\underline{\hat{f}}]_{(p,q)}^{(\bar{p},\bar{q})} = 0$. Here we set $\underline{\hat{f}}(\bar{p},\bar{q}) = \tilde{f}(\bar{r},\bar{p},\bar{q})$, $\underline{\hat{f}}(p,q) = \tilde{f}(r,p,q)$. Hence

$$\left\{ \begin{array}{l} \max_{k \in \{1,\dots,N_\alpha-1\}} (\bar{p}^k - p^k) \cdot \sigma_\alpha^k \cdot \left\{ \hat{f}_\alpha^k(\bar{r},\bar{p}) - \hat{f}_\alpha^k(r,p) \right\} \le 0 \\ \max_{k \in \{1,\dots,N_\beta-1\}} (\bar{q}^k - q^k) \cdot \sigma_\beta^k \cdot \left\{ \hat{f}_\beta^k(\bar{r},\bar{q}) - \hat{f}_\beta^k(r,q) \right\} \le 0 \end{array} \right.$$

Step 6.2: core of the argument

Recall that $\tilde{g}(\bar{r},\bar{p},\bar{q}) = 0 = \tilde{g}(r,p,q)$, and then

$$\left\{ \hat{f}_\alpha^0(\bar{r},\bar{p}) - \hat{f}_\beta^0(\bar{r},\bar{q}) \right\} - \left\{ \hat{f}_\alpha^0(r,p) - \hat{f}_\beta^0(r,q) \right\} = 0 \tag{5.31}$$

which implies (multiplying by $-(\bar{r}-r)$)

$$-(\bar{r}-r) \cdot \left\{ \hat{f}_\alpha^0(\bar{r},\bar{p}) - \hat{f}_\alpha^0(r,p) \right\} + (\bar{r}-r) \cdot \left\{ \hat{f}_\beta^0(\bar{r},\bar{q}) - \hat{f}_\beta^0(r,q) \right\} = 0 \tag{5.32}$$

Hence

$$-(\bar{r}-r) \cdot \left\{ \hat{f}_\alpha^0(\bar{r},\bar{p}) - \hat{f}_\alpha^0(r,p) \right\} \le 0 \quad \text{with} \quad \sigma_\alpha^0 = -1 \tag{5.33}$$

or

$$(\bar{r}-r) \cdot \left\{ \hat{f}_\beta^0(\bar{r},\bar{q}) - \hat{f}_\beta^0(r,q) \right\} \le 0 \quad \text{with} \quad \sigma_\beta^0 = +1 \tag{5.34}$$

If (5.33) holds true, then (5.30) implies $[\hat{f}_\alpha]_{(r,p)}^{(\bar{r},\bar{p})} = 0$. Therefore equation (5.32) implies that inequality (5.34) also hods true, and then $[\hat{f}_\beta]_{(r,q)}^{(\bar{r},\bar{q})} = 0$. This implies $[\underline{\hat{f}}]_{(p,q)}^{(\bar{p},\bar{q})} = 0$ This shows that $\underline{\sigma} \diamond \underline{\hat{f}}$ is Riemann monotone.

Step 7: conclusion

In particular, the Riemann monotonicity of $\underline{\sigma} \diamond \underline{\hat{f}}$ implies the directional monotonicity $\underline{\sigma} \diamond \underline{\hat{f}}$, i.e. for $P := (p_\alpha,p_\beta)$, we deduce that $P^I \mapsto \underline{\sigma}^I \underline{\hat{f}}^I(P)$ is nondecreasing on $[a,b]^I := [a^I,b^I]$. We conclude that $\mathcal{G} := \left\{ P \in [a,b]'_\alpha \times [a,b]'_\beta, \quad \underline{\hat{f}}(P) = \underline{f}(P) \right\}$ is a Riemann germ (because $\underline{\hat{f}}$ is continuous), and $\underline{\hat{f}} = \underline{f}_{\mathcal{G}}$.

Step 8: additional argument for Kružkov germs

Recall from Lemma 5.5 that for Kružkov germs $\mathcal{G}_\gamma$, we have

$$\text{IN} - \text{OUT} = \sum_{k=0,\dots,N_\gamma-1} \sigma_\gamma^k \cdot \operatorname{sign}(p_\gamma^k - q_\gamma^k) \cdot \left\{ \hat{f}_\gamma^k(p_\gamma) - \hat{f}_\gamma^k(q_\gamma) \right\} \geq 0 \quad \text{for all} \quad p_\gamma, q_\gamma \in [a,b]_\gamma$$

which means in particular that

$$\begin{cases} -\operatorname{sign}(\bar{r} - r) \cdot \left\{ \hat{f}_\alpha^0(\bar{r},\bar{p}) - \hat{f}_\alpha^0(r,p) \right\} + S'_\alpha \geq 0 & \text{with} \quad S'_\alpha := \displaystyle\sum_{k=1,\dots,N_\alpha-1} \sigma_\alpha^k \cdot \operatorname{sign}(\bar{p}^k - p^k) \cdot \left\{ \hat{f}_\alpha^k(\bar{r},\bar{p}) - \hat{f}_\alpha^k(r,p) \right\} \\ \operatorname{sign}(\bar{r} - r) \cdot \left\{ \hat{f}_\beta^0(\bar{r},\bar{q}) - \hat{f}_\beta^0(r,q) \right\} + S'_\beta \geq 0 & \text{with} \quad S'_\beta := \displaystyle\sum_{k=1,\dots,N_\beta-1} \sigma_\beta^k \cdot \operatorname{sign}(\bar{q}^k - q^k) \cdot \left\{ \hat{f}_\beta^k(\bar{r},\bar{q}) - \hat{f}_\beta^k(r,q) \right\} \end{cases}$$

Because we have $\tilde{g}(\bar{r},\bar{p},\bar{q}) = 0 = \tilde{g}(r,p,q)$ which implies (5.31), we can now take the sum of both inequalities and get that $S'_\alpha + S'_\beta \geq 0$, i.e. for $\bar{P} := (\bar{p},\bar{q})$ and $P = (p,q)$, that

$$\sum_{I \in \mathcal{I}} \underline{\sigma}^I \cdot \operatorname{sign}(\bar{P}^I - P^I) \cdot \left\{ \underline{\hat{f}}^I(\bar{P}) - \underline{\hat{f}}^I(P) \right\} \geq 0$$

which shows exactly that $\mathcal{G}$ is a Kružkov germ.

Step 9: additional argument for HJ germs

Recall from Lemma 5.5 that for HJ germs $\mathcal{G}_\gamma$, there exists $\hat{h}_\gamma : [a,b]_\gamma \to \mathbb{R}$ such that $\hat{f}_\gamma^k = \hat{h}_\gamma$ for all $k = 0,\dots,N_\gamma - 1$. Recall that $0 = g(r) = \tilde{g}(r,p,q) = \hat{f}_\alpha^0(r,p) - \hat{f}_\beta^0(r,q)$ with

$$\begin{cases} \hat{f}_\alpha^0(r,p) = \hat{h}_\alpha(r,p) = \hat{f}_\alpha^j(r,p) & \text{for all} \quad j \in \{1,\dots,N_\alpha\} \\ \hat{f}_\beta^0(r,q) = \hat{h}_\beta(r,q) = \hat{f}_\beta^k(r,q) & \text{for all} \quad k \in \{1,\dots,N_\beta\} \end{cases}$$

Hence defining $\hat{h}(p,q) := \hat{f}_\alpha^0(r,p) = \hat{f}_\beta^0(r,q)$, we see from Step 2 that this quantity is the same for r' such that $g(r') = 0$, and then is well defined. Moreover, we also have $\underline{\hat{f}}^I = \hat{h}$ for all $I \in \mathcal{I}$. This shows that $\mathcal{G}$ is a HJ germ.

Step 10: additional argument for monotone germs

Recall that $\mathcal{G}_\gamma$ are monotone germs, i.e. that

$$p \mapsto \sigma_\gamma^j \hat{f}_\gamma^j(p) \quad \text{is nonincreasing in } p^k \text{ for all } k \neq j \tag{5.35}$$

Assume by contradiction that $\mathcal{G}$ is not a monotone germ, i.e. that

$$P \mapsto \underline{\sigma}^I \underline{\hat{f}}^I(P) \quad \text{is NOT nonincreasing in } p^K \text{ for all } K \neq I \tag{5.36}$$

i.e. that there exists $P, \tilde{P}$ such that $P - \tilde{P} \in (0,+\infty) \cdot e_K$ and $\underline{\sigma}^I \underline{\hat{f}}^I(P) > \underline{\sigma}^I \underline{\hat{f}}^I(\tilde{P})$.

Case A: $I := (\alpha, j)$ **and** $K := (\alpha, k)$ **with** $k \neq j$

Then we can write

$$P = (p,q), \quad \tilde{P} = (\tilde{p},q), \quad p - \tilde{p} \in (0,+\infty) \cdot e_k, \quad \sigma_\alpha^j \hat{f}_\alpha^j(r,p) > \sigma_\alpha^j \hat{f}_\alpha^j(\tilde{r},\tilde{p}) \tag{5.37}$$

with

$$\begin{cases} 0 = g(r) = \hat{f}_\alpha^0(r,p) - \hat{f}_\beta^0(r,q) \\ 0 = g(\tilde{r}) = \hat{f}_\alpha^0(\tilde{r},\tilde{p}) - \hat{f}_\beta^0(\tilde{r},q) \end{cases} \quad \text{and} \quad \begin{cases} \hat{f}_\alpha^0(\downarrow,\tilde{p}), \quad \hat{f}_\beta^0(\uparrow,q) \\ z \mapsto \sigma_\alpha^0 \hat{f}_\alpha^0(\tilde{r},\tilde{p} + z e_k) \quad \text{nonincreasing with } \sigma_\alpha^0 = -1 \end{cases}$$

We deduce that $r \geq \tilde{r}$, $p \geq \tilde{p}$ with $p^j = \tilde{p}^j$, and then monotonicity (5.35) implies $\sigma_\alpha^j \hat{f}_\alpha^j(r,p) \leq \sigma_\alpha^j \hat{f}_\alpha^j(\tilde{r},\tilde{p})$, which is in contradiction with (5.37).

Case A': $I := (\beta, j)$ **and** $K := (\beta, k)$ **with** $k \neq j$

This case is symmetric to case A (up to a reversion transform (see Definition (4.4)) on component f_β^0).

Case B: $I := (\alpha, j)$ **and** $K := (\beta, k)$

Then we can write

$$P = (p,q), \quad \tilde{P} = (p,\tilde{q}), \quad q - \tilde{q} \in (0,+\infty) \cdot e_k, \quad \sigma_\alpha^j \hat{f}_\alpha^j(r,p) > \sigma_\alpha^j \hat{f}_\alpha^j(\tilde{r},\tilde{p}) \tag{5.38}$$

with

$$\left\{\begin{array}{l} 0=g(r)=\hat f^0_\alpha(r,p)-\hat f^0_\beta(r,q)\\ 0=g(\tilde r)=\hat f^0_\alpha(\tilde r,p)-\hat f^0_\beta(\tilde r,\tilde q)\end{array}\right. \quad \text{and} \quad \left\{\begin{array}{l} \hat f^0_\alpha(\downarrow,p),\quad \hat f^0_\beta(\uparrow,\tilde q)\\ z\mapsto \sigma^0_\beta \hat f^0_\beta(\tilde r,\tilde q+ze_k)\quad \text{nonincreasing with } \sigma^0_\beta=+1\end{array}\right.$$

we get that $r\ge \tilde r$, $q\ge \tilde q$, and then monotonicity (5.35) implies $\sigma^j_\alpha \hat f^j_\alpha(r,p)\le \sigma^j_\alpha \hat f^j_\alpha(\tilde r,p)$, which is in contradiction with (5.38).

Conclusion

Therefore (5.36) is false and this shows that $\mathcal{G}$ is a monotone germ.

Step 11: additional argument for conservative germs

Recall from Lemma 5.5 that $\mathcal{G}_\gamma$ are conservative germs, i.e.

$$\sum_{J^j_\alpha\simeq(-\infty,0)} \hat f^j_\alpha(r,p)=\hat f^0_\alpha(r,p)+\sum_{j\neq 0,\ J^j_\alpha\simeq(0,+\infty)} \hat f^j_\alpha(r,p)\quad \text{for all}\quad (r,p)\in[a^0,b^0]\times[a,b]'_\alpha$$

and

$$\hat f^0_\beta(r,q)+\sum_{j\neq 0,\ J^j_\beta\simeq(-\infty,0)} \hat f^j_\beta(r,q)=\sum_{J^j_\beta\simeq(0,+\infty)} \hat f^j_\beta(r,q)\quad \text{for all}\quad (r,q)\in[a^0,b^0]\times[a,b]'_\beta$$

For r such that $\hat f^0_\alpha(r,p)=\hat f^0_\beta(r,q)$, we deduce that $\sum_{\underline{J}^I\simeq(-\infty,0)} \underline{\hat f}^I(p,q)=\sum_{\underline{J}^I\simeq(0,+\infty)} \underline{\hat f}^I(p,q)$, which shows that $\mathcal{G}$ is conservative. This ends the proof of the lemma.

Then we have the following corollary of Proposition 5.13.

Corollary 5.14 (Gluing of Riemann germs $\mathcal{G}_\gamma$ for $n_\gamma:m_\gamma$ junctions)
For $\gamma=\alpha,\beta$, assume that f_γ satisfies (2.2) for $N_\gamma=n_\gamma+m_\gamma$ and $n_\gamma:m_\gamma$ junctions J_γ with $J^j_\gamma\simeq\sigma^j_\gamma\cdot(-\infty,0)$ and $\sigma_\gamma\in\{\pm1\}^{N_\gamma}$, and consider Riemann germs $\mathcal{G}_\gamma$ with respect to (J_γ,f_γ).

For $\gamma=\alpha$ or β, assume either that 1) f_γ satisfies nondegeneracy condition (2.17), or that 2) $\mathcal{G}_\gamma$ is a Kružkov germ. We allow mixing cases for α and β.

We set $[a,b]^j_\gamma:=[a^j_\gamma,b^j_\gamma]$. We assume that for each $\gamma=\alpha,\beta$, there exists one index $j_\gamma\in\{1,\dots,N_\gamma\}$ such that

$$f^{j_\alpha}_\alpha=f^{j_\beta}_\beta=:f^0\quad \textit{on}\quad [a,b]^{j_\alpha}_\alpha=[a,b]^{j_\beta}_\beta=:[a^0,b^0]\quad \textit{with}\quad J^{j_\alpha}_\alpha\simeq(0,+\infty)\quad \textit{and}\quad J^{j_\beta}_\beta\simeq(-\infty,0)$$

and we glue those two branches. We call $\hat f_\gamma:[a,b]_\gamma\to\mathbb{R}^{n_\gamma}$ the associated fluxes. We set

$$\left\{\begin{array}{l} f'_\gamma:=(f^1_\gamma,\dots,f^{j_\gamma-1}_\gamma,f^{j_\gamma+1}_\gamma,\dots,f^{N_\gamma}_\gamma)\\ \hat f'_\gamma:=(\hat f^1_\gamma,\dots,\hat f^{j_\gamma-1}_\gamma,\hat f^{j_\gamma+1}_\gamma,\dots,\hat f^{N_\gamma}_\gamma)\\ (r,p_\gamma):=(p^1_\gamma,\dots,p^{j_\gamma-1}_\gamma,r,p^{j_\gamma+1}_\gamma,\dots,p^{N_\gamma}_\gamma)\end{array}\right.$$

with an abuse of notation for the last line. Then we define

$$\hat f:=\hat f_\alpha \underset{j_\alpha:j_\beta}{\sharp} \hat f_\beta:[a,b]'_\alpha\times[a,b]'_\beta\to\mathbb{R}^{N_\alpha+N_\beta-2}\quad \textit{with}\quad [a,b]'_\gamma:=\prod_{j\in\{1,\dots,N_\gamma\}\setminus\{j_\gamma\}}[a,b]^j_\gamma \tag{5.39}$$

where for any $p_\alpha\in[a,b]'_\alpha$ and $p_\beta\in[a,b]'_\beta$ (avoiding notation p'_α,p'_β to keep light notations) and

$$\left\{\begin{array}{l} f(p_\alpha,p_\beta):=(f'_\alpha;f'_\beta)(p_\alpha,p_\beta)\\ \hat f(p_\alpha,p_\beta):=(\hat f'_\alpha;\hat f'_\beta)(r,p_\alpha,p_\beta)\quad \textit{for some } r\in[a^0,b^0] \textit{ such that } \hat f^{j_\alpha}_\alpha(r,p_\alpha)=\hat f^{j_\beta}_\beta(r,p_\beta)\end{array}\right.$$

Here such r does exist, and when it is not unique, it does not change the value of $\hat f(p_\alpha,p_\beta)$.
We define $\mathcal{G}:=\mathcal{G}_\alpha\sharp\mathcal{G}_\beta$, the gluing of germs $\mathcal{G}_\alpha$ with $\mathcal{G}_\beta$ along branches $J^{j_\alpha}_\alpha$ and $J^{j_\beta}_\beta$, as

$$\mathcal{G}:=\mathcal{G}_\alpha \underset{j_\alpha:j_\beta}{\sharp} \mathcal{G}_\beta:=\Big\{P\in[a,b]'_\alpha\times[a,b]'_\beta,\quad \hat f(P)=f(P)\Big\} \tag{5.40}$$

Moreover the following holds.

i) (Gluing germs)
Let us define

$$\mathcal{G}_0 := \left\{ (\hat{p}_\alpha, \hat{p}_\beta) \in [a,b]'_\alpha \times [a,b]'_\beta, \quad \text{there exists} \quad \hat{r}_\alpha, \hat{r}_\beta \in [a^0, b^0] \quad s.t. \quad \left\{ \begin{array}{l} (\hat{r}_\alpha, \hat{p}_\alpha) \in \mathcal{G}_\alpha \\ (\hat{r}_\beta, \hat{p}_\beta) \in \mathcal{G}_\beta \\ f^0(\hat{r}_\alpha) = G^{f^0}(\hat{r}_\alpha, \hat{r}_\beta) = f^0(\hat{r}_\beta) \end{array} \right. \right\}$$

where we recall that the standard Godunov flux is given by

$$G^{f^0}(\hat{r}_\alpha, \hat{r}_\beta) = \left\{ \begin{array}{lll} \min\limits_{[\hat{r}_\alpha, \hat{r}_\beta]} f^0 & \text{if} & \hat{r}_\alpha \le \hat{r}_\beta \\ \max\limits_{[\hat{r}_\beta, \hat{r}_\alpha]} f^0 & \text{if} & \hat{r}_\alpha \ge \hat{r}_\beta \end{array} \right.$$

Then we have

$$\mathcal{G} = \mathcal{G}_0 \tag{5.41}$$

ii) (identity element: gluing with standard Godunov flux)
a) *Assume that $N_\beta = 2$ with $1:1$ junction J_β with $\hat{f}^j_\beta(r,s) = G^{f^0}(r,s)$ for $j = 0,1$. Then $\hat{f} = \hat{f}_\alpha$ and $\mathcal{G} = \mathcal{G}_\alpha$.*
b) *Assume that $N_\alpha = 2$ with $1:1$ junction J_α with $\hat{f}^j_\alpha(r,s) = G^{f^0}(r,s)$ for $j = 0,1$. Then $\hat{f} = \hat{f}_\beta$ and $\mathcal{G} = \mathcal{G}_\beta$.*

Proof of Corollary 5.14
Step 0: preliminary remarks
We first notice that from Lemma 5.12 both $\hat{f}_\gamma$ are j_γ-locally quasi-constant. Moreover from Proposition 5.8 and Corollary 5.10, we deduce that $\sigma \diamond \hat{f}$ is Riemann monotone. Those are two key assumptions in Proposition 5.13, that we use below.
Step 1: proof of i)
From definition given in (5.40), we see that

$$(p_\alpha, p_\beta) \in \mathcal{G} \quad \Longleftrightarrow \quad \exists\ r \in [a,b]^{j_\alpha}_\alpha = [a,b]^{j_\beta}_\beta \text{ s.t. } \left\{ \begin{array}{l} (\hat{r}_\alpha, \hat{p}_\alpha) = \pi_{\mathcal{G}_\alpha}(r, p_\alpha) \\ (\hat{r}_\beta, \hat{p}_\beta) = \pi_{\mathcal{G}_\beta}(r, p_\beta) \\ f'_\alpha(\hat{p}_\alpha) = f'_\alpha(\hat{r}_\alpha, \hat{p}_\alpha) = \hat{f}'_\alpha(r, p_\alpha) = f'_\alpha(p_\alpha) \\ f'_\beta(\hat{p}_\beta) = f'_\beta(\hat{r}_\beta, \hat{p}_\beta) = \hat{f}'_\beta(r, p_\beta) = f'_\beta(p_\beta) \\ f^0(\hat{r}_\alpha) = f^{j_\alpha}_\alpha(\hat{r}_\alpha) = \hat{f}^{j_\alpha}_\alpha(r, p_\alpha) = \hat{f}^{j_\beta}_\beta(r, p_\beta) = f^{j_\beta}_\beta(\hat{r}_\beta) = f^0(\hat{r}_\beta) \end{array} \right.$$

Notice that $p_\alpha \in BA(\hat{p}_\alpha)$, $p_\beta \in BA(\hat{p}_\beta)$, $r \in BA_\alpha^{((0,+\infty), f^{j_\alpha}_\alpha)}(\hat{r}_\alpha) \cap BA_\beta^{((-\infty,0), f^{j_\beta}_\beta)}(\hat{r}_\beta)$. From the basins of attraction, we then deduce that

$$p_\alpha = \hat{p}_\alpha, \quad p_\beta = \hat{p}_\beta, \quad f^0(\hat{r}_\alpha) = G^{f^0}(\hat{r}_\alpha, \hat{r}_\beta) = f^0(\hat{r}_\beta), \quad \hat{r}_\alpha, \hat{r}_\beta \in [a^0, b^0]$$

which shows that $(\hat{p}_\alpha, \hat{p}_\beta) \in \mathcal{G}_0$. It is straighforward to check conversely that $(p_\alpha, p_\beta) \in \mathcal{G}_0$ implies $(p_\alpha, p_\beta) \in \mathcal{G}$. Therefore this shows characterization (5.41) of $\mathcal{G}$.
Step 2: proof of ii)
We only prove a) (the proof of b) is similar). Assume that $\hat{f}^j_\beta = G^{f^0}$. Then

$$\hat{f}(p_\alpha, p_\beta) = \tilde{f}(r, p_\alpha, p_\beta) \quad \text{for any } r \in [a^0, b^0] \text{ s.t. } \quad \hat{f}^0_\alpha(p_\alpha, r) = G^{f^0}(r, p_\beta)$$

In particular, if $P = (p_\alpha, p_\beta) \in \mathcal{G}_\alpha$, then we can choose $r = p_\beta$, and we see that $\hat{f}(P) = f(P)$, i.e. $P \in \mathcal{G}$. This shows that $\mathcal{G}_\alpha \subset \mathcal{G}$. Because both are generalized Riemann germs, we deduce that we have equality, i.e. $\mathcal{G}_\alpha = \mathcal{G}$, and then moreover $\hat{f} = \hat{f}_{\mathcal{G}} = \hat{f}_{\mathcal{G}_\alpha} = \hat{f}_\alpha$. This ends the proof.

Lemma 5.15 (Associativity of the gluing)
For $\delta = \alpha, \beta, \gamma$, assume that f_δ satisfies (2.2) for $N_\delta = n_\delta + m_\delta$ and $n_\delta : m_\delta$ junctions J_δ with $J^j_\delta \simeq \sigma^j_\delta \cdot (-\infty, 0)$ and $\sigma_\delta \in \{\pm 1\}^{N_\gamma}$, and consider Riemann germs $\mathcal{G}_\delta$ with respect to (J_δ, f_δ).

For $\delta = \alpha$, β or γ, assume either 1) that f_δ satisfies nondegeneracy condition (2.17), or 2) that $\mathcal{G}_\delta$ is a Kružkov germ. We allow mixing cases for α, β and γ.

We set $[a,b]^j_\delta := [a^j_\delta, b^j_\delta]$*. We also assume that there exists* $j_\alpha, j_\beta, k_\gamma, k_\beta$*, with* $j_\delta, k_\delta \in \{1,\dots,N_\delta\}$ *such that*

$$\begin{cases} f^{j_\alpha}_\alpha = f^{j_\beta}_\beta =: f^A, & [a,b]^{j_\alpha}_\alpha = [a,b]^{j_\beta}_\beta =: [a,b]^A \quad -\sigma^{j_\alpha}_\alpha = 1 = \sigma^{j_\beta}_\beta, \quad J^{j_\alpha}_\alpha \simeq (0,+\infty), \quad J^{j_\beta}_\beta \simeq (-\infty,0), \\ f^{k_\beta}_\beta = f^{k_\gamma}_\gamma =: f^B, & [a,b]^{k_\beta}_\beta = [a,b]^{k_\gamma}_\gamma =: [a,b]^B \quad -\sigma^{k_\beta}_\beta = 1 = \sigma^{k_\gamma}_\gamma, \quad J^{k_\beta}_\beta \simeq (0,+\infty), \quad J^{k_\gamma}_\gamma \simeq (-\infty,0), \quad k_\beta \neq j_\beta \end{cases}$$

With notation of Corollary 5.14, we glue

$$(\mathcal{G}_\alpha \sharp \mathcal{G}_\beta)\sharp \mathcal{G}_\gamma := (\mathcal{G}_\alpha \underset{j_\alpha : j_\beta}{\sharp} \mathcal{G}_\beta) \underset{k_\beta : k_\gamma}{\sharp} \mathcal{G}_\gamma \quad \textit{gluing first } J^{j_\alpha}_\alpha \textit{ with } J^{j_\beta}_\beta, \textit{ and then } J^{k_\beta}_\beta \textit{ with } J^{k_\gamma}_\gamma$$

and also glue

$$\mathcal{G}_\alpha \sharp (\mathcal{G}_\beta \sharp \mathcal{G}_\gamma) := \mathcal{G}_\alpha \underset{j_\alpha : j_\beta}{\sharp} (\mathcal{G}_\beta \underset{k_\beta : k_\gamma}{\sharp} \mathcal{G}_\gamma) \quad \textit{gluing first } J^{k_\beta}_\beta \textit{ with } J^{k_\gamma}_\gamma, \textit{ and then } J^{j_\alpha}_\alpha \textit{ with } J^{j_\beta}_\beta$$

Then we have

$$(\mathcal{G}_\alpha \sharp \mathcal{G}_\beta)\sharp \mathcal{G}_\gamma = \mathcal{G}_\alpha \sharp (\mathcal{G}_\beta \sharp \mathcal{G}_\gamma) \tag{5.42}$$

Proof of Lemma 5.15

We set

$$\begin{cases} [a,b]'_\delta := \prod_{j\in\{1,\dots,N_\delta\}\setminus\{j_\delta\}} [a,b]^j_\delta & \text{for} \quad \delta = \alpha,\beta \\ [a,b]^*_\delta := \prod_{k\in\{1,\dots,N_\delta\}\setminus\{k_\delta\}} [a,b]^k_\delta & \text{for} \quad \delta = \beta,\gamma \\ [a,b]''_\beta := \prod_{j\in\{1,\dots,N_\beta\}\setminus\{j_\beta,k_\beta\}} [a,b]^j_\beta \end{cases}$$

Step 1: first computation

We simply compute

$$\mathcal{G}_{\alpha\beta} := \mathcal{G}_\alpha \sharp \mathcal{G}_\beta := \left\{ \begin{array}{l} (\hat p_\alpha, \hat p_\beta) \in [a,b]'_\alpha \times [a,b]'_\beta, \\ \text{there exists} \quad \hat r_\alpha, \hat r_\beta \in [a,b]^A \quad \text{s.t.} \quad \begin{cases} (\hat p_\alpha, \hat r_\alpha) \in \mathcal{G}_\alpha \\ (\hat p_\beta, \hat r_\beta) \in \mathcal{G}_\beta \\ f^A(\hat r_\alpha) = G^{f^A}(\hat r_\alpha, \hat r_\beta) = f^A(\hat r_\beta) \end{cases} \end{array} \right\}$$

with abuse of notation for $(\hat p_\alpha, \hat r_\alpha)$ and $(\hat p_\beta, \hat r_\beta)$, and similarly

$$\mathcal{G}_{\alpha\beta}\sharp \mathcal{G}_\gamma := \left\{ \begin{array}{l} (\hat p_{\alpha\beta}, \hat p_\gamma) \in \Big([a,b]'_\alpha \times [a,b]''_\beta\Big) \times [a,b]^*_\gamma, \\ \text{there exists} \quad \hat r'_\beta, \hat r_\gamma \in [a,b]^B \quad \text{s.t.} \quad \begin{cases} (\hat p_{\alpha\beta}, \hat r'_\beta) \in \mathcal{G}_{\alpha\beta} \\ (\hat p_\gamma, \hat r_\gamma) \in \mathcal{G}_\gamma \\ f^B(\hat r'_\beta) = G^{f^B}(\hat r'_\beta, \hat r_\gamma) = f^B(\hat r_\gamma) \end{cases} \end{array} \right\}$$

Hence for $\hat p_{\alpha\beta} = (\hat p_\alpha, \hat p'_\beta)$ we get

$$\mathcal{G}_{\alpha\beta}\sharp \mathcal{G}_\gamma := \left\{ \begin{array}{l} (\hat p_\alpha, \hat p'_\beta, \hat p_\gamma) \in [a,b]'_\alpha \times [a,b]''_\beta \times [a,b]^*_\gamma, \\ \begin{cases} \text{there exists} \quad \hat r_\alpha, \hat r_\beta \in [a,b]^A \\ \text{there exists} \quad \hat r'_\beta, \hat r_\gamma \in [a,b]^B \end{cases} \quad \text{s.t.} \quad \begin{cases} (\hat p_\alpha, \hat r_\alpha) \in \mathcal{G}_\alpha \\ (\hat p'_\beta, \hat r'_\beta, \hat r_\beta) \in \mathcal{G}_\beta \\ (\hat p_\gamma, \hat r_\gamma) \in \mathcal{G}_\gamma \\ f^A(\hat r_\alpha) = G^{f^A}(\hat r_\alpha, \hat r_\beta) = f^A(\hat r_\beta) \\ f^B(\hat r'_\beta) = G^{f^B}(\hat r'_\beta, \hat r_\gamma) = f^B(\hat r_\gamma) \end{cases} \end{array} \right\}$$

Step 2: second computation

We simply compute

$$\mathcal{G}_{\beta\gamma} := \mathcal{G}_\beta \sharp \mathcal{G}_\gamma := \left\{ \begin{array}{l} (\hat p_\beta, \hat p_\gamma) \in [a,b]^*_\beta \times [a,b]^*_\gamma, \\ \text{there exists} \quad \hat r'_\beta, \hat r_\gamma \in [a,b]^B \quad \text{s.t.} \quad \begin{cases} (\hat p_\beta, \hat r'_\beta) \in \mathcal{G}_\beta \\ (\hat p_\gamma, \hat r_\gamma) \in \mathcal{G}_\gamma \\ f^B(\hat r'_\beta) = G^{f^B}(\hat r'_\beta, \hat r_\gamma) = f^B(\hat r_\gamma) \end{cases} \end{array} \right\}$$

and then for $\hat{p}_{\beta\gamma} = (\hat{p}'_\beta, \hat{p}_\gamma)$

$$\mathcal{G}_\alpha \sharp \mathcal{G}_{\beta\gamma} := \left\{ \begin{array}{l} (\hat{p}_\alpha, \hat{p}_{\beta\gamma}) \in [a,b]'_\alpha \times [a,b]''_\beta \times [a,b]^*_\gamma, \\ \\ \text{there exists} \quad \hat{r}_\alpha, \hat{r}_\beta \in [a,b]^A \quad \text{s.t.} \quad \left\{ \begin{array}{l} (\hat{p}_\alpha, \hat{r}_\alpha) \in \mathcal{G}_\alpha \\ (\hat{p}'_\beta, \hat{r}_\beta, \hat{p}_\gamma) \in \mathcal{G}_{\beta\gamma} \\ f^A(\hat{r}_\alpha) = G^{f^A}(\hat{r}_\alpha, \hat{r}_\beta) = f^A(\hat{r}_\beta) \end{array} \right. \end{array} \right\}$$

i.e.

$$\mathcal{G}_\alpha \sharp \mathcal{G}_{\beta\gamma} := \left\{ \begin{array}{l} (\hat{p}_\alpha, \hat{p}'_\beta, p_\gamma) \in [a,b]'_\alpha \times [a,b]''_\beta \times [a,b]^*_\gamma, \\ \\ \left\{ \begin{array}{l} \text{there exists} \quad \hat{r}_\alpha, \hat{r}_\beta \in [a,b]^A \\ \text{there exists} \quad \hat{r}'_\beta, \hat{r}_\gamma \in [a,b]^B \end{array} \right. \quad \text{s.t.} \quad \left\{ \begin{array}{l} (\hat{p}_\alpha, \hat{r}_\alpha) \in \mathcal{G}_\alpha \\ (\hat{p}'_\beta, \hat{r}'_\beta, \hat{r}_\beta) \in \mathcal{G}_\beta \\ (\hat{p}_\gamma, \hat{r}_\gamma) \in \mathcal{G}_\gamma \\ f^A(\hat{r}_\alpha) = G^{f^A}(\hat{r}_\alpha, \hat{r}_\beta) = f^A(\hat{r}_\beta) \\ f^B(\hat{r}'_\beta) = G^{f^B}(\hat{r}'_\beta, \hat{r}_\gamma) = f^B(\hat{r}_\gamma) \end{array} \right. \end{array} \right\}$$

Step 3: conclusion
Hence $\mathcal{G}_\alpha \sharp \mathcal{G}_{\beta\gamma} = \mathcal{G}_{\alpha\beta} \sharp \mathcal{G}_\gamma$, which shows (5.42). This ends the proof of the lemma.

Proof of Theorem 2.24
For the proof, we refer to the table of Subsection 2.5. The result follows from Proposition 5.13, Corollary 5.14 and Lemma 5.15. This ends the proof.

5.7 Absence of self-gluing of germs in general

It is natural to ask if the self-gluing of a germ is possible in general.

Let $\mathcal{G}$ be a germ associated to a junction (J,f) with N branches for $j = 0, \dots, N-1$. Assume that $J^{N-1} \simeq (-\infty, 0)$ and $J^0 \simeq (0, +\infty)$ with $f^0 = f^{N-1}$ and $[a^0, b^0] = [a^{N-1}, b^{N-1}]$. If we glue together branch J^0 with branch J^{N-1}, we create a loop in the network. The loop starts from the node 0 goes to J^0 and then to J^{N-1} (because of the gluing) and finally goes back to the node 0.

Assume for instance that we have a nice existence and uniqueness theory for a network with loops of finite length $\varepsilon > 0$. The difficulty is that in general, we can not simply identify the limit behaviour as ε goes to zero, by an effective network with effective germs.

This is already the case for a network composed of a single germ $\mathcal{G}$, with a self-gluing of a single loop of length $\varepsilon > 0$. Contrarily to what could be expected, the limit behaviour may not be described by a self-gluing of the germ for $\varepsilon = 0$. This is already due to the fact that the self-gluing of a germ $\mathcal{G}$ itself is not well-defined in general.

We will give counter-examples in Section 9.5. We show in particular that the the self-gluing of some HJ germs is not well-defined.

5.8 Self-gluing of Kružkov germs and more

In Subsection 5.7, we have seen that the self-gluing of a germ $\mathcal{G}$ itself is not well-defined in general. On the contrary, it is well defined when $\mathcal{G}$ is a Kružkov germ, and this is a non-trivial fact. This obviously extends to networks with loops.

Proposition 5.16 (Self-gluing of a flux function $\hat{f}_\gamma$ for $n_\gamma : m_\gamma$ junction, for a Kružkov germ)
Let γ be a fixed index. Assume that the function f_γ satisfies (2.2) with $N_\gamma = n_\gamma + m_\gamma$ with $n_\gamma, m_\gamma \geq 1$ and $N_\gamma \geq 3$. We consider some $n_\gamma : m_\gamma$ junction J_γ with $J^j_\gamma \simeq \sigma^j_\gamma \cdot (-\infty, 0)$ and $\sigma_\gamma \in \{\pm 1\}^{N_\gamma}$, and a **Kružkov** *germ $\mathcal{G}_\gamma$ with respect to (J_γ, f_γ). We set $[a,b]^j_\gamma := [a^j_\gamma, b^j_\gamma]$. Up to relabel the indices, we assume that the indices go through the values $\{0, \dots, N_\gamma - 1\}$.*

We now assume that there exists two indices $j_1, j_2 \in \{0, \dots, N_\gamma - 1\}$ such that

$$(5.43) \qquad f^{j_1}_\gamma = f^{j_2}_\gamma =: f^0 \quad \text{on} \quad [a,b]^{j_1}_\gamma = [a,b]^{j_2}_\gamma =: [a^0, b^0] \quad \text{with} \quad J^{j_1}_\gamma \simeq (0, +\infty) \quad \text{and} \quad J^{j_2}_\gamma \simeq (-\infty, 0)$$

and we glue those two branches. To simplify the notation, we also assume that $j_1 = 0$ and $j_2 = N_\gamma - 1$. Hence we now have

$$(5.44) \qquad \begin{cases} f_\gamma = (f_\gamma^0, \dots, f_\gamma^{N_\gamma-1}) \\ \mathcal{G}_\gamma \subset [a,b]_\gamma := \prod_{i=0,\dots,N_\gamma-1} [a,b]_\gamma^i \\ [a,b]''_\gamma := \prod_{i=1,\dots,N_\gamma-2} [a,b]_\gamma^i \\ J_\gamma^0 \simeq (0,+\infty) \quad \textit{and} \quad J_\gamma^{N_\gamma-1} \simeq (-\infty,0) \\ f_\gamma^0 = f_\gamma^{N_\gamma-1} =: f^0 \quad \textit{on} \quad [a,b]_\gamma^0 = [a,b]_\gamma^{N_\gamma-1} =: [a^0,b^0] \end{cases}$$

Let the new junction after gluing be defined by

$$\underline{J} := \{0\} \cup \left(\bigcup_{j=1,\dots,N_\gamma-2} J_\alpha^j \right) \quad \textit{and} \quad \underline{\sigma}^k := \sigma_\gamma^k \quad \textit{for} \quad k = 1,\dots,N_\gamma-2$$

We consider $\hat{f}_\gamma : [a,b]_\gamma \to \mathbb{R}^{N_\gamma}$ the Godunov flux associated to the germ $\mathcal{G}_\gamma$, and with notation $p_\gamma = (p_\gamma^1, \dots, p_\gamma^{N_\gamma-2}) \in [a,b]''_\gamma$ (avoiding notation p''_γ to keep light notations), we consider the set

$$(5.45) \qquad R := \left\{ r \in [a^0,b^0], \quad \hat{f}_\gamma^0(r,p_\gamma,r) = \hat{f}_\gamma^{N_\gamma-1}(r,p_\gamma,r) \right\} \quad \textit{with} \quad \hat{f}_\gamma^0(\downarrow,p_\gamma,r), \quad \hat{f}_\gamma^{N_\gamma-1}(r,p_\gamma,\uparrow)$$

Then R is non empty, and define the set

$$\Lambda := \left\{ \lambda = \tilde{f}(r,p_\gamma) \in \mathbb{R}^{N_\gamma-2}, \quad r \in R \right\} \quad \textit{with} \quad \tilde{f}(r,p_\gamma) := (\hat{f}_\gamma^1(r,p_\gamma,r), \dots, \hat{f}_\gamma^{N_\gamma-2}(r,p_\gamma,r))$$

Then Λ is reduced to a singleton $\Lambda = \{\lambda\}$, and this defines the following map

$$\begin{array}{rccc} \underline{\hat{f}}: & [a,b]''_\gamma & \to & \mathbb{R}^{N_\gamma-2} \\ & p_\gamma & \mapsto & \underline{\hat{f}}(p_\gamma) := \lambda \end{array}$$

and we set the map

$$(5.46) \qquad \underline{f} := (f_\gamma^1, \dots, f_\gamma^{N_\gamma-2}) : [a,b]''_\gamma \to \mathbb{R}^{N_\gamma-2}$$

i) (Kružkov germ)
Moreover, the map $\underline{\hat{f}}$ is continuous and the set

$$\mathcal{G} := \left\{ P \in [a,b]''_\gamma, \quad \underline{\hat{f}}(P) = \underline{f}(P) \right\}$$

is a Kružkov germ with respect to $(\underline{J}, \underline{f})$ and $\underline{\hat{f}}$ is the associated Godunov flux at the junction $(n_\gamma - 1) : (m_\gamma - 1)$, i.e.

$$\underline{\hat{f}} = \underline{\hat{f}}_{\mathcal{G}}.$$

We introduce the notation

$$(5.47) \qquad \hat{f}^{j_1 \overset{\sharp}{:} j_2} := \underline{\hat{f}}$$

defined in the special case $j_1 := 0$ and $j_2 := N_\gamma - 1$.

ii) (Kružkov conservative germ)
Assume furthermore that $\mathcal{G}_\gamma$ is a conservative Kružkov germ. Then $\mathcal{G}$ is also a conservative Kružkov germ.

Proof of Proposition 5.16
Step 1: non emptyness of R
We set $p := p_\gamma$, and for $r \in [a^0, b^0]$, we set (with some abuse of notation)

$$g : [a^0, b^0] \to \mathbb{R} \quad \text{with} \quad g(r) := \tilde{g}(r,p) := \hat{f}^0_\gamma(r,p,r) - \hat{f}^{N_\gamma-1}_\gamma(r,p,r)$$

Using the last line of (5.44) and (2.15), we get

$$g(a^0) = \hat{f}^0_\gamma(a^0,p,a^0) - \hat{f}^{N_\gamma-1}_\gamma(a^0,p,a^0) \geq f^0_{\gamma,-}(a^0) - f^{N_\gamma-1}_{\gamma,+}(a^0) = f^0_\gamma(a^0) - f^{N_\gamma-1}_\gamma(a^0) = 0$$

$$g(b^0) = \hat{f}^0_\gamma(b^0,p,b^0) - \hat{f}^{N_\gamma-1}_\gamma(b^0,p,b^0) \leq f^0_{\gamma,+}(b^0) - f^{N_\gamma-1}_{\gamma,-}(b^0) = f^0_\gamma(b^0) - f^{N_\gamma-1}_\gamma(b^0) = 0$$

Therefore $g(a^0) \geq 0 \geq g(b^0)$ and by continuity of g, we get $R \neq \emptyset$.
Step 2: Λ is a singleton
Step 2.1: getting g vanishing on $R \supset [r, \bar{r}]$
Because $\mathcal{G}_\gamma$ is a Kružkov germ, recall that the map

$$[a^0, b^0]^2 \ni (r, r') \mapsto (\hat{f}^0_\gamma(r,p,r'), \hat{f}^{N_\gamma-1}_\gamma(r,p,r')) \tag{5.48}$$

satisfies for all $(\bar{r}, \bar{r}'), (r, r') \in [a^0, b^0]^2$ and all frozen $p \in [a,b]''_\gamma$ (with $\sigma^0_\gamma = -1$ and $\sigma^{N_\gamma-1}_\gamma = 1$)

$$-\text{sign}(\bar{r} - r) \cdot \left\{ \hat{f}^0_\gamma(\bar{r},p,\bar{r}') - \hat{f}^0_\gamma(r,p,r') \right\} + \text{sign}(\bar{r}' - r') \cdot \left\{ \hat{f}^{N_\gamma-1}_\gamma(\bar{r},p,\bar{r}') - \hat{f}^{N_\gamma-1}_\gamma(r,p,r') \right\} \geq 0 \tag{5.49}$$

For $\bar{r} = \bar{r}' > r = r'$, we get $g(\bar{r}) - g(r) \leq 0$, which shows that g is nonincreasing. Hence g is constant (and vanishes) on $[r, \bar{r}]$, if $r, \bar{r} \in R$. We deduce that R is closed interval where g vanishes.
Step 2.2: coincidence of $\hat{f}^0_\gamma$ and $\hat{f}^{N_\gamma-1}_\gamma$
For frozen p, inequality (5.49) shows that the map defined in (5.48) is itself associated to a Kružkov germ $1:1$, and then its flux is such that the matrix

$$\begin{pmatrix} -\dfrac{\partial \hat{f}^0_\gamma}{\partial r} & -\dfrac{\partial \hat{f}^0_\gamma}{\partial r'} \\[2ex] \dfrac{\partial \hat{f}^{N_\gamma-1}_\gamma}{\partial r} & \dfrac{\partial \hat{f}^{N_\gamma-1}_\gamma}{\partial r'} \end{pmatrix} (r,p,r')$$

has nonnegative diagonal and is diagonal column-dominant, i.e.

$$\left\{ \begin{array}{l} \left(-\dfrac{\partial \hat{f}^0_\gamma}{\partial r} - \left| \dfrac{\partial \hat{f}^{N_\gamma-1}_\gamma}{\partial r} \right| \right)(r,p,r') \geq 0 \\[3ex] \left(-\left| \dfrac{\partial \hat{f}^0_\gamma}{\partial r'} \right| + \dfrac{\partial \hat{f}^{N_\gamma-1}_\gamma}{\partial r'} \right)(r,p,r') \geq 0 \end{array} \right. \quad \text{for a.e.} \quad (r,r') \in [a^0,b^0]^2$$

Setting

$$\bar{g}(r,r') := \hat{f}^0_\gamma(r,p,r') - \hat{f}^{N_\gamma-1}_\gamma(r,p,r')$$

we deduce that the map $r \mapsto \bar{g}(r,r')$ is nonincreasing for a.e. r' and the map $r' \mapsto \bar{g}(r,r')$ is nonincreasing for a.e. r. By continuity of $\bar{g}$, we deduce that $\bar{g}$ is nonincreasing everywhere in both variables r, r'. Now if $r < \bar{r}$ with $r, \bar{r} \in R$, we deduce that $\bar{g}(r,r) = \bar{g}(\bar{r},\bar{r}) = 0$. From the monotonicities of $\bar{g}$, we deduce that $\bar{g} = 0$ on $[r,\bar{r}]^2$. This implies more generally that

$$\hat{f}^0_\gamma(\cdot,p,\cdot) = \hat{f}^{N_\gamma-1}_\gamma(\cdot,p,\cdot) \quad \text{on} \quad R^2 \tag{5.50}$$

Step 2.3: concluding that Λ is reduced to a singleton
Step 2.3.1: first unsuccessful try
Because $\mathcal{G}_\gamma$ is a Kružkov germ, we know that $\hat{f}_\gamma$ satisfies $D^{\hat{f}_\gamma} \geq 0$ on $[a,b]^2_\gamma$. We deduce from Proposition 5.9, that $\sigma \diamond \hat{f}_\gamma$ is Riemann monotone. If R is reduced to a singleton, then Λ is also a singleton. Then assume that $r, \bar{r} \in R$ with $\bar{r} > r$. Then (5.50) is not enough to deduce that

$$\{(\bar{r},p,\bar{r}) - (r,p,r)\} \diamond \sigma \diamond \left[\hat{f}_\gamma \right]^{(\bar{r},p,\bar{r})}_{(r,p,r)} = 0$$

and then can not conclude from the Riemann monotonicity of $\sigma \diamond \hat{f}_\gamma$ that $\left[\hat{f}_\gamma\right]_{(r,p,r)}^{(\bar{r},p,\bar{r})} = 0$.

Step 2.3.2: a slightly subtle argument

The way to conclude that Λ is a singleton, is to use the fact that the flux $\hat{f}_\gamma$ is associated to a Kružkov germ, and then it is Lipschitz continuous and satisfies from Proposition 4.20

$$\sigma^i \partial_i \hat{f}^i_\gamma \geq \sum_{j\in\{0,\dots,N_\gamma-1\}\setminus\{i\}} |\partial_i \hat{f}^j_\gamma| \quad \text{a.e. on} \quad [a,b], \quad i=0,\dots,N_\gamma-1$$

which occurs not only for the indices $i,j \in \{0, N_\gamma - 1\}$. In particular this implies

(5.51)
$$\sum_{j\in\{1,\dots,N_\gamma-2\}} |\partial_i \hat{f}^j_\gamma| \leq |\partial_i \hat{f}^i_\gamma| - |\partial_{\bar{i}} \hat{f}^{\bar{i}}_\gamma| \quad \text{for a.e. } (s,p,t)\in R\times\{p\}\times R \quad \text{for} \quad i\in\{0,N_\gamma-1\} \quad \text{and} \quad \bar{i}\in\{0,N_\gamma-1\}\setminus\{i\}$$

This is right here that the argument is slightly subtle. Now let us assume that R has non-empty interior (otherwise the result is trivial). Even if $(\hat{f}^0_\gamma, \hat{f}^{N_\gamma-1}_\gamma)$ is not necessarily constant on $R\times\{p\}\times R$, then (5.50) implies

$$\partial_i \hat{f}^0_\gamma(\cdot,p,\cdot) = \partial_i \hat{f}^{N_\gamma-1}_\gamma(\cdot,p,\cdot) \quad \text{for} \quad i=0,N_\gamma-1 \quad \text{a.e. on} \quad R^2$$

and then (5.51) implies

$$\partial_0 \hat{f}^j_\gamma = 0 = \partial_{N_\gamma-1}\hat{f}^j_\gamma \quad \text{for a.e. } (s,p,t)\in R\times\{p\}\times R \text{ and for all } j=1,\dots,N_\gamma-2$$

We conclude that

$$\hat{f}^j_\gamma = const \quad \text{on } R\times\{p\}\times R \text{ for all } j=1,\dots,N_\gamma-2$$

which shows that Λ is reduced to a singleton.

Step 3: Continuity of $\underline{\hat{f}}$

Consider a sequence $(r_n,p_n)\to(r,p)$ such that

$$\begin{cases} \lambda_n := \underline{\hat{f}}(p_n) = \tilde{f}(r_n,p_n) \to \tilde{f}(r,p) =: \lambda_0, \\ 0 = g(r_n) = \hat{f}^0_\gamma(r_n,p_n,r_n) - \hat{f}^{N_\gamma-1}_\gamma(r_n,p_n,r_n) \to 0 = g(r) = \hat{f}^0_\gamma(r,p,r) - \hat{f}^{N_\gamma-1}_\gamma(r,p,r) \end{cases}$$

The second line shows that $r\in R$, and then $\lambda_0 = \tilde{f}(r,p) = \underline{\hat{f}}(p) =: \lambda$. This shows the continuity of $\underline{\hat{f}}$.

Step 4: bounds on $\underline{\hat{f}}$

We consider $\underline{f}$ as defined in (5.46). By definition of $\underline{\hat{f}}$, we still get that $\underline{f}^I_- \leq \underline{\hat{f}}^I \leq \underline{f}^I_+$ for all $I\in\mathcal{I} := \{1,\dots,N_\gamma-2\}$, i.e. $\underline{f}_- \leq \underline{\hat{f}} \leq \underline{f}_+$.

Step 5: local constancy of $\underline{\hat{f}}$

Let $P := p \in [a,b]''_\gamma$ be such that $\underline{f}(P) \neq \underline{\hat{f}}(P) = \tilde{f}(r,P)$, and let us set $K_P := \left\{I\in\mathcal{I},\ \underline{\hat{f}}^I(P) \neq \underline{f}^I(P)\right\}$. Recall that $\tilde{f}(\cdot,P) = \underline{\hat{f}}(P)$ on R. In particular, for $Q_\varepsilon := P + \sum_{I\in K_P} (-\varepsilon,\varepsilon)e_I$, we get from the continuity and the local constancy of $\hat{f}_\gamma$ that for $\varepsilon>0$ small enough, we have $\tilde{f} = const$ on $R\times\left([a,b]''_\gamma \cap Q_\varepsilon\right)$. Hence $\underline{\hat{f}} = const$ on $[a,b]''_\gamma\cap Q_\varepsilon$, which shows the local constancy of $\underline{\hat{f}}$.

Step 6: directional monotonicity of $\underline{\hat{f}}$

For $P:=p$, we want to show that

$$P^I \mapsto \underline{\sigma}^I \underline{\hat{f}}^I(P) \quad \text{is nondecreasing on} \quad [a,b]^I := [a^I,b^I] \tag{5.52}$$

To this end, we have to take into account the fact that $\underline{\hat{f}}(P) = \tilde{f}(r,P)$ with the dependence $r=r(P)$. To reach our goal, it is more convenient to use directly that $\mathcal{G}_\gamma$ is a Kružkov germ. This means that

$$\begin{aligned} 0 \leq \quad & -\text{sign}(\bar{r}-r)\cdot\left\{\hat{f}^0_\gamma(\bar{r},\bar{p},\bar{r}) - \hat{f}^0_\gamma(r,p,r)\right\} + \text{sign}(\bar{r}-r)\cdot\left\{\hat{f}^{N_\gamma-1}_\gamma(\bar{r},\bar{p},\bar{r}) - \hat{f}^{N_\gamma-1}_\gamma(r,p,r)\right\} \\ & + \sum_{k=1,\dots,N_\gamma-2} \sigma^k_\gamma \cdot \text{sign}(\bar{p}^k - p^k)\cdot\left\{\hat{f}^k_\gamma(\bar{r},\bar{p},\bar{r}) - \hat{f}^k_\gamma(r,p,r)\right\} \end{aligned} \tag{5.53}$$

Using $\tilde{g}(\bar{r},\bar{p}) = 0 = \tilde{g}(r,p)$, we see that (5.53) implies

$$\sum_{k=1,\dots,N_\gamma-2} \underline{\sigma}^I \cdot \operatorname{sign}(\bar{p}^k - p^k) \cdot \left\{ \underline{\hat{f}}^k(\bar{p}) - \underline{\hat{f}}^k(p) \right\} \geq 0$$

which means that $\underline{\hat{f}}$ is associated to a Kružkov germ. In particular, this implies the directional monotonicity (5.52) of $\underline{\hat{f}}$.

Step 7: conclusion

We conclude that $\mathcal{G} := \left\{ P \in [a,b]''_\gamma,\ \underline{\hat{f}}(P) = \underline{f}(P) \right\}$ is a Riemann germ (because $\underline{\hat{f}}$ is continuous), and $\underline{\hat{f}} = \underline{\hat{f}}_{\mathcal{G}}$. Moreover $\mathcal{G}$ is a Kružkov germ.

Step 8: proof of ii)

Recall that $\mathcal{G}_\gamma$ is conservative if and only if its Godunov flux satisfies

$$\sum_{j=0,\dots,N_\gamma-1} \sigma^j_\gamma \hat{f}^j_\gamma = 0$$

The fact that by construction, we have $\hat{f}^0_\gamma(r,p_\gamma,r) = \hat{f}^{N_\gamma-1}_\gamma(r,p_\gamma,r)$ implies that

$$\sum_{j=1,\dots,N_\gamma-2} \underline{\sigma}^j_\gamma \underline{\hat{f}}^j = 0$$

which shows that $\mathcal{G}$ is conservative.

This ends the proof of the Proposition.

Corollary 5.17 (Self-gluing of a Kružkov germ for a $n_\gamma : m_\gamma$ junction)
Let γ be a fixed index. Assume that the function f_γ satisfies (2.2) with $N_\gamma = n_\gamma + m_\gamma$ with $n_\gamma, m_\gamma \geq 1$ and $N_\gamma \geq 3$. We consider some $n_\gamma : m_\gamma$ junction J_γ with $J^j_\gamma \simeq \sigma^j_\gamma \cdot (-\infty, 0)$ and $\sigma_\gamma \in \{\pm 1\}^{N_\gamma}$, and a **Kružkov** *germ $\mathcal{G}_\gamma$ with respect to (J_γ, f_γ). To simplify the presentation, assume that the indices go through $i = 0,\dots,N_\gamma-1$, i.e. $f_\gamma = (f^0_\gamma,\dots,f^{N_\gamma-1}_\gamma)$ and $[a,b]_\gamma := \prod_{i=0,\dots,N_\gamma-1} [a,b]^i_\gamma$ with $[a,b]^i_\gamma := [a^i_\gamma, b^i_\gamma]$. We call $\hat{f}_\gamma : [a,b]_\gamma \to \mathbb{R}^{N_\gamma}$ the Godunov flux associated to $\mathcal{G}_\gamma$. Assume also that*

$$\begin{cases} f^{j_1}_\gamma = f^{j_2}_\gamma =: f^0 \quad \text{defined on} \quad [a,b]^{j_1}_\gamma = [a,b]^{j_2}_\gamma = \left[a^0, b^0\right] \\ \\ J^{j_1}_\gamma = (0,+\infty), \quad J^{j_2}_\gamma = (-\infty, 0) \end{cases}$$

in order to glue branch $J^{j_1}_\gamma$ with branch $J^{j_2}_\gamma$. Then we define for $j_1 := 0$ and $j_2 := N_\gamma - 1$

$$\underline{\hat{f}} := (\hat{f}_\gamma)^{\overset{\sharp}{j_1 : j_2}} : [a,b]''_\gamma \to \mathbb{R}^{N_\gamma-2} \quad \text{with} \quad [a,b]''_\gamma := \prod_{i=1,\dots,N_\gamma-2} [a,b]^i_\gamma$$

where for any $p \in [a,b]''_\gamma$, we have

$$\underline{\hat{f}}(p) := (\hat{f}^1_\gamma,\dots,\hat{f}^{N_\gamma-2}_\gamma)(r,p,r) \quad \text{for some } r \in [a,b]^0_\gamma = [a,b]^{N_\gamma-1}_\gamma =: [a^0,b^0] \text{ such that } \hat{f}^0_\gamma(r,p,r) = \hat{f}^{N_\gamma-1}_\gamma(r,p,r)$$

where such r does exist, and when it is not unique, it does not change the value of $\underline{\hat{f}}(p)$.
We also define

$$\underline{f}(p) = (f^1_\gamma,\dots,f^{N_\gamma-2}_\gamma)(p)$$

We define

$$\mathcal{G} := (\mathcal{G}_\gamma)^{\overset{\sharp}{j_1 : j_2}} := \left\{ P \in [a,b]''_\gamma, \quad \underline{\hat{f}}(P) = \underline{f}(P) \right\} \tag{5.54}$$

Then $\mathcal{G}$ is a Kružkov germ and satisfies

$$\mathcal{G} := \left\{ \hat{p} \in [a,b]''_\gamma, \quad \text{there exists} \quad \hat{r},\hat{r}' \in [a^0,b^0] \quad s.t. \quad \left\{ \begin{array}{l} (\hat{r},\hat{p},\hat{r}') \in \mathcal{G}_\gamma \\ f^0(\hat{r}) = G^{f^0}(\hat{r},\hat{r}') = f^0(\hat{r}') \end{array} \right. \right\} \tag{5.55}$$

where we recall that the standard Godunov flux is given by

$$G^{f^0}(\hat r,\hat r')=\left\{\begin{array}{ll}\min\limits_{[\hat r,\hat r']} f^0 & \text{if}\quad \hat r\le \hat r'\\ \max\limits_{[\hat r',\hat r]} f^0 & \text{if}\quad \hat r\ge \hat r'\end{array}\right.$$

Moreover, if $\mathcal{G}_\gamma$ is conservative Kružkov, then $\mathcal{G}$ is also conservative Kružkov.

Proof of Corollary 5.17
Notice that by definition, the function $\hat f$ is equal to $\underline{\hat f}$ given by Proposition 5.16.
From definition (5.54), we have

$$p\in\mathcal{G}\quad\Longleftrightarrow\quad\left\{\begin{array}{l}\text{there exists } r\in[a,b]^0_\gamma=[a,b]^{N_\gamma-1}_\gamma \text{ s.t.}\\ \\ \left\{\begin{array}{ll}\hat f^i_\gamma(r,p,r)=f^i_\gamma(p), & i=1,\dots,N_\gamma-2\\ \hat f^0_\gamma(r,p,r)=\hat f^{N_\gamma-1}_\gamma(r,p,r)\end{array}\right.\end{array}\right\}$$

and then (using the fact that $f_\gamma(\hat r,\hat p,\hat r')=\hat f_\gamma(\hat r,\hat p,\hat r')=\hat f_\gamma(r,p,r)$ for $(\hat r,\hat p,\hat r')=\pi_{\mathcal{G}_\gamma}(r,p,r)$)

$$p\in\mathcal{G}\quad\Longleftrightarrow\quad\left\{\begin{array}{l}\text{there exists } r\in[a,b]^0_\gamma=[a,b]^{N_\gamma-1}_\gamma \text{ s.t.}\\ \\ \left\{\begin{array}{l}(\hat r,\hat p,\hat r')=\pi_{\mathcal{G}_\gamma}(r,p,r)\\ f(\hat p)=f(p)\\ f^0(\hat r)=f^0_\gamma(\hat r)=f^{N_\gamma-1}_\gamma(\hat r')=f^0(\hat r')\end{array}\right.\end{array}\right\}$$

Notice that

$$p\in BA(\hat p),\quad r\in BA^{((0,+\infty),f^0_\gamma)}(\hat r)\cap BA^{((-\infty,0),f^{N_\gamma-1}_\gamma)}(\hat r')$$

From the basins of attraction with $f^0_\gamma=f^{N_\gamma-1}_\gamma=f^0$, we then deduce that

$$p=\hat p,\quad f^0(\hat r)=G^{f^0}(\hat r,\hat r')=f^0(\hat r'),\quad \hat r,\hat r'\in[a^0,b^0],\quad\text{with}\quad(\hat r,\hat p,\hat r')\in\mathcal{G}_\gamma \tag{5.56}$$

and it is straighforward to check that conversely (5.56) implies that $p\in\mathcal{G}$ in the sense of definition (5.54). Therefore this shows characterization (5.55) of $\mathcal{G}$. This ends the proof.

Similarly to Lemma 5.15, we show the following results

Lemma 5.18 (Associativity of self-gluing for Kružkov germs)
Let γ be a fixed index. Assume that f_γ satisfies (2.2) for $N_\gamma=n_\gamma+m_\gamma\ge 5$ with $n_\gamma,m_\gamma\ge 2$ and some $n_\gamma:m_\gamma$ junction J_γ. Assume that $\mathcal{G}_\gamma$ is a Kružkov germ with respect to (J_γ,f_γ). We also assume that there exists four distinct indices $j_1,j_2,k_1,k_2\in\{1,\dots,N_\gamma\}$ such that

$$\left\{\begin{array}{lll}f^{j_1}_\gamma=f^{j_2}_\gamma=:f^A, & J^{j_1}_\gamma\simeq(0,+\infty), & J^{j_2}_\gamma\simeq(-\infty,0)\\ f^{k_1}_\gamma=f^{k_2}_\gamma=:f^B, & J^{k_1}_\gamma\simeq(0,+\infty), & J^{k_2}_\gamma\simeq(-\infty,0)\end{array}\right.$$

Then with notation of Corollary 5.17, we have the germ equality

$$\left((\mathcal{G}_\gamma)^{j_1\overset{\sharp}{:}j_2}\right)^{k_1\overset{\sharp}{:}k_2}=\left((\mathcal{G}_\gamma)^{k_1\overset{\sharp}{:}k_2}\right)^{j_1\overset{\sharp}{:}j_2}$$

which is associated to a $(n_\gamma-2):(m_\gamma-2)$ junction.

and

Lemma 5.19 (Associativity of the gluing and the self-gluing for Kružkov germs)
For $\gamma=\alpha,\beta$, assume that f_γ satisfies (2.2) for $N_\gamma=n_\gamma+m_\gamma$ for junctions J_γ of type $n_\gamma:m_\gamma$. We consider Kružkov germs $\mathcal{G}_\gamma$ with respect to (J_γ,f_γ).
We also assume that there exists $j_1\in\{1,\dots,N_\alpha\}$ and three distinct indices $j_2,k_1,k_2\in\{1,\dots,N_\beta\}$ such that

$$\left\{\begin{array}{lll}f^{j_1}_\alpha=f^{j_2}_\beta=:f^A, & J^{j_1}_\alpha\simeq(0,+\infty), & J^{j_2}_\beta\simeq(-\infty,0)\\ f^{k_1}_\beta=f^{k_2}_\beta=:f^B, & J^{k_1}_\beta\simeq(0,+\infty), & J^{k_2}_\beta\simeq(-\infty,0)\end{array}\right.$$

Then with notation of Corollaries 5.14 and 5.17, we have the germ equality

$$\mathcal{G}_\alpha \underset{j_1:j_2}{\sharp} \left((\mathcal{G}_\beta)^{\overset{\sharp}{k_1:k_2}} \right) = \left(\mathcal{G}_\alpha \underset{j_1:j_2}{\sharp} \mathcal{G}_\beta \right)^{\overset{\sharp}{k_1:k_2}}$$

which is associated to a $(n_\alpha + n_\beta - 2) : (m_\alpha + m_\beta - 2)$ *junction.*

5.9 Gluing of Godunov quasi-fluxes

Recall that Godunov quasi-fluxes are introduced in Definition 2.22. Here we recall this definition and distinguish some further classes of Godunov quasi-fluxes.

Definition 5.20 (Godunov quasi-flux)
Assume (2.2) with $N \geq 1$ *and* **compact** *box* $[a,b] \subset \mathbb{R}^N$*. Recall that* $\sigma \in \{\pm 1\}^N$ *encodes the orientations of the branches for the junction* J *with fluxes* f*.*
Then a function $\hat{f}_0 : [a,b] \to \mathbb{R}^N$ *is said to be a Godunov quasi-flux with respect to* (J,f)*, if it satisfies the following conditions*

$$(5.57) \qquad \begin{cases} \hat{f}_0 : [a,b] \to \mathbb{R}^N \text{ is continuous} \\ \\ \sigma \diamond \hat{f}_0 : [a,b] \to \mathbb{R}^N \text{ is Riemann monotone in the sense of Definition 2.11} \\ \\ \sigma^j f^j(b^j) \leq \sigma^j \hat{f}_0^j(q)_{|q^j=b^j} \quad \text{and} \quad \sigma^j \hat{f}_0^j(q)_{|q^j=a^j} \leq \sigma^j f^j(a^j) \end{cases}$$

i) (HJ quasi-flux)
We say that the quasi-flux $\hat{f}_0$ *is HJ if*

$$\hat{f}_0^j = g \quad \text{for all} \quad j = 1, \ldots, N, \quad \text{for some function } g : [a,b] \to \mathbb{R}$$

ii) (Kružkov quasi-flux)
We say that the quasi-flux $\hat{f}_0$ *is Kružkov if*

$$(5.58) \qquad 0 \leq D^{\hat{f}_0}(p,q) := \sum_{j=1,\ldots,N} \sigma^j \cdot sign(p^j - q^j) \cdot \left\{ \hat{f}_0^j(p) - \hat{f}_0^j(q) \right\} \quad \text{for all} \quad p,q \in [a,b]$$

ii') (Lipschitz Kružkov quasi-flux)
We say that the quasi-flux $\hat{f}_0$ *is Lipschitz Kružkov if it is Kružkov and if* $\hat{f}_0 : [a,b] \to \mathbb{R}^N$ *is Lipschitz continuous.*
iii) (monotone quasi-flux)
We say that the quasi-flux $\hat{f}_0$ *is monotone if*

$$\text{the maps } p \mapsto \sigma^j \hat{f}_0^j(p) \text{ are nonincreasing in the variable } p^k \text{ for all } k \neq j.$$

iv) (conservative quasi-flux)
We say that the quasi-flux $\hat{f}_0$ *is conservative if*

$$\sum_{j=1,\ldots,N} \sigma^j \cdot \hat{f}_0^j = 0$$

v) (j-local quasi-constancy)
Let $j \in \{1,\ldots,N\}$*. We say that the quasi-flux* $\hat{f}_0$ *is* j*-locally quasi-constant if it satisfies the following. For any* $p \in [a,b]$*, let*

$$\Phi(x) := \hat{f}_0(p + x \cdot e_j) \quad \text{defined for } p^j + x \in [a^j, b^j].$$

Then

$$\Phi = const = \Phi(0) \quad \text{on} \quad \left\{ \Phi^j = \Phi^j(0) \right\}$$

Remark 5.21 (Reformulation of the bounds)
Notice that the last line of (5.57) is equivalent to condition

$$(\sigma \diamond (\hat{f} - f)) \cdot n \geq 0 \quad on \quad \partial[a,b] \quad \text{with outward unit normal } n \tag{5.59}$$

When a point of the boundary $\partial[a,b]$ has several outward normals, then we can take any outward unit normal n.

Remark 5.22 (A technical remark on Lipschitz Kružkov quasi-fluxes)
From Proposition 4.20, we know that a Godunov quasi-flux $\hat{f}_0$ which is Kružkov and satisfies moreover the following partial Lipschitz condition
(5.60)
the map $p \mapsto \hat{f}_0^j(p)$ is locally Lipschitz continuous in p^j uniformly in p^k for $k \neq j$, , and for all $j = 1, \ldots, N$.

is (locally) Lipschitz continuous on the conpact box $[a,b]$, and then (globally) Lipschitz continuous on $[a,b]$. Therefore for Godunov quasi-flux which is Kružkov, it is equivalent to satisfy (5.60) and to be Lispchitz.

Then we have the following result.

Proposition 5.23 (Gluing of Godunov quasi-fluxes)
For $\gamma = \alpha, \beta$, assume that f_γ satisfies (2.2) with compact box $[a,b]_\gamma$ for $N_\gamma = n_\gamma + m_\gamma$ and $n_\gamma : m_\gamma$ junctions J_γ with $J_\gamma^j \simeq \sigma_\gamma^j \cdot (-\infty, 0)$ and $\sigma_\gamma \in \{\pm 1\}^{N_\gamma}$. We set $[a,b]_\gamma^j := [a_\gamma^j, b_\gamma^j]$. We assume that for each $\gamma = \alpha, \beta$, there exists one index $j_\gamma \in \{1, \ldots, N_\gamma\}$ such that

$$f_\alpha^{j_\alpha} = f_\beta^{j_\beta} =: f^0 \quad on \quad [a,b]_\alpha^{j_\alpha} = [a,b]_\beta^{j_\beta} =: [a^0, b^0] \quad with \quad J_\alpha^{j_\alpha} \simeq (0, +\infty) \quad and \quad J_\beta^{j_\beta} \simeq (-\infty, 0) \tag{5.61}$$

and we glue those two branches. To simplify the notation, up to relabel the indices, we now assume that $j_\alpha = 0 = j_\beta$, and the indices now go through the values $\{0, \ldots, N_\gamma - 1\}$. Hence we now have

$$\begin{cases} f_\gamma = (f_\gamma^0, \ldots, f_\gamma^{N_\gamma - 1}) \\ [a,b]'_\gamma := \prod_{i=1,\ldots,N_\gamma - 1} [a,b]_\gamma^i \\ J_\alpha^0 \simeq (0, +\infty) \quad and \quad J_\beta^0 \simeq (-\infty, 0) \end{cases}$$

and consider Godunov quasi-fluxes $\hat{f}_\gamma = (\hat{f}_\gamma^0, \ldots, \hat{f}_\gamma^{N_\gamma - 1})$ with respect to (J_γ, f_γ). For $\gamma = \alpha$ or β, assume either that $\hat{f}_\gamma$ is **0-locally quasi-constant** *(on $[a^0, b^0]$) in the sense of v) if Definition 5.20.*

Let the new junction of type $(n_\alpha + n_\beta - 1) : (m_\alpha + m_\beta - 1)$ after gluing, be defined by

$$\underline{J} := \{0\} \cup \left(\bigcup_{j=1,\ldots,N_\alpha - 1} J_\alpha^j \right) \cup \left(\bigcup_{k=1,\ldots,N_\beta - 1} J_\beta^k \right)$$

and

$$\underline{\sigma}^I := \begin{cases} \sigma_\alpha^k & if \quad I = (k, \alpha) \\ \sigma_\beta^k & if \quad I = (k, \beta) \end{cases}$$

For $p_\gamma = (p_\gamma^1, \ldots, p_\gamma^{N_\gamma - 1}) \in [a,b]'_\gamma$ (avoiding notation p'_γ to keep light notations), let us consider the set

$$R := \left\{ r \in [a^0, b^0], \quad \hat{f}_\alpha^0(r, p_\alpha) = \hat{f}_\beta^0(r, p_\beta) \right\} \quad with \quad \hat{f}_\alpha^0(\downarrow, p_\alpha), \quad \hat{f}_\beta^0(\uparrow, p_\beta) \tag{5.62}$$

Then R is non empty, and define the set

$$\Lambda := \left\{ \lambda = \tilde{f}(r, p_\alpha, p_\beta) \in \mathbb{R}^{N_\alpha + N_\beta - 2}, \quad r \in R \right\}$$

with

$$\tilde{f}(r, p_\alpha, p_\beta) := (\hat{f}_\alpha^1(r, p_\alpha), \ldots, \hat{f}_\alpha^{N_\alpha - 1}(r, p_\alpha); \hat{f}_\beta^1(r, p_\beta), \ldots, \hat{f}_\beta^{N_\beta - 1}(r, p_\beta)) \in \mathbb{R}^{N_\alpha + N_\beta - 2}$$

Then Λ is reduced to a singleton $\Lambda = \{\lambda\}$, and this defines the following map

$$\begin{array}{llll} \underline{\hat{f}}: & [a,b]'_\alpha \times [a,b]'_\beta & \to & \mathbb{R}^{N_\alpha+N_\beta-2} \\ & (p_\alpha,p_\beta) & \mapsto & \underline{\hat{f}}(p_\alpha,p_\beta) := \lambda \end{array}$$

and we set the map

$$\underline{f} := (f_\alpha^1,\dots,f_\alpha^{N_\alpha-1}; f_\beta^1,\dots,f_\beta^{N_\beta-1}) : [a,b]'_\alpha \times [a,b]'_\beta \to \mathbb{R}^{N_\alpha+N_\beta-2}$$

More generally, we also denote by

$$\hat{f}_\alpha \underset{j_\alpha:j_\beta}{\sharp} \hat{f}_\beta$$

the quasi-flux obtained by gluing of quasi-fluxes along indices j_α and j_β, with in particular

$$\hat{f}_\alpha \underset{0:0}{\sharp} \hat{f}_\beta := \underline{\hat{f}}$$

0) (Gluing Godunov quasi-fluxes)
Then $\underline{\hat{f}}$ is continuous, the map $\underline{\sigma} \diamond \underline{\hat{f}}$ is Riemann monotone, and $\underline{\hat{f}}$ is a Godunov quasi-flux with respect to $(\underline{J},\underline{f})$.
i) (Gluing Kružkov quasi-fluxes)
Assume that $\hat{f}_\gamma$ are Kružkov quasi-fluxes for $\gamma = \alpha,\beta$. Then $\underline{\hat{f}}$ is also a Kružkov quasi-flux.
i') (Gluing Lipschitz Kružkov quasi-fluxes)
Assume that $\hat{f}_\gamma$ are Lipschitz Kružkov quasi-fluxes for $\gamma = \alpha,\beta$. Then $\underline{\hat{f}}$ is also a Lipschitz Kružkov quasi-flux.
ii) (Gluing HJ quasi-fluxes)
Assume that $\hat{f}_\gamma$ are HJ quasi-fluxes for $\gamma = \alpha,\beta$. Then $\underline{\hat{f}}$ is also a HJ quasi-flux.
iii) (Gluing monotone quasi-fluxes)
Assume that $\hat{f}_\gamma$ are monotone quasi-fluxes for $\gamma = \alpha,\beta$. Then $\underline{\hat{f}}$ is also a monotone quasi-flux.
iv) (Gluing conservative quasi-fluxes)
Assume that $\hat{f}_\gamma$ are conservative quasi-fluxes for $\gamma = \alpha,\beta$. Then $\underline{\hat{f}}$ is also a conservative quasi-flux.

Proof of Proposition 5.23
We follow the lines of the proof of Proposition 5.13, and adapt it step by steps.
Step 1: non emptyness of R
We replace the a priori bounds, by (5.57) which means

$$\sigma_\gamma^0 f^0(b^0) \le \sigma_\gamma^0 (\hat{f}_\gamma^0)_{|b^0} \quad \text{and} \quad \sigma_\gamma^0 (\hat{f}_\gamma^0)_{|a^0} \le \sigma_\gamma^0 f^0(a^0)$$

i.e. with $\sigma_\beta^0 = 1 = -\sigma_\alpha^0$

$$\left\{ \begin{array}{l} f^0(b^0) \le (\hat{f}_\beta^0)_{|b^0} \\ (\hat{f}_\beta^0)_{|a^0} \le f^0(a^0) \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} f^0(b^0) \ge (\hat{f}_\alpha^0)_{|b^0} \\ (\hat{f}_\alpha^0)_{|a^0} \ge f^0(a^0) \end{array} \right.$$

Hence

$$g(r) := \tilde{g}(r,p,q) := \hat{f}_\alpha^0(r,p) - \hat{f}_\beta^0(r,q)$$

satisfies

$$\left\{ \begin{array}{l} g(a^0) = \hat{f}_\alpha^0(a^0,p) - \hat{f}_\beta^0(a^0,q) \ge f^0(a^0) - f^0(a^0) = 0 \\ g(b^0) = \hat{f}_\alpha^0(b^0,p) - \hat{f}_\beta^0(b^0,q) \le f^0(b^0) - f^0(b^0) = 0 \end{array} \right.$$

which implies again $g(a^0) \ge 0 \ge g(b^0)$, which shows the non emptyness of R.
Step 2: Λ is a singleton
Unchanged.
Step 3: continuity of $\underline{\hat{f}}$
Unchanged.
Step 4: Bounds on $\underline{\hat{f}}$
The bounds correspond here to the last line of (5.57) satisfied by $\hat{f}_\gamma$ and f_γ. By construction, these inequalities are unchanged for $\underline{\hat{f}}$ and $\underline{f}$. Step 5 must be skipped.

Step 6: Riemann monotonicity of $\underline{\sigma} \diamond \underline{\hat{f}}$
The proof is unchanged.
Step 7: conclusion
We conclude here that $\underline{\hat{f}}$ is a Godunov quasi-flux.
Step 8: additional argument for Kružkov quasi-fluxes
Unchanged.
Step 8': additional argument for Lipschitz Kružkov quasi-fluxes
The proof is the same, except that we have furthermore to check that $\underline{\hat{f}}$ is Lipschitz continuous, or which is equivalent satisfies (5.60) (see Remark 5.22). This does not follow immediately from the fact that each quasi-flux $\hat{f}_\gamma$ is itself Lipschitz Kružkov in the sense of Definition 5.20. It requires a proof.

In order to show that $\underline{\hat{f}}$ satisfies (5.60), it is sufficient to estimate $\partial_{p_\gamma^j} \underline{\hat{f}}^{(\gamma,j)}$, say for $\gamma = \alpha$ (the case $\gamma = \beta$ is similar).
Step 8'.1: formal proof
We want to estimate $\partial_{p_\alpha^j} \underline{\hat{f}}^{(\alpha,j)}$. Starting from

$$\hat{f}_\alpha^0(r, p_\alpha) = \hat{f}_\beta^0(r, q_\beta)$$

we get formally by derivation

$$\begin{cases} \left(\partial_0 \hat{f}_\alpha^0 - \partial_0 \hat{f}_\beta^0\right) \cdot \partial_{p_\alpha^j} r + \partial_{p_\alpha^j} \hat{f}_\alpha^0 = 0 \\ \partial_{p_\alpha^j} \underline{\hat{f}}^{(\alpha,j)} = \partial_{p_\alpha^j} \hat{f}_\alpha^j + \partial_0 \hat{f}_\alpha^j \cdot \partial_{p_\alpha^j} r \end{cases}$$

We recall that $\sigma_\alpha^0 = -1$ and $\sigma_\beta^0 = +1$. When $\partial_0 \hat{f}_\alpha^0 < 0$, we get

$$\partial_{p_\alpha^j} \underline{\hat{f}}^{(\alpha,j)} = \partial_{p_\alpha^j} \hat{f}_\alpha^j - \mu \, \partial_{p_\alpha^j} \hat{f}_\alpha^0 \quad \text{with} \quad \mu := \frac{\partial_0 \hat{f}_\alpha^j}{\partial_0 \hat{f}_\alpha^0 - \partial_0 \hat{f}_\beta^0}$$

where $\mu \in [-1, 1]$, because

$$\begin{cases} \partial_0 \hat{f}_\alpha^0 < 0 \\ \partial_0 \hat{f}_\beta^0 \geq 0 \\ \sigma_\alpha^0 \partial_0 \hat{f}_\alpha^0 \geq \sum_{j \neq 0} |\partial_0 \hat{f}_\alpha^j| \end{cases}$$

and then

$$(5.63) \qquad |\partial_{p_\alpha^j} \underline{\hat{f}}^{(\alpha,j)}|_{L^\infty([a_\alpha, b_\alpha])} \leq |\partial_{p_\alpha^j} \hat{f}_\alpha^j|_{L^\infty([a_\alpha, b_\alpha])} + |\partial_{p_\alpha^j} \hat{f}_\alpha^0|_{L^\infty([a_\alpha, b_\alpha])}$$

which provides the desired bound.
Step 8'.2: sketch of the rigorous proof
The difficulty comes from the fact that r is not unique in general, and has no derivatives. One way to make a rigorous proof consists to proceed by approximation.

First, because $\sigma_\gamma \diamond \hat{f}_\gamma$ is Lipschitz and Riemann monotone, we can consider the function for $\varepsilon > 0$

$$\hat{f}_{\gamma,1}(p) := \hat{f}_\gamma(p) + \varepsilon \sigma_\gamma \diamond \left\{ p - \frac{a_\gamma + b_\gamma}{2} \right\}$$

which satisfies

$$(5.64) \qquad (\sigma_\gamma \diamond \left\{ \hat{f}_{\gamma,1} - f_\gamma \right\}) \cdot n \geq C\varepsilon > 0 \quad \text{on} \quad \partial[a, b]$$

for some constant C depending on $b - a$ only. Moreover it has positive derivatives

$$(5.65) \qquad \sigma_\gamma^j \cdot \partial_{p_\gamma^j} \hat{f}_{\gamma,1}^j \geq \varepsilon > 0 \quad \text{on} \quad [a_\gamma, b_\gamma]$$

Now because $\hat{f}_\gamma$ is Kružkov and Lipschitz, we know from Proposition 4.20 that it satisfies

$$(5.66) \qquad \sigma_\gamma^k \partial_k \hat{f}_\gamma^k \geq \sum_{j \in \{0, \dots, N_\gamma - 1\} \setminus \{k\}} |\partial_k \hat{f}_\gamma^j| \quad \text{for all} \quad k$$

and then also $\hat{f}_{\gamma,1}$. Then for any $\delta > 0$ and non-negative mollifyer $\rho_\delta = \delta^{-N_\gamma}\rho(\delta^{-1}\cdot)$ with $\mathrm{supp}(\rho) \subset B_1$, we deduce that

$$\hat{f}^k_{\gamma,2} := \rho_\delta \star \hat{f}^k_{\gamma,1}$$

still satisfies (5.66), but on the reduced box

$$[a_\gamma, b_\gamma]_{-\delta} := \prod_{j=0,\dots,N_\gamma-1} [a^j_\gamma + \delta, b^j_\gamma - \delta]$$

It also satisfies the following variants of (5.65) and (5.64), i.e.

$$\begin{cases} (\sigma_\gamma \diamond \left\{\hat{f}_{\gamma,2} - f_\gamma\right\}) \cdot n \geq C\varepsilon/2 > 0 \quad \text{on} \quad \partial([a,b]_{-\delta}) \\ \sigma^j_\gamma \cdot \partial_{p^j_\gamma} \hat{f}^j_{\gamma,2} \geq \varepsilon/2 > 0 \quad \text{on} \quad [a_\gamma, b_\gamma]_{-\delta} \end{cases}$$

for $\delta > 0$ small enough.
Hence the solution r to

$$\hat{f}^0_{\alpha,2}(r, p_\alpha) = \hat{f}^0_{\beta,2}(r, q_\beta)$$

does exists, is unique and smooth for all $p_\alpha \in [a_\alpha, b_\alpha]_{-\delta}$ and $q_\beta \in [a_\beta, b_\beta]_{-\delta}$. Then the formal proof becomes rigorous and gives (with obvious notation)

$$|\partial_{p^j_\alpha} \underline{\hat{f}}^{(\alpha,j)}_2|_{L^\infty([a_\alpha,b_\alpha]_{-\delta})} \leq |\partial_{p^j_\alpha} \hat{f}^j_{\alpha,2}|_{L^\infty([a_\alpha,b_\alpha]_{-\delta})} + |\partial_{p^j_\alpha} \hat{f}^0_{\alpha,2}|_{L^\infty([a_\alpha,b_\alpha]_{-\delta})}$$

In the limit $\delta \to 0$ and then $\varepsilon \to 0$, we recover (5.63). Therefore we conclude that $\underline{\hat{f}}$ is Lipschitz and then a Lipschitz Kružkov quasi-flux.

Steps 9, 10 and 11: additional arguments for HJ, monotone, and conservative quasi-fluxes
The proof is unchanged. This ends the proof of the proposition.

Lemma 5.24 (Associativity of the gluing of Godunov quasi-fluxes)
For $\delta = \alpha, \beta, \gamma$, assume that f_δ satisfies (2.2) with compact box $[a,b]_\delta$ for $N_\delta = n_\delta + m_\delta$ and $n_\delta : m_\delta$ junctions J_δ with $J^j_\delta \simeq \sigma^j_\delta \cdot (-\infty, 0)$ and $\sigma_\delta \in \{\pm 1\}^{N_\gamma}$. We set $[a,b]^j_\delta := [a^j_\delta, b^j_\delta]$. We also assume that there exists $j_\alpha, j_\beta, k_\gamma, k_\beta$, with $j_\delta, k_\delta \in \{1, \dots, N_\delta\}$ such that

$$\begin{cases} f^{j_\alpha}_\alpha = f^{j_\beta}_\beta =: f^A, \quad [a,b]^{j_\alpha}_\alpha = [a,b]^{j_\beta}_\beta =: [a,b]^A \qquad -\sigma^{j_\alpha}_\alpha = 1 = \sigma^{j_\beta}_\beta, \qquad J^{j_\alpha}_\alpha \simeq (0,+\infty), \qquad J^{j_\beta}_\beta \simeq (-\infty, 0), \\ f^{k_\beta}_\beta = f^{k_\gamma}_\gamma =: f^B, \quad [a,b]^{k_\beta}_\beta = [a,b]^{k_\gamma}_\gamma =: [a,b]^B \qquad -\sigma^{k_\beta}_\beta = 1 = \sigma^{k_\gamma}_\gamma, \qquad J^{k_\beta}_\beta \simeq (0,+\infty), \qquad J^{k_\gamma}_\gamma \simeq (-\infty, 0), \qquad k_\beta \neq j_\beta \end{cases}$$

We consider Godunov quasi-fluxes $\hat{f}_\delta$ with respect to (J_δ, f_δ). For $\delta = \alpha$, β or γ, assume either 1) that $\hat{f}_\alpha$ is j_α-locally quasi-constant, that $\hat{f}_\gamma$ is k_γ-locally quasi-constant, and that $\hat{f}_\beta$ is both j_β and k_β-locally quasi-constant. With notation of Proposition 5.23, we glue

$$(\hat{f}_\alpha \sharp \hat{f}_\beta) \sharp \hat{f}_\gamma := (\hat{f}_\alpha \underset{j_\alpha : j_\beta}{\sharp} \hat{f}_\beta) \underset{k_\beta : k_\gamma}{\sharp} \hat{f}_\gamma \quad \textit{gluing first } J^{j_\alpha}_\alpha \textit{ with } J^{j_\beta}_\beta, \textit{ and then } J^{k_\beta}_\beta \textit{ with } J^{k_\gamma}_\gamma$$

and also glue

$$\hat{f}_\alpha \sharp (\hat{f}_\beta \sharp \hat{f}_\gamma) := \hat{f}_\alpha \underset{j_\alpha : j_\beta}{\sharp} (\hat{f}_\beta \underset{k_\beta : k_\gamma}{\sharp} \hat{f}_\gamma) \quad \textit{gluing first } J^{k_\beta}_\beta \textit{ with } J^{k_\gamma}_\gamma, \textit{ and then } J^{j_\alpha}_\alpha \textit{ with } J^{j_\beta}_\beta$$

Then we have

$$(\hat{f}_\alpha \sharp \hat{f}_\beta) \sharp \hat{f}_\gamma = \hat{f}_\alpha \sharp (\hat{f}_\beta \sharp \hat{f}_\gamma) \tag{5.67}$$

Proof of Lemma 5.24
Here we can not use directly proof of Lemma 5.15, because we do not manipulate germs. Still, we can adapt the proof.

Again, we set

$$\begin{cases} [a,b]'_\delta := \displaystyle\prod_{j\in\{1,\dots,N_\delta\}\backslash\{j_\delta\}} [a,b]^j_\delta & \text{for} \quad \delta=\alpha,\beta \\ [a,b]^*_\delta := \displaystyle\prod_{k\in\{1,\dots,N_\delta\}\backslash\{k_\delta\}} [a,b]^k_\delta & \text{for} \quad \delta=\beta,\gamma \\ [a,b]''_\beta := \displaystyle\prod_{j\in\{1,\dots,N_\beta\}\backslash\{j_\beta,k_\beta\}} [a,b]^j_\beta \end{cases}$$

Step 1: first computation
We simply compute for $(\hat p_\alpha,\hat p_\beta)\in[a,b]'_\alpha\times[a,b]'_\beta$

$$\hat f^I_{\alpha\beta}(\hat p_\alpha,\hat p_\beta) := \begin{cases} \hat f^j_\alpha(\hat p_\alpha,r^A) & \text{if} \quad I=(\alpha,j) \\ \hat f^j_\beta(\hat p_\beta,r^A) & \text{if} \quad I=(\beta,j) \end{cases} \quad \text{with} \quad \begin{cases} \hat f^{j_\alpha}_\alpha(\hat p_\alpha,r^A)=\hat f^{j_\beta}_\beta(\hat p_\beta,r^A) \\ \text{for some} \quad r^A\in[a,b]^A \end{cases}$$

with abuse of notation for $(\hat p_\alpha,r^A)\in[a,b]_\alpha$ and $(\hat p_\beta,r^A)\in[a,b]_\beta$, and similarly for $(\hat p_{\alpha\beta},\hat p_\gamma)\in\Big([a,b]'_\alpha\times[a,b]''_\beta\Big)\times[a,b]^*_\gamma$

$$(\hat f_{\alpha\beta}\sharp\hat f_\gamma)^K(\hat p_{\alpha\beta},\hat p_\gamma) := \begin{cases} \hat f^K_{\alpha\beta}(\hat p_{\alpha\beta},r^B) & \text{if} \quad K=(\alpha,j) \quad \text{or} \quad K=(\beta,j) \\ \hat f^j_\gamma(\hat p_\gamma,r^B) & \text{if} \quad K=(\gamma,j) \end{cases} \quad \text{with} \quad \begin{cases} \hat f^{(\beta,k_\beta)}_{\alpha\beta}(\hat p_{\alpha\beta},r^B)=\hat f^{k_\gamma}_\gamma(\hat p_\gamma,r^B) \\ \text{for some} \quad r^B\in[a,b]^B \end{cases}$$

Hence for $\hat p_{\alpha\beta}=(\hat p_\alpha,\hat p'_\beta)$ (and with $\hat p_\beta=(\hat p'_\beta,r^B)$), we get $(\hat p_\alpha,\hat p'_\beta,\hat p_\gamma)\in[a,b]'_\alpha\times[a,b]''_\beta\times[a,b]^*_\gamma$ and

$$(\hat f_{\alpha\beta}\sharp\hat f_\gamma)^K(\hat p_\alpha,\hat p'_\beta,\hat p_\gamma) = \begin{cases} \hat f^j_\alpha(\hat p_\alpha,r^A) & \text{if} \quad K=(\alpha,j) \\ \hat f^j_\beta(\hat p'_\beta,r^B,r^A) & \text{if} \quad K=(\beta,j) \\ \hat f^j_\gamma(\hat p_\gamma,r^B) & \text{if} \quad K=(\gamma,j) \end{cases} \quad \text{with} \quad \begin{cases} \hat f^{j_\alpha}_\alpha(\hat p_\alpha,r^A)=\hat f^{j_\beta}_\beta(\hat p'_\beta,r^B,r^A) \\ \hat f^{k_\beta}_\beta(\hat p'_\beta,r^B,r^A)=\hat f^{k_\gamma}_\gamma(\hat p_\gamma,r^B) \\ \text{for some} \quad r^A\in[a,b]^A, \quad r^B\in[a,b]^B \end{cases}$$

Step 2: second computation
We simply compute for $(\hat p_\beta,\hat p_\gamma)\in[a,b]^*_\beta\times[a,b]^*_\gamma$

$$\hat f^I_{\beta\gamma}(\hat p_\beta,\hat p_\gamma) = \begin{cases} \hat f^j_\beta(\hat p_\beta,\bar r^B) & \text{if} \quad I=(\beta,j) \\ \hat f^j_\gamma(\hat p_\gamma,\bar r^B) & \text{if} \quad I=(\gamma,j) \end{cases} \quad \text{with} \quad \begin{cases} \hat f^{k_\beta}_\beta(\hat p_\beta,\bar r^B)=\hat f^{k_\gamma}_\gamma(\hat p_\gamma,\bar r^B) \\ \text{for some} \quad \bar r^B\in[a,b]^B \end{cases}$$

and then for $\hat p_{\beta\gamma}=(\hat p'_\beta,\hat p_\gamma)$ (and with $\hat p_\beta=(\hat p'_\beta,\bar r^B)$) and $(\hat p_\alpha,\hat p_{\beta\gamma})\in[a,b]'_\alpha\times[a,b]''_\beta\times[a,b]^*_\gamma$

$$(\hat f_\alpha\sharp\hat f_{\beta\gamma})^K(\hat p_\alpha,\hat p_{\beta\gamma}) = \begin{cases} \hat f^j_\alpha(\hat p_\alpha,\bar r^A) & \text{if} \quad K=(\alpha,j) \\ \hat f^K_{\beta\gamma}(\hat p_{\beta\gamma},\bar r^A) & \text{if} \quad K=(\beta,j) \quad \text{or} \quad K=(\gamma,j) \end{cases} \quad \text{with} \quad \begin{cases} \hat f^{j_\alpha}_\alpha(\hat p_\alpha,\bar r^A)=\hat f^{(\beta,j_\beta)}_{\beta\gamma}(\hat p_{\beta\gamma},\bar r^A) \\ \text{for some} \quad \bar r^A\in[a,b]^A \end{cases}$$

Hence for $(\hat p_\alpha,\hat p'_\beta,p_\gamma)\in[a,b]'_\alpha\times[a,b]''_\beta\times[a,b]^*_\gamma$, we get

$$(\hat f_\alpha\sharp\hat f_{\beta\gamma})^K(\hat p_\alpha,\hat p'_\beta,p_\gamma) = \begin{cases} \hat f^j_\alpha(\hat p_\alpha,\bar r^A) & \text{if} \quad K=(\alpha,j) \\ \hat f^j_\beta(\hat p'_\beta,\bar r^B,\bar r^A) & \text{if} \quad K=(\beta,j) \\ \hat f^j_\gamma(\hat p_\gamma,\bar r^B) & \text{if} \quad K=(\gamma,j) \end{cases} \quad \text{with} \quad \begin{cases} \hat f^{k_\beta}_\beta(\hat p'_\beta,\bar r^B,\bar r^A)=\hat f^{k_\gamma}_\gamma(\hat p_\gamma,\bar r^B) \\ \hat f^{j_\alpha}_\alpha(\hat p_\alpha,\bar r^A)=\hat f^{j_\beta}_\beta(\hat p'_\beta,\bar r^B,\bar r^A) \\ \text{for some} \quad \bar r^A\in[a,b]^A, \quad \bar r^B\in[a,b]^B \end{cases}$$

Step 3: conclusion
From the uniqueness of the values obtained through our construction (whatever are the choices of $r^C,\bar r^C\in[a,b]^C$ for $C=A,B$), we deduce that $\hat f_\alpha\sharp\hat f_{\beta\gamma}=\hat f_{\alpha\beta}\sharp\hat f_\gamma$, which shows (5.67). This ends the proof of the lemma.

Proposition 5.25 (Self-gluing of Kružkov quasi-flux $\hat f_\gamma$ for $n_\gamma : m_\gamma$ junction)
Let γ be a fixed index. Assume that the function f_γ satisfies (2.2) with bounded box $[a,b]_\gamma$ with $N_\gamma=n_\gamma+m_\gamma$ with $n_\gamma,m_\gamma\ge 1$ and $N_\gamma\ge 3$. We consider some $n_\gamma : m_\gamma$ junction J_γ with $J^j_\gamma\simeq\sigma^j_\gamma\cdot(-\infty,0)$ and $\sigma_\gamma\in\{\pm1\}^{N_\gamma}$. We set $[a,b]^j_\gamma := [a^j_\gamma,b^j_\gamma]$. Up to relabel the indices, we assume that the indices go through the values $\{0,\dots,N_\gamma-1\}$.

We now assume that there exists two indices $j_1, j_2 \in \{0, \dots, N_\gamma - 1\}$ *such that*

$$(5.68) \qquad f_\gamma^{j_1} = f_\gamma^{j_2} =: f^0 \quad on \quad [a,b]_\gamma^{j_1} = [a,b]_\gamma^{j_2} =: [a^0, b^0] \quad with \quad J_\gamma^{j_1} \simeq (0, +\infty) \quad and \quad J_\gamma^{j_2} \simeq (-\infty, 0)$$

and we glue those two branches. To simplify the notation, we also assume that $j_1 = 0$ *and* $j_2 = N_\gamma - 1$*. Hence we now have*

$$(5.69) \qquad \begin{cases} f_\gamma = (f_\gamma^0, \dots, f_\gamma^{N_\gamma - 1}) \\ [a,b]_\gamma'' := \prod_{i=1,\dots,N_\gamma-2} [a,b]_\gamma^i \\ J_\gamma^0 \simeq (0, +\infty) \quad and \quad J_\gamma^{N_\gamma-1} \simeq (-\infty, 0) \\ f_\gamma^0 = f_\gamma^{N_\gamma-1} =: f^0 \quad on \quad [a,b]_\gamma^0 = [a,b]_\gamma^{N_\gamma-1} =: [a^0, b^0] \end{cases}$$

We assume that $\hat{f}_\gamma : [a,b]_\gamma \to \mathbb{R}^{N_\gamma}$ *is a Godunov quasi-flux with respect to* (J_γ, f_γ)*, which is assumed to be* **Kružkov (resp. Lipschitz Kružkov)** *in the sense of Definition 5.20.*

Let us consider the new junction of type $(n_\gamma - 1) : (m_\gamma - 1)$ *obtained after gluing, and defined by*

$$\underline{J} := \{0\} \cup \left(\bigcup_{j=1,\dots,N_\gamma-2} J_\alpha^j \right) \quad and \quad \underline{\sigma}^k := \sigma_\gamma^k \quad for \quad k = 1, \dots, N_\gamma - 2$$

We introduce $p_\gamma = (p_\gamma^1, \dots, p_\gamma^{N_\gamma-2}) \in [a,b]_\gamma''$ *(avoiding notation* p_γ'' *to keep light notations), and we consider the set*

$$(5.70) \qquad R := \left\{ r \in [a^0, b^0], \quad \hat{f}_\gamma^0(r, p_\gamma, r) = \hat{f}_\gamma^{N_\gamma-1}(r, p_\gamma, r) \right\} \quad with \quad \hat{f}_\gamma^0(\downarrow, p_\gamma, r), \quad \hat{f}_\gamma^{N_\gamma-1}(r, p_\gamma, \uparrow)$$

Then R *is non empty, and the set*

$$\Lambda := \left\{ \lambda = \tilde{f}(r, p_\gamma) \in \mathbb{R}^{N_\gamma-2}, \quad r \in R \right\} \quad with \quad \tilde{f}(r, p_\gamma) := (\hat{f}_\gamma^1(r, p_\gamma, r), \dots, \hat{f}_\gamma^{N_\gamma-2}(r, p_\gamma, r))$$

is reduced to a singleton $\Lambda = \{\lambda\}$*, and this defines the following map*

$$\begin{array}{rccc} \underline{\hat{f}} : & [a,b]_\gamma'' & \to & \mathbb{R}^{N_\gamma-2} \\ & p_\gamma & \mapsto & \underline{\hat{f}}(p_\gamma) := \lambda \end{array}$$

and we set the map

$$(5.71) \qquad \underline{f} := (f_\gamma^1, \dots, f_\gamma^{N_\gamma-2}) : [a,b]_\gamma'' \to \mathbb{R}^{N_\gamma-2}$$

i) (Kružkov quasi-flux)
Moreover, the map $\underline{\hat{f}}$ *is a Godunov quasi-flux with respect to* $(\underline{J}, \underline{f})$*, which is moreover Kružkov (resp. Lipschitz Kružkov). We introduce the notation*

$$(5.72) \qquad \hat{f}^{j_1 \overset{\sharp}{:} j_2} := \underline{\hat{f}}$$

defined in the special case $j_1 := 0$ *and* $j_2 := N_\gamma - 1$*.*
ii) (Kružkov conservative quasi-flux)
Assume furthermore that $\hat{f}_\gamma$ *is a Lipschitz Kružkov conservative quasi-flux. Then* $\underline{\hat{f}}$ *is also a Kružkov conservative quasi-flux (resp. Lipschitz Kružkov conservative quasi-flux).*

Proof of Proposition 5.25
The proof is a direct adaptation of the proof of Proposition 5.16. We skip the details.

Similarly to Lemmata 5.18 and 5.19, and because the proofs are similar, we state (without proofs) the following results:

Lemma 5.26 (Associativity of self-gluing for Kružkov quasi-fluxes)
Let γ be a fixed index. Assume that f_γ satisfies (2.2) on the bounded box $[a,b]_\gamma$ for $N_\gamma = n_\gamma + m_\gamma \geq 5$ with $n_\gamma, m_\gamma \geq 2$ and some $n_\gamma : m_\gamma$ junction J_γ. Assume that $\hat{f}_\gamma$ is a Kružkov quasi-flux (resp. Lipschitz Kružkov quasi-flux) with respect to (J_γ, f_γ) in the sense of Definition 5.20. We also assume that there exists four distinct indices $j_1, j_2, k_1, k_2 \in \{1, \dots, N_\gamma\}$ such that

$$\begin{cases} f_\gamma^{j_1} = f_\gamma^{j_2} =: f^A, & J_\gamma^{j_1} \simeq (0,+\infty), \qquad J_\gamma^{j_2} \simeq (-\infty, 0) \\ f_\gamma^{k_1} = f_\gamma^{k_2} =: f^B, & J_\gamma^{k_1} \simeq (0,+\infty), \qquad J_\gamma^{k_2} \simeq (-\infty, 0). \end{cases}$$

Then with notation of Proposition 5.25, we have the equality

$$\left((\hat{f}_\gamma)^{\overset{\sharp}{j_1 : j_2}} \right)^{\overset{\sharp}{k_1 : k_2}} = \left((\hat{f}_\gamma)^{\overset{\sharp}{k_1 : k_2}} \right)^{\overset{\sharp}{j_1 : j_2}}$$

which is a Kružkov quasi-flux (resp. Lipschitz Kružkov quasi-flux) associated to a $(n_\gamma - 2) : (m_\gamma - 2)$ junction.

and

Lemma 5.27 (Associativity of the gluing and the self-gluing for Kružkov quasi-fluxes)
For $\gamma = \alpha, \beta$, assume that f_γ satisfies (2.2) on the bounded boxes $[a,b]_\gamma$ for $N_\gamma = n_\gamma + m_\gamma$ for junctions J_γ of type $n_\gamma : m_\gamma$. We consider Kružkov (resp. Lipschitz Kružkov) quasi-fluxes $\hat{f}_\gamma$ with respect to (J_γ, f_γ) in the sense of Definition 5.20.
We also assume that there exists $j_1 \in \{1, \dots, N_\alpha\}$ and three distinct indices $j_2, k_1, k_2 \in \{1, \dots, N_\beta\}$ such that

$$\begin{cases} f_\alpha^{j_1} = f_\beta^{j_2} =: f^A, & J_\alpha^{j_1} \simeq (0,+\infty), \qquad J_\beta^{j_2} \simeq (-\infty, 0) \\ f_\beta^{k_1} = f_\beta^{k_2} =: f^B, & J_\beta^{k_1} \simeq (0,+\infty), \qquad J_\beta^{k_2} \simeq (-\infty, 0). \end{cases}$$

Then with notation of Propositions 5.23 and 5.25, we have the equality

$$\hat{f}_\alpha \underset{j_1 : j_2}{\sharp} \left((\hat{f}_\beta)^{\overset{\sharp}{k_1 : k_2}} \right) = \left(\hat{f}_\alpha \underset{j_1 : j_2}{\sharp} \hat{f}_\beta \right)^{\overset{\sharp}{k_1 : k_2}}$$

which is a Kružkov (resp. Lipschitz Kružkov) quasi-flux associated to a $(n_\alpha + n_\beta - 2) : (m_\alpha + m_\beta - 2)$ junction.

6 Applications

6.1 Restriction of Riemann germs to bounded boxes - a priori L^∞ bounds

We also have the following result (which can also be used to derive a priori L^∞ bounds on solutions with initial values in a bounded set K).

Proposition 6.1 (Restriction of Riemann germs to bounded boxes)
For $N \geq 1$, assume (2.2) and nondegeneracy condition (2.17), and let $\mathcal{G} \subset [a,b]$ be a Riemann germ with respect to (J, f). Let $K \subset [a,b] \cap \mathbb{R}^N$ be a compact set.
i) (Bounded box)
Then there exists a box $[\bar{a}, \bar{b}]$ such that

$$K \subset [\bar{a}, \bar{b}] \subset [a,b] \quad \text{with } [\bar{a}, \bar{b}] \text{ bounded} \quad \text{and} \quad \pi_{\mathcal{G}}([\bar{a}, \bar{b}]) \subset [\bar{a}, \bar{b}]. \tag{6.1}$$

Moreover, for the inclusion there exists a minimal box $[\bar{a}_*, \bar{b}_*]$ satisfying (6.1).
ii) (Restricted Riemann germ)
For any box $[\bar{a}, \bar{b}]$ satisfying (6.1), then the set $\mathcal{G}' := \mathcal{G} \cap [\bar{a}, \bar{b}]$ is a Riemann germ with respect to $(J, f_{|[\bar{a},\bar{b}]})$.

Proof of Lemma 6.1
Up to use reversion transforms, we can assume that the junction is of type $0 : N$. Up to use inversion transforms, we can also assume that $\theta^k = +1$ in (2.2) for each index k. Moreover, up to increase the compact set K, we can assume that $K = [\underline{a}, \underline{b}] \subset [a,b] \cap \mathbb{R}^N$ with $\bar{a}, \bar{b} \in \mathbb{R}^N$ with $\bar{a} < \bar{b}$.
Part 1: case $b = (+\infty, \dots, +\infty)$

Step 1: construction of $\bar{a}$
For each index j, we have the following dichotomy. Either $a^j > -\infty$, and then we set $\bar{a}^j := a^j$. Or we have $a^j = -\infty$, and using the coercivity of f^j at $-\infty$, we choose $\bar{a}^j \in (-\infty, \underline{a}^j]$ such that $f^j(\bar{a}^j) = \inf_{(-\infty,\bar{a}^j]} f^j$. Then (2.14) implies for all index j that

$$\hat{f}^j(p) \geq f^j_-(p) = \inf_{(-\infty,p^j]} f^j = \inf_{[\bar{a}^j,p^j]} f^j \quad \text{for all} \quad p \in [\bar{a}, b]$$

Hence, up to replace $\mathcal{G}$ by $\mathcal{G} \cap [\bar{a}, b]$ which is again a Riemann germ (from ii) of Theorem 2.15 and the continuity of $\hat{f}$), we can assume that $a = \bar{a} \in \mathbb{R}^N$.
Step 2: bound on $\mathcal{G}$ towards $+\infty$
We claim that there exists some

(6.2) $$\text{there exists some } b_* \in [\underline{b}, b] \cap \mathbb{R}^N \text{ such that} \quad \mathcal{G} \cap \left(b_* + [0,+\infty)^N\right) = \emptyset$$

Assume by contradiction that (6.2) is false. Then we deduce that we can construct a sequence $(p_n)_{n\in\mathbb{N}}$ with

(6.3) $$p_n \in \mathcal{G}, \quad p_n < p_{n+1}, \quad p^j_n \to +\infty \quad \text{as} \quad n \to +\infty \quad \text{for all index } j$$

Setting $\hat{p}_n := \pi(p_n) = p_n$, and using the fact that each f^j is coercive, we see that up to extract a subsequence (still denoted by n), we can assume that $\hat{p}_n \leq \hat{p}_{n+1}$ and $\hat{f}(\hat{p}_n) < \hat{f}(\hat{p}_{n+1})$ (using also that $\hat{f} = f$ on $\mathcal{G}$). This means that

(6.4) $$(\hat{p}_{n+1} - \hat{p}_n) \cdot [\hat{f}]^{\hat{p}_{n+1}}_{\hat{p}_n} \geq 0 \quad \text{with} \quad [\hat{f}]^{\hat{p}_{n+1}}_{\hat{p}_n} := \hat{f}^j(\hat{p}_{n+1}) - \hat{f}^j(\hat{p}_n) > 0$$

From Theorem 2.20, we know that $\sigma \diamond \hat{f} = -\hat{f}$ is Riemann monotone with $\sigma^j = -1$ for all j. Therefore

$$-(\hat{p}_{n+1} - \hat{p}_n) \cdot [\hat{f}]^{\hat{p}_{n+1}}_{\hat{p}_n} \leq 0 \quad \text{implies} \quad [\hat{f}]^{\hat{p}_{n+1}}_{\hat{p}_n} = 0$$

Contradiction with (6.4). Therefore (6.3) is false, and this implies (6.2).
Step 3: bound from above on $\hat{f}$ on $[\bar{a}, b]$
We claim that

(6.5) $$\sup_{j=1,\dots,N} \sup_{[\bar{a},b]} \hat{f}^j < +\infty$$

We do the proof by recurrence on the number $N \geq 1$ of branches, using (6.2).
Step 3.1: case $N = 1$
For b_* given in (6.2) for $N = 1$, recall that $b_* \in [\underline{b}, +\infty)$, and consider $\hat{b}_* := \pi_{\mathcal{G}}(b_*) \in \mathcal{G} \subset [a, b]$. Then (6.2) implies that $\hat{b}_* < b_*$ and $BA(\hat{b}_*) \supset \hat{b}_* + [0, +\infty) \ni b_*$. Hence $\hat{f} = \mathit{const}$ on $[b_*, +\infty)$. By assumption, $\mathcal{G}$ is a Riemann germ, and then $\hat{f}$ is continuous, and then bounded on $[\bar{a}, b_*]$. This implies that $\sup_{[\bar{a},b]} \hat{f} < +\infty$, which shows (6.5) for $N = 1$.
Step 3.2: case $N = n + 1$ for $n \geq 1$
Given $n \geq 1$, assume that (6.5) is true for $N - 1 = n$, and let us show it for $N := n + 1$. To this end, we use the slicing lemma 4.12.
We first freeze the last coordinate $\bar{a}^N$ of $\bar{a} = (\bar{a}', \bar{a}^N)$. We consider the germ

$$\mathcal{G}'_{\bar{a}^N} := \left\{p' := (p^1, \dots, p^{N-1}) \in [a', b'], \quad \hat{f}'_{\bar{a}^N}(p') = f'(p')\right\} \quad \text{with} \quad \left\{ \begin{array}{ll} f' & := (f^1, \dots, f^{N-1}) \\ \hat{f}'_{\bar{a}^N}(p') & := (\hat{f}^1, \dots, \hat{f}^{N-1})(p', \bar{a}^N), \end{array} \right.$$

Then from assumption (6.5) at the level $N - 1$, we deduce that

(6.6) $$\sup_{j=1,\dots,N-1} \sup_{[\bar{a}',b']} \hat{f}^j_{\bar{a}^N} < +\infty$$

Because $\mathcal{G}'_{\bar{a}^N} = \left\{p' \in [\bar{a}', b'], \quad \hat{f}'_{\bar{a}^N}(p') = f'(p')\right\}$ and f' is coercive, we deduce from (6.6) that there exists some $R > 0$ such that $\mathcal{G}'_{\bar{a}^N} \subset \bar{a}' + [0, R]^{N-1}$. Now for any $p \in [\bar{a}', b'] \times \left\{\bar{a}^N\right\}$ and $\hat{p} := \pi_{\mathcal{G}}(p)$, using the fact that the function $\hat{f}^N$ is nonincreasing in p^N for $0:N$ junctions, we deduce that $\hat{f}^N(p) = \hat{f}^N(\hat{p}) \le \hat{f}^N(\hat{p}', \bar{a}^N) = \hat{f}^N_{\bar{a}^N}(\hat{p}')$ with $\hat{p}' \in \mathcal{G}'_{\bar{a}^N}$. Therefore

$$\hat{f}^N(p) \le \sup_{[\bar{a}', \bar{a}'+(R,\dots,R)]} \hat{f}^N_{\bar{a}^N} \le C_R \tag{6.7}$$

Hence, using again the monotonicity of $\hat{f}^N$, we get $\sup_{[\bar{a},b]} \hat{f}^N \le \sup_{[\bar{a}',b']\times\{\bar{a}^N\}} \hat{f}^N \le C_R$. Similarly, up to increase the constant C_R, we deduce

$$\sup_{j=1,\dots,N} \sup_{[\bar{a},b]} \hat{f}^j \le C_R.$$

Step 4: construction of $\bar{b}$
Using the coercivity of f^j at $+\infty$, we choose $\bar{b}^j \in [\underline{b}^j, +\infty)$ such that $\sup_{[\bar{a},b]} \hat{f}^j \le f^j(\bar{b}^j) = \sup_{[\bar{a}^j, \bar{b}^j]} f^j$. Then we have

$$\hat{f}^j(p) \le \sup_{[\bar{a},\bar{b}]} \hat{f}^j \le \sup_{[\bar{a},b]} \hat{f}^j \le f^j(\bar{b}^j) \le \sup_{[p^j,\bar{b}^j]} f^j \quad \text{for all} \quad p \in [\bar{a}, \bar{b}]$$

We conclude that

$$\inf_{[\bar{a}^j,p^j]} f^j \le \hat{f}^j(p) \le \sup_{[p^j,\bar{b}^j]} f^j \quad \text{for all} \quad p \in [\bar{a},\bar{b}]$$

and from ii) of Theorem 2.15, we conclude that $\mathcal{G}' := \mathcal{G} \cap [\bar{a},\bar{b}]$ is a generalized germ with respect to $(J, f_{|[\bar{a},\bar{b}]})$ with $\hat{f}_{\mathcal{G}'} = (\hat{f}_{\mathcal{G}})_{|[\bar{a},\bar{b}]}$. Because $\hat{f} = \hat{f}_{\mathcal{G}}$ is continuous, we also deduce that $\mathcal{G}'$ is a Riemann germ.
Step 5: a further property of $[\bar{a},\bar{b}]$
Consider any $p \in [\bar{a},\bar{b}]$, and define $\tilde{p} := \pi_{\mathcal{G}'}(p) \in \mathcal{G}' \subset \mathcal{G}$. Then we have (from the definition of the basins of attraction) $p \in BA^{J,f_{|[\bar{a},\bar{b}]}}(\tilde{p}) = BA^{J,f}(\tilde{p}) \cap [\bar{a},\bar{b}]$, and then $p \in BA^{J,f}(\tilde{p})$ with $\tilde{p} \in \mathcal{G}$. Hence $\tilde{p} = \hat{p} := \pi_{\mathcal{G}}(p)$. In other words, this shows that $(\pi_{\mathcal{G}})_{|[\bar{a},\bar{b}]} = \pi_{\mathcal{G}'}$ with $\mathcal{G}' = \mathcal{G} \cap [\bar{a},\bar{b}]$. In particular, we see that if $\mathcal{G} \cap [\bar{a},\bar{b}]$ is a Riemann germ, then the box $[\bar{a},\bar{b}]$ satisfies $\pi_{\mathcal{G}}([\bar{a},\bar{b}]) \subset [\bar{a},\bar{b}]$.
Step 6: converse property
We now want to show that, under our assumption (2.2), for any bounded box $[\bar{a},\bar{b}] \subset [a,b]$ such that $\pi_{\mathcal{G}}([\bar{a},\bar{b}]) \subset [\bar{a},\bar{b}]$, then $\mathcal{G} \cap [\bar{a},\bar{b}]$ is a Riemann germ. From ii) of Theorem 2.15, we only have to show that for all j

$$\inf_{[\bar{a}^j,p^j]} f^j \le \hat{f}^j(p) \le \sup_{[p^j,\bar{b}^j]} f^j \quad \text{for all} \quad p \in [\bar{a},\bar{b}] \tag{6.8}$$

Indeed, let $p \in [\bar{a},\bar{b}]$. By assumption, we have $\hat{p} := \pi_{\mathcal{G}}(p) \in [\bar{a},\bar{b}]$. From the definition of the basin of attraction $BA(\hat{p}) \ni p \in [\bar{a},\bar{b}]$, and for a junction of type $0:N$, we have $\begin{cases} f^j(p^j) \ge f^j(\hat{p}^j) & \text{if} \quad p^j \ge \hat{p}^j \\ f^j(p^j) \le f^j(\hat{p}^j) & \text{if} \quad p^j \le \hat{p}^j \end{cases}$.
Because $p^j \in BA(\hat{p}^j) \cap [\bar{a}^j,\bar{b}^j] \ni \hat{p}^j$, we deduce that

$$\sup_{[p^j,\bar{b}^j]} f^j \ge \begin{cases} f^j(p^j) \ge f^j(\hat{p}^j) & \text{if} \quad p^j \ge \hat{p}^j \\ f^j(\hat{p}^j) & \text{if} \quad p^j \le \hat{p}^j \end{cases}, \quad \text{and} \quad \inf_{[\bar{a}^j,p^j]} f^j \le \begin{cases} f^j(\hat{p}^j) & \text{if} \quad p^j \ge \hat{p}^j \\ f^j(p^j) \le f^j(\hat{p}^j) & \text{if} \quad p^j \le \hat{p}^j \end{cases}$$

which implies that $\inf_{[\bar{a}^j,p^j]} f^j \le f^j(\hat{p}^j) = \hat{f}^j(\hat{p}) = \hat{f}^j(p) \le \sup_{[p^j,\bar{b}^j]} f^j$, which is exactly (6.8).
Step 7: minimal box
Given the compact set K, consider the set $S := \left\{[\bar{a},\bar{b}] \subset \mathbb{R}^N, \quad K \subset [\bar{a},\bar{b}] \quad \text{and} \quad \pi_{\mathcal{G}}([\bar{a},\bar{b}]) \subset [\bar{a},\bar{b}]\right\}$. Then define the set $K^{\mathcal{G}} := \bigcap_{[\bar{a},\bar{b}]\in S} [\bar{a},\bar{b}]$. By construction, this set is closed and is a box which contains K, i.e. we have $K^{\mathcal{G}} = [\bar{a}_*, \bar{b}_*] \supset K$. Moreover, we have $\pi_{\mathcal{G}}(K^{\mathcal{G}}) \subset K^{\mathcal{G}}$, and then $K^{\mathcal{G}} \in S$ which shows that $K^{\mathcal{G}}$ is the minimal element of S for the inclusion.
Part 2: case $b \neq (+\infty,\dots,+\infty)$
This part is an easy adaptation of Part 1. Step 1 is unchanged. If $b \in \mathbb{R}^N$, then we can choose $\bar{b} := b$. Assume now that $b \notin \mathbb{R}^N$. Then, in Step 2, relation (6.2) has to be replaced by

$$\text{there exists some } b_* \in [\underline{b}, b] \cap \mathbb{R}^N \text{ such that} \quad \mathcal{G} \cap \left(b_* + \sum_{b^j=+\infty} [0,+\infty) e_j \right) = \emptyset \tag{6.9}$$

where $(e_1, \ldots, e_N)$ is the canonical basis of $\mathbb{R}^N$. Step 3.1 is unchanged. In Step 3.2, relation (6.7) has to be modified in

$$\hat{f}^N(p) \le \sup_{[\bar{a}', \bar{a}' + (R, \ldots, R)] \cap [\bar{a}', b']} \hat{f}^N_{\bar{a}^N} \le C_R \tag{6.10}$$

Finally in Step 4, we have to redefine $\bar{b}^j := b^j$ only when $b^j \in \mathbb{R}$. The remaining part of the proof is unchanged. This ends the proof of the proposition.

6.2 Theorem 2.26 and its proof: Kružkov germs

Theorem 2.26 will be a corollary of the following two lemmata.

Lemma 6.2 **(D-maximality of Kružkov germs)**
Assume (2.2) with $N \ge 1$ with $\mathcal{G} \subset [a, b]$ a generalized Riemann germ. Consider the set

$$\mathcal{G}' := \left\{ p \in [a, b], \quad D^f(p, \hat{q}) \ge 0 \quad \text{for all} \quad \hat{q} \in \mathcal{G} \right\} \quad \text{for} \quad D^f \quad \text{defined in (2.8)}$$

If $\mathcal{G}$ is a Kružkov germ, then it satisfies the following D-maximality property: $\mathcal{G}' = \mathcal{G}$.

Proof of Lemma 6.2
Recall that by definition of the Kružkov germ $\mathcal{G}$, it satisfies $D^f \ge 0$ on $\mathcal{G} \times \mathcal{G}$. Hence $\mathcal{G} \subset \mathcal{G}'$. Assume that there exists $p \in \mathcal{G}' \backslash \mathcal{G}$. From Lemma 3.4, we deduce that $D^f(p, \hat{p}) < 0$ for $\hat{p} := \pi(p)$. Contradiction. Therefore $\mathcal{G}' = \mathcal{G}$, and this ends the proof.

Lemma 6.3 **(Stability of Kružkov germs)**
Assume (2.2). Let $\mathcal{G} \subset [a, b]$ be a generalized Riemann germ.
If $\mathcal{G}$ is a Kružkov germ, then $\hat{f}_{\mathcal{G}}$ is continuous.

Proof of Lemma 6.3
We will show the result using Theorem 2.17. We will indeed show that the $\mathcal{G}$-Riemann problem is stable. Let $[a, b] \ni p_n \to p_\infty$ as $n \to +\infty$. From Lemma 3.11, we have for $\hat{p}_n = \pi(p_n)$, that there exists $\tilde{p}_\infty$ such that $p_\infty \in BA(\tilde{p}_\infty)$ and

$$u_n := u^{\mathcal{G}}_{p_n, \hat{p}_n} \to u_\infty := u_{p_\infty, \tilde{p}_\infty} \quad \text{in} \quad L^1_{loc}([0, +\infty) \times J)$$

where we do not know yet that $\tilde{p}_\infty$ belongs to $\mathcal{G}$. This is what we want to prove. For any $0 \le \varphi^k \in C^1_c([0, +\infty) \times \bar{J}^k)$ with $\bar{J}^k := \{0\} \cup J^k \simeq [0, +\infty)$, we define

$$I^k(\varphi, u_n, c) := \int_{(0, +\infty) \times J^k} \left\{ \eta^k(u^k_n, c^k) \varphi^k_t + \psi^k(u^k_n, c^k) \varphi^k_x \right\} \, dt dx + \int_{\{0\} \times J^k} \eta^k(p_n, c^k) \varphi^k \, dx$$

From Definition 2.57 of Kružkov entropy solutions, we have for any $c \in \mathcal{G}$ (which can be seen as a constant $\mathcal{G}$-entropy solution), we have

$$I^k(\phi, u_n, c) \ge 0 \quad \text{for any} \quad 0 \le \phi^k \in C^1_c([0, +\infty) \times J^k)$$

Now for $0 \le \varphi^k \in C^1_c([0, +\infty) \times \bar{J}^k)$, and using the continuity of $\psi^k(\cdot, c^k)$, and the notion of trace at $x = 0$ (which is automatically well-defined for Riemann problem, because of the monotonicity of u^k_n in x), we get (which can be justified, by approximation from C^1 to Lipschitz continuous, with $\phi^k_\varepsilon := \varphi^k - \varphi^k(t, 0)\theta_\varepsilon(x)$ with $\theta_\varepsilon(x) := \max\left\{0, 1 - \varepsilon^{-1} x\right\}$ in the limit $\varepsilon \to 0$), that

$$I^k(\varphi, u_n, c) \ge \int_{(0, +\infty) \times \{0\}} -\psi^k(\hat{p}_n, c^k) \varphi^k$$

Choosing $\varphi^k = \varphi^j =: \varphi^0$ on $(0, +\infty) \times \{0\}$, we get

$$I(\varphi, u_n, c) := \sum_{k=1, \ldots, N} I^k(\varphi, u_n, c) \ge \int_{(0, +\infty) \times \{0\}} D^f(p_n, c) \varphi^0 \ge 0$$

where the last inequality follows from $D^f(p_n, c) \geq 0$ because $p_n, c \in \mathcal{G}$ and $\mathcal{G}$ is a Kružkov germ. Passing to the limit, we get $I(\varphi, u_\infty, c) \geq 0$. Now choosing $\varphi^j_\varepsilon(t,x) = \varphi^0(t,0)\theta_\varepsilon(x)$ for all index j (up to use again approximations from C^1 to Lipschitz continuous), we get in the limit $\varepsilon \to 0$ that

$$0 \leq I(\varphi_\varepsilon, u_\infty, c) \to \int_{(0,+\infty)\times\{0\}} D(\tilde{p}_\infty, c)\varphi^0 = D(\tilde{p}_\infty, c)\left\{\int_{(0,+\infty)\times\{0\}} \varphi^0\right\}$$

Because this is true for all $0 \leq \varphi^0 \in C^1_c([0,+\infty))$, we get $D(\tilde{p}_\infty, c) \geq 0$ for all $c \in \mathcal{G}$. From Lemma 6.2, we deduce that $\tilde{p}_\infty \in \mathcal{G}$ with $u_\infty = u_{p_\infty, \tilde{p}_\infty}$. Hence $\tilde{p}_\infty \in \mathcal{G} \cap \hat{\mathcal{P}}_{p_\infty} = \{\pi(p_\infty)\}$, i.e. $\tilde{p}_\infty = \pi(p_\infty)$ and $u_\infty = u^{\mathcal{G}}_{p_\infty, \pi(p_\infty)}$. This shows the stability of $\mathcal{G}$-Riemann problem. Hence from Theorem 2.17, we deduce that $\hat{f}$ is continuous. This ends the proof of the lemma.

Proof of Theorem 2.26:
For the proof, we refer to the table of Subsection 2.5. This follows from Lemmata 6.2 and Proposition 4.20. This ends the proof of the theorem.

We end this subsection with the following result.

Lemma 6.4 (Convergence of Godunov flux for Kružkov germs)
Assume (2.2) with $N \geq 1$. Let $\mathcal{G}_n \subset [a,b]$ for $n \in \mathbb{N}$ be a sequence of Kružkov germs with respect to (J, f), with associated Godunov flux $\hat{f}_n := \hat{f}_{\mathcal{G}_n}$. Then, up to extract a subsequence (still denoted by n), there exists a Kružkov germ $\mathcal{G}_\infty \subset [a,b]$ with respect to (J, f) and $\hat{f}_\infty := \hat{f}_{\mathcal{G}_\infty}$ such that

$$\hat{f}_{\mathcal{G}_n} \to \hat{f}_{\mathcal{G}_\infty} \quad as \quad n \to +\infty$$

Proof of Lemma 6.4
We know from i) of Proposition 4.20 and vii) of Proposition 2.14 that the sequence $\hat{f}_n$ is locally uniformly Lispchitz. Hence from Ascoli-Arzela theorem, we can extract a convergent subsequence and call $\hat{f}_\infty$ the limit. We define the set $\mathcal{G}_\infty := \left\{\hat{f}_\infty = f\right\}$, which is a closed subset of $[a,b]$. Notice that we have $D^{\hat{f}_n} \geq 0$ on $[a,b]^2$. Passing to the limit, we recover $D^{\hat{f}_\infty} \geq 0$ on $[a,b]^2$. Hence $D^f \geq 0$ on $\mathcal{G}_\infty \times \mathcal{G}_\infty$, which shows that $\mathcal{G}_\infty$ is a Kružkov germ, once we know that $\mathcal{G}_\infty$ is a generalized Riemann germ.

Now recall that $\hat{f}_n$ satisfy (2.14), and also (2.12) which is

$$(6.11) \qquad \begin{cases} \text{For all } p \in [a,b] \text{ and } K_{n,p} := \left\{j \in \{1,\dots,N\}, \quad \hat{f}^j_n(p) \neq f^j(p)\right\}, \qquad \text{there exists } \varepsilon_n > 0 \\ \text{such that for } Q_{n,\varepsilon_n}(p) := p + \displaystyle\sum_{j \in K_{n,p}} (-\varepsilon_n, \varepsilon_n)e_j, \text{ we have} \\ \hat{f}_n = const \quad \text{on} \quad [a,b] \cap Q_{n,\varepsilon_n}(p) \end{cases}$$

Then it is easy to see that condition (6.11) is closed. This follows from the Lipschitz continuity of f which allows to bound from below ε_n for n large enough as $\hat{f}_n \to \hat{f}_\infty$, starting from $K_{\infty,p}$ defined similarly for $\hat{f}_\infty$. We conclude that $\hat{f}_\infty$ satisfies (2.12) and (2.14). Then Theorem 2.15 ii) implies that $\mathcal{G}_\infty$ is a generalized Riemann germ and that $\hat{f}_\infty$ is the Godunov flux associated to $\mathcal{G}_\infty$. We conclude that $\mathcal{G}_\infty$ is a Kružkov germ with $\hat{f}_{\mathcal{G}_\infty} = \hat{f}_\infty$. This ends the proof of the lemma.

6.3 Theorem 2.28 and its proof: D_+-germs

Proposition 6.5 (Characterization of D_+-germs among generalized Riemann germs)
Assume (2.2) with $N \geq 1$. Let $\mathcal{G} \subset [a,b]$ be a generalized Riemann germ, and let $\hat{f} := \hat{f}_{\mathcal{G}}$ be its associated Godunov germ. Then $\mathcal{G}$ is a D_+-germ if and only if $\hat{f}$ is locally Lipschitz continuous on $[a,b]$ and satisfies a.e. on $[a,b]$

$$(6.12) \qquad \partial_i(\sigma^i \hat{f}^i) \geq \sum_{j \in \{1,\dots,N\}\setminus\{i\}} |\partial_i \hat{f}^j|, \quad \textit{for all} \quad i = 1,\dots,N$$

and

$$(6.13) \qquad \partial_i(\sigma^j \hat{f}^j) \leq 0, \quad \textit{for all} \quad i \neq j$$

Proof of Proposition 6.5
The proof follows part of the lines of the proof of Proposition 4.20.
We first recall that from Lemma 5.5, the generalized Riemann germ $\mathcal{G}$ is a D_+-germ if and only if it satisfies $D_+^{\hat{f}} \geq 0$ on $[a,b]^2$ for $\hat{f} := \hat{f}_{\mathcal{G}}$. We only do the proof for $N:0$ junctions. The general case then follows by reversion transforms.
Step 1: necessary condition
Step 1.1: preliminary
We first notice that a D_+-germ is in particular a Kružkov germ (i.e. a D-germ) because $D(p,q) = D_+(p,q) + D_+(q,p)$. Then Proposition 4.20 implies that $\hat{f}$ is locally Lipschitz continuous on $[a,b]$ and satisfies (6.12).
Step 1.2: application
Recall that Corollary 4.19 holds for $N:0$ juntions, and (4.33) means for $\hat{f}$ that a.e. on $[a,b]$ we have

$$(6.14)\qquad \left\{\begin{array}{lll} S_K := \sum_{j\in K} \hat{f}^j & & \\ \partial_k S_K \geq 0 & \text{for all} & k \in K \\ \partial_k S_K \leq 0 & \text{for all} & k \notin K \end{array}\right|, \quad \text{for all} \quad K \subset \{1,\dots,N\}$$

Hence for $K := \{j\}$, this gives (6.13) with $\sigma^\ell = 1$ for all $\ell = 1,\dots,N$.
Step 2: sufficient condition
Assume now that $\hat{f}$ is locally Lipschitz continuous on $[a,b]$ and satisfies (6.12) and (6.13) for $\sigma^j = 1$ for all j. Then (6.12) implies immediately the second line of (6.14), while (6.13) implies the third line of (6.14). Therefore (6.14) holds true, and then Corollary 4.19 implies that $D_+^{\hat{f}} \geq 0$ for a $N:0$ junctions. Finally recall that Lemma 5.5 shows that $\mathcal{G}$ is a D_+-germ. This ends the proof of the proposition.

Proof of Theorem 2.28
For the proof, we refer to the table of Subsection 2.5. The result follows from Proposition 6.5 and i') and iii) of Lemma 5.5. This ends the proof of the theorem.

6.4 Theorem 2.29 and its proof: conservative Riemann germs

Proof of Theorem 2.29
Recall that we assume that $\mathcal{G}$ is a Riemann germ which is conservative. From Lemma 5.5, this means $\hat{f} := \hat{f}_{\mathcal{G}}$ is continuous and satisfies

$$RH^{\hat{f}} = 0 \quad \text{with} \quad RH^{\hat{f}}(p) := \sum_{j=1,\dots,N} \sigma^j \hat{f}^j(p)$$

Still from Lemma 5.5, recall that $\mathcal{G}$ is monotone if and only if

$$(6.15)\qquad p \mapsto \sigma^j \hat{f}^j(p) \quad \text{is nonincreasing in } p^k \text{ for all } k \neq j, \text{ and nondecreasing in } p^j$$

and $\mathcal{G}$ is Kružkov if and only if $D^{\hat{f}} \geq 0$. We first notice that up to apply a suitable reversion transform, we can assume that the junction is of type $N:0$, i.e. that $J^j \simeq (-\infty, 0)$ with $\sigma^j = 1$ for all j.
Step 1: monotone implies Kružkov
Assume that $\hat{f}$ satisfies (6.15) and let us show that it satisfies (4.33). To this end, consider a set $K \subset \{1,\dots,N\}$, and call $S_K := \sum_{j\in K} \hat{f}^j$. Then (6.15) implies

$$(6.16)\qquad \partial_\ell S_K \leq 0 \quad \text{for all} \quad \ell \notin K$$

We also have $S_K = RH^{\hat{f}} - \sum_{j\notin K} \hat{f}^j = -\sum_{j\notin K} \hat{f}^j$. Hence for all $\ell \in K$ and $j \notin K$, we have $\partial_\ell \hat{f}^j \leq 0$, and then

$$(6.17)\qquad \partial_\ell S_K \geq 0 \quad \text{for all} \quad \ell \in K$$

Therefore (6.16)-(6.17) show (4.33). From ii) of Lemma 4.18, this implies $D_+^{\hat{f}} \geq 0$, and then $D^{\hat{f}} \geq 0$ and $\mathcal{G}$ is a Kružkov germ.
Step 2: Kružkov implies monotone

Assume that $\mathcal{G}$ is Kružkov. Then $D^{\hat{f}} \geq 0$, and from i) of Lemma 4.18, this implies that $\bar{S}_K$ satisfies (6.16)-(6.17), with

$$\bar{S}_K := \sum_{j\in K} \hat{f}^j - \sum_{j\notin K} \hat{f}^j = -RH^{\hat{f}} + 2\sum_{j\in K} \hat{f}^j = 2\sum_{j\in K} \hat{f}^j = 2S_K \tag{6.18}$$

Therefore S_K also satisfies (6.16)-(6.17). Given an index k and $K := \{k\}$ and $\ell \notin K$ or $\ell \in K$, we deduce from (6.16)-(6.17) applied to S_K that $\partial_\ell \hat{f}^k \leq 0$ for all $\ell \neq k$ and $\partial_k \hat{f}^k \geq 0$, which is nothing else than condition (6.15). Therefore $\mathcal{G}$ is monotone.

Step 3: equivalence with D_+-germs

Using Theorem 2.28, we deduce that monotone is equivalent to Kružkov which is also equivalent to D_+-germ. This ends the proof of the lemma.

We also have the following result.

Lemma 6.6 (Conservative functions: equivalence of monotonicity and Kružkov property)
Assume (2.2) with $N \geq 1$ and let the associated $\sigma \in \{\pm 1\}^N$ and $[a,b] \subset \mathbb{R}^N$. Let $\hat{f} : [a,b] \to \mathbb{R}^N$ be a continuous function which is conservative in the following sense

$$\sum_{j=1,\dots,N} \sigma^j \hat{f}^j = 0 \quad on \quad [a,b]$$

Then $\hat{f}$ is monotone in the sense of (6.15) if and only if $\hat{f}$ is Kružkov in the following sense

$$D^{\hat{f}} \geq 0 \quad on \quad [a,b]^2$$

which is also equivalent for $\hat{f}$ to be semi Kružkov in the following sense

$$D_+^{\hat{f}} \geq 0 \quad on \quad [a,b]^2$$

Proof of Lemma 6.6

The proof follows the lines of the proof of Theorem 2.29, but using only Lemma 4.18 and (6.18). This ends the proof of the lemma.

6.5 Theorem 2.30 and its proof: Hamilton-Jacobi germs and HJ-relaxation operator

The proof of Theorem 2.30 requires two intermediate results: first Lemma 6.7 shows the existence of some HJ-relaxation operator, which in particular has an action on locally constant functions. Then Proposition 6.9 presents important properties of HJ germs. Finally, we conclude the section with the proof of Theorem 2.30.

Lemma 6.7 (HJ-relaxation operator)
Assume (2.2) with $N \geq 1$. Let h_0 be such that

$$\begin{cases} h_0 : [a,b] \to \mathbb{R} \quad \textit{is continuous} \\ p \mapsto \sigma^j h_0(p) \quad \textit{is nondecreasing in } p^j \textit{ for all } j = 1,\dots,N, \\ f_0 := (h_0,\dots,h_0) \textit{ satisfies the monotone bounds given in the second line of (2.14).} \end{cases} \tag{6.19}$$

Assume moreover that either $[a,b] \subset \mathbb{R}^N$ is compact, or $[a,b] \cap \mathbb{R}^N = \mathbb{R}^N$, or

$$f_0 := (h_0,\dots,h_0) \textit{ is locally constant on } \{f_0 \neq f\} \tag{6.20}$$

Then the following formula defines uniquely $h_1(p)$ for each $p \in [a,b]$

$$\{h_1(p)\} = \bigcup_{q\in[a,b]} \left(\{h_0(q)\} \cap \bigcap_{J^j \simeq (-\infty,0)} \{G^j(p^j,q^j)\} \cap \bigcap_{J^j \simeq (0,+\infty)} \{G^j(q^j,p^j)\} \right). \tag{6.21}$$

Then we set

$$\mathfrak{R} h_0 := h_1 \tag{6.22}$$

and h_1 satisfies (6.19) and moreover $f_1 := (h_1,\dots,h_1)$ is locally constant on $\{f_1 \neq f\}$.

Remark 6.8 *Notice that in Lemma 6.7 and in the special case of assumption (6.20), we do not have to require the continuity of* h_0 *in (6.19). Indeed the continuity of* h_0 *is then automatic, as it will be independently shown later in i) of Proposition 6.9, for the HJ germ* $\mathcal{G}_{f_0} = \{f_0 = f\}$.

Proof of Lemma 6.7
Step 1: h_1 **is well-defined with good properties**
Step 1.1: non emptyness of the intersection
We consider h_0 satisfying (6.19). In particular, for $f_0 := (h_0, \dots, h_0)$ we have $f_- \le f_0 \le f_+$, which means

$$(6.23) \qquad \begin{cases} f_-^j(q) = G^{f^j}(q^j, b^j) \le \quad h_0(q) \quad \le f_+^j(q) = G^{f^j}(q^j, a^j) & \text{if} \quad J^j \simeq (-\infty, 0) \\ f_-^j(q) = G^{f^j}(a^j, q^j) \le \quad h_0(q) \quad \le f_+^j(q) = G^{f^j}(b^j, q^j) & \text{if} \quad J^j \simeq (0, +\infty) \end{cases}$$

Recall that the monotonicities of h_0 are opposite for ingoing and outgoing branches. It is more convenient to have the same monotonicities on each variable. To this end, we define

$$\bar p^j := \sigma^j p^j, \quad [\bar a^j, \bar b^j] := \sigma^j [a^j, b^j] \quad \bar f^j(\bar p) = f^j(p), \quad \bar J^j := \sigma^j(-\infty, 0) \quad \text{for} \quad j = 1, \dots, N \quad \text{and} \qquad \bar h_0(\bar p) := h_0(p)$$

(which can be seen as a composition of some inversion transform and the reversion transform) which satisfies $\bar h_0(\uparrow, \dots, \uparrow)$ and

$$(6.24) \qquad \bar f_-^j(\bar q) = G^{\bar f^j}(\bar q^j, \bar b^j) \le \bar h_0(\bar q) \le \bar f_+^j(\bar q) = G^{\bar f^j}(\bar q^j, \bar a^j) \quad \text{and} \quad \bar J^j \simeq (-\infty, 0) \quad \text{for all} \quad j$$

We also set $\bar J := \{0\} \cup \bigcup_{j=1,\dots,N} \bar J^j$. Then $(\bar J, \bar f)$ satisfies (2.2), with possible coercivity assumption for some $\bar\theta^k \in \{\pm 1\}$. Up to apply some inversion, we can furthermore assume that $\bar\theta^k = 1$.
We distinguish cases.
Case A: $\bar a, \bar b \in \mathbb{R}^N$
Then $\bar f_-^j(\bar a) = \inf_{[\bar a^j, \bar b^j]} \bar f^j$ and $\bar f_+^j(\bar b) = \sup_{[\bar a^j, \bar b^j]} \bar f^j$, $j = 1, \dots, N$. We define the functions

$$\Phi_{\bar p}^j(\bar q) := G^{\bar f^j}(\bar p^j, \bar q^j) - \bar h_0(\bar q), \quad j = 1, \dots, N$$

For any $\bar p \in [\bar a, \bar b]$ and using the monotonicity of $\Phi_{\bar p}^j$, we get for $\iota_{\bar q}^j(\bar a^j) := (\bar q^1, \dots, \bar q^{j-1}, \bar a^j, \bar q^{j+1}, \dots, \bar q^N)$ and for all $\bar q \in [\bar a, \bar b]$ that

$$\Phi_{\bar p}^j(\iota_{\bar q}^j(\bar a^j)) = G^{\bar f^j}(\bar p^j, \bar a^j) - \bar h_0(\iota_{\bar q}^j(\bar a^j)) \ge G^{\bar f^j}(\bar p^j, \bar a^j) - \bar f_+^j(\bar a^j) = \sup_{[\bar a^j, \bar p^j]} \bar f^j - \bar f^j(\bar a^j) \ge 0$$

We also get

$$\Phi_{\bar p}^j(\iota_{\bar q}^j(\bar b^j)) = G^{\bar f^j}(\bar p^j, \bar b^j) - \bar h_0(\iota_{\bar q}^j(\bar b^j)) \le G^{\bar f^j}(\bar p^j, \bar b^j) - \bar f_-^j(\bar b^j) = \inf_{[\bar p^j, \bar b^j]} \bar f^j - \bar f^j(\bar b^j) \le 0$$

Recall that the map $\bar q \mapsto \Phi_{\bar p}(\bar q)$ is continuous. Hence we deduce from Poincaré-Miranda theorem (see [39]) that there exists $\bar q \in [\bar a, \bar b]$ such that

$$(6.25) \qquad \Phi_{\bar p}(\bar q) = 0$$

This implies that

$$(6.26) \qquad \left\{\bar h_0(\bar q)\right\} \cap \bigcap_{j=1,\dots,N} \left\{G^{\bar f^j}(\bar p^j, \bar q^j)\right\} \ne \emptyset$$

Case B: $-\bar a^j = \bar b^j = +\infty$ **for all** j
Here we do not use Poincaré-Miranda theorem, but make a direct proof. We first notice that (6.24) implies for the special choice $\bar q := \bar p$ $G^{\bar f^j}(\bar p^j, \bar b^j) \le \bar h_0(\bar p) \le G^{\bar f^j}(\bar p^j, \bar a^j)$ for all j, i.e. (using the coercivity of $\bar f^j$ to bound $\underline{\lambda}_{\bar p}$ from below)

$$-\infty < \underline{\lambda}_{\bar p} := \max_{j=1,\dots,N} G^{\bar f^j}(\bar p^j, \bar b^j) = \max_{j=1,\dots,N} \bar f_-^j(\bar p^j) \le \bar h_0(\bar p) \le \min_{j=1,\dots,N} \bar f_+^j(\bar p^j) = \min_{j=1,\dots,N} G^{\bar f^j}(\bar p^j, \bar a^j) =: \overline{\lambda}_{\bar p} = +\infty$$

Consider now for $\varepsilon > 0$, the functions

$$G_\varepsilon^{\bar f^j}(\bar p^j,\bar q^j) := G^{\bar f^j}(\bar p^j,\bar q^j) - \varepsilon(\bar q^j - \bar p^j), \quad j = 1,\dots,N$$

Then $\bar q^j \mapsto G_\varepsilon^{\bar f^j}(\bar p^j,\bar q^j)$ is locally Lipschitz continuous and satisfies $\frac{\partial}{\partial \bar q^j} G_\varepsilon^{\bar f^j}(\bar p^j,\bar q^j) \le -\varepsilon$ a.e.. Moreover $G_\varepsilon^{\bar f^j}(\bar p^j,\bar a^j) \ge \bar\lambda_{\bar p} - \varepsilon(\bar a^j - \bar p^j) \ge \bar\lambda_{\bar p} = +\infty$ and $G_\varepsilon^{\bar f^j}(\bar p^j,\bar b^j) \le \underline{\lambda}_{\bar p} - \varepsilon(\bar b^j - \bar p^j) \le \underline{\lambda}_{\bar p}$. Hence for any $\lambda \in \mathbb{R}$, there exists a unique $\bar q^j_\varepsilon(\lambda) \in \mathbb{R}$ such that

$$G_\varepsilon^{\bar f^j}(\bar p^j,\bar q^j_\varepsilon(\lambda)) = \lambda \tag{6.27}$$

and moreover the map $\lambda \mapsto \bar q^j_\varepsilon(\lambda)$ is nonincreasing and continuous. Now define $\Theta_\varepsilon(\lambda) := \lambda - \bar h_0(\bar q_\varepsilon(\lambda))$, which by construction is also nondecreasing in λ, and satisfies $\Theta_\varepsilon(\underline{\lambda}_{\bar p}) \le 0 \le \Theta_\varepsilon(\bar\lambda_{\bar p}) = +\infty$. By continuity of Θ_ε, we deduce that there exists $\lambda^\varepsilon \in [\underline{\lambda}_{\bar p}, +\infty)$ such that

$$\Theta_\varepsilon(\lambda^\varepsilon) = 0 \tag{6.28}$$

Up to extract a subsequence assume that $\lambda^\varepsilon \to \lambda^* \in [\underline{\lambda}_{\bar p}, +\infty]$ as $\varepsilon \to 0$. Assume by contradiction that $\lambda^* = +\infty$. From (6.27) and (6.28), we have

$$\bar h_0(\bar q^\varepsilon) = G^{\bar f^j}(\bar p^j,\bar q^{\varepsilon,j}) - \varepsilon(\bar q^{\varepsilon,j} - \bar p^j) = \lambda^\varepsilon \to +\infty \quad \text{with} \quad \bar q^{\varepsilon,j} := \bar q^j_\varepsilon(\lambda^\varepsilon)$$

where the second equality implies that $\bar q^{\varepsilon,j} \to -\infty$. This is in contradiction with the first equality and the monotonicity of $\bar h_0$. We conclude that $\lambda^* \in [\underline{\lambda}_{\bar p}, +\infty)$ and then $\bar q^\varepsilon \to \bar q^*$ which satisfies $\bar h_0(\bar q^*) = G^{\bar f^j}(\bar p^j, \bar q^{*,j})$ for all j, which is (6.26) for $\bar q := \bar q^*$.

Case C: $\bar f_0 := (\bar h_0,\dots,\bar h_0)$ is locally constant on $\{\bar f_0 \ne \bar f\}$

From ii) of Theorem 2.15, we deduce that $\mathcal{G}_{\hat{\bar f}_0} := \left\{\hat{\bar f}_0 = \bar f\right\}$ is a generalized Riemann germ. Setting $\pi := \pi_{\mathcal{G}_{\hat{\bar f}_0}}$, we define

$$\hat{\bar p} := \pi(\bar p), \quad \bar p \in BA^{(\bar J,\bar f)}(\hat{\bar p}), \quad \lambda_0 := \bar h_0(\hat{\bar p}) = \bar h_0(\bar p)$$

where in the last equality we have used the fact that $\bar h_0$ is constant on $BA^{(\bar J,\bar f)}(\hat{\bar p})$. Notice now that $\hat{\bar p} \in \mathcal{G}_{\bar f_0}$ and then $G^{\bar f^j}(\bar p^j, \hat{\bar p}^j) = \bar f^j(\hat{\bar p}^j) = \bar h_0(\hat{\bar p}) = \lambda_0$ which is (6.26) with $\bar q := \hat{\bar p}$.

Step 1.2: uniqueness of the common value

Now assume that there exist two values $\bar q, \bar q' \in [\bar a, \bar b]$ such that $\begin{cases} \lambda = \bar h_0(\bar q) = G^{\bar f^j}(\bar p^j,\bar q^j) \\ \lambda' = \bar h_0(\bar q') = G^{\bar f^j}(\bar p^j,\bar q'^j) \end{cases}$. Assume by contradiction that $\lambda \ne \lambda'$. Up to exchange $\bar q$ and $\bar q'$, we can assume $\lambda' > \lambda$. From the monotonicities of the standard Godunov fluxes, we deduce that $\bar q'^j < \bar q^j$. From the monotonicies of $\bar h_0(\uparrow,\dots,\uparrow)$, we deduce that $\lambda' = \bar h_0(\bar q') \le \bar h_0(\bar q) = \lambda$. Contradiction. Hence $\lambda' = \lambda$, and we have uniqueness of the value λ and we call $\bar h_1(\bar p) := \lambda$. We also define $h_1(p) := \bar h_1(\bar p)$.

Step 2: h_1 satisfies condition (6.19)

It is sufficient to check that $\bar h_1$ satisfies condition (6.19) with h_0 replaced by $\bar h_0$ and $\sigma^j = 1$ for all j.

Step 2.1: continuity of $\bar h_1$

The continuity of $\bar h_1$ follows from the compactness of the set of solutions $\bar q$ to (6.25). Indeed consider a sequence $[\bar a, \bar b] \ni \bar p_n \to \bar p \in [\bar a, \bar b]$ as $n \to +\infty$, and an associated sequence $\bar q_n \in [\bar a, \bar b]$ such that

$$\bar h_1(\bar p_n) = \bar h_0(\bar q_n) = G^{\bar f^j}(\bar p^j_n, \bar q^j_n) \quad \text{for all} \quad j \tag{6.29}$$

If $[\bar a, \bar b] \subset \mathbb{R}^N$ is compact, then the result follows from the continuity of $\bar h_0, \bar f$.
We will use coercivity in assumption (2.2) in order to show compactness of the sequence $\bar q_n$. Then assume by contradiction that $|\bar q^k_n| \to +\infty$ as $n \to +\infty$ for at least some index k. Notice that

$$G^{\bar f^k}(\bar p^k_n, \bar q^k_n) = \inf_{[\bar p^k_n, \bar q^k_n]} \bar f^k \le \bar f^k(\bar p^k_n) \quad \text{and} \quad \bar h_0(\bar q_n) \ge \bar f^k_-(\bar q_n) = \inf_{[\bar q^k_n, \bar b^k]} \bar f^k \to +\infty \quad \text{if} \quad \bar q^k_n \to +\infty$$

and this leads to a contradiction with equality (6.29). Therefore we deduce that $\bar q^k_n \to -\infty$, and then $\bar h_0(\bar q_n) = G^{\bar f^k}(\bar p^k_n, \bar q^k_n) = \sup_{[\bar q^k_n, \bar p^k_n]} \bar f^k \to +\infty$. Now (6.29) implies that $\bar q^j_n \to -\infty$ for all $j = 1,\dots,N$.

Consider any $\bar c \in [\bar a, \bar b]$. The monotonicitiy of $\bar h_0$ then implies that $\bar h_0(\bar c) \ge \bar h_0(\bar q_n) \to +\infty$. Contradiction. Therefore we conclude that $\bar q_n$ stays bounded, and we can extract a convergent subsequence that we still denote $(\bar{q}_n)_n$ such that $\bar{q}_n \to \bar{q}$. Using the continuities of $\bar{h}, \bar{f}$, we can pass to the limit in (6.29) which gives $\bar{h}_1(\bar{p}_n) \to \bar{h}_0(\bar{q}) = G^{\bar{f}^j}(\bar{p}^j, \bar{q}^j) = \bar{h}_1(\bar{p})$ for all j, which shows the continuity of $\bar{h}_1$.

Step 2.2: checking other properties of $\bar{h}_1$

Let us now show the monotonicities of $\bar{h}_1$. We write $\bar{h}_1(\bar{p}) = \lambda = \bar{h}_0(\bar{q}) = G^{\bar{f}^j}(\bar{p}^j, \bar{q}^j)$, $j = 1, \dots, N$, and consider $\bar{p}'$ with $\bar{p}'^1 = \bar{p}^1$ and $\bar{p}' \geq \bar{p}$, and $\bar{h}_1(\bar{p}') = \lambda' = \bar{h}_0(\bar{q}') = G^{\bar{f}^j}(\bar{p}^j, \bar{q}'^j)$.

Assume by contradiction that $\lambda' < \lambda$. Then the monotonicities of the standard Godunov fluxes imply $\bar{q}' > \bar{q}$ and the monotonicity of $\bar{h}_0$ implies $\lambda' = \bar{h}_0(\bar{q}') \geq \bar{h}_0(\bar{q}) = \lambda$. Contradiction. We deduce that $\lambda' \geq \lambda$, and then $\bar{h}_1$ is nonincreasing in $\bar{p}^1$. Similarly, we show that $\bar{h}_1$ is nondecreasing in each coordinate $\bar{p}^j$.

We now check that $\bar{h}_1$ satisfies the bounds given in the second line of (2.14) for h_0 replaced by $\bar{h}_0$ and $\sigma^j = 1$ for all j. Recall that $\bar{h}_0$ satisfies (6.24), i.e.

$$\bar{f}^j_-(\bar{q}) = G^{\bar{f}^j}(\bar{q}^j, \bar{b}^j) \leq \bar{h}_0(\bar{q}) \leq \bar{f}^j_+(\bar{q}) = G^{\bar{f}^j}(\bar{q}^j, \bar{a}^j) \quad \text{for all } j$$

Then for $\bar{h}_1(\bar{p}) = \lambda = \bar{h}_0(\bar{q}) = G^{\bar{f}^j}(\bar{p}^j, \bar{q}^j)$ for all j, we get

$$\bar{f}^j_-(\bar{p}^j) = G^{\bar{f}^j}(\bar{p}^j, \bar{b}^j) \leq \lambda = G^{\bar{f}^j}(\bar{p}^j, \bar{q}^j) \leq G^{\bar{f}^j}(\bar{p}^j, \bar{a}^j) = \bar{f}^j_+(\bar{p}^j)$$

which shows exactly that $\bar{h}_1$ satisfies (6.23) which are the bounds given in the second line of (2.14).

Step 3: local constancy of $f_1 := (h_1, \dots, h_1)$

Again, we show it for $\bar{f}_1 := (\bar{h}_1, \dots, \bar{h}_1)$ with respect to $\bar{f}$ and this gives the result for $f_1 = (h_1, \dots, h_1)$ with respect to f. Assume that $\bar{h}_1(\bar{p}) \neq \bar{f}^j(\bar{p})$ for all $j \in I \subset \{1, \dots, N\}$, and recall the argument of the proof of Proposition 3.10, in Step 2. We have here

$$\bar{f}^j(\bar{p}) \neq \bar{h}_1(\bar{p}) = \bar{h}_0(\bar{q}) = G^{\bar{f}^j}(\bar{p}^j, \bar{q}^j) = \begin{cases} \sup\limits_{[\bar{q}^j, \bar{p}^j]} \bar{f}^j = \sup\limits_{[\bar{q}^j, \bar{p}^j)} \bar{f}^j & \text{if } \quad \bar{q}^j < \bar{p}^j \\ \inf\limits_{[\bar{p}^j, \bar{q}^j]} \bar{f}^j = \inf\limits_{(\bar{p}^j, \bar{q}^j]} \bar{f}^j & \text{if } \quad \bar{q}^j > \bar{p}^j \end{cases}$$

where we have used the fact that the inf / sup can not be reached at $\bar{p}^j$, because $\bar{f}^j(\bar{p}^j) \neq \bar{h}_1(\bar{p})$. Notice also that we can not have $\bar{q}^j = \bar{p}^j$, otherwise we would get $\bar{h}_1(\bar{p}) = \bar{h}_0(\bar{p}) = G^{\bar{f}^j}(\bar{p}^j, \bar{p}^j) = \bar{f}^j(\bar{p}^j) = \bar{f}^j(\bar{p})$, which is impossible by assumption. This shows now that for $\bar{p}_\varepsilon$ close to $\bar{p}$ in $\left(\bar{p} + \sum\limits_{j \in I} \mathbb{R} e_j\right) \cap [\bar{a}, \bar{b}]$, we also have (by continuity of $\bar{f}, \bar{h}_0$)

$$\bar{h}_0(\bar{q}) = G^{\bar{f}^j}(\bar{p}^j_\varepsilon, \bar{q}^j) \begin{cases} = \left\{ \begin{array}{lll} \sup\limits_{[\bar{q}^j, \bar{p}^j_\varepsilon]} \bar{f}^j = \sup\limits_{[\bar{q}^j, \bar{p}^j)} \bar{f}^j & \text{if} & \bar{q}^j < \bar{p}^j_\varepsilon \\ \inf\limits_{[\bar{p}^j_\varepsilon, \bar{q}^j]} \bar{f}^j = \inf\limits_{(\bar{p}^j, \bar{q}^j]} \bar{f}^j & \text{if} & \bar{q}^j > \bar{p}^j_\varepsilon \end{array} \right\} = \bar{h}_1(\bar{p}) & \text{for all} \quad j \in I \\ \\ = \bar{h}_1(\bar{p}) \quad \text{because} \quad \bar{p}^j_\varepsilon = \bar{p}^j & \text{for all} \quad j \notin I \end{cases}$$

This justifies a posteriori that we can choose $\bar{q}_\varepsilon := \bar{q}$ in $\bar{h}_1(\bar{p}_\varepsilon) = \bar{h}_0(\bar{q}_\varepsilon) = G^{\bar{f}^j}(\bar{p}^j_\varepsilon, \bar{q}^j_\varepsilon)$. Therefore $\bar{f}^j(\bar{p}_\varepsilon) \neq \bar{h}_1(\bar{p}_\varepsilon) = \bar{h}_0(\bar{q}) = \bar{h}_1(\bar{p})$ for all $j \in I$. Hence $\bar{f}_1$ is locally constant on $\{\bar{f}_1 \neq \bar{f}\}$.

This ends the proof of the lemma. Then we have the following result.

Proposition 6.9 (Properties of HJ germs)

Assume (2.2) with $N \geq 1$. Let $\mathcal{G}$ be a generalized Riemann germ which is a HJ germ.

i) (Regularity properties of $\hat{f}$)

Then $\mathcal{G}$ is a Riemann germ, i.e. $\hat{f} = \hat{f}_{\mathcal{G}}$ is continuous, and $\hat{f}$ is locally Lipschitz continuous on $[a, b]$.

ii) (HJ-relaxation formula)

Moreover there exists some continuous function $\hat{h} : [a, b] \to \mathbb{R}$ such that for all index j, we have $\hat{f}^j = \hat{h}$. And the function $\hat{h}$ satisfies the following HJ-relaxation formula

$$\text{(6.30)} \qquad \text{for all } p \in [a, b], \text{ there exists } q \in [a, b] \text{ s.t. } \quad \hat{h}(p) = \hat{h}(q) = \begin{cases} G^{f^j}(p^j, q^j) & \text{if} \quad J^j \simeq (-\infty, 0) \\ G^{f^j}(q^j, p^j) & \text{if} \quad J^j \simeq (0, +\infty) \end{cases}$$

Proof of Proposition 6.9
Step 1: proof of i)
By assumption, we know that there exists $h : \mathcal{G} \to \mathbb{R}$ such that $f = (h, \dots, h)$ on $\mathcal{G}$. Define $\hat{h} := h \circ \pi_{\mathcal{G}} : [a, b] \to \mathbb{R}$. This implies that $\hat{f} = (\hat{h}, \dots, \hat{h}) : [a, b] \to \mathbb{R}^N$. Up to apply a reversion transform, we can assume that we work on $N : 0$ junction. From (4.14), we deduce that for each $j \in \{1, \dots, N\}$, the map $p \mapsto \hat{h}(p)$ is locally Lipschitz continuous in p^j, uniformly in p^k for $k \neq j$. Because this is true for any index j, we deduce that $\hat{h}$ and then $\hat{f}$ is continuous. Moreover, from (4.14) we have $|\partial_{p^j} \hat{h}(p)| \leq |(f^j)'(p^j)|$ for a.e. $p \in [a, b]$.
Step 2: proof of ii)
From Step 1, we know that $\hat{f}$ is locally constant on $\left\{\hat{f} \neq f\right\}$ and $\mathcal{G} = \left\{\hat{f} = f\right\}$. Hence any $\hat{p} \in \mathcal{G}$ satisfies $\hat{h}(\hat{p}) = G^{f^j}(\hat{p}^j, \hat{p}^j)$ for all $j = 1, \dots, N$. Therefore, setting $h_0 := \hat{h}$, we see that the function h_1 given by relaxation formula (6.21) satisfies

$$h_1(\hat{p}) = h_0(q) = G^{f^j}(\hat{p}^j, q^j) = \hat{h}(\hat{p}) \quad \text{with} \quad q = \hat{p}$$

i.e. $h_1 = \hat{h}$ on $\mathcal{G}$. Therefore $f_1 = (h_1, \dots, h_1)$ satisfies $f_1 = f$ on $\mathcal{G}$, and from Lemma 6.7, we know that f_1 is locally constant on $\{f_1 \neq f\}$ and also satisfies (2.14). Therefore ii) of Theorem 2.15 shows that $f_1 = \hat{f}_{\mathcal{G}} = \hat{f}$. Moreover we have $(\hat{h}, \dots, \hat{h}) = \hat{f} = f_1 = (h_1, \dots, h_1)$ with $h_1 = \mathfrak{R}h_0 = \mathfrak{R}\hat{h}$, where $\mathfrak{R}h_0$ is defined in (6.22). This shows that $\hat{h} = \mathfrak{R}\hat{h}$ which means exactly (6.30). This ends the proof of the lemma.

Proof of Theorem 2.30
For the proof, we refer to the table of Subsection 2.5. Theorem 2.30 follows from Proposition 6.9, and the two last lines of (2.23) follow from ii) of Theorem 2.15. This ends the proof of the theorem.

6.6 Theorem 2.34 and its proof: HJ germ $\mathcal{G}$ determined by $\chi\mathcal{G}$

We now start with the following result.

Lemma 6.10 (Key reduction for HJ germs)
Assume (2.2) with $N \geq 1$. Let $\mathcal{G}$ be a generalized Riemann germ which is a HJ germ with $\hat{f}_{\mathcal{G}} = (\hat{h}, \dots, \hat{h})$. We consider any $y = (y^1, \dots, y^N)$ with $y^j \in J^j$ for all j. Let $p \in [a, b]$ be such that there exists $\lambda \in \mathbb{R}$ with

$$(6.31) \qquad f^j(p^j) = \lambda \quad \textit{for all} \quad j = 1, \dots, N$$

Assume the following

$$(6.32) \qquad \textit{for all} \quad q \in \underline{\chi}\mathcal{G}, \quad \Big(q \diamond y \geq p \diamond y \quad \Longrightarrow \quad \hat{h}(q) \leq \lambda\Big)$$

and

$$(6.33) \qquad \textit{for all} \quad q \in \overline{\chi}\mathcal{G}, \quad \Big(q \diamond y \leq p \diamond y \quad \Longrightarrow \quad \hat{h}(q) \geq \lambda\Big)$$

with sub/super characteristic sets $\underline{\chi}\mathcal{G}$ and $\overline{\chi}\mathcal{G}$ given in Definition 2.33 and Hadamard product $\diamond$ given in Definition 2.11. Then

$$\lambda = \hat{h}(p)$$

Proof of Lemma 6.10
Let $p \in [a, b]$ satisfying (6.31) and set

$$v^j(t, x) = -\lambda t + p^j x, \quad j = 1, \dots, N$$

Then v^j is a viscosity solution of $v^j_t + f^j(v^j_x) = 0$ on $\mathbb{R} \times J^j$, $j = 1, \dots, N$. We claim that $v = (v^1, \dots, v^N)$ is a $\hat{h}$-viscosity solution on $\{x = 0\}$ if $v^0(t, 0) := v^j(t, 0)$ satisfies

$$(6.34) \qquad v^0_t + \hat{h}(v^1_x, \dots, v^N_x)(t, 0)) = 0 \quad \text{for all} \quad t \in \mathbb{R}$$

and we show it in the next steps. To this end, for $q \in [a, b]$, we set $w = (w^1, \dots, w^N)$ with

$$w^j(t, x) := -\lambda t + q^j x, \quad j = 1, \dots, N$$

Step 1: checking that v is a $\hat{h}$-viscosity subsolution on $\{x=0\}$
Notice that for all $q \in [a,b]$, inequality $q \diamond y \geq p \diamond y$ is equivalent to

$$w \geq v \quad \text{on} \quad \mathbb{R} \times J \quad \text{with equality on} \quad \mathbb{R} \times \{0\} \tag{6.35}$$

Set $I := \left\{j \in \{1,\dots,N\}, \quad \sigma^j = 1\right\}$. Now from Lemma 10.4 (and using I-reversion composed with I-inversion), we get that v is a $\hat{h}$-viscosity subsolution on $\{x=0\}$ if and only if for all $q \in \underline{\chi}\mathcal{G}$, relation (6.35) implies the subsolution viscosity inequality $-\lambda + \hat{h}(q) \leq 0$. In other words, condition (6.32) is equivalent to the fact that v is a $\hat{h}$-viscosity subsolution on $\{x=0\}$.
Step 2: checking that v is a $\hat{h}$-viscosity supersolution on $\{x=0\}$
Similarly notice that for all $q \in [a,b]$, inequality $q \diamond y \leq p \diamond y$ is equivalent to

$$w \leq v \quad \text{on} \quad \mathbb{R} \times J \quad \text{with equality on} \quad \mathbb{R} \times \{0\} \tag{6.36}$$

Now from Lemma 10.4 (and using I-reversion composed with I-inversion), we get that v is a $\hat{h}$-viscosity supersolution on $\{x=0\}$ if and only if for all $q \in \overline{\chi}\mathcal{G}$, relation (6.36) implies the supersolution viscosity inequality $-\lambda + \hat{h}(q) \geq 0$. In other words, condition (6.33) is equivalent to the fact that v is a $\hat{h}$-viscosity supersolution on $\{x=0\}$.
Step 3: conclusion
We conclude that v is both a $\hat{h}$-viscosity subsolution and supersolution on $\{x=0\}$. Therefore v is a $\hat{h}$-viscosity solution and satisfies (6.34), i.e. $-\lambda + \hat{h}(p) = 0$, which ends the proof of the lemma.

Proof of Theorem 2.34
Let $\hat{f}_{\mathcal{G}} = (\hat{h},\dots,\hat{h})$ and $\hat{f}_{\mathcal{G}_0} = (\hat{h}_0,\dots,\hat{h}_0)$ be the associated Godunov fluxes to the germs $\mathcal{G}$ and $\mathcal{G}_0$. Let $p \in \mathcal{G}_0 \supset \chi\mathcal{G}$ and let us show that $p \in \mathcal{G}$. We set $\lambda := \hat{h}_0(p) = f^j(p^j)$.
Step 1: proof of (6.32)
Let q be such that

$$q \in \underline{\chi}\mathcal{G} \quad \text{and} \quad q \diamond y \geq p \diamond y \tag{6.37}$$

Because $\underline{\chi}\mathcal{G} \subset \mathcal{G}_0$, we deduce that $q \in \mathcal{G}_0$ and then $\hat{h}(q) = f^j(q) = \hat{h}_0(q) \leq \hat{h}_0(p) = \lambda$, where we have used the monotonicities of $\hat{h}_0$ and inequality in (6.37). This shows (6.32).
Step 2: proof of (6.33)
The proof is similar to the one of Step 1.
Step 3: conclusion
From Lemma 6.10, we deduce that $f^j(p^j) = \hat{h}_0(p) = \lambda = \hat{h}(p)$. Therefore $p \in \mathcal{G}$. This ends the proof of the lemma.

6.7 Theorem 2.35 and its proof: conservative $1:1$ germs

The proof of Theorem 2.35 requires the following intermediate result.

Lemma 6.11 (Conservative $1:1$ junctions)
Assume (2.2) for $N=2$ and $1:1$ junctions. Let $\mathcal{G} \subset [a,b]$ be a generalized Riemann germ.
i) (Equivalence of conservative and HJ germs)
Then $\mathcal{G}$ is conservative, i.e. satisfies

$$f^1 = f^2 \quad \text{on} \quad \mathcal{G} \tag{6.38}$$

if and only if it is a HJ germ.
ii) (Further properties of the germ)
When $\mathcal{G}$ is conservative, then $\mathcal{G}$ is also a Riemann germ which is at the same time a Kružkov germ, a HJ germ and a monotone germ.

Proof of Lemma 6.11
Point i) follows from the definitions. We now focus on the proof of point ii).
Relation (6.38) shows that $\mathcal{G}$ is a HJ germ, and then a Riemann germ from Proposition 6.9. Defining $h : \mathcal{G} \to \mathbb{R}$ by $h(p) := f^1(p) = f^2(p)$ for $p \in \mathcal{G}$, and setting $\hat{h} := h \circ \pi_{\mathcal{G}}$, we see that $\hat{f} = (\hat{f}^1, \hat{f}^2) = (\hat{h}, \hat{h})$

with monotonicities $\hat{h}(\uparrow,\downarrow)$. This shows that $\mathcal{G}$ is a monotone germ.
We now set for $\sigma^1 = 1 = -\sigma^2$

$$D^{\hat{f}}(p,q) := \sum_{k=1,2} D^{\hat{f}^k}(p,q) = \text{IN} - \text{OUT} \quad \text{with} \quad D^{\hat{f}^k}(p,q) := \sigma^k \cdot \text{sign}(p^k - q^k) \cdot \left\{\hat{f}^k(p) - \hat{f}^k(q)\right\}$$

Hence we have $D^{\hat{f}}(p,q) = \left\{\text{sign}(p^1 - q^1) - \text{sign}(p^2 - q^2)\right\} \cdot \left\{\hat{h}(p) - \hat{h}(q)\right\}$ and the monotonicities of $\hat{h}$ show that $D^{\hat{f}} \geq 0$ on $[a,b]^2$. Because $D^{\hat{f}}$ and D^f coincide on $\mathcal{G} \times \mathcal{G}$, we deduce that $D^f \geq 0$ on $\mathcal{G} \times \mathcal{G}$. Therefore $\mathcal{G}$ is a Kružkov germ. This ends the proof of the lemma.

Proof of Theorem 2.35
For the proof, we refer to the table of Subsection 2.5. The result follows from Lemma 6.11, from ii) of Proposition 6.9 for the relaxation formula and from Theorem 2.34 for the characterization of the germ by $\chi\mathcal{G}$. Notice that inequalities (2.27) then follow automatically from ii) of Theorem 2.15. The independent proof of relation (2.25) (which gives explicitly a way to recover the full germ from its characteristic subset) is postponed to Lemma 8.17. This ends the proof of the theorem.

7 Riemann relaxation of Godunov quasi-fluxes

This section has been strongly inspired from discussions with B. Andreianov in connection with [3].

7.1 Riemann relaxation operator and proof of Theorem 2.23

We first state and prove the following result, and then, as a corollary give the proof of Theorem 2.23.

Proposition 7.1 (Riemann relaxation operator, compact box)
Assume (2.2) for $N \geq 1$ with compact box $[a,b] \subset \mathbb{R}^N$ and orientation of branches $\sigma \in \{\pm 1\}^N$. Let us consider a function $g_0 : [a,b] \to \mathbb{R}^N$ which is a Godunov quasi-flux in the sense of Definition 2.22, i.e. satisfying

$$\text{(7.1)} \qquad \begin{cases} g_0 : [a,b] \to \mathbb{R}^N \ \textit{continuous} \\ \\ \sigma \diamond g_0 : [a,b] \to \mathbb{R}^N \ \textit{Riemann monotone} \\ \\ \sigma^j f^j(b^j) \leq \sigma^j g_0^j(q)_{|q^j = b^j} \quad \textit{and} \quad \sigma^j g_0^j(q)_{|q^j = a^j} \leq \sigma^j f^j(a^j) \end{cases}$$

We define for $p,q \in [a,b]$ the vectorial standard Godunov flux with orientation σ

$$G_\sigma^f(p,q) := (G_{\sigma^1}^{f^1}(p^1,q^1), \dots, G_{\sigma^N}^{f^N}(p^N,q^N)) \quad \textit{with} \quad G_{\sigma^j}^{f^j}(p^j,q^j) = \begin{cases} G^{f^j}(p^j,q^j) & \textit{if} \quad \sigma^j = 1 \\ G^{f^j}(q^j,p^j) & \textit{if} \quad \sigma^j = -1 \end{cases}$$

Then the following formula defines uniquely $g_1(p) \in \mathbb{R}^N$ for all $p \in [a,b]$, given by the singleton

$$\text{(7.2)} \qquad \{g_1(p)\} := \bigcup_{q \in [a,b]} \left(\{G_\sigma^f(p,q)\} \cap \{g_0(q)\}\right)$$

i) (Riemann relaxation operator)
Then we set

$$\text{(7.3)} \qquad \mathfrak{R} g_0 := g_1$$

and the function g_1 satisfies (7.1).
ii) (Godunov flux properties)
Moreover g_1 satisfies the following properties

$$\text{(7.4)} \qquad \begin{cases} g_1 \ \textit{is locally constant on} \ \{g_1 \neq f\} \\ \\ \min_{[p^j,b^j]} \sigma^j f^j \leq \sigma^j g_1^j(p) \leq \max_{[a^j,p^j]} \sigma^j f^j \quad \textit{for all } j \textit{ and } p \in [a,b] \end{cases}$$

and the set

$$\mathcal{G}_1 := \{p \in [a,b], \quad g_1(p) = f(p)\} \tag{7.5}$$

is a Riemann germ.
iii) (Projection property)
We also have $\mathfrak{R}g_1 = g_1$. More generally, if g_0 satisfies (7.1) and (7.4), then $\mathfrak{R}g_0 = g_0$.

Remark 7.2 (Weak Riemann monotonicity)
It is possible to see that the proof of Proposition 7.1 can be adapted to the case where the second line of (7.1) is replaced by the assumption that $h := \sigma \diamond g_0 : [a,b] \to \mathbb{R}^N$ is only assumed to be weakly Riemann monotone, which means that

$$(p-q) \diamond [h]_q^p \le 0 \quad \Longrightarrow \quad (p-q) \diamond [h]_q^p = 0. \tag{7.6}$$

Then $\sigma \diamond \mathfrak{R}g_0$ is not only weakly Riemann monotone, but is Riemann monotone.

Proof of Proposition 7.1
Step 1: reduction to $N:0$ junctions
Recall that for $n:m$ junctions, we can apply a reversion for the m indices, defining

$$(\tilde{f}^j, \tilde{\sigma}^j) = \begin{cases} (-f^j, -\sigma^j) & \text{if} \quad \sigma^j = -1 \\ (f^j, \sigma^j) & \text{if} \quad \sigma^j = 1 \end{cases}$$

Then for

$$(\tilde{p}^j, \tilde{q}^j) := \begin{cases} (q^j, p^j) & \text{if} \quad \sigma^j = -1 \\ (p^j, q^j) & \text{if} \quad \sigma^j = 1 \end{cases}$$

we get for $\sigma^j = -1$

$$G^{\tilde{f}^j}(\tilde{p}^j, \tilde{q}^j) = \left\{ \begin{array}{ll} \displaystyle\min_{[\tilde{p}^j, \tilde{q}^j]} \tilde{f}^j & \text{if} \quad \tilde{p}^j \le \tilde{q}^j \\ \displaystyle\max_{[\tilde{q}^j, \tilde{p}^j]} \tilde{f}^j & \text{if} \quad \tilde{p}^j \ge \tilde{q}^j \end{array} \right| = \left\{ \begin{array}{ll} \displaystyle\min_{[q^j, p^j]} -f^j & \text{if} \quad q^j \le p^j \\ \displaystyle\max_{[p^j, q^j]} -f^j & \text{if} \quad q^j \ge p^j \end{array} \right| = -G^{f^j}(p^j, q^j)$$

while $G^{\tilde{f}^j} = G^{f^j}$ if $\sigma^j = 1$. We conclude that, up to reversions, we can assume that $\sigma = (1, \dots, 1) \in \mathbb{R}^N$. Therefore, it is sufficient to do the reasonings for $N:0$ junctions, a situation that we assume up to the end of the proof.
Step 2: Non emptyness of R_p
We set

$$G^f = G^f_\sigma \quad \text{with} \quad \sigma = (1, \dots, 1)$$

Then, given $p \in [a,b]$, consider the set

$$R_p := \left\{ q \in [a,b], \quad G^f(p,q) = g_0(q) \right\}$$

We set

$$\Phi : [a,b] \to \mathbb{R}^N \quad \text{with} \quad \Phi(q) := g_0(q) - G^f(p,q)$$

Recall that g_0 is Riemann monotone, that is

$$(q-r) \diamond [g_0]_r^q \le 0 \quad \Longrightarrow \quad [g_0]_r^q = 0$$

Step 2.1: Riemann monotonicity of $-G^f(p,\cdot)$
Assume that for $q, r \in [a,b]$, we have

$$(q-r) \diamond [-G^f(p,\cdot)]_r^q \le 0$$

Then the monotonicity of the standard Godunov flux $G^{f^j}(p^j, \downarrow)$ implies $(q-r) \diamond [-G^f(p,\cdot)]_r^q = 0$, which itself implies $[-G^f(p,\cdot)]_r^q = 0$. This means exactly that $-G^f(p,\cdot)$ is Riemann monotone.
Step 2.2: Riemann monotonicity of Φ
Now, assume that for $q, r \in [a,b]$, we have

$$(q-r) \diamond [\Phi]_r^q \le 0 \tag{7.7}$$

Still the monotonicity of the standard Godunov flux $G^{f^j}(p^j,\downarrow)$ implies

$$(q-r)\diamond [g_0]_r^q \le (q-r)\diamond [G^f(p,\cdot)]_r^q \le 0$$

Then Riemann monotonicity of g_0 implies $[g_0]_r^q = 0$. Hence from (7.7), we deduce that

$$(q-r)\diamond [-G^f(p,\cdot)]_r^q \le 0$$

and the Riemann monotonicity of $-G^f(p,\cdot)$ implies $[-G^f(p,\cdot)]_r^q = 0$. We conclude that

$$[\Phi]_r^q = [g_0]_r^q + [-G^f(p,\cdot)]_r^q = 0$$

which shows that Φ is Riemann monotone.
Step 2.3: $R_p \neq \emptyset$
For $q \in [a,b]$, we get

$$\Phi^j(q)_{|q^j=b^j} = g_0^j(q)_{|q^j=b^j} - G^{f^j}(p^j,b^j) = g_0^j(q)_{|q^j=b^j} - \min_{[p^j,b^j]} f^j \ge f^j(b^j) - \min_{[p^j,b^j]} f^j \ge 0$$

where in the first inequality, we have used assumption (7.1) on the bounds on the boundary of the box $[a,b]$. Similarly, we show that

$$\Phi^j(q)_{|q^j=a^j} \le 0$$

Using the continuity of Φ, we deduce from the Poincaré-Miranda theorem theorem (see [39]) that there exists some $q_* \in [a,b]$ such that

$$\Phi(q_*) = 0$$

which shows that $\emptyset \neq R_p \ni q_*$. Indeed, more precisely, this follows from Poincaré-Miranda theorem applied to $\Phi_\varepsilon := \Phi + \varepsilon(Id - \frac{a+b}{2})$, giving some root $\Phi_\varepsilon(q_*^\varepsilon) = 0$, and then taking the limit $q_*^\varepsilon \to q_*$ as $\varepsilon \to 0$.
Step 3: Singleton property of $g_0(R_p)$
Assume now by contradiction that there exist $q_a, q_b \in R_p$ such that

$$g_0(q_a) \neq g_0(q_b) \tag{7.8}$$

Then the Riemann monotonicity implies that

$$(q_a - q_b)\diamond [g_0]_{q_b}^{q_a} \not\le 0$$

otherwise this would imply $[g_0]_{q_b}^{q_a} = 0$ which is not the case. Therefore there exists some index j such that

$$(q_a^j - q_b^j)\cdot [g_0^j]_{q_b}^{q_a} > 0$$

Up to exchange q_a and q_b, we can assume that

$$q_a^j - q_b^j > 0. \tag{7.9}$$

Hence $[g_0^j]_{q_b}^{q_a} > 0$, i.e. using the fact that $q_a, q_b \in R_p$, this implies

$$G^{f^j}(p^j, q_a^j) = g_0^j(q_a) > g_0^j(q_b) = G^{f^j}(p^j, q_b^j)$$

The monotonicity $G^{f^j}(p,\downarrow)$ then implies that

$$q_a^j < q_b^j$$

Contradiction with (7.9). We deduce that (7.8) is false and then that g_0 is constant on R_p. Therefore, this common value defines $g_1(p) := g_0(q)$ for any $q \in R_p$.
Step 4: proof of i)
We want to show that g_1 satisfies (7.1).
Step 4.1: continuity of g_1
Consider any sequence $[a,b] \ni p_n \to p$. Let $q_n \in R_{p_n}$. Then we have

$$g_1(p_n) = g_0(q_n) = G^f(p_n, q_n) = f(q_n)$$

Then up to extract a subsequence (still denoted by (n)), we can assume that $q_n \to q \in [a,b]$. By continuity of g_0 and G^f, we deduce that

$$g_1(p_n) \to g_0(q) = G^f(p,q) = f(q)$$

i.e. that $q \in R_p$, and then $g_0(q) = g_1(p)$ which shows that $g_1(p_n) \to g_1(p)$. This implies the continuity of the map g_1.

Step 4.2: Riemann monotonicity of g_1

Let $p, p' \in [a,b]$, and assume that

$$(p'-p) \diamond [g_1]_p^{p'} \le 0 \tag{7.10}$$

Let $q \in R_p$ and $q' \in R_{p'}$. They satisfy

$$g_1(p) = G^f(p,q) = g_0(q) \quad \text{and} \quad g_1(p') = G^f(p',q') = g_0(q') \tag{7.11}$$

Step 4.2.1: effect of a j-reversion

Recall that a j-inversion consists to define

$$(\tilde f^k, \tilde\sigma^k) := \begin{cases} (f^k, \sigma^k) & \text{if} \quad k \ne j \\ (-f^k(-\cdot), \sigma^j) & \text{if} \quad k = j \end{cases}$$

and

$$\tilde g_0^k(r^1, \dots, r^j, \dots, r^N) := \begin{cases} g_0^k(r^1, \dots, -r^j, \dots, r^N) & \text{if} \quad k \ne j \\ -g_0^k(r^1, \dots, -r^j, \dots, r^N) & \text{if} \quad k = j \end{cases}$$

and similarly for $\tilde g_1$. For

$$(\tilde p^k, \tilde q^k) := \begin{cases} (p^k, q^k) & \text{if} \quad k \ne j \\ (-p^k, -q^k) & \text{if} \quad k = j \end{cases}$$

(and similarly for $(\tilde q', \tilde q')$ in terms of (p', q')), we get

$$G^{\tilde f^j}(\tilde p^j, \tilde q^j) = \left\{ \begin{array}{ll} \min\limits_{[\tilde p^j, \tilde q^j]} \tilde f^j & \text{if} \quad \tilde p^j \le \tilde q^j \\ \max\limits_{[\tilde q^j, \tilde p^j]} \tilde f^j & \text{if} \quad \tilde p^j \ge \tilde q^j \end{array} \right| = \left\{ \begin{array}{ll} \min\limits_{[-p^j, -q^j]} -f^j(-\cdot) & \text{if} \quad -p^j \le -q^j \\ \max\limits_{[-q^j, -p^j]} -f^j(-\cdot) & \text{if} \quad -p^j \ge -q^j \end{array} \right| = -G^{f^j}(p^j, q^j)$$

Then it is easy to check that (7.11) becomes

$$\tilde g_1(\tilde p) = G^{\tilde f}(\tilde p, \tilde q) = \tilde g_0(\tilde q) \quad \text{and} \quad \tilde g_1(\tilde p') = G^{\tilde f}(\tilde p', \tilde q') = \tilde g_0(\tilde q')$$

with moreover

$$(\tilde p'^k - \tilde p^k) = \begin{cases} (p'^k - p^k) & \text{if} \quad k \ne j \\ -(p'^k - p^k) & \text{if} \quad k = j \end{cases}$$

and similarly for $\tilde q' - \tilde q$ with respect to $q' - q$.

Step 4.2.2: application

Using Step 4.2.1, we see that, up to apply suitable inversions, we can assume that

$$q' - q \ge 0.$$

Fix an index j. By assumption (7.10), we have

$$(p'^j - p^j)[g_1^j]_p^{p'} \le 0 \tag{7.12}$$

Hence (7.11) implies the first equality

$$[g_0^j]_q^{q'} = [g_1^j]_p^{p'} = \begin{cases} (p'^j - p^j)^{-1} \cdot (p'^j - p^j)[g_1^j]_p^{p'} \le 0 & \text{if} \quad p'^j - p^j > 0 \\ G^{f^j}(p'^j, q'^j) - G^{f^j}(p^j, q^j) \le 0 & \text{if} \quad p'^j - p^j \le 0 \end{cases}$$

where in the first line we have used (7.12), while in the second line, we have used the monotonicities $G^{f^j}(\uparrow, \downarrow)$ with $q'^j - q^j \ge 0$. We then conclude that

$$(q'^j - q^j)[g_0^j]_q^{q'} \le 0$$

We conclude that

$$(q' - q) \diamond [g_0]_q^{q'} \leq 0$$

and because g_0 is Riemann monotone, this implies

$$[g_1]_p^{p'} = [g_0]_q^{q'} = 0$$

which shows that g_1 itself is Riemann monotone.

Step 4.3: bounds on g_1

Let us show that

$$\min_{[p^j,b^j]} f^j \leq g_1^j(p) \tag{7.13}$$

By definition of $g_1^j(p)$, there exists $q \in R_p$ such that

$$g_1^j(p) = G^{f^j}(p^j, q^j) \geq G^{f^j}(p^j, b^j) = \min_{[p^j,b^j]} f^j$$

which shows (7.13). The proof for g_1 of the other inequality of the second line of (7.4) is similar. Moreover the second line of (7.4) implies that g_1 satisfies the third line of (7.1). We conclude that g_1 satisfies (7.1).

Step 5: proof of ii)

We have already seen in Step 4.3 that g_1 satisfies the second line of (7.4). It then remains to prove the first line of (7.4), i.e. that g_1 is locally constant on $\{g_1 \neq f\}$.

Consider now a point $p \in [a, b]$ such that $g_1(p) \neq f(p)$, and let $q \in R_p$ such that

$$g_1(p) = G^f(p, q) = g_0(q)$$

We set

$$K_p := \left\{ j \in \{1, \ldots, N\}, \quad g_1^j(p) \neq f^j(p) \right\}$$

Then for any $j \in K_p$, we have

$$f^j(p^j) \neq G^{f^j}(p^j, q^j) = \left\{ \begin{array}{ll} \min\limits_{[p^j,q^j]} f^j & \text{if} \quad p^j < q^j \\ \max\limits_{[q^j,p^j]} f^j & \text{if} \quad p^j > q^j \end{array} \right.$$

where we notice that the case $p^j = q^j$ is excluded here.

Then there exists $r^j \in [a^j, b^j]$ such that

$$f^j(p^j) \quad \left\{ \begin{array}{llll} > \min\limits_{[p^j,q^j]} f^j = f^j(r^j) & \text{with} & r^j \in (p^j, q^j] & \text{if} \quad p^j < q^j \\ < \max\limits_{[q^j,p^j]} f^j = f^j(r^j) & \text{with} & r^j \in [q^j, p^j) & \text{if} \quad p^j > q^j \end{array} \right.$$

Because $r^j \neq p^j$, we see that there exists some $\varepsilon > 0$ small enough such that for all $p'^j \in (p^j - \varepsilon, p^j + \varepsilon) \cap [a^j, b^j]$ we have

$$f^j(p'^j) \neq G^{f^j}(p'^j, q^j) = f^j(r^j)$$

Defining

$$p'^k = p^k \quad \text{for all} \quad k \notin K_p$$

we deduce that

$$G^{f^k}(p'^k, q^k) = \left\{ \begin{array}{ll} G^{f^k}(p^k, q^k) = g_0^k(q) & \text{if} \quad k \notin K_p \\ f^k(r^k) = G^{f^k}(p^k, q^k) = g_0^k(q) & \text{if} \quad k \in K_p \end{array} \right.$$

Hence

$$G^f(p', q) = g_0(q)$$

which shows that $q \in R_{p'}$ and then

$$g_1(p') = g_0(q) = g_1(p) \quad \text{for all} \quad p' \in \left\{ p + \sum_{j \in K_p} (-\varepsilon, \varepsilon) e_j \right\} \cap [a, b]$$

which shows that g_1 is locally constant on $\{g_1 \neq f\}$. From ii) of Theorem 2.15, we deduce that the set $\mathcal{G}_1$ defined in (7.5) is a Riemann germ.
Step 6: proof of iii)
Step 6.1: when g_0 satisfies (7.1) and (7.4)
We set $g_1 := \mathfrak{R} g_0$ and want to prove that $g_1 = g_0$. Because g_0 satisfies both (7.1) and (7.4), we know that the set $\mathcal{G}_1 = \{g_0 = f\}$ defined in (7.5) is a Riemann germ. From i) of Theorem 2.15, we have

$$[a,b] = \bigcup_{\hat{p} \in \mathcal{G}_1} BA(\hat{p}) \tag{7.14}$$

where the basin of attraction BA is defined in Definition 2.10. Now for any $p \in BA(\hat{p})$ with $J^j \simeq (-\infty, 0)$ for all indices j, we have

$$G^f(p, \hat{p}) = f(\hat{p}) = g_0(\hat{p})$$

Therefore $\hat{p} \in R_p$, and then

$$g_1(p) = g_0(\hat{p}) = g_0(p) \quad \text{because} \quad g_0 = const \quad \text{on} \quad BA(\hat{p}) \ni p$$

From (7.14), we deduce that $g_1 = g_0$, which shows that $\mathfrak{R} g_0 = g_0$.
Step 6.2: $\mathfrak{R} g_1 = g_1$
We set $g_2 := \mathfrak{R} g_1$ and want to show that $g_2 = g_1$, using the fact that $g_1 = \mathfrak{R} g_0$. Indeed points i) and ii) show that g_1 satisfies both (7.1) and (7.4). Then Step 6.1 implies that $\mathfrak{R} g_1 = g_1$.
This ends the proof of the proposition.

Proof of Theorem 2.23
The proof follows from Proposition 7.1 for the characterization of $\mathfrak{R} g_0$ as a Godunov flux associated to a Riemann germ.
Conversely assume that f satisfies furthermore nondegeneracy condition (2.17), and consider any Riemann germ $\mathcal{G}$. Then i) of Theorem 2.20 shows that its Godunov flux $\hat{f} := \hat{f}_{\mathcal{G}}$ is such that $\sigma \diamond \hat{f}$ is Riemann monotone (and continuous). Moreover ii) of Theorem 2.15 show that $\hat{f}$ satisfies (2.14) with in particular monotonicity bounds. Therefore we conclude that $\hat{f}$ satisfies (7.1). Now we can consider $g_1 := \mathfrak{R} \hat{f}$ provided by Proposition 7.1. Because $\mathcal{G} := \left\{p \in [a,b],\ \hat{f}(p) = f(p)\right\}$, we deduce that

$$g_1(\hat{p}) = G^f_\sigma(\hat{p}, \hat{p}) = f(\hat{p}) = \hat{f}(\hat{p}) \quad \text{for all} \quad \hat{p} \in \mathcal{G}$$

By construction g_1, is locally constant on $\{g_1 \neq f\}$, and we deduce that

$$g_1 = g_1(\hat{p}) = \hat{f}(\hat{p}) = \hat{f} \quad \text{on} \quad BA^{J,f}(\hat{p})$$

Because from i) of Theorem 2.15, we know that the collection $(BA^{J,f}(\hat{p}))_{\hat{p} \in \mathcal{G}}$ forms a partition of $[a,b]$, we deduce that $g_1 = \hat{f}$ on $[a,b]$. Then $\hat{f} = g_1 = \mathfrak{R}(\hat{f})$ as desired, i.e. that $\hat{f}$ is self-Riemann relaxed. This ends the proof of the theorem.

The following result will be used later on (see Step 3 of the proof of Proposition 18.7).

Lemma 7.3 (A property of relaxation, compact box)
We work under the assumptions of Proposition 7.1. Let $\hat{p} \in \mathcal{G}_1$. Then there exists $q \in [a,b]$ such that $f(\hat{p}) = G^f_\sigma(\hat{p}, q) = g_0(q)$. Let $\tau \in \{\pm 1\}^N$ be such that $\tau \diamond (\hat{p} - q) \geq 0$.
Let $\hat{r} \in [a,b]$ be such that

$$\tau \diamond (\hat{r} - q) \geq 0 \quad \textit{is the smallest (componentwise) under the constraint that } f(\hat{r}) = f(\hat{p}).$$

i) (Alternative germ point $\hat{r}$)
Then $\hat{r} \in \mathcal{G}_1$ and we have the following property

$$\left.\begin{array}{l} u \in BA(\hat{p}) \\ S := \left\{j \in \{1, \dots, N\}, \quad u^j \neq \hat{p}^j\right\} \\ 0 \leq \max\limits_{j \in S} D^{f^j}(u, \hat{r}) \end{array}\right\} \quad \textit{implies} \quad S = \emptyset$$

ii) (Wave connecting q to $\hat{r}$)

Now consider $U : J^* \to \mathbb{R}$ with trace $U(0) \in \mathbb{R}^N$ such that

$$U(0) := q \quad \text{and} \quad U' = f(U) - f(\hat{r}) \quad \text{on} \quad J^* \tag{7.15}$$

Then U^j is monotone on J^j with $U^j(-\sigma^j\infty) = \hat{r}^j$.

Proof of Lemma 7.3
In order to simplify the presentation of the reasoning, we come back to a normalized situation. Up to suitable reversions, we can assume that $\sigma = (-1, \dots, -1) \in \mathbb{R}^N$, and up to suitable inversions, we can assume that $q \le \hat{p}$.
Step 1: proof of i)
Then $\hat{r}$ is the smallest element such that $f(\hat{r}) = f(\hat{p})$ and $q \le \hat{r}$. Moreover we have

$$g_0^j(q) = f^j(\hat{p}^j) = G^{f^j}(q^j, \hat{p}^j) = \min_{[q^j, \hat{p}^j]} f^j = \min_{[q^j, \hat{r}^j]} f^j = G^{f^j}(q^j, \hat{r}^j) = g_1^j(\hat{r}) = f^j(\hat{r})$$

Therefore $\hat{r} \in \mathcal{G}_1$. Now let $u \in BA(\hat{p})$ be such that

$$(u^j - \hat{r}^j)\left\{f^j(u^j) - f^j(\hat{r}^j)\right\} \le 0$$

Then either $\hat{r}^j = \hat{p}^j$, and then $BA(\hat{p}^j) = BA(\hat{r}^j)$ which implies $u^j = \hat{r}^j$. Or $q^j \le \hat{r}^j < \hat{p}^j$ and then $\min_{[q^j, \hat{p}^j]} f^j = f^j(\hat{p}^j)$ implies that $u^j \in BA(\hat{p}^j) \subset [\hat{p}^j, b^j]$, which implies $u^j = \hat{p}^j$. Therefore in all cases, we get $u = \hat{p}$.
Therefore, if $0 \le D^{f^j}(u, \hat{r})$, we deduce that $u^j = \hat{p}^j$. This implies point i).
Step 2: proof of ii)
Either $f^j(q^j) = f^j(\hat{p}^j)$, and then $\hat{r}^j = q^j$ which implies $U^j \equiv \hat{r}^j$, or $f^j(q^j) \ne f^j(\hat{p}^j) = f^j(\hat{r}^j)$ and then

$$f^j(s) \ne f^j(\hat{r}^j) \quad \text{for all} \quad \tau^j(\hat{r}^j - s) > 0 \quad \text{with} \quad \tau^j(s - q^j) \ge 0$$

Therefore U^j is monotone with limit at infinity along J^j equal to $\hat{r}^j$. This ends the proof of the lemma.

7.2 Further properties of Riemann relaxation

Proposition 7.4 (Transfer of properties by Riemann relaxation)
Assume (2.2) for $N \ge 1$ with compact box $[a, b] \subset \mathbb{R}^N$ and orientation of branches $\sigma \in \{\pm 1\}^N$. Let us consider a Godunov quasi-flux g_0, i.e. satisfying (7.1), and let $g_1 := \mathfrak{R} g_0$ be the Riemann relaxation of g_0 as defined in (7.3).
Let

$$D^{g_0}(p, q) := \sum_{j=1,\dots,N} \sigma^j \, sign(p^j - q^j) \cdot \left\{ g_0^j(p) - g_0^j(q) \right\} \quad \text{and} \quad D_+^{g_0}(p, q) := \sum_{j=1,\dots,N} \sigma^j \, sign^+(p^j - q^j) \cdot \left\{ g_0^j(p) - g_0^j(q) \right\}$$

i) (Kružkov)
Assume that g_0 satisfies $D^{g_0} \ge 0$ on $[a, b]^2$. Then g_1 also satisfies $D^{g_1} \ge 0$ on $[a, b]^2$. Moreover $\mathcal{G}_1 := \{g_1 = f\}$ is a Kružkov Riemann germ, and g_1 is in particular Lipschitz continuous.
ii) (Monotone Kružkov)
Assume that g_0 satisfies $D_+^{g_0} \ge 0$ on $[a, b]^2$. Then g_1 also satisfies $D_+^{g_1} \ge 0$ on $[a, b]^2$. Moreover $\mathcal{G}_1 := \{g_1 = f\}$ is a monotone Kružkov Riemann germ, and g_1 is in particular Lipschitz continuous.
iii) (Monotonicity)
Assume that g_0 satisfies the following monotonicity property

$$p \mapsto \sigma^j g_0^j(p) \quad \text{is nonincreasing in } p^k \text{ for all } k \ne j, \quad k, j \in \{1, \dots, n\} \tag{7.16}$$

Then g_1 also satisfies monotonicity property (7.16).
iv) (Hamilton-Jacobi)
Assume that g_0 satisfies the following HJ condition

$$g_0^j(p) = h_0(p) \quad \text{for all} \quad j = 1, \dots, N \quad \text{and all } p \in [a, b] \quad \text{for some function } h_0 : [a, b] \to \mathbb{R} \tag{7.17}$$

Then g_1 also satisfies HJ condition (7.17) for some function $h_1 : [a, b] \to \mathbb{R}$, and g_1 is in particular Lipschitz continuous.

v) (Conservative)
Assume that g_0 satisfies the conservative relation

$$\sum_{j=1,\dots,N} \sigma^j g_0^j(p) = 0 \quad \text{for all} \quad p \in [a,b] \tag{7.18}$$

Then g_1 also satisfies the conservative relation (7.18).

Proof of Proposition 7.4
Up to apply some reversions, we can assume that $\sigma = (1,\dots,1)$, and that the junction is of type $N:0$. From Proposition 7.1, we know that $g_1 := \mathfrak{R}g_0$ is such that $\mathcal{G}_1 := \{p \in [a,b],\ g_1(p) = f(p)\}$ is a Riemann germ. In particular, from ii) of Theorem 2.15, we know that

$$g_1 = \hat{f}_{\mathcal{G}_1}$$

Now let $\hat{p}, \hat{p}' \in \mathcal{G}_1$. We want to show that

$$D^f(\hat{p}', \hat{p}) \geq 0 \quad \text{in case i)} \quad \Big(\text{resp. } D^f_+(\hat{p}', \hat{p}) \geq 0 \text{ in case ii)}\Big) \tag{7.19}$$

Setting for some fixed index j, the quantity

$$D^{f^j}(q^{j\prime}, q^j) := \text{sign}(q^{j\prime} - q^j) \cdot \left\{f^j(q^{j\prime}) - f^j(q^j)\right\} \quad \text{and} \quad D^{f^j}_\pm(q^{j\prime}, q^j) := \text{sign}^\pm(q^{j\prime} - q^j) \cdot \left\{f^j(q^{j\prime}) - f^j(q^j)\right\}$$

and for

$$(\bar{q}^{j\prime}, \bar{q}^j) = (-q^{j\prime}, -q^j) \quad \text{and} \quad \bar{f}^j(r) = -f^j(-r)$$

we get

$$D^{f^j}(q', q) = D^{\bar{f}^j}(\bar{q}', \bar{q}), \quad D^{f^j}_\pm(q', q) = D^{\bar{f}^j}_\mp(\bar{q}', \bar{q}) = D^{\bar{f}^j}_\pm(\bar{q}, \bar{q}')$$

Therefore, up to suitable inversions, we can also assume that

$$\hat{p}' \geq p$$

where in case ii) we have to check $D^{f^j}_\pm$ for both signs.
Step 1: proof of i)
From Step 2 of the proof of Proposition 7.1, there exist $q \in R_{\hat{p}}$ and $q' \in R_{\hat{p}'}$. They satisfy

$$f(\hat{p}) = g_1(\hat{p}) = G^f(\hat{p}, q) = g_0(q) \quad \text{and} \quad f(\hat{p}') = g_1(\hat{p}') = G^f(\hat{p}', q') = g_0(q') \tag{7.20}$$

Now fix some $j \in \{1,\dots,N\}$. We now distinguish several cases.
Case A: $q^{j\prime} - q^j \leq 0$ and $\hat{p}^{j\prime} - \hat{p}^j \geq 0$
Assume by contradiction that

$$f^j(\hat{p}') < f^j(\hat{p}) \tag{7.21}$$

Then we have

$$G^{f^j}(\hat{p}^{j\prime}, q^{j\prime}) = f^j(\hat{p}') < f^j(\hat{p}) = G^{f^j}(\hat{p}^j, q^j)$$

But the monotonicities of the standard Godunov flux $G^{f^j}(\uparrow, \downarrow)$ imply $G^{f^j}(\hat{p}^{j\prime}, q^{j\prime}) \geq G^{f^j}(\hat{p}^j, q^j)$. Contradiction. Therefore (7.21) is false, and we conclude that $f^j(\hat{p}') \geq f^j(\hat{p})$. Hence, using (7.20), we get

$$[g_0^j]_q^{q'} = [f^j]_{\hat{p}}^{\hat{p}'} \geq 0$$

and

$$\begin{aligned} D^{f^j}(\hat{p}', \hat{p}) &= \text{sign}(\hat{p}^{j\prime} - \hat{p}^j) \cdot [f^j]_{\hat{p}}^{\hat{p}'} \\ &\geq \text{sign}(q^{j\prime} - q^j) \cdot [f^j]_{\hat{p}}^{\hat{p}'} \\ &= \text{sign}(q^{j\prime} - q^j) \cdot [g_0^j]_q^{q'} \\ &= D^{g_0^j}(q', q) \end{aligned}$$

Case B: $q^{j\prime} - q^j > 0$ and $\hat{p}^{j\prime} - \hat{p}^j = 0$

Then

$$[g_0^j]_q^{q'} = [f^j]_{\hat{p}}^{\hat{p}'} = 0$$

which implies

$$D^{f^j}(\hat{p}', \hat{p}) = D^{g_0^j}(q', q) = 0$$

Case C: $q^{j\prime} - q^j > 0$ **and** $\hat{p}^{j\prime} - \hat{p}^j > 0$
Then we have

$$\begin{aligned} D^{f^j}(\hat{p}', \hat{p}) &= \text{sign}(\hat{p}^{j\prime} - \hat{p}^j) \cdot \{f^j(\hat{p}') - f^j(\hat{p})\} \\ &= \text{sign}(\hat{p}^{j\prime} - \hat{p}^j) \cdot [f^j]_{\hat{p}}^{\hat{p}'} \\ &= \text{sign}(\hat{p}^{j\prime} - \hat{p}^j) \cdot [g_0^j]_q^{q'} \end{aligned}$$

where in the third line we have used (7.20). Here $\text{sign}(\hat{p}^{j\prime} - \hat{p}^j) = \text{sign}(q^{j\prime} - q^j) = 1$ and then

$$D^{f^j}(\hat{p}', \hat{p}) = D^{g_0^j}(q', q)$$

Conclusion
We deduce that in all cases, we have $D^{f^j}(\hat{p}', \hat{p}) \geq D^{g_0^j}(q', q)$. Therefore

$$D^f(\hat{p}', \hat{p}) \geq D^{g_0}(q', q) \geq 0 \quad \text{for all} \quad \hat{p}', \hat{p} \in \mathcal{G}_1$$

by assumption on g_0. We conclude that $\mathcal{G}_1$ is a Kružkov germ. Therefore i) of Lemma 5.5 implies that $\hat{f}_{\mathcal{G}_1}$ satisfies

$$D^{\hat{f}_{\mathcal{G}_1}} \geq 0 \quad \text{on} \quad [a, b]^2$$

Therefore $g_1 = \hat{f}_{\mathcal{G}_1}$ satisfies $D^{g_1} \geq 0$ on $[a, b]^2$.
Step 2: proof of ii)
The same reasoning applies with in particular $\text{sign}^\pm(\hat{p}^{j\prime} - \hat{p}^j) \geq \text{sign}^\pm(q^{j\prime} - q^j)$ in Case A. We then conclude that $D^f_\pm(\hat{p}', \hat{p}) \geq D^{g_0}_\pm(q', q) \geq 0$. Independently, notice that the result also follows from points i) and iii).
Step 3: proof of iv) and v)
The results of ii) and iv) both follow from the fact that for all $p \in [a, b]$, there exists $q \in R_p$ such that

$$g_1(p) = g_0(q).$$

Step 4: proof of iii)
We only do the proof for $\sigma = (1, \ldots, 1) \in \mathbb{R}^N$ for a $N : 0$ junction, and the general case can be recovered using suitable reversions, as usual. First we start with a formal proof which is natural (but can still be made rigorous in the special case where g_0 is Lipschitz continuous), and then give a rigorous proof which works for continuous g_0.
Step 4.1: Formal proof
Consider $q = q(p)$ solution of

$$G^f(p, q) = g_0(q)$$

Then the derivative with respect to p gives

$$M \partial_p q = \partial_p G^f(p, q) \quad \text{with} \quad M := -\partial_q G^f(p, q) + \partial_q g_0$$

where M has nonpositive terms outside the diagonal, and nonnegative terms on the diagonal. In the case where the diagonal is positive and M is invertible, then up to a change of variables, we can assume that we can write

$$M = Id - A \quad \text{with } A_{ij} \geq 0 \text{ and } A_{ii} = 0 \text{ for all indices } i, j$$

Then we have (sometimes called Leontiev inversion)

$$M^{-1} = (Id - A)^{-1} = Id + A + A^2 + \ldots$$

i.e.

$$(M^{-1})_{ij} \geq 0 \quad \text{for all indices } i, j$$

and $\partial_p q = M^{-1} \cdot \partial_p G^f(p, q)$ with $\partial_{p^i} G^{f^i}(p, q) \geq 0$ implies

$$\partial_{p^i} q^j \geq 0 \quad \text{for all} \quad i, j$$

Because $g_1(p) = G^f(p,q)$, we get for $i \neq j$

$$\partial_{p^i} g_1^j(p) = \left(\partial_{q^j} G^{f^j}(p^j, q^j)\right) \partial_{p^i} q^j \leq 0$$

which ends the formal proof.

Step 4.2: Rigorous proof

Step 4.2.a: uniqueness of q_ε

For $p \in [a,b]$, we first have to face the fact that $q \in R_p$ is not unique in general. To remedy to this difficulty, we have to come back to Step 2.3 of the proof of Proposition 7.1. We consider a slight perturbation of the problem for $\varepsilon > 0$, and we define for frozen p, the functions of q

$$\Phi_\varepsilon := g_0 + \varphi_\varepsilon, \quad \varphi_\varepsilon(q) := \varepsilon \left\{ q - \frac{a+b}{2} \right\} - G^f(p,q)$$

where Φ_ε has the same monotonicities as g_0, is still Riemann monotone, with moreover $\partial_{q^j} \Phi_\varepsilon^j \geq \partial_{q^j} \varphi_\varepsilon^j \geq \varepsilon > 0$, because of the monotonicity of the Godunov flux $G^{f^j}(p^j, \downarrow)$. Then Poincaré-Miranda theorem theorem (see [39]) still shows the existence of some $q_\varepsilon \in [a,b]$ such that $\Phi_\varepsilon(q_\varepsilon) = 0$. Assume by contradiction that this equation has two solutions q_ε and q'_ε. Then $\Phi_\varepsilon(q'_\varepsilon) - \Phi_\varepsilon(q_\varepsilon) = 0$ implies

$$g_0(q'_\varepsilon) - g_0(q_\varepsilon) = -\left\{ \varphi(q'_\varepsilon) - \varphi(q_\varepsilon) \right\}$$

The monotonicity of φ_ε implies

$$(q'_\varepsilon - q_\varepsilon) \diamond \left\{ g_0(q'_\varepsilon) - g_(q_\varepsilon) \right\} \leq 0$$

and the Riemann monotonicity of g_0 implies that $g_0(q'_\varepsilon) - g_0(q_\varepsilon) = 0$. Therefore $\varphi(q'_\varepsilon) - \varphi(q_\varepsilon) = 0$ and this implies $q'_\varepsilon = q_\varepsilon$. Hence we can write $q_\varepsilon = q_\varepsilon(p)$.

Step 4.2.b: monotonicity of $q_\varepsilon(p)$

Given $\bar{q}_\varepsilon := q_\varepsilon(\bar{p})$ and $q_\varepsilon := q_\varepsilon(p)$ for $p, \bar{p} \in [a,b]$ with $p \geq \bar{p}$, we first want to show the following monotonicity

$$q_\varepsilon \geq \bar{q}_\varepsilon \tag{7.22}$$

To this end, let us make apparent the dependence on p, writting

$$\Phi_\varepsilon(p,q) := g_0(q) + \varepsilon \left\{ q - \frac{a+b}{2} \right\} - G^f(p,q)$$

which has monotonicity $\Phi_\varepsilon(\downarrow, q)$. Hence

$$0 = \Phi_\varepsilon(p, q_\varepsilon) \leq \Phi_\varepsilon(\bar{p}, q_\varepsilon)$$

Therefore for $q \in [a, q_\varepsilon]$, we have from the monotonicity of g_0 (non-positive off-diagonal terms in the Jacobian matrix Dg_0)

$$\Phi_\varepsilon^j(\bar{p}, q)_{q^j = q_\varepsilon^j} \geq \Phi_\varepsilon^j(\bar{p}, q_\varepsilon) \geq 0$$

and still

$$\Phi_\varepsilon^j(\bar{p}, q)_{q^j = a^j} \leq 0$$

both valid for all j. Applying again Poincaré-Miranda theorem theorem (see [39]), we get that $\bar{q}_\varepsilon \in [a, q_\varepsilon]$, which shows (7.22). Hence

$$\partial_{p^i} q_\varepsilon^j(p) \geq 0 \quad \text{for all indices } i, j$$

Step 4.2.c: monotonicity of $\tilde{g}_\varepsilon$

Recall that $g_1(p) = G^f(p,q) = g_0(q)$. We then define approximations of g_0 and of g_1, respectively as

$$g_\varepsilon := g_0 + \varepsilon \left\{ q - \frac{a+b}{2} \right\} \quad \text{and} \quad \tilde{g}_\varepsilon(p) := G^f(p, q_\varepsilon) = g_\varepsilon(q_\varepsilon) \quad \text{with} \quad q_\varepsilon = q_\varepsilon(p)$$

Recall that g_ε (as g_0) has the following monotonicities

$$\partial_{q^j} g_\varepsilon^k(q) \leq 0 \quad \text{for all} \quad j \neq k$$

Hence for $i \neq k$, we recover the following monotonicities

$$\partial_{p^i} \tilde{g}^k_\varepsilon(p) = \left(\partial_{q^k} G^{f^k}(p^k, q^k_\varepsilon)\right) \partial_{p^i} q^k_\varepsilon(p) \leq 0$$

Step 4.2.d: conclusion
Now passing to the limit $\varepsilon \to 0$, from the uniqueness of the limit, we get $g_\varepsilon \to g_0$, $\tilde{g}_\varepsilon \to g_1$, and this shows that we recover the following monotonicities

$$\partial_{p^i} g^k_1(p) \leq 0 \quad \text{for all} \quad i \neq k$$

Recall that Riemann monotonicity always implies the diagonal monotonicity $\partial_{p^i} g^i_1 \geq 0$. Therefore g_1 satisfies monotonicity property (7.16). This ends the proof of the lemma.

7.3 Examples of Riemann monotone functions

In this subsection, we indicate very briefly a class of examples of Riemann monotone functions.

Then have for instance the following result.

Lemma 7.5 (Example of Riemann monotone functions)
Let $[a, b] \subset \mathbb{R}^N$ be a compact box and consider a continuous function $g : [a, b] \to \mathbb{R}^N$ satisfying for some $r \in [1, +\infty)$

$$\sum_{j=1,\dots,N} (p^j - q^j)|p^j - q^j|^{r-1} \left\{g^j(p) - g^j(q)\right\} \geq 0 \quad \textit{for all} \quad p, q \in [a, b] \tag{7.23}$$

with the convention that

$$(p^j - q^j)|p^j - q^j|^{r-1} := sign(p^j - q^j) \quad \textit{for} \quad r = 1.$$

Then g is weakly Riemann monotone in the sense of Remark 7.2. Moreover, for every $\eta > 0$, the function $g + \eta \cdot id$ is Riemann monotone.

Proof of Lemma 7.5
The weak Riemann monotonicity is straightforward. We then notice that $g_\eta := g + \eta \cdot id$ satisfies

$$\sum_{j=1,\dots,N} (p^j - q^j)|p^j - q^j|^{r-1} \left\{g^j_\eta(p) - g^j_\eta(q)\right\} \geq \eta \sum_{j=1,\dots,N} |p^j - q^j|^{r+1}$$

This implies that g_η is (in particular) Riemann monotone and ends the proof of the lemma.

More generally, the interested reader can consult the book Facchinei, Pang [18]. From this book, in Subsection 3.5.2, on page 298, recall that $M \in \mathbb{R}^{N\times N}$ is P-matrix, if it satisfies for all $x \in \mathbb{R}^N$

$$x \diamond Mx \leq 0 \quad \Longrightarrow \quad x = 0$$

(which implies that $Mx = 0$ and then that M is Riemann monotone in the sense of Definition 8.1, given later below).
Similarly, a function $g : [a, b] \to \mathbb{R}^N$ is said to be a P-function if

$$(x - y) \diamond (g(x) - g(y)) \leq 0 \quad \Longrightarrow \quad x = y$$

(which is an equivalent formulation of condition c) in Definition 3.5.8 on page 299 in [18]). In particular, such a function g is Riemann monotone.
Moreover, it is known (see condition a) in Proposition 3.5.9 in [18]) that for a C^1 function $g : [a, b] \to \mathbb{R}^N$, to be a P-function (and then a Riemann monotone function), it is sufficient that it satisfies

$$Dg(x) \text{ is a } P\text{-matrix for all } x \in [a, b].$$

Hence this last condition can be used to prove that some functions are Riemann monotone.

7.4 Commutation of Riemann relaxation and gluing

Proposition 7.6 (Riemann relaxation of glued Godunov quasi-fluxes)
For $\gamma = \alpha, \beta$, assume that f_γ satisfies (2.2) with compact box $[a,b]_\gamma$ for $N_\gamma = n_\gamma + m_\gamma$ and $n_\gamma : m_\gamma$ junctions J_γ with $J^j_\gamma \simeq \sigma^j_\gamma \cdot (-\infty, 0)$ and $\sigma_\gamma \in \{\pm 1\}^{N_\gamma}$. We set $[a,b]^j_\gamma := [a^j_\gamma, b^j_\gamma]$. We assume that for each $\gamma = \alpha, \beta$, there exists one index $j_\gamma \in \{1, \dots, N_\gamma\}$ such that

$$(7.24) \qquad f^{j_\alpha}_\alpha = f^{j_\beta}_\beta =: f^0 \quad \text{on} \quad [a,b]^{j_\alpha}_\alpha = [a,b]^{j_\beta}_\beta =: [a^0, b^0] \quad \text{with} \quad J^{j_\alpha}_\alpha \simeq (0, +\infty) \quad \text{and} \quad J^{j_\beta}_\beta \simeq (-\infty, 0)$$

and we glue those two branches. To simplify the notation, up to relabel the indices, we now assume that $j_\alpha = 0 = j_\beta$, and the indices now go through the values $\{0, \dots, N_\gamma - 1\}$. Hence we now have

$$\begin{cases} f_\gamma = (f^0_\gamma, \dots, f^{N_\gamma - 1}_\gamma) \\ [a,b]'_\gamma := \prod_{i=1,\dots,N_\gamma-1} [a,b]^i_\gamma \\ J^0_\alpha \simeq (0, +\infty) \quad \text{and} \quad J^0_\beta \simeq (-\infty, 0) \end{cases}$$

and consider Godunov quasi-fluxes $\hat{f}_\gamma = (\hat{f}^0_\gamma, \dots, \hat{f}^{N_\gamma - 1}_\gamma)$ with respect to (J_γ, f_γ). Assume also that $\hat{f}_\gamma$ are both **0-locally quasi-constant** *(on $[a^0, b^0]$) in the sense of v) if Definition 5.20. Then consider the glued Godunov quasi-flux*

$$\underline{\hat{f}} := \hat{f}_\alpha \underset{0:0}{\sharp} \hat{f}_\beta$$

which is defined in Proposition 5.23.
Then we have the following commutation of Riemann relaxation and gluing

$$\mathfrak{R}(\hat{f}_\alpha \underset{0:0}{\sharp} \hat{f}_\beta) = (\mathfrak{R}\hat{f}_\alpha) \underset{0:0}{\sharp} (\mathfrak{R}\hat{f}_\beta)$$

where Riemann relaxation operator $\mathfrak{R}$ is defined in Proposition 7.1.

Proof of Proposition 7.6
We just check the result by computation.
In order to simplify the presentation, notice up to suitable reversions, we can assume that $\sigma^j_\alpha = 1$ for $j = 1, \dots, N_\alpha - 1$, and $\sigma^j_\beta = -1$ for $j = 1, \dots, N_\beta - 1$.
We also set

$$\underline{\sigma}^I := \begin{cases} \sigma^j_\alpha & \text{if} \quad I = (\alpha, j) \quad \text{with} \quad j \in \{1, \dots, N_\alpha - 1\} \\ \sigma^j_\beta & \text{if} \quad I = (\beta, j) \quad \text{with} \quad j \in \{1, \dots, N_\beta - 1\} \end{cases}$$

Step 1: first computation
We set the standard Godunov flux

$$G^{f^j_\gamma}(p_\gamma, \bar{p}_\gamma) := G^{f^j_\gamma}(p^j_\gamma, \bar{p}^j_\gamma)$$

and

$$G^{f^j_\gamma}_{\sigma_\gamma}(p_\gamma, \bar{p}_\gamma) := \begin{cases} G^{f^j_\gamma}(p_\gamma, \bar{p}_\gamma) & \text{if} \quad \sigma^j_\gamma = 1 \\ G^{f^j_\gamma}(\bar{p}_\gamma, p_\gamma) & \text{if} \quad \sigma^j_\gamma = -1 \end{cases}$$

Then we have the relaxation of the Godunov quasi-flux $\hat{f}_\alpha$ which is given by

$$\mathfrak{R}\hat{f}^j_\alpha(p) = \begin{cases} \hat{f}^j_\alpha(\bar{p}) = G^{f^j_\alpha}(p, \bar{p}) & \text{for} \quad j \neq 0 \\ \hat{f}^j_\alpha(\bar{p}) = G^{f^j_\alpha}(\bar{p}, p) & \text{for} \quad j = 0 \end{cases}$$

and similarly

$$\mathfrak{R}\hat{f}^j_\beta(q) = \begin{cases} \hat{f}^j_\beta(\bar{q}) = G^{f^j_\beta}(q, \bar{q}) & \text{for} \quad j = 0 \\ \hat{f}^j_\beta(\bar{q}) = G^{f^j_\beta}(\bar{q}, q) & \text{for} \quad j \neq 0 \end{cases}$$

We also have the gluing of the two Godunov fluxes $\mathfrak{R}\hat{f}_\alpha$ and $\mathfrak{R}\hat{f}_\beta$ along the indices $0:0$ given by

$$\underline{\hat{f}}^I_1(\hat{p}_\alpha, \hat{q}_\beta) := (\mathfrak{R}\hat{f}_\alpha \underset{0:0}{\sharp} \mathfrak{R}\hat{f}_\beta)^I(\hat{p}_\alpha, \hat{q}_\beta) = \begin{cases} (\mathfrak{R}\hat{f}^j_\alpha)(\hat{p}_\alpha, r) & \text{if} \quad I = (\alpha, j) \\ (\mathfrak{R}\hat{f}^j_\beta)(\hat{q}_\beta, r) & \text{if} \quad I = (\beta, j) \end{cases} \quad \text{with} \quad \begin{cases} (\mathfrak{R}\hat{f}^0_\alpha)(\hat{p}_\alpha, r) = (\mathfrak{R}\hat{f}^0_\beta)(\hat{q}_\beta, r) \\ \text{for} \quad r \in [a^0, b^0] \end{cases}$$

i.e.

$$\underline{\hat{f}}_1^I(\hat{p}_\alpha,\hat{q}_\beta)=\left\{\begin{array}{lll}\hat{f}^j_\alpha(\bar{p}_\alpha,\bar{r}_\alpha)=G^{f^j_\alpha}(\hat{p}_\alpha,r;\bar{p}_\alpha,\bar{r}_\alpha) & \text{if} \quad I=(\alpha,j) & \text{with} \quad j\neq 0\\ \hat{f}^j_\alpha(\bar{p}_\alpha,\bar{r}_\alpha)=G^{f^j_\beta}(\bar{p}_\alpha,\bar{r}_\alpha;\hat{p}_\alpha,r) & \text{if} \quad I=(\alpha,j) & \text{with} \quad j=0\\ & & \\ \hat{f}^j_\beta(\bar{q}_\beta,\bar{r}_\beta)=G^j_\beta(\hat{q}_\beta,r;\bar{q}_\beta,\bar{r}_\beta) & \text{if} \quad I=(\beta,j) & \text{with} \quad j=0\\ \hat{f}^j_\beta(\bar{q}_\beta,\bar{r}_\beta)=G^j_\beta(\bar{q}_\beta,\bar{r}_\beta;\hat{q}_\beta,r) & \text{if} \quad I=(\beta,j) & \text{with} \quad j\neq 0\end{array}\right. \quad \text{with} \quad \left\{\begin{array}{l}(\mathfrak{R}\hat{f}^0_\alpha)(\hat{p}_\alpha,r)=(\mathfrak{R}\hat{f}^0_\beta)(\hat{q}_\beta,r)\\ \text{for} \quad r,\bar{r}_\alpha,\bar{r}_\beta\in[a^0,b^0]\end{array}\right.$$

i.e.

$$\underline{\hat{f}}_1^I(\hat{p}_\alpha,\hat{q}_\beta)=\left\{\begin{array}{lll}\hat{f}^j_\alpha(\bar{p}_\alpha,\bar{r}_\alpha)=G^{f^j_\alpha}(\hat{p}^j_\alpha,\bar{p}^j_\alpha) & \text{if} \quad I=(\alpha,j) & \text{with} \quad j\neq 0\\ \hat{f}^0_\alpha(\bar{p}_\alpha,\bar{r}_\alpha)=G^{f^0}(\bar{r}_\alpha,r) & \text{if} \quad I=(\alpha,j) & \text{with} \quad j=0\\ & & \\ \hat{f}^0_\beta(\bar{q}_\beta,\bar{r}_\beta)=G^{f^0}(r,\bar{r}_\beta) & \text{if} \quad I=(\beta,j) & \text{with} \quad j=0\\ \hat{f}^j_\beta(\bar{q}_\beta,\bar{r}_\beta)=G^{f^j_\beta}(\bar{q}^j_\beta,\hat{q}^j_\beta) & \text{if} \quad I=(\beta,j) & \text{with} \quad j\neq 0\end{array}\right. \quad \text{with} \quad \left\{\begin{array}{l}G^{f^0}(\bar{r}_\alpha,r)=G^{f^0}(r,\bar{r}_\beta)\\ \text{for} \quad r,\bar{r}_\alpha,\bar{r}_\beta\in[a^0,b^0]\end{array}\right.$$

i.e.

$$\left\{\begin{array}{l}\underline{\hat{f}}_1^I(\hat{p}_\alpha,\hat{q}_\beta)=\left\{\begin{array}{lll}\hat{f}^j_\alpha(\bar{p}_\alpha,\bar{r}_\alpha)=G^{f^j_\alpha}(\hat{p}^j_\alpha,\bar{p}^j_\alpha) & \text{if} \quad I=(\alpha,j) & \text{with} \quad j\neq 0\\ \hat{f}^j_\beta(\bar{q}_\beta,\bar{r}_\beta)=G^{f^j_\beta}(\bar{q}^j_\beta,\hat{q}^j_\beta) & \text{if} \quad I=(\beta,j) & \text{with} \quad j\neq 0\end{array}\right.\\ \text{with}\\ \hat{f}^0_\alpha(\bar{p}_\alpha,\bar{r}_\alpha)=G^{f^0}(\bar{r}_\alpha,r)=G^{f^0}(r,\bar{r}_\beta)=\hat{f}^0_\beta(\bar{q}_\beta,\bar{r}_\beta) \quad \text{for} \quad r,\bar{r}_\alpha,\bar{r}_\beta\in[a^0,b^0]\end{array}\right. \tag{7.25}$$

Step 2: second computation
We first glue together the Godunov quasi-fluxes to get

$$(\hat{f}_\alpha \underset{0:0}{\sharp} \hat{f}_\beta)^I(\bar{p}_\alpha,\bar{q}_\beta):=\left\{\begin{array}{ll}\hat{f}^j_\alpha(\bar{p}_\alpha,r_0) & \text{if} \quad I=(\alpha,j)\\ \hat{f}^j_\beta(\bar{q}_\beta,r_0) & \text{if} \quad I=(\beta,j)\end{array}\right. \quad \text{with} \quad \left\{\begin{array}{l}f^0_\alpha(\bar{p}_\alpha,r_0)=f^0_\beta(\bar{q}_\beta,r_0)\\ \text{for} \quad r_0\in[a^0,b^0]\end{array}\right.$$

and then we relax this Godunov quasi-flux to

$$\underline{\hat{f}}_2^I(\hat{p}_\alpha,\hat{q}_\beta):=(\mathfrak{R}(\hat{f}_\alpha \underset{0:0}{\sharp} \hat{f}_\beta))^I(\hat{p}_\alpha,\hat{q}_\beta)=(\hat{f}_\alpha \underset{0:0}{\sharp} \hat{f}_\beta)^I(\bar{p}_\alpha,\bar{q}_\beta)=\left\{\begin{array}{ll}G^{f^j_\alpha}_{\underline{\sigma}^I}(\hat{p}_\alpha,\bar{p}_\alpha) & \text{if} \quad I=(\alpha,j),\\ G^{f^j_\beta}_{\underline{\sigma}^I}(\hat{q}_\beta,\bar{q}_\beta) & \text{if} \quad I=(\beta,j),\end{array}\right.$$

i.e.

$$\left\{\begin{array}{l}\underline{\hat{f}}_2^I(\hat{p}_\alpha,\hat{q}_\beta)=\left\{\begin{array}{ll}\hat{f}^j_\alpha(\bar{p}_\alpha,r)=G^{f^j_\alpha}(\hat{p}^j_\alpha,\bar{p}^j_\alpha) & \text{if} \quad I=(\alpha,j),\\ \hat{f}^j_\beta(\bar{q}_\beta,r)=G^{f^j_\beta}(\bar{q}^j_\beta,\hat{q}^j_\beta) & \text{if} \quad I=(\beta,j),\end{array}\right.\\ \text{with}\\ m:=\hat{f}^0_\alpha(\bar{p}_\alpha,r_0)=\hat{f}^0_\beta(\bar{q}_\beta,r_0) \quad \text{for} \quad r_0\in[a^0,b^0] \quad \text{and} \quad f^0_\alpha(\bar{p}_\alpha,\downarrow), \quad f^0_\beta(\bar{q}_\beta,\uparrow)\end{array}\right. \tag{7.26}$$

Notice that (7.25) and (7.26) are close to each other.
Step 3: final argument
It remains to explain why (7.25) and (7.26) are equivalent. This is due to the fact that $\hat{f}_\gamma$ are both 0-locally constant for $\gamma=\alpha,\beta$. Indeed, let $(\hat{p}_\alpha,\hat{q}_\beta)$, and associated $(\bar{p}_\alpha,\bar{q}_\beta,r_0)$ as introduced in (7.26). Let us now associate some values $(\bar{r}_\alpha,r,\bar{r}_\beta)$.
Case 1: $m=f^0(r^0)$
Then we can choose

$$(\bar{r}_\alpha,r,\bar{r}_\beta):=(r_0,r_0,r_0)$$

and this implies that

$$\underline{\hat{f}}_2(\hat{p}_\alpha,\hat{q}_\beta)=\underline{\hat{f}}_1(\hat{p}_\alpha,\hat{q}_\beta). \tag{7.27}$$

Case 2: $m<f^0(r^0)$
Because $\hat{f}_\gamma$ are 0-locally constant, we deduce that both $\hat{f}^0_\alpha(\bar{p}_\alpha,\cdot)$ and $\hat{f}^0_\beta(\bar{q}_\beta,\cdot)$ are both locally constant around r_0 (and below f^0). We then consider the largest interval $[r_0^-,r_0^+]\subset[a^0,b^0]$ such that

$$f^0\geq m \quad \text{on} \quad [r_0^-,r_0^+]\ni r_0$$

In particular, we have

$$\hat{f}^0_\alpha(\bar{p}_\alpha,\cdot) = m = \hat{f}^0_\beta(\bar{q}_\beta,\cdot) \quad \text{on} \quad [r_0^-,r_0^+] \tag{7.28}$$

Case 2.1: $r_0^+ < b^0$, or $r_0^+ = b^0$ and $f(b^0) = m$
Then

$$m = f^0(r_0^+) = G^{f^0}(r_0^-,r_0^+) = G^{f^0}(r_0^+,r_0^+)$$

Therefore

$$\hat{f}^0_\alpha(\hat{p}_\alpha,r_0^-) = m = G^{f^0}(r_0^-,r_0^+) = G^{f^0}(r_0^+,r_0^+) = \hat{f}^0_\beta(\bar{q}_\beta,r_0^+)$$

and we can choose

$$(\bar{r}_\alpha,r,\bar{r}_\beta) := (r_0^-,r_0^+,r_0^+)$$

which implies again (7.27).
Case 2.2: $a^0 < r_0^-$, or $r_0^- = a^0$ and $f(a^0) = m$
This case is symmetric of Case 2.1. Here

$$m = f^0(r_0^-) = G^{f^0}(r_0^-,r_0^-) = G^{f^0}(r_0^-,r_0^+)$$

Therefore

$$\hat{f}^0_\alpha(\hat{p}_\alpha,r_0^-) = m = G^{f^0}(r_0^-,r_0^-) = G^{f^0}(r_0^-,r_0^+) = \hat{f}^0_\beta(\bar{q}_\beta,r_0^+)$$

and we can choose

$$(\bar{r}_\alpha,r,\bar{r}_\beta) := (r_0^-,r_0^-,r_0^+)$$

which implies again (7.27).
Case 2.3: $a^0 = r_0^-$ and $r_0^+ = b^0$, and $f^0(a^0), f^0(b^0) > m$
Then the last line of (7.25) is never satisfied for any triple $(\bar{r}_\alpha,r,\bar{r}_\beta)$. But the fact that $\hat{f}_\gamma$ are both Godunov quasi-fluxes for $\gamma=\alpha,\beta$, implies that for all $(\bar{p}_\alpha,\bar{q}_\beta)$ we have (see (5.59))

$$\left\{\begin{array}{l} \sigma^0_\alpha\left\{\hat{f}^0_\alpha(\bar{p}_\alpha,\cdot) - f^0\right\}\cdot n \ge 0 \quad \text{on} \quad \partial[a^0,b^0] \\ \sigma^0_\beta\left\{\hat{f}^0_\beta(\bar{q}_\beta,\cdot) - f^0\right\}\cdot n \ge 0 \quad \text{on} \quad \partial[a^0,b^0] \end{array}\right.$$

where n is the outward unit normal to the interval $[a^0,b^0]$. But (7.28) means that both maps $\hat{f}^0_\alpha(\bar{p}_\alpha,\cdot) = \hat{f}^0_\beta(\bar{q}_\beta,\cdot) = m$ on $[a^0,b^0]$. Because $\sigma^0_\alpha = -\sigma^0_\beta$, we deduce that

$$(m - f^0) = 0 \quad \text{on} \quad \partial[a^0,b^0]$$

Contradiction. This shows that Case 2.3 never happens.
Case 3: $m > f^0(r^0)$
This case is symmetric of Case 2, and then implies (7.27).
Conclusion
Hence in all cases, we get (7.27). This ends the proof of the proposition.

8 Complementary results

8.1 More on Riemann monotonicity

Our goal is now to show that for locally Lipschitz continuous maps $h:\mathbb{R}^N\to\mathbb{R}^N$ which are Riemann monotone, their Jacobian matrices have a particular monotonicity property, called P_0-monotonicity. In order to describe this property, it is useful to focus first on the linear case $h(x)=Mx$, and to transfer our notions of monotonicity for the functions $h=\hat{f}_\mathcal{G}$, to the square matrices M.

Definition 8.1 (P_0-monotone, Riemann monotone, Kružkov monotone)
A principal minor of a $n\times n$ matrix $M=(M_{ij})_{i,j\in\{1,\dots,n\}}$ is the determinant of a submatrix $M_I := (M_{ij})_{i,j\in I}$ for a subset $\emptyset\ne I\subset\{1,\dots,n\}$, obtained by depletion of columns and of lines with the same labels. Let M be a real $n\times n$ matrix.
i) (P_0-monotone)

A matrix M is said to be a P_0-monotone, if all its principal minors are nonnegative.
ii) (Riemann monotone)
A matrix M is said to be a Riemann monotone, if for all $x \in \mathbb{R}^n$, it satisfies

$$x \diamond Mx \le 0 \quad \Longrightarrow \quad Mx = 0$$

iii) (Kružkov monotone)
A matrix M is said to be Kružkov monotone if it satisfies

$$M_{jj} \ge \sum_{i\in\{1,\dots,n\}\backslash\{j\}} |M_{ij}| \quad \textit{for all} \quad j = 1,\dots,n$$

(i.e. if M is (non-negatively) diagonally column-dominant).

Remark 8.2 (Related references)
For associated notions of P_0-functions and P-functions used in the complementarity problem, see the book of FACCHINEI, PANG *[18] (Subsection 3.5.2, on page 298). For the relation between P-functions and global univalence theorems, see the book of* PARTHASARATHY *[44] (for instance on page 20 in Chapter III). We also refer the reader to the book of* JOHNSON, SMITH, TSATSOMEROS *[34] (see Chapter 4), for the properties of related P-matrices. For instance if M is a P_0-(monotone) matrix, then $M + \varepsilon\ id$ is a P-matrix for all $\varepsilon > 0$.*

Then we have the following result.

Lemma 8.3 (Hierarchy of monotonicities)
For a real $n \times n$ matrix M, we have the following hierarchy

i) *M is Kružkov monotone* $\Longrightarrow$ **ii)** *M is Riemann monotone* $\Longrightarrow$ **iii)** *M is P_0-monotone*

Proof of Lemma 8.3
Step 1: i) implies ii)
Assume that M is Kružkov monotone in the sense of Definition 8.1, and consider the linear map $h(x) := Mx$. Then ii) of Proposition 4.20 shows that $D^h \ge 0$. Then Proposition 5.9 implies that h is Riemann monotone. Because h is linear, this means that M is Riemann monotone in the sense of Definition 8.1.
Step 2: ii) implies iii)
We only show that the determinant of the full matrix M is nonnegative, and the argument is the same for all principal submatrices. The argument of the proof of Lemma 5.3 shows that the linear map of matrix $M + \varepsilon id_{\mathbb{R}^n}$ is injective for all $\varepsilon > 0$. For large $\varepsilon > 0$, the determinant is then positive, hence by continuity in $\varepsilon > 0$, we deduce that $det(M + \varepsilon id_{\mathbb{R}^n}) > 0$. In the limit $\varepsilon \to 0^+$, this shows that $det(M) \ge 0$. This ends the proof of the lemma.

Then we have the following result.

Proposition 8.4 (Jacobian of Riemann monotone maps)
Assume (2.2) for $N \ge 1$. Assume that $h : [a,b] \to \mathbb{R}^N$ is Riemann monotone and locally Lipschitz continuous. Then the Jacobian matrix $(\partial_j h^i)_{i,j\in\{1,\dots,N\}}$ is a P_0-monotone matrix a.e. on $[a,b]$.

Proof of Proposition 8.4
The proof follows closely the proof for matrices (see Step 2 of the proof of Lemma 8.3).
We only show that the determinant of the full Jacobian matrix is nonnegative, and the argument is the same for all principal submatrices. Assume by contradiction that there exists a Lebesgue point $p_0 \in [a,b]$ such that

$$det(A) < 0 \quad \text{with} \quad A := Dh(p_0) \tag{8.1}$$

Then Lemma 5.3 shows that $h+\varepsilon id_{\mathbb{R}^n}$ is injective for all $\varepsilon > 0$. For large $\varepsilon > 0$, the determinant $det(A+\varepsilon id_{\mathbb{R}^n})$ is positive, hence by continuity in $\varepsilon > 0$, we deduce that $det(A + \varepsilon id_{\mathbb{R}^n}) \ge 0$ for all $\varepsilon > 0$. In the limit $\varepsilon \to 0^+$, this gives a contradiction with (8.1). Therefore (8.1) is false, and we conclude that $det(Dh) \ge 0$ a.e. on $[a,b]$. This ends the proof of the proposition.

The following result shows that P_0-monotonicity is indeed very close to Riemann monotonicity.

Lemma 8.5 (A property of P_0-monotone matrices)
Let M be a P_0-monotone $n\times n$ matrix. Then for all $\varepsilon > 0$, the matrix $M^\varepsilon := M+\varepsilon Id$ is Riemann monotone.

Proof of Lemma 8.5
Consider some $x \in \mathbb{R}^n$ such that $x \diamond M^\varepsilon x \le 0$. We now use an argument introduced in Fiedler and Pták [19], in the proof of their Theorem 1.3. We deduce that there exists some diagonal matrix $\Delta \ge 0$, with nonnegative diagonal such that $M^\varepsilon x = -\Delta x$. Let us denote $\Delta^\varepsilon := \Delta + \varepsilon Id$, and M_I is the principal submatrix of M of indices I, and similarly for $\Delta^\varepsilon_{\bar I}$ the submatrix of Δ^ε of indices $\bar I := \{1,\dots,N\}\backslash I$. If $x \ne 0$, then x is a 0-eigenvector of the matrix $M + \Delta^\varepsilon$, and using the fact that M is P_0-monotone, we get

$$0 = det(M + \Delta^\varepsilon) = \sum_{I\subset\{1,\dots,N\}} det(M_I)det(\Delta^\varepsilon_{\bar I}) \ge det(\Delta^\varepsilon) \ge \varepsilon^n > 0$$

Contradiction. Hence we deduce that $x = 0$ and then $M^\varepsilon x = 0$. Hence we have shown that $x \diamond M^\varepsilon x \le 0$ implies $M^\varepsilon x = 0$, i.e. that M^ε is Riemann monotone. This ends the proof of the lemma.

We finish this subsection with two counter-examples, which show that we can not hope the Jacobian matrix of Riemann monotone maps, to be Riemann monotone everywhere.

Lemma 8.6 (Counter-example for matrices)
For $n = 2$, consider $A := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then A is P_0-monotone, but is not Riemann monotone.

Proof of Lemma 8.6
Consider $p := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then we have $p \diamond Ap = 0_{\mathbb{R}^2}$ and $Ap \ne 0_{\mathbb{R}^2}$. This ends the proof of the lemma.

Lemma 8.7 (Counter-example for maps)
For $n = 2$ and $x, y \in \mathbb{R}$, consider the map $h : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$h = (h^x, h^y) \quad \textit{with} \quad \left\{ \begin{array}{l} h^x(x,y) = x^3 + y \\ h^y(x,y) = -x + y^3 \end{array} \right.$$

Then the map h is Riemann monotone, but its Jacobian matrix $Dh(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is not Riemann monotone as a matrix (i.e. in the sense of Definition 8.1).

Proof of Lemma 8.7
For $p = (x,y)$ and $\bar p := (\bar x, \bar y)$, we see that $(\bar p - p) \diamond [h]^{\bar p}_p \le 0$ means

$$\left\{ \begin{array}{l} (\bar x - x)\cdot\left\{(\bar x^3 - x^3) + (\bar y - y)\right\} \le 0 \\ (\bar y - y)\cdot\left\{-(\bar x - x) + (\bar y^3 - y^3)\right\} \le 0 \end{array} \right.$$

Taking the sum, we get $0 \le (\bar x - x)(\bar x^3 - x^3) + (\bar y - y)(\bar y^3 - y^3) \le 0$, which implies $\bar x - x = 0 = \bar y - y$, i.e. $\bar p = p$ and then $[h]^{\bar p}_p = 0$. Hence $(\bar p - p) \diamond [h]^{\bar p}_p \le 0$ implies $[h]^{\bar p}_p = 0$, which shows that h is Riemann monotone. This ends the proof of the lemma.

8.2 Adding an $(N + 1)$-th branch to get conservative germs

In what follows, RH refers to the Rankine-Hugoniot relation.

Lemma 8.8 (Adding an $(N + 1)$-th branch)
Assume (2.2) with $N \ge 1$, and let $\mathcal{G} \subset [a,b]$ be a set. Define

$$(RH^f)(p) := \sum_{j=1,\dots,N} \sigma^j f^j(p^j)$$

Consider the smallest one-dimensional box $I_{N+1} := [a_{N+1}, b_{N+1}] \subset \mathbb{R}$ such that

$$I_{N+1} \supset \; -RH^f(\mathcal{G})$$

Let us consider any locally Lipschitz continuous decreasing bijective function $f^{N+1} : I_{N+1} \to I_{N+1}$. We then define

(8.2)
$$\tilde{f} := (f, f^{N+1}), \quad \tilde{a} := (a, a_{N+1}), \quad \tilde{b} := (b, b_{N+1}), \quad \tilde{J} := J \cup J^{N+1} \quad \textit{with} \quad J^{N+1} \simeq (-\infty, 0), \quad \sigma^{N+1} = 1$$

and the set

$$\tilde{\mathcal{G}} := \left\{\tilde{p} := (p, p^{N+1}) \in [\tilde{a}, \tilde{b}], \quad p^{N+1} := (f^{N+1})^{-1}(-RH^f(p)) \quad \textit{with} \quad p \in \mathcal{G}\right\} \tag{8.3}$$

i) (Construction of a conservative germ $\tilde{\mathcal{G}}$)
Then $\tilde{\mathcal{G}} \subset [\tilde{a}, \tilde{b}]$ is a generalized Riemann germ with respect to $(\tilde{J}, \tilde{f})$ if and only if $\mathcal{G}$ is a generalized Riemann germ with respect to (J, f). When it is the case, then $\tilde{\mathcal{G}}$ is moreover conservative. More precisely if $\hat{f} = \hat{f}_{\mathcal{G}}$ (resp. $\hat{\tilde{f}} := \hat{f}_{\tilde{\mathcal{G}}}$) is the flux associated to the Riemann germ $\mathcal{G}$ (resp. $\tilde{\mathcal{G}}$), then

$$\hat{\tilde{f}} = \hat{\tilde{f}}_{\tilde{\mathcal{G}}} = (\hat{f}, -RH^{\hat{f}}) \quad \textit{and} \quad RH^{\hat{\tilde{f}}} = 0 \quad \textit{on} \quad [\tilde{a}, \tilde{b}]$$

ii) (Case of Riemann germs)
Then $\tilde{\mathcal{G}} \subset [\tilde{a}, \tilde{b}]$ is a Riemann germ if and only if $\mathcal{G} \subset [a, b]$ is a Riemann germ.
iii) (Case of D_+-germs)
Then $\mathcal{G}$ is a D_+^f-germ if and only if $\tilde{\mathcal{G}}$ is a $D_+^{\tilde{f}}$-germ.
iv) (Case of conservative Kružkov sets)
Let $\mathcal{G} \subset [a, b] \subset \mathbb{R}^N$ be a set and $\tilde{\mathcal{G}} \subset [\tilde{a}, \tilde{b}] \subset \mathbb{R}^{1+N}$ be the set defined in (8.3). Then

$$\begin{cases} D^f \geq 0 & \textit{on} \quad \mathcal{G} \times \mathcal{G} \\ RH^f = 0 & \textit{on} \quad \mathcal{G} \end{cases} \tag{8.4}$$

implies

$$\begin{cases} D^{\tilde{f}} \geq 0 & \textit{on} \quad \tilde{\mathcal{G}} \times \tilde{\mathcal{G}} \\ RH^{\tilde{f}} = 0 & \textit{on} \quad \tilde{\mathcal{G}} \end{cases} \tag{8.5}$$

for $\tilde{f}$ defined in (8.2).

Proof of Lemma 8.8
Points i) and ii) of the lemma are straightforward. We now prove iii). Up to apply a suitable reversion transform, we can assume that J is of type $N : 0$.
Step 1: from $D_+^{\tilde{f}}$-germ to D_+^f-germ
Consider $\tilde{p} = (p, p^{N+1}), \tilde{q} = (q, q^{N+1}) \in [\tilde{a}, \tilde{b}]$.
We have
$$D_+^{\hat{\tilde{f}}}(\tilde{p}, \tilde{q}) = D_+^{\hat{f}}(p, q) + \left\{\text{sign}^+(p^{N+1} - q^{N+1})\right\} \cdot \left\{-RH^{\hat{f}}(p) - \left(-RH^{\hat{f}}(q)\right)\right\}$$
with $D_+^{\hat{f}}(p, q) = \sum_{j=1,\dots,N} \left\{\text{sign}^+(p^j - q^j)\right\} \cdot [\hat{f}^j]_q^p$. Either $p^{N+1} \leq q^{N+1}$ and then $D_+^{\hat{\tilde{f}}}(\tilde{p}, \tilde{q}) = D_+^{\hat{f}}(p, q)$. Or $p^{N+1} > q^{N+1}$ and (using $1 - \text{sign}^+(x) = -\text{sign}^-(x)$ if $x \neq 0$), we get, if $p^j \neq q^j$ for all $j = 1, \dots, N$, that

$$\begin{aligned} D_+^{\hat{\tilde{f}}}(\tilde{p}, \tilde{q}) &= D_+^{\hat{f}}(p, q) - [RH^{\hat{f}}]_q^p \\ &= \sum_{j=1,\dots,N} \left\{\text{sign}^-(p^j - q^j)\right\} \cdot [\hat{f}^j]_q^p \\ &= \sum_{j=1,\dots,N} \left\{\text{sign}^+(q^j - p^j)\right\} \cdot [\hat{f}^j]_p^q \\ &= D_+^{\hat{f}}(q, p) \end{aligned}$$

For Ω defined in (4.22), this means $D_+^{\hat{\tilde{f}}}(\tilde{p}, \tilde{q}) = D_+^{\hat{f}}(q, p)$ if $(q, p) \in \Omega$. Hence if $D_+^{\hat{\tilde{f}}} \geq 0$, then $\tilde{\mathcal{G}}$ is Kružkov and then $\hat{\tilde{f}}$ is continuous and $\hat{f}$ is also continuous. Moreover we have $D_+^{\hat{f}} \geq 0$ on Ω. From Lemma 4.17, we deduce that $D_+^{\hat{f}} \geq 0$ on $\overline{\Omega} = [a, b]^2$. Therefore, if $\tilde{\mathcal{G}}$ is a $D_+^{\tilde{f}}$-germ, then $\mathcal{G}$ is a D_+^f-germ.

Step 2: from D_+^f-germ to $D_+^{\tilde f}$-germ
Conversely, if $D_+^{\hat f} \geq 0$, then

$$0 \leq D_+^{\hat{\tilde f}}(\tilde p, \tilde q) = \begin{cases} D_+^{\hat f}(p,q) & \text{if} \quad p^{N+1} < q^{N+1} \\ D_+^{\hat f}(q,p) & \text{if} \quad p^{N+1} > q^{N+1} \end{cases}$$

Hence for

$$\tilde\Omega := \left\{(\tilde p, \tilde q) \in [\tilde a, \tilde b], \quad \tilde p^j \neq \tilde q^j \quad \text{for all} \quad j = 1, \ldots, N+1\right\}$$

we see that $D_+^{\hat{\tilde f}} \geq 0$ on $\tilde\Omega$. From Lemma 4.17, we deduce that $D_+^{\hat{\tilde f}} \geq 0$ on $\overline{\tilde\Omega} = [\tilde a, \tilde b]^2$. Therefore, if $\mathcal{G}$ is a D_+^f-germ, then $\tilde{\mathcal{G}}$ is a $D_+^{\tilde f}$-germ.
Step 3: proof of iv)
Assume that the set $\mathcal{G}$ satisfies (8.4). For all $p, q \in \mathcal{G}$, consider $p^{N+1} := (f^{N+1})^{-1}(-RH^f(p)) = (f^{N+1})^{-1}(0)$ and $q^{N+1} := (f^{N+1})^{-1}(-RH^f(p)) = (f^{N+1})^{-1}(0)$. Then for $\tilde p = (p, p^{N+1}), \tilde q = (q, q^{N+1}) \in \tilde{\mathcal{G}} \subset [\tilde a, \tilde b]$, we have

$$D^{\tilde f}(\tilde p, \tilde q) = D^f(p,q) + \left\{\text{sign}(p^{N+1} - q^{N+1})\right\} \cdot \left\{-RH^f(p) - \left(-RH^f(q)\right)\right\} = D^f(p,q) \geq 0$$

From the definition of $\tilde{\mathcal{G}}$, we deduce (8.5).
This ends the proof of the lemma.

8.3 Duality for D_+-germs

In this section, we study the following notion of duality, in particular useful for D_+-germs.

Definition 8.9 (Left-dual and right-dual)
Assume (2.2) with $N \geq 1$, and let $\mathcal{G} \subset [a,b]$ be a set.
Then we define the left-dual of the set $\mathcal{G}$ as

$${}^*\mathcal{G} := {}^{\overset{*}{f}}\mathcal{G} := \left\{p' \in [a,b], \quad D_+^f(p',p) \geq 0 \quad \textit{for all} \quad p \in \mathcal{G}\right\}$$

and its right-dual

$$\mathcal{G}^* := \mathcal{G}^{\overset{*}{f}} := \left\{p' \in [a,b], \quad D_+^f(p,p') \geq 0 \quad \textit{for all} \quad p \in \mathcal{G}\right\}$$

where D_+^f is defined in (2.9).

Remark 8.10 (A model case)
Before to start with duality, it is instructive to keep in mind the following model case for $1:0$ junctions. For $N = 1$, we consider $f^1(u) = g(u) = u(1-u)$ on $[0,1]$, with $A_{max} := \sup_{[0,1]} g = \frac{1}{4} = g(u_0)$ with $u_0 := \frac{1}{2}$. We set the monotone envelopes of g

$$g^+(p) := \begin{cases} g(u) & \textit{for} \quad p \in [0,u_0] \\ g(u_0) & \textit{for} \quad p \in [u_0,1] \end{cases} \quad \textit{and} \quad g^-(p) := \begin{cases} g(u_0) & \textit{for} \quad p \in [0,u_0] \\ g(u) & \textit{for} \quad p \in [u_0,1] \end{cases}$$

and the monotone inverse functions $u_+(\lambda) \leq u_-(\lambda)$ defined by

$$[0,u_0] \ni u_+(\lambda) := \begin{cases} (g_+)^{-1}(\lambda) & \textit{for} \quad \lambda \in [0, A_{max}) \\ u_0 & \textit{for} \quad \lambda = A_{max} \end{cases} \quad \textit{and} \quad [u_0,1] \ni u_-(\lambda) := \begin{cases} (g_-)^{-1}(\lambda) & \textit{for} \quad \lambda \in [0, A_{max}) \\ u_0 & \textit{for} \quad \lambda = A_{max} \end{cases}$$

For a $1:0$ junction, and for the parameter $A \in [0, A_{max}]$, we can consider the following germ $\mathcal{G}_A \subset [0,1]$ defined by

$$\mathcal{G}_A := \{p \in [0,1], \quad \hat g_A(p) = g(p)\} \quad \textit{with flux} \quad \hat g_A(p) := \min\left\{A, g^+(p)\right\}$$

Then for $D_+^g(p,q) := \textit{sign}^+(p-q) \cdot \{g(p) - g(q)\}$, a direct computation gives

$$\begin{cases} \mathcal{G}_A & = [0, u_+(A)] \cup \{u_-(A)\} \\ {}^*(\mathcal{G}_A) & = [0, u_-(A)] \\ (\mathcal{G}_A)^* & = [0, u_+(A)] \cup [u_-(A), 1] \end{cases}$$

Here the left-dual behaves like a sort of left completion of $\mathcal{G}_A$, while the right dual behaves like a sort of right completion.

We now start with the following straightforward result (with $D_+^f(p,q) := \sum_{j=1,\dots,N} \sigma^j \cdot \text{sign}^+(p^j - q^j) \cdot \left\{f^j(p^j) - f^j(q^j)\right\}$).

Lemma 8.11 (Explicit characterization of duality)
Assume (2.2) with $N \geq 1$. Then for any sets $P, Q \subset [a,b]$, we have $D_+^f(P,Q) \geq 0$ if and only if
(8.6)
$$\sum_{j\in K} \sigma^j \cdot \left\{f^j(p) - f^j(q)\right\} \geq 0 \quad \text{for all} \quad (p,q) \in P\times Q, \quad \text{for all} \quad K \subset \{1,\dots,N\} \quad \text{such that} \quad p-q \in \tilde{E}_K,$$
with
$$\tilde{E}_K := \sum_{j\in K} (0,+\infty)e_j - \sum_{j\notin K} [0,+\infty)e_j. \tag{8.7}$$

Remark 8.12 *Notice that quantity $\tilde{E}_K$ shares some similarities with quantity E_K defined in (4.25).*

We also notice the following straightforward result about left and right duals.

Lemma 8.13 (Exchanging left and right-duals by inversion transform)
Assume (2.2) with $N \geq 1$, and let $\mathcal{G} \subset [a,b]$ be a set.
Consider the full inversion transform
$$\bar{p} := -p, \quad \bar{f}(\bar{p}) := -f(-\bar{p}), \quad \bar{\mathcal{G}} := -\mathcal{G}$$
Then we have
$$D_+^f(p,q) = D_+^{\bar{f}}(\bar{q},\bar{p})$$
and
$$\overset{*}{{}_{\bar{f}}}\bar{\mathcal{G}} = \overline{\mathcal{G}^{\overset{*}{f}}} \quad \text{and} \quad \overline{{}^*\mathcal{G}} = \bar{\mathcal{G}}^{\overset{*}{\bar{f}}}$$

We now have the following result.

Lemma 8.14 (Characterization of the duals of D_+-germs)
Assume (2.2) with $N \geq 1$, and let $\mathcal{G} \subset [a,b]$ be a D_+-germ. Then we have for $\hat{f} = \hat{f}_{\mathcal{G}}$
$${}^*\mathcal{G} = \mathcal{G}^{SUB} := \left\{p \in [a,b], \quad \sigma \diamond (\hat{f} - f)(p) \leq 0\right\} \quad \text{and} \quad D_+^f(\mathcal{G}^{SUB}, \mathcal{G}) \geq 0 \tag{8.8}$$
and
$$\mathcal{G}^* = \mathcal{G}^{SUP} := \left\{p \in [a,b], \quad \sigma \diamond (\hat{f} - f)(p) \geq 0\right\} \quad \text{and} \quad D_+^f(\mathcal{G}, \mathcal{G}^{SUP}) \geq 0 \tag{8.9}$$

Proof of Lemma 8.14
We only prove (8.8), because (8.9) follows from (8.8) and Lemma 8.13.
Step 1: proof that $\mathcal{G}^{SUB} \subset {}^*\mathcal{G}$
Fix some $p \in \mathcal{G}^{SUB}$, and let us consider any $q \in \mathcal{G}$ such that $p - q \in \check{E}_K$ with $\check{E}_K$ defined in (8.7). Then we get
$$\begin{aligned} D_+^f(p,q) &= \sum_{j\in K} \sigma^j \cdot \left\{f^j(p) - f^j(q)\right\} \\ &= \sum_{j\in K} \sigma^j \cdot \left\{f^j(p) - \hat{f}^j(q)\right\} \\ &\geq \sum_{j\in K} \sigma^j \cdot \left\{\hat{f}^j(p) - \hat{f}^j(q)\right\} \\ &= D_+^{\hat{f}}(p,q) \\ &\geq 0 \end{aligned}$$
because $\mathcal{G}$ is a D_+-germ. Therefore $p \in {}^*\mathcal{G}$.
Step 2: proof that $\mathcal{G}^{SUB} \supset {}^*\mathcal{G}$
Conversely, consider $p \in {}^*\mathcal{G}$ and for $\hat{p} := \pi_{\mathcal{G}}(p)$, let us set
$$K := \left\{k \in \{1,\dots,N\}, \quad \sigma^k \cdot \left\{f^k(p^k) - f^k(\hat{p}^k)\right\} < 0\right\}. \tag{8.10}$$

Because p^k belongs to the basin of attraction $BA^k(\hat{p}^k)$ on the branch $J^k \simeq \sigma^k \cdot (-\infty, 0)$, we deduce that

$$p^k > \hat{p}^k \quad \text{for all} \quad k \in K$$

Similarly, for all $k \in \{1, \dots, N\} \setminus K$, we have $\sigma^k \cdot \left\{f^k(p^k) - f^k(\hat{p}^k)\right\} \geq 0$ and then

$$p^k \leq \hat{p}^k \quad \text{for all} \quad k \in \{1, \dots, N\} \setminus K$$

Therefore $\hat{p} \in \mathcal{G}$ with $p - \hat{p} \in \check{E}_K$. Because $p \in {}^*\mathcal{G}$, we deduce that

$$0 \leq D_+^f(p, \hat{p}) = \sum_{j \in K} \sigma^j \cdot \left\{f^j(p) - f^j(\hat{p})\right\} \underset{(8.10)}{<} 0$$

if $K \neq \emptyset$. Contradiction. Therefore $K = \emptyset$, and $\sigma \diamond (f - \hat{f})(p) \geq 0$, i.e. $p \in \mathcal{G}^{SUB}$.
This shows (8.8) and ends the proof of the lemma.

Corollary 8.15 (Key dissipation of the duals)
Assume (2.2) with $N \geq 1$, and let $\mathcal{G} \subset [a, b]$ be a D_+-germ. Then we have

$$D_+({}^*\mathcal{G}, \mathcal{G}^*) \geq 0, \quad \textit{i.e.} \quad D_+(\mathcal{G}^{SUB}, \mathcal{G}^{SUP}) \geq 0$$

and

$${}^*\mathcal{G} \cap \mathcal{G}^* = \mathcal{G}$$

Proof of Corollary 8.15
Let $p \in {}^*\mathcal{G} = \mathcal{G}^{SUB} = \left\{\sigma \diamond (\hat{f} - f) \leq 0\right\}$ and $q \in \mathcal{G}^* = \mathcal{G}^{SUP} = \left\{\sigma \diamond (\hat{f} - f) \geq 0\right\}$ such that $p - q \in \check{E}_K$. Then we have

$$\begin{aligned} D_+^f(p, q) &= \sum_{j \in K} \sigma^j \cdot \left\{f^j(p) - f^j(q)\right\} \\ &\geq \sum_{j \in K} \sigma^j \cdot \left\{\hat{f}^j(p) - \hat{f}^j(q)\right\} \\ &= D_+^{\hat{f}}(p, q) \\ &\geq 0 \end{aligned}$$

because $\mathcal{G}$ is a D_+-germ. This ends the proof of the corollary.

Lemma 8.16 (Max and Min of duals for D_+-germs)
Assume (2.2) with $N \geq 1$, and let $\mathcal{G} \subset [a, b]$ be a D_+-germ. Then we have componentwisely

$$\max\left\{\mathcal{G}^{SUB}, \mathcal{G}^{SUB}\right\} \subset \mathcal{G}^{SUB}$$

and

$$\min\left\{\mathcal{G}^{SUP}, \mathcal{G}^{SUP}\right\} \subset \mathcal{G}^{SUP}$$

Proof of Lemma 8.16
For $\gamma = \alpha, \beta$, let us consider $p_\gamma \in \mathcal{G}^{SUB}$, which then satisfies $\sigma \diamond (\hat{f} - f)(p_\gamma) \leq 0$. Then consider $p := \max\{p_\alpha, p_\beta\}$ componentwisely, i.e.

$$p^j := \max\left\{p_\alpha^j, p_\beta^j\right\}, \quad j = 1, \dots, N$$

Recall that we have the monotonicities $(\sigma^1 \hat{f}^1)(\uparrow, \downarrow, \dots, \downarrow)$. Let $\gamma_1 \in \{\alpha, \beta\}$ be such that $p^1 = p_{\gamma_1}^1$. Then, using the monotonicities of $\sigma^1 \hat{f}^1$, we deduce that

$$\sigma^1 \hat{f}^1(p) = \sigma^1 \hat{f}^1(p_{\gamma_1}^1, p') \leq \sigma^1 \hat{f}^1(p_{\gamma_1}) = \sigma^1 f^1(p_{\gamma_1}^1) = \sigma^1 f^1(p)$$

Similarly, we get $\sigma \diamond (\hat{f} - f)(p) \leq 0$, which shows that $p \in \mathcal{G}^{SUB}$. The proof for the minimum is similar. This ends the proof of the lemma.

8.4 Duality and characteristic subsets for conservative $1:1$ germs

Lemma 8.17 (Duality and characteristic subsets for conservative $1:1$ germs)
Assume (2.2) with $N=2$ for $1:1$ junction with $f=(f^L,f^R)$ for indices $j=L,R$ (for left and right) with $\sigma^L=1$ and $\sigma^R=-1$. Let $\mathcal{G}\subset[a,b]$ be a conservative Kružkov germ. Following Definition 2.33, we recall the following sets of characteristic points of $\mathcal{G}$ (the sub-charcateristic set $\underline{\chi}\mathcal{G}$ and super-characteristic set $\overline{\chi}\mathcal{G}$)

$$\left\{\begin{array}{l}
\underline{\chi}\mathcal{G}:=\left\{\hat p=(\hat p^L,\hat p^R)\in\mathcal{G},\quad \left|\begin{array}{lll} f^L<f^L(\hat p^L) & \text{on} & (\hat p^L,\hat p^L+\varepsilon)\subset[a^L,b^L]\\ f^R<f^R(\hat p^R) & \text{on} & (\hat p^R-\varepsilon,\hat p^R)\subset[a^R,b^R]\end{array}\right|\quad \text{for some } \varepsilon>0\right\}\\[2ex]
\overline{\chi}\mathcal{G}:=\left\{\hat p=(\hat p^L,\hat p^R)\in\mathcal{G},\quad \left|\begin{array}{lll} f^L>f^L(\hat p^L) & \text{on} & (\hat p^L-\varepsilon,\hat p^L)\subset[a^L,b^L]\\ f^R>f^R(\hat p^R) & \text{on} & (\hat p^R,\hat p^R+\varepsilon)\subset[a^R,b^R]\end{array}\right|\quad \text{for some } \varepsilon>0\right\}\\[2ex]
\chi\mathcal{G}:=\underline{\chi}\mathcal{G}\cup\overline{\chi}\mathcal{G}
\end{array}\right.$$

For $p=(p^L,p^R)$ and $q=(q^L,q^R)$, we recall

$$RH^f(p):=f^L(p^L)-f^R(p^R),\quad D^f_+(p,q)=\text{sign}^+(p^L-q^L)\cdot\left\{f^L(p^L)-f^L(q^L)\right\}-\text{sign}^+(p^R-q^R)\cdot\left\{f^R(p^R)-f^R(q^R)\right\}$$

Then we have

(8.11)
$$\left\{\begin{array}{l}
{}''(\underline{\chi}\mathcal{G}):=\left\{p\in[a,b],\quad RH^f(p)\ge 0,\quad D^f_+(p,q)\ge 0\quad \text{for all}\quad q\in\underline{\chi}\mathcal{G}\right\}=\mathcal{G}^{SUB}=\left\{\sigma\diamond(\hat f-f)\le 0\right\}\\
(\overline{\chi}\mathcal{G})'':=\left\{p\in[a,b],\quad RH^f(p)\le 0,\quad D^f_+(q,p)\ge 0\quad \text{for all}\quad q\in\overline{\chi}\mathcal{G}\right\}=\mathcal{G}^{SUP}=\left\{\sigma\diamond(\hat f-f)\ge 0\right\}
\end{array}\right.$$

and

$$(\chi\mathcal{G})':=\left\{p\in[a,b],\quad RH^f(p)=0,\quad D^f(p,q)\ge 0\quad \text{for all}\quad q\in\chi\mathcal{G}\right\}=\mathcal{G}\tag{8.12}$$

Proof of Lemma 8.17
Step 1: proof of (8.11)
We prove the first line of (8.11) (the proof of the second line is similar).
Consider $p\in{}''(\underline{\chi}\mathcal{G})$. Then we have

$$f^L(p^L)-f^R(p^R)\ge 0\tag{8.13}$$

and

$$\text{sign}^+(p^L-q^L)\cdot\left\{f^L(p^L)-f^L(q^L)\right\}-\text{sign}^+(p^R-q^R)\cdot\left\{f^R(p^R)-f^R(q^R)\right\}\ge 0\quad\text{for all}\quad q\in\underline{\chi}\mathcal{G}\tag{8.14}$$

Assume by contradiction that $p\notin\mathcal{G}^{SUB}=\left\{\hat f^L-f^L\le 0,\ -(\hat f^R-f^R)\le 0\right\}$.
Case A: $f^L(p^L)<f^L(\hat p^L)$
Then we get $p^L\in BA^L(\hat p)\cap(\hat p^L,+\infty)$. Moreover, using (8.13), we get

$$f^R(p^R)\le f^L(p^L)<f^L(\hat p^L)=f^R(\hat p^R)$$

and then $p^R\in BA^R(\hat p)\cap(-\infty,\hat p^R)$, which shows that $\hat p\in\underline{\chi}\mathcal{G}$. Then the first term in (8.14) leads to a contradiction for the choice of $q:=\hat p$.
Case B: $f^R(p^R)>f^R(\hat p^R)$
Similarly to Case A, we get $p^R\in BA^R(\hat p)\cap(\hat p^R,+\infty)$. Moreover, using (8.13), we get now

$$f^L(p^L)\ge f^R(p^R)>f^R(\hat p^R)=f^L(\hat p^L)$$

and then $p^L\in BA^L(\hat p)\cap(-\infty,\hat p^L)$, which shows that $\hat p\in\overline{\chi}\mathcal{G}$. Then the second term in (8.14) leads to a contradiction for the choice of $q:=\hat p$.
Conclusion and consequences
Therefore $p\in\mathcal{G}^{SUB}$. Hence we have shown that

$${}''(\overline{\chi}\mathcal{G})\subset\mathcal{G}^{SUB}={}^*\mathcal{G}\tag{8.15}$$

But, by duality, the inclusion $\underline{\chi}\mathcal{G} \subset \mathcal{G}$ implies the reverse inclusion

$$''(\underline{\chi}\mathcal{G}) \supset {}''\mathcal{G} := {}^*\mathcal{G} \cap \left\{ RH^f \geq 0 \right\} \tag{8.16}$$

Moreover using the fact that $RH^{\hat{f}} = 0$, we deduce that

$$\left\{ RH^f \geq 0 \right\} \supset \left\{ \sigma \diamond (\hat{f} - f) \leq 0 \right\} = \mathcal{G}^{SUB} = {}^*\mathcal{G}$$

Hence (8.16) shows that $''(\underline{\chi}\mathcal{G}) \supset {}^*\mathcal{G}$ and the reverse inclusion (8.15) implies the equality, i.e. the first line of (8.11).
Step 2: proof of (8.12)
Step 2.1: preliminaries
Let $p \in (\chi\mathcal{G})'$, i.e. satisfying

$$f^L(p^L) - f^R(p^R) = 0 \tag{8.17}$$

and

$$\mathrm{sign}(p^L - q^L) \cdot \left\{ f^L(p^L) - f^L(q^L) \right\} - \mathrm{sign}(p^R - q^R) \cdot \left\{ f^R(p^R) - f^R(q^R) \right\} \geq 0 \quad \text{for all} \quad q \in \chi\mathcal{G} \tag{8.18}$$

Again assume by contradiction that $p \notin \mathcal{G}^{SUB} = \left\{ \hat{f}^L - f^L \leq 0,\ -(\hat{f}^R - f^R) \leq 0 \right\}$.
Case A: $f^L(p^L) < f^L(\hat{p}^L)$
Because (8.17) implies (8.13), then Case A of Step 1 shows that $\hat{p} \in \underline{\chi}\mathcal{G}$ with

$$\left\{ \begin{array}{rl} f^L(p^L) & < f^L(\hat{p}^L) \\ p^L & > \hat{p}^L, \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{rl} f^R(p^R) & < f^R(\hat{p}^R) \\ p^R & < \hat{p}^R \end{array} \right. .$$

Then both terms in (8.18) lead to a contradiction for $q := \hat{p}$.
Case B: $f^R(p^R) > f^R(\hat{p}^R)$
Similarly, because (8.17) implies (8.13), then Case B of Step 1 shows that $\hat{p} \in \overline{\chi}\mathcal{G}$ with

$$\left\{ \begin{array}{rl} f^L(p^L) & > f^L(\hat{p}^L) \\ p^L & < \hat{p}^L \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{rl} f^R(p^R) & > f^R(\hat{p}^R) \\ p^R & > \hat{p}^R \end{array} \right.$$

and both terms (8.18) lead to a contradiction for $q := \hat{p}$.
Conclusion: $p \in \mathcal{G}^{SUB}$
Step 2.2: Further conclusion
Similarly we show that $p \in \mathcal{G}^{SUP}$. Hence $p \in \mathcal{G}^{SUB} \cap \mathcal{G}^{SUP} = \mathcal{G}$. This shows that

$$(\chi\mathcal{G})' \subset \mathcal{G} \tag{8.19}$$

Conversely, notice that $\chi\mathcal{G} \subset \mathcal{G}$, and then by duality, we get

$$(\chi\mathcal{G})' \supset \mathcal{G}' = \mathcal{G}^D \cap \left\{ RH^f = 0 \right\} \tag{8.20}$$

with $\mathcal{G}^D := \left\{ p \in [a,b],\ D^f(p,q) \geq 0 \text{ for all } q \in \mathcal{G} \right\}$. From Lemma 6.2 on the D-maximality of Kružkov germs, we deduce that

$$\mathcal{G}^D = \mathcal{G} \subset \left\{ RH^f = 0 \right\}$$

where the last inclusion follows from the fact that $\mathcal{G}$ is a conservative (Kružkov) germ. Then (8.20) gives $(\chi\mathcal{G})' \supset \mathcal{G}' = \mathcal{G}$, and the reverse inclusion (8.19) implies the equality, i.e. (8.12). This ends the proof of the lemma.

Remark 8.18 *For $N = 1$ and for a junction of type $1:0$ or of type $0:1$ (where no Rankine-Hugoniot relation is required), results similar to Lemma 8.17 still hold. Precisely, we have*

$$\left\{ p \in [a,b], \quad D^f_+(p,q) \geq 0 \quad \textit{for all} \quad q \in \underline{\chi}\mathcal{G} \right\} = \mathcal{G}^{SUB}$$

$$\left\{ p \in [a,b], \quad D^f_+(q,p) \geq 0 \quad \textit{for all} \quad q \in \overline{\chi}\mathcal{G} \right\} = \mathcal{G}^{SUP}$$

and

$$\left\{ p \in [a,b], \quad D^f(p,q) \geq 0 \quad \textit{for all} \quad q \in \chi\mathcal{G} \right\} = \mathcal{G}$$

9 General examples and counter-examples

9.1 A monotone generalized germ which is not a Riemann germ

Lemma 9.1 (A monotone generalized germ which is not a Riemann germ on a $1:1$ junction) *Assume (2.2) with $N=2$ for $j=L,R$ with $J^L \simeq (-\infty,0)$ and $J^R \simeq (0,+\infty)$ and $[a,b]=[0,1]^2$. We consider Lipschitz continuous functions $f^j:[0,1]\to\mathbb{R}$ for $j=L,R$ with*

$$\left\{\begin{array}{l} 0<\hat{p}^L<\hat{q}^L<1\\ f^L \text{ decreasing}\end{array}\right. \quad \text{and} \quad \left\{\begin{array}{lll} 0<\hat{p}^R<\hat{q}^R<1 & & \\ f^R(\hat{p}^R)=f^R(\hat{q}^R)=0 & & \\ f^R<0 & \text{on} & [0,\hat{p}^R)\\ f^R>0 & \text{on} & (\hat{p}^R,\hat{q}^R)\cup(\hat{q}^R,1]\end{array}\right.$$

We set

$$\mathcal{G}=\{\hat{p},\hat{q}\} \quad \text{with} \quad \hat{p}=(\hat{p}^L,\hat{p}^R) \quad \text{and} \quad \hat{q}=(\hat{q}^L,\hat{q}^R)$$

Then $\mathcal{G}\subset[a,b]$ is a generalized Riemann germ with respect to (J,f), which is also a monotone germ. Moreover $\hat{f}=\hat{f}_{\mathcal{G}}$ satisfies

$$\left\{\begin{array}{lll} \hat{f}^R=0 & \text{on} & [a,b]=[0,1]^2\\ \hat{f}^L=\left\{\begin{array}{l} f^L(\hat{p}^L)\\ f^L(\hat{q}^L)\end{array}\right. & \begin{array}{l}\text{on}\\ \text{on}\end{array} & \begin{array}{l} BA(\hat{p})=[0,1]\times[0,\hat{q}^R)\\ BA(\hat{q})=[0,1]\times[\hat{q}^R,1]\end{array}\end{array}\right.$$

Hence $\hat{f}^L$ is discontinuous on $[0,1]\times\left\{\hat{q}^R\right\}$ with $\partial_{p^R}\hat{f}^L\le 0$ and $\mathcal{G}$ is not a Riemann germ.

9.2 A monotone germ with non Lipschitz Godunov flux

Lemma 9.2 (A monotone germ on a $2:0$ junction, with $\hat{f}$ continuous but not locally Lipschitz) *Set $N=2$ and $[a,b]=[0,1]^2$ and three C^1 functions*

$$\left\{\begin{array}{l} g:[0,1]\to[0,1] \text{ an increasing bijection (hence with } g(0)=0 \text{ and } g(1)=1)\\ f^1:[0,1]\to\mathbb{R} \text{ decreasing with } (f^1)'\le-\delta<0 \text{ on } [0,1]\\ f^2:[0,1]\to\mathbb{R} \text{ increasing with } (f^2)'\ge\delta>0 \text{ on } [0,1]\end{array}\right.$$

Then the set

$$\mathcal{G}:=\left\{p=(p^1,p^2)\in[a,b], \quad p^2=g(p^1)\right\}$$

is a Riemann germ, and satisfies $\mathcal{G}=\left\{\hat{f}=f\right\}$ with continuous function $\hat{f}=\hat{f}_{\mathcal{G}}$ given by

$$\left\{\begin{array}{l} \hat{f}^2(p):=f^2(p^2)\\ \hat{f}^1(p):=f^1(g^{-1}(p^2))\end{array}\right.$$

Then $\mathcal{G}$ is a monotone germ for a $2:0$ junction.
Moreover $\mathcal{G}$ is a Kružkov germ (for f) if and only if

$$\text{the map } (f^1+f^2\circ g):[0,1]\to\mathbb{R} \text{ is nondecreasing} \tag{9.1}$$

In particular, if $g'(0)=0$, then $\mathcal{G}$ is a monotone germ, is not a Kružkov germ and $\hat{f}$ is not locally Lipschitz.

Proof of Lemma 9.2
We just compute $D^f(p,q)=\text{sign}(p^1-q^1)\cdot\left\{f^1(p^1)-f^1(q^1)\right\}+\text{sign}(p^2-q^2)\cdot\left\{f^2(p^2)-f^2(q^2)\right\}$. Then for $p,q\in\mathcal{G}$, i.e. for $p^2=g(p^1)$ and $q^2=g(q^1)$, we get (with g increasing) that $D^f(p,q)=\text{sign}(p^1-q^1)\cdot\left[f^1+f^2\circ g\right]_{q^1}^{p^1}$, and then $D^f\ge 0$ on $\mathcal{G}^2$ if and only if (9.1) holds true. Moreover, if $g'(0)=0$, then $(f^1+f^2\circ g)'(0)=(f^1)'(0)\le-\delta$ and (9.1) does not hold true, which shows that $\mathcal{G}$ is not a Kružkov germ. This ends the proof of the lemma.

9.3 Counter-example to Riemann monotonicity for degenerate f

Lemma 9.3 (Counter-example to Riemann monotonicity for degenerate f on a $2:0$ junction) *Assume (2.2) with $N=2$, $J^j \simeq (-\infty,0)$ for $j=1,2$ and $[a,b]:=[0,1]^2$. We consider Lipschitz continuous functions $f^j:[0,1]\to\mathbb{R}$ for $j=1,2$ with*

$$\left\{\begin{array}{l} f^2=0\\ f^1 \quad \textit{decreasing}\\ h:[0,1]\to[0,1] \quad \textit{continuous increasing and bijective}\end{array}\right.$$

We set

$$\mathcal{G} := \left\{p=(p^1,p^2)\in[a,b], \quad p^1=h(p^2)\right\}$$

Then $\mathcal{G}$ is a Riemann germ with respect to (J,f) with

$$\left\{\begin{array}{l} \hat f^2=0\\ \hat f^1(p)=(f^1\circ h)(p^2)\end{array}\right.$$

Here $\hat f$ is continuous but not Riemann monotone. Moreover $\mathcal{G}$ is a monotone germ.

Proof of Lemma 9.3
Consider $p,q\in[a,b]$ with $p>q$, $\hat f^1(p)<\hat f^1(q)$, $\hat f^2(p)=0=\hat f^2(q)$. Hence $(p-q)\diamond[\hat f]_q^p\le 0$ but $[\hat f]_q^p\ne 0$. This ends the proof of the lemma.

9.4 Counter-example to gluing without Riemann monotonicity

Lemma 9.4 (Counter-example to gluing Riemann germs without Riemann monotonicity) *Assume (2.2) with $N=2$, $J^j\simeq(-\infty,0)$ for $j=1,2$ and $[a,b]:=[0,1]^2$. We consider Lipschitz continuous functions $f^j:[0,1]\to\mathbb{R}$ for $j=1,2$ with*

$$\left\{\begin{array}{l} f^2=0\\ f^1 \quad \textit{decreasing}\\ h:[0,1]\to[0,1] \quad \textit{continuous increasing and bijective}\end{array}\right.$$

We set

$$\mathcal{G} := \left\{p=(p^1,p^2)\in[a,b], \quad p^1=h(p^2)\right\} =: \bar{\mathcal{G}}$$

Let us define

$$\bar f=(\bar f^1,\bar f^2):=(-f^1,-f^2) \quad \textit{and} \quad \bar J:=\{0\}\cup\bigcup_{j=1,2}\bar J^2 \quad \textit{with} \quad \bar J^j:=-J^j \quad \textit{for} \quad j=1,2$$

Then $\mathcal{G}$ is a monotone Riemann germ with respect to (J,f) and $\bar{\mathcal{G}}$ is a monotone Riemann germ with respect to $(\bar J,\bar f)$ with $\hat{\bar f}=-\hat f$. Moreover both $\mathcal{G}$ and $\bar{\mathcal{G}}$ are not Riemann monotone and the set

$$\mathcal{G}_0 := \bar{\mathcal{G}}\sharp\mathcal{G} \quad \textit{for the gluing of branch } \bar J^2\simeq(0,+\infty) \textit{ with } J^2\simeq(-\infty,0)$$

defined as in (5.41) is such that $\mathcal{G}_0$ is a not a generalized Riemann germ.

Proof of Lemma 9.4
From Lemma 9.3, we know that $\mathcal{G}$ is a monotone Riemann germ with $\left\{\begin{array}{l} \hat f^2=0\\ \hat f^1(p)=(f^1\circ h)(p^2)\end{array}\right.$. Moreover, by construction, $\bar{\mathcal{G}}$ is obtained by reversion transform of $\mathcal{G}$ (see Definition 4.4), and then is also a monotone Riemann germ. Define the set for $f^0:=0=f^2=\bar f^2$

$$\mathcal{G}_0 := \left\{ (p^1,\bar p^1)\in[a^1,b^1]^2, \quad \text{there exists } (\bar p^2,p^2)\in[a^2,b^2]^2 \text{ s.t.} \quad \left|\begin{array}{l} (\bar p^1,\bar p^2)\in\bar{\mathcal{G}}\\ (p^1,p^2)\in\mathcal{G}\\ f^0(\bar p^2)=G^{f^0}(\bar p^2,p^2)=f^0(p^2)\end{array}\right.\right\}$$

By definition, we have $\mathcal{G}_0=\bar{\mathcal{G}}\sharp\mathcal{G}$ for the gluing of branch $\bar J^2\simeq(0,+\infty)$ with $J^2\simeq(-\infty,0)$. If $\hat f$ and $\hat{\bar f}$ would be Riemann monotone, then $\mathcal{G}_0\subset[a^1,b^1]^2$ would be a Riemann germ with respect to (J_0,f_0) with $f_0:=(f^1,\bar f^1)$ and $J_0:=\{0\}\cup J^1\cup\bar J^1$. But we have $\mathcal{G}_0=[a^1,b^1]^2$, and for any $q\in[a^1,b^1]^2$, we have

$$BA^{(J_0,f_0)}(q)=[a^1,b^1]^2$$

Therefore $(BA(\hat q))_{\hat q\in\mathcal{G}_0}$ is not a partition of $[a^1,b^1]^2$, which shows that $\mathcal{G}_0$ is a not a generalized Riemann germ. This ends the proof of the lemma.

9.5 Counter-examples to self-gluing of HJ germs

9.5.1 Self-gluing of a $2:1$ HJ germ

Lemma 9.5 (Counter-example to self-gluing of HJ germs)
Consider $g:[0,1]\to\mathbb{R}$ with $g(u):=u(1-u)$. For $N=3$, consider $f^0=f^1=f^2:=g$, $[a,b]:=[0,1]^3$ and branches $J^0\simeq(-\infty,0)$ and $J^1\simeq J^2\simeq(0,+\infty)$ and $J:=\{0\}\cup\bigcup_{j=0,1,2}J^j$. We set the nondecreasing and nonincreasing envelopes of g

$$g^+(u):=\max_{[0,u]}g \quad \text{and} \quad g^-(u):=\max_{[u,1]}g \quad \text{for all} \quad u\in[0,1]$$

Then the set

$$\mathcal{G}:=\left\{p=(p^0,p^1,p^2)\in[0,1]^3, \quad \min\left\{g^+(p^0),g^-(p^1),g^-(p^2)\right\}=g(p^0)=g(p^1)=g(p^2)\right\}$$

is a HJ germ with respect to (J,f), which is not a Kružkov germ.
Let $\mathcal{G}^{\sharp}_{2:0}$ be the self-gluing of $\mathcal{G}$ along branches J^2 and J^0 for the flux $f^2=f^0=g$. Then $\mathcal{G}^{\sharp}_{2:0}$ is not a generalized Riemann germ with respect to (J^1,f^1).

Proof of Lemma 9.5
Here $g:[0,1]\to\mathbb{R}$ is a bell-shaped flux. For $\gamma=(\gamma^0,\gamma^1,\gamma^2)$ and $N=3$, define the function $\gamma:[0,+\infty)^N\to[0,+\infty)^N$ as $\hat\gamma^j(\gamma):=\min\left\{\gamma^0,\gamma^1,\gamma^2\right\}$ for $j=0,1,2$. Then it is easy to check that $\hat\gamma$ is a HJ preflux in the sense of Definition 11.1. For the capacity $\bar\gamma(p):=(g^+(p^0),g^-(p^1),g^-(p^2))$, we then deduce that $\gamma:=\hat\gamma\circ\bar\gamma$ is a Godunov flux, and that $\mathcal{G}$ is the associated HJ germ. We set $G^g:[0,1]^2\to\mathbb{R}$ for the standard Godunov flux associated to the flux g, which satisfies here $G^g(p^L,p^R)=\min(g^+(p^L),g^-(p^R))$. Then recall that the standard Godunov germ is defined as $\mathcal{G}^g:=\left\{p=(p^L,p^R)\in[0,1]^2,\ g(p^L)=G^g(p^L,p^R)=g(p^R)\right\}$. Then, by definition, we have

$$\mathcal{G}^{\sharp}_{2:0}=\left\{p^1\in[0,1], \quad \text{there exists } (p^2,p^0)\in\mathcal{G}^g \text{ such that } p=(p^0,p^1,p^2)\in\mathcal{G}\right\}$$

For $\lambda\in[0,\frac14]$, we define $u_\pm(\lambda):=(g^\pm)^{-1}(\lambda)$. Now for any $p^1\in[0,1]$, set $\lambda:=g(p^1)$ and $p^0:=u_+(\lambda)=:p^2$. Hence it is straightforward to check that $(p^2,p^0)\in\mathcal{G}^g$ and $(p^0,p^1,p^2)\in\mathcal{G}$, which implies that $p^1\in\mathcal{G}^{\sharp}_{0:2}$. Therefore $\mathcal{G}^{\sharp}_{2:0}=[0,1]$ and $\left(BA^{(J^1,f^1)}(\hat p^1)\right)_{\hat p^1\in\mathcal{G}^{\sharp}_{0:2}}$ is not a partition of $[0,1]$, by definition of the Basin of Attraction. Therefore Theorem 2.15 implies that $\mathcal{G}^{\sharp}_{2:0}$ is not a generalized Riemann germ (with respect to (J^1,f^1)). Finally Corollary 5.17 implies that $\mathcal{G}$ is not a Kružkov germ. This ends the proof of the lemma.

9.5.2 Self-gluing of a $2:2$ HJ germ

Consider a function g as follows

$$(9.2)\qquad g:[0,1]\to[0,+\infty) \quad \text{stricly concave with} \quad \left\{\begin{array}{l} g(0)=0=g(1)\\ \text{and maximum at } c\in(0,1) \text{ with } g_{\max}:=g(c)\end{array}\right.$$

We will also need the monotone envelopes

$$g^+(u):=\left\{\begin{array}{lll} g(u) & \text{for} & u\in[0,c]\\ g(c) & \text{for} & u\in[c,1]\end{array}\right. \quad \text{and} \quad g^-(u):=\left\{\begin{array}{lll} g(c) & \text{for} & u\in[0,c]\\ g(u) & \text{for} & u\in[c,1]\end{array}\right.$$

We will consider the self-gluing of some Hamilton-Jacobi germ $\mathcal{G}$ for a $2:2$ junction. For $(u^1,u^2,u^3,u^4)\in[0,1]^4$, let us define

$$(9.3)\qquad F(u^1,u^2,u^3,u^4)=\min\left\{g^+(u^1),g^+(u^2),g^-(u^3),g^-(u^4)\right\}$$

with indices $1,2$ for ingoing branches and indices $3,4$ for outgoing branches. Then $\mathcal{G}$ is a Hamilton-Jacobi germ given by

$$\mathcal{G}:=\left\{(u^1,u^2,u^3,u^4)\in[0,1]^4, \quad F(u^1,u^2,u^3,u^4)=g(u^1)=g(u^2)=g(u^3)=g(u^4)\right\}$$

with flux $f=(g,g,g,g)$ and Godunov flux $\hat f_{\mathcal{G}}=(F,F,F,F)$.

Now imagine that we want to realize the self-gluing set $\mathcal{G}^\sharp$ of the germ $\mathcal{G}$, gluing the outgoing branch 4 with the ingoing branch 2, i.e. gluing J^4 with J^2. This can for instance be done for some common state $u^2 = u^4$ and gives a passing flux equal to

$$F^\sharp(u^1,u^3) := F(u^1,u^2,u^3,u^2) = \min\left\{g^+(u^1), g^-(u^3), A\right\} \quad \text{with} \quad A := g(u^2)$$

which appears to be a junction with flux limiter depending on the internal state u^2. If such self-gluing set $\mathcal{G}^\sharp$ is a germ, then it has to be independent on any admissible choice of $u^2 = u^4$, which is not the case here. The situation is similar if we do the gluing with $u^2 \neq u^4$ but such that (u^4, u^2) is a stationary shock solution of

$$g(u^4) = G^g(u^4,u^2) = g(u^2)$$

where G^g is the Godunov flux associated to g. Because of the special shape of g, we have in particular $G^g(u^L,u^R) = \min(g^+(u^L), g^-(u^R))$.

An alternative way to realize that the self-gluing is not well-defined, is to consider $r = u^2 = u^4$ such that

$$(\hat{f}^2_{\mathcal{G}} - \hat{f}^4_{\mathcal{G}})(u^1,r,u^3,r) = 0$$

which does not impose any condition on r, because the fact that $\mathcal{G}$ is a Hamilton-Jacobi germ precisely implies that $\hat{f}_{\mathcal{G}} = (F,F,F,F)$. Nevertheless $F(u^1,r,u^3,r)$ still depends on r, and then does not define uniquely what should be the Godunov flux of a self-glued germ if it would exist.

Remark 9.6 (Construction of $\mathcal{G}$ by gluing two different germs)
It is furthermore possible to see that the germ $\mathcal{G}$ is itself already obtained by gluing of two Hamilton-Jacobi germs $\mathcal{G}_\alpha$ and $\mathcal{G}_\beta$, i.e. that $\mathcal{G} = \mathcal{G}_\alpha \sharp \mathcal{G}_\beta$. Here $\mathcal{G}_\alpha$ is of type $2:1$ with fluxes $(f^1_\alpha, f^2_\alpha, f^0_\alpha) = (g,g,2g)$ (where $2g$ is the flux on the outgoing branch) and $\mathcal{G}_\beta$ of type $1:2$ with fluxes $(f^0_\beta, f^3_\beta, f^4_\beta) = (2g,g,g)$ (where $2g$ is the flux on the ingoing branch). We set

$$\left\{\begin{array}{l} \mathcal{G}_\alpha := \left\{(u^1,u^2,u^0) \in [0,1]^3, \quad F_\alpha(u^1,u^2,u^0) = g(u^1) = g(u^2) = 2g(u^0)\right\} \\[2mm] \text{with} \quad F_\alpha(u^1,u^2,u^0) := \min\left\{g^+(u^1), g^+(u^2), 2g^-(u^0)\right\} \end{array}\right.$$

and

$$\left\{\begin{array}{l} \mathcal{G}_\beta := \left\{(u^0,u^3,u^4) \in [0,1]^3, \quad F_\beta(u^0,u^3,u^4) = 2g(u^0) = g(u^3) = g(u^4)\right\} \\[2mm] \text{with} \quad F_\beta(u^0,u^3,u^4) := \min\left\{2g^+(u^0), g^-(u^3), g^-(u^4)\right\} \end{array}\right.$$

Here we have the associated Godunov fluxes $\hat{f}_{\mathcal{G}_\alpha} = (F_\alpha, F_\alpha, F_\alpha)$ and $\hat{f}_{\mathcal{G}_\beta} = (F_\beta, F_\beta, F_\beta)$. Then we define for $(u^1,u^2,u^3,u^4) \in [0,1]^4$

$$F(u^1,u^2,u^3,u^4) := F_\alpha(u^1,u^2,u^0) = F_\beta(u^0,u^3,u^4) \quad \text{for some } u^0 \in [0,1]$$

where from Proposition 5.13, we know that the value of F is independent on the admissible value of u^0. It is furthermore easy to see that F satisfies (9.3).

Remark 9.7 (A different situation for loops in particular Hamilton-Jacobi networks)
Notice that for networks with Hamilton-Jacobi germs at each node, it is possible to restrict the framework of solutions u (as usual solution to conservation laws) such that it satisfies moreover locally $u = v_x$ for some function v which is continuous at every node of the network. Under this additional assumption, we see that if the function u is constant on a loop of the network, then this implies that $u = 0$ on this loop. Therefore in this particular and more restrictive framework, we see that the value of u on each (arbitrarily small) loop may be no longer free, but has to be taken equal to zero.

9.6 Strange germs for $2:0$ junctions and classification

Lemma 9.8 (Strange germs for $2:0$ junctions and classification; $f^1\downarrow, f^2\uparrow$)
Assume (2.2) for a with $N = 2$, $J^j \simeq (-\infty,0)$ for $j = 1,2$ and $[a,b] := [0,1]^2$. We set $J := \{0\} \cup J^1 \cup J^2$. We consider Lipschitz continuous functions $f^j : [0,1] \to \mathbb{R}$ for $j = 1,2$ with

$$f^1 \quad \textit{decreasing and} \quad f^2 \quad \textit{increasing.}$$

i) (Classification of Riemann germs)
Then the set $\mathcal{G} \subset [a,b]$ *is a Riemann germ with respect to* (J,f) *if and only if there exists* $h : [0,1] \to [0,1]$ *continuous such that*

$$\mathcal{G} = \mathcal{G}_h \quad \text{with} \quad \mathcal{G}_h := \left\{p = (p^1,p^2) \in [a,b], \quad p^1 = h(p^2)\right\} \tag{9.4}$$

We also have $\mathcal{G}_h = \left\{\hat{f}_h = f\right\}$ *with continuous function* $\hat{f}_h = \hat{f}_{\mathcal{G}_h}$ *given by*

$$\begin{cases} \hat{f}^2_h(p) := f^2(p^2) \\ \hat{f}^1_h(p) := f^1(h(p^2)) \end{cases} \tag{9.5}$$

ii) (Existence of strange germs, nonconservative Kružkov germs)
If h *is not monotone, then the Riemann germ* $\mathcal{G}$ *is not monotone, not HJ, neither conservative. Moreover, if* f^1, f^2, h *are Lipschitz continuous, then the Riemann germ* $\mathcal{G}$ *is Kružkov if and only if*

$$(f^2)' + \left\{(f^1)' \circ h\right\} \cdot |h'| \geq 0 \quad a.e. \ on \quad [0,1] \tag{9.6}$$

with the convention that $g \cdot |h'| = 0$ *if* $h' = 0$*, even where* g *is not defined. In particular* $\mathcal{G}$ *is not Kružkov if* $|h'|$ *is large enough.*

On the contrary, if $(f^2)' \geq \delta > 0$ *a.e. on* $[0,1]$*, and for* $|h'|$ *small enough with* h *non monotone, then* $\mathcal{G}$ *is a nonmonotone nonconservative Kružkov germ.*

Moreover if h *is nondecreasing such that the function* $f^2 + f^1 \circ h$ *is nondecreasing and non identically equal to zero, then* $\mathcal{G}$ *is a monotone nonconservative Kružkov germ.*

Proof of Lemma 9.8
Step 1: proof of i)
Step 1.1: necessary inclusion of $\mathcal{G}$
From the slicing lemma 4.12, notice that if $\mathcal{G} \subset [a,b]$ is a Riemann germ with respect to (J,f) with $\hat{f} := \hat{f}_{\mathcal{G}}$, then for any fixed $p^2 \in [0,1]$, the set

$$\mathcal{G}_{p^2} := \left\{p^1 \in [0,1], \quad \hat{f}^1(p^1,p^2) = f^1(p^1)\right\}$$

is a generalized Riemann germ with respect to (J^1, f^1), which is also a Riemann germ because $\hat{f}$ is continuous. Because f^1 is decreasing and the map $\hat{f}^1_{p^2} : p^1 \mapsto \hat{f}^1(p^1,p^2)$ is nondecreasing and locally constant on $\left\{\hat{f}^1_{p^2} \neq f^1\right\}$, we deduce that $\hat{f}^1_{p^2}$ is constant on $[0,1]$. Moreover, we know that we have $\hat{f}^1_{p^2} = f^1 \circ \pi_{\mathcal{G}_{p^2}}$. Because f^1 is decreasing, we deduce that $\pi_{\mathcal{G}_{p^2}} : [0,1] \to [0,1]$ is constant and set $h(p^2) := \pi_{\mathcal{G}_{p^2}}(p^1)$ for any $p^1 \in [0,1]$. Hence we have $\hat{f}^1(p^1,p^2) := f^1(h(p^2))$. Still because f^1 is decreasing, and $\hat{f}^1$ is continuous, we deduce that $h : [0,1] \to [0,1]$ is continuous. Therefore

$$\mathcal{G} \subset \left\{\hat{f}^1 = f^1\right\} = \mathcal{G}_h \quad \text{with continuous} \quad h : [0,1] \to [0,1].$$

It is also easy to check that $\mathcal{G}_h = \left\{\hat{f}_h = f\right\}$.
Step 1.2: property of $\mathcal{G}_h$
Conversely, consider $\mathcal{G}_h = \left\{\hat{f}_h = f\right\}$ with $\hat{f}_h$ given in (9.5) for continuous $h : [0,1] \to [0,1]$. Notice also that because f^2 is increasing, we see that $\hat{f}^j_h$ has expected monotonicities in p^j. Moreover it is easy to check that $\hat{f}_h$ is locally constant on $\left\{\hat{f}_h \neq f\right\}$. Now from (2.15) and from the monotonicities of the f^j's, we get

$$f^1_- = f^1 \leq f^1(0) \equiv f^1_+ \quad \text{and} \quad f^2_- \equiv f^2(0) \leq f^2 = f^2_+$$

Hence from ii) of Theorem 2.15, we deduce that $\mathcal{G}_h$ is a generalized Riemann germ, and from i) of Theorem 2.17, we deduce that $\mathcal{G}_h$ is a Riemann germ.
Step 1.3: conclusion for $\mathcal{G}$
Now from Step 1.1, we have $\mathcal{G} \subset \mathcal{G}_h \subset [a,b]$ with both $\mathcal{G}$ and $\mathcal{G}_h$ generalized Riemann germs (in $[a,b]$) with respect to (J,f). From i) of Theorem 2.15, we conclude that $\mathcal{G} = \mathcal{G}_h$.
Step 2: proof of ii)

For conservative, HJ and monotone germs, we use the definitions of the classes of germs and also Lemma 5.5 for their characterization in term of their Godunov fluxes $\hat{f}$.

Hence the Riemann germ $\mathcal{G}$ is conservative if and only if $\hat{f}^1 + \hat{f}^2 = 0$, i.e. $-f^1(h(p^2)) = f^2(p^2)$, i.e. the (continuous) function $h := (-f^1)^{-1} \circ f^2$ is increasing.

Similarly, the Riemann germ $\mathcal{G}$ is HJ if and only if $\hat{f}^1 = \hat{f}^2$ and is nondecreasing in each coordinate p^j, i.e. $h := (f^1)^{-1} \circ f^2$ is decreasing.

Similarly, the Riemann germ $\mathcal{G}$ is monotone if and only if $p^2 \mapsto \hat{f}^1(p)$ and $p^1 \mapsto \hat{f}^2(p)$ are nonincreasing, i.e. h is nondecreasing.

By definition, the Riemann germ $\mathcal{G}$ is Kružkov if and only if we have $D^f(\bar{p}, p) \geq 0$ for all $p, \bar{p} \in \mathcal{G}$, i.e.

(9.7)
$$\text{sign}(\bar{p}^1 - p^1)\cdot\left\{f^1(\bar{p}^1) - f^1(p^1)\right\} + \text{sign}(\bar{p}^2 - p^2)\cdot\left\{f^2(\bar{p}^2) - f^2(p^2)\right\} \geq 0 \quad \text{for all} \quad p^1 := h(p^2), \quad \bar{p}^1 = h(\bar{p}^2), \quad p^2, \bar{p}^2 \in [0,1]$$

Because the composition of Lipschitz functions is Lipschitz, and using Rademacher's theorem, we deduce (9.6) in the limit $|\bar{p}^2 - p^2| \to 0$. Conversely, the integration of (9.6) implies (9.7).

In particular, if h is nondecreasing such that the function $f^2 + f^1 \circ h$ is also nondecreasing and non identically equal to zero, then $\mathcal{G}$ is a monotone nonconservative Kružkov germ. This ends the proof of the lemma.

Lemma 9.9 (Classification for $2:0$ junctions; $f^1 \downarrow, f^2 \downarrow$)
Assume (2.2) for a with $N = 2$, $J^j \simeq (-\infty, 0)$ for $j = 1, 2$ and $[a, b] := [0, 1]^2$. We set $J := \{0\} \cup J^1 \cup J^2$. We consider Lipschitz continuous functions $f^j : [0, 1] \to \mathbb{R}$ for $j = 1, 2$ with

$$f^1 \textit{ and } f^2 \textit{ decreasing.}$$

Then the set $\mathcal{G} \subset [a, b]$ is a Riemann germ with respect to (f, J) if and only if there exists $\hat{p} \in [a, b]$ such that $\mathcal{G} = \{\hat{p}\}$. When it is the case, then we also have $\mathcal{G} = \left\{\hat{f} = f\right\}$ with the function $\hat{f} = \hat{f}_{\mathcal{G}}$ given by $\hat{f} = const = f(\hat{p})$.

Proof of Lemma 9.9
Consider some $p \in [a, b]$, and set $\hat{p} := \pi(p)$ with $\pi := \pi_{\mathcal{G}} : [a, b] \to \mathcal{G}$ the natural projection map. Because f^1, f^2 are both decreasing for a $2:0$ junction, we deduce that $BA(\hat{p}) = [a, b]$, and then $\pi = const = \hat{p}$, which shows that $\mathcal{G} = \{\hat{p}\}$. Moreover $\hat{f}_{\mathcal{G}} = f(\hat{p})$. Conversely, for any $\hat{p} \in [a, b]$, it is straightforward to check that $\mathcal{G} = \{\hat{p}\}$ is a Riemann germ. This ends the proof of the lemma.

Lemma 9.10 (Classification for $2:0$ junctions; $f^1 \uparrow, f^2 \uparrow$)
Assume (2.2) for a with $N = 2$, $J^j \simeq (-\infty, 0)$ for $j = 1, 2$ and $[a, b] := [0, 1]^2$. We set $J := \{0\} \cup J^1 \cup J^2$. We consider Lipschitz continuous functions $f^j : [0, 1] \to \mathbb{R}$ for $j = 1, 2$ with

$$f^1 \textit{ and } f^2 \textit{ increasing.}$$

i) (Classification)
Then the set $\mathcal{G} \subset [a, b]$ is a Riemann germ with respect to (f, J) if and only if there exist continuous maps $h^j_\pm : [0, 1] \to [0, 1]$ for $j = 1, 2$ such that $0 \leq h^j_- \leq h^j_+ \leq 1$, and for $\bar{j} \in \{1, 2\} \setminus \{j\}$, we have

$$\mathcal{G} = K^1 \cap K^2 \quad \textit{with} \quad K^j := \left\{p \in [a, b], \quad h^j_-(p^{\bar{j}}) \leq p^j \leq h^j_+(p^{\bar{j}})\right\}$$

with moreover $\mathcal{G} = \left\{\hat{f} = f\right\}$, with $\hat{f}$ locally constant on $\left\{\hat{f} \neq f\right\}$, where $\hat{f} : [a, b] \to \mathbb{R}^2$ is continuous and satisfies for $j = 1, 2$

$$\hat{f}^j(p) = f^j(T^{h^j_+(p^{\bar{j}})}_{h^j_-(p^{\bar{j}})}(p^j)) \quad \textit{with} \quad T^y_x(z) = x \vee z \wedge y \quad \textit{for} \quad x \leq y \tag{9.8}$$

and $x \vee z := \max(x, z)$, $y \wedge z := \min(y, z)$.
ii) (Further properties)

Moreover, we have the following properties for any $p \in [0,1]^2$ *with* $\hat{p} := \pi_{\mathcal{G}}(p) \in [0,1]^2$*, we have*

$$(9.9) \quad \begin{cases} \left((\hat{f}-f)^1(p) > 0 \quad \textit{and} \quad (\hat{f}-f)^2(p) > 0\right) & \textit{implies} \quad BA(\hat{p}) \supset \quad [(0,0),\hat{p}] & \ni p \\ \left((\hat{f}-f)^1(p) > 0 \quad \textit{and} \quad (\hat{f}-f)^2(p) < 0\right) & \textit{implies} \quad BA(\hat{p}) \supset \quad [(0,\hat{p}^2),(\hat{p}^1,1)] & \ni p \\ \left((\hat{f}-f)^1(p) < 0 \quad \textit{and} \quad (\hat{f}-f)^2(p) > 0\right) & \textit{implies} \quad BA(\hat{p}) \supset \quad [(\hat{p}^1,0),(1,\hat{p}^2)] & \ni p \\ \left((\hat{f}-f)^1(p) < 0 \quad \textit{and} \quad (\hat{f}-f)^2(p) < 0\right) & \textit{implies} \quad BA(\hat{p}) \supset \quad [\hat{p},(1,1)] & \ni p \end{cases}$$

Proof of Lemma 9.10
Part 1: proof of i)
Step 1: necessary conditions on $\mathcal{G}$
Step 1.1: freezing p^2
From the slicing lemma 4.12, notice that if $\mathcal{G} \subset [a,b]$ is a Riemann germ with respect to (J,f) with $\hat{f} := \hat{f}_{\mathcal{G}}$, then for any fixed $p^2 \in [0,1]$, the set

$$\mathcal{G}_{p^2} := \left\{p^1 \in [0,1], \quad \hat{f}^1(p^1,p^2) = f^1(p^1)\right\}$$

is a generalized Riemann germ with respect to (J^1,f^1), which is also a Riemann germ because $\hat{f}$ is continuous. Because f^1 is increasing and the map $\hat{f}^1_{p^2} : p^1 \mapsto \hat{f}^1(p^1,p^2)$ is nondecreasing and locally constant on $\left\{\hat{f}^1_{p^2} \neq f^1\right\}$, we deduce that $\hat{f}^1_{p^2}$ coincides with f^1 only on a subinterval $[h^1_-(p^2), h^1_+(p^2)] \subset [0,1]$, and satisfies

$$\hat{f}^1_{p^2}(p^1) = f^1(T^{z_+}_{z_-}(p^1)) \quad \text{with} \quad \begin{cases} z_- := h^1_-(p^2) \\ z_+ := h^1_+(p^2) \end{cases}$$

Because $\hat{f}^1$ is continuous, we deduce that $\left\{\hat{f}^1 = f^1\right\}$ is a closed set, and then h^1_- is lower semicontinuous and h^1_+ is upper semicontinuous. Moreover, it is easy to see that the strict monotonicity of f^1 and the continuity of $\hat{f}^1$ also imply the continuity of $h^1_\pm$.
Step 1.2: freezing p^1 and first consequences
By symmetry from Step 1, we get a similar result, exchanging indices 1 and 2. This shows (9.8) with continuous maps $h^j_\pm : [0,1] \to [0,1]$. Moreover, this implies that $\mathcal{G} = \left\{\hat{f} = f\right\} = K^1 \cap K^2$, and $\hat{f}$ is locally constant on $\left\{\hat{f} \neq f\right\}$ and $\hat{f} : [a,b] \to \mathbb{R}^2$ is continuous from i) of Theorem 2.17, because $\mathcal{G}$ is a Riemann germ.
Step 2: sufficient conditions for $\mathcal{G}$
From ii) of Theorem 2.14, we only have to check the second line of (2.14). Recall that

$$f^j(p^j) = \inf_{[p^j,1]} f^j = f^j_-(p^j) \le f^j_+(p^j) = \sup_{[0,p^j]} f^j = f^j(1)$$

and because f^j is increasing. We deduce from the expression of $\hat{f}^j$ that $f^j_- \le \hat{f}^j \le f^j_+$, and Theorem 2.14 implies that $\mathcal{G} = \left\{\hat{f} = f\right\}$ is a generalized Riemann germ. Because $\hat{f}$ is continuous, $\mathcal{G}$ is then a Riemann germ.
Part 2: proof of ii)
We only do the proof for the first line of (9.9) (the other cases are similar). Assume that $(\hat{f}^1 - f^1)(p) > 0$, $(\hat{f}^2 - f^2)(p) > 0$. Because f^1, f^2 are increasing, we deduce that $BA(\hat{p}) \cap [0_{\mathbb{R}^2}, \hat{p}] = [0_{\mathbb{R}^2}, \hat{p}]$, which shows the result with moreover $\hat{f} = f(\hat{p})$ on $[0_{\mathbb{R}^2}, \hat{p}]$. This ends the proof of the lemma.

9.7 An explicit example of gluing without cancellation property

Lemma 9.11 (A $1:1$ explicit example of gluing without cancellation property)
Assume (2.2) with $N=2$*,* $J^L \simeq (-\infty,0)$ *and* $J^R \simeq (0,+\infty)$ *and* $f^j = g$ *with* $[a^j,b^j] = [0,1]$ *for* $j = L,R$*, with* $g : [0,1] \to \mathbb{R}$ *strictly concave with* $g(0) = 0 = g(1)$*. Let* $A_0 := \max_{[0,1]} g = g(p_0) > 0$ *with* $p_0 \in (0,1)$ *and* $A \in [0,A_0]$*. We define*

$$g^+(x) := \begin{cases} g(x) & \textit{for} \quad x \in [0,p_0] \\ g(p_0) & \textit{for} \quad x \in (p_0,1] \end{cases} \quad \textit{and} \quad g^-(x) := \begin{cases} g(p_0) & \textit{for} \quad x \in [0,p_0] \\ g(x) & \textit{for} \quad x \in (p_0,1] \end{cases}$$

and

$$\mathcal{G}_A := \left\{(p^L,p^R)\in[0,1]^2, \quad \min\left\{A, g^+(p^L), g^-(p^R)\right\} = g(p^L) = g(p^R)\right\}$$

Then for any $A, B \in [0, A_0]$*, we have*

$$\mathcal{G}_A \sharp \mathcal{G}_B = \mathcal{G}_{\min\{A,B\}} \quad \text{for the gluing of } J^R_{\mathcal{G}_A} \simeq (0,+\infty) \text{ with } J^L_{\mathcal{G}_B} \simeq (-\infty, 0)$$

In particular we always have

$$\mathcal{G}_{A_0}\sharp\mathcal{G}_A = \mathcal{G}_A\sharp\mathcal{G}_A \quad \text{and} \quad \mathcal{G}_A\sharp\mathcal{G}_{A_0} = \mathcal{G}_A\sharp\mathcal{G}_A$$

which does not imply the cancellation property that $\mathcal{G}_A = \mathcal{G}_{A_0}$*.*

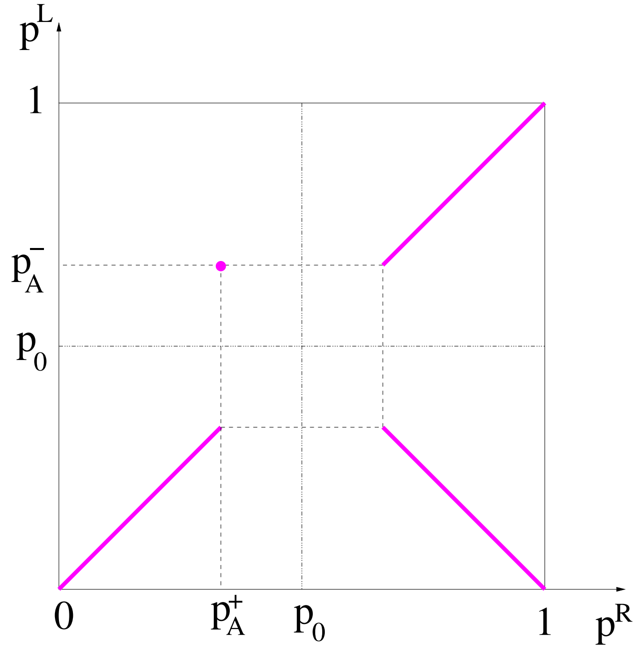

Figure 2: The germ $\mathcal{G}_A$ for $A \in (0, A_0)$

Proof of Lemma 9.11

For $\lambda \in [0, A_0]$, let $p^\pm_\lambda \in [0,1]$ be such that $g(p^\pm_\lambda) = \lambda = g^\pm(p^\pm_\lambda)$. Notice that $\mathcal{G}_\lambda$ is a HJ germ. From [29], we know that all HJ germs with all convex fluxes (and then also with all concave fluxes) are classified by a flux limiter, which is λ, and particularly by a single point $(p^-_\lambda, p^+_\lambda) \in \mathcal{G}_\lambda$, because $\chi\mathcal{G}_\lambda = \left\{(p^-_\lambda, p^+_\lambda)\right\}$, where $\chi\mathcal{G}_\lambda$ is the characteristic subset of the HJ germ $\mathcal{G}_\lambda$ (see Theorem 2.34). Let

$$\tilde{\mathcal{G}_0} := \mathcal{G}_A\sharp\mathcal{G}_B = \left\{ (p^L,\tilde{p}^R)\in[0,1]^2, \quad \text{there exists } (\tilde{p}^L, p^R)\in[0,1]^2 \text{ s.t.} \quad \left| \begin{array}{l} (p^L,p^R)\in\mathcal{G}_A \\ (\tilde{p}^L,\tilde{p}^R)\in\mathcal{G}_B \\ g(p^R) = G^g(p^R,\tilde{p}^L) = g(\tilde{p}^L) \end{array} \right. \right\}$$

From Theorem 2.24, we know that $\tilde{\mathcal{G}_0}$ is a HJ germ, and then of the form $\tilde{\mathcal{G}_0} = \mathcal{G}_\lambda$ for some $\lambda \in [0, A_0]$. Moreover $(p^L, \tilde{p}^R) \in G_0$ if and only if (using $G^g(p^R, \tilde{p}^L) = \min\left\{g^+(p^R), g^-(\tilde{p}^L)\right\}$)

$$\left\{ \begin{array}{l} \min\left\{A, g^+(p^L), g^-(p^R)\right\} = g(p^L) = g(p^R) \\ \min\left\{B, g^+(\tilde{p}^L), g^-(\tilde{p}^R)\right\} = g(\tilde{p}^L) = g(\tilde{p}^R) \\ g(p^R) = \min\left\{g^+(p^R), g^-(\tilde{p}^L)\right\} = g(\tilde{p}^L) \end{array} \right.$$

For $C := \min\{A, B\}$, we have (with obvious notation for the gluing $(p^L,p^R)\sharp(\tilde{p}^L,\tilde{p}^R) := (p^L,\tilde{p}^R)$)

$$\tilde{\mathcal{G}_0} \ni (p^-_C, p^+_C) = \left\{ \begin{array}{ll} (p^-_C,p^+_C)\sharp(p^+_C,p^+_C) & \text{if} \quad C = A \\ (p^-_C,p^-_C)\sharp(p^-_C,p^+_C) & \text{if} \quad C = B \end{array} \right.$$

We deduce that $\lambda = C$ and $\tilde{\mathcal{G}_0} = \mathcal{G}_C$ which ends the proof of the lemma.

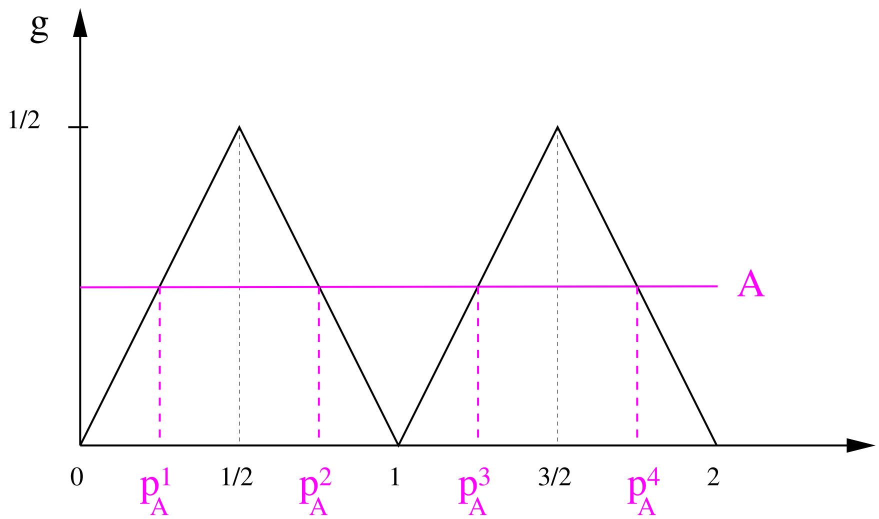

Figure 3: Graph of g

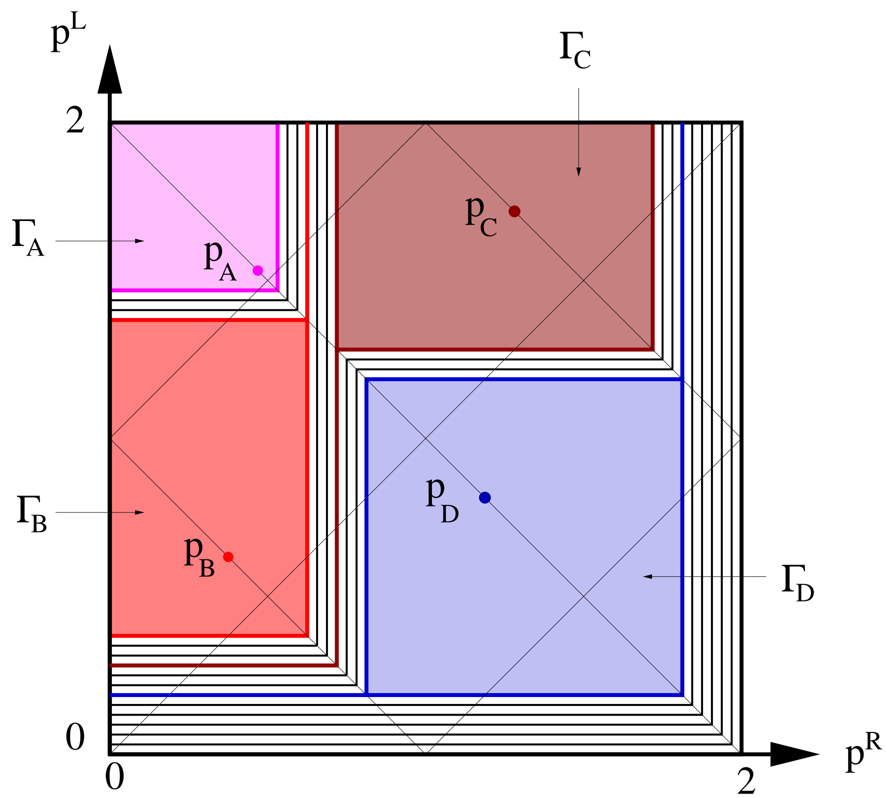

Figure 4: Level sets of $\hat{h}$ with four plateaux (see (9.10))

9.8 An example of non commutativity of the gluing

Lemma 9.12 (Explicit example of a HJ germ for $1:1$ junction)
We set $J := \{0\} \cup J^L \cup J^R$ with $J^L \simeq (-\infty, 0)$ and $J^R \simeq (0, +\infty)$. We also set $f = (f^L, f^R) := (g, g)$ and $[a, b] := [0, 2]^2$, with

$$g : [0,2] \to \mathbb{R} \quad \text{with} \quad g(u) := g_0(u) + g_0(u-1) \quad \text{and} \quad g_0(u) := \max\{0, \min\{u, 1-u\}\}$$

Given $\mathcal{A} := (A, B, C, D)$ with $\frac{1}{2} > A > B > C > D > 0$, we want to define a germ $\mathcal{G}_{\mathcal{A}}$. To this end, given

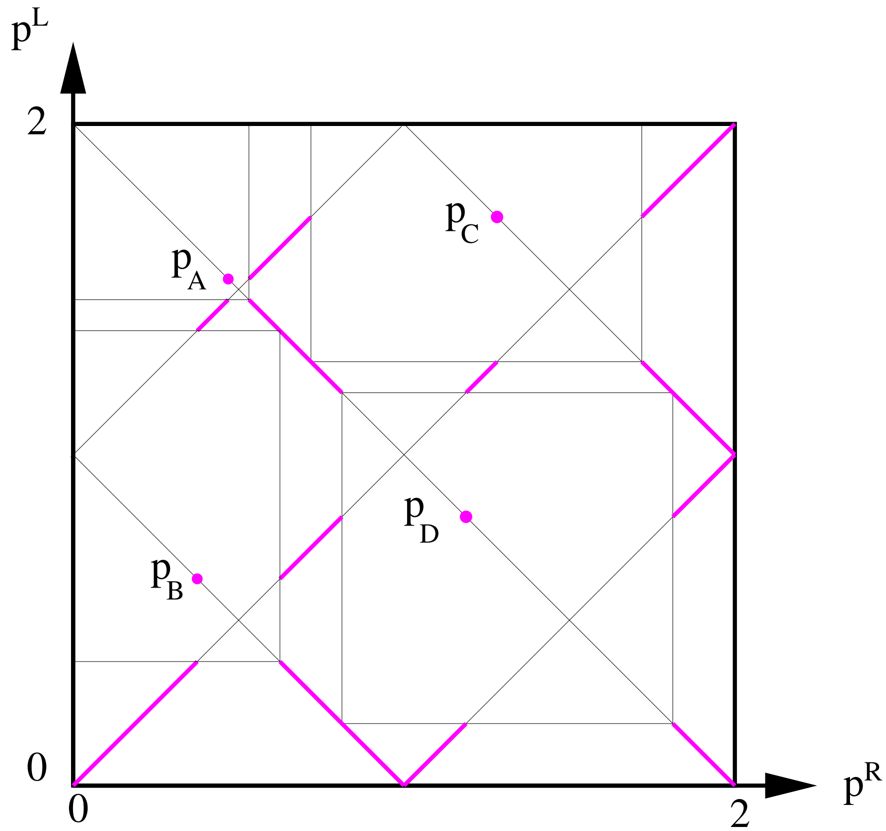

Figure 5: HJ Germ $\mathcal{G}_{\mathcal{A}} \subset \big\{g(p^L) = g(p^R)\big\}$ and characteristic subset $\chi\mathcal{G}_{\mathcal{A}} = \{p_A, p_B, p_C, p_D\}$

$\lambda \in [0, \frac{1}{2}]$, *we set*

$$p^1_\lambda := \lambda, \quad p^2_\lambda = 1-\lambda, \quad p^3_\lambda := 1+\lambda, \quad p^4_\lambda := 2-\lambda \quad \textit{with} \quad \begin{cases} g(p^k_\lambda) = \lambda \quad \textit{for} \quad k = 1, \dots, 4 \\ 0 \le p^1_\lambda \le p^2_\lambda \le p^3_\lambda \le p^4_\lambda \le 2 \end{cases}$$

We also set for $p = (p^L, p^R)$

$$\begin{cases} p_A := (p^4_A, p^1_A) \\ p_B := (p^2_B, p^1_B) \\ p_C := (p^4_C, p^3_C) \\ p_D := (p^2_D, p^3_D) \end{cases}$$

We then define for $p = (p^L, p^R) \in [a, b] := [0, 2]^2$

$$\hat{h}(p) := \lambda \quad \textit{for} \quad p \in \Gamma_\lambda \tag{9.10}$$

with

$$\Gamma_\lambda := \begin{cases} \big[p^3_A, 2\big] \times \big[0, p^1_A\big] & \textit{for} \quad \lambda = A \\ \Gamma^{AB}_\lambda := \big(\{p^3_\lambda\} \times \big[0, p^2_\lambda\big]\big) \cup \big(\big[p^3_\lambda, 2\big] \times \{p^2_\lambda\}\big) & \textit{for} \quad \lambda \in (A, B) \\ \big(\big[p^1_B, p^3_B\big] \times \big[0, p^2_B\big]\big) \cup \Gamma^{AB}_B & \textit{for} \quad \lambda = B \\ \Gamma^{BC}_\lambda := \big(\{p^1_\lambda\} \times \big[0, p^2_\lambda\big]\big) \cup \big(\big[p^1_\lambda, 2\big] \times \{p^2_\lambda\}\big) & \textit{for} \quad \lambda \in (B, C) \\ \big(\big[p^3_C, 2\big] \times \big[p^2_C, p^4_C\big]\big) \cup \Gamma^{BC}_C & \textit{for} \quad \lambda = C \\ \Gamma^{CD}_\lambda := \big(\{p^1_\lambda\} \times \big[0, p^2_\lambda\big]\big) \cup \big(\big[p^1_\lambda, p^3_\lambda\big] \times \{p^2_\lambda\}\big) \cup \big(\{p^3_\lambda\} \times \big[p^2_\lambda, p^4_\lambda\big]\big) \cup \big(\big[p^3_\lambda, 2\big] \times \{p^4_\lambda\}\big) & \textit{for} \quad \lambda \in (C, D) \\ \big(\big[p^1_\lambda, p^3_\lambda\big] \times \big[p^2_\lambda, p^4_\lambda\big]\big) \times \Gamma^{CD}_D & \textit{for} \quad \lambda = D \\ \Gamma^{D0}_\lambda := \big(\{p^1_\lambda\} \times \big[p^1_\lambda, p^4_\lambda\big]\big) \cup \big(\big[p^1_\lambda, 2\big] \times \{p^4_\lambda\}\big) & \textit{for} \quad \lambda \in (D, 0] \end{cases}$$

which is a continuous function $\hat{h} : [0,2]^2 \to \mathbb{R}$. Let $\hat{f} = (\hat{h}, \hat{h})$. Then

$$\mathcal{G}_{\mathcal{A}} := \left\{ p \in [a,b], \quad \hat{f}(p) = f(p) \right\}$$

is a generalized Riemann germ which is a HJ germ (and then also a conservative germ and a Kružkov germ) with respect to (J,f). Moreover we have

$$\chi \mathcal{G}_{\mathcal{A}} = \{p_A, p_B, p_C, p_D\} \tag{9.11}$$

Proof of Lemma 9.12

It is easy to check that $\hat{h} : [0,2]^2 \to \mathbb{R}$ is continuous. Moreover by construction $\hat{f} := (\hat{h}, \hat{h})$ is locally constant on $\left\{ \hat{f} \neq f \right\}$, with $\hat{h}$ nonincreasing in p^R and nondecreasing in p^L. Moreover, we have the following monotone bounds

$$\max\left\{ f_-^L(p^L), f_-^R(p^R) \right\} = 0 \le \hat{h}(p) \le \min\left\{ f_+^L(p^L), f_+^R(p^R) \right\} = \min\left\{ \frac{1}{2}, p^L, 2 - p^R \right\}$$

with

$$\begin{cases} f_+^L(p^L) := \sup\limits_{[0,p^L]} g = \min\left\{ p^L, \dfrac{1}{2} \right\} \\ f_-^L(p^L) := \inf\limits_{[p^L,2]} g = 0 \\ f_-^R(p^R) := \inf\limits_{[0,p^R]} g = 0 \\ f_+^R(p^R) := \sup\limits_{[p^R,2]} g = \min\left\{ 2 - p^R, \dfrac{1}{2} \right\} \end{cases}$$

Then from ii) of Theorem 2.15, we deduce that $\mathcal{G}_{\mathcal{A}}$ is a generalized Riemann germ with respect to (J,f). Moreover by construction it is a HJ germ. Finally it is easy to check (9.11). This ends the proof of the lemma.

Lemma 9.13 (An example of non commutativity of the gluing)
We work with notation of Lemma 9.12. Let

$$\mathcal{A} := (A,B,C,D) \quad \textit{with} \quad \frac{1}{2} > A > B > C > D > 0$$

and

$$\mathcal{A}' := (A',B',C,D) \quad \textit{with} \quad \frac{1}{2} > A' > B' > C > D > 0$$

and

$$A' > A > B' > B > C > D$$

Then the gluing satisfies

$$\tilde{\mathcal{G}}' := \mathcal{G}_{\mathcal{A}} \sharp \mathcal{G}_{\mathcal{A}'} = \mathcal{G}_{(B',B,C,D)} \neq \mathcal{G}_{(B,B,C,D)} = \mathcal{G}_{\mathcal{A}'} \sharp \mathcal{G}_{\mathcal{A}} =: \tilde{\mathcal{G}} \tag{9.12}$$

Proof of Lemma 9.13

First notice that by gluing HJ germs are preserved. Hence both $\tilde{\mathcal{G}}$ and $\tilde{\mathcal{G}}'$ are HJ germs, and are then characterized by their characteristic subsets that we expect to satisfy

$$\chi \tilde{\mathcal{G}} \supset \{p_{\tilde{A}}, p_{\tilde{B}}, p_{\tilde{C}}, p_{\tilde{D}}\} \quad \text{and} \quad \chi \tilde{\mathcal{G}}' \supset \{p_{\tilde{A}'}, p_{\tilde{B}'}, p_{\tilde{C}'}, p_{\tilde{D}'}\} \tag{9.13}$$

Precisely we compute (with obvious notation for $p = (p^L, p^R)$ and the gluing $(p^L, p^R) \sharp (p^{L'}, p^{R'}) := (p^L, p^{R'})$)

$$\begin{cases} p_{\tilde{A}'} := (p^4_{B'}, p^1_{B'}) & = (p^4_{B'} p^2_{B'}) \sharp (p^2_{B'}, p^1_{B'}) & \in \mathcal{G}_{\mathcal{A}} \sharp \mathcal{G}_{\mathcal{A}'} & \text{with} \quad \tilde{A}' := B' \\ p_{\tilde{B}'} := (p^2_B, p^1_B) & = (p^2_B, p^1_B) \sharp (p^1_B, p^1_B) & \in \mathcal{G}_{\mathcal{A}} \sharp \mathcal{G}_{\mathcal{A}'} & \text{with} \quad \tilde{B}' := B \\ p_{\tilde{C}'} := (p^4_C, p^3_C) & = (p^4_C, p^4_C) \sharp (p^4_C, p^3_C) & \in \mathcal{G}_{\mathcal{A}} \sharp \mathcal{G}_{\mathcal{A}'} & \text{with} \quad \tilde{C}' := C \\ p_{\tilde{D}'} := (p^2_D, p^3_D) & = (p^2_D, p^2_D) \sharp (p^2_D, p^3_D) & \in \mathcal{G}_{\mathcal{A}} \sharp \mathcal{G}_{\mathcal{A}'} & \text{with} \quad \tilde{D}' := D \\ & & & \\ p_{\tilde{A}} := (p^4_B, p^1_B) & = (p^4_B, p^2_B) \sharp (p^2_B, p^1_B) & \in \mathcal{G}_{\mathcal{A}'} \sharp \mathcal{G}_{\mathcal{A}} & \text{with} \quad \tilde{A} := B \\ p_{\tilde{B}} := (p^2_B, p^1_B) & = (p^2_B, p^2_B) \sharp (p^2_B, p^1_B) & \in \mathcal{G}_{\mathcal{A}'} \sharp \mathcal{G}_{\mathcal{A}} & \text{with} \quad \tilde{B} := B \\ p_{\tilde{C}} := (p^4_C, p^3_C) & = (p^4_C, p^4_C) \sharp (p^4_C, p^3_C) & \in \mathcal{G}_{\mathcal{A}'} \sharp \mathcal{G}_{\mathcal{A}} & \text{with} \quad \tilde{C} := C \\ p_{\tilde{D}} := (p^2_D, p^3_D) & = (p^2_D, p^2_D) \sharp (p^2_D, p^3_D) & \in \mathcal{G}_{\mathcal{A}'} \sharp \mathcal{G}_{\mathcal{A}} & \text{with} \quad \tilde{D} := D \end{cases}$$

This shows (9.13), and then Theorem 2.34 shows (9.12). This ends the proof of the lemma.

9.9 D-maximality does not imply completeness

We give two examples. The first example is explicit for $2:0$ junction, while the second is less explicit for junctions $0:3$.

Lemma 9.14 (Explicit conservative D-maximal set, which is not complete, for 2:0 junctions) *Let $g(x) := |x|-1$ with $g:\mathbb{R}\to\mathbb{R}$, and a $2:0$ junction J with $f^1 = f^2 := g$. There exists an explicit set $\mathcal{G}\subset\mathbb{R}^2$ satisfying*

$$\left\{\begin{array}{lll} D^f \geq 0 & on & \mathcal{G}\times\mathcal{G} \\ f^1+f^2=0 & on & \mathcal{G}\end{array}\right. \tag{9.14}$$

Then the set $\mathcal{G}$ is conservative D-maximal in the following sense: if some set $\mathcal{G}'\subset\mathbb{R}^2$ satisfies (9.14) and $\mathcal{G}\subset\mathcal{G}'$, then $\mathcal{G}'=\mathcal{G}$. Moreover $\mathcal{G}$ is not complete, i.e. $\bigcup_{\hat p\in\mathcal{G}} BA(\hat p)\neq\mathbb{R}^2$.

Proof of Lemma 9.14
Step 0: preliminaries
We consider the following three points $A:=(2,0)$, $B:=(0,2)$, $B':=(0,-2)$, and define the subset $\mathcal{G}\subset\mathbb{R}^2$ as $\mathcal{G}:=S_{AB}\cup S_{AB'}$, where S_{AB} is the closed segment joining A to B in $\mathbb{R}^2$. For all $p,q\in\mathbb{R}^2$, recall the dissipation

$$D^f := D^f(p,q) := \sum_{j=1,2} \text{sign}(p^j-q^j)\cdot\left[f^j\right]_{q^j}^{p^j} \quad \text{with} \quad \left[f^j\right]_{q^j}^{p^j} := f^j(p^j)-f^j(q^j)$$

Step 1: proof that $D^f\geq 0$ on $\mathcal{G}\times\mathcal{G}$
We have

$$D^f = \left\{\begin{array}{llllll} \left[f^1\right]_{q^1}^{p^1} + \left[f^2\right]_{q^2}^{p^2} \geq 0 & \text{if} & p,q\in S_{AB}, & & \text{with} & p^2=p^1 \geq q^1 = q^2, \\ \left[f^1\right]_{q^1}^{p^1} - \left[f^2\right]_{q^2}^{p^2} \geq 0 & \text{if} & p,q\in S_{AB'}, & & \text{with} & -p^2=p^1 \geq q^1 = -q^2, \\ \left[f^1\right]_{q^1}^{p^1} + \left[f^2\right]_{q^2}^{p^2} \geq 0, & \text{if} & p\in S_{AB}, & q\in S_{AB'}, & \text{with} & p^2=p^1 \geq q^1 = -q^2 \geq 0, \\ -\left[f^1\right]_{q^1}^{p^1} + \left[f^2\right]_{q^2}^{p^2} \geq 0, & \text{if} & p\in S_{AB}, & q\in S_{AB'}, & \text{with} & 0\leq p^2=p^1 < q^1 = -q^2. \end{array}\right.$$

Step 2: maximality of $\mathcal{G}$
Notice first that it is straightforward to check that $f^1+f^2=0$ on $\mathcal{G}$.
Assume now that there exists $p\in\mathbb{R}^2$ such that $\mathcal{G}':=\{p\}\cup\mathcal{G}$ satisfies (9.14). Then we have $(f^1+f^2)(p)=0$, i.e. $|p^1|+|p^2|=2$. If $p^1<0$, then $|p^2|<2$ and we get

$$D^f(B',p) = \left\{f^1(0)-f^1(p^1)\right\} - \left\{f^2(-2)-f^2(p^2)\right\} = -|p^1| - \left\{2-|p^2|\right\} = 2|p^2|-4<0$$

Contradiction. Therefore $p^1\geq 0$ and $p\in\mathcal{G}$, i.e. $\mathcal{G}'=\mathcal{G}$. We conclude that $\mathcal{G}\subset\mathbb{R}^2$ satisfying (9.14) is maximal for the inclusion.
Step 3: basin of attraction of $\mathcal{G}$
We have

$$\left\{\begin{array}{lll} BA(p) = (-\infty,0]\times(-\infty,2), & \text{for} & p=B' \\ BA(p) = \left\{p^1\right\}\times(\infty,-p^2) & \text{for all} & p\in S_{AB'}\backslash\{B'\} \\ BA(p) = \{p\} & \text{for all} & p\in S_{AB}\backslash\{A,B\} \\ BA(p) = (-\infty,0]\times\{2\}, & \text{for} & p=B \end{array}\right.$$

Then

$$\bigcup_{\hat p\in\mathcal{G}} BA(\hat p) = \mathbb{R}^2\backslash\Omega_0 \quad \text{with the open set} \quad \Omega_0 := \left\{p=(p^1,p^2)\in\mathbb{R}^2, \quad p^2>h(p^1)\right\}\neq\emptyset$$

where $h(p^1):=\min\left\{2,2-p^1,-\infty\cdot 1_{\{p^1>2\}}\right\}$. This shows that $\mathcal{G}$ is not complete and ends the proof of the lemma.

Lemma 9.15 (A conservative D-maximal set, which is not complete, for 3:0 junctions)
There exists a $3:0$ junction J with $f^j = g$ for $j = 1,2,3$ for some Lipschitz continuous function $g : [\alpha,\beta] \to \mathbb{R}$ for some $\alpha,\beta \in \mathbb{R}$ with $\alpha < \beta$. We set $a := (\alpha,\alpha,\alpha)$ and $b := (\beta,\beta,\beta)$. Then there exists a set $\mathcal{G} \subset [a,b] \subset \mathbb{R}^N$ for $N = 3$ satisfying

$$(9.15)\qquad \left\{\begin{array}{lll} D^f \geq 0 & \text{on} & \mathcal{G}\times\mathcal{G} \\ \displaystyle\sum_{j=1,\ldots,N} f^j = 0 & \text{on} & \mathcal{G} \end{array}\right.$$

Moreover $\mathcal{G}$ is conservative D-maximal in the following sense: for every set $\mathcal{G}' \subset [a,b]$ satisfying (9.15) such that $\mathcal{G} \subset \mathcal{G}'$, then $\mathcal{G}' = \mathcal{G}$. Moreover $\mathcal{G}$ is not complete, i.e. $\displaystyle\bigcup_{\hat{p}\in\mathcal{G}} BA(\hat{p}) \neq [a,b]$.

Proof of Lemma 9.15
Step 1: properties of $\mathcal{G}_0$
Let $\varepsilon,\delta,\eta \in (0,1)$ be such that $\alpha < -1 < \varepsilon+\delta < \beta$ and g such that

$$g(\alpha) = 0, \quad g(-1) = \eta, \quad g(\varepsilon) = -(1+\eta), \quad g(\varepsilon+\delta) = 1, \quad g(\beta) = 0$$

We set

$$\left\{\begin{array}{l} U_1 := -e_1 + \varepsilon e_2 + (\varepsilon+\delta)e_3 \\ U_2 := -e_2 + \varepsilon e_3 + (\varepsilon+\delta)e_1 \\ U_3 = -e_3 + \varepsilon e_1 + (\varepsilon+\delta)e_2 \end{array}\right.$$

and consider the set

$$\mathcal{G}_0 := \{a,b,U_1,U_2,U_3\} \subset [a,b].$$

Using $g(-1)+g(\varepsilon)+g(\varepsilon+\delta) = 0$, it is straightforward to check that $\mathcal{G}_0$ satisfies the second line of (9.15). Now let us check that we have

$$(9.16)\qquad D^f_+ \geq 0 \quad \text{on} \quad \mathcal{G}_0\times\mathcal{G}_0$$

which implies that $\mathcal{G}_0$ satisfies the first line of (9.15), because $D^f(p,q) = D^f_+(p,q) + D^f_+(q,p)$. In order to check (9.16), now consider the matrix E whose lines are vectors U_1,U_2,U_3, i.e.

$$E = \begin{pmatrix} U_1^1 & U_1^2 & U_1^3 \\ U_2^1 & U_2^2 & U_2^3 \\ U_3^1 & U_3^2 & U_3^3 \end{pmatrix} = \begin{pmatrix} -1 & \varepsilon & \varepsilon+\delta \\ \varepsilon+\delta & -1 & \varepsilon \\ \varepsilon & \varepsilon+\delta & -1 \end{pmatrix}$$

Then we consider $B = g(E)$ componentwise, we get with $g_i^j := g(U_i^j)$

$$B = \begin{pmatrix} g_1^1 & g_1^2 & g_1^3 \\ g_2^1 & g_2^2 & g_2^3 \\ g_3^1 & g_3^2 & g_3^3 \end{pmatrix} = \begin{pmatrix} \eta & -(1+\eta) & 1 \\ 1 & \eta & -(1+\eta) \\ -(1+\eta) & 1 & \eta \end{pmatrix}$$

Recall that

$$D_+(U_1,U_2) = \text{sign}^+(U_1^1-U_2^1)\cdot\left\{g_1^1-g_2^1\right\} + \text{sign}^+(U_1^2-U_2^2)\cdot\left\{g_1^2-g_2^2\right\} + \text{sign}^+(U_1^3-U_2^3)\cdot\left\{g_1^3-g_2^3\right\}$$

We get

$$\left\{\begin{array}{l} D_+(U_1,U_2) = \left\{g_1^2-g_2^2\right\} + \left\{g_1^3-g_2^3\right\} = -\left\{g_1^1-g_2^1\right\} = D_+(U_2,U_1) \\ D_+(U_2,U_3) = \left\{g_2^1-g_3^1\right\} + \left\{g_2^3-g_3^3\right\} = -\left\{g_2^2-g_3^2\right\} = D_+(U_3,U_2) \\ D_+(U_3,U_1) = \left\{g_3^1-g_1^1\right\} + \left\{g_3^2-g_1^2\right\} = -\left\{g_3^3-g_1^3\right\} = D_+(U_1,U_3) \end{array}\right.$$

Hence $D_+(U_i,U_j) \geq 0$ for $i,j = 1,\ldots,3$ if and only if $g_1^1 \leq g_2^1$, $g_2^2 \leq g_3^2$, $g_3^3 \leq g_1^3$, which is the case because $g_1^1 = g_2^2 = g_3^3 = \eta \leq 1 = g_2^1 = g_3^2 = g_1^3$. We also have $D_+(a,b) = 0 = D_+(b,a)$ because $g(\alpha) = 0 = g(\beta)$. Moreover Rankine-Hugoniot relation $f^1+f^2+f^3 = 0$ implies

$$D_+(U_i,a) = 0 = D_+(a,U_i), \quad D_+(U_i,b) = 0 = D_+(b,U_i), \quad \text{for} \quad i = 1,2,3$$

Hence (9.16) holds true, and then $\mathcal{G}_0$ satisfies (9.15).
Step 2: definition of $\tilde{\mathcal{G}}$

We now consider a set $\tilde{\mathcal{G}} \subset [a,b]$ satisfying (9.15) with $\mathcal{G}_0 \subset \tilde{\mathcal{G}}$, and such that $\tilde{\mathcal{G}}$ is maximal for the inclusion.
Step 3: uncompleteness of $\tilde{\mathcal{G}}$
Assume by contradiction that $\tilde{\mathcal{G}}$ is complete, i.e. that $\bigcup_{\hat{p}\in\tilde{\mathcal{G}}} BA(\hat{p}) = [a,b]$. From Lemma 3.4, we deduce that $(BA(\hat{p}))_{\hat{p}\in\tilde{\mathcal{G}}}$ is a partition of $[a,b]$, and then $\tilde{\mathcal{G}}$ is a generalized Riemann germ. Because $\tilde{\mathcal{G}}$ satisfies (9.15), we deduce that $\tilde{\mathcal{G}}$ is a conservative Kružkov germ, hence $\hat{f} = \hat{f}_{\tilde{\mathcal{G}}}$ satisfies for $N = 3$

$$\left\{\begin{array}{lll} D^{\hat{f}} \geq 0 & \text{on} & [a,b]^2 \\ \displaystyle\sum_{j=1,\ldots,N} \hat{f}^j = 0 & \text{on} & [a,b] \end{array}\right. \tag{9.17}$$

We set $\hat{f}_0^j := \hat{f}^j(0)$ for $j = 1,2,3$ and $0 = 0_{\mathbb{R}^3}$. We compute

$$\left\{\begin{array}{l} D^{\hat{f}}(U_1,0) = \left\{g_1^2 - \hat{f}_0^2\right\} + \left\{g_1^3 - \hat{f}_0^3\right\} = -\left\{g_1^1 - \hat{f}_0^1\right\} = D^{\hat{f}}(0,U_1) \\ D^{\hat{f}}(U_2,0) = \left\{g_2^1 - \hat{f}_0^1\right\} + \left\{g_2^3 - \hat{f}_0^3\right\} = -\left\{g_2^2 - \hat{f}_0^2\right\} = D^{\hat{f}}(0,U_2) \\ D^{\hat{f}}(U_3,0) = \left\{g_3^1 - \hat{f}_0^1\right\} + \left\{g_3^2 - \hat{f}_0^2\right\} = -\left\{g_3^3 - \hat{f}_0^3\right\} = D^{\hat{f}}(0,U_3) \end{array}\right.$$

From the first line of (9.17), we deduce $g_1^1 \leq \hat{f}_0^1$, $g_2^2 \leq \hat{f}_0^2$, $g_3^3 \leq \hat{f}_0^3$, which gives

$$0 < 3\eta = g_1^1 + g_2^2 + g_3^3 \leq \hat{f}_0^1 + \hat{f}_0^2 + \hat{f}_0^3 = 0$$

where the last equality follows from the second line of (9.17). Contradiction. Hence $\tilde{\mathcal{G}}$ is not complete and this ends the proof.

Corollary 9.16 (Counter-example to completeness for $N \geq 3$)
Let $N \geq 3$. Then there exists a $N:0$ junction and particular Lipschitz continuous functions $(f^j)_{j=1,\ldots,N}$ and a set $\mathcal{G} \subset [a,b] \subset \mathbb{R}^N$ satisfying (9.15). Moreover $\mathcal{G}$ is conservative D-maximal in the following sense: for every set $\mathcal{G}' \subset [a,b]$ satisfying (9.15) such that $\mathcal{G} \subset \mathcal{G}'$, then $\mathcal{G}' = \mathcal{G}$. Moreover $\mathcal{G}$ is not complete, i.e. $\bigcup_{\hat{p}\in\mathcal{G}} BA(\hat{p}) \neq [a,b]$.

Proof of Corollary 9.16
For $N = 3$, the result follows from Lemma 9.15 for some set $\mathcal{G}_3$ satisfying (9.15) which is maximal for the inclusion, and the contradiction to the completeness of $\mathcal{G}_3$ precisely follows from the evaluation of

$$D^{\hat{f}_{\mathcal{G}_3}}(0_{\mathbb{R}^3}, U_j) \geq 0, \quad j = 1,2,3. \tag{9.18}$$

For $N \geq 3$, we set

$$p'_N = (p_N^4, \ldots, p_N^N) \quad \text{with} \quad p_N^k := (f^k)^{-1}(0), \quad k = 4,\ldots,N. \tag{9.19}$$

Now fix some $N \geq 3$, and assume that there exists some set $\mathcal{G}_N$ satisfying (9.15) which is maximal for the inclusion. Let us now show that we can transfer the result to the level $N+1$. Indeed, applying Lemma 8.8, we see that we can construct a set $\tilde{\mathcal{G}}_N \subset [\tilde{a},\tilde{b}] \subset \mathbb{R}^{N+1}$ which satisfies (9.15), hence at the level $N+1$. Then consider a set $\mathcal{G}_{N+1}$ with $\tilde{\mathcal{G}}_N \subset \mathcal{G}_{N+1} \subset [\tilde{a},\tilde{b}] \subset \mathbb{R}^{N+1}$ satisfying (9.15) at the level $N+1$, and which is maximal for the inclusion.

Assume now by contradiction that the set $\mathcal{G}_{N+1}$ is complete. Then the argument of Step 3 of the proof of Lemma 9.15 applies and shows that $\mathcal{G}_{N+1}$ is indeed a conservative Kružkov germ. In particular $\hat{f}_{\mathcal{G}_{N+1}}$ is continuous, and using definition (9.19) at level $N+1$, we get

$$D^{\hat{f}_{\mathcal{G}_3}}(0_{\mathbb{R}^3}, U_j) = D^{\hat{f}_{\mathcal{G}_{N+1}}}((0_{\mathbb{R}^3}, p'_{N+1}), (U_j, p'_{N+1})) \geq 0, \quad j = 1,2,3$$

which leads to the same contradiction as (9.18) did. Therefore $\mathcal{G}_{N+1} \subset \mathbb{R}^{N+1}$ must be not complete. This ends the proof of the corollary.

10 Appendix of Part I

10.1 Standard Riemann problem

We consider the entropy solution $u = u(t,x)$ to the following Riemann problem

$$\left\{\begin{array}{l} u_t + (g(u))_x = 0 \quad \text{on} \quad (0,+\infty)\times\mathbb{R} \\ u(0,x) = u_0(x) := \left\{\begin{array}{lll} p_L & \text{if} & x<0 \\ p_R & \text{if} & x>0 \end{array}\right. \end{array}\right. \tag{10.1}$$

Lemma 10.1 (Explicit solution to Riemann's problem on the real line)
Assume that $g:\mathbb{R}\to\mathbb{R}$ is continuous, locally Lipschitz. The for any $p_L, p_R \in \mathbb{R}$, there exists a unique entropy solution u to (10.1). It satisfies $u(t,x) = U(x/t)$ for all $t>0$. Moreover, let us define $I := [\min(p_L,p_R), \max(p_L,p_R)]$ and

$$\tilde{g} := \left\{\begin{array}{lll} \textit{convex envelop of } g^I & \textit{if} & p_L < p_R \\ \textit{concave envelop of } g^I & \textit{if} & p_L > p_R \end{array}\right. \Bigg| \quad \textit{with} \quad g^I(a) := \left\{\begin{array}{llll} g(a) & \textit{if} & a\in I & \\ +\infty & \textit{if} & a\in\mathbb{R}\backslash I, & p_L<p_R \\ -\infty & \textit{if} & a\in\mathbb{R}\backslash I, & p_L>p_R \end{array}\right.$$

and set

$$\xi_L \le \xi_R \quad \textit{with} \quad (\xi_L,\xi_R) := \left\{\begin{array}{lll} (\ \tilde{g}'(p_L^+)\ ,\ \tilde{g}'(p_R^-)\ ) & \textit{if} & p_L<p_R \\ (\ 0\ ,\ 0\ ) & \textit{if} & p_L = p_R \\ (\ \tilde{g}'(p_L^-)\ ,\ \tilde{g}'(p_R^+)\ ) & \textit{if} & p_L > p_R \end{array}\right.$$

Then

$$U(\xi) = \left\{\begin{array}{lll} p_L & \textit{if} & \xi<\xi_L \\ p_R & \textit{if} & \xi>\xi_R \\ ((\tilde{g}_{|I})')^{-1}(\xi) & \textit{if} & \xi\in[\xi_L,\xi_R] \end{array}\right. \tag{10.2}$$

where the map $U:\mathbb{R}\to\mathbb{R}$ is monotone and is uniquely defined outside a countable set.

Remark 10.2 *Notice that $((\tilde{g}_{|I})')^{-1}$ is not defined for $\xi_L = 0 = \xi_R$, which only arises when $p_L = p_R$. When $p_L \neq p_L$, as a help, the function $((\tilde{g}_{|I})')^{-1}$ is better understood as the inverse maximal monotone graph of the maximal monotone graph $(\tilde{g}_{|I})'$.*

proof of Lemma 10.1
For the proof, we refer to the textbook Serre [45], where it is done for C^∞ functions g. Indeed, only the regularity C^2 is used there. This can easily be extended to the case of g continuous and locally Lipschitz, by approximation, and stability of entropy solutions. On the same topic, the reader can also consult textbooks Dafermos [16] and Holden, Risebro [27]. This ends the proof of the lemma.

Remark 10.3 *Notice that the result of Lemma 10.1 also follows from HJ-SCL relations (for instance justified using vanishing viscosity method, and BV bounds for scalar conservation laws). Indeed it is also a straightforward consequence of Hopf formula (see Theorem 3.1 in [6]) for convex initial data of Hamilton-Jacobi equations. Here the convex initial data is $W_0(x) = p_L x\cdot 1_{\{x<0\}} + p_R x \cdot 1_{\{x>0\}}$ when $p_L<p_R$, for non convex Hamiltonian g.*

10.2 Reduction of test functions for viscosity solutions on junctions

Consider the problem

$$\left\{\begin{array}{lll} v_t^j + f^j(v_x^j) = 0 & \text{on} & \mathbb{R}\times J^j, \quad j=1,\dots,N \\ v^0(t,0) := v^j(t,0) & \text{on} & \mathbb{R}\times\{0\}, \quad j=1,\dots,N \\ v_t^0 + \hat{h}(v_x^1,\dots,v_x^N) = 0 & \text{on} & \mathbb{R}\times\{0\} \end{array}\right. \tag{10.3}$$

We also define the half-relaxation operators

$$\left\{\begin{array}{l} (\underline{R}\hat{h})(p) := \sup\limits_{q\in[p,b]} \min\left\{\hat{h}, f_{\min}\right\}(q) \quad \text{with} \quad f_{\min}(q) := \min\limits_{j=1,\dots,N} f^j(q^j) \\ (\overline{R}\hat{h})(p) := \inf\limits_{q\in[a,p]} \max\left\{\hat{h}, f_{\max}\right\}(q) \quad \text{with} \quad f_{\max}(q) := \max\limits_{j=1,\dots,N} f^j(q^j) \end{array}\right. \tag{10.4}$$

Lemma 10.4 (Reducing the set of test functions)
Assume (2.2) with $N \geq 1$ with a junction J of type $0 : N$. Let $\mathcal{G} \subset [a,b]$ be a generalized Riemann germ which is a HJ germ. Then the associated Godunov flux is $\hat{f}_{\mathcal{G}} = (\hat{h}, \ldots, \hat{h})$ with $\hat{h} : [a,b] \to \mathbb{R}$ given by i) of Theorem 2.30. For any $p \in [a,b]$, consider the function $v = (v^1, \ldots, v^N)$ defined by

$$v^j(t,x) := -\lambda t + p^j x \quad \textit{for all} \quad (t,x) \in \mathbb{R} \times J^j, \quad j = 1, \ldots, N$$

i) (Viscosity subsolutions)
Then $\hat{h}$ satisfies

$$\hat{h} = \underline{R}\hat{h} \quad \textit{with } \underline{R} \textit{ defined in (10.4)} \tag{10.5}$$

Moreover v is a viscosity subsolution of (10.3) if and only if $-\lambda + f^j(p^j) \leq 0$ and

$$\textit{for all} \quad q \in \underline{\chi}\mathcal{G}, \quad \Big(q \geq p \quad \Longrightarrow \quad -\lambda + \hat{h}(q) \leq 0\Big)$$

ii) (Viscosity supersolutions)
Then $\hat{h}$ satisfies

$$\hat{h} = \overline{R}\hat{h} \quad \textit{with } \overline{R} \textit{ defined in (10.4)} \tag{10.6}$$

Moreover v is a viscosity supersolution of (10.3) if and only if $-\lambda + f^j(p^j) \geq 0$ and

$$\textit{for all} \quad q \in \overline{\chi}\mathcal{G}, \quad \Big(q \leq p \quad \Longrightarrow \quad -\lambda + \hat{h}(q) \geq 0\Big)$$

Proof of Lemma 10.4
Step 1: proof of (10.5)
By definition of $\underline{R}\hat{h}$ and by monotonicity of $\hat{h}$, we have $\underline{R}\hat{h} \leq \hat{h}$. Now for $p \in [a,b]$, let $\hat{p} := \pi_{\mathcal{G}}(p)$. Then

$$(\underline{R}\hat{h})(p) := \sup_{q \in [p,b]} \min \left\{\hat{h}, f_{\min}\right\}(q) \geq \sup_{q \in [p,b] \cap BA(\hat{p})} \min \left\{\hat{h}, f_{\min}\right\}(q)$$

By construction, there exists $q \in [p,b] \cap BA(\hat{p})$ such that $f^j(q^j) \geq f^j(\hat{p}^j) = \hat{h}(\hat{p}) = \hat{h}(q)$, $j = 1, \ldots, N$. Therefore $(\underline{R}\hat{h})(p) \geq \hat{h}(\hat{p}) = \hat{h}(p)$, which implies the equality and then shows (10.5).
Step 2: proof of (10.6)
The proof is similar to Step 1.
Step 3: proof of i)
The proof of i) is a simple variant of the proof of Theorem 2.7 in [29]. An easy adaptation of the proof is done for instance in [23] in the case $[a,b] \cap \mathbb{R}^N = \mathbb{R}^N$, in the subsection on the reduction of test functions. The adaptation to the general case $[a,b]$ is straightforward (and indeed easier).
Step 4: proof of ii)
The proof is similar to Step 3. This ends the proof of the lemma.

10.3 Gluing of matrices

In Subsection 5.6, we have studied the gluing of two germs $\mathcal{G}_\alpha \sharp \mathcal{G}_\beta$. We have also seen that naturally is associated the gluing of their Godunov fluxes $\hat{f} = \hat{f}_\alpha \sharp \hat{f}_\beta$.

In this section, we are interested in the algebra giving the Jacobian matrix $D\hat{f}$ in terms of the two Jacobian matrices $D\hat{f}_\alpha$ and $D\hat{f}_\beta$. This is given by the following result.

Lemma 10.5 (Formal Jacobian matrix after gluing)
For $\gamma = \alpha, \beta$, let some integers $n_\gamma \geq 2$. By abuse of notation, let us also allow the indices α, β to denote two sets of indices with $\alpha \simeq \{1, \ldots, n_\alpha - 1\}$ and $\beta \simeq \{1, \ldots, n_\beta - 1\}$. Now let $\hat{f}_\gamma = (\hat{f}^0_\gamma, \hat{f}^1_\gamma, \ldots, \hat{f}^{n_\gamma - 1}_\gamma) : \mathbb{R}^{n_\gamma} \supset [a,b]_\gamma \to \mathbb{R}^{n_\gamma}$. For $p_\gamma = (p^1_\gamma, \ldots, p^{n_\gamma - 1}_\gamma) \in \mathbb{R}^{n_\gamma - 1}$, let us set $\hat{f} = (-\hat{f}^0_\alpha, \hat{f}^1_\alpha, \ldots, \hat{f}^{n_\alpha - 1}_\alpha) \sharp \hat{f}_\beta$ defined formally by

$$\hat{f}^j(p_\alpha, p_\beta) := \left\{ \begin{array}{lll} \hat{f}^j_\alpha(r, p_\alpha) & \textit{if} & j \in \alpha \\ \hat{f}^j_\beta(r, p_\beta) & \textit{if} & j \in \beta \end{array} \right. \quad \Bigg| \quad \textit{with } r \textit{ satisfying} \quad -\hat{f}^0_\alpha(r, p_\alpha) = \hat{f}^0_\beta(r, p_\beta) \tag{10.7}$$

(having in mind $J^0_\alpha \simeq (0,+\infty)$ and $J^0_\beta \simeq (-\infty,0)$, and then here $(-\hat{f}^0_\alpha)(\downarrow,p_\alpha)$ and $\hat{f}^0_\beta(\uparrow,p_\beta)$, i.e. $\hat{f}^0_\alpha(\uparrow,p_\alpha)$). We set (with index i for the line and j for the column)

$$\hat{B}' := (\partial_j \hat{f}^i_\alpha)_{i,j\in\{0\}\cup\alpha} = \begin{pmatrix} \partial_0 \hat{f}^0_\alpha & (\partial_j \hat{f}^0_\alpha)_{j\in\alpha} \\ (\partial_0 \hat{f}^i_\alpha)_{i\in\alpha} & (\partial_j \hat{f}^i_\alpha)_{i,j\in\alpha} \end{pmatrix} = \begin{pmatrix} B'_0 & D' \\ C' & B' \end{pmatrix}$$

and

$$\hat{B} := (\partial_j \hat{f}^i_\beta)_{i,j\in\{0\}\cup\beta} = \begin{pmatrix} \partial_0 \hat{f}^0_\beta & (\partial_j \hat{f}^0_\beta)_{j\in\beta} \\ (\partial_0 \hat{f}^i_\beta)_{i\in\beta} & (\partial_j \hat{f}^i_\beta)_{i,j\in\beta} \end{pmatrix} = \begin{pmatrix} B_0 & D \\ C & B \end{pmatrix}$$

Then we have formally

$$\hat{B}' \star \hat{B} := \lambda_0 (\partial_j \hat{f}^i)_{i,j\in\alpha\cup\beta} = \lambda_0 \begin{pmatrix} B' & 0 \\ 0 & B \end{pmatrix} - \begin{pmatrix} C' \\ C \end{pmatrix} \cdot (D' \quad D) \quad \text{with} \quad \lambda_0 := B'_0 + B_0$$

When $\lambda_0 > 0$, we set $\hat{B}' \sharp \hat{B} := \lambda_0^{-1}(\hat{B}' \star \hat{B})$.

Proof of Lemma 10.5
Taking the derivative of the last equation of (10.7), we get easily

$$\begin{cases} \partial_j \hat{f}^0_\alpha + \lambda_0 \partial_j r = 0 & \text{if} \quad j \in \alpha \\ \partial_j \hat{f}^0_\beta + \lambda_0 \partial_j r = 0 & \text{if} \quad j \in \beta \end{cases}$$

The elimination of $\partial_j r$ then gives

$$\lambda_0 (\partial_j \hat{f}^i)_{i,j\in\alpha\cup\beta} = \begin{pmatrix} (\lambda_0 \partial_j \hat{f}^i_\alpha - \partial_0 \hat{f}^i_\alpha \partial_j \hat{f}^0_\alpha)_{i,j\in\alpha} & (0 - \partial_0 \hat{f}^i_\alpha \partial_j \hat{f}^0_\beta)_{(i,j)\in\beta\times\alpha} \\ (0 - \partial_0 \hat{f}^i_\beta \partial_j \hat{f}^0_\alpha)_{(i,j)\in\alpha\times\beta} & (\lambda_0 \partial_j \hat{f}^i_\beta - \partial_0 \hat{f}^i_\beta \partial_j \hat{f}^0_\beta)_{i,j\in\beta} \end{pmatrix}$$

and the result follows. This ends the proof of the lemma.

Then we have the following result about the new algebra of gluing of matrices.

Proposition 10.6 (Properties of the gluing of matrices)
Let M_n denote the set of $n\times n$ matrices, with the convention that $M_n = \{0\}$ for $n \le 0$. For $m,n \ge 0$ and $m+n \ge 1$, the gluing map

$$\begin{array}{llll} \star: & M_{m+1} \times M_{n+1} & \to & M_{m+n} \\ & (\hat{B}', \hat{B}) & \mapsto & \hat{B}' \star \hat{B} \end{array}$$

is quadratic. Moreover if the square matrices $\hat{B}', \hat{B}$ are P_0-monotone (resp. Riemann monotone, resp. Kružkov monotone, in the sense of Definition 8.1), then $\hat{B}' \star \hat{B}$ is also a P_0-monotone (resp. Riemann monotone, resp. Kružkov monotone).

For the proof of Proposition 10.6, we need the following easy result.

Lemma 10.7 (A property of Riemann monotone matrices)
Assume that the following matrix $\hat{B} = \begin{pmatrix} B_0 & D \\ C & B \end{pmatrix}$ is Riemann monotone (in the sense of Definition 8.1), where the block decomposition is for $B_0 \in \mathbb{R}$ and $B \in \mathbb{R}^n$. We have $B_0 \ge 0$. Moreover $B_0 = 0$ implies $C = 0$.

Proof of Lemma 10.7
Recall that by assumption, for all $x \in \mathbb{R}^{1+n}$, we know that $x \diamond (\hat{B}\cdot x) \le 0$ implies $\hat{B}\cdot x = 0$. Now for $x = (a,0,\dots,0)^T$ with $a \ne 0$, we get $x \diamond (\hat{B}\cdot x) = 0$ with $B_0 = 0$, and then $0 = \hat{B}\cdot x = a\begin{pmatrix} 0 \\ C \end{pmatrix}$ implies $C = 0$. The fact that $B_0 \ge 0$ is general and follows from the fact that $\hat{B}$ is in particular P_0-monotone. This ends the proof of the lemma.

Proof of Proposition 10.6
Step 1: Proof of P_0-monotonicity
We claim that

$$\delta := det(\hat{B}' \star \hat{B}) = \lambda_0^{(m+n-1)} \left\{ det(B')det(\hat{B}) + det(\hat{B}')det(B) \right\} \tag{10.8}$$

Notice that $\delta \geq 0$ if both matrices $\hat{B}', \hat{B}$ are P_0-monotone. This property also passes to minors of $\hat{B}' \star \hat{B}$, because they are then expressed as functions of minors of $\hat{B}'$ and of $\hat{B}$.

Now let us show (10.8). Denoting by $\check{B}'_l$ the matrix B' whose column l has been suppressed (and similarly $\check{B}_q$ the matrix B whose column q has been suppressed), we get

$$\begin{aligned}
\lambda_0^{-(m+n-1)}\delta &:= \lambda_0^{-(m+n-1)}\det(\hat{B}' \star \hat{B}) \\
&= \lambda_0^{-(m+n-1)} \begin{vmatrix} 1 & 0 & 0 \\ 0 & \lambda_0 B' - C'D' & -C'D \\ 0 & -CD' & \lambda_0 B - CD \end{vmatrix} \\
&= \lambda_0^{-(m+n-1)} \begin{vmatrix} 1 & 0 & 0 \\ C'/\lambda_0 & \lambda_0 B' - C'D' & -C'D \\ C/\lambda_0 & -CD' & \lambda_0 B - CD \end{vmatrix} \\
&= \lambda_0^{-(m+n-1)} \begin{vmatrix} 1 & \lambda_0 D' & \lambda_0 D \\ C'/\lambda_0 & \lambda_0 B' & 0 \\ C/\lambda_0 & 0 & \lambda_0 B \end{vmatrix} \\
&= \begin{vmatrix} \lambda_0 & D' & D \\ C' & B' & 0 \\ C & 0 & B \end{vmatrix} \\
&= \lambda_0 \det(B')\det(B) + \det(B) \cdot \sum_{l=1}^{m} (-1)^l D'_l \det(C', \check{B}'_l) + (-1)^m \sum_{q=1}^{n} (-1)^q D_q \begin{vmatrix} C' & B' & 0 \\ C & 0 & \check{B}_q \end{vmatrix} \\
&= \lambda_0 \det(B')\det(B) + \det(B) \cdot \sum_{l=1}^{m} (-1)^l D'_l \det(C', \check{B}'_l) + \sum_{q=1}^{n} (-1)^q D_q \begin{vmatrix} B' & C' & 0 \\ 0 & C & \check{B}_q \end{vmatrix} \\
&= \lambda_0 \det(B')\det(B) + \det(B) \cdot \sum_{l=1}^{m} (-1)^l D'_l \det(C', \check{B}'_l) + \det(B') \cdot \sum_{q=1}^{n} (-1)^q D_q \det(C, \check{B}_q)
\end{aligned}$$

where, in the fifth line we have factorized the first column by λ_0^{-1} and the other columns by λ_0, in the sixth line we have used expansion along the first row. Again using expansion of the determinant on the first row, recall that

$$\begin{cases} \det(\hat{B}') = B'_0 \det(B') + \sum_{l=1}^{m} (-1)^l D'_l \det(C', \check{B}'_l) \\ \det(\hat{B}) = B_0 \det(B) + \sum_{q=1}^{q} (-1)^l D_q \det(C, \check{B}_q) \\ \lambda_0 = B'_0 + B_0 \end{cases}$$

Hence $\lambda_0^{-(m+n-1)}\delta = \det(\hat{B}')\det(B) + \det(B')\det(\hat{B})$ which shows (10.8).

Step 2: Proof of Riemann monotonicity

Case A: $\lambda_0 = 0$

The case $\lambda_0 = B'_0 + B_0 = 0$ implies $\hat{B}' \star \hat{B} = -\check{C}\check{D}$ with $\check{C} := \begin{pmatrix} C' \\ C \end{pmatrix}$ and $\check{D} := (D' \quad D)$.

Recall that $B'_0, B_0 \geq 0$. Hence $\lambda_0 = 0$ also implies $B'_0 = 0 = B_0$. Moreover Lemma 10.7 shows that $\check{C} = 0$. Hence $\hat{B}' \star \hat{B} = 0$ which is in particular Riemann monotone.

Case B: $\lambda_0 := B'_0 + B_0 > 0$

Then in statement of Proposition 5.13, we can consider the functions

$$\begin{cases} \hat{f}^0_\alpha(r, p_\alpha) & := B'_0 r + D' \cdot p_\alpha \\ \hat{f}^i_\alpha(r, p_\alpha) & := (C')^i r + (B' \cdot p_\alpha)^i \\ & \\ \hat{f}^0_\beta(r, p_\beta) & := B_0 r + D \cdot p_\beta \\ \hat{f}^i_\beta(r, p_\beta) & := C^i r + (B \cdot p_\beta)^i \end{cases}$$

They are not locally constant, but there is a unique solution $r \in \mathbb{R}$ of

$$-\hat{f}^0_\alpha(r, p_\alpha) = \hat{f}^0_\beta(r, p_\beta).$$

When $\hat{B}', \hat{B}$ are Riemann monotone matrices, then $\hat{f}_\alpha, \hat{f}_\beta$ are Riemann monotone maps. Then Step 6 of the proof of Proposition 5.13 applies and shows that $\hat{f}(p_\alpha, p_\beta)$ is Riemann monotone, which means precisely that the matrix $\hat{B}' \sharp \hat{B} := \lambda_0^{-1}(\hat{B}' \star \hat{B})$ is a Riemann monotone matrix.

Step 3: Proof of Kružkov monotonicity

If $\lambda_0 = 0$, then Step 2 shows that $\hat{B}' \star \hat{B} = 0$ which is in particular Kružkov monotone. If $\lambda_0 > 0$, the proof follows the lines of Step 2, replacing Riemann monotonicity by Kružkov monotonicity, and using Step 8 of the proof of Proposition 5.13 instead of Step 6. This ends the proof of the lemma.

Remark 10.8 *Our calculations are related to the Schur complement (see [34]) for* $\hat{B} := \begin{pmatrix} B_0 & D \\ C & B \end{pmatrix}$, *which is* $\hat{B}/B_0 := B - CB_0^{-1}D$ *and satisfies classically* $\det \hat{B} = \det B_0 \cdot \det(\hat{B}/B_0)$.

Lemma 10.9 (Formal Jacobian matrix after self-gluing)

Let γ *be a fixed index, and some integers* $n_\gamma \geq 3$. *Let* $I_\gamma := \{0, \dots, n_\gamma - 1\}$ *and* $I'' := \{1, \dots, n_\gamma - 2\}$. *Now let* $\hat{f}_\gamma = (\hat{f}^0_\gamma, \hat{f}^1_\gamma, \dots, \hat{f}^{n_\gamma - 1}_\gamma) : \mathbb{R}^{n_\gamma} \supset [a,b]_\gamma \to \mathbb{R}^{n_\gamma}$. *For* $p_\gamma = (p^1_\gamma, \dots, p^{n_\gamma-2}_\gamma) \in \mathbb{R}^{n_\gamma - 2}$, *let us set* $\hat{f} = (-\hat{f}^{k_1}_\gamma, \hat{f}^1_\gamma, \dots, \hat{f}^{n_\gamma-2}_\gamma, \hat{f}^{k_2}_\gamma)^{k_1 \overset{\sharp}{:} k_2}$ *with* $k_1 := 0$ *and* $k_2 := n_\gamma - 1$, *defined formally for* $j \in I''$ *by*

$$\hat{f}^j(p_\gamma) := \hat{f}^j_\gamma(r, p_\gamma, r) \quad \text{with } r \text{ satisfying} \quad -\hat{f}^0_\gamma(r, p_\gamma, r) = \hat{f}^{n_\gamma-1}_\gamma(r, p_\gamma, r). \tag{10.9}$$

(having in mind $J^{k_1}_\gamma \simeq (0, +\infty)$ *and* $J^{k_2}_\gamma \simeq (-\infty, 0)$, *and then here* $(-\hat{f}^{k_1}_\gamma)(\downarrow, p_\gamma, r)$ *and* $\hat{f}^{k_2}_\gamma(r', p_\gamma, \uparrow)$, *i.e.* $\hat{f}^{k_1}_\gamma(\uparrow, p_\gamma, r)$*). We set (with index* i *for the line and* j *for the column)*

$$\hat{B} := (\partial_j \hat{f}^i_\gamma)_{i,j \in \{k_1\} \cup I'' \cup \{k_2\}} = \begin{pmatrix} (\partial_{k_1} \hat{f}^{k_1}_\gamma) & (\partial_j \hat{f}^{k_1}_\gamma)_{j \in I''} & (\partial_{k_2} \hat{f}^{k_1}_\gamma) \\ (\partial_{k_1} \hat{f}^i_\gamma)_{i \in I''} & (\partial_j \hat{f}^i_\gamma)_{i,j \in I''} & (\partial_{k_2} \hat{f}^i_\gamma)_{i \in I''} \\ (\partial_{k_1} \hat{f}^{k_2}_\gamma) & (\partial_j \hat{f}^{k_2}_\gamma)_{j \in I''} & (\partial_{k_2} \hat{f}^{k_2}_\gamma) \end{pmatrix} = \begin{pmatrix} B'_0 & D' & F'_0 \\ C' & B & C \\ E'_0 & D & B_0 \end{pmatrix}$$

Then we have formally

$$\hat{B}^{j_1 \overset{\star}{:} j_2} := \lambda_0 (\partial_j \hat{f}^i)_{i,j \in I''} = \lambda_0 B - (C' + C) \cdot (D + D') \quad \text{with} \quad \lambda_0 := \{B'_0 + F'_0\} + \{E'_0 + B_0\}$$

When $\lambda_0 > 0$, *we moreover set* $\hat{B}^{j_1 \overset{\sharp}{:} j_2} := \lambda_0^{-1} \hat{B}^{j_1 \overset{\star}{:} j_2}$.

Proof of Lemma 10.9

Taking the derivative of the last equality of (10.9), we get easily

$$-\left\{\partial_j \hat{f}^{j_1}_\gamma + (\partial_j r)\left\{\partial_{j_1} \hat{f}^{j_1} + \partial_{j_2} \hat{f}^{j_1}_\gamma\right\}\right\} = \partial_j \hat{f}^{j_2}_\gamma + (\partial_j r)\left\{\partial_{j_1} \hat{f}^{j_2} + \partial_{j_2} \hat{f}^{j_2}_\gamma\right\}$$

i.e.

$$\partial_j \hat{f}^{j_1}_\gamma + \partial_j \hat{f}^{j_2}_\gamma + \lambda_0 \partial_j r = 0$$

The elimination of $\partial_j r$ then gives

$$\lambda_0 (\partial_j \hat{f}^i)_{i,j \in I''} = \left(\lambda_0 \partial_j \hat{f}^i_\gamma - \left\{\partial_j \hat{f}^{j_1}_\gamma + \partial_j \hat{f}^{j_2}_\gamma\right\} \cdot \left\{\partial_{j_1} \hat{f}^i_\gamma + \partial_{j_2} \hat{f}^i_\gamma\right\}\right)_{i,j \in I''}$$

and the result follows. This ends the proof of the lemma.

Proposition 10.10 (Properties of the self-gluing of a matrix)

Let M_n *denote the set of* $n \times n$ *matrices, with the convention that* $M_n = \{0\}$ *for* $n \leq 0$. *For* $n \geq 3$, *the self-gluing map*

$$\begin{array}{llll} \star: & M_n & \to & M_{n-2} \\ & \hat{B} & \mapsto & \hat{B}^\star := \hat{B}^{j_1 \overset{\star}{:} j_2} \quad \text{with} \quad j_1 := 0, \ j_2 := n-1 \end{array}$$

is quadratic. Moreover if the square matrix $\hat{B}$ *is Kružkov monotone (in the sense of Definition 8.1), then* $\hat{B}^\star$ *is also Kružkov monotone.*

Proof of Proposition 10.10
Case A: $\lambda_0 := \{B_0' + F_0'\} + \{E_0' + B_0\} = 0$
Because $\hat{B}$ is Kružkov monotone, we know with $I'' := \{1, \ldots, n-2\}$ that

$$\begin{cases} B_0' \geq |E_0'| + \sum_{i \in I''} |(C')^i| \\ B_0 \geq |F_0'| + \sum_{i \in I''} |C^i| \end{cases}$$

Hence $\lambda_0 = 0$ implies $C' = 0 = C$ and then $\hat{B}^* = 0$, which is in particular a Kružkov monotone matrix.
Case B: $\lambda_0 > 0$
Then in statement of Proposition 5.16, for $n_\gamma := n$, we can consider the functions for $p_\gamma = (p_\gamma^1, \ldots, p_\gamma^{n_\gamma - 2}) \in \mathbb{R}^{n_\gamma - 2}$ with $k_1 := 0$, $k_2 := n_\gamma - 1$

$$\begin{cases} \hat{f}_\gamma^{k_1}(r', p_\gamma, r) & := B_0' r' + D' \cdot p_\gamma + F_0' r \\ \hat{f}_\gamma^{i}(r', p_\gamma, r) & := (C')^i r' + (B \cdot p_\gamma)^i + C^i r, \quad i \in I'' \\ \hat{f}_\gamma^{k_2}(r', p_\gamma, r) & := E_0' r' + D \cdot p_\gamma + B_0 r \end{cases}$$

They are not locally constant, but there is a unique solution $r \in \mathbb{R}$ of

$$-\hat{f}_\gamma^{k_1}(r, p_\gamma, r) = \hat{f}_\gamma^{k_2}(r, p_\gamma, r).$$

The proof follows the lines of Step 6 of the proof of Proposition 5.13, which shows that $\hat{f}$ defined in (10.9) satisfies $D^{\hat{f}} \geq 0$, which means exactly that $\lambda_0^{-1} \hat{B}^\star$ is Kružkov monotone. This ends the proof of the lemma.

Part II
Theory of prefluxes for bell-shaped fluxes

11 Prefluxes for bell-shaped fluxes

In most of the applications, the fluxes are bell-shaped. In this special case, the Godunov flux at the junction enjoys a nice structure: it admits a natural polar decomposition involving a preflux which is easier to analyse. We develop those ideas in the following subsections.

11.1 Preflux and capacity for bell-shaped fluxes

In the special case of bell-shaped fluxes, it is possible to show that Godunov flux at the junction splits in a preflux and a capacity, that we introduce below. The capacity is explicit, while the preflux encodes the structure of the Godunov flux.

Definition 11.1 (Preflux)
Let $N \geq 1$ and $\sigma \in \{\pm 1\}^N$.
0) (Preflux)
We say that $\hat{\gamma}$ is a **preflux** *if it satisfies the following set of conditions*
(11.1)

$$\begin{cases} \hat{\gamma} : [0, +\infty)^N \to [0, +\infty)^N \quad \text{is continuous} & \textbf{(Continuity)} \\ \hat{\gamma} \text{ is locally constant on } \left\{\hat{\gamma} \neq id_{[0,+\infty)^N}\right\} \text{ in the sense of Definition 2.13} & \textbf{(Local constancy)} \\ 0 \leq \hat{\gamma} \leq id_{[0,+\infty)^N} & \textbf{(Bounds)} \end{cases}$$

Recall that the local constancy of $\hat{\gamma}$ means that for all $\gamma_* \in [0, +\infty)^N$, and $I := \left\{j \in \{1, \ldots, N\},\ \hat{\gamma}^j(\gamma_*) \neq \gamma_*^j\right\}$, there exists $\varepsilon > 0$ such that

$$\hat{\gamma} = const = \hat{\gamma}(\gamma_*) \quad on \quad Q_\varepsilon(\gamma_*) := \left(\gamma_* + \sum_{j \in I} (-\varepsilon, \varepsilon) e_j\right) \cap [0, +\infty)^N$$

i) (quasi HJ preflux)
We say that the preflux $\hat{\gamma}$ *is quasi HJ if there exists* $\omega = (\omega^1, \dots, \omega^N) \in (0,+\infty)^N$ *such that*

$$\hat{\gamma}^j = \omega^j g \quad \text{for all} \quad j = 1,\dots,N, \quad \text{for some function } g : [0,+\infty)^N \to [0,+\infty)$$

i') (HJ preflux)
We say that the preflux $\hat{\gamma}$ *is HJ if it is quasi HJ with* $\omega^j = 1$ *for all* $j = 1,\dots,N$*.*
ii) (Kružkov preflux)
We say that the preflux $\hat{\gamma}$ *is Kružkov if*

$$0 \le D_*^{\hat{\gamma}}(\bar{\gamma},\gamma) := \sum_{j=1,\dots,N} sign(\bar{\gamma}^j - \gamma^j) \cdot \left\{\hat{\gamma}^j(\bar{\gamma}) - \hat{\gamma}^j(\gamma)\right\} \quad \text{for all} \quad \bar{\gamma},\gamma \in [0,+\infty)^N$$

iii) (σ-monotone Kružkov preflux)
We say that the preflux $\hat{\gamma}$ *is* σ*-monotone Kružkov if*

$$0 \le D_{*+}^{\hat{\gamma}}(\bar{\gamma},\gamma) := \sum_{j=1,\dots,N} sign^{\sigma^j}(\bar{\gamma}^j - \gamma^j) \cdot \left\{\hat{\gamma}^j(\bar{\gamma}) - \hat{\gamma}^j(\gamma)\right\} \quad \text{for all} \quad \bar{\gamma},\gamma \in [0,+\infty)^N$$

(where we make some abuse of notation for $sign^{\sigma^j} = sign^+$ *or* $sign^-$*).*
iv) (σ-monotone preflux)
We say that the preflux $\hat{\gamma}$ *is* σ*-monotone if*

$$\text{the maps } \gamma \mapsto \sigma^j\hat{\gamma}^j(\gamma) \text{ are nonincreasing in the variable } \sigma^k\gamma^k \text{ for all } k \neq j.$$

v) (conservative preflux)
We say that the preflux $\hat{\gamma}$ *is conservative if*

$$\sum_{j=1,\dots,N} \sigma^j \cdot \hat{\gamma}^j = 0 \tag{11.2}$$

We also say that the preflux $\hat{\gamma}$ *is n:m conservative if the n is the number of indices j such that* $\sigma^j = -1$ *and m is the number of indices j such that* $\sigma^j = +1$*.*

Remark 11.2 *Notice that equivalently to point 0) of Definition 11.1, a preflux on* $Q := [0,+\infty)^N$ *can be defined as a continuous function as a function* $\hat{\gamma} : Q \to Q$ *such that* $0 \le \hat{\gamma} \le id_Q$ *and satisfying moreover the following local constancy condition*

$$\partial_j \hat{\gamma} = 0 \quad \text{on} \quad \left\{(\hat{\gamma} - id_Q)^j < 0\right\}, \quad \text{for all} \quad j = 1,\dots,N \tag{11.3}$$

Notice that the derivative applies not only to component $\hat{\gamma}^j$*, but to the whole vector* $\hat{\gamma}$*. This is a very strong coupling condition between the components.*

Remark 11.3 *Notice that with our definition v), we see that n:m conservative prefluxes and m:n conservative prefluxes do coincide, because relation (11.2) does not distinguish between* σ *and* $-\sigma$*.*

Lemma 11.4 (Basic monotonicity of prefluxes)
Let $N \ge 1$ *and* $\hat{\gamma} : [0,+\infty)^N \to [0,+\infty)^N$ *be a preflux in the sense of Definition 11.1. Then it satisfies*

$$\gamma \mapsto \hat{\gamma}^j(\gamma) \text{ is nondecreasing in } \gamma^j\text{, for all } j = 1,\dots,N \qquad \textbf{(Basic monotonicity)}$$

Proof of Lemma 11.4
Let us fix all coordinates $\gamma^j = \gamma_0^j$ for $j = 2,\dots,N$. Then the map $\hat{f}^1(\gamma^1) := \hat{\gamma}^1(\gamma^1,\gamma_0^2,\dots,\gamma_0^N)$ is a preflux for $N' = 1$ branch. This means that $\hat{f}^1$ is continuous, locally constant on $\left\{\hat{f}^1 \neq id_{[0,+\infty)}\right\}$ and satisfies $0 \le \hat{f}^1(\gamma^1) \le \gamma^1$ for all $\gamma^1 \in [0,+\infty)$. Let us define

$$A^1 := \inf\left\{\gamma^1 > 0, \quad \hat{f}^1(\gamma^1) < \gamma^1\right\}$$

with the convention that $A^1 = +\infty$ for the infimum of the empty set. Then we get easily that

$$\hat{f}^1(\gamma^1) = \min\left\{\gamma^1, A^1\right\}$$

In particular, this shows that $\hat{f}^1$ is nondecreasing, i.e. the map $\gamma \mapsto \hat{\gamma}^1(\gamma)$ is nondecreasing in γ^1. The same reasoning also works for all indices $j = 1,\dots,N$ and gives the result. This ends the proof of the lemma.

Remark 11.5 *Notice that condition (11.1) defining a preflux, does NOT mean exactly that $\hat\gamma$ is a Godunov flux for a junction $N:0$ with function $f := id_{[0,+\infty)^N}$ associated to some Riemann germ*[4]. *Indeed let us consider $\sigma = (1,\dots,1) \in \mathbb{R}^N$, $a = 0_{\mathbb{R}^N}$ and $b := (+\infty,\dots,+\infty)$. Then we see that $\hat\gamma$ satisfies all properties of a Godunov flux associated to a generalized Riemann germ given in ii) of Theorem 2.15 with $f_+ = id_{[0,+\infty)^N} = f_-$, but in the second line of (2.14), inequality $f_- \le \hat\gamma$ is not satisfied in general (except in the special case where $\hat\gamma \equiv id_{[0,+\infty)^N}$). In particular (from Theorem 2.17), we deduce that the set $\mathcal{G}_{\hat\gamma} := \left\{\hat\gamma = id_{[0,+\infty)^N}\right\} \subset [0,+\infty)^N$ is not a Riemann germ when $\hat\gamma \not\equiv id_{[0,+\infty)^N}$.*

Definition 11.6 (Capacity, for bell-shaped fluxes)
Assume (2.2) for $N \ge 1$.
i) (Bell-shaped)
We say that $f = (f^1,\dots,f^N)$ is bell-shaped, if each scalar function $f^k : [a^k,b^k] \to [0,+\infty)$ is continuous, satisfies $f^k(a^k) = 0 = f^k(b^k)$, has a maximum value $f^k_{\max} := f^k(c^k)$ at $c^k \in (a^k,b^k)$ and is increasing on (a^k,c^k) and decreasing on (c^k,b^k) for $k = 1,\dots,N$. We set the monotone functions

$$(11.4)\qquad f^{k,+}(q) = \begin{cases} f^k(q) & \text{for } q \in [a^k,c^k] \\ f^k(c^j) & \text{for } q \in [c^k,b^k] \end{cases} \quad \text{and} \quad f^{k,-}(q) = \begin{cases} f^k(c^k) & \text{for } q \in [a^k,c^k] \\ f^k(q) & \text{for } q \in [c^k,b^k] \end{cases}$$

ii) (Capacity)
We recall that the orientations of each branch J^k is encoded in $\sigma^k = +1$ if $J^k \simeq (-\infty,0)$ and $\sigma^k = -1$ if $J^k \simeq (0,+\infty)$. We define the capacity $\bar\gamma = (\bar\gamma^1,\dots,\bar\gamma^N) : [a,b] \to [0,+\infty)^N$ as the following function for $p \in [a,b]$

$$(11.5)\qquad \bar\gamma^k(p) := \bar\gamma^k(p^k) := f^{k,\sigma^k}(p^k) \quad \text{for all} \quad k = 1,\dots,N$$

(with a slight abuse of notation).

Remark 11.7 *As an example, consider some $1:1$ junction with indices $j = L,R$ (for left and right) with $\sigma^L = 1$ and $\sigma^R = -1$. Then the standard Godunov flux $\hat f = (\hat f^L, \hat f^R)$ associated to some HJ germ with flux limiter A, is given by $\hat f^L(p) = \hat f^R(p) = \min\left\{A, f^{L,+}(p^L), f^{R,-}(p^-)\right\}$. Then the capacity is $\bar\gamma(p) = (f^{L,+}(p^L), f^{R,-}(p^R))$ for $p = (p^L,p^R)$ and the preflux is $\hat\gamma = (\hat\gamma^L,\hat\gamma^R)$ with $\hat\gamma^L(\gamma) = \hat\gamma^R(\gamma) = \min\left\{A,\gamma^L,\gamma^R\right\}$ for $\gamma = (\gamma^L,\gamma^R)$. Here the preflux $\hat\gamma$ has all the properties: $\hat\gamma$ is HJ, conservative, σ-monotone, Kružkov, and σ-monotone Kružkov.*

11.2 Polar decomposition of Godunov flux: Demand and Supply interpretation

We show that the Godunov flux has a unique polar decomposition in a preflux composed with a capacity. The preflux is unique on the image of the capacity.

Theorem 11.8 (Polar decomposition of Godunov flux, for bell-shaped fluxes)
Assume (2.2) for $N \ge 1$ for a junction (J,f) with $\sigma \in \{\pm1\}^N$. Assume that f is bell-shaped in the sense of Definition 11.6, and call $\bar\gamma : [a,b] \to [0,+\infty)^N$ the capacity given by Definition 11.6.
i) (Polar decomposition)
Let $\mathcal{G} \subset [a,b]$ be a Riemann germ with respect to (J,f). Then the Godunov flux $\hat f_{\mathcal{G}}$ associated to $\mathcal{G}$ has the following polar decomposition

$$(11.6)\qquad \hat f_{\mathcal{G}} = \hat\gamma \circ \bar\gamma$$

where $\hat\gamma : [0,+\infty)^N \to [0,+\infty)^N$ is a preflux (as in Definition 11.1) and $\bar\gamma$ is the capacity. Moreover the preflux $\hat\gamma$ is unique on the image of the capacity

$$\hat K := Im(\bar\gamma) = \prod_{k=1,\dots,N} [0, f^k_{\max}]$$

ii) (Riemann germ construction)
Given any preflux $\hat\gamma : [0,+\infty)^N \to [0,+\infty)^N$, we define

$$(11.7)\qquad \mathcal{G} := \mathcal{G}_{\hat f} := \left\{p \in [a,b], \quad \hat f(p) = f(p)\right\} \quad \text{with} \quad \hat f := \hat\gamma \circ \bar\gamma : [a,b] \to [0,+\infty)^N.$$

[4]And contrarily to what the author thought and stated erroneously in a preliminary version of this work.

Then $\mathcal{G}$ is a Riemann germ.

iii) (Further properties of the germ)

In the previous construction i)-ii), the preflux $\hat{\gamma}$ is such that the restriction $\hat{\gamma}_{|\hat{K}}$ is HJ (resp. σ-monotone, conservative, conservative Kružkov), if and only if the germ $\mathcal{G}$ is HJ (resp. monotone, conservative, conservative Kružkov).

Moreover, if the Riemann germ $\mathcal{G}$ is Kružkov (resp. monotone Kružkov), then the restriction of the preflux $\hat{\gamma}_{|\hat{K}}$ is also Kružkov (resp. σ-monotone Kružkov).

iv) (Counter-example: Kružkov preflux $\not\Longrightarrow$ Kružkov germ)

There are examples where the restriction of the preflux $\hat{\gamma}_{|\hat{K}}$ is Kružkov (resp. σ-monotone Kružkov), and where the associated Riemann germ $\mathcal{G}_{\hat{f}}$ given in (11.7) is not Kružkov (resp. not monotone Kružkov).

v) (Conservative preflux)

Let us consider a conservative preflux $\hat{\gamma} : [0,+\infty)^N \to [0,+\infty)^N$. Then $\hat{\gamma}$ is σ-monotone if and only if it is Kružkov.

In particular Theorem 11.8 implies Theorem 2.40 of the Introduction.

Remark 11.9 (Mathematical proof of Demand and Supply interpretation of Lebacque) *The polar decomposition (11.6) is a mathematical result for bell-shaped fluxes which can be interpretated as in Lebacque [37], published in 1996: here the capacity denotes either the Demand of ingoing branches or the Supply of outgoing branches.*

Remark 11.10 *Notice that Kružkov property for preflux $\hat{\gamma}$ is not transferable in general to the germ $\mathcal{G}_{\hat{f}}$, because the Kružkov property does not behave well by composition by functions in general, contrarily to monotonicity properties.*

Proof of Theorem 11.8

Step 1: proof of ii)

Step 1.1: continuity and basic monotonicity

We notice that the map $\bar{\gamma} : [a,b] \to [0,+\infty)^N$ defined in (11.5) is continuous and each map $p^j \mapsto \sigma^j \bar{\gamma}^j(p^j)$ is nondecreasing. This implies that $\hat{f}$ is continuous and the map $p \mapsto \sigma^j \hat{f}^j(p)$ is nondecreasing in p^j for all indices j.

Step 1.2: local constancy

Let us now check that $\hat{f}$ is locally constant on $\left\{\hat{f} \neq f\right\}$. Fix some $p \in [a,b]$, and let

$$\tilde{I} := \left\{ j \in \{1,\dots,N\},\ \hat{f}^j(p) \neq f^j(p) \right\} \quad \text{and} \quad I := \left\{ j \in \{1,\dots,N\},\ \hat{\gamma}^j(\bar{\gamma}(p)) \neq \bar{\gamma}^j(p) \right\}$$

Consider $q \in [a,b]$ such that

$$q^j \quad \begin{cases} = p^j & \text{if} \quad j \notin \tilde{I} \\ \in (p^j - \varepsilon, p^j + \varepsilon) \cap \left[a^j, b^j\right] & \text{if} \quad j \in \tilde{I} \end{cases}$$

Now, consider some $j \in \tilde{I}$ and assume that $\sigma^j = +1$ (the case $\sigma^j = -1$ is similar). We distinguish two cases.

Case A: $p^j \in [a^j, c^j]$

Then we have $\hat{\gamma}^j(\bar{\gamma}(p)) = \hat{f}^j(p) \neq f^j(p) = f^{j,+}(p^j) = \bar{\gamma}^j(p)$, which shows that $j \in I$. Moreover $\bar{\gamma}^j(q) \in \left(\bar{\gamma}^j(p) - \delta, \bar{\gamma}^j(p) + \delta\right) \cap [0,+\infty)$ for some $\delta > 0$ small enough.

Case B: $p^j \in (c^j, b^j]$

Then for $\varepsilon > 0$ small enough, we have $q^j > c^j$ and then $\bar{\gamma}^j(q) = f^{j,+}(q^j) = f^j(c^j) = \bar{\gamma}^j(p)$.

Conclusion

Using both cases A and B, and the local constancy of $\hat{\gamma}$ on $\left\{\hat{\gamma} \neq id_{[0,+\infty)^N}\right\}$, we deduce that for $\varepsilon > 0$ small enough, we have $\hat{f}(q) = \hat{\gamma}(\bar{\gamma}(q)) = \hat{\gamma}(\bar{\gamma}(p)) = \hat{f}(p)$, which means exactly that $\hat{f}$ is locally constant on $\left\{\hat{f} \neq f\right\}$.

Step 1.3: bounds

We also have for $p \in [a,b]$

$$\begin{cases} f^j_-(p^j) = \inf\limits_{[p^j,b^j]} f^j = 0 \le \hat{f}^j(p) \le \bar{\gamma}^j(p) = f^{j,+}(p^j) = \sup\limits_{[a^j,p^j]} f^j = f^j_+(p^j) & \text{if} \quad J^j \simeq (-\infty,0) \\ \\ f^j_-(p^j) = \inf\limits_{[a^j,p^j]} f^j = 0 \le \hat{f}^j(p) \le \bar{\gamma}^j(p) = f^{j,-}(p^j) = \sup\limits_{[p^j,b^j]} f^j = f^j_+(p^j) & \text{if} \quad J^j \simeq (0,+\infty) \end{cases}$$

From the first three steps and the characterization of generalized Riemann germs (see ii) of Theorem 2.15), and of Riemann germs (see i) of Theorem 2.17), we deduce that $\mathcal{G}$ is a Riemann germ.

Step 2: proof of i)

Let $\mathcal{G}$ be a Riemann germ, and set $\hat{f} := \hat{f}_{\mathcal{G}}$ which is known to be continuous.

Step 2.1: unique decomposition of $\hat{f}$

Notice that the capacity $\bar{\gamma}$ genuinely varies on the set

$$K := \prod_{j=1,\dots,N} K^j \quad \text{with} \quad K^j := \begin{cases} \left[a^j, c^j\right] & \text{if} \quad \sigma^j = 1 \\ \\ \left[c^j, b^j\right] & \text{if} \quad \sigma^j = -1 \end{cases}$$

We then define $\rho = (\rho^1, \dots, \rho^N) : [a,b] \to K$ as

$$\rho^j(p^j) = \begin{cases} \max\left\{p^j, c^j\right\} & \text{if} \quad \sigma^j = 1 \\ \min\left\{p^j, c^j\right\} & \text{if} \quad \sigma^j = -1 \end{cases}, \quad \text{for all} \quad p^j \in [a^j, b^j]$$

which is such that $f^{j,\sigma^j} = f^j \circ \rho^j$, and then $\bar{\gamma} = f \circ \rho$. Because ρ is a projection, we deduce in particular that

$$\bar{\gamma} = \bar{\gamma}_{|K} \circ \rho. \tag{11.8}$$

Now for $p \in [a,b]$, we first distinguish the first index and set $p =: (p^1, p')$, and consider $g := \hat{f}^1(\cdot, p') : [a^1, b^1] \to \mathbb{R}$. Assume also that $\sigma^1 = 1$ (the case $\sigma^1 = -1$ is similar). From Theorem 2.15 and the slicing Lemma 4.12, we know that g is nondecreasing and is locally constant on $\left\{g \neq f^1\right\}$. Because f^1 is decreasing on (c^1, b^1), we deduce that g is constant on $[c^1, b^1]$. Hence $g \neq f^1$ a.e. on $[c^1, b^1]$. Because $\hat{f}$ is locally constant on $\left\{\hat{f} \neq f\right\}$, we deduce that the whole function $\hat{f}(\cdot, p') : [a^1, b^1] \to \mathbb{R}^N$ is locally constant on $[c^1, b^1]$. Therefore, we have $\hat{f}(\cdot, p') = \hat{f}(\cdot, p') \circ \rho^1$. The the same raisonning with all indices j, shows that

$$\hat{f} = \hat{f} \circ \rho \tag{11.9}$$

We now define

$$\hat{\gamma} := \hat{\gamma}_{|\hat{K}} := \hat{f} \circ (\bar{\gamma}_{|K})^{-1} : \Gamma_0(c) \to \Gamma_0(c) \quad \text{with} \quad \Gamma_0(c) := [0, \bar{\gamma}(c)] = \prod_{j=1,\dots,N} \left[0, f^j(c^j)\right]. \tag{11.10}$$

Hence we get $\hat{f}_{|K} = \hat{\gamma} \circ \bar{\gamma}_{|K}$ and then using (11.9), we get

$$\begin{aligned} \hat{f} &= \hat{f}_{|K} \circ \rho \\ &= \hat{\gamma} \circ \bar{\gamma}_{|K} \circ \rho \\ &\overset{(11.8)}{=} \hat{\gamma} \circ \bar{\gamma}. \end{aligned}$$

Notice that the invertibility of $\bar{\gamma}_{|K}$ shows that the function $\hat{\gamma}$ is unique on the image K of the capacity $\bar{\gamma}$ and given by (11.10).

When necessary, we can define

$$\hat{\gamma} := \hat{\gamma}_{|\hat{K}} \circ \hat{\rho}_{\hat{K}} : \qquad [0, +\infty)^N \to [0, +\infty)^N$$

where

$$\begin{array}{rlcl} \hat{\rho}_{\hat{K}} : & [0, +\infty)^N & \to & \hat{K} := \Gamma_0(c) \\ & (\lambda^1, \dots, \lambda^N) & \mapsto & \left(\min\left\{\lambda^1, f^1(c^1)\right\}, \dots, \min\left\{\lambda^N, f^N(c^N)\right\}\right) \end{array}$$

is the projection on $\hat{K}$. Then it is easy to check that now $\hat{\gamma}$ has the same properties as its restriction $\hat{\gamma}_{|\hat{K}}$, but is extended to the whole $[0, +\infty)^N$.

Step 2.2: properties of the preflux

The fact that $\hat{\gamma}$ is a preflux in the sense of point 0) of Definition 11.1 (i.e. continuity, bounds, basic monotonicity and local constancy) follows immediately from its expression (11.10), and from the similar properties of $\hat{f}$.

Step 3: proof of iii)

Step 3.1: σ-monotone preflux
For a σ-monotone preflux, we know for $k \neq j$ that $\sigma^j \hat{\gamma}^j$ is nonincreasing in $\sigma^k \bar{\gamma}^k$, which is itself nondecreasing in p^k, by definition of $\bar{\gamma}$. By composition, we deduce that $\sigma^j \hat{f}^j$ is nonincreasing in p^k for all $k \neq j$. This shows that if the preflux $\hat{\gamma}$ is σ-monotone, then (from Lemma 5.5) the Riemann germ $\mathcal{G}$ is monotone. Conversely, we similarly get that if $\mathcal{G}$ is monotone, then $\hat{\gamma}$ is σ-monotone.
Step 3.2: HJ and conservative properties
It is straightforward to check that the preflux $\hat{\gamma}$ is HJ (resp. conservative) if and only if $\hat{f}$ satisfies the same properties, which from Lemma 5.5 is equivalent to the similar properties for the germ $\mathcal{G}$.
Step 3.3: Kružkov germ $\Longrightarrow$ Kružkov preflux
The result follows from the restriction $\hat{f}_{|K} = \hat{\gamma} \circ \bar{\gamma}_{|K}$, and the fact that $\bar{\gamma}_{|K} : K \to [0, \bar{\gamma}(c)]$ is bijective. We also use the change of variables $\text{sign}(\bar{\gamma}^j(\bar{p}^j) - \bar{\gamma}^j(p^j)) = \sigma^j \text{sign}(\bar{p}^j - p^j)$ for $\bar{p}, p \in K$. The similar result holds true for monotone Kružkov germ which implies σ-monotone Kružkov preflux, using the change of variables $\text{sign}^{\sigma^j}(\bar{\gamma}^j(\bar{p}^j) - \bar{\gamma}^j(p^j)) = \sigma^j \text{sign}^+(\bar{p}^j - p^j)$.
Step 3.4: conservative Kružkov
If $\mathcal{G}_{\hat{f}}$ is conservative Kružkov, then Steps 3.2 and 3.3 imply that the preflux $\hat{\gamma}$ is conservative Kružkov. Conversely, if the preflux $\hat{\gamma}$ is conservative Kružkov, we can only indirectly show that it transfers to the germ. We first show the following result.
Step 3.4.1: $\hat{\gamma}$ is conservative Kružkov $\Longrightarrow$ $\hat{\gamma}$ is σ-monotone
We adapt Step 2 of the proof of Theorem 2.29, in a much more direct way.
We first notice that for $p = (p^1, p')$ and $\hat{\gamma}_0^1(p^1) := \hat{\gamma}^1(p^1, p')$, the continuous map $\hat{\gamma}_0^1$ satisfies $0 \leq \hat{\gamma}_0^1 \leq id_{[0,+\infty)}$ and is locally constant on $\left\{\hat{\gamma}_0^1 \neq id_{[0,+\infty)}\right\}$. This implies that $\hat{\gamma}_0^1(p^1) = \min\left\{p^1, A\right\}$ for some constant A, and in particular this map is Lipschitz continuous and satisfies $(\hat{\gamma}_0^1)'(p^1) \in \{0, 1\}$ a.e.. The same result also holds true for each map $p^j \mapsto \hat{\gamma}^j(p)$. Then from Proposition 4.20 ii) and the fact that $\hat{\gamma}$ is Kružkov, we deduce that $\hat{\gamma}$ is Lipschitz continuous on $[0, +\infty)^N$ and satisfies

$$\partial_j \hat{\gamma}^j \geq \sum_{k \in \{1,\ldots,N\}\setminus\{j\}} |\partial_j \hat{\gamma}^k| \quad \text{a.e.}$$

On the other hand, $\hat{\gamma}$ is conservative, i.e. satisfies $\sum_k \sigma^k \hat{\gamma}^k = 0$. Hence we get

$$\sigma^j \partial_j \hat{\gamma}^j + \sum_{k \in \{1,\ldots,N\}\setminus\{j\}} \sigma^k \partial_j \hat{\gamma}^k = 0 \quad \text{a.e.}$$

Because $\partial_j \hat{\gamma}^j \in \{0, 1\}$, we deduce that

$$\sigma^j \sigma^k \partial_j \hat{\gamma}^k \leq 0 \quad \text{a.e.}$$

i.e. that $\sigma^k \hat{\gamma}^k$ is nonincreasing in $\sigma^j p^j$, i.e. that $\hat{\gamma}$ is σ-monotone.
Step 3.4.2: core of the proof
Now we know that $\hat{\gamma}$ is σ-monotone and conservative. Because those properties are transferable in general to the germ $\mathcal{G}_{\hat{f}}$ (from Steps 3.1 and 3.2), we deduce that $\mathcal{G}_{\hat{f}}$ is monotone conservative, and then from Theorem 2.29, $\mathcal{G}_{\hat{f}}$ is also conservative Kružkov. This ends the proof of Step 3.4 for the equivalence of conservative Kružkov preflux $\hat{\gamma}$ and conservative Kružkov germ $\mathcal{G}_{\hat{f}}$.
Step 4: proof of iv), a counter-example
We build a counter-example for some $1:1$ junction, with indices $j = L, R$ (for left and right). We build some preflux $\hat{\gamma}$ which is Kružkov, but non conservative. We will then show that $\mathcal{G}_{\hat{f}}$ is not Kružkov for $\hat{f} := \hat{\gamma} \circ \bar{\gamma}$, with the capacity $\bar{\gamma}(p) = (f^{L,+}(p^L), f^{R,-}(p^R))$ for $p = (p^L, p^R) \in [a, b]$, where $f = (f^L, f^R)$ is bell-shaped. Precisely, consider $\phi : [0, +\infty) \to [0, +\infty)$ which is a continuous increasing and bijective function. We set

$$\hat{\gamma} : [0, +\infty)^2 \to [0, +\infty)^2 \quad \text{with} \quad \hat{\gamma}(\gamma) := \left(\min\left\{\gamma^L, \phi(\gamma^R)\right\},\ \min\left\{\gamma^R, \phi^{-1}(\gamma^L)\right\}\right) \quad \text{for} \quad \gamma := (\gamma^L, \gamma^R)$$

Then it is easy to check that $\hat{\gamma} = (\hat{\gamma}^L, \hat{\gamma}^R)$ is a preflux. Moreover for $\sigma = (\sigma^L, \sigma^R) = (1, -1)$, the preflux $\hat{\gamma}$ is σ-monotone, Kružkov, and σ-monotone Kružkov. Moreover $\hat{\gamma}$ is conservative (or equivalently HJ) if and only if $\phi = id_{[0,+\infty)}$. Now assume that $\phi \neq id_{[0,+\infty)}$, such that $\hat{\gamma}$ is not conservative, say with $\phi(\gamma_0^R) > \gamma_0^R$ for some $\gamma_0^R > 0$. Then assume that the maxima of the bell-shaped functions satisfy $f^L(c^L) > \phi(\gamma_0^R)$ and $f^R(c^R) > \gamma_0^R$. Then consider $p_0^R \in (c^R, b^R)$ such that $f^R(p_0^R) = \gamma_0^R$, and $p_0^L \in (a^L, c^L)$ such that $f^L(p_0^L) = \phi(\gamma_0^R)$, and call $p_0 := (p_0^L, p_0^R)$. Then we get

$$\bar{\gamma}(b) = (f^{L,+}(b^L), f^{R,-}(b^R)) = (f^L(c^L), 0), \quad \bar{\gamma}(p_0) = (f^{L,+}(p_0^L), f^{R,-}(p_0^R)) = (\phi(\gamma_0^R), \gamma_0^R)$$

and using $\hat{f} := \hat{\gamma} \circ \bar{\gamma}$, we get $\hat{f}(b) = (0,0)$ and $\hat{f}(p_0) = (\phi(\gamma_0^R), \gamma_0^R)$. Hence

$$\begin{aligned} D^{\hat{f}}(b, p_0) &= \text{sign}(b^L - p_0^L) \cdot \left\{ \hat{f}^L(b) - \hat{f}^L(p_0) \right\} - \text{sign}(b^R - p_0^R) \cdot \left\{ \hat{f}^R(b) - \hat{f}^R(p_0) \right\} \\ &= \left\{ \hat{f}^L(b) - \hat{f}^L(p_0) \right\} - \left\{ \hat{f}^R(b) - \hat{f}^R(p_0) \right\} \\ &= \gamma_0^R - \phi(\gamma_0^R) \\ &< 0 \end{aligned}$$

The case $\phi(\gamma_0^R) < \gamma_0^R$ leads similarly to $D^{\hat{f}}(a, p_0) = \phi(\gamma_0^R) - \gamma_0^R < 0$, using $\hat{f}(a) = (0,0)$. Then from Lemma 5.5, we deduce that the Riemann germ $\mathcal{G}_{\hat{f}} \subset [a,b]$ is not Kružkov. A fortiori, $\mathcal{G}_{\hat{f}}$ is neither monotone Kružkov.

Step 5: proof of v)

From point iii), we know that $\hat{\gamma}$ is conservative Kružkov, if and only if the germ $\mathcal{G}_{\hat{f}}$ is conservative Kružkov, with $\hat{f} := \hat{\gamma} \circ \bar{\gamma}$. From Theorem 2.29, we know that a germ is conservative Kružkov, if and only if it is conservative monotone. Finally from point ii), we deduce that $\mathcal{G}_{\hat{f}}$ is conservative monotone if and only if the preflux $\hat{\gamma}$ is conservative σ-monotone. This finally shows that the preflux is conservative Kružkov is and only if it is conservative σ-monotone. This ends the proof of the theorem.

11.3 Further properties of prefluxes

Lemma 11.11 (Characterizations and properties of prefluxes)
Assume $N \geq 1$ and $\sigma \in \{\pm 1\}^N$. Let $\hat{\gamma} : [0, +\infty)^N \to [0, +\infty)^N$ be a preflux.

i) (Riemann monotonicity and projection property of prefluxes)
Then the preflux $\hat{\gamma}$ is Riemann monotone and is a nonlinear projection, i.e. satisfies $\hat{\gamma} \circ \hat{\gamma} = \hat{\gamma}$.
In particular, we have

$$\hat{\gamma} : [0, +\infty)^N \to \mathcal{G}_{\hat{\gamma}} := \left\{ \hat{\gamma} = id_{[0,+\infty)^N} \right\}$$

where the set $\mathcal{G}_{\hat{\gamma}}$ is not a generalized Riemann germ in general (except if $\hat{\gamma} \equiv id_{[0,+\infty)^N}$).

ii) (Characterization of Kružkov prefluxes)
Then the preflux $\hat{\gamma}$ is Kružkov if and only if $\hat{\gamma}$ is Lipschitz continuous and satisfies

$$\partial_j \hat{\gamma}^j \geq \sum_{k \in \{1,\dots,N\} \setminus \{j\}} |\partial_j \hat{\gamma}^k| \quad \textit{for all indices} \quad j \tag{11.11}$$

iii) (Characterization of conservative Kružkov prefluxes)
Then the preflux $\hat{\gamma}$ is conservative Kružkov if and only if it satisfies

$$\left\{ \begin{array}{l} \displaystyle\sum_{j=1,\dots,N} \sigma^j \hat{\gamma}^j = 0 \\ \\ \sigma^k \sigma^j \partial_k \hat{\gamma}^j \leq 0 \quad \textit{in} \quad \mathcal{D}'((0,+\infty)^N) \qquad \textit{for all} \quad k \neq j \end{array} \right. \tag{11.12}$$

Proof of Lemma 11.11

Step 0: preliminaries

Even if $\hat{\gamma}$ is not the Godunov flux associated to a Riemann germ, it is almost the case. To see (and use) it, consider any $b \in (0, +\infty)^N$ and $a := 0_{\mathbb{R}^N}$, and $f^j(p^j) := \min\left\{p^j, 2b^j - p^j\right\}$. Then f is bell-shaped on $[0, 2b]$. For $\sigma = (1, \dots, 1) \in \mathbb{R}^N$, we set $f^{j,+}(p^j) := \min\left\{p^j, b^j\right\}$. Then, from Theorem 11.8, we deduce that the function $\hat{f} := \hat{\gamma} \circ f^+$ is a Godunov flux associated to a Riemann germ, with moreover

$$\hat{f} = \hat{\gamma} \quad \text{on} \quad [a, b]$$

Hence all the properties of $\hat{f}$ can be transfered to γ, in the limit $b \to (+\infty, \dots, +\infty)$.

Step 1: proof of i)

From i) of Theorem 2.20, we deduce that $\hat{f}$ is Riemann monotone on $[a, 2b]$, and then

$$\hat{\gamma} \text{ is Riemann monotone on } [a, b]. \tag{11.13}$$

Moreover, let us define the associated Riemann germ

$$\mathcal{G}_{2b} := \left\{ p \in [a, 2b], \quad \hat{f}(p) = f(p) \right\}$$

and the part

$$\mathcal{G}_b := \mathcal{G}_{2b} \cap [a,b] = \left\{p \in [a,b], \quad \hat{f}(p) = f(p)\right\} = \{p \in [a,b], \quad \hat{\gamma}(p) = p\}$$

Hence $\hat{\gamma}_{|\mathcal{G}_b} = id_{\mathcal{G}_b}$ and

$$\hat{\gamma} \circ \hat{\gamma} = \hat{\gamma} \quad \text{on} \quad [a,b] \tag{11.14}$$

Because (11.13) and (11.14) are true for any $b \in (0,+\infty)^N$, we deduce that $\hat{\gamma}$ is Riemann monotone and satisfies $\hat{\gamma} \circ \hat{\gamma} = \hat{\gamma}$ on $[a,+\infty)^N = [0,+\infty)^N$.

Step 2: proof of ii)

The result follows directly from Proposition 4.20 ii). Indeed, we have to notice that $\hat{\gamma}^j(\bar{\gamma}) = \min\left\{\bar{\gamma}^j, A^j(\bar{\gamma})\right\}$ where the function A^j does not depend on $\bar{\gamma}^j$. This implies in particular that the map $\bar{\gamma} \mapsto \hat{\gamma}^j(\bar{\gamma})$ is 1-Lipschitz in the variable $\bar{\gamma}^j$, and then allows us to apply ii) of Proposition 4.20.

Step 3: proof of iii)

We use point v) of Theorem 11.8 to deduce that conservative Kružkov prefluxes are conservative σ-monotone prefluxes. Then the result follows from Definition 11.1. This ends the proof of the lemma.

We finish this section with examples.

Lemma 11.12 (Example of the truncation preflux)

Assume $N \geq 1$, and let $\bar{\lambda} = (\bar{\lambda}^1, \ldots, \bar{\lambda}^N) \in [0,+\infty]^N$, and the truncation function

$$T_{\bar{\lambda}} : [0,+\infty)^N \to [0,+\infty)^N \quad \textit{with} \quad T_{\bar{\lambda}}(\gamma) = \left(\min\left\{\gamma^1, \bar{\lambda}^1\right\}, \ldots, \min\left\{\gamma^N, \bar{\lambda}^N\right\}\right) \tag{11.15}$$

Then $T_{\bar{\lambda}}$ is a (Kružkov) preflux.

Proof of Lemma 11.12

The function $T_{\bar{\lambda}}$ is continuous. We obviously have $0 \leq T_{\bar{\lambda}}(\gamma) \leq \gamma$. Moreover if $T^j_{\bar{\lambda}}(\gamma) < \gamma^j$, then all coordinates of $T_{\bar{\lambda}}$ are locally independent on γ^j, which means that $T_{\bar{\lambda}}$ is locally constant on $\left\{T_{\bar{\lambda}} \neq id_{[0,+\infty)^N}\right\}$. Finally each map $\gamma \mapsto T^j_{\bar{\lambda}}(\gamma)$ is nondecreasing in γ^j. This shows that $T_{\bar{\lambda}}$ is a preflux. Moreover its is straightforward to check that it is a Kružkov preflux. This ends the proof of the lemma.

Lemma 11.13 (Prefluxes composed with a truncation from the right)

Assume $N \geq 1$, and let $\hat{\gamma}_0 : [0,+\infty)^N \to [0,+\infty)^N$ be a preflux, and for $\bar{\lambda} \in [0,+\infty]^N$, let $T_{\bar{\lambda}} : [0,+\infty)^N \to [0,+\infty)^N$ be the truncation preflux defined in (11.15). Then $\hat{\gamma} := \hat{\gamma}_0 \circ T_{\bar{\lambda}}$ is also a preflux.
Moreover, if $\hat{\gamma}_0$ is HJ (resp. σ-monotone, conservative; Kružkov), then $\hat{\gamma}$ has the same property.

Proof of Lemma 11.12

Step 1: $\hat{\gamma}$ is a preflux

The function $\hat{\gamma}$ is continuous by composition. Recall that $0 \leq \hat{\gamma}_0(\gamma), T_{\bar{\lambda}}(\gamma) \leq \gamma$, an inequality which is also preserved by composition. Assume that for some $\gamma_* \in [0,+\infty)^N$, we have $\hat{\gamma}^j_0(T_{\bar{\lambda}}(\gamma_*)) = \hat{\gamma}^j(\gamma_*) < \gamma^j_*$. We know that $T^j_{\bar{\lambda}}(\gamma_*) \leq \gamma^j_*$.

Case A: $T^j_{\bar{\lambda}}(\gamma_*) = \gamma^j_*$

Recall that $\hat{\gamma}_0(\gamma)$ is independent on γ^j close to $T^j_{\bar{\lambda}}(\gamma_*) = \gamma^j_*$. Now because

$$\text{the components of } T^k_{\bar{\lambda}}(\gamma) \text{ do not depend on } \gamma^j \text{ for } k \neq j, \tag{11.16}$$

this implies that $\hat{\gamma} = \hat{\gamma}_0 \circ T_{\bar{\lambda}}$ is also independent on γ^j close to γ^j_*.

Case B: $T^j_{\bar{\lambda}}(\gamma_*) < \gamma^j_*$

Then the whole vector $T_{\bar{\lambda}}$ is independent on γ^j for γ^j close to γ^j_*, and then $\hat{\gamma} = \hat{\gamma}_0 \circ T_{\bar{\lambda}}$ is also independent on γ^j for γ^j close to γ^j_*.

This shows that $\hat{\gamma}$ is locally constant on $\left\{\hat{\gamma} \neq id_{[0,+\infty)^N}\right\}$. Finally the fact that $\gamma \mapsto \hat{\gamma}^j_0(\gamma)$ is nondecreasing in γ^j, and the same property for $T_{\bar{\lambda}}$ and property (11.16) imply that $\gamma \mapsto (\hat{\gamma}^j_0 \circ T_{\bar{\lambda}})(\gamma)$ is nondecreasing in γ^j. Therefore $\hat{\gamma}$ is also a preflux.

Step 2: further properties of $\hat{\gamma}$

Notice that if $\hat{\gamma}_0$ is HJ (resp. σ-monotone, conservative), then it is directly transferable by composition to $\hat{\gamma} = \hat{\gamma}_0 \circ T_{\bar{\lambda}}$. Kružkov property is not transferable directly, but here follows from the characterization of Kružkov prefluxes by ii) of Lemma 11.11. This ends the proof of the lemma.

We also have the following extension result whose the proof is straightforward.

Lemma 11.14 (Extension of prefluxes outside a box $[0,\bar\lambda]$)
Let $N\ge 1$ and let $\bar\lambda\in(0,+\infty)^N$. Assume that $\hat\gamma_0:[0,\bar\lambda]\to[0,+\infty)^N$ satisfies preflux condition only on the box $[0,\bar\lambda]$, i.e.
(11.17)
$$\begin{cases} \hat\gamma:[0,\bar\lambda]\to[0,+\infty)^N \quad \text{is continuous} & \textbf{(Continuity)}\\ \hat\gamma \text{ is locally constant on } \left\{\bar\gamma\in[0,\bar\lambda],\quad \hat\gamma(\bar\gamma)\neq\bar\gamma\right\} \text{ in the sense of Definition 2.13} & \textbf{(Local constancy)}\\ 0\le\hat\gamma\le id_{[0,\bar\lambda]} & \textbf{(Bounds)}\end{cases}$$
Consider the truncation operator $T_{\bar\lambda}:[0,+\infty)^N\to[0,\bar\lambda]$ defined in (11.15). Then $\hat\gamma_0\circ T_{\bar\lambda}:[0,+\infty)^N\to[0,+\infty)^N$ is a preflux.

Remark 11.15 (Loosing preflux property by composition from the left)
Notice that $\hat\gamma:=T_{\bar\lambda}\circ\hat\gamma_0$ is not a preflux in general. It is continuous and satisfies $0\le\hat\gamma\le id_{[0,+\infty)^N}$, but is not locally constant in general. Indeed

$$(T_{\bar\lambda}\circ\hat\gamma_0)^j(\bar\gamma)<\bar\gamma^j \tag{11.18}$$

means $\min\left\{\bar\lambda^j,\hat\gamma_0^j(\bar\gamma)\right\}<\bar\gamma^j$*. And if for some $k\neq j$, we know that $\hat\gamma_0^k(\bar\gamma)$ does depend on $\bar\gamma^j$, then locally we have*
$$\begin{cases}\hat\gamma_0^j(\bar\gamma)=\bar\gamma^j\\ \hat\gamma_0^k(\bar\gamma)<\bar\gamma^k\end{cases}$$
Therefore, we can choose $\bar\lambda^k$ large enough, and $\bar\lambda^j$ small enough such that
$$\begin{cases}\bar\lambda^j<\hat\gamma_0^j(\bar\gamma)=\bar\gamma^j\\ \hat\gamma_0^k(\bar\gamma)<\min\left\{\bar\lambda^k,\bar\gamma^k\right\}\end{cases}$$
and then this implies (11.18), while $(T_{\bar\lambda}\circ\hat\gamma_0)^k(\bar\gamma)=\hat\gamma_0^k(\bar\gamma)$ is not locally constant in $\bar\gamma^j$.

We also have

Lemma 11.16 (Gluing with a flux limiter and the truncation preflux)
Assume (2.2) for $N\ge1$ for a junction (J,f). Assume that f is bell-shaped in the sense of Definition 11.6, and call $\bar\gamma:[a,b]\to[0,+\infty)^N$ the capacity given by Definition 11.6.
Let $\mathcal{G}\subset[a,b]$ be a Riemann germ with respect to (J,f), and let $\hat\gamma_{\mathcal{G}}:[0,+\infty)^N\to[0,+\infty)^N$ be its associated preflux in the polar decomposition of its Godunov flux $\hat f_{\mathcal{G}}=\hat\gamma_{\mathcal{G}}\circ\bar\gamma$. For $j\in\{1,\dots,N\}$, and $\bar\lambda^j\in[0,+\infty]$, let us define define the following flux limited $1:1$ germ

$$\mathcal{G}^j_{\bar\lambda^j}:=\left\{(p^L,p^R)\in[a^j,b^j]^2,\quad \min\left\{\bar\lambda^j,G^{f^j}(p^L,p^R)\right\}=f^j(p^L)=f^j(p^R)\right\}$$

where G^{f^j} is the standard Godunov germ associated to the flux f^j.
Then for every fixed $j_0\in\{1,\dots,N\}$, we have

$$\hat\gamma_{\tilde{\mathcal{G}}}=\hat\gamma_{\mathcal{G}}\circ T_{\bar\lambda}\quad\text{with}\quad\begin{cases}\tilde{\mathcal{G}}:=\mathcal{G}\underset{j_0:\alpha}{\sharp}\mathcal{G}^{j_0}_{\bar\lambda^{j_0}}\quad\text{with}\quad \alpha=\begin{cases}L & \text{if}\quad J^{j_0}\simeq(0,+\infty)\\ R & \text{if}\quad J^{j_0}\simeq(-\infty,0)\end{cases}\\ \bar\lambda:=(+\infty,\dots,+\infty,\bar\lambda^{j_0},+\infty,\dots,+\infty)\end{cases}$$

which holds at least on the image of the capacity $\bar\gamma$, which is $\displaystyle\prod_{j=1,\dots,N}[0,\max_{[a^j,b^j]}f^j]$.
Here the Riemann germ $\tilde{\mathcal{G}}$ is the gluing (in the sense of Theorem 2.24) of $\mathcal{G}$ along its branch j_0 with the branch α of $\mathcal{G}^{j_0}_{\bar\lambda^{j_0}}$.

Proof of Lemma 11.16
We set $f^{j,+}(q):=\sup\limits_{[a^1,q]}f^j$ and $f^{j,-}(q):=\inf\limits_{[q,b^1]}f^j$, and we have

$$\hat f^L_{\mathcal{G}^j_{\bar\lambda^j}}(p^L,p^R)=\min\left\{\bar\lambda^j,G^{f^j}(p^L,p^R)\right\}\quad\text{with}\quad G^{f^j}(p^L,p^R)=\min\left\{f^{j,+}(p^L),f^{j,-}(p^R)\right\}$$

Up to relabel the indices, let us assume that $j_0 = 1$ and that $\alpha = L$ (the case $\alpha = R$ is similar). From Corollary 5.14, we know that there exists some $r \in [a^1, b^1]$ such that

$$\hat{f}^j_{\tilde{\mathcal{G}}}(p^1,\dots,p^N) = \hat{f}^j_{\mathcal{G}}(r,p^2,\dots,p^N) \quad \text{for all } j=1,\dots,N, \text{ where } r \text{ satisfies} \quad \hat{f}^1_{\mathcal{G}}(r,p^2,\dots,p^N) = \hat{f}^L_{\mathcal{G}^1_{\bar{\lambda}^1}}(r,p^1)$$

Then we have

$$\hat{\gamma}^1_{\mathcal{G}}(f^{1,-}(r),\bar{\gamma}^2(p^2),\dots,\bar{\gamma}^N(p^N)) = \min\left\{\bar{\lambda}^1, f^{1,+}(r), f^{1,-}(p^1)\right\}$$

By assumption, we know that we can write $\hat{\gamma}^1_{\mathcal{G}}(\gamma) = \min\left\{\gamma^1, A^1(\gamma^2,\dots,\gamma^N)\right\}$. Hence

$$\Lambda^1 := \min\left\{f^{1,-}(r), \bar{A}^1\right\} = \min\left\{\bar{\lambda}^1, f^{1,+}(r), f^{1,-}(p^1)\right\} \quad \text{with} \quad \bar{A}^1 := A^1(\bar{\gamma}^2(p^2),\dots,\bar{\gamma}^N(p^N))$$

and then $\Lambda^1 = \min\left\{\bar{A}^1, \bar{\lambda}^1, f^{1,-}(p^1)\right\}$. Therefore everything is equivalent to replace everywhere in $\hat{\gamma}_{\mathcal{G}}(f^{1,-}(r),\bar{\gamma}^2(p^2),\dots,\bar{\gamma}^N(p^N))$ the value $f^{1,-}(r)$ by $\min\left\{\bar{\lambda}^1, \bar{\gamma}^1(p^1)\right\}$ with $\bar{\gamma}^1(p^1) := f^{1,-}(p^1)$. This means that

$$(\hat{\gamma}_{\tilde{\mathcal{G}}} \circ \bar{\gamma})(p) = \hat{f}_{\tilde{\mathcal{G}}}(p) = \hat{\gamma}_{\mathcal{G}}(f^{1,-}(r),\bar{\gamma}^2(p^2),\dots,\bar{\gamma}^N(p^N)) = (\hat{\gamma}_{\mathcal{G}} \circ T_{\bar{\lambda}} \circ \bar{\gamma})(p)$$

and then

$$\hat{\gamma}_{\tilde{\mathcal{G}}} = \hat{\gamma}_{\mathcal{G}} \circ T_{\bar{\lambda}} \quad \text{on} \quad \mathrm{Im}(\bar{\gamma}) = \prod_{j=1,\dots,N} [0, \max_{[a^j,b^j]} f^j]$$

This ends the proof of the lemma.

11.4 Gluing prefluxes

Theorem 11.17 (Gluing of prefluxes $\hat{\lambda}_\gamma$ for junctions with N_γ branches)
For $\gamma = \alpha, \beta$, assume that $\hat{\lambda}_\gamma : [0,+\infty)^{N_\gamma} \to [0,+\infty)^{N_\gamma}$ is a preflux with $N_\gamma \geq 1$, satisfying $\hat{\lambda}_\gamma = \hat{\lambda}_\gamma \circ T_{\bar{\lambda}_\gamma}$ with $\bar{\lambda}_\gamma \in (0,+\infty)^{N_\gamma}$ and where $T_{\bar{\lambda}_\gamma}$ is the truncation operator defined in (11.15) for $N = N_\gamma$ and $\bar{\lambda} = \bar{\lambda}_\gamma$. To simplify the notation, we label the indices with $j_\gamma \in \{0,\dots,N_\gamma - 1\}$. Hence we now have

$$\hat{\lambda}_\gamma = (\hat{\lambda}^0_\gamma,\dots,\hat{\lambda}^{N_\gamma-1}_\gamma)$$

In order to glue indices $j_\alpha = 0$ and $j_\beta = 0$, we assume

$$\bar{\lambda}^0_\alpha = \bar{\lambda}^0_\beta =: \bar{\lambda}^0 \in (0,+\infty)$$

and define the doubling set

$$\mathbb{D}_{\bar{\lambda}^0} := \left\{(\lambda^{L,0},\lambda^{R,0}) \in [0,\bar{\lambda}^0]^2, \quad \max\left\{\lambda^{L,0},\lambda^{R,0}\right\} = \bar{\lambda}^0\right\}$$

Now for $p_\gamma = (p^1_\gamma,\dots,p^{N_\gamma-1}_\gamma) \in [0,+\infty)^{N_\gamma-1}$ (avoiding notation p'_γ to keep light notations), we define the set

$$R := \left\{(\lambda^{L,0},\lambda^{R,0}) \in \mathbb{D}_{\bar{\lambda}^0}, \quad \hat{\lambda}^0_\alpha(\lambda^{L,0},p_\alpha) = \hat{\lambda}^0_\beta(\lambda^{R,0},p_\beta)\right\} \tag{11.19}$$

Then R is non empty, and we define the set

$$\Lambda := \left\{\lambda = \tilde{\lambda}(\lambda^{L,0},\lambda^{R,0},p_\alpha,p_\beta) \in \mathbb{R}^{N_\alpha+N_\beta-2}, \quad (\lambda^{L,0},\lambda^{R,0}) \in R\right\}$$

with

$$\tilde{\lambda}(\lambda^{L,0},\lambda^{R,0},p_\alpha,p_\beta) := ((\hat{\lambda}^1_\alpha,\dots,\hat{\lambda}^{N_\alpha-1}_\alpha)(\lambda^{L,0},p_\alpha);(\hat{\lambda}^1_\beta,\dots,\hat{\lambda}^{N_\beta-1}_\beta)(\lambda^{R,0},p_\beta)) \in \mathbb{R}^{N_\alpha+N_\beta-2}$$

Then Λ is reduced to a singleton $\Lambda = \{\lambda\}$, and this defines the following map

$$\begin{array}{rlcl} \underline{\hat{\lambda}}: & [0,+\infty)^{N_\alpha-1} \times [0,+\infty)^{N_\beta-1} & \to & \mathbb{R}^{N_\alpha+N_\beta-2} \\ & (p_\alpha,p_\beta) & \mapsto & \underline{\hat{\lambda}}(p_\alpha,p_\beta) := \lambda \end{array}$$

0) (Gluing prefluxes)

Then $\hat{\underline{\lambda}}$ is a preflux and satisfies $\hat{\underline{\lambda}} = \hat{\underline{\lambda}} \circ T_{\bar{\underline{\lambda}}}$ with $\bar{\underline{\lambda}} := (\bar{\lambda}_\alpha^1, \dots, \bar{\lambda}_\alpha^{N_\alpha - 1}; \bar{\lambda}_\beta^1, \dots, \bar{\lambda}_\beta^{N_\beta - 1})$, with $T_{\bar{\underline{\lambda}}}$ defined in (11.15).
We use the notation

$$\hat{\lambda}_\alpha \underset{j_\alpha : j_\beta}{\sharp} \hat{\lambda}_\beta := \hat{\underline{\lambda}} \tag{11.20}$$

which is defined here for $j_\alpha = 0 = j_\beta$ (and that can be easily generalized for indices $j_\alpha \in \{0, \dots, N_\alpha - 1\}$ and $j_\beta \in \{0, \dots, N_\beta - 1\}$ such that $\bar{\lambda}_\alpha^{j_\alpha} = \bar{\lambda}_\beta^{j_\beta} \in (0, +\infty)$).
i) (Gluing Kružkov prefluxes)
Assume that $\hat{\lambda}_\gamma$ are Kružkov prefluxes for $\gamma = \alpha, \beta$. Then $\hat{\underline{\lambda}}$ is also a Kružkov preflux.
ii) (Gluing Hamilton-Jacobi prefluxes)
Assume that $\hat{\lambda}_\gamma$ are HJ prefluxes for $\gamma = \alpha, \beta$. Then $\hat{\lambda}$ is also a HJ preflux.
iii) (Gluing monotone prefluxes)
Assume that $\hat{\lambda}_\gamma$ are σ_γ-monotone prefluxes with $\sigma_\gamma = (\sigma_\gamma^0, \dots, \sigma_\gamma^{N_\gamma - 1}) \in \{\pm 1\}^{N_\gamma}$ for $\gamma = \alpha, \beta$. Then $\hat{\lambda}$ is a $\underline{\sigma}$-monotone preflux with

$$\underline{\sigma} := (\sigma_\alpha^1, \dots, \sigma_\alpha^{N_\alpha - 1}; \sigma_\beta^1, \dots, \sigma_\beta^{N_\beta - 1}). \tag{11.21}$$

iv) (Gluing conservative prefluxes)
Assume that $\hat{\lambda}_\gamma$ are conservative prefluxes for σ_γ and $\gamma = \alpha, \beta$, satisfying moreover

$$\sigma_\alpha^0 = -\sigma_\beta^0 \tag{11.22}$$

Then $\hat{\lambda}$ is also a conservative preflux for $\underline{\sigma}$ defined in (11.21).

Remark 11.18 (Symmetry of gluing for prefluxes)
Let for $j_\alpha := 0 =: j_\beta$, and $p_\gamma := (p_\gamma^1, \dots, p_\gamma^{N_\gamma - 1})$, let us consider

$$\begin{cases} \hat{\underline{\lambda}}_L(p_\alpha, p_\beta) := \hat{\lambda}_\alpha \underset{0:0}{\sharp} \hat{\lambda}_\beta & = (\hat{\underline{\lambda}}_L^{(\alpha,1)}, \dots, \hat{\underline{\lambda}}_L^{(\alpha, N_\alpha - 1)}; \hat{\underline{\lambda}}_L^{(\beta,1)}, \dots, \hat{\underline{\lambda}}_L^{(\beta, N_\beta - 1)})(p_\alpha, p_\beta) \\ \hat{\underline{\lambda}}_R(p_\beta, p_\alpha) := \hat{\lambda}_\beta \underset{0:0}{\sharp} \hat{\lambda}_\alpha & = (\hat{\underline{\lambda}}_R^{(\beta,1)}, \dots, \hat{\underline{\lambda}}_R^{(\beta, N_\beta - 1)}; \hat{\underline{\lambda}}_R^{(\alpha,1)}, \dots, \hat{\underline{\lambda}}_R^{(\alpha, N_\alpha - 1)})(p_\beta, p_\alpha), \end{cases}$$

Here the letters L, R have nothing to do with left or right branches. There arise here in order to distinguish left-gluing of prefluxes and right-gluing of prefluxes. Then we have the following symmetry of the gluing of prefluxes (contrarily to the gluing of Godunov fluxes):

$$\hat{\underline{\lambda}}_R^K(p_\beta, p_\alpha) = \hat{\underline{\lambda}}_L^K(p_\alpha, p_\beta) \quad \text{for all} \quad K \in \mathcal{N}_\alpha \cup \mathcal{N}_\beta$$

with $\mathcal{N}_\gamma := \{(\gamma, 1), \dots, (\gamma, N_\gamma - 1)\}$.

Remark 11.19 *Notice that in point iii) of Lemma 11.17, we do not require $\sigma_\alpha^0 = -\sigma_\beta^0$ for the monotonicity of the glued prefluxes.*

Proof of Theorem 11.17
The proof follows from Proposition 5.13 for the gluing of fluxes. We introduce the piecewise linear functions

$$\left\{ \begin{array}{l} g_\gamma^j(p^j) := \min\{p^j, 2\bar{\lambda}_\gamma^j - p^j\} \\ g_\gamma^{j,\pm}(p^j) \quad \text{similar to notation (11.4) of Definition 11.6} \end{array} \right| \quad \text{for } p^j \in [0, 2\bar{\lambda}_\gamma^j] \text{ and } j = j_\gamma \in \{0, \dots, N_\gamma - 1\}.$$

Given some $\tilde{\sigma}_\gamma \in \{\pm 1\}^{N_\gamma}$, we define $\hat{f}_\gamma := \hat{\lambda}_\gamma \circ g_\gamma^{\tilde{\sigma}_\gamma}$, and then consider the gluing $\underline{\hat{f}} := \hat{f}_\alpha \sharp \hat{f}_\beta$ along the indices $j_\alpha = 0 = j_\beta$. We then consider the preflux associated to $\underline{\hat{f}}$, that we write

$$\underline{\hat{f}} = \hat{\underline{\lambda}} \circ \underline{g}^{\tilde{\underline{\sigma}}}$$

with

$$\tilde{\underline{\sigma}} := (\tilde{\sigma}_\alpha^1, \dots, \tilde{\sigma}_\alpha^{N_\alpha - 1}; \tilde{\sigma}_\beta^1, \dots, \tilde{\sigma}_\beta^{N_\beta - 1})$$

and $\underline{g} := (g_\alpha^1, \dots, g_\alpha^{N_\alpha-1}; g_\beta^1, \dots, g_\beta^{N_\beta-1})$. Then it is easy to check that $\hat{\underline{\lambda}}$ is given as the statement of the lemma, because (say for $\tilde{\sigma}_\alpha^0 = -1$ and $\tilde{\sigma}_\beta^0 = 1$, we glue $J_\alpha^0 \simeq (0,+\infty)$ on the left to $J_\beta^0 \simeq (-\infty, 0)$ on the right)

(11.23)

$$\text{(left gluing)} := \hat{\lambda}_\alpha^0(g^{0,-}(r), p_\alpha^1, \dots, p_\alpha^{N_\alpha-1}) = \hat{\lambda}_\beta^0(g^{0,+}(r), p_\beta^1, \dots, p_\beta^{N_\beta-1}) =: \text{(right gluing)} \quad \text{for} \quad r \in [0, 2\bar{\lambda}^0]$$

and all the results follow with $(\lambda^{L,0}, \lambda^{R,0}) = (g^{0,-}(r), g^{0,+}(r))$. Notice that we have

$$\begin{cases} \text{(left gluing)} = \text{a right branch} \\ \text{(right gluing)} = \text{a left branch} \end{cases}$$

which explains the signs in (11.23). In point iv) for conservative prefluxes, we need furthermore to choose $\tilde{\sigma}_\gamma := \sigma_\gamma$. This ends the proof of the lemma.

Corollary 11.20 (Compatibility of gluing for prefluxes and fluxes)
For $\gamma = \alpha, \beta$, we assume $N_\gamma \geq 1$ and orientations of the branches $\sigma_\gamma \in \{\pm 1\}^{N_\gamma}$. We assume moreover that the fluxes f_γ are bell-shaped in the sense of Definition 11.6 with $f_\gamma^j(c_\gamma^j) = \bar{\lambda}_\gamma^j$. We assume moreover that

$$\begin{cases} \hat{f}_\gamma = \hat{\lambda}_\gamma \circ f_\gamma^{\sigma_\gamma} \quad \text{for } f_\gamma^{\sigma_\gamma} \text{ with notation (11.4)} \\ f_\alpha^{j_\alpha} \equiv f_\beta^{j_\beta}, \\ \bar{\lambda}_\alpha^{j_\alpha} = \bar{\lambda}_\beta^{j_\beta} \in (0,+\infty) \end{cases}$$

Then we have for the gluing along indices j_α and j_β

$$\hat{f}_\alpha \underset{j_\alpha : j_\beta}{\sharp} \hat{f}_\beta = (\hat{\lambda}_\alpha \underset{j_\alpha : j_\beta}{\sharp} \hat{\lambda}_\beta) \circ \underline{f}^{\underline{\sigma}} \quad \text{for } \underline{f}^{\underline{\sigma}} \text{ with notation (11.4)}$$

where $\hat{f}_\alpha \underset{j_\alpha : j_\beta}{\sharp} \hat{f}_\beta$ is defined in (5.39), and $\hat{\lambda}_\alpha \underset{j_\alpha : j_\beta}{\sharp} \hat{\lambda}_\beta$ is defined in (11.20). Moreover, in the special case $j_\alpha = 0 = j_\beta$ (in order to simplify notations), we recall that the following quantities are simply defined by concatenation

$$\begin{cases} \underline{f} := (f_\alpha^1, \dots, f_\alpha^{N_\alpha-1}; f_\gamma^1, \dots, f_\gamma^{N_\gamma-1}) \\ \underline{\sigma} := (\sigma_\alpha^1, \dots, \sigma_\alpha^{N_\alpha-1}; \sigma_\gamma^1, \dots, \sigma_\gamma^{N_\gamma-1}). \end{cases}$$

Proof of Corollary 11.20
The result follows from an easy adaptation of the proof of Theorem 11.17, with bell-shaped functions g_γ replaced by bell-shaped functions f_γ. We skip the details. This ends the proof of the corollary.

Lemma 11.21 (Self-gluing of Kružkov prefluxes)
Let γ be a fixed index and let $N_\gamma \geq 3$ and let us consider some **Kružkov** *preflux $\hat{\lambda}_\gamma : [0,+\infty)^{N_\gamma} \to [0,+\infty)^{N_\gamma}$. Let $\bar{\lambda}_\gamma \in (0,+\infty)^{N_\gamma}$ and assume that $\hat{\lambda}_\gamma = \hat{\lambda}_\gamma \circ T_{\bar{\lambda}_\gamma}$ and where $T_{\bar{\lambda}_\gamma}$ is the truncation operator defined in (11.15) for $N = N_\gamma$ and $\bar{\lambda} = \bar{\lambda}_\gamma$. To simplify the notation, we label the indices with $j_\gamma \in \{0, \dots, N_\gamma - 1\}$. Hence we now have*

$$\hat{\lambda}_\gamma = (\hat{\lambda}_\gamma^0, \dots, \hat{\lambda}_\gamma^{N_\gamma-1})$$

In order to glue indices $j_1 := 0$ and $j_2 := N_\gamma - 1$, we assume

$$\bar{\lambda}_\gamma^0 = \bar{\lambda}_\gamma^{N_\gamma-1} =: \bar{\lambda}^0 \in (0,+\infty)$$

and define the doubling set

$$\mathbb{D}_{\bar{\lambda}^0} := \left\{(\lambda^{L,0}, \lambda^{R,0}) \in [0, \bar{\lambda}^0]^2, \quad \max\left\{\lambda^{L,0}, \lambda^{R,0}\right\} = \bar{\lambda}^0\right\}$$

Now for $p_\gamma = (p_\gamma^1, \dots, p_\gamma^{N_\gamma-2}) \in [0,+\infty)^{N_\gamma-2}$ (avoiding notation p_γ'' to keep light notations), we define the set

$$R := \left\{(\lambda^{L,0}, \lambda^{R,0}) \in \mathbb{D}_{\bar{\lambda}^0}, \quad \hat{\lambda}_\gamma^0(\lambda^{L,0}, p_\gamma, \lambda^{R,0}) = \hat{\lambda}_\gamma^0(\lambda^{L,0}, p_\gamma, \lambda^{R,0})\right\} \tag{11.24}$$

Then R is non empty, and we define the set

$$\Lambda := \left\{\lambda = \tilde{\lambda}(\lambda^{L,0}, \lambda^{R,0}, p_\gamma) \in \mathbb{R}^{N_\gamma - 2}, \quad (\lambda^{L,0}, \lambda^{R,0}) \in R\right\}$$

with

$$\tilde{\lambda}(\lambda^{L,0}, \lambda^{R,0}, p_\gamma) := (\hat{\lambda}^1_\gamma, \dots, \hat{\lambda}^{N_\gamma-2}_\gamma)(\lambda^{L,0}, p_\gamma, \lambda^{R,0}) \in \mathbb{R}^{N_\gamma-2}$$

Then Λ is reduced to a singleton $\Lambda = \{\lambda\}$, and this defines the following map

$$\begin{array}{llll} \hat{\underline{\lambda}}: & [0,+\infty)^{N_\gamma-2} & \to & \mathbb{R}^{N_\gamma-2} \\ & p_\gamma & \mapsto & \hat{\underline{\lambda}}(p_\gamma) := \lambda \end{array}$$

i) (Kružkov preflux)

Then $\hat{\underline{\lambda}} : [0,+\infty)^{N_\gamma-2} \to [0,+\infty)^{N_\gamma-2}$ is a Kružkov preflux and satisfies $\hat{\underline{\lambda}} = \hat{\underline{\lambda}} \circ T_{\bar{\underline{\lambda}}}$ and where $T_{\bar{\underline{\lambda}}}$ is the truncation operator defined in (11.15) for $N := N_\gamma - 2$ and $\bar{\lambda} := \bar{\underline{\lambda}}$. We also set

$$\hat{\lambda}^{j_1 \overset{\sharp}{:} j_2} := \hat{\underline{\lambda}} \tag{11.25}$$

that we have described in the special case $j_1 := 0$ and $j_2 := N_\gamma - 1$.

ii) (Conservative Kružkov preflux)

Moreover if the preflux $\hat{\lambda}_\gamma$ is conservative Kružkov, then $\hat{\underline{\lambda}}$ is also conservative Kružkov.

Proof of Lemma 11.21

The proof is based on Proposition 5.16, and follows the lines of the proof of Theorem 11.17. Precisely, from (5.45), we get with $\bar{\lambda}^0_\gamma = \bar{\lambda}^{N_\gamma-1}_\gamma$ and $g^0_\gamma = g^{N_\gamma-1}_\gamma$

$$\hat{\lambda}^0_\gamma(g^{0,-}(r), p_\gamma, g^{0,+}(r)) = \hat{\lambda}^{N_\gamma-1}_\gamma(g^{0,-}(r), p_\gamma, g^{0,+}(r)) \quad \text{for some} \quad r \in [0, 2\bar{\lambda}^0]$$

The remaining part of the proof is easy, and we skip the details. This ends the proof of the lemma.

Similarly, we also have the following results (whose we skip the proofs and detailed statements).

Lemma 11.22 (Compatibility of self-gluing for prefluxes and fluxes)

We work under assumptions of Lemmata 11.21 for the preflux $\hat{\lambda}_\gamma$ and Proposition 5.16 for the Godunov flux $\hat{f}_\gamma$ which is assumed to be modeled on bel–shaped fluxes f_γ, i.e.

$$\hat{f}_\gamma = \hat{\lambda}_\gamma \circ f^{\sigma_\gamma}_\gamma \quad \textit{for } f^{\sigma_\gamma}_\gamma \textit{ with notation (11.4}$$

with usual notations. Then we have the following compatibility relation for gluing along indices $j_1 \neq j_2$

$$\hat{f}^{j_1 \overset{\sharp}{:} j_2} := \hat{\lambda}^{j_1 \overset{\sharp}{:} j_2} \circ \underline{f}^{\underline{\sigma}} \quad \textit{for } \underline{f}^{\underline{\sigma}} \textit{ with notation (11.4)}$$

with $\hat{f}^{j_1 \overset{\sharp}{:} j_2}$ defined in (5.47), and $\hat{\lambda}^{j_1 \overset{\sharp}{:} j_2}$ defined in (11.25). Moreover, in the special case $j_1 = 0$ and $j_2 := N_\gamma - 1$ (in order to simplify notations), we have

$$\left\{ \begin{array}{l} \underline{f} := (f^1_\gamma, \dots, f^{N_\gamma-2}_\gamma) \\ \underline{\sigma} := (\sigma^1_\gamma, \dots, \sigma^{N_\gamma-2}_\gamma). \end{array} \right.$$

Lemma 11.23 (Associativity of gluing for prefluxes)

As in Lemmata 5.15, 5.18, 5.19, the associativity of gluing is also true for prefluxes. For associativity of self-gluing and self-gluing (or of self-gluing and gluing), the Kružkov property is moreover required for the prefluxes.

11.5 Further monotonicities of glued prefluxes

Lemma 11.24 (Monotonicities of glued prefluxes)
For $\gamma = \alpha, \beta$, assume that $\hat\lambda_\gamma : [0,+\infty)^{N_\gamma} \to [0,+\infty)^{N_\gamma}$ is a preflux with $N_\gamma \geq 1$, satisfying $\hat\lambda_\gamma = \hat\lambda_\gamma \circ T_{\bar\lambda_\gamma}$ with $\bar\lambda_\gamma \in (0,+\infty)^{N_\gamma}$ and where $T_{\bar\lambda_\gamma}$ is the truncation operator defined in (11.15) for $N = N_\gamma$ and $\bar\lambda = \bar\lambda_\gamma$. To simplify the notation, we label the indices with $j_\gamma \in \{0, \dots, N_\gamma - 1\}$. Hence we now have

$$\hat\lambda_\gamma = (\hat\lambda^0_\gamma, \dots, \hat\lambda^{N_\gamma - 1}_\gamma).$$

In order to glue indices $j_\alpha = 0$ and $j_\beta = 0$, we assume

$$\bar\lambda^0_\alpha = \bar\lambda^0_\beta =: \bar\lambda^0 \in (0,+\infty).$$

Then let

$$\hat\lambda := \hat\lambda_\alpha \underset{j_\alpha : j_\beta}{\sharp} \hat\lambda_\beta \quad \text{defined in (11.20).} \tag{11.26}$$

Then $\hat\lambda$ is a preflux. Moreover, assume that there exists $\varepsilon^{(\gamma,\ell)}_{(\gamma,k)} \in \{\pm 1\}$ such that

$$\varepsilon^{(\gamma,\ell)}_{(\gamma,k)} \partial_{p^k_\gamma} \hat\lambda^\ell_\gamma \geq 0 \quad \text{in} \quad \mathcal{D}'((0,+\infty)^{N_\gamma}), \quad k, \ell \in \{0, \dots, N_\gamma - 1\}, \quad \gamma = \alpha, \beta. \tag{11.27}$$

Then we have

$$\underline{\varepsilon}^K_I \ \partial_{\underline{p}^I} \hat\lambda^K \geq 0 \quad \text{in} \quad \mathcal{D}'((0,+\infty)^{N_\alpha + N_\beta - 2}) \tag{11.28}$$

with $\underline{p}^I = p^k_\gamma$ and $I = (\gamma, k)$, with for certain indices $I, K \in \mathcal{N}_\alpha \cup \mathcal{N}_\beta$ and values $\underline{\varepsilon}^K_I \in \{\pm 1\}$ that will be precised below, and with $\mathcal{N}_\gamma := \{(\gamma, 1), \dots, (\gamma, N_\gamma - 1)\}$.
i) (General cross monotonicities)
Then (11.28) holds true for all $(I,K) \in (\mathcal{N}_\alpha \times \mathcal{N}_\beta) \cup (\mathcal{N}_\beta \times \mathcal{N}_\alpha)$ with the following values

$$\left\{ \begin{aligned} \underline{\varepsilon}^I_K &:= +\varepsilon^{(\alpha,0)}_K \cdot \varepsilon^I_{(\beta,0)}, \\ \underline{\varepsilon}^K_I &:= +\varepsilon^{(\beta,0)}_I \cdot \varepsilon^K_{(\alpha,0)}, \end{aligned} \right| \quad K := (\alpha, k) \in \mathcal{N}_\alpha, \quad I := (\beta, \ell) \in \mathcal{N}_\beta.$$

ii) (Additional block monotonicities)
Assume moreover that

$$\varepsilon^I_K = -\varepsilon^{(\gamma,0)}_K \cdot \varepsilon^I_{(\gamma,0)}, \quad K = (\gamma,k), I = (\gamma, \ell) \in \mathcal{N}_\gamma, \quad K \neq I, \quad \text{for } \gamma = \alpha \text{ (resp. } \gamma = \beta). \tag{11.29}$$

Then (11.28) holds true for all the following values

$$\underline{\varepsilon}^I_K := \varepsilon^I_K \quad K = (\gamma,k), I = (\gamma,\ell) \in \mathcal{N}_\gamma, \quad K \neq I, \quad \text{for } \gamma = \alpha \text{ (resp. } \gamma = \beta).$$

iii) (Full monotonicities)
Moreover under (11.29) for both $\gamma = \alpha$ and $\gamma = \beta$, then we have

$$\underline{\varepsilon}^I_K = -\delta_{KI} \cdot \varepsilon^{(\gamma,0)}_K \cdot \varepsilon^I_{(\gamma',0)} \quad \text{for all} \quad K := (\gamma,k), I := (\gamma', \ell) \in \mathcal{N}_\alpha \cup \mathcal{N}_\beta \quad \text{with} \quad K \neq I. \tag{11.30}$$

with

$$\delta_{KI} := \left\{ \begin{array}{ll} 1 & \text{if} \quad \gamma = \gamma', \\ -1 & \text{if} \quad \gamma \neq \gamma'. \end{array} \right.$$

Remark 11.25 *Notice that (11.30) means that outside the diagonal, the Jacobian matrix of the preflux $\hat\lambda$ has a block structure, where each block (up to a fixed sign factor) can be seen as the tensor product of two signed vectors.*

Remark 11.26 a) *Notice also that for a σ_γ-monotone preflux $\hat{\lambda}_\gamma$ associated to a junction with orientations σ_γ, we have in particular*

$$-\sigma_\gamma^i \sigma_\gamma^j \partial_i \hat{\lambda}_\gamma^j \geq 0 \quad \textit{for all} \quad i \neq j \tag{11.31}$$

for $i,j \in \mathcal{N}_\gamma$, which coincides with condition (11.29) of point ii) in the special case where

$$\varepsilon_{(\gamma,\ell)}^{(\gamma,0)} = \varepsilon_{(\gamma,0)}^{(\gamma,\ell)} = \eta_\gamma \sigma_\gamma^\ell \quad \textit{for all} \quad \ell \in \mathcal{N}_\gamma \tag{11.32}$$

for some fixed $\eta_\gamma \in \{\pm 1\}$.
b) *Moreover, for a σ_γ-monotone preflux $\hat{\lambda}_\gamma$, notice that (11.31) holds not only for all $i,j \in \mathcal{N}_\gamma$ but for $i,j \in \{0\} \cup \mathcal{N}_\gamma$, which corresponds furthermore to $-\sigma_\gamma^\ell \sigma_\gamma^0 \partial_\ell \hat{\lambda}_\gamma^0 \geq 0$ for all $\ell \in \mathcal{N}_\gamma$, i.e. to*

$$\varepsilon_{(\gamma,\ell)}^{(\gamma,0)} = \eta_\gamma \sigma_\gamma^\ell \quad \textit{with} \quad \eta_\gamma := -\sigma_\gamma^0 \in \{\pm 1\}\,. \tag{11.33}$$

Now if we have furthermore

$$\sigma_\alpha^0 = -\sigma_\beta^0 \tag{11.34}$$

then point iii) of Theorem 11.17 implies that $\hat{\lambda}$ is $\underline{\sigma}$-monotone with $\underline{\sigma}_{(\gamma,k)} := \sigma_\gamma^k$. Therefore we have

$$-\sigma_\gamma^k \sigma_{\gamma'}^\ell \partial_{(\gamma,k)} \hat{\lambda}^{(\gamma',\ell)} \geq 0 \quad \textit{for all} \quad K := (\gamma,k) \neq (\gamma',\ell) =: I$$

and we recover (11.30) where

$$\delta_{KI} = \eta_\gamma \eta_{\gamma'} = \sigma_\gamma^0 \sigma_{\gamma'}^0 \quad \textit{and} \quad -\delta_{KI} \cdot \varepsilon_K^{(\gamma,0)} \cdot \varepsilon_{(\gamma',0)}^I = -\sigma_\gamma^k \sigma_{\gamma'}^\ell .$$

Hence points i) and ii) appear here at least as new results with partial monotonicities. Moreover point iii) is new when we do not assume both (11.33) and (11.34).

Proof of Lemma 11.24
Step 1: preliminaries
Recall that we define the doubling set

$$\mathbb{D}_{\bar{\lambda}^0} := \left\{ (\lambda^{L,0}, \lambda^{R,0}) \in [0, \bar{\lambda}^0]^2, \quad \max\left\{\lambda^{L,0}, \lambda^{R,0}\right\} = \bar{\lambda}^0 \right\}$$

Now for $p_\gamma = (p_\gamma^1, \ldots, p_\gamma^{N_\gamma - 1}) \in [0,+\infty)^{N_\gamma - 1}$ (avoiding notation p'_γ to keep light notations), we define the set

$$R := \left\{ (\lambda^{L,0}, \lambda^{R,0}) \in \mathbb{D}_{\bar{\lambda}^0}, \quad \hat{\lambda}_\alpha^0(\lambda^{L,0}, p_\alpha) = \hat{\lambda}_\beta^0(\lambda^{R,0}, p_\beta) \right\} \tag{11.35}$$

Then R is non empty, and we define the set

$$\Lambda := \left\{ \lambda = \tilde{\lambda}(\lambda^{L,0}, \lambda^{R,0}, p_\alpha, p_\beta) \in \mathbb{R}^{N_\alpha + N_\beta - 2}, \quad (\lambda^{L,0}, \lambda^{R,0}) \in R \right\}$$

with

$$\tilde{\lambda}(\lambda^{L,0}, \lambda^{R,0}, p_\alpha, p_\beta) := ((\hat{\lambda}_\alpha^1, \ldots, \hat{\lambda}_\alpha^{N_\alpha - 1})(\lambda^{L,0}, p_\alpha); (\hat{\lambda}_\beta^1, \ldots, \hat{\lambda}_\beta^{N_\beta - 1})(\lambda^{R,0}, p_\beta)) \in \mathbb{R}^{N_\alpha + N_\beta - 2}$$

Then Λ is reduced to the singleton $\Lambda = \left\{ \hat{\lambda}(p_\alpha, p_\beta) \right\}$.
Step 2: proof of i)
Let us consider the indices $K := (\alpha, k) \in \mathcal{N}_\alpha$ and $I := (\beta, \ell) \in \mathcal{N}_\beta$. Consider $\hat{\lambda}^{\alpha,k}$ as a function of $\underline{p}^I = p_\beta^\ell$, which arises then only along the values of $\lambda^{L,0}$ such that

$$\hat{\lambda}_\alpha^0(\lambda^{L,0}, p_\alpha) = \hat{\lambda}_\beta^0(\lambda^{R,0}, p_\beta) \tag{11.36}$$

As in (11.23), recall that we can also think to $(\lambda^{L,0}, \lambda^{R,0})$ as

$$(\lambda^{L,0}, \lambda^{R,0}) := (\min(\bar{\lambda}^0, 2\bar{\lambda}^0 - r), \min(r, \bar{\lambda}^0)) =: (g^-(r), g^+(r))$$

Hence (11.36) becomes

$$\hat{\lambda}^0_\alpha(g^-(r),p_\alpha) = \hat{\lambda}^0_\beta(g^+(r),p_\beta) \tag{11.37}$$

Formally, we have

$$\partial_{\underline{p}^I}\hat{\lambda}^K(p_\alpha,p_\beta) = \partial_{p^\ell_\beta}\hat{\lambda}^{\alpha,k}(p_\alpha,p_\beta) = \partial_{p^0_\alpha}\hat{\lambda}^{\alpha,k}(g^-,p_\alpha)\cdot(g^-)'\cdot\partial_{p^\ell_\beta} r \quad \text{with} \quad (g^-)' \le 0$$

where (11.37) gives formally

$$\partial_{p^\ell_\beta}\hat{\lambda}^0_\beta = A\cdot\partial_{p^\ell_\beta} r \quad \text{with} \quad A := (g^-)'\partial_{p^0_\alpha}\hat{\lambda}^0_\alpha - (g^+)'\partial_{p^0_\beta}\hat{\lambda}^0_\beta \le 0$$

Hence formally, we get that

$$\text{sign}\left(\partial_{p^\ell_\beta}\hat{\lambda}^{\alpha,k}\right) = \text{sign}\left(\partial_{p^0_\alpha}\hat{\lambda}^{\alpha,k}\right)\cdot\text{sign}\left(\partial_{p^\ell_\beta}\hat{\lambda}^0_\beta\right)$$

which shows formally point i). We skip the rigorous proof which can be made similarly to the remaining part of this work.
Step 3: proof of ii)
We choose $K := (\beta,k)$ and $I := (\beta,\ell)$ with $\ell \neq k$. We get formally

$$\partial_{\underline{p}^I}\hat{\lambda}^K(p_\alpha,p_\beta) = \partial_{\underline{p}^I}\hat{\lambda}^{\beta,k}(p_\alpha,p_\beta) = \frac{d}{dp^\ell_\beta}\left[\hat{\lambda}^{\beta,k}(g^+(r),p_\alpha)\right] = \partial_{p^\ell_\beta}\hat{\lambda}^{\beta,k}(g^+,p_\beta) + \partial_{p^0_\beta}\hat{\lambda}^{\beta,k}(g^+,p_\alpha)\cdot(g^+)'\cdot\partial_{p^\ell_\beta} r$$

where $\frac{d}{dp^\ell_\beta}$ stands for the total derivative, and then assumption (11.29) implies formally point ii). Again we skip the rigorous proof.
Step 4: proof of iii)
Assume that (11.29) holds true for $\gamma = \alpha,\beta$, and we want to show relation (11.30). We then just notice that point iii) follows immediately from points i) and ii). This ends the proof of the lemma.

11.6 Sufficient conditions for 1:2 conservative prefluxes

Lemma 11.27 (1:2 conservative prefluxes: (I) the central case)
Let us consider continuous functions $A^j_0 : [0,+\infty) \to [0,+\infty)$ for $j = 0,1,2$, satisfying

$$A^0_0 = id_{[0,+\infty)} = A^1_0 + A^2_0$$

and for $\bar{\gamma} = (\bar{\gamma}^0,\bar{\gamma}^1,\bar{\gamma}^2) \in [0,+\infty)^3$, let us define the function $\hat{\lambda} = (\hat{\lambda}^0,\hat{\lambda}^1,\hat{\lambda}^2) : [0,+\infty)^3 \to [0,+\infty)^3$ as

$$\begin{cases} \hat{\lambda}^1(\bar{\gamma}) := \min\left\{\bar{\gamma}^1, \max\left\{A^1_0(\bar{\gamma}^0), \bar{\gamma}^0 - \bar{\gamma}^2\right\}\right\}, \\ \hat{\lambda}^2(\bar{\gamma}) := \min\left\{\bar{\gamma}^2, \max\left\{A^2_0(\bar{\gamma}^0), \bar{\gamma}^0 - \bar{\gamma}^1\right\}\right\}, \\ \hat{\lambda}^0 = \hat{\lambda}^1 + \hat{\lambda}^2 \end{cases}$$

Then $\hat{\lambda} : [0,+\infty)^3 \to [0,+\infty)^3$ is a 1:2 conservative preflux. Moreover we have

$$\hat{\lambda}^0(\bar{\gamma}) = \min\left\{\bar{\gamma}^0, \bar{\gamma}^1 + \bar{\gamma}^2\right\} \tag{11.38}$$

Remark 11.28 *We notice that the functions A^j_0 for $j = 1,2$ are not assumed to be monotone, but only the sum $A^1_0 + A^2_0$ is monotone.*

Proof of Lemma 11.27
The proof is simple, but slightly tricky and has to be done in the right way.
Step 1: proof of (11.38)
We distinguish several cases.
Case A: $\bar{\gamma}^1 < A^1_0$ **and** $\bar{\gamma}^2 < A^2_0$
Then $\hat{\lambda}^0 = \bar{\gamma}^1 + \bar{\gamma}^2 < A^1_0 + A^2_0 = \bar{\gamma}^0$.
Case B: $\bar{\gamma}^1 < A^1_0$ **and** $\bar{\gamma}^2 \ge A^2_0$

Then $\hat{\lambda}^1 = \bar{\gamma}^1$. Moreover

$$\bar{\gamma}^0 - \bar{\gamma}^1 > \bar{\gamma}^0 - A_0^1 = A_0^2$$

which implies $\max\left\{A_0^2, \bar{\gamma}^0 - \bar{\gamma}^1\right\} = \bar{\gamma}^0 - \bar{\gamma}^1$. Therefore $\hat{\lambda}^2 = \min\left\{\bar{\gamma}^2, \bar{\gamma}^0 - \bar{\gamma}^1\right\}$ and $\hat{\lambda}^0 = \min\left\{\bar{\gamma}^0, \bar{\gamma}^1 + \bar{\gamma}^2\right\}$.

Case B': $\bar{\gamma}^1 \geq A_0^1$ and $\bar{\gamma}^2 < A_0^2$

This case is symmetric of Case B.

Case C: $\bar{\gamma}^1 \geq A_0^1$ and $\bar{\gamma}^2 \geq A_0^2$

Then we deduce that

$$\bar{\gamma}^0 - \bar{\gamma}^1 \leq \bar{\gamma}^0 - A_0^1 = A_0^2$$

and then

$$\hat{\lambda}^2 = \min\left\{\bar{\gamma}^2, A_0^2\right\} = A_0^2$$

and similarly $\hat{\lambda}^1 = A_0^1$, which gives

$$\hat{\lambda}^0 = \hat{\lambda}^1 + \hat{\lambda}^2 = A_0^1 + A_0^2 = \bar{\gamma}^0 \leq \bar{\gamma}^1 + \bar{\gamma}^2.$$

Conclusion

In all cases, we get (11.38).

Step 2: proof that $\hat{\lambda}$ is a preflux

By definition, the map $\hat{\lambda} : [0, +\infty)^3 \to [0, +\infty)^3$ is continuous and satisfies

$$\hat{\lambda}^j(\bar{\gamma}) = 0 \quad \text{if} \quad \bar{\gamma}^j = 0, \quad \text{if } j \in \{0, 1, 2\}$$

It remains to show that $\hat{\lambda}$ is locally constant. We distinguish the indices $j = 0, 1, 2$.

Case A: $\hat{\lambda}^0(\bar{\gamma}) < \bar{\gamma}^0$

From (11.38), we deduce that

$$\hat{\lambda}^1 + \hat{\lambda}^2 = \hat{\lambda}^0 = \bar{\gamma}^1 + \bar{\gamma}^2$$

Because $\hat{\lambda}^k \leq \bar{\gamma}^k$ for $k = 1, 2$, we deduce that $\hat{\lambda}^k = \bar{\gamma}^k$ for $k = 1, 2$. Therefore

$$\hat{\lambda} = (\bar{\gamma}^1 + \bar{\gamma}^2, \bar{\gamma}^1, \bar{\gamma}^2) \quad \text{while} \quad \hat{\lambda}^0 < \bar{\gamma}^0$$

which shows that $\hat{\lambda}$ is locally constant in $\bar{\gamma}^0$ on $\left\{\hat{\lambda}^0(\bar{\gamma}) < \bar{\gamma}^0\right\}$.

Case B: $\hat{\lambda}^1(\bar{\gamma}) < \bar{\gamma}^1$

Then

$$\hat{\lambda}^1 = \max\left\{A_0^1, \bar{\gamma}^0 - \bar{\gamma}^2\right\}$$

Assume by contradiction that $\hat{\lambda}^2$ locally depends on $\bar{\gamma}^1$. This implies locally at those points that

$$\hat{\lambda}^2 = \bar{\gamma}^0 - \bar{\gamma}^1$$

Hence

$$\hat{\lambda}^0 = \hat{\lambda}^1 + \hat{\lambda}^2 < \bar{\gamma}^0 \tag{11.39}$$

We deduce from Case A that

$$\hat{\lambda} = (\bar{\gamma}^1 + \bar{\gamma}^2, \bar{\gamma}^1, \bar{\gamma}^2)$$

and then

$$\bar{\gamma}^0 - \bar{\gamma}^1 = \hat{\lambda}^2 = \bar{\gamma}^2$$

On the other hand (11.39) gives

$$\bar{\gamma}^1 + \bar{\gamma}^2 = \hat{\lambda}^1 + \hat{\lambda}^2 < \bar{\gamma}^0$$

which leads to a contradiction. Therefore $\hat{\lambda}^2$ is locally independent of $\bar{\gamma}^1$. Therefore

$$\hat{\lambda}^1 = \max\left\{A_0^1, \bar{\gamma}^0 - \bar{\gamma}^2\right\}, \quad \hat{\lambda}^2 = \min\left\{\bar{\gamma}^2, A_0^2\right\}, \quad \hat{\lambda}^0 = \hat{\lambda}^1 + \hat{\lambda}^2 \quad \text{while} \quad \hat{\lambda}^1 < \bar{\gamma}^1$$

which shows that $\hat{\lambda}$ is locally constant in $\bar{\gamma}^1$ on $\left\{\hat{\lambda}^1(\bar{\gamma}) < \bar{\gamma}^1\right\}$.

Case B': $\hat{\lambda}^2(\bar{\gamma}) < \bar{\gamma}^2$

This case is symmetric of Case B.

Conclusion

We conclude that $\hat{\lambda}$ is locally constant on $\left\{\hat{\lambda} \neq id_{[0,+\infty)^3}\right\}$.

This ends the proof of the lemma.

Lemma 11.29 (1:2 conservative prefluxes: (II) fundamental limiters)
Let us consider continuous functions $A_k = (A_k^0, A_k^1, A_k^2) : [0,+\infty) \to [0,+\infty)^3$ *for* $k = 0,1,2$, *satisfying*

$$\left\{\begin{array}{ll} A_k^0 = A_k^1 + A_k^2, & k = 0,1,2, \\ A_j^j = id_{[0,+\infty)}, & j = 0,1,2 \end{array}\right.$$

Let us consider the three curves for $j = 0,1,2$

$$\Gamma_j := \left\{(A_j^1, A_j^2)(\bar\gamma^j), \quad \bar\gamma^j \in [0,+\infty)\right\}.$$

and the following monotone (possibly discontinuous) envelopes of Γ_0 *for* $\bar\gamma^1, \bar\gamma^2 \in [0,+\infty)$

$$\left\{\begin{array}{l} \underline{A}_1^2(\bar\gamma^1) := \sup\limits_{A_0^1(\bar\gamma^0)\le\bar\gamma^1} A_0^2(\bar\gamma^0), \\ \underline{A}_2^1(\bar\gamma^2) := \sup\limits_{A_0^2(\bar\gamma^0)\le\bar\gamma^2} A_0^1(\bar\gamma^0), \end{array}\right.$$

Then we have

$$\underline{A}_1^2 \circ \underline{A}_2^1 \ge id_{[0,+\infty)} \quad \textit{and} \quad \underline{A}_2^1 \circ \underline{A}_1^2 \ge id_{[0,+\infty)} \tag{11.40}$$

We assume the following epigraph conditions

$$\left\{\begin{array}{l} A_1^2 \ge \underline{A}_1^2 \\ A_2^1 \ge \underline{A}_2^1 \end{array}\right. \tag{11.41}$$

i) (Expression of the preflux)
For $\bar\gamma = (\bar\gamma^0, \bar\gamma^1, \bar\gamma^2) \in [0,+\infty)^3$, *we define the function* $\hat\lambda = (\hat\lambda^0, \hat\lambda^1, \hat\lambda^2) : [0,+\infty)^3 \to [0,+\infty)^3$ *as*

$$\left\{\begin{array}{l} \hat\lambda^1(\bar\gamma) := \min\left\{\bar\gamma^1, A_2^1(\bar\gamma^2), \max\left\{A_0^1(\bar\gamma^0), \bar\gamma^0 - \bar\gamma^2\right\}\right\}, \\ \hat\lambda^2(\bar\gamma) := \min\left\{\bar\gamma^2, A_1^2(\bar\gamma^1), \max\left\{A_0^2(\bar\gamma^0), \bar\gamma^0 - \bar\gamma^1\right\}\right\}, \\ \hat\lambda^0 = \hat\lambda^1 + \hat\lambda^2 \end{array}\right. \tag{11.42}$$

Then $\hat\lambda : [0,+\infty)^3 \to [0,+\infty)^3$ *is a 1:2 conservative preflux. Moreover we have*

$$\hat\lambda^0(\bar\gamma) = \min\left\{\bar\gamma^0, \bar\gamma^1 + \bar\gamma^2, A_1^0(\bar\gamma^1), A_2^0(\bar\gamma^2)\right\} \tag{11.43}$$

Remark 11.30 *Notice that in Lemma 11.29 the functions* $A_k^j : [0,+\infty) \to [0,+\infty)$ *for* $k \neq j$ *are not assumed to be monotone in general. Moreover the particular shape (11.42) of the preflux* $\hat\lambda$ *has been guessed from the necessary conditions derived in Lemma 12.5 below.*

Proof of Lemma 11.29
Step 1: proof of (11.40)
By definition of $\underline{A}_2^1(\bar\gamma^2)$, we have

$$\underline{A}_2^1(\bar\gamma^2) = \sup_{A_0^2(p^0)\le\bar\gamma^2} A_0^1(p^0)$$

and we have

$$\underline{A}_1^2(\underline{A}_2^1(\bar\gamma^2)) = \sup_{A_0^1(q^0)\le\underline{A}_2^1(\bar\gamma^2)} A_0^2(q^0)$$

Hence

$$\underline{A}_1^2(\underline{A}_2^1(\bar\gamma^2)) = \sup_{\left\{A_0^1(q^0) \le \sup\limits_{A_0^2(p^0)\le\bar\gamma^2} A_0^1(p^0)\right\}} A_0^2(q^0)$$

Because we can always choose at least $q^0 = p^0$, we get

$$\underline{A}_1^2(\underline{A}_2^1(\bar\gamma^2)) \ge \sup_{A_0^2(p^0)\le\bar\gamma^2} A_0^2(p^0) = \bar\gamma^2$$

which shows the first inequality of (11.40). The proof of the second inequality in (11.40) is similar.

Step 2: proof of (11.43)
We distinguish several cases.
Case A: $\bar\gamma^1 < A^1_0$ **and** $\bar\gamma^2 < A^2_0$
Then
$$\hat\lambda^0 = \min\left\{\bar\gamma^1, A^1_2(\bar\gamma^2)\right\} + \min\left\{\bar\gamma^2, A^2_1(\bar\gamma^1)\right\}$$
Hence
$$\hat\lambda^0 = \begin{cases} \bar\gamma^1+\bar\gamma^2 < A^1_0 + A^2_0 = A^0_0 & \text{if} \quad \bar\gamma^1 \le A^1_2(\bar\gamma^2) \quad \text{and} \quad \bar\gamma^2 \le A^2_1(\bar\gamma^1),\\ A^1_2 + \bar\gamma^2 = A^0_2 & \text{if} \quad \bar\gamma^1 > A^1_2(\bar\gamma^2) \quad \text{and} \quad \bar\gamma^2 \le A^2_1(\bar\gamma^1),\\ \bar\gamma^1 + A^2_1 = A^0_1 & \text{if} \quad \bar\gamma^1 \le A^1_2(\bar\gamma^2) \quad \text{and} \quad \bar\gamma^2 > A^2_1(\bar\gamma^1),\\ \text{something} & \text{if} \quad \bar\gamma^1 > A^1_2(\bar\gamma^2) \quad \text{and} \quad \bar\gamma^2 > A^2_1(\bar\gamma^1),\end{cases} \tag{11.44}$$
but the last condition is empty. More precisely, we have sligthly more, we have
$$\left\{\bar\gamma^1 > A^1_2(\bar\gamma^2), \quad \bar\gamma^2 \ge A^2_1(\bar\gamma^1)\right\} = \emptyset \tag{11.45}$$
and also
$$\left\{\bar\gamma^1 \ge A^1_2(\bar\gamma^2), \quad \bar\gamma^2 > A^2_1(\bar\gamma^1)\right\} = \emptyset \tag{11.46}$$
Indeed, let us prove (11.45). Using our epigraph condition (11.41), the left hand side of (11.45) implies
$$\begin{cases} \bar\gamma^1 > \underline{A}^1_2(\bar\gamma^2),\\ \bar\gamma^2 \ge \underline{A}^2_1(\bar\gamma^1),\end{cases}$$
and then (11.40) implies by composition
$$\bar\gamma^1 > \underline{A}^1_2(\bar\gamma^2) \ge (\underline{A}^1_2 \circ \underline{A}^2_1)(\bar\gamma^1) \ge \bar\gamma^1$$
which leads to a contradiction, which shows (11.45).
Hence we conclude (in the first three cases of (11.44)) that (11.43) holds true.
Case B: $\bar\gamma^1 < A^1_0$ **and** $\bar\gamma^2 \ge A^2_0$
Then $\hat\lambda^1 = \min\left\{\bar\gamma^1, A^1_2(\bar\gamma^2)\right\}$. Moreover
$$\bar\gamma^0 - \bar\gamma^1 > \bar\gamma^0 - A^1_0 = A^2_0$$
which implies $\max\left\{A^2_0, \bar\gamma^0-\bar\gamma^1\right\} = \bar\gamma^0 - \bar\gamma^1$. Therefore $\hat\lambda^2 = \min\left\{\bar\gamma^2, A^2_1(\bar\gamma^1), \bar\gamma^0-\bar\gamma^1\right\}$ and
$$\hat\lambda^0 = \min\left\{\bar\gamma^1, A^1_2(\bar\gamma^2)\right\} + \min\left\{\bar\gamma^2, A^2_1(\bar\gamma^1), \bar\gamma^0-\bar\gamma^1\right\} \le \min\left\{\bar\gamma^0, \bar\gamma^1+\bar\gamma^2, A^0_1(\bar\gamma^1), A^0_2(\bar\gamma^2)\right\} \tag{11.47}$$
Hence
(11.48)
$$\hat\lambda^0 = \begin{cases} \bar\gamma^1+\bar\gamma^2 & \text{if} \quad \bar\gamma^1 \le A^1_2(\bar\gamma^2) \quad \text{and} \quad \bar\gamma^2 \le A^2_1(\bar\gamma^1) \quad \text{and} \quad \bar\gamma^2 \le \bar\gamma^0-\bar\gamma^1,\\ \bar\gamma^1 + (\bar\gamma^0-\bar\gamma^1) = \bar\gamma^0 & \text{if} \quad \bar\gamma^1 \le A^1_2(\bar\gamma^2) \quad \text{and} \quad \bar\gamma^2 \le A^2_1(\bar\gamma^1) \quad \text{and} \quad \bar\gamma^2 > \bar\gamma^0-\bar\gamma^1,\\ & \\ A^1_2 + \bar\gamma^2 = A^0_2 & \text{if} \quad \bar\gamma^1 > A^1_2(\bar\gamma^2) \quad \text{and} \quad \bar\gamma^2 \le A^2_1(\bar\gamma^1) \quad \text{and} \quad \bar\gamma^2 \le \bar\gamma^0-\bar\gamma^1,\\ \text{something} & \text{if} \quad \bar\gamma^1 > A^1_2(\bar\gamma^2) \quad \text{and} \quad \bar\gamma^2 \le A^2_1(\bar\gamma^1) \quad \text{and} \quad \bar\gamma^2 > \bar\gamma^0-\bar\gamma^1,\\ & \\ \bar\gamma^1 + A^2_1 = A^0_1 & \text{if} \quad \bar\gamma^1 \le A^1_2(\bar\gamma^2) \quad \text{and} \quad \bar\gamma^2 > A^2_1(\bar\gamma^1) \quad \text{and} \quad \bar\gamma^2 \le \bar\gamma^0-\bar\gamma^1,\\ \bar\gamma^1 + \min\left\{A^2_1, \bar\gamma^0-\bar\gamma^1\right\} = \min\left\{A^0_1, \bar\gamma^0\right\} & \text{if} \quad \bar\gamma^1 \le A^1_2(\bar\gamma^2) \quad \text{and} \quad \bar\gamma^2 > A^2_1(\bar\gamma^1) \quad \text{and} \quad \bar\gamma^2 > \bar\gamma^0-\bar\gamma^1,\\ & \\ \text{something} & \text{if} \quad \bar\gamma^1 > A^1_2(\bar\gamma^2) \quad \text{and} \quad \bar\gamma^2 > A^2_1(\bar\gamma^1) \quad ,\end{cases}$$
where the condition in the last line is again empty. Let us now show that the condition in the fourth line is also empty. Indeed, we have
$$\begin{cases} \bar\gamma^1 < A^1_0(\bar\gamma^0),\\ \bar\gamma^2 \ge A^2_0(\bar\gamma^0),\end{cases}$$
Hence
$$\bar\gamma^1 > A^1_2(\bar\gamma^2) \ge \underline{A}^1_2(\bar\gamma^2) = \sup_{A^2_0(p^0)\le\bar\gamma^2} A^1_0(p^0) \ge A^1_0(\bar\gamma^0) > \bar\gamma^1$$

Contradiction.
Therefore, we see using (11.47) that the five remaining lines of (11.48) show that (11.43) holds true.
Case B': $\bar\gamma^1 \ge A_0^1$ and $\bar\gamma^2 < A_0^2$
This case is symmetric of Case B.
Case C: $\bar\gamma^1 \ge A_0^1$ and $\bar\gamma^2 \ge A_0^2$
Then we deduce that
$$\bar\gamma^0 - \bar\gamma^1 \le \bar\gamma^0 - A_0^1 = A_0^2$$
and then
$$\hat\lambda^2 = \min\left\{\bar\gamma^2, A_0^2(\bar\gamma^0), A_1^2(\bar\gamma^1)\right\} = \min\left\{A_0^2(\bar\gamma^0), A_1^2(\bar\gamma^1)\right\}$$
and similarly $\hat\lambda^1 = \min\left\{A_0^1(\bar\gamma^0), A_2^1(\bar\gamma^2)\right\}$, which gives (using assumptions of Case C)
$$\hat\lambda^0 = \min\left\{A_0^1(\bar\gamma^0), A_2^1(\bar\gamma^2)\right\} + \min\left\{A_0^2(\bar\gamma^0), A_1^2(\bar\gamma^1)\right\} \le \min\left\{A_1^0(\bar\gamma^1), A_2^0(\bar\gamma^2), \bar\gamma^0, \bar\gamma^1 + \bar\gamma^2\right\}$$
Precisely, we get
$$\hat\lambda^0 = \begin{cases} \bar\gamma^0 & \text{if} \quad A_0^1 \le A_2^1 \quad \text{and} \quad A_0^2 \le A_1^2, \\ \text{something} & \text{if} \quad A_0^1 > A_2^1 \quad \text{or} \quad A_0^2 > A_1^2, \end{cases} \tag{11.49}$$
Let us show that the assumption of the second line is empty, i.e. that $A_0^1(\bar\gamma^0) > A_2^1(\bar\gamma^2)$ or $A_0^2(\bar\gamma^0) > A_1^2(\bar\gamma^1)$ is empty. Indeed, assume for instance that we have the first strict inequality
$$A_0^1(\bar\gamma^0) > A_2^1(\bar\gamma^2) \tag{11.50}$$
Then, using $\bar\gamma^2 \ge A_0^2(\bar\gamma^0)$, we get that
$$A_2^1(\bar\gamma^2) \ge \underline{A}_2^1(\bar\gamma^2) = \sup_{A_0^2(p^0)\le\bar\gamma^2} A_0^1(p^0) \ge A_0^1(\bar\gamma^0) > A_2^1(\bar\gamma^2)$$
where we have used (11.50) in the last strict inequality. This leads to a contradiction. We get a similar contradiction with the second strict inequality, and conclude the the expected condition is empty. Therefore, the first line of (11.49) gives the result (11.43).
Conclusion
In all cases, we get (11.43).
Step 3: proof that $\hat\lambda$ is a preflux
By definition, the map $\hat\lambda : [0,+\infty)^3 \to [0,+\infty)^3$ is continuous and satisfies
$$\hat\lambda^j(\bar\gamma) = 0 \quad \text{if} \quad \bar\gamma^j = 0, \quad \text{if } j \in \{0,1,2\}$$
It remains to show that $\hat\lambda$ is locally constant. We distinguish the indices $j = 0,1,2$.
Case A: $\hat\lambda^0(\bar\gamma) < \bar\gamma^0$
From (11.43), we deduce that
$$\hat\lambda^1 + \hat\lambda^2 = \hat\lambda^0 = \min\left\{\bar\gamma^1 + \bar\gamma^2, A_1^0(\bar\gamma^1), A_2^0(\bar\gamma^2)\right\} = \min\left\{\bar\gamma^1 + \bar\gamma^2, \bar\gamma^1 + A_1^2(\bar\gamma^1), \bar\gamma^2 + A_2^1(\bar\gamma^2)\right\} \tag{11.51}$$
We know that
$$\begin{cases} \hat\lambda^1 \le \min\left\{\bar\gamma^1, A_2^1(\bar\gamma^2)\right\}, \\ \hat\lambda^2 \le \min\left\{\bar\gamma^2, A_1^2(\bar\gamma^1)\right\}, \end{cases}$$
and introduce $\varepsilon_1, \varepsilon_2 \ge 0$ such that
$$\begin{cases} \hat\lambda^1 + \varepsilon_1 = \min\left\{\bar\gamma^1, A_2^1(\bar\gamma^2)\right\}, \\ \hat\lambda^2 + \varepsilon_2 = \min\left\{\bar\gamma^2, A_1^2(\bar\gamma^1)\right\}, \end{cases}$$
we deduce that
$$\begin{aligned} \hat\lambda^0 + \varepsilon_1 + \varepsilon_2 &= \min\left\{\bar\gamma^1, A_2^1(\bar\gamma^2)\right\} + \min\left\{\bar\gamma^2, A_1^2(\bar\gamma^1)\right\} \\ &= \begin{cases} \bar\gamma^1 + \bar\gamma^2 & \le \hat\lambda^0 \quad \text{if} \quad \bar\gamma^1 \le A_2^1(\bar\gamma^2) \quad \text{and} \quad \bar\gamma^2 \le A_1^2(\bar\gamma^1), \\ A_2^1 + \bar\gamma^2 = A_2^0 & \le \hat\lambda^0 \quad \text{if} \quad \bar\gamma^1 > A_2^1(\bar\gamma^2) \quad \text{and} \quad \bar\gamma^2 \le A_1^2(\bar\gamma^1), \\ \bar\gamma^1 + A_1^2 = A_1^0 & \le \hat\lambda^0 \quad \text{if} \quad \bar\gamma^1 \le A_2^1(\bar\gamma^2) \quad \text{and} \quad \bar\gamma^2 > A_1^2(\bar\gamma^1), \\ \text{something} = A_2^1 + A_1^2 & \le \hat\lambda^0 \quad \text{if} \quad \bar\gamma^1 > A_2^1(\bar\gamma^2) \quad \text{and} \quad \bar\gamma^2 > A_1^2(\bar\gamma^1), \end{cases} \end{aligned}$$

where we have used (11.51). Notice also that the condition in the last line is empty because of (11.45). We deduce that $\varepsilon_1 + \varepsilon_2 = 0$ and then

$$(\hat\lambda^0, \hat\lambda^1, \hat\lambda^2) = (\min\left\{\bar\gamma^1 + \bar\gamma^2, \bar\gamma^1 + A_1^2(\bar\gamma^1), \bar\gamma^2 + A_2^1(\bar\gamma^2)\right\}, \min\left\{\bar\gamma^1, A_2^1(\bar\gamma^2)\right\}, \min\left\{\bar\gamma^2, A_1^2(\bar\gamma^1)\right\}) \tag{11.52}$$

This shows that $\hat\lambda(\bar\gamma)$ is locally constant in $\bar\gamma^0$ on $\left\{\hat\lambda^0(\bar\gamma) < \bar\gamma^0\right\}$.

Case B: $\hat\lambda^1(\bar\gamma) < \bar\gamma^1$

Then

$$\hat\lambda^1 = \min\left\{A_2^1(\bar\gamma^2), \max\left\{A_0^1(\bar\gamma^0), \bar\gamma^0 - \bar\gamma^2\right\}\right\}$$

Assume by contradiction that

$$\hat\lambda^2 \text{ locally depends on } \bar\gamma^1. \tag{11.53}$$

This implies locally at those points that

$$\hat\lambda^2 = \min\left\{A_1^2(\bar\gamma^1), \bar\gamma^0 - \bar\gamma^1\right\}$$

Hence

$$\hat\lambda^0 = \hat\lambda^1 + \hat\lambda^2 < \bar\gamma^0 \tag{11.54}$$

We deduce from Case A that $\hat\lambda$ is locally constant in $\bar\gamma^0$, and is given by (11.52). Therefore

$$\begin{cases} \hat\lambda^1 = \min\left\{\bar\gamma^1, A_2^1(\bar\gamma^2)\right\}, \\ \hat\lambda^2 = \min\left\{\bar\gamma^2, A_1^2(\bar\gamma^1)\right\} = \min\left\{A_1^2(\bar\gamma^1), \bar\gamma^0 - \bar\gamma^1\right\} \end{cases}$$

Because $\hat\lambda^1 < \bar\gamma^1$ and $\hat\lambda^2$ is independent on $\bar\gamma^0$, we get locally

$$\begin{cases} \hat\lambda^1 = A_2^1(\bar\gamma^2) < \bar\gamma^1, \\ \hat\lambda^2 = A_1^2(\bar\gamma^1) \le \bar\gamma^2 \end{cases}$$

which is an empty condition by (11.45). Contradiction. Hence (11.53) is false and $\hat\lambda^2$ is locally independent on $\bar\gamma^1$, and we deduce that $\hat\lambda$ is locally constant in $\bar\gamma^1$ on $\left\{\hat\lambda^1(\bar\gamma) < \bar\gamma^1\right\}$.

Case B': $\hat\lambda^2(\bar\gamma) < \bar\gamma^2$

This case is symmetric of Case B.

Conclusion

We conclude that $\hat\lambda$ is locally constant on $\left\{\hat\lambda \neq id_{[0,+\infty)^3}\right\}$.

This ends the proof of the lemma.

Lemma 11.31 (1:2 conservative prefluxes: (III) truncation)
Under the assumptions of Lemma 11.29, let us consider the 1:2 conservative preflux $\hat\lambda$ given in (11.42). Let $\bar\lambda^0 \in (0, +\infty)$, and $\bar\lambda = (\bar\lambda^0, +\infty, +\infty)$ and the truncation $T_{\bar\lambda}$. Then

$$\hat\lambda_0 := \hat\lambda \circ T_{\bar\lambda}$$

is a 1:2 conservative preflux which writes explicitly with

$$\bar{\bar\gamma}^0 := \min\left\{\bar\gamma^0, \bar\lambda^0\right\}$$

as

$$\begin{cases} \hat\lambda_0^1(\bar\gamma) := \min\left\{\bar\gamma^1, A_2^1(\bar\gamma^2), \max\left\{A_0^1(\bar{\bar\gamma}^0), \bar{\bar\gamma}^0 - \bar\gamma^2\right\}\right\}, \\ \hat\lambda_0^2(\bar\gamma) := \min\left\{\bar\gamma^2, A_1^2(\bar\gamma^1), \max\left\{A_0^2(\bar{\bar\gamma}^0), \bar{\bar\gamma}^0 - \bar\gamma^1\right\}\right\}, \\ \hat\lambda_0^0 = \hat\lambda_0^1 + \hat\lambda_0^2 = \min\left\{\bar{\bar\gamma}^0, \bar\gamma^1 + \bar\gamma^2, A_1^0(\bar\gamma^1), A_2^0(\bar\gamma^2)\right\} \end{cases} \tag{11.55}$$

We set

$$A_*^0 := \bar\lambda^0, \quad A_*^1 := A_0^1(\bar\lambda^0), \quad A_*^2 := A_0^2(\bar\lambda^0)$$

Assume moreover that

$$\left\{\begin{array}{l} A_1^2(A_*^1) = A_*^2, \\ A_1^0(\bar\gamma^1) \le \bar\lambda^0 \quad \textit{i.e.} \quad A_1^2(\bar\gamma^1) \le \bar\lambda^0 - \gamma^1 \quad \textit{if} \quad \bar\gamma^1 \le A_*^1 \\ A_2^1(A_*^2) = A_*^1, \\ A_2^0(\bar\gamma^2) \le \bar\lambda^0 \quad \textit{i.e.} \quad A_2^1(\bar\gamma^2) \le \bar\lambda^0 - \gamma^2 \quad \textit{if} \quad \bar\gamma^2 \le A_*^2 \end{array}\right.$$

and define

$$\left\{\begin{array}{l}
\bar A_1^2(\bar\gamma^1) := \left\{\begin{array}{ll} A_1^2(\bar\gamma^1) \le \bar\lambda^0 - \bar\gamma^1, & \textit{if} \quad \bar\gamma^1 < A_*^1 \\ A_*^2 & \textit{if} \quad \bar\gamma^1 \ge A_*^1 \end{array}\right. \\
\bar A_2^1(\bar\gamma^2) := \left\{\begin{array}{ll} A_2^1(\bar\gamma^2) \le \bar\lambda^0 - \bar\gamma^2, & \textit{if} \quad \bar\gamma^2 < A_*^2 \\ A_*^1 & \textit{if} \quad \bar\gamma^2 \ge A_*^2 \end{array}\right. \\
\bar A_1^0(\bar\gamma^1) := \left\{\begin{array}{ll} \bar\gamma^1 + \bar A_1^2(\bar\gamma^1) = A_1^0(\bar\gamma^1) \le \bar\lambda^0, & \textit{if} \quad \bar\gamma^1 < A_*^1 \\ A_*^0 & \textit{if} \quad \bar\gamma^1 \ge A_*^1 \end{array}\right. \\
\bar A_2^0(\bar\gamma^2) := \left\{\begin{array}{ll} \bar\gamma^2 + \bar A_2^1(\bar\gamma^2) = A_2^0(\bar\gamma^2) \le \bar\lambda^0, & \textit{if} \quad \bar\gamma^2 < A_*^2 \\ A_*^0 & \textit{if} \quad \bar\gamma^2 \ge A_*^2 \end{array}\right. \\
\bar A_0^1(\bar\gamma^0) := \left\{\begin{array}{ll} A_0^1(\bar\gamma^0), & \textit{if} \quad \bar\gamma^0 < A_*^0 \\ A_*^1 & \textit{if} \quad \bar\gamma^0 \ge A_*^0 \end{array}\right. \\
\bar A_0^2(\bar\gamma^0) := \left\{\begin{array}{ll} A_0^2(\bar\gamma^0), & \textit{if} \quad \bar\gamma^0 < A_*^0 \\ A_*^2 & \textit{if} \quad \bar\gamma^0 \ge A_*^0 \end{array}\right.
\end{array}\right.$$

Then we have

$$\left\{\begin{array}{l} \hat\lambda_0^1(\bar\gamma) := \min\left\{\bar\gamma^1, \bar A_2^1(\bar\gamma^2), \max\left\{\bar A_0^1(\bar\gamma^0), \bar\gamma^0 - \bar\gamma^2\right\}\right\}, \\ \hat\lambda_0^2(\bar\gamma) := \min\left\{\bar\gamma^2, \bar A_1^2(\bar\gamma^1), \max\left\{\bar A_0^2(\bar\gamma^0), \bar\gamma^0 - \bar\gamma^1\right\}\right\}, \\ \hat\lambda_0^0 = \hat\lambda_0^1 + \hat\lambda_0^2 = \min\left\{\bar\gamma^0, \bar\gamma^1 + \bar\gamma^2, \bar A_1^0(\bar\gamma^1), \bar A_2^0(\bar\gamma^2)\right\} \end{array}\right. \tag{11.56}$$

Proof of Lemma 11.31

Notice that (because Γ_0 is sandwiched in between Γ_1 and Γ_2), we have

$$\left\{\begin{array}{lll} A_1^2(\bar\gamma^1) \ge A_*^2 & \text{for} & \bar\gamma^1 \ge A_*^1, \\ A_2^1(\bar\gamma^2) \ge A_*^1 & \text{for} & \bar\gamma^2 \ge A_*^2 \end{array}\right.$$

and then

$$\left\{\begin{array}{lll} A_1^0(\bar\gamma^1) \ge A_*^0 & \text{for} & \bar\gamma^1 \ge A_*^1, \\ A_2^0(\bar\gamma^2) \ge A_*^0 & \text{for} & \bar\gamma^2 \ge A_*^2 \end{array}\right.$$

which justifies the expression of $\hat\lambda_0^0$.

Let us now set

$$F^1(\bar\gamma^0, \bar\gamma^2) := \min\left\{A_2^1(\bar\gamma^2), \max\left\{A_0^1(\bar{\bar\gamma}^0), \bar{\bar\gamma}^0 - \bar\gamma^2\right\}\right\}$$

and

$$\bar F^1(\bar\gamma^0, \bar\gamma^2) := \min\left\{\bar A_2^1(\bar\gamma^2), \max\left\{\bar A_0^1(\bar\gamma^0), \bar\gamma^0 - \bar\gamma^2\right\}\right\}$$

Hence $F^1 = \bar F^1$ because we have

$$\left\{\begin{array}{lll}
\text{no truncation at all} & \text{if} \quad \bar\gamma^0 < A_*^0, & \bar\gamma^2 < A_*^2, \\
\max\left\{\bar A_0^1(\bar\gamma^0), \bar\gamma^0 - \bar\gamma^2\right\} = \max\left\{A_0^1(\bar\gamma^0), \bar\gamma^0 - \bar\gamma^2\right\} = A_0^1(\bar\gamma^0) \le A_*^1 = \bar A_2^1(\bar\gamma^2) \le A_2^1(\bar\gamma^2) & \text{if} \quad \bar\gamma^0 < A_*^0, & \bar\gamma^2 \ge A_*^2, \\
\bar A_2^1(\bar\gamma^2) = A_2^1(\bar\gamma^2) \le \bar\lambda^0 - \bar\gamma^2 \le \bar\gamma^0 - \bar\gamma^2 & \text{if} \quad \bar\gamma^0 \ge A_*^0, & \bar\gamma^2 < A_*^2, \\
\left.\begin{array}{l} \max\left\{A_0^1(\bar{\bar\gamma}^0), \bar{\bar\gamma}^0 - \bar\gamma^2\right\} = A_*^1 \le A_2^1(\bar\gamma^2) \\ \max\left\{\bar A_0^1(\bar\gamma^0), \bar\gamma^0 - \bar\gamma^2\right\} \ge A_*^1 = \bar A_2^1(\bar\gamma^2) \end{array}\right\} & \text{if} \quad \bar\gamma^0 \ge A_*^0, & \bar\gamma^2 \ge A_*^2,
\end{array}\right.$$

Similarly, we have $F^2 = \bar F^2$. We conclude that (11.56) holds true. This ends the proof of the lemma.

Lemma 11.32 (1:2 conservative prefluxes: (IV) extension of the limiters)
Under the assumptions of Lemma 11.31, we now extend the limiters $\bar{A}_2^1$ and $\bar{A}_1^2$. Precisely, we consider continuous functions $\tilde{A}_k^j : [0,+\infty) \to [0,+\infty)$ satisfying

$$\tilde{A}_k^j(\bar{\gamma}^k) = A_*^j \quad \textit{for} \quad \bar{\gamma}^k \geq A_*^k \quad \textit{for all} \quad (j,k) \in \{(1,2),(2,1),(0,1),(0,2)\}$$

such that

$$\begin{cases} \bar{A}_1^2(\bar{\gamma}^1) = \min\left\{\tilde{A}_1^2(\bar{\gamma}^1), \max\left\{A_*^2, \bar{\lambda}^0 - \bar{\gamma}^1\right\}\right\} \\ \\ \bar{A}_2^1(\bar{\gamma}^2) = \min\left\{\tilde{A}_2^1(\bar{\gamma}^2), \max\left\{A_*^1, \bar{\lambda}^0 - \bar{\gamma}^2\right\}\right\} \\ \\ \tilde{A}_0^1 = \bar{A}_0^1 \\ \\ \tilde{A}_0^2 = \bar{A}_0^2 \end{cases}$$

Then we can also write

$$\begin{cases} \hat{\lambda}_0^1(\bar{\gamma}) := \min\left\{\bar{\gamma}^1, \tilde{A}_2^1(\bar{\gamma}^2), \max\left\{\tilde{A}_0^1(\bar{\gamma}^0), \bar{\gamma}^0 - \bar{\gamma}^2\right\}\right\}, \\ \hat{\lambda}_0^2(\bar{\gamma}) := \min\left\{\bar{\gamma}^2, \tilde{A}_1^2(\bar{\gamma}^1), \max\left\{\tilde{A}_0^2(\bar{\gamma}^0), \bar{\gamma}^0 - \bar{\gamma}^1\right\}\right\}, \\ \hat{\lambda}_0^0 = \hat{\lambda}_0^1 + \hat{\lambda}_0^2 \end{cases} \tag{11.57}$$

which is a 1:2 conservative preflux.

Proof of Lemma 11.32
Let us consider for instance $\hat{\lambda}_0^2$ (the reasoning is similar for $\hat{\lambda}_0^1$). We have

$$\tilde{A}_1^2(\bar{\gamma}^1) = A_*^2 = \bar{A}_1^2(\bar{\gamma}^1) \quad \text{if} \quad \bar{\gamma}^1 \geq A_*^1$$

If $\bar{\gamma}^1 < A_*^1$, then we have

$$\max\left\{\tilde{A}_0^2(\bar{\gamma}^0), \bar{\gamma}^0 - \bar{\gamma}^1\right\} \leq \max\left\{A_*^2, \bar{\lambda}^0 - \bar{\gamma}^1\right\}$$

which implies using $\bar{A}_1^2 \leq \tilde{A}_1^2$ and $\bar{A}_2^0 = \tilde{A}_0^2$

$$\begin{aligned} I_1 := \min\left\{\bar{A}_1^2(\bar{\gamma}^1), \max\left\{\bar{A}_0^2(\bar{\gamma}^0), \bar{\gamma}^0 - \bar{\gamma}^1\right\}\right\} \quad &\leq \min\left\{\tilde{A}_1^2(\bar{\gamma}^1), \max\left\{\tilde{A}_0^2(\bar{\gamma}^0), \bar{\gamma}^0 - \bar{\gamma}^1\right\}\right\} \\ &\leq \min\left\{\tilde{A}_1^2(\bar{\gamma}^1), \max\left\{A_*^2, \bar{\lambda}^0 - \bar{\gamma}^1\right\}\right\} \\ &= \bar{A}_1^2(\bar{\gamma}^1) \end{aligned}$$

whose we deduce that

$$\begin{aligned} I_1 \quad &\leq \min\left\{\tilde{A}_1^2(\bar{\gamma}^1), \max\left\{\tilde{A}_0^2(\bar{\gamma}^0), \bar{\gamma}^0 - \bar{\gamma}^1\right\}\right\} =: \tilde{I}_1 \\ &\leq \min\left\{\bar{A}_1^2(\bar{\gamma}^1), \max\left\{\tilde{A}_0^2(\bar{\gamma}^0), \bar{\gamma}^0 - \bar{\gamma}^1\right\}\right\} \\ &= \min\left\{\bar{A}_1^2(\bar{\gamma}^1), \max\left\{\bar{A}_0^2(\bar{\gamma}^0), \bar{\gamma}^0 - \bar{\gamma}^1\right\}\right\} \\ &= I_1 \end{aligned}$$

Therefore $I_1 = \tilde{I}_1$, and this justifies the expression of $\hat{\lambda}_0^2$. This ends the proof of the lemma.

As a complement, we now deliver the following result.

Lemma 11.33 (More on 1:2 conservative prefluxes)
Under the assumptions of Lemma 11.29, the following holds.
i) (Preflux zone by zone)
For any (possibly empty) subset $I \subset \{0,1,2\}$ and $\bar{I} := \{0,1,2\} \backslash I$, we set

$$E_I := \left\{\bar{\gamma} \in [0,+\infty)^3, \quad \left\{ \begin{array}{lll} \hat{\lambda}^j(\bar{\gamma}) < \bar{\gamma}^j & \textit{for all} & j \in I, \\ \hat{\lambda}^j(\bar{\gamma}) = \bar{\gamma}^j & \textit{for all} & j \in \bar{I}, \end{array} \right. \middle| \right\}$$

Assume furthermore that $A_0^j > 0$ on $(0,+\infty)$ for $j = 1,2$, and let

$$\begin{cases} U_1 = \textit{connected component of } [0,+\infty)^2 \backslash \Gamma_0 \textit{ below } \Gamma_0, \\ U_2 = \textit{connected component of } [0,+\infty)^2 \backslash \Gamma_0 \textit{ above } \Gamma_0, \\ \bar{U}_j = U_j \cup \Gamma_0, \quad j = 1,2, \end{cases}$$

and

$$\left\{\begin{array}{l} K_0 := \left\{(p^1,p^2)\in[0,+\infty)^2, \quad (p^1,p^2)\le (A_2^1(p^2),A_1^2(p^1))\right\},\\ K_1 := \left\{(p^1,p^2)\in \bar U_1 \quad with \quad p^1\le A_2^1(p^2)\right\},\\ K_2 := \left\{(p^1,p^2)\in \bar U_2 \quad with \quad p^2\le A_1^2(p^1)\right\},\end{array}\right.$$

which satisfy

$$K_1\cap K_2=\Gamma_0 \quad and \quad K_0=K_1\cup K_2$$

with K_1 below Γ_0 and K_2 above Γ_0.
Then we have explicitly

$$\hat\lambda(\bar\gamma)=\left\{\begin{array}{llll} \bar\gamma & on & E_\emptyset & =\left\{(p^1+p^2,p^1,p^2),\quad (p^1,p^2)\in K_0\right\},\\ (\bar\gamma^1+\bar\gamma^2,\bar\gamma^1,\bar\gamma^2) & on & E_{\{0\}} & =(0,+\infty)e_0+\left\{(p^1+p^2,p^1,p^2),\quad (p^1,p^2)\in K_0\right\},\\ (\bar\gamma^0,\bar\gamma^0-\bar\gamma^2,\bar\gamma^2) & on & E_{\{1\}} & =(0,+\infty)e_1+\left\{(p^1+p^2,p^1,p^2),\quad (p^1,p^2)\in K_1)\right\}\\ (\bar\gamma^0,\bar\gamma^1,\bar\gamma^0-\bar\gamma^1) & on & E_{\{2\}} & =(0,+\infty)e_2+\left\{(p^1+p^2,p^1,p^2),\quad (p^1,p^2)\in K_2\right\}\\ & & & \\ A_2(\bar\gamma^2) & on & E_{\{0,1\}} & =\left\{(\bar\gamma^0,\bar\gamma^1)>(A_2^0,A_2^1)(\bar\gamma^2)\right\}\\ A_1(\bar\gamma^1) & on & E_{\{0,2\}} & =\left\{(\bar\gamma^0,\bar\gamma^2)>(A_1^0,A_1^2)(\bar\gamma^1)\right\}\\ A_0(\bar\gamma^0) & on & E_{\{1,2\}} & =\left\{(\bar\gamma^1,\bar\gamma^2)>(A_0^1,A_0^2)(\bar\gamma^0)\right\}\\ & & & \\ nothing & on & E_{\{0,1,2\}} & =\emptyset \end{array}\right.$$

with the partition

$$[0,+\infty)^3=E_\emptyset\cup E_{\{0\}}\cup E_{\{1\}}\cup E_{\{2\}}\cup E_{\{0,1\}}\cup E_{\{0,2\}}\cup E_{\{1,2\}}\cup E_{\{0,1,2\}}.$$

Proof of Lemma 11.33
Even if it is probably possible to check it by hands, we prefer to see this statement as a consequence of necessary conditions for 1:2 conservative prefluxes, as it is stated in Lemma 12.5, whose independent proof will be given later below. This ends the proof of the lemma.

12 Some characterizations of prefluxes

12.1 Explicit characterization of prefluxes for $N=1,2$

In this subsection, we give explicit charcaterizations of prefluxes in some special cases.

Proposition 12.1 (Explicit characterization of prefluxes, $N=1,2$)
Let us consider functions $\hat\gamma:[0,+\infty)^N\to[0,+\infty)^N$ for $N\ge 1$.
i) (Characterization of prefluxes for $N=1$)
A function $\hat\gamma$ is a preflux for $N=1$ if and only if there exists a constant $A_*^1\in[0,+\infty]$ such that

$$\hat\gamma^1(\bar\gamma^1)=\min\left\{\bar\gamma^1,A_*^1\right\} \quad for\ all \quad \bar\gamma^1\in[0,+\infty)$$

Moreover all such $\hat\gamma^1$ are Kružkov prefluxes.
ii) (Characterization of prefluxes for $N=2$)
A function $\hat\gamma$ is a preflux for $N=2$ if and only if there exists a constant $(A_*^1,A_*^2)\in[0,+\infty)^2\cup\{(+\infty,+\infty)\}$ and two continuous functions $A^1,A^2:[0,+\infty)\to[0,+\infty]$ such that

$$\left\{\begin{array}{l} \left.\begin{array}{l}\hat\gamma^1(\bar\gamma)=\min\left\{\bar\gamma^1,A^1(\bar\gamma^2)\right\}\\ \hat\gamma^2(\bar\gamma)=\min\left\{\bar\gamma^2,A^2(\bar\gamma^1)\right\}\end{array}\right. \quad for\ all \quad \bar\gamma=(\bar\gamma^1,\bar\gamma^2)\in[0,+\infty)^2\\ \\ A^1=const=A_*^1 \quad on \quad [A_*^2,+\infty)\\ A^2=const=A_2^* \quad on \quad [A_*^1,+\infty)\end{array}\right.$$

with the epigraph compatibility condition

$$\left\{\bar\gamma^1>A^1(\bar\gamma^2)\right\}\cap\left\{\bar\gamma^2>A^2(\bar\gamma^1)\right\}=\left\{(\bar\gamma^1,\bar\gamma^2)\in(A_*^1,+\infty)\times(A_*^2,+\infty)\right\}$$

Then

$$\hat\gamma([0,+\infty)^2)=\left\{(\bar\gamma^1,\bar\gamma^2)\in[0,+\infty)^2,\quad \bar\gamma^1\le A^1(\bar\gamma^2),\quad \bar\gamma^2\le A^2(\bar\gamma^1)\right\}$$

iii) (Characterization of Kružkov prefluxes for $1:1$ junctions)
For $\sigma = (\sigma^1, \sigma^2) = (1, -1)$, then the preflux $\hat{\gamma} : [0, +\infty)^2 \to [0, +\infty)^2$ is Kružkov if and only if it is as in point ii) where A^j are 1-Lipschitz functions for $j = 1, 2$ (including some possible constant functions A^j identically equal to $+\infty$).
iv) (Characterization of monotone prefluxes for $1:1$ junctions)
For $\sigma = (\sigma^1, \sigma^2) = (1, -1)$, then the preflux $\hat{\gamma} : [0, +\infty)^2 \to [0, +\infty)^2$ is σ-monotone if and only if it is as in point ii) with moreover

$$(A^1)' \geq 0, \quad (A^2)' \geq 0 \quad \text{in} \quad \mathcal{D}'(0, +\infty)$$

v) (Characterization of conservative prefluxes for $1:1$ junctions)
A preflux $\hat{\gamma} : [0, +\infty)^2 \to [0, +\infty)^2$ is conservative in the sense of Definition 11.1 if and only if there exists some constant $A^0_* \in [0, +\infty]$ such that

$$\hat{\gamma}^1(\bar{\gamma}) = \bar{\gamma}^2(\bar{\gamma}) = \min\left\{\bar{\gamma}^1, \bar{\gamma}^2, A^0_*\right\}$$

Proposition 12.1 implies in particular Propositions 2.48 and 2.49 of the Introduction.

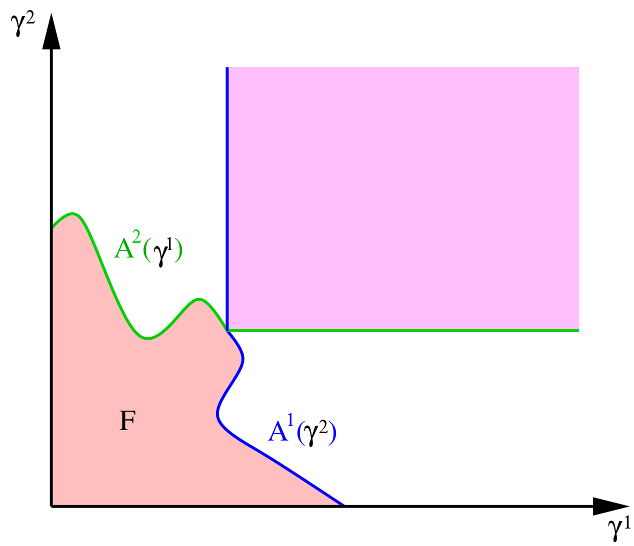

Figure 6: Compatibility for A^1 and A^2 with $F := \hat{\gamma}([0, +\infty)^2)$.

Remark 12.2 (Formal Bassins of Attraction)
For $N = 1$, notice that the Bassins of Attraction of $\hat{\gamma}^1$ for $f := Id_{[0,+\infty)}$ and $\sigma^1 = 1$ are the singletons $\left\{p^1\right\}_{p^1 \in [0, A^1_*]}$ and formally $(A^1_*, +\infty]$.
For $N = 2$, the Bassins of Attraction of $\hat{\gamma}$ for $f := Id_{[0,+\infty)^2}$ and $(\sigma^1, \sigma^2) = (1, 1)$ are the singletons $\left\{p = (p^1, p^2)\right\}_{p^1 < A^1(p^2),\ p^2 < A^2(p^1)}$, formally the rectangle $(A^1_*, +\infty] \times [A^2_*, +\infty]$, and also formally the horizontal lines $(A^1(p^2), +\infty] \times \left\{p^2\right\}$ for $p^2 < A^2_*$ and the vertical lines $\left\{p^1\right\} \times (A^1(p^2), +\infty]$ for $p^1 < A^1_*$.

Proof of Proposition 12.1
Step 1: proof of i)
Let us consider the set

$$S_0 := \left\{\bar{\gamma}^1 \in [0, +\infty), \quad \hat{\gamma}^1(\bar{\gamma}^1) < \bar{\gamma}^1\right\}$$

which is an open set because $\hat{\gamma}^1$ is continuous. Because $\hat{\gamma}^1$ is locally constant on $\left\{\hat{\gamma}^1 < Id_{[0,+\infty)}\right\}$, we deduce that $\hat{\gamma}^1$ is constant on S_0. Because $0 \leq \hat{\gamma}^1 \leq Id_{[0,+\infty)}$, we deduce that there exists $A^1_* \in [0, +\infty]$ such that $S_0 = (A^1_*, +\infty)$. This implies easily the result.
Step 2: proof of ii)
Step 2.1: preliminary remarks

We first notice that by definition of the preflux $\hat{\gamma}$, we can write

$$\left\{\begin{array}{l}\hat{\gamma}^1(\bar{\gamma}) = \min\left\{\bar{\gamma}^1, \tilde{A}^1(\bar{\gamma})\right\} \\ \hat{\gamma}^2(\bar{\gamma}) = \min\left\{\bar{\gamma}^2, \tilde{A}^2(\bar{\gamma})\right\}\end{array}\right.$$

with continuous functions $\tilde{A}^j : [0,+\infty)^2 \to [0,+\infty)$ satisfying $\tilde{A}^j(\bar{\gamma}) \le \bar{\gamma}^j$.
If we freeze $\bar{\gamma}^2 := \bar{\gamma}_0^2$, then notice that $\bar{\gamma}^1 \mapsto \hat{\gamma}^1(\bar{\gamma}^1, \bar{\gamma}_0^2)$ is a preflux for $N = 1$. Then Step 1 implies that $\tilde{A}^1(\bar{\gamma}^1, \bar{\gamma}_0^2)$ is independent on $\bar{\gamma}^1$. Hence we can write $\tilde{A}^1(\bar{\gamma}) = A^1(\bar{\gamma}^2)$, and similarly $\tilde{A}^2(\bar{\gamma}) = A^2(\bar{\gamma}^1)$, i.e.

$$\left\{\begin{array}{l}\hat{\gamma}^1(\bar{\gamma}) = \min\left\{\bar{\gamma}^1, A^1(\bar{\gamma}^2)\right\} \\ \hat{\gamma}^2(\bar{\gamma}) = \min\left\{\bar{\gamma}^2, A^2(\bar{\gamma}^1)\right\}\end{array}\right.$$

Step 2.2: intersection of the epigraphs
We consider the following set

$$S := \left\{\bar{\gamma} \in [0,+\infty)^2, \quad \left| \begin{array}{l}\bar{\gamma}^1 > A^1(\bar{\gamma}^2) \\ \bar{\gamma}^2 > A^2(\bar{\gamma}^1)\end{array}\right.\right\}$$

Then $\hat{\gamma}$ is locally constant on S, which implies that $A(\bar{\gamma}) := (A^1(\bar{\gamma}^2), A^2(\bar{\gamma}^1))$ is also locally constant on S. In particular A is constant on each connected component of S.
Then we have two cases.
Case A: $S \neq \emptyset$
If $\bar{\gamma} \in S$, then A is locally constant and we deduce that

$$\bar{\gamma} + [0,+\infty)^2 \subset S$$

Then S is connected and A is constant on S. Let us call $A_* := (A_*^1, A_*^2) \in [0,+\infty)^2$ this constant. Then we have

$$A = const = A_* \quad \text{on} \quad S = (A_*^1, +\infty) \times (A_*^2, +\infty)$$

By continuity of A, this equality is still true on $\bar{S} := [A_*^1, +\infty) \times [A_*^2, +\infty)$.
Case B: $S = \emptyset$
Then we set $A_* = (+\infty, +\infty)$.
We see that in both Cases A and B, we recover the desired result.
Step 2.3: proof of iii): Kružkov condition
Assume that $\hat{\gamma}$ is a Kružkov preflux. From Definition 11.1 of Kružkov prefluxes, and the characterization of continuous Kružkov functions which are partially Lipschitz (Proposition 4.20), we deduce that $\hat{\gamma}$ is Lipschitz continuous and satisfies

$$\left\{\begin{array}{l}|\partial_1 \hat{\gamma}^2| \le \partial_1 \hat{\gamma}^1 \le \partial_1 Id = 1 \\ |\partial_2 \hat{\gamma}^1| \le \partial_2 \hat{\gamma}^2 \le \partial_2 Id = 1\end{array}\right.$$

Hence the functions A^j can be chosen locally Lipschitz continuous (possibly equal to the constant $+\infty$) and satisfy

$$\left\{\begin{array}{lll}|(A^2)'(\bar{\gamma}^1)| \le 1 & \text{a.e. on} & \left\{A^2(\bar{\gamma}^1) < \bar{\gamma}^2\right\} =: S_2 \\ |(A^1)'(\bar{\gamma}^2)| \le 1 & \text{a.e. on} & \left\{A^1(\bar{\gamma}^2) < \bar{\gamma}^1\right\} =: S_1\end{array}\right.$$

We define the projection

$$P_1 S_2 := \left\{\bar{\gamma}^1 \in [0,+\infty) \quad \text{such that } \bar{\gamma} = (\bar{\gamma}^1, \bar{\gamma}^2) \in S_2 \text{ for some } \bar{\gamma}^2\right\}$$

Then each set $P_1 S_2$ is either empty with $A^2 \equiv +\infty$, or non empty and equal to $[0,+\infty)$ with A^2 which is 1-Lipschitz. We get a similar result for the function A^1.
Conversely, it is straighforward to check that when the functions A^j are both 1-Lipschitz, then the preflux $\hat{\gamma}$ is Kružkov.
Step 3: proof of iv)
Recall that for some $1:1$ junction, we have $\sigma = (\sigma^1, \sigma^2) = (+1, -1)$, and that $\hat{\gamma}$ is σ-monotone if and only if

$$\sigma^j \sigma^k \partial_j \hat{\gamma}^k \le 0 \quad \text{for all} \quad j \neq k$$

i.e.

$$\partial_1 \hat{\gamma}^2 \ge 0, \quad \partial_2 \hat{\gamma}^1 \ge 0$$

Step 4: proof of v)
Recall that a preflux for a junction $1:1$ is conservative if and only if $\hat{\gamma}^1 = \hat{\gamma}^2$. From Step 2, we deduce that

$$\hat{\gamma}^1(\bar{\gamma}) = \min\left\{\bar{\gamma}^1, A^1(\bar{\gamma}^2)\right\} = \min\left\{\bar{\gamma}^2, A^2(\bar{\gamma}^1)\right\} = \hat{\gamma}^2(\bar{\gamma})$$

Hence when $\bar{\gamma}^1 > A^1(\bar{\gamma}^2)$, we deduce that

$$A^1(\bar{\gamma}^2) = \min\left\{\bar{\gamma}^2, A^2(\bar{\gamma}^1)\right\} \quad \text{on} \quad \left\{\bar{\gamma}^1 > A^1(\bar{\gamma}^2)\right\}$$

and

$$A^1(\bar{\gamma}^2) = A^2(\bar{\gamma}^1) = A^2_* = A^1_* =: A^0_* \quad \text{on} \quad \left\{\bar{\gamma}^1 > A^1(\bar{\gamma}^2)\right\} \cap \left\{\bar{\gamma}^2 > A^2(\bar{\gamma}^1\right\} = (A^1_*, +\infty) \times (A^2_*, +\infty)$$

Therefore $A^1(\bar{\gamma}^2) = \min\left\{\bar{\gamma}^2, A^0_*\right\}$ and similarly $A^2(\bar{\gamma}^1) = \min\left\{\bar{\gamma}^1, A^0_*\right\}$. Therefore

$$\hat{\gamma}^1(\bar{\gamma}) = \hat{\gamma}^2(\bar{\gamma}) = \min\left\{\bar{\gamma}^1, \bar{\gamma}^2, A^0_*\right\}$$

Conversely, this is straightforward to check that such a function $\hat{\gamma}$ is a conservative preflux. This ends the proof of the proposition.

12.2 Characterization of prefluxes for N branches

Lemma 12.3 (Characterization of prefluxes for N branches)
Let $N \geq 1$ and $\hat{\gamma} : [0, +\infty)^N \to [0, +\infty)^N$ be a function. For a subset $I \subset \{1, \ldots, N\}$, and $\bar{\gamma} \in [0, +\infty]^N$, we set

$$\begin{array}{rlll} \pi_I : & [0, +\infty]^N & \to & [0, +\infty]^N \\ & \bar{\gamma} & \mapsto & \pi_I(\bar{\gamma}) := \displaystyle\sum_{j \in I} \bar{\gamma}^j e_j \end{array}$$

For each subset $I \subset \{1, \ldots, N\}$, and any function

$$\hat{A}_I : [0, +\infty)^N \to [0, +\infty]^N \quad \text{which satisfies} \quad \hat{A}_I = \hat{A}_I \circ \pi_{\bar{I}} = \pi_I \circ \hat{A}_I \tag{12.1}$$

and we define the epigraph

$$E_I := \left\{ z + \hat{A}_I(z) + \sum_{j \in I} (0, +\infty) e_j, \quad z \in \pi_{\bar{I}}([0, +\infty)^N) \right\} = E_I + \sum_{j \in I} (0, +\infty) e_j \tag{12.2}$$

i) (Characterization)
Then $\hat{\gamma} : [0, +\infty)^N \to [0, +\infty)^N$ is a preflux if and only if for any subset $I \subset \{1, \ldots, N\}$ with Card$(I) = 1, 2$, there exist continuous functions $\hat{A}_I$ satisfying (12.1) such that

$$\hat{\gamma}^k(\bar{\gamma}) = \min\left\{\bar{\gamma}^k, \hat{A}^k_{\{k\}}(\bar{\gamma})\right\}, \quad \text{for all} \quad k = 1, \ldots, N \tag{12.3}$$

and the following epigraph compatibility condition holds with Card$(I) = 2$

$$\bigcap_{j \in I} E_{\{j\}} = E_I \quad \text{for all subsets} \quad I \subset \{1, \ldots, N\}. \tag{12.4}$$

Then we moreover have

$$\hat{A}^j_{\{j\}} = \hat{A}^j_I \quad \text{on} \quad E_I \quad \text{for all} \quad j \in I \tag{12.5}$$

ii) (Additional properties)
Moreover, conditions (12.1), (12.4) and (12.5) still hold true for any subset $I \subset \{1, \ldots, N\}$ with $1 \leq$ Card$(I) \leq N$.

Remark 12.4 *Notice that condition (12.4) for all sets I of cardinal 2 means*

$$E_{\{j\}} \cap E_{\{k\}} = E_{\{j,k\}} \quad \text{for all} \quad j \neq k$$

i.e.

$$\left\{\bar{\gamma}^j > \hat{A}^j_{\{j\}}(\bar{\gamma})\right\} \cap \left\{\bar{\gamma}^k > \hat{A}^k_{\{k\}}(\bar{\gamma})\right\} = \left\{(\bar{\gamma}^j, \bar{\gamma}^k) > (\hat{A}^j_{\{j,k\}}, \hat{A}^k_{\{j,k\}})(\bar{\gamma})\right\} \quad \text{for all} \quad j \neq k$$

Proof of Lemma 12.3
Step 1: sufficient condition
Assume the existence of continuous function $\hat{A}_I$ for all $I \subset \{1, \dots, N\}$ with $\text{Card}(I) = 1, 2$, satisfying (12.1), (12.4). Let us show that $\hat{\gamma} : [0, +\infty)^N \to [0, +\infty)^N$ defined by (12.3) is a preflux. By definition $\hat{\gamma}$ is continuous and satisfies $0 \le \hat{\gamma} \le id_{[0,+\infty)^N}$. Let us show that $\hat{\gamma}$ is locally constant in $\bar{\gamma}^k$ on $\left\{\hat{\gamma}^k(\bar{\gamma}) < \bar{\gamma}^k\right\}$. We set

$$h^k(\bar{\gamma}) := \hat{A}^k_{\{k\}}(\bar{\gamma})$$

We do the proof for $k = 1$ (the proof is similar for any other index k). Let $\bar{\gamma}_*$ be such that

$$h^1(\bar{\gamma}_*) < \bar{\gamma}^1_*$$

By assumption, $h^1(\bar{\gamma})$ is independent on $\bar{\gamma}^1$. Let us show that each $\hat{\gamma}^j(\bar{\gamma}) = \min\left\{\bar{\gamma}^j, h^j(\bar{\gamma})\right\}$ are also independent on $\bar{\gamma}^1$ for each $j = 2, \dots, N$, locally around $\bar{\gamma}_*$. It is sufficient to do it for $j = 2$ (the cases $j > 2$ are similar).
Case A: $h^2(\bar{\gamma}_*) > \bar{\gamma}^2_*$
Then locally, we have $\hat{\gamma}^2(\bar{\gamma}) = \bar{\gamma}^2$ in a neighborhood of $\bar{\gamma}_*$, and then is locally independent on $\bar{\gamma}^1$.
Case B: $h^2(\bar{\gamma}_*) < \bar{\gamma}^2_*$
Then

$$\bar{\gamma}_* \in \left\{\bar{\gamma}^1 > h^1(\bar{\gamma})\right\} \cap \left\{\bar{\gamma}^2 > h^2(\bar{\gamma})\right\} = E_{\{1,2\}} = \left\{(\bar{\gamma}^1, \bar{\gamma}^2) > g(\bar{\gamma})\right\} \quad \text{with} \quad g(\bar{\gamma}) := (A^1_{\{1,2\}}, A^2_{\{1,2\}})(\pi_{\{1,2\}}(\bar{\gamma}))$$

Hence

$$(h^1, h^2)(\bar{\gamma}) = g(\bar{\gamma}) \quad \text{for all} \quad (\bar{\gamma}^1, \bar{\gamma}^2) > g(\bar{\gamma})$$

Therefore $(h^1, h^2)(\bar{\gamma})$ are constant in $(\bar{\gamma}^1, \bar{\gamma}^2)$ on $\left\{(\bar{\gamma}^1, \bar{\gamma}^2) > g(\bar{\gamma})\right\}$.
In particular $\hat{\gamma}^2(\bar{\gamma}) = h^2(\bar{\gamma})$ in independent on $\bar{\gamma}^1$ in a neighborhood of $\bar{\gamma}_*$.
Case C: $h^2(\bar{\gamma}_*) = \bar{\gamma}^2_*$
Recall that $h^2(\bar{\gamma})$ is independent on $\bar{\gamma}^2$. Moreover

$$\left\{\bar{\gamma}^1 > h^1(\bar{\gamma})\right\} \cap \left\{\bar{\gamma}^2 > h^2(\bar{\gamma})\right\} = E_{\{1,2\}}$$

and the argument of Case B shows that h^2 is locally independent on $\bar{\gamma}^1$, which is then also the case for $\hat{\gamma}^2(\bar{\gamma}) = \min\left\{\bar{\gamma}^2, h^2(\bar{\gamma})\right\}$.
Step 2: necessary condition
Step 2.1: first properties
Assume that $\hat{\gamma}$ is a preflux. Because $0 \le \hat{\gamma} \le id_{[0,+\infty)^N}$, we deduce the existence of functions $h^k := \hat{A}^k_{\{k\}}$: $[0, +\infty)^N \to [0, +\infty]$ such that

$$\hat{\gamma}^k(\bar{\gamma}) = \min\left\{\bar{\gamma}^k, h^k(\bar{\gamma})\right\}, \quad \text{for all} \quad k = 1, \dots, N$$

Moreover, because $\hat{\gamma}^k(\bar{\gamma})$ is independent on $\bar{\gamma}^k$ on $\left\{\bar{\gamma}^k > \hat{\gamma}^k(\bar{\gamma})\right\}$, we deduce that h^k is also independent on $\bar{\gamma}^k$. Moreover h^k is then continuous on $\left\{h^k < +\infty\right\}$, because it coincides there with $\hat{\gamma}^k$ for $\bar{\gamma}^k$ large enough. Moreover, it is easy to deduce that the continuity of $\hat{\gamma}^k$ also implies the full continuity of $h^k : [0, +\infty)^N \to [0, +\infty]$ (as shows an argument by contradiction). We then define

$$\hat{A}_{\{k\}} := h^k(\bar{\gamma}) \cdot e_k$$

which then satisfies the required properties in (12.1) for $\text{Card}(I) = 1$. In particular, we have the following epigraph relation

$$E_{\{k\}} := \left\{\bar{\gamma}^k > h^k(\bar{\gamma})\right\}$$

Step 2.2: further properties
Let us show (12.4) for $I := \{j, k\}$ with $j \ne k$. It is sufficient to do it for $(j, k) = (1, 2)$ (all the other cases are similar). We define

$$Z_{\{1,2\}} := E_{\{1\}} \cap E_{\{2\}}$$

and let us show that $Z_I = E_I$ for some continuous function $\hat{A}_I$ satisfying (12.1) and $I := \{1, 2\}$. Freezing the coordinates $\tilde{\gamma}_* := (\bar{\gamma}^3_*, \dots, \bar{\gamma}^N_*)$, we can consider the map

$$[0, +\infty)^2 \ni (\bar{\gamma}^1, \bar{\gamma}^2) \mapsto (\hat{\gamma}^1, \hat{\gamma}^2)(\bar{\gamma}^1, \bar{\gamma}^2, \tilde{\gamma}_*) \in [0, +\infty)^2$$

which is also a preflux for 2 branches, by restriction of some preflux for N branches. In particular Proposition 12.1 applies. This shows the existence of quantities $(\ell^1,\ell^2)(\bar\gamma_*)\in[0,+\infty)^2\cup\{(+\infty,+\infty)\}$ such that

$$Z=\left\{(\bar\gamma^1,\bar\gamma^2)>(\ell^1,\ell^2)(\bar\gamma^3,\dots,\bar\gamma^N)\right\}$$

and also such that

$$\text{(12.6)}\qquad \left\{\begin{array}{lll} h^2(\bar\gamma)=\ell^2(\bar\gamma^3,\dots,\bar\gamma^N) & \text{on} & \left\{\bar\gamma^1>\ell^1(\bar\gamma^3,\dots,\bar\gamma^N)\right\},\\ h^1(\bar\gamma)=\ell^1(\bar\gamma^3,\dots,\bar\gamma^N) & \text{on} & \left\{\bar\gamma^2>\ell^2(\bar\gamma^3,\dots,\bar\gamma^N)\right\}.\end{array}\right.$$

We then set

$$\left\{\begin{array}{l}\hat A^1_{\{1,2\}}(\bar\gamma):=\ell^1(\bar\gamma^3,\dots,\bar\gamma^N),\\ \hat A^2_{\{1,2\}}(\bar\gamma):=\ell^2(\bar\gamma^3,\dots,\bar\gamma^N),\\ \hat A^j_{\{1,2\}}(\bar\gamma):=0,\quad j=3,\dots,N,\\ \hat A_{\{1,2\}}:=\displaystyle\sum_{j=1,\dots,N}\hat A^j_{\{1,2\}}\cdot e_j\end{array}\right.$$

which satisfies (12.1) and

$$E_{\{1\}}\cap E_{\{2\}}=E_{\{1,2\}}$$

In particular, from (12.6), we get

$$\hat A^j_{\{j\}}=\hat A^j_{\{1,2\}}\quad\text{on}\quad E_{\{1,2\}}\quad\text{for all}\quad j=1,2$$

which shows (12.5) for $\mathrm{Card}(I)=2$.
It remains to show that $\hat A_{\{1,2\}}:[0,+\infty)^N\to[0,+\infty]^N$ is continuous, i.e. that the maps ℓ^1,ℓ^2 are continuous. Indeed, by definition, we have

$$\hat\gamma^j(\bar\gamma)=\ell^j(\bar\gamma^3,\dots,\bar\gamma^N)\quad\text{for } j=1,2\quad\text{in}\quad\left\{(\bar\gamma^1,\bar\gamma^2)>(h^1,h^2)(\bar\gamma)\right\}=\left\{(\bar\gamma^1,\bar\gamma^2)>(\ell^1,\ell^2)(\bar\gamma^3,\dots,\bar\gamma^N)\right\}$$

Because of the continuity of the preflux $\hat\gamma$, we deduce the continuity of $\hat\gamma^j(\bar\gamma):=\min\left\{\bar\gamma^j,\ell^j(\bar\gamma^3,\dots,\bar\gamma^N)\right\}$, and then the continuity of ℓ^j on $\left\{\ell^j<+\infty\right\}$ for $j=1,2$.
Finally, using the continuity of the preflux, it is easy to show that $\ell^j:[0,+\infty)^{N-2}\to[0,+\infty]$ is continuous (as shows a direct argument by contradiction). We conclude that $\hat A_I$ is continuous for $I:=\{1,2\}$ and then for $\mathrm{Card}(I)=2$.
Step 3: Additional properties and proof of ii)
We do the proof by induction on $m:=\mathrm{Card}(I)\ge 2$. We assume that

(12.7) (12.1), (12.4) and (12.5) still hold true for any subset $I\subset\{1,\dots,N\}$ with $1\le\mathrm{Card}(I)\le m$.

We already that this is true for $m=2$, and assume that $m<N$. Our goal is to prove (12.7) at the level $m+1$. To this end, consider a set $I\subset\{1,\dots,N\}$ with $\mathrm{Card}(I)=m+1$. Up to relabel the indices, we can assume that

$$I:=\{1,\dots,m+1\}$$

For $I'\subset I$ and $\bar\gamma\in[0,+\infty)^N$, we set

$$\bar\gamma^{I'}:=(\bar\gamma^j)_{j\in I'}$$

For any $j\in I$, we also set

$$I_j:=I\backslash\{j\}$$

Then we set

$$Z_I:=\bigcap_{j\in I}E_{\{j\}}=\bigcap_{j\in I}E_{I_j}$$

where, using induction hypothesis, we have

$$E_{I_j}=\left\{\bar\gamma^{I_j}>(\hat A_{I_j})^{I_j}(\bar\gamma)\right\}\quad\text{with}\quad\hat A_{I_j}=\hat A_{I_j}\circ\pi_{\overline{I_j}}$$

i.e. that $\hat A_{I_j}(\bar\gamma)$ only depends on the components $\bar\gamma^{I_j}$ of $\bar\gamma$.
Case A: $m=2$

Setting

$$\begin{cases} \ell^j(\bar\gamma^3) := \hat A^j_{\{1,2\}}(0,0,\bar\gamma^3), & j=1,2, \\ \tilde\ell^j(\bar\gamma^2) := \hat A^j_{\{1,3\}}(0,\bar\gamma^2,0), & j=1,3, \\ \bar\ell^j(\bar\gamma^1) := \hat A^j_{\{2,3\}}(\bar\gamma^1,0,0), & j=2,3, \end{cases}$$

we already know from (12.5) that

$$\begin{cases} E_{\{1,2\}} := \left\{(\bar\gamma^1,\bar\gamma^2) > (\ell^1,\ell^2)(\bar\gamma^3)\right\}, & \text{with } \ell^j = \hat A^j_{\{1,2\}} = \hat A^j_{\{j\}} \text{ on } E_{\{1,2\}}, \quad j=1,2 \\ E_{\{1,3\}} := \left\{(\bar\gamma^1,\bar\gamma^3) > (\tilde\ell^1,\tilde\ell^3)(\bar\gamma^2)\right\}, & \text{with } \tilde\ell^j = \hat A^j_{\{1,3\}} = \hat A^j_{\{j\}} \text{ on } E_{\{1,3\}}, \quad j=1,3 \\ E_{\{2,3\}} := \left\{(\bar\gamma^2,\bar\gamma^3) > (\bar\ell^2,\bar\ell^3)(\bar\gamma^1)\right\}, & \text{with } \bar\ell^j = \hat A^j_{\{2,3\}} = \hat A^j_{\{j\}} \text{ on } E_{\{2,3\}}, \quad j=2,3 \end{cases}$$

Notice that

$$\begin{cases} \hat A^1_{\{1\}}(\bar\gamma) = \ell^1(\bar\gamma^3) & \text{on} \quad E_{\{1,2\}}, \\ \hat A^1_{\{1\}}(\bar\gamma) = \tilde\ell^1(\bar\gamma^2) & \text{on} \quad E_{\{1,3\}}. \end{cases}$$

Hence, with some abuse of notation, we get the first line (and the next two lines are obtained similarly)

$$\begin{cases} \hat A^1_{\{1\}} = \ell^1 = \tilde\ell^1 = const =: \hat A^1_{\{1,2,3\}} & \text{on} \quad E_{\{1,2\}}\cap E_{\{1,3\}} = Z_{\{1,2,3\}}, \\ \hat A^2_{\{2\}} = \ell^2 = \bar\ell^2 = const =: \hat A^2_{\{1,2,3\}} & \text{on} \quad E_{\{1,2\}}\cap E_{\{2,3\}} = Z_{\{1,2,3\}}, \\ \hat A^3_{\{3\}} = \tilde\ell^3 = \bar\ell^3 = const =: \hat A^3_{\{1,2,3\}} & \text{on} \quad E_{\{1,3\}}\cap E_{\{2,3\}} = Z_{\{1,2,3\}}, \end{cases} \tag{12.8}$$

Hence

$$Z_{\{1,2,3\}} = \left\{(\bar\gamma^1,\bar\gamma^2,\bar\gamma^3) > (\hat A^1_{\{1,2,3\}},\hat A^2_{\{1,2,3\}},\hat A^3_{\{1,2,3\}})\right\} = E_{\{1,2,3\}} \tag{12.9}$$

for $E_{\{1,2,3\}}$ defined in (12.2). Therefore (12.8) and (12.9) show respectively (12.4) and (12.5), and also imply (12.1) for $I := \{1,2,3\}$.

Case B: $m > 2$

The proof is similar to Case A. For $\bar\gamma \in [0,+\infty)^N$, we define

$$\bar\gamma' := (\bar\gamma^4,\dots,\bar\gamma^N), \quad \bar\gamma'' := (\bar\gamma^{m+2},\dots,\bar\gamma^N) = \bar\gamma^{\bar I}$$

We then set

$$\begin{cases} \ell^j(\bar\gamma^3,\bar\gamma'') := \hat A^j_{I_3}(0,0,\bar\gamma^3,0_{\mathbb R^{m-2}},\bar\gamma''), & j=1,2, \\ \tilde\ell^j(\bar\gamma^2,\bar\gamma'') := \hat A^j_{I_2}(0,\bar\gamma^2,0,0_{\mathbb R^{m-2}},\bar\gamma''), & j=1,3, \\ \bar\ell^j(\bar\gamma^1,\bar\gamma'') := \hat A^j_{I_1}(\bar\gamma^1,0,0,0_{\mathbb R^{m-2}},\bar\gamma''), & j=2,3, \end{cases}$$

we already know from (12.5) that

$$\begin{cases} E_{I_3} := \left\{\bar\gamma^j > \ell^j(\bar\gamma^3,\bar\gamma''), \quad j\in I_3\right\}, & \text{with} \quad \ell^j = \hat A^j_{I_3} = \hat A^j_{\{j\}} \quad \text{on} \quad E_{I_3}, \quad j\in I_3 \\ E_{I_2} := \left\{\bar\gamma^j > \tilde\ell^j(\bar\gamma^2,\bar\gamma''), \quad j\in I_2\right\}, & \text{with} \quad \tilde\ell^j = \hat A^j_{I_2} = \hat A^j_{\{j\}} \quad \text{on} \quad E_{I_2}, \quad j\in I_2 \\ E_{I_1} := \left\{\bar\gamma^j > \bar\ell^j(\bar\gamma^1,\bar\gamma''), \quad j\in I_1\right\}, & \text{with} \quad \bar\ell^j = \hat A^j_{I_1} = \hat A^j_{\{j\}} \quad \text{on} \quad E_{I_1}, \quad j\in I_1 \end{cases}$$

Notice that

$$\begin{cases} \hat A^j_{\{j\}}(\bar\gamma) = \ell^j(\bar\gamma^3,\bar\gamma'') & \text{on} \quad E_{I_1}, \quad j\in I_3 \\ \hat A^j_{\{j\}}(\bar\gamma) = \tilde\ell^j(\bar\gamma^2,\bar\gamma'') & \text{on} \quad E_{I_2}, \quad j\in I_2. \end{cases}$$

Hence, with some abuse of notation, we get the first line (and the next two lines are obtained similarly)

$$\begin{cases} \hat A^j_{\{j\}}(\bar\gamma) = \ell^j(\bar\gamma^3,\bar\gamma'') = \tilde\ell^j(\bar\gamma^2,\bar\gamma'') =: \hat A^j_I(0_{\mathbb R^{m+1}},\bar\gamma'') & \text{on} \quad E_{I_3}\cap E_{I_2} = Z_I, \quad j\in I_3\cap I_2 \\ \hat A^j_{\{j\}}(\bar\gamma) = \ell^j(\bar\gamma^3,\bar\gamma'') = \bar\ell^2(\bar\gamma^1,\bar\gamma'') =: \hat A^j_I(0_{\mathbb R^{m+1}},\bar\gamma'') & \text{on} \quad E_{I_3}\cap E_{I_1} = Z_I, \quad j\in I_3\cap I_1 \\ \hat A^j_{\{j\}}(\bar\gamma) = \tilde\ell^j(\bar\gamma^2,\bar\gamma'') = \bar\ell^j(\bar\gamma^1,\bar\gamma'') =: \hat A^j_I(0_{\mathbb R^{m+1}},\bar\gamma'') & \text{on} \quad E_{I_2}\cap E_{I_1} = Z_I, \quad j\in I_2\cap I_1 \end{cases} \tag{12.10}$$

Extending the function $\hat A_I$ such that

$$\hat A_I := \hat A_I \circ \pi_{\bar I} = \pi_I \circ \hat A_I$$

we get

$$Z_I = E_{I_1}\cap E_{I_2}\cap E_{I_3} = \left\{\bar\gamma^I > \hat A^I_I(\bar\gamma)\right\} = E_I \tag{12.11}$$

for E_I defined in (12.2). Therefore (12.10) and (12.11) show respectively (12.4) and (12.5), and also imply (12.1).

This ends the proof of the lemma.

12.3 Necessary conditions for 1:2 conservative prefluxes

Lemma 12.5 (Necessary conditions for 1:2 conservative prefluxes)
Let us consider functions $\hat{\gamma} = (\hat{\gamma}^0, \hat{\gamma}^1, \hat{\gamma}^2) : [0,+\infty)^3 \to [0,+\infty)^3$ which are 1:2 conservative, i.e. satisfy

$$\hat{\gamma}^0 = \hat{\gamma}^1 + \hat{\gamma}^2 \tag{12.12}$$

1) (General conservative preflux)
Then the 1:2 conservative preflux $\hat{\gamma}$ satisfies the following.
i) (Existence of the constant A_* and the three functions A_0, A_1, A_2)
There exist

$$\begin{cases} \textit{a constant} & A_* := (A_*^0, A_*^1, A_*^2) & \in [0,+\infty]^3 & \\ \textit{three continuous functions} & A_j = (A_j^0, A_j^1, A_j^3) : & [0,+\infty) \to [0,+\infty]^3 & \textit{for } j = 0,1,2 \\ \textit{with} & A_j = const & \textit{on} \quad [A_*^j, +\infty) & \\ \textit{and} & A_j^j(\bar{\gamma}^j) := \min\left\{\bar{\gamma}^j, A_*^j\right\} & \textit{for all} \quad \bar{\gamma}^j \in [0,+\infty) & \end{cases}$$

satisfying the conservation relations

$$A_j^0 = A_j^1 + A_j^2 \quad \textit{on} \quad [0,+\infty)$$

with moreover one the following fives cases

$$\begin{cases} \textbf{(Case 1)} & A_* \in [0,+\infty)^3 & A_j(A_*^j) = A_*, \quad j = 0,1,2 \\ \textbf{(Case 2)} & A_* \in \{(+\infty,+\infty,+\infty)\} & \\ \textbf{(Case 3)} & A_* \in \{+\infty\} \times [0,+\infty) \times \{+\infty\}, & A_1^0(A_*^1) = +\infty = A_1^2(A_*^1) \\ \textbf{(Case 4)} & A_* \in \{+\infty\} \times \{+\infty\} \times [0,+\infty), & A_2^0(A_*^2) = +\infty = A_2^1(A_*^2) \\ \textbf{(Case 5)} & A_* \in \{+\infty\} \times [0,+\infty) \times [0,+\infty), & \begin{cases} A_2^0(A_*^2) = +\infty = A_2^1(A_*^2) \\ A_1^0(A_*^1) = +\infty = A_1^2(A_*^1) \end{cases} \end{cases} \tag{12.13}$$

Moreover, we have

$$\left.\begin{cases} A_*^0 = A_*^1 + A_*^2, \\ A_j = const = A_* \quad \textit{on} \quad [A_*^j, +\infty), \quad j = 0,1,2 \end{cases}\right| \quad \textit{in Cases 1 to 4}$$

ii) (Three characteristic curves: Γ_0 sandwiched in between Γ_1 and Γ_2)
Let us consider the L-shape set

$$L := \left\{(\bar{\gamma}^1, \bar{\gamma}^2) \in [0,+\infty)^2, \quad \bar{\gamma}^1 \le A_*^1 \quad \textit{or} \quad \bar{\gamma}^2 \le A_*^2\right\}$$

and the three continuous planar curves

$$\begin{cases} \Gamma_0 := \left\{(A_0^1, A_0^2)(\bar{\gamma}^0) \in [0,+\infty)^2, \quad \bar{\gamma}^0 \in [0, A_*^0]\right\} \\ \Gamma_2 := \left\{(\bar{\gamma}^1, \bar{\gamma}^2) \in [0,+\infty] \times [0, A_*^2], \quad \bar{\gamma}^1 = A_2^1(\bar{\gamma}^2)\right\} \\ \Gamma_1 := \left\{(\bar{\gamma}^1, \bar{\gamma}^2) \in [0, A_*^1] \times [0,+\infty], \quad \bar{\gamma}^2 = A_1^2(\bar{\gamma}^1)\right\} \end{cases}$$

which characterize completely the preflux.
Then

$$\Gamma_0 \subset L \quad \textit{and} \quad L \backslash \Gamma_0 = \Omega_1 \cup \Omega_2 \quad \textit{where} \quad \Omega_1 \cap \Omega_2 = \emptyset$$

with the sets

$$\begin{cases} \Omega_2 := \textit{union of connected components of } L\backslash\Gamma_0 \textit{ intersecting the axis } [0,+\infty) \times \{0\} \\ \Omega_1 := \textit{union of connected components of } L\backslash\Gamma_0 \textit{ intersecting the axis } \{0\} \times [0,+\infty) \end{cases}$$

Then we have the epigraphs compatibility conditions associated to the two curves Γ_1 and Γ_2

$$\begin{cases} E_2 := \left\{(\bar{\gamma}^1, \bar{\gamma}^2) \in [0,+\infty) \times [0, A_*^2], \quad \bar{\gamma}^1 > A_2^1(\bar{\gamma}^2)\right\} & \subset \Omega_2 \\ E_1 := \left\{(\bar{\gamma}^1, \bar{\gamma}^2) \in [0, A_*^1] \times [0,+\infty), \quad \bar{\gamma}^2 > A_1^2(\bar{\gamma}^1)\right\} & \subset \Omega_1 \end{cases} \tag{12.14}$$

In other words, the curve Γ_0 is sandwiched in between the two curves Γ_1 and Γ_2. These curves can be tangential but with no crossing points.

iii) (Expression of $\hat{\gamma}$)

We define the sets

$$\begin{cases} \hat{\Sigma}_0 := \left\{(\bar{\gamma}^1,\bar{\gamma}^2) > (A_0^1,A_0^2)(\bar{\gamma}^0)\right\} \\ \hat{\Sigma}_1 := \left\{(\bar{\gamma}^0,\bar{\gamma}^2) > (A_1^0,A_1^2)(\bar{\gamma}^1)\right\} \\ \hat{\Sigma}_2 := \left\{(\bar{\gamma}^0,\bar{\gamma}^1) > (A_2^0,A_2^1)(\bar{\gamma}^2)\right\} \\ \hat{\Sigma} := \hat{\Sigma}_0 \cup \hat{\Sigma}_1 \cup \hat{\Sigma}_2 \\ S := (A_*^0,+\infty)\times(A_*^1,+\infty)\times(A_*^2,+\infty) \\ \hat{\Sigma}_j^* := \Sigma_j \backslash S \\ H := \left\{\bar{\gamma}\in[0,+\infty)^3, \quad \bar{\gamma}^0 = \bar{\gamma}^1+\bar{\gamma}^2\right\} \\ H^* := H\backslash\hat{\Sigma} \\ F_j := (\Omega_j\cup\Gamma_0)\backslash E_j \quad j=1,2 \\ F_0 = F_1\cup F_2 \\ \hat{F}_0 := \left\{(p^0+\varepsilon,p^1,p^2)\in[0,+\infty)^3, \quad (p^1,p^2)\in F_0, \quad p^0=p^1+p^2, \quad \varepsilon\in(0,+\infty)\right\} \\ \hat{F}_1 := \left\{(p^0,p^1+\varepsilon,p^2)\in[0,+\infty)^3, \quad (p^1,p^2)\in F_2, \quad p^0=p^1+p^2, \quad \varepsilon\in(0,+\infty)\right\} \\ \hat{F}_2 := \left\{(p^0,p^1,p^2+\varepsilon)\in[0,+\infty)^3, \quad (p^1,p^2)\in F_1, \quad p^0=p^1+p^2, \quad \varepsilon\in(0,+\infty)\right\} \\ D_j^* := \hat{F}_j, \quad j=0,1,2 \end{cases}$$

where the indices are mixed for $\hat{F}_j$ and F_j. Then we have a partition

$$[0,+\infty)^3 = D_0^*\cup D_1^*\cup D_2^*\cup H^*\cup\hat{\Sigma}_0^*\cup\hat{\Sigma}_1^*\cup\hat{\Sigma}_2^*\cup S.$$

We also define the injection map

$$\begin{array}{rccc} \iota: & [0,+\infty)^2 & \to & [0,+\infty)^3 \\ & (\bar{\gamma}^1,\bar{\gamma}^2) & \to & (\bar{\gamma}^1+\bar{\gamma}^2,\bar{\gamma}^1,\bar{\gamma}^2) \end{array}$$

and

$$\tilde{F}_j := \iota(F_j), \quad \tilde{\Gamma}_j := \iota(\Gamma_j), \quad j=0,1,2$$

and the preflux is given by

(12.15)

$$\hat{\gamma}(\bar{\gamma}) = \begin{cases} \tilde{\gamma}_0(\bar{\gamma}) := (\bar{\gamma}^1+\bar{\gamma}^2,\bar{\gamma}^1,\bar{\gamma}^2) & \in\tilde{F}_0 & on & D_0^* & = \left\{\hat{\gamma}^0(\bar{\gamma})<\bar{\gamma}^0\right\}\cap\left\{\hat{\gamma}^1(\bar{\gamma})=\bar{\gamma}^1\right\}\cap\left\{\hat{\gamma}^2(\bar{\gamma})=\bar{\gamma}^2\right\} \\ \tilde{\gamma}_1(\bar{\gamma}) := (\bar{\gamma}^0,\bar{\gamma}^0-\bar{\gamma}^2,\bar{\gamma}^2) & \in\tilde{F}_2 & on & D_1^* & = \left\{\hat{\gamma}^0(\bar{\gamma})=\bar{\gamma}^0\right\}\cap\left\{\hat{\gamma}^1(\bar{\gamma})<\bar{\gamma}^1\right\}\cap\left\{\hat{\gamma}^2(\bar{\gamma})=\bar{\gamma}^2\right\} \\ \tilde{\gamma}_2(\bar{\gamma}) := (\bar{\gamma}^0,\bar{\gamma}^1,\bar{\gamma}^0-\bar{\gamma}^1) & \in\tilde{F}_1 & on & D_2^* & = \left\{\hat{\gamma}^0(\bar{\gamma})=\bar{\gamma}^0\right\}\cap\left\{\hat{\gamma}^1(\bar{\gamma})=\bar{\gamma}^1\right\}\cap\left\{\hat{\gamma}^2(\bar{\gamma})<\bar{\gamma}^2\right\} \\ \bar{\gamma} & \in\tilde{F}_0 & on & H^* & = \left\{\hat{\gamma}^0(\bar{\gamma})=\bar{\gamma}^0\right\}\cap\left\{\hat{\gamma}^1(\bar{\gamma})=\bar{\gamma}^1\right\}\cap\left\{\hat{\gamma}^2(\bar{\gamma})=\bar{\gamma}^2\right\} \\ A_0(\bar{\gamma}^0) & \in\tilde{\Gamma}_0 & on & \hat{\Sigma}_0^* & = \left\{\hat{\gamma}^0(\bar{\gamma})=\bar{\gamma}^0\right\}\cap\left\{\hat{\gamma}^1(\bar{\gamma})<\bar{\gamma}^1\right\}\cap\left\{\hat{\gamma}^2(\bar{\gamma})<\bar{\gamma}^2\right\} \\ A_1(\bar{\gamma}^1) & \in\tilde{\Gamma}_1 & on & \hat{\Sigma}_1^* & = \left\{\hat{\gamma}^0(\bar{\gamma})<\bar{\gamma}^0\right\}\cap\left\{\hat{\gamma}^1(\bar{\gamma})=\bar{\gamma}^1\right\}\cap\left\{\hat{\gamma}^2(\bar{\gamma})<\bar{\gamma}^2\right\} \\ A_2(\bar{\gamma}^2) & \in\tilde{\Gamma}_2 & on & \hat{\Sigma}_2^* & = \left\{\hat{\gamma}^0(\bar{\gamma})<\bar{\gamma}^0\right\}\cap\left\{\hat{\gamma}^1(\bar{\gamma})<\bar{\gamma}^1\right\}\cap\left\{\hat{\gamma}^2(\bar{\gamma})=\bar{\gamma}^2\right\} \\ A_* & & on & S & = \left\{\hat{\gamma}^0(\bar{\gamma})<\bar{\gamma}^0\right\}\cap\left\{\hat{\gamma}^1(\bar{\gamma})<\bar{\gamma}^1\right\}\cap\left\{\hat{\gamma}^2(\bar{\gamma})<\bar{\gamma}^2\right\} \end{cases}$$

where the indices are mixed for $\tilde{F}_j$ and D_j^*.

iv) (Further properties)

Moreover we have

$$\begin{cases} \hat{\Sigma}_0 = \left\{\hat{\gamma}^1(\bar{\gamma})<\bar{\gamma}^1\right\}\cap\left\{\hat{\gamma}^2(\bar{\gamma})<\bar{\gamma}^2\right\} \\ \hat{\Sigma}_1 = \left\{\hat{\gamma}^0(\bar{\gamma})<\bar{\gamma}^0\right\}\cap\left\{\hat{\gamma}^2(\bar{\gamma})<\bar{\gamma}^2\right\} \\ \hat{\Sigma}_2 = \left\{\hat{\gamma}^0(\bar{\gamma})<\bar{\gamma}^0\right\}\cap\left\{\hat{\gamma}^1(\bar{\gamma})<\bar{\gamma}^1\right\} \\ S = \hat{\Sigma}_0\cap\hat{\Sigma}_1\cap\hat{\Sigma}_2 \\ D_0 := \left\{\hat{\gamma}^1(\bar{\gamma})=\bar{\gamma}^1\right\}\cap\left\{\hat{\gamma}^2(\bar{\gamma})=\bar{\gamma}^2\right\} \\ D_1 := \left\{\hat{\gamma}^0(\bar{\gamma})=\bar{\gamma}^0\right\}\cap\left\{\hat{\gamma}^2(\bar{\gamma})=\bar{\gamma}^2\right\} \\ D_2 := \left\{\hat{\gamma}^0(\bar{\gamma})=\bar{\gamma}^0\right\}\cap\left\{\hat{\gamma}^1(\bar{\gamma})=\bar{\gamma}^1\right\} \\ D_j = D_j^*\cup H^*, \quad j=0,1,2 \\ D_0^* = \left\{\bar{\gamma}^0>\bar{\gamma}^1+\bar{\gamma}^2\right\}\backslash\hat{\Sigma} \\ H^* := H\backslash\hat{\Sigma} = D_0\cap D_1\cap D_2 = \tilde{F}_0 = \left\{\hat{\gamma}=id_{[0,+\infty)^3}\right\} \\ [0,+\infty)^3\backslash\hat{\Sigma} = D_0\cup D_1\cup D_2 = D_0^*\cup D_1^*\cup D_2^*\cup H^* \end{cases}$$

and

$$\begin{cases} \Gamma_j \subset F_j, \quad j = 1,2 \\ F_1 \cap F_2 = \Gamma_0 \\ \tilde{\Gamma}_j := \left\{ A_j(\bar{\gamma}^j), \quad \bar{\gamma}^j \in [0, A_*^j] \right\}, \quad j = 0,1,2 \\ \tilde{\Gamma}_j \subset \hat{\Sigma}_j \quad \textit{for} \quad j = 0,1,2 \end{cases}$$

and

$$\hat{\gamma}([0,+\infty)^3) = H^*$$

with

$$\hat{\gamma}(\hat{\Sigma}_j^*) = \tilde{\Gamma}_j, \quad j = 0,1,2 \tag{12.16}$$

and also for $A'_* := (A_*^1, A_*^2)$

$$\begin{array}{ccc} F_0 & = & F_0 \\ \cup & & \cup \\ F_1 & & F_2 \\ \cup & & \cup \\ \Gamma_1, \Gamma_0 & & \Gamma_0, \Gamma_2 \\ \cup & & \cup \\ \{A'_*\} & = & \{A'_*\} \end{array}$$

and

$$\partial F_j \subset \Gamma_0 \cup \Gamma_j \cup [0,+\infty]e_{\bar{j}}, \quad j = 1,2 \quad \textit{with} \quad \bar{j} \in \{1,2\} \setminus \{j\}$$

2) (Kružkov conservative preflux)
If $\hat{\gamma}$ is Kružkov conservative then in the previous description, the functions A_0, A_1, A_2 are Lipschitz continuous and satisfy

$$\begin{cases} |(A_0^1)'| + |(A_0^2)'| \le 1, & (A_0^1)' \ge 0, & (A_0^2)' \ge 0 & a.e. \ on \quad [0, A_*^0] \\ |(A_1^0)'| + |(A_1^2)'| \le 1, & (A_1^0)' \ge 0, & -(A_1^2)' \ge 0 & a.e. \ on \quad [0, A_*^1] \\ |(A_2^0)'| + |(A_2^1)'| \le 1 & (A_2^0)' \ge 0, & -(A_2^1)' \ge 0 & a.e. \ on \quad [0, A_*^2] \end{cases} \tag{12.17}$$

Remark 12.6 *In Lemma 12.5, notice that a conservative preflux $\hat{\gamma}$ is fully charcaterized by the three curves $\Gamma_0, \Gamma_1, \Gamma_2 \subset [0,+\infty)^2$ which merge at the point $(A_*^1, A_*^2) \in [0,+\infty)^2$. The curve Γ_0 is sandwiched in between Γ_1 and Γ_2, and $(0,0) \in \Gamma_0$. Moreover Γ_1 is a graph in direction e_2 (parametrized by coordinate $\bar{\gamma}^1$), Γ_2 is a graph in direction e_1 (parametrized by coordinate $\bar{\gamma}^2$), and Γ_0 is a graph in direction $e_2 - e_1$.*

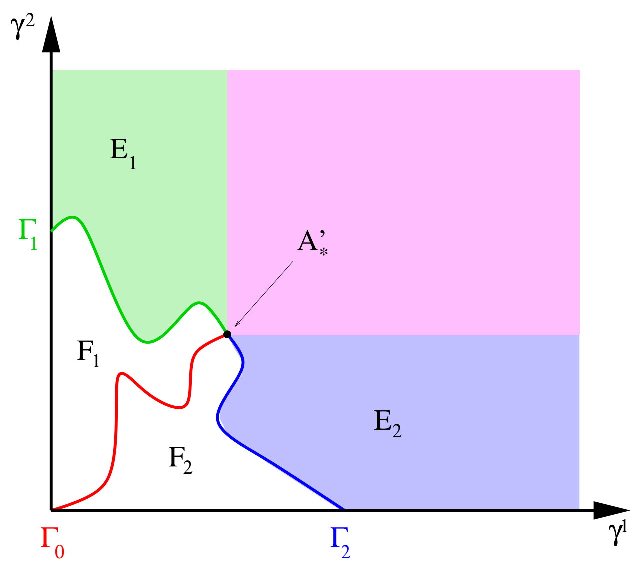

Figure 7: The curve Γ_0 sandwiched in between Γ_1 and Γ_2, merging at A'_*

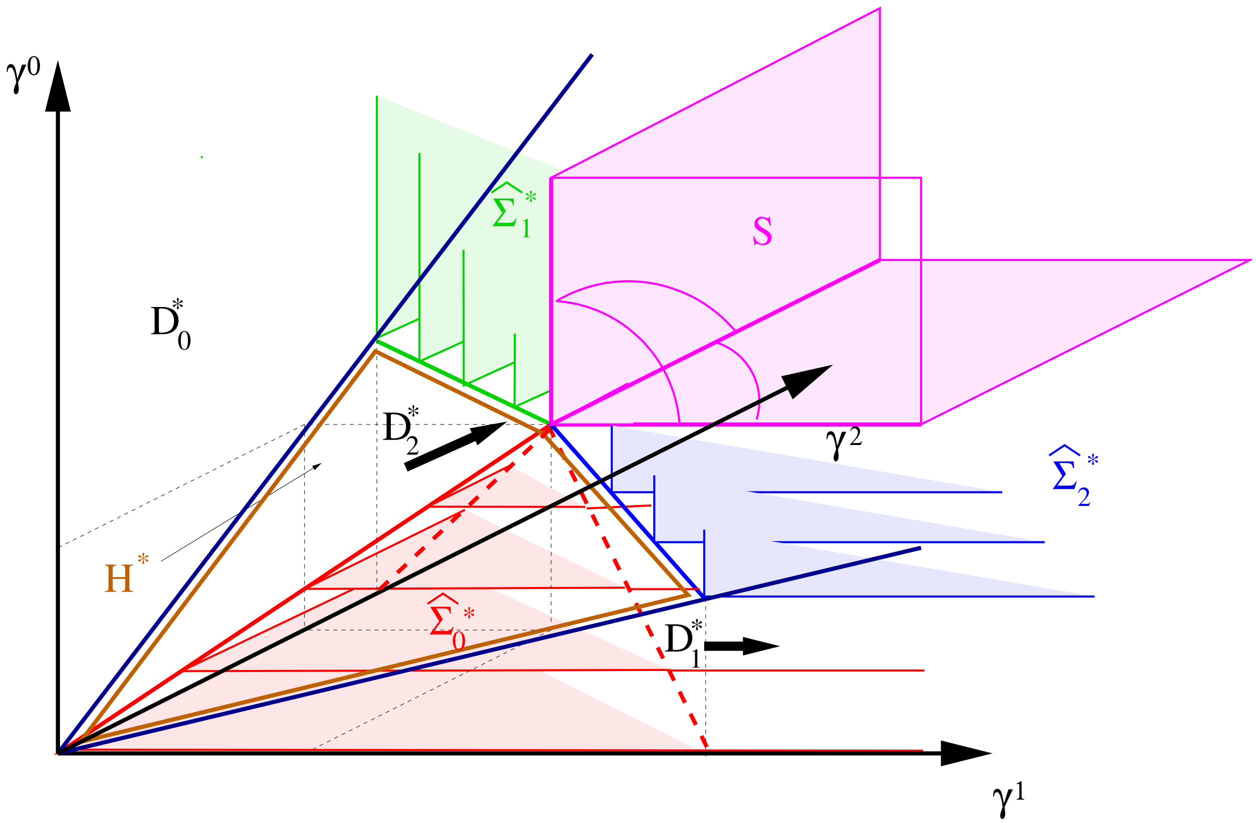

Figure 8: The partition of $[0,+\infty)^3$

Proof of Lemma 12.5

The result and its proof are not particularly easy to guess and this is the main difficulty of the proof. Nevertheless, the core of the proof consists in an elementary but lengthful analysis using topology and geometry in dimensions 2 and 3. Let (e_0, e_1, e_2) be an orthonormal basis of $\mathbb{R}^3$.

Step 1: preliminary remarks

Step 1.1: freezing $\bar\gamma_0^0$

We first freeze the value $\bar\gamma^0 := \bar\gamma_0^0$ and notice that $(\bar\gamma^1,\bar\gamma^2) \mapsto (\hat\gamma^1,\hat\gamma^2)(\bar\gamma_0^0,\bar\gamma^1,\bar\gamma^2)$ is a preflux for $N=2$ branches. Then from ii) of Proposition 12.1, there exists two continuous functions $B^1, B^2 : [0,+\infty) \to [0,+\infty]$ and a constant $B_* := (B_*^1, B_*^2) \in [0,+\infty)^2 \cup \{(+\infty,+\infty)\}$ such that

$$(12.18) \qquad \left\{ \begin{array}{l} \hat\gamma^1(\bar\gamma_0^0,\bar\gamma^1,\bar\gamma^2) = \min\left\{\bar\gamma^1, B^1(\bar\gamma^2)\right\}, \\ \hat\gamma^2(\bar\gamma_0^0,\bar\gamma^1,\bar\gamma^2) = \min\left\{\bar\gamma^2, B^2(\bar\gamma^1)\right\}, \\ \\ B^1 = const = B_*^1 \quad \text{on} \quad [B_*^2,+\infty) \\ B^2 = const = B_*^2 \quad \text{on} \quad [B_*^1,+\infty) \\ \\ \left\{\bar\gamma^1 > B^1(\bar\gamma^2)\right\} \cap \left\{\bar\gamma^2 > B^2(\bar\gamma^1)\right\} = \left\{(\bar\gamma^1,\bar\gamma^2) \in (B_*^1,+\infty)\times(B_*^2,+\infty)\right\} \end{array}\right.$$

We know that

$$\bar\gamma_0^0 \ge \hat\gamma^0(\bar\gamma_0^0, R, R) = \min\left\{R, B^1(R)\right\} + \min\left\{R, B^2(R)\right\}$$

as $R \to +\infty$, and then both B^1 and B^2 are bounded close to $+\infty$. This shows that the set defined in the last line of (12.18) is non empty, which implies

$$B_* \in [0,+\infty)^2$$

with moreover $\bar\gamma_0^0 \ge \hat\gamma^0(\bar\gamma_0^0, B_*^1, B_*^2) = B_*^1 + B_*^2$. Moreover $\hat\gamma$ is locally constant in $\bar\gamma_0^0$ on $\left\{\hat\gamma^0(\bar\gamma_0^0,\cdot,\cdot) < \bar\gamma_0^0\right\}$. Recall that $B_* = B_*(\bar\gamma_0^0)$ and the function $B(\bar\gamma^1,\bar\gamma^2) := (B^1(\bar\gamma^2), B^2(\bar\gamma^1)) = B_{\bar\gamma_0^0}(\bar\gamma^1,\bar\gamma^2)$ also depends on $\bar\gamma_0^0$

in general. Because the preflux is conservative, we set

$$A_0(\bar\gamma_0^0) := (B_*^1 + B_*^2, B_*^1, B_*^2)(\bar\gamma_0^0)$$

and then for $\bar\gamma^0 = \bar\gamma_0^0$ and $\bar\gamma = (\bar\gamma^0, \bar\gamma^1, \bar\gamma^2)$, we get

$$(12.19)\qquad \left\{\begin{array}{ll} \hat\gamma(\bar\gamma) = A_0(\bar\gamma^0) & \text{on}\quad \Sigma_0(\bar\gamma^0) := A_0(\bar\gamma^0) + \{0\}\times(0,+\infty)^2\\ \left\{\hat\gamma^1(\bar\gamma) < \bar\gamma^1\right\} \cap \left\{\hat\gamma^2(\bar\gamma) < \bar\gamma^2\right\} = & \left\{\bar\gamma \in \Sigma_0(\bar\gamma^0)\right\} =: \hat\Sigma_0\\ & \\ A_0 : [0,+\infty) \to [0,+\infty)^3 & \text{continuous},\\ A_0 \text{ locally constant} & \text{on } \left\{A_0^0 < id_{[0,+\infty)}\right\},\\ A_0^0 = A_0^1 + A_0^2 & \text{on}\quad [0,+\infty) \end{array}\right.$$

Step 1.2: freezing $\bar\gamma_0^1$
Similarly we freeze $\bar\gamma_0^1$, and get from ii) of Proposition 12.1, there exists two continuous functions C^1, C^2 : $[0,+\infty) \to [0,+\infty]$ and a constant $C_* := (C_*^0, C_*^2) \in [0,+\infty)^2 \cup \{(+\infty,+\infty)\}$ such that

$$\left\{\begin{array}{l} \hat\gamma^0(\bar\gamma^0, \bar\gamma_0^1, \bar\gamma^2) = \min\left\{\bar\gamma^0, C^0(\bar\gamma^2)\right\},\\ \hat\gamma^2(\bar\gamma^0, \bar\gamma_0^1, \bar\gamma^2) = \min\left\{\bar\gamma^2, C^2(\bar\gamma^1)\right\},\\ \\ C^0 = const = C_*^0 \quad \text{on} \quad [C_*^2, +\infty)\\ C^2 = const = C_*^2 \quad \text{on} \quad [C_*^0, +\infty)\\ \\ \left\{\bar\gamma^0 > C^0(\bar\gamma^2)\right\} \cap \left\{\bar\gamma^2 > C^2(\bar\gamma^0)\right\} = \left\{(\bar\gamma^0, \bar\gamma^2) \in (C_*^0, +\infty) \times (C_*^2, +\infty)\right\} \end{array}\right.$$

We then set

$$A_1(\bar\gamma_0^1) := (C_*^0, C_*^0 - C_*^2, C_*^2)(\bar\gamma_0^1)$$

and then for $\bar\gamma^1 = \bar\gamma_0^1$ and $\bar\gamma = (\bar\gamma^0, \bar\gamma^1, \bar\gamma^2)$ we have
(12.20)

$$\left\{\begin{array}{ll} \hat\gamma(\bar\gamma) = A_1(\bar\gamma^1) & \text{on}\quad \Sigma_1(\bar\gamma^0) := A_1(\bar\gamma^1) + (0,+\infty)\times\{0\}\times(0,+\infty)\\ \left\{\hat\gamma^0(\bar\gamma) < \bar\gamma^0\right\} \cap \left\{\hat\gamma^2(\bar\gamma) < \bar\gamma^2\right\} = & \left\{\bar\gamma \in \Sigma_1(\bar\gamma^1)\right\} =: \hat\Sigma_1\\ & \\ A_1 : [0,+\infty) \to [0,+\infty] \times [0,+\infty) \times [0,+\infty] & \text{continuous},\\ A_1 \text{ locally constant} & \text{on } \left\{A_1^1 < id_{[0,+\infty)}\right\},\\ A_1^0 = A_1^1 + A_1^2 & \text{on}\quad [0,+\infty) \end{array}\right.$$

Step 1.3: freezing $\bar\gamma_0^2$
Similarly, for $\bar\gamma^2 = \bar\gamma_0^2$ and $\bar\gamma = (\bar\gamma^0, \bar\gamma^1, \bar\gamma^2)$, we also have

$$(12.21)\qquad \left\{\begin{array}{ll} \hat\gamma(\bar\gamma) = A_2(\bar\gamma^2) & \text{on}\quad \Sigma_2(\bar\gamma^2) := A_2(\bar\gamma^2) + (0,+\infty)^2\times\{0\}\\ \left\{\hat\gamma^0(\bar\gamma) < \bar\gamma^0\right\} \cap \left\{\hat\gamma^1(\bar\gamma) < \bar\gamma^1\right\} = & \left\{\bar\gamma \in \Sigma_2(\bar\gamma^2)\right\} =: \hat\Sigma_2\\ & \\ A_2 : [0,+\infty) \to [0,+\infty]^2 \times [0,+\infty) & \text{continuous},\\ A_2 \text{ locally constant} & \text{on } \left\{A_2^2 < id_{[0,+\infty)}\right\},\\ A_2^0 = A_2^1 + A_2^2 & \text{on}\quad [0,+\infty) \end{array}\right.$$

Step 1.4: the set S
Let us consider the set

$$S := \left\{\hat\gamma^0(\bar\gamma) < \bar\gamma^0\right\} \cap \left\{\hat\gamma^1(\bar\gamma) < \bar\gamma^1\right\} \cap \left\{\hat\gamma^2(\bar\gamma) < \bar\gamma^2\right\}$$

Then S is an open set, and recall $\hat\gamma$ is locally constant on S. This implies that $\bar\gamma + [0,+\infty)^3 \subset S$ for all $\bar\gamma \in S$. Therefore S is connected and $\hat\gamma$ is constant on S.
Step 2: case $S \neq \emptyset$
We will see that this corresponds to Case 1 in (12.13).
Step 2.1: general structure
From (12.19)-(12.20)-(12.21), we deduce that $S = \hat\Sigma_0 \cap \hat\Sigma_1 \cap \hat\Sigma_2$ and $\hat\gamma(\bar\gamma) = A_0(\bar\gamma^0) = A_1(\bar\gamma^1) = A_2(\bar\gamma^2)$ for all $\bar\gamma \in S$. Therefore

$$[0,+\infty)^3 \ni A_0(+\infty) = A_1(+\infty) = A_2(+\infty) =: A_* = (A_*^0, A_*^1, A_*^2)$$

with $A^0_* = A^1_* + A^2_*$ and for $j = 0, 1, 2$

$$A_j = const = A_* \quad \text{on} \quad (A^j_*, +\infty).$$

Hence

$$\hat{\gamma} = A_* \quad \text{on} \quad S = (A^0_*, +\infty) \times (A^1_*, +\infty) \times (A^2_*, +\infty)$$

Moreover the fact that $0 \le \hat{\gamma} \le Id_{[0,+\infty)^3}$ implies also $0 \le A^j_j \le Id_{[0,+\infty)}$ and then

$$A^j_j(\bar{\gamma}^j) = \min\left\{\bar{\gamma}^j, A^j_*\right\}$$

In particular, we get

$$\begin{cases} \bar{\gamma}^0 & = A^1_0(\bar{\gamma}^0) & + A^2_0(\bar{\gamma}^0) & \text{for} \quad \bar{\gamma}^0 \in [0, A^0_*] \\ A^0_1(\bar{\gamma}^1) & = \bar{\gamma}^1 & + A^2_1(\bar{\gamma}^1) & \text{for} \quad \bar{\gamma}^1 \in [0, A^1_*] \\ A^0_2(\bar{\gamma}^2) & = A^1_2(\bar{\gamma}^2) & + \bar{\gamma}^2 & \text{for} \quad \bar{\gamma}^2 \in [0, A^2_*] \end{cases}$$

We also have

$$(12.22) \qquad \hat{\gamma}(\bar{\gamma}) = \begin{cases} A_0(\bar{\gamma}^0) & \text{on} \quad \left\{\hat{\gamma}^1(\bar{\gamma}) < \bar{\gamma}^1\right\} \cap \left\{\hat{\gamma}^2(\bar{\gamma}) < \bar{\gamma}^2\right\} = \left\{(\bar{\gamma}^1, \bar{\gamma}^2) > (A^1_0, A^2_0)(\bar{\gamma}^0)\right\} = \hat{\Sigma}_0 \\ A_1(\bar{\gamma}^1) & \text{on} \quad \left\{\hat{\gamma}^0(\bar{\gamma}) < \bar{\gamma}^0\right\} \cap \left\{\hat{\gamma}^2(\bar{\gamma}) < \bar{\gamma}^2\right\} = \left\{(\bar{\gamma}^0, \bar{\gamma}^2) > (A^0_1, A^2_1)(\bar{\gamma}^1)\right\} = \hat{\Sigma}_1 \\ A_2(\bar{\gamma}^2) & \text{on} \quad \left\{\hat{\gamma}^0(\bar{\gamma}) < \bar{\gamma}^0\right\} \cap \left\{\hat{\gamma}^1(\bar{\gamma}) < \bar{\gamma}^1\right\} = \left\{(\bar{\gamma}^0, \bar{\gamma}^1) > (A^0_2, A^1_2)(\bar{\gamma}^2)\right\} = \hat{\Sigma}_2 \end{cases}$$

and set

$$\hat{\Sigma} := \bigcup_{j=0,1,2} \hat{\Sigma}_j$$

and

$$\hat{\gamma}(\bar{\gamma}) = \begin{cases} \tilde{\gamma}_0(\bar{\gamma}) := (\bar{\gamma}^1 + \bar{\gamma}^2, \bar{\gamma}^1, \bar{\gamma}^2) & \text{on} \quad \left\{\hat{\gamma}^1(\bar{\gamma}) = \bar{\gamma}^1\right\} \cap \left\{\hat{\gamma}^2(\bar{\gamma}) = \bar{\gamma}^2\right\} =: D_0 \\ \tilde{\gamma}_1(\bar{\gamma}) := (\bar{\gamma}^0, \bar{\gamma}^0 - \bar{\gamma}^2, \bar{\gamma}^2) & \text{on} \quad \left\{\hat{\gamma}^0(\bar{\gamma}) = \bar{\gamma}^0\right\} \cap \left\{\hat{\gamma}^2(\bar{\gamma}) = \bar{\gamma}^2\right\} =: D_1 \\ \tilde{\gamma}_2(\bar{\gamma}) := (\bar{\gamma}^0, \bar{\gamma}^1, \bar{\gamma}^0 - \bar{\gamma}^1) & \text{on} \quad \left\{\hat{\gamma}^0(\bar{\gamma}) = \bar{\gamma}^0\right\} \cap \left\{\hat{\gamma}^1(\bar{\gamma}) = \bar{\gamma}^1\right\} =: D_2 \end{cases}$$

with

$$[0, +\infty)^3 \backslash \hat{\Sigma} = D_0 \cup D_1 \cup D_2$$

Let us define the hyperplane

$$H := \left\{\bar{\gamma} \in [0, +\infty)^3, \quad \bar{\gamma}^0 = \bar{\gamma}^1 + \bar{\gamma}^2\right\}$$

and

$$D^*_j := D_j \backslash H \quad \text{and} \quad H^* := H \backslash \hat{\Sigma}$$

which implies $D_j \subset D^*_j \cup H^*$. Notice that

$$(12.23) \qquad \tilde{\gamma}_j(\bar{\gamma}) = \tilde{\gamma}_k(\bar{\gamma}) \quad \text{for some } j \neq k, \text{ if and only if} \quad \bar{\gamma}^0 = \bar{\gamma}^1 + \bar{\gamma}^2$$

which implies that $D^*_j \cap D^*_k = \emptyset$ for all $j \neq k$. We deduce the partition

$$[0, +\infty)^3 \backslash \hat{\Sigma} = D^*_0 \cup D^*_1 \cup D^*_2 \cup H^*$$

with

$$(12.24) \qquad \hat{\gamma}(\bar{\gamma}) = \begin{cases} \tilde{\gamma}_j(\bar{\gamma}) & \text{on} \quad D^*_j, \quad j = 0, 1, 2 \\ \bar{\gamma} & \text{on} \quad H^* \end{cases}$$

Therefore the definition of D_j implies that $H^* \subset D_j$, and then we get the partition

$$D_j = D^*_j \cup H^*, \quad j = 0, 1, 2$$

Moreover, using $\hat{\gamma} \le Id_{[0,+\infty)^3}$, we get

$$\begin{cases} D^*_0 & \subset \left\{\bar{\gamma}^0 > \bar{\gamma}^1 + \bar{\gamma}^2\right\} \\ D^*_1, D^*_2 & \subset \left\{\bar{\gamma}^0 < \bar{\gamma}^1 + \bar{\gamma}^2\right\} \end{cases}$$

and then

$$(12.25) \qquad D^*_0 = \left\{\bar{\gamma}^0 > \bar{\gamma}^1 + \bar{\gamma}^2\right\} \backslash \hat{\Sigma}$$

Step 2.2: a cleaning claim
We claim that if $p,q\in[0,+\infty)^3$ satisfy

$$\text{(12.26)}\qquad \left\{\begin{array}{l} p^0=p^1+p^2\\ q^0=q^1+q^2\\ p\in\overline{\{\hat\gamma^1(\bar\gamma)<\bar\gamma^1,\ \hat\gamma^2(\bar\gamma)<\bar\gamma^2\}}\\ q\in\overline{\{\hat\gamma^0(\bar\gamma)<\bar\gamma^0\}}\\ p^0>q^0\\ p^j<q^j\quad\text{for some index } j\in\{1,2\}\end{array}\right.$$

then

$$\text{(12.27)}\qquad \hat\gamma=const=\hat\gamma(p)=\hat\gamma(q)\quad\text{on}\quad r+[0,+\infty)^3\quad\text{with}\quad r^k:=\max\left\{p^k,q^k\right\},\quad k=0,1,2$$

Let us prove the claim for $j=2$ (the proof is similar for $j=1$). Then $p^0>q^0$, $p^1>q^1$, $p^2<q^2$ and $r=(p^0,p^1,q^2)$. We can approximate (p,q) by $(p_\varepsilon,q_\varepsilon)$ with the similar strict inequalities

$$\text{(12.28)}\qquad p^0_\varepsilon>q^0_\varepsilon,\quad p^1_\varepsilon>q^1_\varepsilon,\quad p^2_\varepsilon<q^2_\varepsilon\quad\text{and}\quad r_\varepsilon:=(p^0_\varepsilon,p^1_\varepsilon,q^2_\varepsilon)$$

and moreover
$$\left\{\begin{array}{l} p_\varepsilon\in\left\{\hat\gamma^1(\bar\gamma)<\bar\gamma^1,\ \hat\gamma^2(\bar\gamma)<\bar\gamma^2\right\}\\ q_\varepsilon\in\left\{\hat\gamma^0(\bar\gamma)<\bar\gamma^0\right\}\end{array}\right.$$

Then
$$\left\{\begin{array}{lll} \hat\gamma=const & \text{on} & p_\varepsilon+\{0\}\times[0,+\infty)^2=:E_{p_\varepsilon}\\ \hat\gamma=const & \text{on} & q_\varepsilon+[0,+\infty)\times\{0\}^2=:E_{q_\varepsilon}\end{array}\right.$$

and inequalities (12.28) imply that

$$r_\varepsilon\in E_{p_\varepsilon}\cap E_{q_\varepsilon}\subset\left\{\hat\gamma^0(\bar\gamma)<\bar\gamma^0,\ \hat\gamma^1(\bar\gamma)<\bar\gamma^1,\ \hat\gamma^2(\bar\gamma)<\bar\gamma^2\right\}=S$$

which implies $\hat\gamma=const=\hat\gamma(p_\varepsilon)=\hat\gamma(q_\varepsilon)$ on $r_\varepsilon+[0,+\infty)^3$. We then get the result of the claim, passing to the limit $\varepsilon\to0$ and using the continuity of $\hat\gamma$.
Step 2.3: curves $\Gamma_0,\Gamma_1,\Gamma_2$
Let us consider the change of variables

$$\begin{array}{llll}\Phi: & [0,+\infty)^2 & \to & [0,+\infty)\times\mathbb{R}\\ & (\bar\gamma^1,\bar\gamma^2) & \mapsto & \Phi(\bar\gamma^1,\bar\gamma^2):=(\bar\gamma^1+\bar\gamma^2,\bar\gamma^2-\bar\gamma^1)=(X,Y)\end{array}$$

Then the map
$$\begin{array}{llll}\Gamma: & [0,A^0_*] & \to & [0,+\infty)^2\\ & \bar\gamma^0 & \mapsto & (A^1_0,A^2_0)(\bar\gamma^0)\end{array}$$

satisfies $\underline{\Gamma}(\bar\gamma^0):=\Phi\circ\Gamma(\bar\gamma^0)=(\bar\gamma^0,\underline{\Gamma}_Y(\bar\gamma^0))$ with $|\underline{\Gamma}_Y(\bar\gamma^0)|\le\bar\gamma^0$ and $\underline{\Gamma}_Y(\bar\gamma^0):=(A^2_0-A^1_0)(\bar\gamma^0)$, which shows that the image of $[0,A^0_*]$ by the map $\underline{\Gamma}$ is the graph of a continuous map with value in

$$\underline{\Delta}:=\left\{(X,Y)\in[0,A^0_*]\times\mathbb{R},\quad |Y|\le X\right\}$$

with $\underline{\Gamma}(0)=(0,0)$ and $\underline{\Gamma}(A^0_*)=(A^0_*,Y_0)$ where $Y_0:=A^2_*-A^1_*\in[-A^0_*,A^0_*]$. Because $S\neq\emptyset$, we first deduce that $A_*\in[0,+\infty)^3$. We set $\Gamma_0=\Gamma([0,A^0_*])$ and $\underline{\Gamma}_0:=\Phi(\Gamma_0)$. We now consider the two closed connected components
$$\left\{\begin{array}{l}\underline{\Delta}_1:=\{(X,Y)\in\underline{\Delta},\quad Y\ge\underline{\Gamma}_Y(X)\},\\ \underline{\Delta}_2:=\{(X,Y)\in\underline{\Delta},\quad Y\le\underline{\Gamma}_Y(X)\},\end{array}\right.$$

Therefore for $\alpha=1,2$, we set

$$\Delta_\alpha:=\Phi^{-1}(\underline{\Delta}_\alpha)\quad\text{and}\quad\Delta:=\Phi^{-1}(\underline{\Delta})=\left\{(\bar\gamma^1,\bar\gamma^2)\in[0,+\infty)^2,\quad\bar\gamma^1+\bar\gamma^2\le A^0_*\right\}$$

and get that the curve Γ_0 is a common boundary in Δ of the two closed connected sets Δ_α for $\alpha=1,2$, with

$$\Gamma_0=\Delta_2\cap\Delta_1\quad\text{and}\quad\Delta=\Delta_2\cup\Delta_1$$

Moreover Δ_1 is connected to

$$L_1 := \left\{(\bar\gamma^1,\bar\gamma^2)\in[0,A^1_*]\times\mathbb{R},\quad \bar\gamma^1+\bar\gamma^2\geq A^0_*\right\}$$

and Δ_2 is connected to

$$L_2 := \left\{(\bar\gamma^1,\bar\gamma^2)\in\mathbb{R}\times[0,A^2_*],\quad \bar\gamma^1+\bar\gamma^2\geq A^0_*\right\}$$

Let

$$L := \left\{(\bar\gamma^1,\bar\gamma^2)\in[0,+\infty)^2,\quad \bar\gamma^1\leq A^1_*\quad\text{or}\quad \bar\gamma^2\leq A^2_*\right\}$$

Then each connected component of $L\backslash\Gamma_0$ is in only one of the following three cases: either it is equal to one of the two connected components of $L\backslash\Delta$, or it intersects $\Delta_2\backslash\Gamma_0$, or it intersects $\Delta_1\backslash\Gamma_0$. In all cases, each connected component of $L\backslash\Gamma_0$ only intersects either the horizontal axis $[0,+\infty)\times\{0\}$ or the vertical axis $\{0\}\times[0,+\infty)$, but not both.

We then call Ω_1 the union of connected components of $L\backslash\Gamma_0$ which intersect the vertical axis $\{0\}\times[0,+\infty)$, and Ω_2 the union of connected components of $L\backslash\Gamma_0$ which intersect the horizontal axis $[0,+\infty)\times\{0\}$. Then we have the partition

$$L=\Omega_1\cup\Omega_2\cup\Gamma_0.$$

Let us consider the continuous curves

$$\begin{cases} (A^1_*,A^2_*)\in\Gamma_0\\ \Gamma_2 := \left\{(\bar\gamma^1,\bar\gamma^2)\in[0,+\infty]\times[0,A^2_*],\quad \bar\gamma^1=A^1_2(\bar\gamma^2)\right\}\\ \Gamma_1 := \left\{(\bar\gamma^1,\bar\gamma^2)\in[0,A^1_*]\times[0,+\infty],\quad \bar\gamma^2=A^2_1(\bar\gamma^1)\right\}\end{cases}$$

and let us consider the (associated) epigraphs

$$\begin{cases} E_0 := (A^1_*,+\infty)\times(A^2_*,+\infty)\\ E_2 := \left\{(\bar\gamma^1,\bar\gamma^2)\in[0,+\infty)\times[0,A^2_*],\quad \bar\gamma^1>A^1_2(\bar\gamma^2)\right\}\\ E_1 := \left\{(\bar\gamma^1,\bar\gamma^2)\in[0,A^1_*]\times[0,+\infty),\quad \bar\gamma^2>A^2_1(\bar\gamma^1)\right\}\end{cases}$$

Now consider the injection map

$$\begin{array}{llll}\iota: & [0,+\infty)^2 & \to & [0,+\infty)^3\\ & (\bar\gamma^1,\bar\gamma^2) & \to & (\bar\gamma^1+\bar\gamma^2,\bar\gamma^1,\bar\gamma^2)\end{array}$$

and the projection maps for $j=0,1,2$

$$\begin{array}{llll}\pi_j: & [0,+\infty)^3 & \to & [0,+\infty)^2\\ & \bar\gamma:=(\bar\gamma^0,\bar\gamma^1,\bar\gamma^2) & \mapsto & \pi_j(\bar\gamma)=\begin{cases}(\bar\gamma^1,\bar\gamma^2) & \text{if}\quad j=0\\ (\bar\gamma^0-\bar\gamma^2,\bar\gamma^2) & \text{if}\quad j=1\\ (\bar\gamma^1,\bar\gamma^0-\bar\gamma^1) & \text{if}\quad j=2\end{cases}\end{array}$$

which satisfy $(\iota\circ\pi_j)_{|H}=id_H$ and are such that $\pi_j(p)$ is independent on the component p^j of p. We define the sets

$$\tilde E_j := \iota(E_j)\quad j=0,1,2$$

From (12.22), we then deduce that $\tilde E_0\subset S$ and then

$$\hat\Sigma_j\cap H=\tilde E_j,\quad j=0,1,2 \tag{12.29}$$

Notice also that by construction, we have

$$\begin{cases}\hat\Sigma_j+[0,+\infty)e_0=\hat\Sigma_j\quad\text{for}\quad j=1,2\\ S+[0,+\infty)e_0=S\end{cases}$$

Hence

$$\pi_0(\tilde E_j)=E_j,\quad j=0,1,2 \tag{12.30}$$

and we deduce from (12.24) that

$$H^*=H\cap\left\{\hat\gamma=id_{[0,+\infty)^3}\right\}=\iota(F_0)\quad\text{with}\quad F_0:=[0,+\infty)^2\backslash(E_0\cup E_1\cup E_2)=L\backslash(E_1\cup E_2) \tag{12.31}$$

Moreover (12.25) implies $D_0^* + [0,+\infty)e_0 = D_0^*$ and then

$$\pi_0(D_0^*) = F_0$$

Because $D_0 = D_0^* \cup H^*$, we deduce that

$$D_0^* = \left\{(p^0+\varepsilon, p^1, p^2) \in [0,+\infty)^3, \quad (p^1,p^2) \in F_0, \quad p^0 = p^1+p^2, \quad \varepsilon \in (0,+\infty)\right\} \tag{12.32}$$

Step 2.4: compatibility of epigraphs
Let Ω_j^∞ be the unique unbounded connected component of Ω_j for $j=1,2$.
Assume by contradiction that

$$E_2 \not\subset \Omega_2^\infty \tag{12.33}$$

Then $A_*^2 > 0$ and the two curves Γ_0 and Γ_2 do cross each other with the meaning that there exists some point

$$(\bar\gamma_0^1, \bar\gamma_0^2) \in E_2 \cap (\Delta_2 \backslash \Gamma_0)$$

This implies that there exists two sequences of points $P_\varepsilon = (P_\varepsilon^1, P_\varepsilon^2) \in \Gamma_2$ and $Q_\varepsilon = (Q_\varepsilon^1, Q_\varepsilon^2) \in \Gamma_0$ with

$$P_\varepsilon^1 + P_\varepsilon^2 > Q_\varepsilon^1 + Q_\varepsilon^2, \quad P_\varepsilon^2 < Q_\varepsilon^2, \quad P_\varepsilon, Q_\varepsilon \to P_0 = (P_0^1, P_0^2) \in [0,+\infty) \times [0, A_*^2)$$

Then we can apply the cleaning claim of Step 2.1.2 to $p = (P_\varepsilon^1 + P_\varepsilon^2, P_\varepsilon)$ and $q = (Q_\varepsilon^1 + Q_\varepsilon^2, Q_\varepsilon)$ to deduce in the limit that

$$(P_0^1 + P_0^2, P_0) + [0,+\infty)^3 \subset S = A_* + [0,+\infty)^3$$

This leads to a contradiction with $P_0^2 < A_*^2$. Hence (12.33) is false, and we deduce that $E_2 \subset \Omega_2^\infty \subset \Omega_2$, and similarly that $E_1 \subset \Omega_1^\infty \subset \Omega_1$.

In other words, this means that the curves Γ_0 and Γ_1 (resp. Γ_0 and Γ_2) can not cross each other, even if they may be "tangential" to each other. Notice that the curve Γ_0 is sandwiched in between the two curves Γ_1 and Γ_0. As a consequence Γ_1 and Γ_2 do not cross neither, even if they may be tangential (which then implies that the three curves coincide there).

Step 2.5: identification of D_1^* and D_2^*
We define sets

$$F_j := (\Gamma_0 \cup \Omega_j) \backslash E_j, \quad j=1,2$$

which are closed sets, and we have the partition $L = E_2 \cup F_0 \cup E_1$ and the relations

$$F_0 = F_2 \cup F_1 \quad \text{and} \quad F_1 \cap F_2 = \Gamma_0 \quad \text{with} \quad F_j \supset \Gamma_j, \quad j=1,2$$

Now define the sets

$$\begin{cases} \hat F_1 := \left\{(p^0, p^1+\varepsilon, p^2) \in [0,+\infty)^3, \quad (p^1,p^2) \in F_2, \quad p^0 = p^1+p^2, \quad \varepsilon \in (0,+\infty)\right\} \\ \hat F_2 := \left\{(p^0, p^1, p^2+\varepsilon) \in [0,+\infty)^3, \quad (p^1,p^2) \in F_1, \quad p^0 = p^1+p^2, \quad \varepsilon \in (0,+\infty)\right\} \end{cases}$$

where the indices are mixed for $\hat F_j$ and F_j. Let us show that

$$D_j^* = \hat F_j, \quad j=1,2 \tag{12.34}$$

We do the proof for $j=1$ (the case $j=2$ is similar).

Step 2.5.a: first inclusions
Let $p \in D_1^*$ with $D_1^* = \left\{\hat\gamma^0(\bar\gamma) = \bar\gamma^0\right\} \cap \left\{\hat\gamma^2(\bar\gamma) = \bar\gamma^2\right\} \cap \left\{\bar\gamma^0 < \bar\gamma^1 + \bar\gamma^2\right\}$ and let

$$\varepsilon_* := \sup\left\{\varepsilon > 0, \quad p^0 < (p^1 - \varepsilon) + p^2\right\}$$

Because of the local constancy of $\hat\gamma$ in $\bar\gamma^1$ on $\left\{\hat\gamma^1(\bar\gamma) < \hat\gamma^1\right\}$, we deduce that $p_* := (p^0, p^1 - \varepsilon_*, p^2) = \iota \circ \pi_1(p)$ satisfies

$$\hat\gamma(p_*) = p_* = (p_*^0, p_*^1, p_*^2) \quad \text{and} \quad p_* + (0,+\infty)e_1 \subset D_1^*$$

Moreover from (12.31), we get $(p_*^1, p_*^2) \in L \backslash (E_1 \cup E_2) = F_0$, i.e.

$$\pi_1(D_1^*) \subset F_0 \quad \text{and} \quad D_1^* = \iota \circ \pi_1(D_1^*) + (0,+\infty)e_1$$

Step 2.5.b: excluded intersections
Now consider some point $(p^1, p^2) \in \Omega_1$, and set $p^0 := p^1 + p^2$.
Case 1: $p^0 \in [0, A^0_*]$
Then there exists $\delta > 0$ such that the line $(p^1, p^2) + \mathbb{R}(e_1 - e_2)$ intersects the curve Γ_0 in a single point $(p^1, p^2) + \delta(e_1 - e_2) = (A^1_0(p^0), A^2_0(p^0))$. Then the point $p := (p^0, p^1, p^2)$ is such that the "line" passing through p intersects the "plane" passing through $A_0(p^0)$, i.e. precisely

$$(p + (0, +\infty)e_1) \cap (A_0(p^0) + \Pi_0) \neq \emptyset \quad \text{with} \quad \Pi_0 := \{0\} \times (0, +\infty)^2 \tag{12.35}$$

But $(A_0(p^0) + \Pi_0) \subset \hat{\Sigma}_0$ with $\hat{\Sigma}_0 \cap D^*_1 = \emptyset$, and the relation $D^*_1 = D^*_1 + (0, +\infty)e_1$ implies that $(p + (0, +\infty)e_1) \cap D^*_1 = \emptyset$ and then

$$p \notin \pi_1(D^*_1). \tag{12.36}$$

Case 2: $p^0 > A^0_*$
Then there exists $\delta > 0$ such that the line $(p^1, p^2) + \mathbb{R}(e_1 - e_2)$ intersects the vertical "line" $(A^1_*, A^2_*) + (0, +\infty)e_2$ in a single point $(A^1_*, A^2_*) + \delta e_2$. Because A_0 is constant on $[A^0_*, +\infty)$, we deduce that (12.35) still holds true, which again implies (12.36). Hence $\Omega_1 \cap \pi_1(D^*_1) = \emptyset$.
Step 2.5.c: precised inclusion
We deduce that $\pi_1(D^*_1) \subset F_1$ and then

$$D^*_1 \subset \hat{F}_2 \tag{12.37}$$

Step 2.5.d: reversed inclusion
Conversely, let

$$(p^1, p^2) \in F_2$$

and let $p := (p^0, p^1, p^2)$ with $p^0 := p^1 + p^2$. For $\varepsilon > 0$, we set

$$p_\varepsilon := p + \varepsilon e_1$$

and we want to show that $p_\varepsilon \in D^*_1$. Recall that we have the following partition

$$[0, +\infty)^3 = D^*_0 \cup D^*_1 \cup D^*_2 \cup H^* \cup \hat{\Sigma}^*_0 \cup \hat{\Sigma}^*_1 \cup \hat{\Sigma}^*_2 \cup S$$

with $\hat{\Sigma}^*_j := \hat{\Sigma}_j \backslash S$ for $j = 0, 1, 2$, and that $p_\varepsilon \notin D^*_0 \cup H^*$. We distinguish cases.
Case A: $p_\varepsilon \in D^*_2$
Then we know that $\pi_2(D^*_2) \subset F_1$. Hence if $p_\varepsilon \in D^*_2$, we deduce that

$$q_\varepsilon := (p^1 + \varepsilon, p^2 - \varepsilon) = \pi_2(p_\varepsilon) \in F_1$$

Because $q := (p^1, p^2) \in F_2$, we deduce that there exists a point in $F_1 \cap F_2 = \Gamma_0$ which belongs to the the segment of extremities q and q_ε. But recall that Γ_0 is the graph in the direction $e_2 - e_1$, with $F_1 \backslash \Gamma_0$ above and $F_2 \backslash \Gamma_0$ below. This leads to a contradiction, and Case A is then impossible.
Case B: $p_\varepsilon \in \hat{\Sigma}_0$
Then we deduce $(p^1 + \varepsilon, p^2) > (A^1_0, A^2_0)(p^0)$. At the limit $\varepsilon \to 0$, this gives

$$p^1 \geq A^1_0(p^0), \quad p^2 > A^2_0(p^0)$$

Hence $p^1 + p^2 > A^1_0(p^0) + A^2_0(p^0) = p^0$. Contradiction.
Case C: $p_\varepsilon \in \hat{\Sigma}_2$
Then we deduce

$$(p^0_\varepsilon, p^1_\varepsilon) = (p^0, p^1 + \varepsilon) > (A^0_2, A^1_2)(p^2) = (\hat{\gamma}^0, \hat{\gamma}^1)(p_\varepsilon) \tag{12.38}$$

Assume that there exists $\varepsilon_* \geq 0$ such that

$$p^1 + \varepsilon_* = A^1_2(p^2)$$

Because $\hat{\gamma}$ is locally constant on $\left\{\hat{\gamma}^1(\bar{\gamma}) < \bar{\gamma}^1\right\}$, we deduce that $\hat{\gamma}(p_{\varepsilon_*}) = \hat{\gamma}(p_\varepsilon) = A_2(p^2)$. Hence

$$p^1 + \varepsilon_* + p^2 = A^1_2(p^2) + p^2 = A^0_2(p^2) < p^0 = p^1 + p^2$$

Contradiction. Therefore (12.38) is still true for $\varepsilon = 0$, i.e. $(p^0, p^1) > (A_2^0, A_2^1)(p^2) = (\hat{\gamma}^0, \hat{\gamma}^1)(p)$. Contradiction with the fact that $\hat{\gamma}(p) = p$.

Case D: $p_\varepsilon \in \hat{\Sigma}_1$

Then we deduce that

$$(p_\varepsilon^0, p_\varepsilon^2) = (p^0, p^2) > (A_1^0, A_1^2)(p^1 + \varepsilon) = (\hat{\gamma}^0, \hat{\gamma}^2)(p_\varepsilon)$$

Because $\hat{\gamma}$ is locally constant on $\left\{\hat{\gamma} \neq Id_{[0,+\infty)^3}\right\}$, we deduce that we can decrease the value of ε up to zero without changing the value of $\hat{\gamma}$. Therefore at the limit, we get $(p^0, p^2) > (\hat{\gamma}^0, \hat{\gamma}^2)(p)$. Contradiction with the fact that $\hat{\gamma}(p) = p$.

Conclusion

We then conclude that there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, we have $p_\varepsilon \in D_1^*$. Because $D_1^* = D_1^* + (0, +\infty)e_1$, we deduce that

$$p + (0, +\infty)e_1 \subset D_1^*$$

Therefore $\hat{F}_1 = \iota(F_2) + (0, +\infty)e_1 \subset D_1^*$. With (12.37), we deduce that $D_1^* = \hat{F}_1$, which shows (12.34).

Step 2.6: $\mathbf{Im}(\hat{\gamma}) = H^*$

We already know that $\hat{\gamma}(\bar{\gamma}) = \bar{\gamma}$ for all $\bar{\gamma} \in H^* = \iota(F_0) =: \tilde{F}_0$. We also have (using (12.32) and (12.34))

$$\hat{\gamma}(\bar{\gamma}) = \tilde{\gamma}_j(\bar{\gamma}) \in \iota(F_j) =: \tilde{F}_j \subset H^* \quad \text{for} \quad \bar{\gamma} \in D_j^*, \quad j = 0, 1, 2$$

From (12.22), we deduce that

$$\hat{\gamma}(\hat{\Sigma}_j) \subset \iota(\Gamma_j) =: \tilde{\Gamma}_j \subset H^* \quad \text{for} \quad j = 0, 1, 2$$

Therefore

$$\hat{\gamma}([0, +\infty)^3) = H^*$$

We moreover have

$$\hat{\gamma}(\hat{\Sigma}_j^*) = \tilde{\Gamma}_j, \quad j = 0, 1, 2$$

which shows (12.16).

Step 3: case $S = \emptyset$

Because each A_j is locally constant on $\left\{A_j^j < id_{[0,+\infty)}\right\}$, we deduce the existence of some $A_*^j \in [0, +\infty]$ such that

$$A_j^j(\bar{\gamma}^j) = \min\left\{\bar{\gamma}^j, A_*^j\right\}, \quad j = 0, 1, 2.$$

We first claim that

$$A_*^0 = +\infty \tag{12.39}$$

Indeed, assume by contradiction that $A_*^0 < +\infty$. This implies that $A_0(A_*^0) \in \mathbb{R}$, and then (12.19) implies that

$$\hat{\gamma} = A_0(A_*^0) \quad \text{on} \quad A_0(A_*^0) + [0, +\infty)^3 \subset S$$

Contradiction with $S = \emptyset$. This implies (12.39).

We claim that

$$A_*^1 = +\infty \quad \text{or} \quad \left\{\begin{array}{l} A_*^1 < \infty \\ A_1^j(A_*^1) = +\infty \quad \text{for} \quad j = 0, 2 \end{array}\right. \tag{12.40}$$

where we have used $A_1^0 = A_1^1 + A_1^2$. Indeed, assume by contradiction that $A_*^1 < +\infty$ and $A_1(A_*^1) \in \mathbb{R}^3$. Then we also have from (12.20) that

$$\hat{\gamma} = A_1(A_*^1) \quad \text{on} \quad A_1(A_*^1) + [0, +\infty)^3 \subset S$$

Again contradiction with $S = \emptyset$. This shows (12.40). Similarly to (12.40), we also show

$$A_*^2 = +\infty \quad \text{or} \quad \left\{\begin{array}{l} A_*^2 < \infty \\ A_2^j(A_*^2) = +\infty \quad \text{for} \quad j = 0, 1 \end{array}\right. \tag{12.41}$$

Therefore we have (12.39), (12.40) and (12.41) and then

$$A_j(A_*^j) \notin \mathbb{R}^3 \quad \text{for each} \quad j = 0, 1, 2$$

This corresponds to Cases 2 to 5 in (12.13). Except for this small change, the whole proof remains true, and then the curve Γ_0 is also sandwiched in between the two curves Γ_1 and Γ_2, as in Step 2.

Step 4: Kružkov conservative case

Recall that a conservative preflux $\hat\gamma$ is Kružkov if and only if it is σ-monotone (see v) of Theorem 11.8). For $\sigma = (\sigma^0, \sigma^1, \sigma^2) = (1, -1, -1)$, recall that $\hat\gamma$ is σ-monotone if and only if we have in the sense of distributions

$$\sigma^j \sigma^k \partial_j \hat\gamma^k \le 0 \quad \text{for all} \quad k \ne j$$

i.e.

$$\left\{\begin{array}{ll} \partial_0 \hat\gamma^1 \ge 0, & \partial_0 \hat\gamma^2 \ge 0 \\ \partial_1 \hat\gamma^0 \ge 0, & -\partial_1 \hat\gamma^2 \ge 0 \\ \partial_2 \hat\gamma^0 \ge 0, & -\partial_2 \hat\gamma^1 \ge 0. \end{array}\right.$$

Here this implies

$$\left\{\begin{array}{llll} (A_0^1)' \ge 0, & (A_0^2)' \ge 0 & \text{in} & \mathcal{D}'(0, A_*^0) \\ (A_1^0)' \ge 0, & -(A_1^2)' \ge 0 & \text{in} & \mathcal{D}'(0, A_*^1) \\ (A_2^0)' \ge 0, & -(A_2^1)' \ge 0 & \text{in} & \mathcal{D}'(0, A_*^2) \end{array}\right. \tag{12.42}$$

Indeed, we can be more precise, because (12.15) joint to the characterization of Kružkov functions (see Proposition 4.20), also implies that $\hat\gamma$ is Lipschitz continuous and satisfies (12.17). This ends the proof of the lemma.

12.4 Characterization of 1:2 conservative prefluxes

Theorem 12.7 (Characterization of 1:2 conservative prefluxes)

We consider a $1:2$ *junction with* $\sigma = (\sigma^0, \sigma^1, \sigma^2) = (1, -1, -1)$, *or* $2:1$ *junction with* $\sigma = (\sigma^0, \sigma^1, \sigma^2) = (-1, 1, 1)$.

i) (Explicit general conservative preflux)

Then every preflux $\hat\gamma = (\hat\gamma^0, \hat\gamma^1, \hat\gamma^2) : [0, +\infty)^3 \to [0, +\infty)^3$ *is 1:2 conservative if and only if we have for for all* $\bar\gamma = (\bar\gamma^0, \bar\gamma^1, \bar\gamma^2) \in [0, +\infty)^3$

$$\left\{\begin{array}{l} \hat\gamma^1(\bar\gamma) = \min\left\{\bar\gamma^1, A_2^1(\bar\gamma^2), \max\left\{A_0^1(\bar\gamma^0), \bar\gamma^0 - \bar\gamma^2\right\}\right\} \\ \hat\gamma^2(\bar\gamma) = \min\left\{\bar\gamma^2, A_1^2(\bar\gamma^1), \max\left\{A_0^2(\bar\gamma^0), \bar\gamma^0 - \bar\gamma^1\right\}\right\} \\ \hat\gamma^0 = \hat\gamma^1 + \bar\gamma^2 \end{array}\right. \tag{12.43}$$

where there exists $A_* = (A_*^0, A_*^1, A_*^2) \in [0, +\infty]^3$ *and the continuous functions* A_0, A_1, A_2 *satisfying*

$$\left\{\begin{array}{lll} A_j = (A_j^0, A_j^1, A_j^2) : & & [0, +\infty) \to [0, +\infty]^3, \\ & & \\ A_j^0 = A_j^1 + A_j^2, & & \\ & & \\ A_j = const & on & [A_*^j, +\infty), \\ A_j^j(\bar\gamma^j) = \min\left\{\bar\gamma^j, A_*^j\right\} & for & \bar\gamma^j \in [0, +\infty), \end{array}\right| \quad j = 0, 1, 2$$

with moreover one the following fives cases

$$\left\{\begin{array}{lll} \textbf{(Case 1)} & A_* \in [0, +\infty)^3 & A_j(A_*^j) = A_*, \quad j = 0, 1, 2 \\ & & \\ \textbf{(Case 2)} & A_* \in \{(+\infty, +\infty, +\infty)\} & \\ & & \\ \textbf{(Case 3)} & A_* \in \{+\infty\} \times [0, +\infty) \times \{+\infty\}, & A_1^0(A_*^1) = +\infty = A_1^2(A_*^1) \\ & & \\ \textbf{(Case 4)} & A_* \in \{+\infty\} \times \{+\infty\} \times [0, +\infty), & A_2^0(A_*^2) = +\infty = A_2^1(A_*^2) \\ & & \\ \textbf{(Case 5)} & A_* \in \{+\infty\} \times [0, +\infty) \times [0, +\infty), & \left\{\begin{array}{l} A_2^0(A_*^2) = +\infty = A_2^1(A_*^2) \\ A_1^0(A_*^1) = +\infty = A_1^2(A_*^1) \end{array}\right. \end{array}\right. \tag{12.44}$$

Moreover, we have

$$\left\{\begin{array}{l} A_*^0 = A_*^1 + A_*^2, \\ A_j = const = A_* \quad on \quad [A_*^j, +\infty), \quad j = 0, 1, 2 \end{array}\right| \quad \text{in Cases 1 to 4}$$

Moreover the curves

$$\Gamma_j := (A_j^1, A_j^2)([0, A_*^j]) \subset [0, +\infty]^2, \quad j = 0, 1, 2$$

are such that Γ_0 *is sandwiched in between* Γ_1 *and* Γ_2 *in the following sense*

$$\Gamma_0 \subset \left\{ (\bar{\gamma}^1, \bar{\gamma}^2) \in [0, +\infty)^2 \backslash ([A_*^1, +\infty) \times [A_*^2, +\infty)), \quad \bar{\gamma}^1 \le A_2^1(\bar{\gamma}^2), \quad \bar{\gamma}^2 \le A_1^2(\bar{\gamma}^1) \right\}.$$

We also have

$$\hat{\gamma}^0(\bar{\gamma}) \ge \min \left\{ \bar{\gamma}^0, \bar{\gamma}^1 + \bar{\gamma}^2, A_1^0(\bar{\gamma}^1), A_2^0(\bar{\gamma}^2) \right\}$$

ii) (Simplified $\hat{\gamma}^0$ under an additional assumption)
Moreover, under assumption

$$A_j^0(\cdot) \le A_*^0 \quad \text{for} \quad j = 1, 2 \tag{12.45}$$

we have

$$\hat{\gamma}^0(\bar{\gamma}) = \min \left\{ \bar{\gamma}^0, \bar{\gamma}^1 + \bar{\gamma}^2, A_1^0(\bar{\gamma}^1), A_2^0(\bar{\gamma}^2) \right\} \tag{12.46}$$

iii) (Kružkov conservative preflux)
Then $\hat{\gamma}$ *is a Kružkov conservative if only if in the previous description, the functions* A_0, A_1, A_2 *are Lipschitz continuous and satisfy*

$$\left\{ \begin{array}{llll} |(A_0^1)'| + |(A_0^2)'| \le 1, & (A_0^1)' \ge 0, & (A_0^2)' \ge 0 & a.e. \ on \ [0, A_*^0] \\ |(A_1^0)'| + |(A_1^2)'| \le 1, & (A_1^0)' \ge 0, & -(A_1^2)' \ge 0 & a.e. \ on \ [0, A_*^1] \\ |(A_2^0)'| + |(A_2^1)'| \le 1 & (A_2^0)' \ge 0, & -(A_2^1)' \ge 0 & a.e. \ on \ [0, A_*^2] \end{array} \right. \tag{12.47}$$

Theorem 12.7 implies in particular Theorem 2.50 of the Introduction.

Remark 12.8 *Notice that condition (12.45) is automatically satisfied for Kružkov prefluxes in case of* $1:2$ *or* $2:1$ *junctions.*

Proof of Theorem 12.7
The necessary conditions are given in iii) of Lemma 12.5 for $1:2$ junctions, and our goal is to translate it in a single formula. The proof consists in three parts, where the core of the work is contained in Part II.
Part I: the Kružkov case
Assuming point i) of Theorem 12.7 already proved, we know that $\hat{\gamma}$ is given by (12.43). Moreover if $\hat{\gamma}$ is Kružkov, from point 2) of Lemma 12.5, we know that (12.47) holds true. Conversely, if (12.47) holds true, then the implied monotonicities of the 1:2 conservative preflux $\hat{\gamma}$ imply that $\hat{\gamma}$ is Kružkov, from iii) of Lemma 11.11. Notice also that from Definition 11.1 v), conservative prefluxes for $1:2$ junctions coincide with conservative prefluxes for $2:1$ junctions.
Part II: necessary expression of the preflux
This part deals with the proof deal with Case 1 where $A_* \in [0, +\infty)^3$. For the other cases, we notice that for any conservative preflux $\hat{\gamma}$ and for the truncation operator $T_{(R,R,R)}$ with $R > 0$, the new preflux $\hat{\gamma} \circ T_{(R,R,R)}$ is a conservative preflux with finite values of A_* and A_j, i.e. as in Case 1. Therefore, in the limit $R \to +\infty$, we recover the expression of the preflux in the remaining Cases 2 to 5.
Step 1: reformulation
We take the convention that

$$\bar{\gamma} \in [0, +\infty)^3.$$

We recall (12.15), namely
(12.48)

$$\hat{\gamma}(\bar{\gamma}) = \left\{ \begin{array}{llll} \tilde{\gamma}_0(\bar{\gamma}) := (\bar{\gamma}^1 + \bar{\gamma}^2, \bar{\gamma}^1, \bar{\gamma}^2) & \in \tilde{F}_0 & \text{on} \ D_0^* & = \left\{ \hat{\gamma}^0(\bar{\gamma}) < \bar{\gamma}^0 \right\} \cap \left\{ \hat{\gamma}^1(\bar{\gamma}) = \bar{\gamma}^1 \right\} \cap \left\{ \hat{\gamma}^2(\bar{\gamma}) = \bar{\gamma}^2 \right\} \\ \tilde{\gamma}_1(\bar{\gamma}) := (\bar{\gamma}^0, \bar{\gamma}^0 - \bar{\gamma}^2, \bar{\gamma}^2) & \in \tilde{F}_2 & \text{on} \ D_1^* & = \left\{ \hat{\gamma}^0(\bar{\gamma}) = \bar{\gamma}^0 \right\} \cap \left\{ \hat{\gamma}^1(\bar{\gamma}) < \bar{\gamma}^1 \right\} \cap \left\{ \hat{\gamma}^2(\bar{\gamma}) = \bar{\gamma}^2 \right\} \\ \tilde{\gamma}_2(\bar{\gamma}) := (\bar{\gamma}^0, \bar{\gamma}^1, \bar{\gamma}^0 - \bar{\gamma}^1) & \in \tilde{F}_1 & \text{on} \ D_2^* & = \left\{ \hat{\gamma}^0(\bar{\gamma}) = \bar{\gamma}^0 \right\} \cap \left\{ \hat{\gamma}^1(\bar{\gamma}) = \bar{\gamma}^1 \right\} \cap \left\{ \hat{\gamma}^2(\bar{\gamma}) < \bar{\gamma}^2 \right\} \\ \bar{\gamma} & \in \tilde{F}_0 & \text{on} \ H^* & = \left\{ \hat{\gamma}^0(\bar{\gamma}) = \bar{\gamma}^0 \right\} \cap \left\{ \hat{\gamma}^1(\bar{\gamma}) = \bar{\gamma}^1 \right\} \cap \left\{ \hat{\gamma}^2(\bar{\gamma}) = \bar{\gamma}^2 \right\} \\ A_0(\bar{\gamma}^0) & \in \tilde{\Gamma}_0 & \text{on} \ \hat{\Sigma}_0^* & = \left\{ \hat{\gamma}^0(\bar{\gamma}) = \bar{\gamma}^0 \right\} \cap \left\{ \hat{\gamma}^1(\bar{\gamma}) < \bar{\gamma}^1 \right\} \cap \left\{ \hat{\gamma}^2(\bar{\gamma}) < \bar{\gamma}^2 \right\} \\ A_1(\bar{\gamma}^1) & \in \tilde{\Gamma}_1 & \text{on} \ \hat{\Sigma}_1^* & = \left\{ \hat{\gamma}^0(\bar{\gamma}) < \bar{\gamma}^0 \right\} \cap \left\{ \hat{\gamma}^1(\bar{\gamma}) = \bar{\gamma}^1 \right\} \cap \left\{ \hat{\gamma}^2(\bar{\gamma}) < \bar{\gamma}^2 \right\} \\ A_2(\bar{\gamma}^2) & \in \tilde{\Gamma}_2 & \text{on} \ \hat{\Sigma}_2^* & = \left\{ \hat{\gamma}^0(\bar{\gamma}) < \bar{\gamma}^0 \right\} \cap \left\{ \hat{\gamma}^1(\bar{\gamma}) < \bar{\gamma}^1 \right\} \cap \left\{ \hat{\gamma}^2(\bar{\gamma}) = \bar{\gamma}^2 \right\} \\ A_* & & \text{on} \ S & = \left\{ \hat{\gamma}^0(\bar{\gamma}) < \bar{\gamma}^0 \right\} \cap \left\{ \hat{\gamma}^1(\bar{\gamma}) < \bar{\gamma}^1 \right\} \cap \left\{ \hat{\gamma}^2(\bar{\gamma}) < \bar{\gamma}^2 \right\} \end{array} \right.$$

Recall that the planar sets $\tilde{F}_0, \tilde{F}_1, \tilde{F}_2$ are the lifting in the plane $H := \left\{\bar{\gamma}^0 = \bar{\gamma}^1 + \bar{\gamma}^2\right\}$ of the planar faces F_0, F_1, F_2 defined in Lemma 12.5. We also have

$$\left\{\begin{array}{lcl} \partial F_0 & \supset & \Gamma_1 \cup \Gamma_2 \\ \partial F_2 & \supset & \Gamma_0 \cup \Gamma_2 \\ \partial F_1 & \supset & \Gamma_0 \cup \Gamma_1. \end{array}\right.$$

where the curves $\Gamma_0, \Gamma_1, \Gamma_2$ are defined in Lemma 12.5 and illustrated on Figure 7. Recall also that each set $E \in \left\{S, H^*, \hat{\Sigma}_0^*, \hat{\Sigma}_1^*, \hat{\Sigma}_2^*, D_0^*, D_1^*, D_2^*\right\}$ satisfies

$$E = \hat{\gamma}(E) + \sum_{j \in I_E} (0, +\infty) e_j \quad \text{for} \quad I_E := \left\{j \in \{0,1,2\}, \quad \hat{\gamma}^j(\bar{\gamma}) < \bar{\gamma}^j \quad \text{for all} \quad \bar{\gamma} \in E\right\}$$

Hence we have

$$\left\{\begin{array}{l} S = \{\bar{\gamma} > A_*\} \\ H^* = \tilde{F}_0 = \left\{\bar{\gamma}^0 = \bar{\gamma}^1 + \bar{\gamma}^2, \quad \bar{\gamma}^1 \leq A_2^1(\bar{\gamma}^2), \quad \bar{\gamma}^2 \leq A_1^2(\bar{\gamma}^1)\right\} \\ \hat{\Sigma}_0^* = \left\{\bar{\gamma}^1 > A_0^1(\bar{\gamma}^0), \quad \bar{\gamma}^2 > A_0^2(\bar{\gamma}^0), \quad \bar{\gamma}^0 \leq A_*^0\right\} \\ \hat{\Sigma}_1^* = \left\{\bar{\gamma}^0 > A_1^0(\bar{\gamma}^1), \quad \bar{\gamma}^2 > A_1^2(\bar{\gamma}^1), \quad \bar{\gamma}^1 \leq A_*^1\right\} \\ \hat{\Sigma}_2^* = \left\{\bar{\gamma}^0 > A_2^0(\bar{\gamma}^2), \quad \bar{\gamma}^1 > A_2^1(\bar{\gamma}^2), \quad \bar{\gamma}^2 \leq A_*^2\right\} \end{array}\right.$$

where we have used the partition illustrated on Figure 8. We also have

$$\begin{array}{rl} D_0^* & = \left\{\bar{\gamma} = \tilde{\gamma} + (\varepsilon, 0, 0), \quad \varepsilon > 0, \quad \tilde{\gamma} \in \tilde{F}_0\right\} \\ & = \left\{\bar{\gamma}^0 > \bar{\gamma}^1 + \bar{\gamma}^2, \quad (\bar{\gamma}^1, \bar{\gamma}^2) \in \text{Proj}^{\perp}_{\mathbb{R}e_1 + \mathbb{R}e_2}(\tilde{F}_2) = F_2\right\} \\ & = \left\{\bar{\gamma}^0 > \bar{\gamma}^1 + \bar{\gamma}^2, \quad \bar{\gamma}^1 \leq A_2^1(\bar{\gamma}^2), \quad \bar{\gamma}^2 \leq A_1^2(\bar{\gamma}^1)\right\} \end{array}$$

and

$$\begin{array}{rl} D_1^* & = \left\{\bar{\gamma} = \tilde{\gamma} + (0, \varepsilon, 0), \quad \varepsilon > 0, \quad \tilde{\gamma} \in \tilde{F}_2\right\} \\ & = \left\{\bar{\gamma}^1 > \bar{\gamma}^0 - \bar{\gamma}^2, \quad (\bar{\gamma}^0, \bar{\gamma}^2) \in \text{Proj}^{\perp}_{\mathbb{R}e_0 + \mathbb{R}e_2}(\tilde{F}_2)\right\} \\ & = \left\{\bar{\gamma}^0 < \bar{\gamma}^1 + \bar{\gamma}^2, \quad \bar{\gamma}^0 \leq A_2^0(\bar{\gamma}^2), \quad \bar{\gamma}^2 \leq A_0^2(\bar{\gamma}^0)\right\} \end{array}$$

and similarly

$$\begin{array}{rl} D_2^* & = \left\{\bar{\gamma} = \tilde{\gamma} + (0, 0, \varepsilon), \quad \varepsilon > 0, \quad \tilde{\gamma} \in \tilde{F}_1\right\} \\ & = \left\{\bar{\gamma}^2 > \bar{\gamma}^0 - \bar{\gamma}^1, \quad (\bar{\gamma}^0, \bar{\gamma}^1) \in \text{Proj}^{\perp}_{\mathbb{R}e_0 + \mathbb{R}e_1}(\tilde{F}_1)\right\} \\ & = \left\{\bar{\gamma}^0 < \bar{\gamma}^1 + \bar{\gamma}^2, \quad \bar{\gamma}^0 \leq A_1^0(\bar{\gamma}^1), \quad \bar{\gamma}^1 \leq A_0^1(\bar{\gamma}^0)\right\} \end{array}$$

We can rewrite with explicit domains for $\hat{\gamma}$

(12.49)

$$\hat{\gamma}(\bar{\gamma}) = \left\{\begin{array}{llll} (\bar{\gamma}^1 + \bar{\gamma}^2, \bar{\gamma}^1, \bar{\gamma}^2) & \in \tilde{F}_0 & \text{on} & D_0^* = \left\{\bar{\gamma}^0 > \bar{\gamma}^1 + \bar{\gamma}^2, \quad \bar{\gamma}^1 \leq A_2^1(\bar{\gamma}^2), \quad \bar{\gamma}^2 \leq A_1^2(\bar{\gamma}^1)\right\} \\ (\bar{\gamma}^0, \bar{\gamma}^0 - \bar{\gamma}^2, \bar{\gamma}^2) & \in \tilde{F}_2 & \text{on} & D_1^* = \left\{\bar{\gamma}^0 < \bar{\gamma}^1 + \bar{\gamma}^2, \quad \bar{\gamma}^0 \leq A_2^0(\bar{\gamma}^2), \quad \bar{\gamma}^2 \leq A_0^2(\bar{\gamma}^0)\right\} \\ (\bar{\gamma}^0, \bar{\gamma}^1, \bar{\gamma}^0 - \bar{\gamma}^1) & \in \tilde{F}_1 & \text{on} & D_2^* = \left\{\bar{\gamma}^0 < \bar{\gamma}^1 + \bar{\gamma}^2, \quad \bar{\gamma}^0 \leq A_1^0(\bar{\gamma}^1), \quad \bar{\gamma}^1 \leq A_0^1(\bar{\gamma}^0)\right\} \\ \bar{\gamma} & \in \tilde{F}_0 & \text{on} & H^* = \left\{\bar{\gamma}^0 = \bar{\gamma}^1 + \bar{\gamma}^2, \quad \bar{\gamma}^1 \leq A_2^1(\bar{\gamma}^2), \quad \bar{\gamma}^2 \leq A_1^2(\bar{\gamma}^1)\right\} \\ A_0(\bar{\gamma}^0) & \in \tilde{\Gamma}_0 & \text{on} & \hat{\Sigma}_0^* = \left\{\bar{\gamma}^1 > A_0^1(\bar{\gamma}^0), \quad \bar{\gamma}^2 > A_0^2(\bar{\gamma}^0), \quad \bar{\gamma}^0 \leq A_*^0\right\} \\ A_1(\bar{\gamma}^1) & \in \tilde{\Gamma}_1 & \text{on} & \hat{\Sigma}_1^* = \left\{\bar{\gamma}^0 > A_1^0(\bar{\gamma}^1), \quad \bar{\gamma}^2 > A_1^2(\bar{\gamma}^1), \quad \bar{\gamma}^1 \leq A_*^1\right\} \\ A_2(\bar{\gamma}^2) & \in \tilde{\Gamma}_2 & \text{on} & \hat{\Sigma}_2^* = \left\{\bar{\gamma}^0 > A_2^0(\bar{\gamma}^2), \quad \bar{\gamma}^1 > A_2^1(\bar{\gamma}^2), \quad \bar{\gamma}^2 \leq A_*^2\right\} \\ A_* & & \text{on} & S = \{\bar{\gamma} > A_*\} \end{array}\right.$$

Moreover

$$\hat{\gamma}^1(\bar{\gamma}) \in \left\{\bar{\gamma}^1, \quad \bar{\gamma}^0 - \bar{\gamma}^2, \quad A_0^1(\bar{\gamma}^0), \quad A_2^1(\bar{\gamma}^2), \quad A_*^1\right\}$$

i.e. $\hat{\gamma}^1$ can only take five different values.

Step 2: ordering

We then order these five values (when we can and when it is useful) on each set. In what follows, we get on each line the value $\hat{\gamma}^1(\bar{\gamma})$, the set and some ordering

$$(12.50)\qquad \begin{cases} \bar{\gamma}^1 & : & D_0^* & : & \bar{\gamma}^1 < \bar{\gamma}^0 - \bar{\gamma}^2, & \bar{\gamma}^1 \le A_2^1(\bar{\gamma}^2) \\ \bar{\gamma}^0 - \bar{\gamma}^2 & : & D_1^* & : & \bar{\gamma}^1 > \bar{\gamma}^0 - \bar{\gamma}^2 \ge A_0^1(\bar{\gamma}^0), & \bar{\gamma}^0 - \bar{\gamma}^2 \le A_2^1(\bar{\gamma}^2) \\ \bar{\gamma}^1 & : & D_2^* & : & A_2^1(\bar{\gamma}^2) \boxed{\ge} A_0^1(\bar{\gamma}^0) \ge \bar{\gamma}^1 > \bar{\gamma}^0 - \bar{\gamma}^2 & \\ \bar{\gamma}^1 & : & H^* & : & A_2^1(\bar{\gamma}^2) \ge \bar{\gamma}^1 = \bar{\gamma}^0 - \bar{\gamma}^2 & \\ A_0^1(\bar{\gamma}^0) & : & \hat{\Sigma}_0^* & : & \bar{\gamma}^1 > A_0^1(\bar{\gamma}^0) > \bar{\gamma}^0 - \bar{\gamma}^2 & A_2^1(\bar{\gamma}^2) \boxed{\ge} A_0^1(\bar{\gamma}^0) \\ \bar{\gamma}^1 & : & \hat{\Sigma}_1^* & : & \bar{\gamma}^1 \boxed{\boxed{\le}} A_2^1(\bar{\gamma}^2) & \bar{\gamma}^1 \boxed{\boxed{\le}} A_0^1(\bar{\gamma}^0) \\ A_2^1(\bar{\gamma}^2) & : & \hat{\Sigma}_2^* & : & \gamma^1 > A_2^1(\bar{\gamma}^2) & \bar{\gamma}^0 - \bar{\gamma}^2 > A_2^1(\bar{\gamma}^2) \\ A_*^1 & : & S & : & A_2^1(\bar{\gamma}^2) = A_*^1 = A_0^1(\bar{\gamma}^0) < \bar{\gamma}^1 & \end{cases}$$

where the explanations follow.

Case A: Terms in $\bar{\gamma}^0$

For $\hat{\Sigma}_0^*$, we notice that $\gamma^2 > A_0^2(\bar{\gamma}^0)$ means (because $\bar{\gamma}^0 \le A_*^0$)

$$\bar{\gamma}^0 - \bar{\gamma}^2 < \bar{\gamma}^0 - A_0^2(\bar{\gamma}^0) = A_0^1(\bar{\gamma}^0)$$

Similarly for D_1^*, we notice that $\bar{\gamma}^2 \le A_0^2(\bar{\gamma}^0)$ means

$$\bar{\gamma}^0 - \bar{\gamma}^2 \ge \bar{\gamma}^0 - A_0^2(\bar{\gamma}^0) \ge \min\left\{\bar{\gamma}^0, A_*^0\right\} - A_0^2(\bar{\gamma}^0) = A_0^1(\bar{\gamma}^0)$$

Case B: Terms in $\bar{\gamma}^2$

For $\hat{\Sigma}_2^*$, we have $\bar{\gamma}^0 > A_2^0(\bar{\gamma}^2)$ and then (using $\bar{\gamma}^2 \le A_*^2$)

$$\bar{\gamma}^0 - \bar{\gamma}^2 > A_2^0(\bar{\gamma}^2) - \bar{\gamma}^2 = A_2^1(\bar{\gamma}^2)$$

Similarly, for D_1^*, we also have $\bar{\gamma}^0 \le A_2^0(\bar{\gamma}^2)$ and then (using $\bar{\gamma}^2 \ge \min\left\{\bar{\gamma}^2, A_*^2\right\}$)

$$\bar{\gamma}^0 - \bar{\gamma}^2 \le A_2^0(\bar{\gamma}^2) - \bar{\gamma}^2 \le A_2^0(\bar{\gamma}^2) - \min\left\{\bar{\gamma}^2, A_*^2\right\} = A_2^1(\bar{\gamma}^2)$$

Case C: Terms using comparison of Γ_1 and Γ_0

Notice that

$$(12.51)\qquad \hat{\Sigma}_1^* = \tilde{\Gamma}_1 + (0,+\infty)e_0 + (0,+\infty)e_2$$

Let $\bar{\gamma} \in \hat{\Sigma}_1^*$ and let us use (12.51) with $\tilde{\gamma} \in \tilde{\Gamma}_1$ and $\bar{\gamma} - \tilde{\gamma} \in (0,+\infty)e_0 + (0,+\infty)e_2$. Then we have $\tilde{\gamma}^1 := \bar{\gamma}^1 \in [0, A_*^1]$ and

$$\bar{\gamma}^0 > A_1^0(\tilde{\gamma}^1) =: \tilde{\gamma}^0, \quad \bar{\gamma}^2 > A_1^2(\tilde{\gamma}^1) =: \tilde{\gamma}^2, \quad \tilde{\gamma}' := (\tilde{\gamma}^1, \tilde{\gamma}^2)$$

and

$$(12.52)\qquad \tilde{\gamma}' + (0,+\infty)e_2 \subset E_1 \quad \text{with} \quad E_1 := \left\{(p^1,p^2) \in [0, A_*^1] \times [0,+\infty), \quad p^2 > A_1^2(p^1)\right\}$$

and then

$$(12.53)\qquad (\tilde{\gamma}' + (0,+\infty)e_2) \cap \Gamma_0 = \emptyset$$

Case C.1: $\tilde{\gamma}^0 \le A_*^0$

Then there exists $\varepsilon \ge 0$ such that

$$(\tilde{\gamma}^1, A_1^2(\tilde{\gamma}^1)) = \varepsilon(e_2 - e_1) + (A_0^1, A_0^2)(\tilde{\gamma}^0)$$

and

$$\tilde{\gamma}^1 \le A_0^1(\tilde{\gamma}^0)$$

Case C.1.a: $\bar{\gamma}^0 \le A_*^0$

Because Γ_0 is a graph in direction $e_2 - e_1$, we deduce from (12.53) that Γ_0 does not cross the line $\tilde{\gamma}' + (0,+\infty)e_2$, and then by continuity, we still have

$$\bar{\gamma}^1 = \tilde{\gamma}^1 \le A_0^1(\bar{\gamma}^0) \quad \text{for} \quad \bar{\gamma}^0 > \tilde{\gamma}^0$$

Case C.1.b: $\bar{\gamma}^0 > A^0_*$
Then
$$\bar{\gamma}^1 = \tilde{\gamma}^1 \le A^1_* = A^1_0(\bar{\gamma}^0)$$

Case C.2: $\tilde{\gamma}^0 > A^0_*$
Then for $\bar{\gamma}^0 > \tilde{\gamma}^0$, we also get
$$\bar{\gamma}^1 = \tilde{\gamma}^1 \le A^1_* = A^1_0(\tilde{\gamma}^0) = A^1_0(\bar{\gamma}^0)$$

In all cases, we get
$$\bar{\gamma}^1 \le A^1_0(\bar{\gamma}^0)$$

which is used for $\hat{\Sigma}^*_1$.
Case D: Terms using comparison of Γ_1 and Γ_2
Let $\bar{\gamma} \in \hat{\Sigma}^*_1$ with notation of Case C. From (12.52), we deduce that

$$(\tilde{\gamma}' + (0, +\infty)e_2) \cap \Gamma_2 = \emptyset \quad \text{with} \quad \tilde{\gamma}' = (\tilde{\gamma}^1, \tilde{\gamma}^2), \quad \tilde{\gamma}^2 = A^2_1(\tilde{\gamma}^1)$$

Case D.1: $\tilde{\gamma}^2 \le A^2_*$
Because Γ_0 is sandwiched in between Γ_1 and Γ_2, we deduce that

$$\tilde{\gamma}^1 \le A^1_2(\tilde{\gamma}^2)$$

Case D.1.a: $\bar{\gamma}^2 \le A^2_*$
Because Γ_2 is a graph in direction e_1, we deduce from (12.52) that Γ_2 does not cross the line $\tilde{\gamma}' + (0, +\infty)e_2$, and then by continuity, we still have

$$\bar{\gamma}^1 = \tilde{\gamma}^1 \le A^1_2(\bar{\gamma}^2) \quad \text{for} \quad \bar{\gamma}^2 > \tilde{\gamma}^2$$

Case D.1.b: $\bar{\gamma}^2 > A^2_*$
Then
$$\bar{\gamma}^1 = \tilde{\gamma}^1 \le A^1_* = A^1_2(\bar{\gamma}^2)$$

Case D.2: $\tilde{\gamma}^2 > A^2_*$
Then for $\bar{\gamma}^2 > \tilde{\gamma}^2$, we also get
$$\bar{\gamma}^1 = \tilde{\gamma}^1 \le A^1_* = A^1_2(\tilde{\gamma}^2) = A^1_2(\bar{\gamma}^2)$$

In all cases, we get
$$\bar{\gamma}^1 \le A^1_* \le A^1_2(\bar{\gamma}^2)$$

that we use for $\hat{\Sigma}^*_1$.
Conclusion
All together this shows (12.50). From (12.50), we deduce that

$$\hat{\gamma}^1(\bar{\gamma}) = \min\left\{\bar{\gamma}^1, A^1_2(\bar{\gamma}^2), \max\left\{A^1_0(\bar{\gamma}^0), \bar{\gamma}^0 - \bar{\gamma}^2\right\}\right\}$$

which is the first line of (12.43). By symmetry, we get the second line, and the third line follows from the fact that $\hat{\gamma}$ is conservative.
Step 3: proof of (12.46)
From (12.49), we get for each set, the value $\hat{\gamma}^0(\bar{\gamma})$ and the previous ordering (except for S)

$$(12.54) \quad \left\{\begin{array}{lllll}
\bar{\gamma}^1 + \bar{\gamma}^2 & : & D^*_0 & : & \bar{\gamma}^1 + \bar{\gamma}^2 < \bar{\gamma}^0, \qquad\qquad \bar{\gamma}^1 \le A^1_2(\bar{\gamma}^2) \\
\bar{\gamma}^0 & : & D^*_1 & : & \bar{\gamma}^0 < \bar{\gamma}^1 + \bar{\gamma}^2, \quad \bar{\gamma}^1 > \bar{\gamma}^0 - \bar{\gamma}^2 \ge A^1_0(\bar{\gamma}^0), \qquad \bar{\gamma}^0 - \bar{\gamma}^2 \le A^1_2(\bar{\gamma}^2) \\
\bar{\gamma}^0 & : & D^*_2 & : & \bar{\gamma}^0 < \bar{\gamma}^1 + \bar{\gamma}^2, \quad A^1_2(\bar{\gamma}^2) \ge A^1_0(\bar{\gamma}^0) \ge \bar{\gamma}^1 > \bar{\gamma}^0 - \bar{\gamma}^2 \\
\bar{\gamma}^0 & : & H^* & : & \bar{\gamma}^0 = \bar{\gamma}^1 + \bar{\gamma}^2, \quad A^1_2(\bar{\gamma}^2) \ge \bar{\gamma}^1 = \bar{\gamma}^0 - \bar{\gamma}^2 \\
\bar{\gamma}^0 & : & \hat{\Sigma}^*_0 & : & \bar{\gamma}^1 > A^1_0(\bar{\gamma}^0) > \bar{\gamma}^0 - \bar{\gamma}^2 \qquad A^1_2(\bar{\gamma}^2) \ge A^1_0(\bar{\gamma}^0) \\
A^0_1(\bar{\gamma}^1) & : & \hat{\Sigma}^*_1 & : & \bar{\gamma}^1 \le A^1_2(\bar{\gamma}^2) \qquad \bar{\gamma}^1 \le A^1_0(\bar{\gamma}^0) \\
A^0_2(\bar{\gamma}^2) & : & \hat{\Sigma}^*_2 & : & \gamma^1 > A^1_2(\bar{\gamma}^2) \qquad \bar{\gamma}^0 - \bar{\gamma}^2 > A^1_2(\bar{\gamma}^2) \\
A^0_* & : & S & : & A^0_1(\bar{\gamma}^1) = A^0_* = A^0_2(\bar{\gamma}^2) < \bar{\gamma}^0, \bar{\gamma}^1 + \bar{\gamma}^2
\end{array}\right.$$

Hence we see that
$$\hat{\gamma}^0(\bar{\gamma}) \in \left\{\bar{\gamma}^0, \quad \bar{\gamma}^1 + \bar{\gamma}^2, \quad A^0_1(\bar{\gamma}^1), \quad A^0_2(\bar{\gamma}^2), \quad A^0_*\right\}$$

i.e. that $\hat{\gamma}^0$ takes only five values, and distinguishing the values on S and outside S, we get also

$$\hat{\gamma}^0(\bar{\gamma}) \ge \min\left\{\bar{\gamma}^0, \quad \bar{\gamma}^1 + \bar{\gamma}^2, \quad A_1^0(\bar{\gamma}^1), \quad A_2^0(\bar{\gamma}^2)\right\}$$

We now need to work out the ordering.
Step 3.1: First remarks using (12.45)
We get in particular

$$(12.55) \qquad \left\{ \begin{array}{lllllll}
\bar{\gamma}^1 + \bar{\gamma}^2 & : & D_0^* & : & \bar{\gamma}^1 + \bar{\gamma}^2 < \bar{\gamma}^0, & & \bar{\gamma}^1 + \bar{\gamma}^2 \boxed{\le} A_1^0(\bar{\gamma}^1), A_2^0(\bar{\gamma}^2) \\
\bar{\gamma}^0 & : & D_1^* & : & \bar{\gamma}^0 < \bar{\gamma}^1 + \bar{\gamma}^2 & & \bar{\gamma}^0 \le A_1^0(\bar{\gamma}^1), A_2^0(\bar{\gamma}^2) \\
\bar{\gamma}^0 & : & D_2^* & : & \bar{\gamma}^0 < \bar{\gamma}^1 + \bar{\gamma}^2 & & \\
\bar{\gamma}^0 & : & H^* & : & \bar{\gamma}^0 = \bar{\gamma}^1 + \bar{\gamma}^2 & & \bar{\gamma}^0 \boxed{\le} A_1^0(\bar{\gamma}^1), A_2^0(\bar{\gamma}^2) \\
 & & & & & & \\
\bar{\gamma}^0 & : & \hat{\Sigma}_0^* & : & \bar{\gamma}^0 < \bar{\gamma}^1 + \bar{\gamma}^2 & & \bar{\gamma}^0 \le A_2^0(\bar{\gamma}^2) \\
A_1^0(\bar{\gamma}^1) & : & \hat{\Sigma}_1^* & : & A_1^0(\bar{\gamma}^1) \boxed{\le} A_2^0(\bar{\gamma}^2) & & \\
 & & & & & & \\
A_2^0(\bar{\gamma}^2) & : & \hat{\Sigma}_2^* & : & \bar{\gamma}^1 + \bar{\gamma}^2, \bar{\gamma}^0 > A_2^0(\bar{\gamma}^2) & & \\
A_*^0 & : & S & : & A_1^0(\bar{\gamma}^1) = A_*^0 = A_2^0(\bar{\gamma}^2) < \bar{\gamma}^0, \bar{\gamma}^1 + \bar{\gamma}^2 & &
\end{array} \right.$$

that we justify below.
Case A: Terms for $\hat{\Sigma}_2^*$
We notice for $\bar{\gamma} \in \hat{\Sigma}_2^*$, we have $\bar{\gamma}^2 \le A_*^2$ and then

$$\bar{\gamma}^2 + A_2^1(\bar{\gamma}^2) = A_2^0(\bar{\gamma}^2)$$

Case B: Terms for $\hat{\Sigma}_1^*$
Case B.1: $\bar{\gamma}^2 \le A_*^2$
Then

$$\bar{\gamma}^1 + \bar{\gamma}^2 \le A_2^1(\bar{\gamma}^2) + \bar{\gamma}^2 = A_2^0(\bar{\gamma}^2)$$

Moreover, by symmetry of the case of $\hat{\Sigma}_2^*$, we then know that

$$\bar{\gamma}^1 + \bar{\gamma}^2 > A_1^0(\bar{\gamma}^1)$$

and then

$$A_1^0(\bar{\gamma}^1) < A_2^0(\bar{\gamma}^2)$$

Case B.2: $\bar{\gamma}^2 > A_*^2$
Then we have

$$A_2^0(\bar{\gamma}^2) = A_*^0 \ge A_1^0(\bar{\gamma}^1)$$

where we have used (12.45). Hence in all Cases B, we deduce that

$$A_2^0(\bar{\gamma}^2) \ge A_1^0(\bar{\gamma}^1).$$

Case C: Term D_1^*
Then $\bar{\gamma}^2 \le A_*^2$ and we get

$$\bar{\gamma}^0 \le \bar{\gamma}^2 + A_2^1(\bar{\gamma}^2) = A_2^0(\bar{\gamma}^2) \boxed{\le} A_*^0$$

where the last inequality follows from (12.45). This also implies that

$$\bar{\gamma}^0 < \bar{\gamma}^1 + \bar{\gamma}^2 \le \bar{\gamma}^1 + A_1^2(\bar{\gamma}^1) = A_1^0(\bar{\gamma}^1) \quad \text{if} \quad \bar{\gamma}^1 \le A_*^1$$

while

$$\bar{\gamma}^0 \le A_*^0 = A_1^0(\bar{\gamma}^1) \quad \text{if} \quad \bar{\gamma}^1 > A_*^1$$

Hence in all cases, we get

$$\bar{\gamma}^0 \boxed{\le} A_1^0(\bar{\gamma}^1)$$

Case D: Term $\hat{\Sigma}_0^*$

We have

$$\bar{\gamma}^0 < \bar{\gamma}^2 + A_0^1(\bar{\gamma}^0) \leq \bar{\gamma}^2 + A_2^1(\bar{\gamma}^2)$$

If $\bar{\gamma}^2 \leq A_*^2$, then we get $\bar{\gamma}^2 + A_2^1(\bar{\gamma}^2) = A_2^0(\bar{\gamma}^2)$. And if $\bar{\gamma}^2 > A_*^2$, then $A_2^0(\bar{\gamma}^2) = A_*^0 \geq \bar{\gamma}^0$ for $\bar{\gamma} \in \hat{\Sigma}_0^*$. Therefore, in all cases, we get

$$\bar{\gamma}^0 \leq A_0^2(\bar{\gamma}^2)$$

Case E: Term $H^* \cup D_0^*$
For $\bar{\gamma} \in H^* \cup D_0^*$, we have

$$\bar{\gamma}^1 \leq A_1^* \quad \text{or} \quad \bar{\gamma}^2 \leq A_2^*$$

We also have $\bar{\gamma}^2 \leq A_1^2(\bar{\gamma}^1)$ and $\bar{\gamma}^1 \leq A_2^1(\bar{\gamma}^2)$. Hence

$$\bar{\gamma}^1 + \bar{\gamma}^2 \leq \left\{ \begin{array}{ll} \bar{\gamma}^1 + A_1^2(\bar{\gamma}^1) = A_1^0(\bar{\gamma}^1) & \text{if} \quad \bar{\gamma}^1 \leq A_*^1 \\ \bar{\gamma}^2 + A_2^1(\bar{\gamma}^2) = A_2^0(\bar{\gamma}^2) & \text{if} \quad \bar{\gamma}^2 \leq A_*^2 \end{array} \right\} \boxed{\leq} A_*^0$$

where we have used (12.45) for the last inequality. Hence this implies in all cases

$$\bar{\gamma}^1 + \bar{\gamma}^2 \boxed{\leq} A_1^0(\bar{\gamma}^1), A_2^0(\bar{\gamma}^2)$$

Step 3.2: conclusion
Using (12.55), and by symmetry in indices 1 and 2, we get

$$\left\{ \begin{array}{lllllll} \bar{\gamma}^1 + \bar{\gamma}^2 & : & D_0^* & : & \bar{\gamma}^1 + \bar{\gamma}^2 < \bar{\gamma}^0, & & \bar{\gamma}^1 + \bar{\gamma}^2 \boxed{\leq} A_1^0(\bar{\gamma}^1), A_2^0(\bar{\gamma}^2) \\ \bar{\gamma}^0 & : & D_1^* & : & \bar{\gamma}^0 < \bar{\gamma}^1 + \bar{\gamma}^2 & & \bar{\gamma}^0 \boxed{\leq} A_1^0(\bar{\gamma}^1), \quad \bar{\gamma}^0 \leq A_2^0(\bar{\gamma}^2) \\ \bar{\gamma}^0 & : & D_2^* & : & \bar{\gamma}^0 < \bar{\gamma}^1 + \bar{\gamma}^2 & & \bar{\gamma}^0 \leq A_1^0(\bar{\gamma}^1), \quad \bar{\gamma}^0 \boxed{\leq} A_2^0(\bar{\gamma}^2) \\ \bar{\gamma}^0 & : & H^* & : & \bar{\gamma}^0 = \bar{\gamma}^1 + \bar{\gamma}^2 & & \bar{\gamma}^0 \boxed{\leq} A_1^0(\bar{\gamma}^1), A_2^0(\bar{\gamma}^2) \\ & & & & & & \\ \bar{\gamma}^0 & : & \hat{\Sigma}_0^* & : & \bar{\gamma}^0 < \bar{\gamma}^1 + \bar{\gamma}^2 & & \bar{\gamma}^0 \leq A_1^0(\bar{\gamma}^1), A_2^0(\bar{\gamma}^2) \\ & & & & & & \\ A_1^0(\bar{\gamma}^1) & : & \hat{\Sigma}_1^* & : & A_1^0(\bar{\gamma}^1) \boxed{\leq} A_2^0(\bar{\gamma}^2), & & A_1^0(\bar{\gamma}^1) < \bar{\gamma}^1 + \bar{\gamma}^2, \bar{\gamma}^0 \\ A_2^0(\bar{\gamma}^2) & : & \hat{\Sigma}_2^* & : & A_2^0(\bar{\gamma}^2) \boxed{\leq} A_1^0(\bar{\gamma}^1), & & A_2^0(\bar{\gamma}^2) < \bar{\gamma}^1 + \bar{\gamma}^2, \bar{\gamma}^0 \\ A_*^0 & : & S & : & A_1^0(\bar{\gamma}^1) = A_*^0 = A_2^0(\bar{\gamma}^2) < \bar{\gamma}^0, \bar{\gamma}^1 + \bar{\gamma}^2 & & \end{array} \right.$$

Hence, under assumption (12.45), we get

$$\hat{\gamma}^0(\bar{\gamma}) = \min \left\{ \bar{\gamma}^0, \bar{\gamma}^1 + \bar{\gamma}^2, A_1^0(\bar{\gamma}^1), A_2^0(\bar{\gamma}^2) \right\}$$

which is (12.46).
Part III: sufficient conditions
We finally have to justify that if $\hat{\gamma}$ is given by expression (12.43), then $\hat{\gamma}$ is a 1:2 conservative preflux. This follows from Lemma 11.32 in the subcase of Case 1) when we assume moreover that

$$A_* \in (0, +\infty)^3 \quad \text{and} \quad A_j : [0, +\infty) \to [0, +\infty)^3 \tag{12.56}$$

The general Case 1), and all other Cases 2),3),4),5) are then obtained as limit (for the uniform convergence on compact sets of $[0, +\infty)^3$) of prefluxes in Case 1) satisfying (12.56). Notice that the limit $\hat{\gamma}$ remains continuous, and that the local constancy property is a closed property under such limits. This shows that $\hat{\gamma}$ is still a 1:2 conservative preflux at the limit. This ends the proof of the theorem.

Lemma 12.9 (Sanity check for a quasi Hamilton-Jacobi preflux)
Given $\theta^1, \theta^2 \in (0,1)$ with $\theta^1 + \theta^2 = 1$, let us consider the following quasi Hamilton-Jacobi preflux $\hat{\gamma} = (\hat{\gamma}^0, \hat{\gamma}^1, \hat{\gamma}^2) : [0, +\infty)^N \to [0, +\infty)^N$ for $N = 3$ defined by

$$\left\{ \begin{array}{l} \hat{\gamma}^0(\bar{\gamma}) = \min \left\{ \bar{\gamma}^0, \dfrac{\bar{\gamma}^1}{\theta^1}, \dfrac{\bar{\gamma}^1}{\theta^1} \right\}, \\ \hat{\gamma}^j = \theta^j \hat{\gamma}^0, \quad j = 1, 2 \end{array} \right. \tag{12.57}$$

Then $\hat{\gamma}$ has expression (12.43) with

$$\text{(12.58)}\qquad \begin{cases} A_0^j(\bar{\gamma}^0) = \theta^j \bar{\gamma}^0, \quad j = 1,2, \\ A_2^1(\bar{\gamma}^2) = \dfrac{\theta^1}{\theta^2}\bar{\gamma}^2, \\ \\ A_1^2(\bar{\gamma}^1) = \dfrac{\theta^2}{\theta^1}\bar{\gamma}^1, \end{cases}$$

and the curves $\Gamma_0, \Gamma_1, \Gamma_2$ defined in Theorem 12.7 do coincide.

Proof of Lemma 12.9
Step 1: preliminaries
Let us call $\hat{\gamma}_0$ the preflux given by expression (12.43) with (12.58). We define $\theta^0 := 1$ and

$$\begin{cases} \breve{\gamma}^j := \dfrac{\bar{\gamma}^j}{\theta^j}, \\ \check{\gamma}^j := (\theta^j)^{-1}\hat{\gamma}^j(\bar{\gamma}), \\ \check{\gamma}_0^j := (\theta^j)^{-1}\hat{\gamma}_0^j(\bar{\gamma}), \end{cases} \qquad j = 0,1,2$$

and

$$X_j := \frac{\breve{\gamma}^0 - \theta^j \breve{\gamma}^j}{1-\theta^j}, \quad j = 1,2$$

From (12.57), we have

$$\check{\gamma}^j = \min\left\{\breve{\gamma}^0, \breve{\gamma}^1, \breve{\gamma}^2\right\}, \quad j = 0,1,2$$

and

$$\begin{cases} \check{\gamma}_0^1 = \min\left\{\breve{\gamma}^1, \breve{\gamma}^2, \max\left\{\breve{\gamma}^0, X_2\right\}\right\}, \\ \check{\gamma}_0^2 = \min\left\{\breve{\gamma}^1, \breve{\gamma}^2, \max\left\{\breve{\gamma}^0, X_1\right\}\right\}, \\ \check{\gamma}_0^0 = \theta^1 \check{\gamma}_0^1 + \theta^2 \check{\gamma}_0^2 \end{cases}$$

Step 2: comparison of $\check{\gamma}$ and $\check{\gamma}_0$
Let

$$m := \min\left\{\breve{\gamma}^1, \breve{\gamma}^2\right\}$$

Case A: $\breve{\gamma}^0 \geq m$
Then

$$\check{\gamma}_0^1 = m = \check{\gamma}_0^2 = \check{\gamma}_0^0$$

Case B: $\breve{\gamma}^0 < m$
Then notice that

$$\breve{\gamma}^0 \geq X_j \quad \Longleftrightarrow \quad \breve{\gamma}^j \geq \breve{\gamma}^0, \quad j = 1,2$$

where the right hand side is true, because we assume $\breve{\gamma}^0 < m$. Therefore

$$\max\left\{\breve{\gamma}^0, X_j\right\} = \breve{\gamma}^0, \quad j = 1,2$$

and

$$\check{\gamma}_0^1 = \breve{\gamma}^0 = \check{\gamma}_0^2 = \check{\gamma}_0^0$$

Conclusion
We have shown that $\check{\gamma} = \check{\gamma}_0$, and this ends the proof of the lemma.

12.5 Explicit characterization of conservative germs for $1:2$ junctions

Lemma 12.10 (Explicit characterization of conservative germs, in a special case)
Let us consider a conservative preflux $\hat{\gamma} : [0,+\infty)^3 \to [0,+\infty)^3$ for $1:2$ junctions (i.e. with $\sigma_0 = (\sigma_0^0, \sigma_0^1, \sigma_0^2) = (+1,-1,-1)$). Assume the following bound

$$\text{(12.59)}\qquad \left\{\hat{\gamma} = id_{[0,+\infty)^3}\right\} \subset [0, R'], \quad \text{with} \quad 0 \leq R' := (R^{0\prime}, R^{1\prime}, R^{2\prime}) < R := (R^0, R^1, R^2)$$

Consider bell-shaped functions $f^j : [a^j, b^j] \to [0,+\infty)$ in the sense of Definition 11.6, such that

$$f^j_{\max} = f^j(c^j) = R^{j\prime}, \quad j = 0,1,2$$

Then the following set for $p = (p^0, p^1, p^2)$

$$\mathcal{G} := \left\{ p \in [a,b], \quad \hat{f}(p) = f(p) \right\} \quad \text{with} \quad \hat{f} := \hat{\gamma} \circ \bar{\gamma} \quad \text{and} \quad \bar{\gamma}(p) = (f^{0,+}(p^0), f^{1,-}(p^1), f^{2,-}(p^2))$$

is a conservative Riemann germ, where

$$f^{j,+}(p^j) = f^j(\min\left\{p^j, c^j\right\}) \quad \text{and} \quad f^{j,-}(p^j) = f^j(\max\left\{p^j, c^j\right\})$$

i) (Notation)
Let (e_0, e_1, e_2) *be a basis of* $\mathbb{R}^3$ *and* $Q' := [0,+\infty)e_1 + [0,+\infty)e_2$. *Now following statement of Lemma 12.5, let us consider the three curves* $\Gamma_0, \Gamma_1, \Gamma_2 \subset Q'$ *with merge at the point*

$$\Gamma_0 \cap \Gamma_1 \cap \Gamma_2 = A'_* := (A^1_*, A^2_*) \in Q'$$

and the three closed faces F_0, F_1, F_2 *with* $F_0 = F_1 \cup F_2$ *and* $F_1 \cap F_2 = \Gamma_0$ *and*

$$\partial F_j \subset \Gamma_0 \cup \Gamma_j \cup [0,+\infty)e_{\bar{j}}, \quad j = 1,2 \quad \text{with} \quad \bar{j} \in \{1,2\} \setminus \{j\}$$

We then define the lifting from $[0,+\infty)^2$ *to the hyperplane* $\left\{\bar{\gamma}^0 = \bar{\gamma}^1 + \bar{\gamma}^2\right\}$ *as*

$$\begin{array}{rccc} \iota: & [0,+\infty)^2 & \to & [0,+\infty)^3 \\ & (\bar{\gamma}^1, \bar{\gamma}^2) & \mapsto & (\bar{\gamma}^1 + \bar{\gamma}^2, \bar{\gamma}^1, \bar{\gamma}^2) \end{array}$$

and set

$$\left\{ \begin{array}{ll} \tilde{\Gamma}_j := \iota(\Gamma_j), & j = 0,1,2, \\ \tilde{F}_j := \iota(F_j), & j = 0,1,2 \\ A_* := \iota(A'_*), & \end{array} \right.$$

Now we define the three "reflexion" maps for $j = 0,1,2$

$$\begin{array}{rccc} \tilde{\tau}_j: & [a^j, b^j] & \to & [a^j, b^j] \\ & p^j & \mapsto & \left\{ \begin{array}{ll} (f^{j,-})^{-1}(f^{j,+}(p^j)) & \text{if} \quad p^j \in [a^j, c^j] \\ (f^{j,+})^{-1}(f^{j,-}(p^j)) & \text{if} \quad p^j \in [c^j, b^j] \end{array} \right. \end{array}$$

and the associated three "reflexion" maps for $j = 0,1,2$

$$\begin{array}{rccc} \tau_j: & [a,b] & \to & [a,b] \\ & p & \mapsto & p + (\tilde{\tau}_j(p^j) - p^j)e_j \end{array}$$

Now for $\sigma = (\sigma^0, \sigma^1, \sigma^2) \in \left\{\pm 1\right\}^3$, *we define the map*

$$\tau^\sigma := \tau_0^{n_0} \tau_1^{n_1} \tau_2^{n_2} \quad \text{with} \quad n_j := \left\{ \begin{array}{ll} 0 & \text{if} \quad \sigma^j = +1 \\ 1 & \text{if} \quad \sigma^j = -1 \end{array} \right. , \quad j = 0,1,2$$

and for a set $B \subset [a,b]$, *we define*

$$B^\sigma := \tau^\sigma(B)$$

ii) (Explicit characterization of $\mathcal{G}$)
We set

$$Q := \prod_{j=0,1,2} [a^j, c^j] = [a,c]$$

Then

$$\mathcal{G} \cap Q^\sigma = \left\{ \begin{array}{ll} f^{-1}(\{A_*\}) \cap Q^\sigma & \text{for} \quad \sigma = (-;+,+) \\ f^{-1}(\tilde{\Gamma}_0) \cap Q^\sigma & \text{for} \quad \sigma = (+;+,+) \\ & \\ f^{-1}(\tilde{\Gamma}_1) \cap Q^\sigma & \text{for} \quad \sigma = (-;-,+) \\ f^{-1}(\tilde{\Gamma}_2) \cap Q^\sigma & \text{for} \quad \sigma = (-;+,-) \\ & \\ f^{-1}(\tilde{F}_1) \cap Q^\sigma & \text{for} \quad \sigma = (+;-,+) \\ f^{-1}(\tilde{F}_2) \cap Q^\sigma & \text{for} \quad \sigma = (+;+,-) \\ & \\ f^{-1}(\tilde{F}_0) \cap Q^\sigma & \text{for} \quad \sigma = (+;-,-) \\ f^{-1}(\tilde{F}_0) \cap Q^\sigma & \text{for} \quad \sigma = (-;-,-) \end{array} \right.$$

iii) (Extension to the case $R = R'$)
Points i) and ii) are still valid if $R' = R$.

Remark 12.11 *Notice that point iii) with $\hat{\gamma}([0,+\infty)^3) \subset [0,R]$ is not a restriction in general, because we can always replace $\hat{\gamma}$ by the preflux $\hat{\gamma} \circ T_R$ where T_R is the truncation preflux defined in (11.15). In particular, in such a case, we see that the infinite possible values of the functions A_j, can be replaced by finite ones and do not matter.*

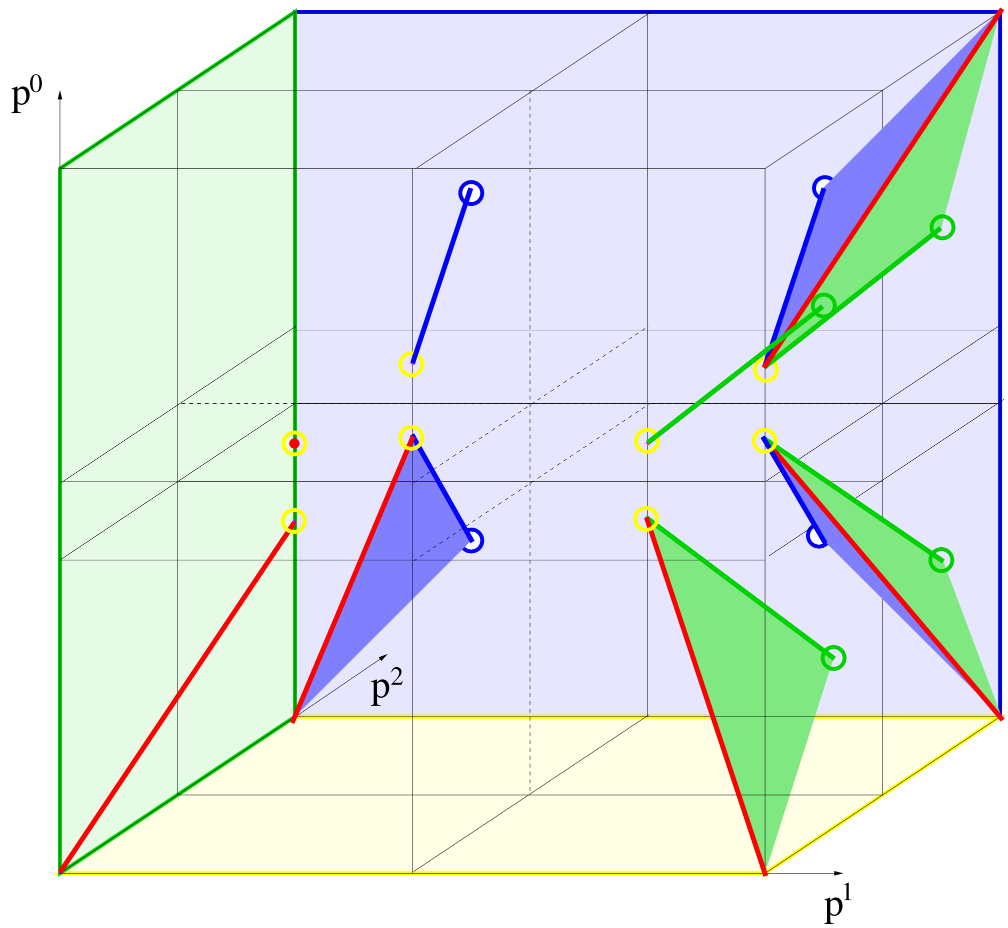

Figure 9: Sketch of a generic conservative germ for 1 : 2 junctions

Proof of Lemma 12.10
We use notation of Lemma 12.5.
Step 1: preliminaries
From Lemma 12.5, recall that

$$\hat{\gamma}([0,+\infty)^3) \subset H^* = \tilde{F}_0 \subset [0,R'] \subset [0,R]$$

Notice that $f \circ \tau^\sigma = f$ on $[a,b]$. Then we get for $p \in Q^\sigma = [a,c]^\sigma$

$$\bar{\gamma}(p) = f^{(+;-,-)}(p) = \begin{cases} f(p) & \text{if} \quad \sigma = (+;-,-) \\ \\ (R^0, f^1, f^2)(p) & \text{if} \quad \sigma = (-;-,-) \\ (f^0, R^1, f^2)(p) & \text{if} \quad \sigma = (+;+,-) \\ (f^0, f^1, R^2)(p) & \text{if} \quad \sigma = (+;-,+) \\ \\ (f^0, R^1, R^2)(p) & \text{if} \quad \sigma = (+;+,+) \\ (R^0, f^1, R^2)(p) & \text{if} \quad \sigma = (-;-,+) \\ (R^0, R^1, f^2)(p) & \text{if} \quad \sigma = (-;+,-) \\ \\ (R^0, R^1, R^2) & \text{if} \quad \sigma = (-;+,+) \end{cases} \tag{12.60}$$

Now for $\lambda \in [0, R]$, we define the quantity

$$(12.61)\qquad \bar{\lambda}^\sigma := \hat{\tau}^\sigma(\lambda) := \begin{cases} \lambda & \text{if} \quad \sigma = (+;-,-) \\ \\ (R^0, \lambda^1, \lambda^2) & \text{if} \quad \sigma = (-;-,-) \\ (\lambda^0, R^1, \lambda^2) & \text{if} \quad \sigma = (+;+,-) \\ (\lambda^0, \lambda^1, R^2) & \text{if} \quad \sigma = (+;-,+) \\ \\ (\lambda^0, R^1, R^2) & \text{if} \quad \sigma = (+;+,+) \\ (R^0, \lambda^1, R^2) & \text{if} \quad \sigma = (-;-,+) \\ (R^0, R^1, \lambda^2) & \text{if} \quad \sigma = (-;+,-) \\ \\ (R^0, R^1, R^2) & \text{if} \quad \sigma = (-;+,+) \end{cases}$$

Then for $p = \tau^\sigma(q)$ with $q \in Q$, we see from (12.60) that

$$(12.62)\qquad \bar{\gamma}(\tau^\sigma(q)) = \bar{\gamma}(p) = \hat{\tau}^\sigma(\lambda) = \bar{\lambda}^\sigma \quad \text{with} \quad \lambda := f(q)$$

Let us now introduce the quantity $c^{j\prime}$ uniquely defined by

$$f^j(c^{j\prime}) = R^{j\prime} < R^j, \quad c^{j\prime} \in [a^j, c^j), \quad j = 0, 1, 2$$

and

$$Q' := [a, c']$$

Then for any $q \in Q'$ and $\lambda := f(q)$, we have $\lambda \leq \bar{\lambda}^\sigma$, and moreover (because $R' < R$)

$$\text{for } j = 0, 1, 2, \text{ we have:} \qquad (\bar{\lambda}^\sigma)^j = \lambda^j \quad \text{if and only if} \quad \sigma^j = \sigma_0^j$$

i.e.

$$\text{for all } q \in Q' \text{ and } \lambda := f(q), \quad \bar{\lambda}^\sigma \in \begin{cases} \{\hat{\gamma}(\bar{\gamma}) = \bar{\gamma}\} = H^* & \text{if} \quad \sigma = (+;-,-) \\ \\ \left\{\hat{\gamma}^0 < \bar{\gamma}^0,\ \hat{\gamma}^1 = \bar{\gamma}^1,\ \hat{\gamma}^2 = \bar{\gamma}^2\right\} = D_0^* & \text{if} \quad \sigma = (-;-,-) \\ \left\{\hat{\gamma}^0 = \bar{\gamma}^0,\ \hat{\gamma}^1 < \bar{\gamma}^1,\ \hat{\gamma}^2 = \bar{\gamma}^2\right\} = D_1^* & \text{if} \quad \sigma = (+;+,-) \\ \left\{\hat{\gamma}^0 = \bar{\gamma}^0,\ \hat{\gamma}^1 = \bar{\gamma}^1,\ \hat{\gamma}^2 < \bar{\gamma}^2\right\} = D_2^* & \text{if} \quad \sigma = (+;-,+) \\ \\ \left\{\hat{\gamma}^0 = \bar{\gamma}^0,\ \hat{\gamma}^1 < \bar{\gamma}^1,\ \hat{\gamma}^2 < \bar{\gamma}^2\right\} = \hat{\Sigma}_0^* & \text{if} \quad \sigma = (+;+,+) \\ \left\{\hat{\gamma}^0 < \bar{\gamma}^0,\ \hat{\gamma}^1 = \bar{\gamma}^1,\ \hat{\gamma}^2 < \bar{\gamma}^2\right\} = \hat{\Sigma}_1^* & \text{if} \quad \sigma = (-;-,+) \\ \left\{\hat{\gamma}^0 < \bar{\gamma}^0,\ \hat{\gamma}^1 < \bar{\gamma}^1,\ \hat{\gamma}^2 = \bar{\gamma}^2\right\} = \hat{\Sigma}_2^* & \text{if} \quad \sigma = (-;+,-) \\ \\ \{\hat{\gamma}(\bar{\gamma}) < \bar{\gamma}\} = S & \text{if} \quad \sigma = (-;+,+) \end{cases}$$

From the expression of $\hat{\gamma}$ in Lemma 12.5 and $H^* \subset [0, R']$, we deduce that

$$(12.63)\qquad \text{for all} \quad \lambda \in H^*, \quad \hat{\gamma}(\bar{\lambda}^\sigma) = \begin{cases} \lambda & \in \tilde{F}_0 & =: M_\sigma & \text{if} \quad \sigma = (+;-,-) \\ \\ \tilde{\gamma}_0(\bar{\lambda}^\sigma) = \tilde{\gamma}_0(\lambda) & \in \tilde{F}_0 & =: M_\sigma & \text{if} \quad \sigma = (-;-,-) \\ \tilde{\gamma}_1(\bar{\lambda}^\sigma) = \tilde{\gamma}_1(\lambda) & \in \tilde{F}_1 & =: M_\sigma & \text{if} \quad \sigma = (+;+,-) \\ \tilde{\gamma}_2(\bar{\lambda}^\sigma) = \tilde{\gamma}_2(\lambda) & \in \tilde{F}_2 & =: M_\sigma & \text{if} \quad \sigma = (+;-,+) \\ \\ A_0((\bar{\lambda}^\sigma)^0) = A_0(\lambda^0) & \in \tilde{\Gamma}_0 & =: M_\sigma & \text{if} \quad \sigma = (+;+,+) \\ A_1((\bar{\lambda}^\sigma)^1) = A_0(\lambda^1) & \in \tilde{\Gamma}_1 & =: M_\sigma & \text{if} \quad \sigma = (-;-,+) \\ A_2((\bar{\lambda}^\sigma)^2) = A_0(\lambda^2) & \in \tilde{\Gamma}_2 & =: M_\sigma & \text{if} \quad \sigma = (-;+,-) \\ \\ A_* = \hat{\gamma}(A_*) & \in \{A_*\} & =: M_\sigma & \text{if} \quad \sigma = (-;+,+) \end{cases}$$

Furthermore, because $\hat{\gamma}(\lambda) = \lambda$ for $\lambda \in H^* = \tilde{F}_0 \supset M_\sigma$ for all σ, we deduce from (12.63)

$$(12.64)\qquad \hat{\gamma}(\bar{\lambda}^\sigma) = \hat{\gamma}(\lambda) = \lambda \quad \text{for all} \quad \lambda \in M_\sigma$$

Step 2: application
We have

$$\mathcal{G}_\sigma := \mathcal{G} \cap Q^\sigma = \{p \in Q^\sigma, \quad (\hat{\gamma} \circ \bar{\gamma})(p) = f(p)\}$$

i.e. using $p = \tau^\sigma(q)$ and $\bar{\lambda}^\sigma$ defined in (12.61) with $\lambda := f(q)$, we deduce from (12.62)

$$(\tau^\sigma)^{-1}(\mathcal{G}_\sigma) = \left\{q \in Q, \quad \lambda := f(q), \quad \hat{\gamma}(\bar{\lambda}^\sigma) = \lambda\right\} \subset \hat{\gamma}([0,+\infty)^3) = H^* \subset [0, R']$$

i.e.

$$(\tau^\sigma)^{-1}(\mathcal{G}_\sigma) = \left\{q \in Q', \quad \lambda := f(q) \in H^*, \quad \hat{\gamma}(\bar{\lambda}^\sigma) = \lambda\right\} = f^{-1}(M_\sigma) \cap Q$$

where we have used (12.63) and (12.64) for the last equality. Therefore

$$\mathcal{G}_\sigma = f^{-1}(M_\sigma) \cap \tau^\sigma(Q)$$

which shows point ii).
Step 3: proof of iii)
It is easy to see that the result iii) follows from the limit $R' < R \to R'$.
This ends the proof of the lemma.

12.6 Proposition 2.47 and its proof: intersections of conservative lines

We first prove two lemmata, and then deduce Proposition 2.47 at the very end of this subsection.

Lemma 12.12 (Intersection of two conservative lines for bell-shaped fluxes)
Assume (2.2) for $N = 4$ with $2:2$ junctions, and call f^{jL} and f^{jR} respectively the j-th ingoing and j-the outgoing fluxes for $j = 1,2$. Assume that each $f^{j\alpha}$ is strictly concave with maximum at $c^{j\alpha} \in (a^{j\alpha}, b^{j\alpha})$ and $f^{j\alpha}(a^{j\alpha}) = 0 = f^{j\alpha}(b^{j\alpha})$. Let $\mathcal{G} \subset [a,b] \subset \mathbb{R}^4$ be a Riemann germ satisfying for $p = (p^{1L}, p^{1R}, p^{2L}, p^{2R}) = (p^1, p^2)$

$$\text{(12.65)} \qquad \mathcal{G} \subset \bigcap_{j=1,2} \Sigma^j, \quad \text{with} \quad \Sigma^j := \left\{p \in [a,b], \quad f^{jL}(p^{jL}) = f^{jR}(p^{jR})\right\}$$

which means that each j-th line $J^{jL} \cup \{0\} \cup J^{jR} \simeq \mathbb{R}$ is conservative. We set

$$\begin{cases} G^j(p^j) := \min\left\{f^{jL,+}(p^{jL}), f^{jR,-}(p^{jR})\right\} \\ G^j_{\max} := G^j(c^{jL}, c^{jR}) \\ \bar{j} \in \{1,2\} \setminus \{j\} \end{cases}$$

Let $\hat{f} = (\hat{f}^{1L}, \hat{f}^{1R}, \hat{f}^{2L}, \hat{f}^{2R})$ be the Godunov flux associated to the Riemann germ $\mathcal{G}$.
i) (Characterization of Godunov flux)
Then there exists two constants $G^j_0 \in [0, G^j_{\max}]$ and two continuous functions $A^j : [0, G^{\bar{j}}_{\max}] \to [0, G^j_{\max}]$ for $j = 1,2$, such that

$$\text{(12.66)} \qquad \begin{cases} \hat{f}^{1L}(p) = \hat{f}^{1R}(p) = \min\left\{G^1, A^1(G^2)\right\} & \text{with} \quad G^j = G^j(p^j), \quad j = 1,2 \\ \hat{f}^{2L}(p) = \hat{f}^{2R}(p) = \min\left\{A^2(G^1), G^2\right\} \\ A^1 = const = G^1_0 \quad \text{on} \quad (G^2_0, G^2_{\max}] \\ A^2 = const = G^2_0 \quad \text{on} \quad (G^1_0, G^1_{\max}] \\ \\ \left\{G^1 > A^1(G^2)\right\} \cap \left\{G^2 > A^2(G^1)\right\} = \left\{G^1 > G^1_0, \ G^2 > G^2_0\right\} \end{cases}$$

where the last line is a geometric compatibility condition on the intersection of epigraphs (that can be empty). Moreover G^j_0 and A^j are unique.
Conversely, if $\hat{f}$ satisfies (12.66), then $\mathcal{G} := \left\{\hat{f} = f\right\}$ is a Riemann germ satisfying (12.65).
ii) (Identification of the monotone case)
Let $\mathcal{G}$ be a Riemann germ as in i). Then $\mathcal{G}$ is monotone if and only if the functions A^j can be chosen constant for $j = 1,2$.
iii) (Identification of Kružkov case)
Moreover the germ $\mathcal{G}$ is Kružkov if an only if each of the functions A^1 and A^2 can be chosen constant.

Remark 12.13 *Notice that if $G_0^j = G_{\max}^j$, then $(G_0^j, G_{\max}^j] = \emptyset$ and the condition for $A^{\bar{j}}$ is empty.*

Proof of Lemma 12.12
Step 1: first identification of the flux
Let us freeze $p^2 = p_0^2$. Then assumption (12.65) implies that the flux

$$p^1 \mapsto (\hat{f}^{1L}(p^1, p_0^2), \hat{f}^{1R}(p^1, p_0^2))$$

is a flux associated to a conservative germ, hence also a Hamilton-Jacobi germ for 1 : 1 junction. From [29] for convex fluxes (and then also for concave fluxes), we know that such a germ is fully characterized by a single real number A^1, called a flux limiter (see also Proposition 12.1 for charcaterization of conservative prefluxes, joint to the polar decomposition Theorem 11.8). Precisely, we have
(12.67)

$$\hat{f}^{1L}(p^1, p_0^2) = \hat{f}^{1R}(p^1, p_0^2) = \min\left\{\bar{\gamma}^{1L}, \bar{\gamma}^{1R}, A^1\right\} = \min\left\{G^1(p^1), A^1\right\} \quad \text{with} \quad \left\{ \begin{array}{l} \bar{\gamma}^{1L}(p^1) := f^{1L,+}(p^{1L}) \\ \bar{\gamma}^{1R}(p^1) := f^{1R,-}(p^{1R}) \end{array} \right.$$

where the constant A^1 depends on $p_0^2 = p^2$. We get a similar expression for $\hat{f}^{2L} = \hat{f}^{2R}$ and a constant A^2 depending on p^1.
Step 2: using bell-shaped fluxes
Notice that all fluxes $f^{j\alpha}$ have bell-shaped. Hence from Theorem 11.8 on polar decomposition, we know that

$$\hat{f} = \hat{\gamma} \circ \bar{\gamma}$$

with a preflux function $\hat{\gamma} : [0, +\infty)^4 \to [0, +\infty)^4$ and the capacity $\bar{\gamma} = (\bar{\gamma}^1, \bar{\gamma}^2)$ where $\bar{\gamma}^j := (\bar{\gamma}^{jL}, \bar{\gamma}^{jR})$ is defined as in (12.67). This implies that

$$A^j = \tilde{A}^j(\bar{\gamma}^{\bar{j}})$$

Step 3: intersection of the epigraphs
The proof is now a variant of Step 2 of the proof of Proposition 12.1. We consider the following set (also modelled on the last line of (12.66))

$$\tilde{S} := \left\{ \bar{\gamma} \in \tilde{\Gamma}, \quad \left| \begin{array}{l} \min\left\{\bar{\gamma}^{1L}, \bar{\gamma}^{1R}\right\} > \tilde{A}^1(\bar{\gamma}^2) \\ \min\left\{\bar{\gamma}^{2L}, \bar{\gamma}^{2R}\right\} > \tilde{A}^2(\bar{\gamma}^1) \end{array} \right. \right\} \quad \text{with} \quad \tilde{\Gamma} := \prod_{j=1,2} [0, f^{jL}(c^{jL})] \times [0, f^{jR}(c^{jR})]$$

Recall that by definition, the preflux $\hat{\gamma}$ is then locally constant on $\left\{\hat{\gamma} \neq Id_{[0,+\infty)^4}\right\}$. This implies that

$$\hat{\gamma}(\bar{\gamma}) = (\tilde{A}^1(\bar{\gamma}^2), \tilde{A}^1(\bar{\gamma}^2), \tilde{A}^2(\bar{\gamma}^1), \tilde{A}^2(\bar{\gamma}^1)) < \bar{\gamma} \quad \text{for all} \quad \bar{\gamma} \in \tilde{S}$$

and then $\hat{\gamma}$ is locally constant on $\tilde{S}$. Then $\tilde{A}(\bar{\gamma}) := (\tilde{A}^1(\bar{\gamma}^2), \tilde{A}^2(\bar{\gamma}^1))$ is also locally constant on $\tilde{S}$. In particular $\tilde{A}$ is constant on each connected component of $\tilde{S}$.
Then we have two cases.
Case A: $\tilde{S} \neq \emptyset$
If $\bar{\gamma} \in \tilde{S}$, then $\tilde{A}$ is locally constant and we deduce that

$$\tilde{\Gamma} \cap (\bar{\gamma} + [0, +\infty)^4) \subset \tilde{S}$$

Then $\tilde{S}$ is connected and $\tilde{A}$ is constant on $\tilde{S}$. This implies that

$$\tilde{S} = \left\{ \bar{\gamma} \in \tilde{\Gamma}, \quad \left| \begin{array}{l} \min\left\{\bar{\gamma}^{1L}, \bar{\gamma}^{1R}\right\} > (\tilde{A}^1 \circ \bar{\gamma}^2)(c^{2L}, c^{2R}) =: G_0^1 \\ \min\left\{\bar{\gamma}^{2L}, \bar{\gamma}^{2R}\right\} > (\tilde{A}^2 \circ \bar{\gamma}^1)(c^{1L}, c^{1R}) =: G_0^2 \end{array} \right. \right\}$$

which implies

$$0 \le G_0^j < G_{\max}^j := \min\left\{f^{jL}(c^{jL}), f^{jR}(c^{jR})\right\}$$

Case B: $\tilde{S} = \emptyset$
Then, for later use, we set

(12.68) $$G_0^j := G_{\max}^j \quad \text{for} \quad j = 1, 2$$

Step 4: testing a condition

Let

$$\tilde{S}^2 := \left\{ \bar\gamma \in \tilde\Gamma, \quad \left| \begin{array}{l} \min\left\{\bar\gamma^{1L},\bar\gamma^{1R}\right\} > \tilde{A}^1(\bar\gamma^2) \\ \min\left\{\bar\gamma^{2L},\bar\gamma^{2R}\right\} \le \tilde{A}^2(\bar\gamma^1), \end{array} \right. \right\} \quad \text{and} \quad \tilde{S}^2_* := \left\{\bar\gamma \in \tilde{S}^2, \quad \bar\gamma^{2L} \neq \bar\gamma^{2R}\right\}$$

Then for $j=2$ and for $\bar\gamma \in \tilde{S}^2_*$, we get that

$$\hat\gamma^{1L}(\bar\gamma) = \tilde{A}^1(\bar\gamma^2)$$

is locally constant in $\max\left\{\bar\gamma^{2L},\bar\gamma^{2R}\right\}$, and then only depends on $G^2 := \min\left\{\bar\gamma^{2L},\bar\gamma^{2R}\right\}$. For $\bar\gamma^1$ fixed, we deduce by continuity that $\tilde{A}^1$ only depends on $G^2 := \min\left\{\bar\gamma^{2L},\bar\gamma^{2R}\right\}$. Hence

$$\tilde{A}^1(\bar\gamma^2) =: A^1(G^2) \quad \left\{ \begin{array}{llll} & & \text{if} \quad \tilde{S}^2 \neq \emptyset & \\ \text{with} & A^1 \equiv G^1_0 & \text{if} \quad \tilde{S}^2 = \emptyset & \text{and } \tilde{S} \neq \emptyset \\ \text{with} & A^1 \equiv G^1_{\max} & \text{if} \quad \tilde{S}^2 = \emptyset & \text{and } \tilde{S} = \emptyset \end{array} \right.$$

and similarly for $G^1 := \min\left\{\bar\gamma^{1L},\bar\gamma^{1R}\right\}$, we have

$$\tilde{A}^2(\bar\gamma^1) =: A^2(G^1) \quad \left\{ \begin{array}{llll} & & \text{if} \quad \tilde{S}^1 \neq \emptyset & \\ \text{with} & A^2 \equiv G^2_0 & \text{if} \quad \tilde{S}^1 = \emptyset & \text{and } \tilde{S} \neq \emptyset \\ \text{with} & A^2 \equiv G^2_{\max} & \text{if} \quad \tilde{S}^1 = \emptyset & \text{and } \tilde{S} = \emptyset \end{array} \right.$$

Step 5: conclusion
In all cases, we have

$$\left\{ \begin{array}{l} \hat\gamma^{1\alpha}(\bar\gamma) = \min\left\{G^1, A^1(G^2)\right\} \\ \hat\gamma^{2\alpha}(\bar\gamma) = \min\left\{A^2(G^1), G^2\right\} \end{array} \right. \quad \text{with} \quad G^j := \min\left\{\bar\gamma^{jL},\bar\gamma^{jR}\right\}$$

and

$$\tilde{S} = \left\{ \bar\gamma \in \tilde\Gamma, \quad \left| \begin{array}{l} G^1 > A^1(G^2) \\ G^2 > A^2(G^1) \end{array} \right. \right\} = \left\{ \bar\gamma \in \tilde\Gamma, \quad \left| \begin{array}{l} G^1 > G^1_0 \\ G^2 > G^2_0 \end{array} \right. \right\}$$

for some $G^j_0 \in [0, G^j_{\max}]$. On the model of $\tilde{S}$, we now define the reduced sets

$$S := \left\{ (G^1,G^2) \in \Gamma, \quad \left| \begin{array}{l} G^1 > A^1(G^2) \\ G^2 > A^2(G^1) \end{array} \right. \right\} = \left\{ (G^1,G^2) \in \Gamma, \quad \left| \begin{array}{l} G^1 > G^1_0 \\ G^2 > G^2_0 \end{array} \right. \right\} \quad \text{with} \quad \Gamma := [0,G^1_{\max}]\times[0,G^2_{\max}]$$

and

$$S^2 := \left\{ (G^1,G^2) \in \Gamma, \quad \left| \begin{array}{l} G^1 > A^1(G^2) \\ G^2 \le A^2(G^1), \end{array} \right. \right\}$$

and

$$S^1 := \left\{ (G^1,G^2) \in \Gamma, \quad \left| \begin{array}{l} G^1 \le A^1(G^2) \\ G^2 > A^2(G^1), \end{array} \right. \right\}$$

Case A: $S \neq \emptyset$
Then we know that $G^j_0 < G^j_{\max}$ and

$$\left\{ \begin{array}{lll} A^1 = const = G^1_0 & \text{on} & (G^2_0, G^2_{\max}] \\ A^2 = const = G^2_0 & \text{on} & (G^1_0, G^1_{\max}] \end{array} \right.$$

which shows (12.66).
Case B: $S = \emptyset$
Hence

$$\left\{ \begin{array}{ll} \left\{G^1 > A^1(G^2)\right\} \subset \left\{G^2 \le A^2(G^1)\right\} & \text{and then} \quad S^2 \neq \emptyset \\ \left\{G^2 > A^2(G^1)\right\} \subset \left\{G^1 \le A^1(G^2)\right\} & \text{and then} \quad S^1 \neq \emptyset \end{array} \right.$$

Then the choice (12.68) insures that (12.66) also holds true.
Step 6: conversely
Conversely, assume that $\hat f$ is given by (12.66), hence with in particular for $G^j = G^j(p^j_0)$

$$\left\{ \begin{array}{l} \hat f^{1L}(p) = \hat f^{1R}(r) = \min\left\{G^1, A^1(G^2)\right\} \\ \hat f^{2L}(p) = \hat f^{2R}(r) = \min\left\{G^2, A^2(G^1)\right\} \end{array} \right.$$

Let us check that $\hat{f}$ is locally constant. Fix p_0 such that

$$\hat{f}^{1R}(p_0) = \min\left\{G^1, A^1(G^2)\right\} < f^{1R}(p_0^{1R}) \quad \text{with} \quad G^1 = \min\left\{f^{1L,+}(p_0^{1L}), f^{1R,-}(p_0^{1R})\right\}$$

then let us show that $\hat{f}$ is locally constant in p^{1R}. Because $f^{1R}(p_0^{1R}) \le f^{1R,-}(p_0^{1R})$, we deduce that

$$\min\left\{G^1, A^1(G^2)\right\} < f^{1R,-}(p_0^{1R})$$

Case A: $f^{1L,+}(p_0^{1L}) < f^{1R,-}(p_0^{1R})$
Then G^1 is locally constant in p^{1R}, and this is also the case of $\hat{f}$.
Case B: $f^{1L,+}(p_0^{1L}) \ge f^{1R,-}(p_0^{1R})$
Then

$$G^1 > A^1(G^2)$$

and $\hat{f}^{1\alpha}$ is locally constant in p^{1R}.
Case B.1: $G^2 < A^2(G_1)$
Then $\hat{f}^{2\alpha}$ is locally constant in p^{1R}.
Case B.2: $G^2 > A^2(G_1)$
Then $(G^1, G^2) \in S$. Because A^2 is constant on S, we deduce that $\hat{f}^{2\alpha}$ is locally constant in p^{1R}.
Case B.3: $G^2 = A^2(G_1)$
For frozen $p^2 := p_0^2$ and $p^{1L} := p_0^{1L}$, if $p^{1R} \mapsto \hat{f}^{2L}(p) = \hat{f}^{2R}(p)$ is not locally constant, then $A^2(G_1)$ with $G^1 := G^1(p_0^{1L}, p^{1R})$ takes a different value than $G^2(p_0^2)$. Hence we enter in Cases B.1 or B.2. But in each of those two cases, this implies that $\hat{f}^{2L}$ is locally constant in p^{1R}. By continuity of $\hat{f}^{1L}$, we deduce that $\hat{f}^{1L} = \hat{f}^{1R}$ is still locally constant in Case B.3.
Notice that the same reasoning also shows that

$$\begin{cases} \hat{\gamma}^{1\alpha}(\bar{\gamma}) := (\min\left\{G^1, A^1(G^2)\right\}, \\ \hat{\gamma}^{2\alpha}(\bar{\gamma}) := (\min\left\{G^2, A^2(G^1)\right\}, \end{cases} \quad \text{with} \quad G^j := \min\left\{\bar{\gamma}^{jL}, \bar{\gamma}^{jR}\right\}$$

is locally constant on $\left\{\bar{\gamma} \neq Id_{[0,+\infty)^4}\right\}$ and then is a preflux. Then polar decomposition Theorem 11.8 implies that $\hat{f}$ is a Godunov flux associated to a Riemann germ $\mathcal{G} := \left\{\hat{f} = f\right\}$.
Step 7: the monotone case
Recall, from Lemma 5.5, that $\mathcal{G}$ is monotone if and only if $\hat{f} = \hat{f}_{\mathcal{G}}$ satisfies (say in the sense of monotone functions)

$$\sigma^\alpha \partial_\beta \hat{f}^\alpha \le 0 \quad \text{for all} \quad \alpha \neq \beta$$

with $\sigma^{jL} = 1 = -\sigma^{jR}$, i.e.

$$\begin{cases} \qquad\qquad \partial_{2L}\hat{f}^{1L} \le 0, \quad \partial_{1R}\hat{f}^{1L} \le 0, \quad \partial_{2R}\hat{f}^{1L} \le 0, \\ \\ \partial_{1L}\hat{f}^{2L} \le 0, \qquad\qquad \partial_{1R}\hat{f}^{2L}, \le 0 \quad \partial_{2R}\hat{f}^{2L} \le 0, \end{cases}$$

where $\partial_{1R}\hat{f}^{1L} \le 0$ and $\partial_{2R}\hat{f}^{2L} \le 0$ are automatically satisfied. Because G^2 has opposite monotonicities in p^{2L} and p^{2R}, we deduce that A^1 can be chosen constant (and similarly for A^2). Conversely, if A^j are constant functions, then it is straightforward that $\mathcal{G}$ is monotone.
Step 8: the Kružkov case
Recall from Theorem 2.29, that for conservative germs, monotone germs coincide with Kružkov germs. This gives the result. This ends the proof of the lemma.

Lemma 12.14 (Intersection of n conservative lines for bell-shaped fluxes)
Assume (2.2) for $N = 2n$ with $n : n$ junctions, and call f^{jL} and f^{jR} respectively the j-th ingoing and j-the outgoing fluxes for $j = 1, \dots, n$. Assume that each $f^{j\alpha}$ is strictly concave with maximum at $c^{j\alpha} \in (a^{j\alpha}, b^{j\alpha})$ and $f^{j\alpha}(a^{j\alpha}) = 0 = f^{j\alpha}(b^{j\alpha})$. Let $\mathcal{G} \subset [a, b]$ be a closed generalized Riemann germ satisfying for $p = (p^{1L}, p^{1R}, \dots, p^{nL}, p^{nR})$ with $p^j := (p^{jL}, p^{jR})$

(12.69) $$\mathcal{G} \subset \bigcap_{j=1,\dots,n} \Sigma^j, \quad \text{with} \quad \Sigma^j := \left\{p \in [a, b], \quad f^{jL}(p^{jL}) = f^{jR}(p^{jR})\right\}$$

which means that each j-th line $J^{jL} \cup \{0\} \cup J^{jR} \simeq \mathbb{R}$ is conservative. We set

$$G^j(p) := G^j(p^j) := \min\left\{f^{jL,+}(p^{jL}), f^{jR,-}(p^{jR})\right\}, \quad G^j_{\max} := \min\left\{f^{jL}(c^{jL}), f^{jR}(c^{jR})\right\}$$

Let $\hat{f} = (\hat{f}^{1L}, \hat{f}^{1R}, \dots, \hat{f}^{nL}, \hat{f}^{nR})$ be the Godunov flux associated to the Riemann germ $\mathcal{G}$.

i) (Characterization of Godunov flux)

Then there exists a function $\hat{A}$ such that we have for $\alpha = L, R$

(12.70)

$$\hat{f}^{j\alpha} = \hat{A}^j \circ G \quad \text{with} \quad G := (G^1, \dots, G^n) \quad \text{with a preflux} \quad \hat{A} = (\hat{A}^1, \dots, \hat{A}^n) : [0, +\infty)^n \to [0, +\infty)^n$$

Moreover the function $\hat{A}$ has a unique restriction to $\prod_{j=1,\dots,n} [0, G^j_{\max}]$. In particular $0 \le \hat{A}^j(G) \le G^j$, and $\hat{A}^j$ is locally constant in the variable G^j on the set $\left\{\hat{A}^j(G) < G^j\right\}$.

Conversely, if $\hat{f}$ satisfies (12.70), then $\hat{f}$ is the Godunov flux associated to a Riemann germ $\mathcal{G} = \left\{\hat{f} = f\right\}$ which satisfies (12.69).

ii) (Identification of monotone case)

Moreover $\mathcal{G}$ is monotone if and only if the function $\hat{A}$ satisfies on $\prod_{j=1,\dots,n} [0, G^j_{\max}]$

$$\hat{A}^j(G) = \min\left\{G^j, A^j\right\} \quad \text{for some constant } A^j \text{ for } j = 1, \dots, n \tag{12.71}$$

iii) (Identification of Kružkov case)

Moreover $\mathcal{G}$ is Kružkov if and only if the function $\hat{A}$ satisfies (12.71).

Remark 12.15 *Notice that (12.70) is a special sort of polar decomposition of $\hat{f}$.*

Proof of Lemma 12.14

Step 1: special case $n = 2$

Step 1.1: identification of the flux $\hat{f}$

Then we can apply Lemma 12.12. It shows that

$$\left\{\begin{array}{l} \hat{f}^{1\alpha} = \min\left\{G^1, A^1(G^2)\right\} =: \hat{A}^1(G) \\ \hat{f}^{2\alpha} = \min\left\{G^2, A^2(G^1)\right\} =: \hat{A}^2(G) \end{array}\right. \quad \text{with} \quad G := (G^1, G^2)$$

We know that $A^j : [0, G^j_{\max}] \to [0, G^j_{\max}]$. Up to extend A^1, A^1 by continuity on $[0, +\infty)$ as constant functions where they are extended, we see that $\hat{A} := (\hat{A}^1, \hat{A}^2)$ is a continuous function defined on $[0, +\infty)^2$. By definition, we see that $\hat{A}^1$ is nondecreasing in G^1. Moreover $0 \le \hat{A}^1(G) \le G^1$ and $\hat{A}^1$ is locally constant in G^1 on $\left\{\hat{A}^1(G) < G^1\right\}$. Moreover generally, Step 6 of the proof of Lemma 12.12 shows that $\hat{A}_{|[0,G^1_{\max}] \times [0,G^2_{\max}]}$ is a preflux, and then is locally constant. It is then easy to see that this is still true for the full function $\hat{A}$.

Step 1.2: conversely

Conversely, let us assume that

$$\left\{\begin{array}{l} \hat{f}^{1\alpha} = \hat{A}^1(G) \\ \hat{f}^{2\alpha} = \hat{A}^2(G) \end{array}\right. \quad \text{with} \quad G := (G^1, G^2), \quad G^j := \min\left\{f^{jL,+}(p^{jL}), f^{jR,-}(p^{jR})\right\}$$

where $\hat{A} = (\hat{A}^1, \hat{A}^2) : [0, +\infty)^2 \to [0, +\infty)^2$ is a preflux. Let us show that $\hat{f}$ is a Godunov flux associated to a Riemann germ. To this end, it is sufficient to check that

$$\left\{\begin{array}{l} \hat{\gamma}^{1L}(\bar{\gamma}) = \hat{\gamma}^{1R}(\bar{\gamma}) := \hat{A}^1(G^1, G^2) \\ \hat{\gamma}^{2L}(\bar{\gamma}) = \hat{\gamma}^{2R}(\bar{\gamma}) := \hat{A}^2(G^1, G^2) \end{array}\right| \quad \text{with} \quad G^j := \min\left\{\bar{\gamma}^{jL}, \bar{\gamma}^{jR}\right\}$$

is such that $\hat{\gamma}$ is a preflux.

First, $\hat{\gamma} : [0, +\infty)^4 \to [0, +\infty)^4$ is continuous, and using the fact that $\hat{A} : [0, +\infty)^2 \to [0, +\infty)^2$ is a preflux, we deduce that $0 \le \hat{\gamma} \le Id_{[0,+\infty)^4}$, and that $\hat{\gamma}^{j\alpha}$ is nondecreasing in $\bar{\gamma}^{j\alpha}$.

It remains to check that $\hat{\gamma}$ is locally constant on $\left\{\hat{\gamma} \ne Id_{[0,+\infty)^4}\right\}$. Assume that $\hat{\gamma}^{1R}(\bar{\gamma}) < \bar{\gamma}^{1R}$. Then

$$\hat{A}^1(G^1, G^2) < \bar{\gamma}^{1R} \le G^1$$

Because $\hat{A}$ is preflux and then locally constant, we deduce that $\hat{A}(G)$ is locally constant in G^1 on $\left\{\hat{A}^1(G) < G^1\right\}$, and then $\hat{\gamma}$ locally constant in $\bar{\gamma}^{1R}$ on $\left\{\hat{\gamma}^{1R}(\bar{\gamma}) < \bar{\gamma}^{1R}\right\}$. The reasoning with the other components of $\hat{\gamma}$ is exactly the same. Therefore $\hat{\gamma}$ is a locally constant on $\left\{\hat{\gamma} \neq Id_{[0,+\infty)^4}\right\}$, and is then a preflux. We conclude that $\hat{f} = \hat{\gamma} \circ \bar{\gamma}$ is a Godulov flux associated to a Riemann germ $\mathcal{G} := \left\{\hat{f} = f\right\}$ with $f = (f^{1L}, f^{1R}, f^{2L}, f^{2R})$.

Step 2: special case $n \geq 3$

Step 2.1: identification of the flux $\hat{f}$

We already assume that freezing one variable $p^k := (p^{kL}, p^{k,R})$, for the $n-1$ variables p^j for $j \neq k$, the statement of Lemma 12.14 holds true. Hence there exists a map $\hat{A}_k : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ such that for $j \neq k$, we have

$$\hat{f}^{jL}(p) = \hat{f}^{jR}(p) = \hat{A}^j_k(G^1, \ldots, G^{k-1}, G^{k+1}, \ldots, G^n; p^k), \quad j \neq k$$

Because $n \geq 3$, we can freeze another variable $\ell \in \{1, \ldots, n\} \setminus \{k\}$, and get similarly

$$\hat{f}^{jL}(p) = \hat{f}^{jR}(p) = \hat{A}^j_k(G^1, \ldots, G^{k-1}, G^{k+1}, \ldots, G^n; p^k) = \hat{A}^j_\ell(G^1, \ldots, G^{\ell-1}, G^{\ell+1}, \ldots, G^n; p^\ell), \quad j \neq k, \ell$$

This implies that the dependence of $\hat{A}^j_k$ in p^k is done through

$$G^k = \min\left\{\bar{\gamma}^{kL}, \bar{\gamma}^{kR}\right\}, \quad \bar{\gamma}^{kL} := f^{kL,+}(p^{kL}), \quad \bar{\gamma}^{kR} := f^{kR,-}(p^{kR})$$

Therefore

$$\hat{f}^{jL}(p) = \hat{f}^{jR}(p) = \hat{A}^j(G) \quad \text{with} \quad G = (G^1, \ldots, G^n), \quad j = 1, \ldots, n$$

Moreover $\hat{A}^j$ inherits the properties of $\hat{A}^j_k$ and $\hat{A}^j_\ell$. In particular

$$0 \leq \hat{A}^j(G) \leq G^j, \quad G^j \mapsto \hat{A}^j \quad \text{is nondecreasing}$$

and $\hat{A} = (\hat{A}^1, \ldots, \hat{A}^n) : [0, +\infty)^n \to [0, +\infty)^n$ is continuous. Moreover $\hat{A}$ is locally constant in G^j on $\left\{\hat{A}^j(G) < G^j\right\}$. All these properties just show that $\hat{A}$ is a preflux.

Step 2.2: conversely

Conversely, let us assume that for $\alpha = L, R$

$$\hat{f}^{j\alpha}(p) = \hat{A}^j \circ G \quad \text{with} \quad G := (G^1, \ldots, G^n), \quad G^j := \min\left\{f^{jL,+}(p^{jL}), f^{jR,-}(p^{jR})\right\}$$

where $\hat{A} = (\hat{A}^1, \ldots, \hat{A}^n) : [0, +\infty)^n \to [0, +\infty)^n$ is a preflux. We show that $\mathcal{G} := \left\{\hat{f} = f\right\}$ is a Riemann germ proving that the map $\hat{\gamma} := (\hat{\gamma}^{1L}, \hat{\gamma}^{1R}, \ldots, \hat{\gamma}^{1n}, \hat{\gamma}^{nR}) : [0, +\infty)^{2n} \to [0, +\infty)^{2n}$ is a preflux, where for $\alpha = L, R$, we define

$$\hat{\gamma}^{j\alpha}(\bar{\gamma}) := \hat{A}^j(G) \quad \text{with} \quad G^j := \min\left\{\bar{\gamma}^{1L}, \bar{\gamma}^{jR}\right\}$$

It is straightforward to deduce from the properties of $\hat{A}$ that $\hat{\gamma}$ is continuous, satisfies $0 \leq \hat{\gamma} \leq Id_{[0,+\infty)^{2n}}$ and $\hat{\gamma}^{j\alpha}$ is nondecreasing in $\bar{\gamma}^{j\alpha}$. Let us now check that $\hat{\gamma}$ is locally constant on $\left\{\hat{\gamma} \neq Id_{[0,+\infty)^{2n}}\right\}$. Consider $\hat{\gamma}^{1R}(\bar{\gamma}) < \bar{\gamma}^{1R} \leq G^1$. Then the local constancy of $\hat{A}$ implies that $\hat{\gamma}$ is locally constant in $\bar{\gamma}^{1R}$ on $\left\{\hat{\gamma}^{1R}(\bar{\gamma}) < \bar{\gamma}^{1R}\right\}$. The same property also holds for all other components than $\bar{\gamma}^{1R}$. This shows the expected local constancy of $\hat{\gamma}$.

Step 3: identification of monotone property

Recall, from Lemma 5.5, that $\mathcal{G}$ is monotone if and only if $\hat{f} = \hat{f}_{\mathcal{G}}$ satisfies (say in the sense of monotone functions)

$$\sigma^\alpha \partial_\beta \hat{f}^\alpha \leq 0 \quad \text{for all} \quad \alpha \neq \beta$$

with $\sigma^{jL} = 1 = -\sigma^{jR}$, i.e.

$$\partial_{jL} \hat{f}^{kL} \leq 0, \quad \partial_{jR} \hat{f}^{kL} \leq 0 \quad \text{for all} \quad j \neq k$$

Because G^j has opposite monotonicities in p^{jL} and p^{jR}, we deduce from (12.70) that $\hat{A}^k$ is independent on G^j for $j \neq k$. Hence

$$\hat{A}^k(G) = \min\left\{G^k, A^k\right\}$$

for some constant A^k.

Step 4: identification of Kružkov property

Recall from Theorem 2.29, that for conservative germs, monotone germs coincide with Kružkov germs. This gives the result. This ends the proof of the lemma.

Proof of Proposition 2.47

We just apply ii) of Lemma 12.14.

12.7 Refined structure of prefluxes

Lemma 12.16 (Pseudo-stratification of prefluxes)
Let $N \geq 1$ and $\hat{\gamma} : [0,+\infty)^N \to [0,+\infty)^N$ be a preflux. We set

$$X := \left\{\hat{\gamma} = id_{[0,+\infty)^N}\right\}$$

For a subset $I \subset \{1,\dots,N\}$, we define the sets

$$\Omega_I := \left\{\bar{\gamma} \in [0,+\infty)^N, \quad \left| \begin{array}{lll} \hat{\gamma}^j(\bar{\gamma}) < \bar{\gamma}^j & \text{for all} & j \in I \\ \hat{\gamma}^j(\bar{\gamma}) = \bar{\gamma}^j & \text{for all} & j \notin I \end{array} \right. \right\}$$

and

$$X_I := \{p \in X, \quad p + \Pi_I \subset \Omega_I\} \quad \text{with} \quad \Pi_I := \left\{ \begin{array}{ll} \{0\} & \text{if} \quad I = \emptyset \\ \sum_{j\in I}(0,+\infty)e_j & \text{if} \quad I \neq \emptyset \end{array} \right. \tag{12.72}$$

i) (Projection onto X_I)
Then $\Omega_I = X_I + \Pi_I$. We also have the following partition

$$[0,+\infty)^N = \bigcup_{I\subset\{1,\dots,N\}} \Omega_I, \quad \text{with} \quad \Omega_I = X = X_I \quad \text{for} \quad I = \emptyset$$

and the projection map

$$\hat{\gamma}_{|\Omega_I} : \Omega_I \to X_I$$

where X_I is a closed set.
ii) (Graph property of X_I)
Let

$$V_I := \left\{ \begin{array}{ll} \{0\} & \text{if} \quad I = \emptyset \\ \sum_{j\in I}\mathbb{R}e_j & \text{if} \quad I \neq \emptyset \end{array} \right.$$

Then consider the following orthogonal projection $Proj^{\perp}_{V_I^{\perp}} : [0,+\infty)^N \to V_I^{\perp}$ onto the vector space $V_I^{\perp}$ (i.e. the orthogonal to V_I for the standard scalar product). Let $K_{V_I^{\perp}} := Proj^{\perp}_{V_I^{\perp}}(X_I)$.
Then X_I is the graph of a continuous map

$$K_{V_I^{\perp}} \to V_I$$

where $K_{V_I^{\perp}}$ is a closed set.
iii) (Inclusions)
For all $I, J \subset \{1,\dots,N\}$, we have

$$I \subset J \quad \Longrightarrow \quad X_I \supset X_J$$

Remark 12.17 *Notice that we may have $X_I \cap X_J \neq \emptyset$ with $I \cap J = \emptyset$. We may also have $X_I \cap X_K \neq \emptyset$ but $X_K \not\subset X_I$ neither $X_I \not\subset X_K$. In order to illustrate those two cases, see Lemma 12.5 for conservative prefluxes for $1:2$ junctions (with $I = \{0\}$, $J = \{1\}$ for the first example, and $I = \{0,2\}$, $K = \{0,1\}$ for the second example).*

Proof of Lemma 12.16
Step 1: proof of i)
Step 1.1: projection of a partition
By definition, we first notice that

$$\Omega_I \cap \Omega_J = \emptyset \quad \text{if} \quad I \neq J \tag{12.73}$$

Moreover, by definition, we have $X_I + \Pi_I \subset \Omega_I$. Because $\hat{\gamma}$ is locally constant on $\left\{\hat{\gamma} \neq id_{[0,+\infty)^N}\right\}$, we deduce that $\Omega_I + \overline{\Pi}_I \subset \Omega_I$, i.e.

$$\Omega_I = \Omega_I + \bar{\Pi}_I$$

Then for any $q \in \Omega_I$, one has

$$\hat{\gamma}(q) \in X \quad \text{and} \quad q \in \hat{\gamma}(q) + \Pi_I \subset \Omega_I$$

Therefore $\hat{\gamma}(q) \in X_I$ and this defines a projection map

$$\hat{\gamma}_{|\Omega_I} : \Omega_I \to X_I$$

We also have the following partition

$$[0,+\infty)^N = \bigcup_{I\subset\{1,\ldots,N\}} \Omega_I, \quad \text{with} \quad \Omega_I = X = X_I \quad \text{for} \quad I = \emptyset$$

while the following

$$X = \bigcup_{I\subset\{1,\ldots,N\}} X_I$$

is not a partition of X.

Step 1.2: closedness

Consider a sequence of points $p_n \in X_I$ with $p_n \to p \in X$ as $n \to +\infty$. By assumption, we have $p_n + \Pi_I \subset \Omega_I$. Therefore, for all $q \in \Pi_I$, we get

$$\hat{\gamma}(p_n + q) = p_n$$

By continuity of the preflux, we get at the limit

$$\hat{\gamma}(p + q) = p$$

which shows that $p + q \in \Omega_I$ for all $q \in \Pi_I$. Hence

$$p \in X_I$$

and this shows that X_I is a closed set.

Step 2: proof of ii)

Consider the following orthogonal projection $\Phi_I := \text{Proj}^{\perp}_{V_I^{\perp}} : [0,+\infty)^N \to V_I^{\perp}$, and set

$$K_{V_I^{\perp}} := \Phi_I(X_I) \subset \quad V_I^{\perp} \cap [0,+\infty)^N$$

First, assume by contradiction, that there exist two points $p_a, p_b \in X_I$ with the same image $q := \Phi_I(p_c)$ for $c = a, b$. By definition of X_I, we have

$$p_c + \Pi_I \subset \Omega_I, \quad c = a, b$$

Then

$$\hat{\gamma} = const \quad \text{on} \quad (p_a + \Pi_I) \cap (p_b + \Pi_I) \neq \emptyset$$

and this constant is equal to $\hat{\gamma}(p_c) = p_c$ for $c = a, b$. Therefore $p_a = p_b$. Hence the map

$$(\Phi_I)_{|X_I} : X_I \to K_{V_I^{\perp}}$$

is injective. Hence Φ_I^{-1} is well defined on the set $K_{V_I^{\perp}}$, and shows that X_I is a graph above $K_{V_I^{\perp}}$.
Moreover, because the projection Φ_I is continuous, we deduce that the set $K_{V_I^{\perp}} = \Phi_I(X_I)$ is a closed set as the image of a closed set. Using now the orthogonal projection onto V_I instead onto $V_I^{\perp}$, let us consider the map

$$\text{Proj}^{\perp}_{V_I} \circ (\Phi_I)^{-1} : K_{V_I^{\perp}} \to V_I$$

whose graph is precisely X_I. Because the set X_I is closed, we deduce that the map $\text{Proj}^{\perp}_{V_I} \circ (\Phi_I)^{-1}$ is continuous. Therefore X_I is the graph of a continuous map $K_{V_I^{\perp}} \to V_I$.

Step 3: proof of iii)

Let $I \subset J$ and let us show that $X_J \subset X_I$. Assume that $I \neq J$, otherwise the result is trivial. Let $p \in X_J$ and $q \in \Pi_I$. Then there exists a sequence

$$\Pi_J \ni q^{\varepsilon} \to q \in \Pi_I$$

By assumption, we have $p + q^{\varepsilon} \in \Omega_J$ and then

$$\hat{\gamma}(p + q^{\varepsilon}) = p$$

At the limit $\varepsilon \to 0$, we get

$$\hat{\gamma}(p+q) = p \quad \text{for all} \quad q \in \Pi_I$$

This means that $p + \Pi_I \subset \Omega_I$, i.e. that $p \in X_I$, and then $X_J \subset X_I$. This ends the proof of the lemma.

The following result generalizes Lemma 12.10 for more branches, but is illustrated on a particular flux, which is piecewise linear (and then simplifies the presentation).

Lemma 12.18 (General properties of a germ for bell-shaped fluxes, in a particular case)
Let $N \geq 1$ and let us consider a preflux $\hat{\gamma} : [0,+\infty)^N \to [0,+\infty)^N$ for a junction characterized by some $\sigma_0 \in \{\pm 1\}^N$. Assume the following bound

$$X := \left\{\hat{\gamma} = id_{[0,+\infty)^N}\right\} \subset [0,R']^N, \quad R > R' \tag{12.74}$$

for some $R' \geq 0$ and set $b := R(1,\dots,1) \in \mathbb{R}^N$ and $a := 0_{\mathbb{R}^N}$. Consider the functions

$$f^j(p^j) := \min\left\{p^j, 2R - p^j\right\}, \quad j = 1,\dots,N$$

Then the following set

$$\mathcal{G} := \left\{p \in [a,b], \quad \hat{f}(p) = f(p)\right\} \quad \text{with} \quad \hat{f} := \hat{\gamma} \circ \bar{\gamma} \quad \text{and} \quad \bar{\gamma}(p) = (f^{1,\sigma_0^1}(p^1),\dots,f^{N,\sigma_0^N}(p^N))$$

is a conservative Riemann germ, where

$$f^{j,+}(p^j) = f^j(\min\left\{p^j, R\right\}) \quad \text{and} \quad f^{j,-}(p^j) = f^j(\max\left\{p^j, R\right\})$$

i) (Reflexion maps)
Let $(e_1,\dots,e_N)$ be a basis of $\mathbb{R}^N$. Now we define the reflexion maps for $j = 1,\dots,N$

$$\begin{array}{rccc} \tau_j : & [0,+\infty)^N & \to & [0,+\infty)^N \\ & \bar{\gamma} & \mapsto & \bar{\gamma} + 2(R - \bar{\gamma}^j)e_j \end{array}$$

Now for $\sigma = (\sigma^1,\dots,\sigma^N) \in \{\pm 1\}^N$, we define the map

$$\tau^\sigma := \tau_1^{n_1} \dots \tau_N^{n_N} \quad \text{with} \quad n_j := \left\{ \begin{array}{lll} 0 & \text{if} & \sigma^j = +1 \\ 1 & \text{if} & \sigma^j = -1 \end{array} \right. , \quad j = 0,1,2$$

ii) (Explicit characterization of $\mathcal{G}$)
We set

$$Q_R := [0,R]^N$$

Then we have for all $\sigma \in \{\pm 1\}^N$

$$\mathcal{G}_\sigma := \mathcal{G} \cap \tau^\sigma(Q_R) = \tau^\sigma(X_{I_\sigma}) \quad \text{with} \quad I_\sigma := \left\{j \in \{1,\dots,N\}, \quad \sigma^j \neq \sigma_0^j\right\}$$

and where X_I is a closed subset of X, as defined in (12.72).
In particular, we have the following inclusions for all $\sigma, \sigma' \in \{\pm 1\}^N$

$$(\tau^\sigma)^{-1}(\mathcal{G}_\sigma) \subset (\tau^{\sigma'})^{-1}(\mathcal{G}_{\sigma'}) \quad \text{if} \quad I_\sigma \supset I_{\sigma'}$$

Proof of Lemma 12.18
For a set $B \subset [0,+\infty)^N$, we define

$$B^\sigma := \tau^\sigma(B)$$

Step 1: preliminaries
Notice that $f \circ \tau^\sigma = f$ on Q_{2R}. Then we get for $p \in Q_R^\sigma$

$$\bar{\gamma}^j(p) = \left\{ \begin{array}{lll} f^j(p) & \text{if} & \sigma_0^j \sigma^j = +1 \\ R & \text{if} & \sigma_0^j \sigma^j = -1 \end{array} \right.$$

i.e. for $p = \tau^\sigma(q)$, we get $q \in Q_R$ and

$$(\bar{q}^\sigma)^j := (\bar{\gamma} \circ \tau^\sigma)^j(q) = \begin{cases} q^j & \text{if} \quad \sigma_0^j \sigma^j = +1 \\ R & \text{if} \quad \sigma_0^j \sigma^j = -1 \end{cases} \tag{12.75}$$

Moreover $q^j \le (\bar{q}^\sigma)^j$. Hence

$$\text{for} \quad q \in Q_{R'}, \quad \bar{q}^\sigma \in \left\{ \bar{\gamma} \in [0,+\infty)^N, \quad \left| \begin{array}{lll} \hat{\gamma}^j(\bar{\gamma}) = \bar{\gamma}^j & \text{if} & \sigma_0^j \sigma^j = +1 \\ \hat{\gamma}^j(\bar{\gamma}) < \bar{\gamma}^j & \text{if} & \sigma_0^j \sigma^j = -1 \end{array} \right. \right\} = \Omega_{I_\sigma} \quad \text{with} \quad I_\sigma := \left\{ j, \ \sigma^j \ne \sigma_0^j \right\}$$

We deduce that

$$\text{for} \quad q \in Q_{R'}, \quad \hat{\gamma}(\bar{q}^\sigma) \in X_{I_\sigma} \quad \text{with} \quad I_\sigma := \left\{ j, \ \sigma^j \ne \sigma_0^j \right\} \tag{12.76}$$

Furthermore, because $\hat{\gamma}(q) = q$ for $q \in X = \left\{ \hat{\gamma} = id_{[0,+\infty)^N} \right\}$, we deduce

$$\hat{\gamma}(\bar{q}^\sigma) = q \quad \text{for all} \quad q \in X_{I_\sigma} \quad \text{with} \quad I_\sigma := \left\{ j, \ \sigma^j \ne \sigma_0^j \right\} \tag{12.77}$$

Step 2: application
We have

$$\mathcal{G}_\sigma := \mathcal{G} \cap Q_R^\sigma = \{ p \in Q_R^\sigma, \quad (\hat{\gamma} \circ \bar{\gamma})(p) = f(p) \}$$

i.e. using $\bar{q}^\sigma$ defined in (12.75)

$$(\tau^\sigma)^{-1}(\mathcal{G}_\sigma) = \{ q \in Q_R, \quad \hat{\gamma}(\bar{q}^\sigma) = q \} \subset \hat{\gamma}([0,+\infty)^N) = X$$

i.e.

$$(\tau^\sigma)^{-1}(\mathcal{G}_\sigma) = \{ q \in X, \quad \hat{\gamma}(\bar{q}^\sigma) = q \} = X_{I_\sigma}$$

where we have used (12.76) and (12.77) for the last equality. Therefore

$$\mathcal{G}_\sigma = \tau^\sigma(X_{I_\sigma})$$

which shows point ii). This ends the proof of the lemma.

13 Quasi-prefluxes and their relaxation as prefluxes

13.1 Quasi-prefluxes and Riemann relaxation

In practice some functions are candidate for prefluxes, but sometimes are not, because they miss the local constancy property (we will call them quasi-preflux). One way to remedy to this difficulty consists to do some suitable Riemann relaxation of the candidate to finally get a genuine preflux.

On the model of Definition 11.1 for prefluxes, we introduce the following definition.

Definition 13.1 (Quasi-preflux)
Let $N \ge 1$. Let $\bar{\lambda} = (\bar{\lambda}^1, \dots, \bar{\lambda}^N) \in (0,+\infty)^N \cup \{+\infty\}^N$. A function $\hat{\gamma}_0 : [0,+\infty)^N \to \mathbb{R}^N$ is said to be a **$\bar{\lambda}$-quasi-preflux** *(or simply quasi-preflux) if it satisfies the following conditions*

(13.1)

$$\begin{cases} \hat{\gamma}_0 : [0,+\infty)^N \to \mathbb{R}^N \quad \text{is continuous,} \qquad \text{with moreover } \hat{\gamma}_0 : [0,+\infty)^N \to [0,+\infty)^N \text{ if } \bar{\lambda} = (+\infty, \dots, +\infty), \\ \\ \hat{\gamma}_0 \quad \text{is Riemann monotone in the sense of Definition 2.12,} \\ \\ (\hat{\gamma}_0^j(q))_{|\{q^j=0\}} \le 0 \quad \text{and} \quad (\hat{\gamma}_0^j(q))_{|\left\{q^j \ge \bar{\lambda}^j\right\}} \ge 0 \quad \text{for all} \quad j = 1, \dots, N \end{cases}$$

i) (HJ quasi-preflux)
We say that the quasi-preflux $\hat{\gamma}_0$ is HJ if

$$\hat{\gamma}_0^j = g \quad \text{for all} \quad j = 1, \dots, N, \quad \text{for some function } g : [0,+\infty)^N \to \mathbb{R}$$

ii) (Kružkov quasi-preflux)
We say that the quasi-preflux $\hat{\gamma}_0$ *is Kružkov if*

$$(13.2) \qquad 0 \le D_*^{\hat{\gamma}_0}(\bar{\gamma},\gamma) := \sum_{j=1,\dots,N} sign(\bar{\gamma}^j - \gamma^j)\cdot\left\{\hat{\gamma}_0^j(\bar{\gamma}) - \hat{\gamma}_0^j(\gamma)\right\} \quad \text{for all} \quad \bar{\gamma},\gamma \in [0,+\infty)^N$$

ii') (Lipschitz Kružkov quasi-preflux)
We say that the quasi-preflux $\hat{\gamma}_0$ *is Lipschitz Kružkov if satisfies (13.2) and if furthermore* $\hat{\gamma}_0 : [0,+\infty)^N \to \mathbb{R}^N$ *is globally Lipschitz continuous.*
iii) (σ-monotone Kružkov quasi-preflux)
We say that the quasi-preflux $\hat{\gamma}_0$ *is* σ*-monotone Kružkov if*

$$0 \le D_{*+}^{\hat{\gamma}_0}(\bar{\gamma},\gamma) := \sum_{j=1,\dots,N} sign^{\sigma^j}(\bar{\gamma}^j - \gamma^j)\cdot\left\{\hat{\gamma}_0^j(\bar{\gamma}) - \hat{\gamma}_0^j(\gamma)\right\} \quad \text{for all} \quad \bar{\gamma},\gamma \in [0,+\infty)^N$$

(where we make some abuse of notation for $sign^{\sigma^j} = sign^+$ *or* $sign^-$*).*
iv) (σ-monotone quasi-preflux)
We say that the quasi-preflux $\hat{\gamma}_0$ *is* σ*-monotone if*

$$(13.3) \qquad \text{the maps } \gamma \mapsto \sigma^j \hat{\gamma}_0^j(\gamma) \text{ are nonincreasing in the variable } \sigma^k\gamma^k \text{ for all } k \ne j.$$

v) (conservative quasi-preflux)
We say that the quasi-preflux $\hat{\gamma}_0$ *is conservative if*

$$(13.4) \qquad \sum_{j=1,\dots,N} \sigma^j \cdot \hat{\gamma}_0^j = 0$$

We also say that the quasi-preflux $\hat{\gamma}_0$ *is n:m conservative if n is the number of indices j such that* $\sigma^j = -1$ *and m is the number of indices j such that* $\sigma^j = +1$*.*
vi) (Bounded continuity on the box $[0,+\infty]^N$)
We say that the quasi-preflux $\hat{\gamma}_0$ *is boundedly continuous on the box* $[0,+\infty]^N$*, if for any* $\rho > 0$*, it admits an extension (still denoted by* $\hat{\gamma}_0$*) such that*

$$\hat{\gamma}_0 : \overline{\hat{\gamma}_0^{-1}([-\rho,\rho]^N)}^{[0,+\infty]^N} \to [-\rho,\rho]^N \quad \text{is continuous}$$

where $\overline{\hat{\gamma}_0^{-1}([-\rho,\rho]^N)}^{[0,+\infty]^N}$ *is the closure in the box* $[0,+\infty]^N$ *of the set* $\hat{\gamma}_0^{-1}([-\rho,\rho]^N) \subset [0,+\infty)^N$*.*
vi') (Bounded local constancy at infinity)
For quasi-preflux $\hat{\gamma}_0$ *which is boundedly continuous on the box* $[0,+\infty]^N$*, we say that* $\hat{\gamma}_0$ *is boundedly locally constant at infinity if for any* $\bar{\gamma} \in [0,+\infty]^N$ *with*

$$I := \left\{ j \in \{1,\dots,N\}, \quad \bar{\gamma}^j = +\infty \quad \text{with } \hat{\gamma}_0(\bar{\gamma}) \text{ bounded}\right\} \ne \emptyset$$

there exists $\rho > 0$ *such that*

$$\hat{\gamma}_0 = const = \hat{\gamma}_0(\bar{\gamma}) \quad \text{on} \quad \bar{\gamma}_\rho + \sum_{j\in I}[0,+\infty)e_j \quad \text{with} \quad \bar{\gamma}_\rho^j := \begin{cases} \rho & \text{if} \quad j \in I \\ \bar{\gamma}^j & \text{if} \quad j \notin I \end{cases}$$

vii) (Uniform local bound)
We say that a quasi-preflux $\hat{\gamma}_0 : [0,+\infty)^N \to \mathbb{R}^N$ *is* **uniformly locally bounded** *if there exists a continuous map* $F : [0,+\infty) \to [0,+\infty)$ *such that*

$$|\hat{\gamma}_0^j(\bar{\gamma})| \le F(\bar{\gamma}^j), \quad j = 1,\dots,N$$

Remark 13.2 (A sufficient condition)
1) (Generalities)
If $\hat{\gamma}_0$ *satisfies* $0 \le \hat{\gamma}_0 \le id_{[0,+\infty)^N}$*, then this implies the third line of (13.1).*
Notice also that any preflux is a quasi-preflux.
2) (Conditions at infinity)

Notice that part vi), vi') of Definition 13.1 will be in particular used in Proposition 13.8 for Riemann relaxation $\mathfrak{R}_\infty$, while condition vii) will be additionally required in part v) of Lemma 13.12 for the gluing with $\bar{\lambda}^0 = +\infty$ (i.e. with no flux limiter on the link between two junctions).

3) (Lipschitz Kružkov versus Kružkov)

Notice that a quasi-preflux can be Kružkov without being Lipschitz continuous (examples are easy to construct by composition from the right of Lipschitz Kružkov quasi-prefluxes by Hölder functions).

On the other hand, Proposition 5.9 shows that any Kružkov function which is Lipschitz, is automatically Riemann monotone, which is a key property to check that a function is a quasi-preflux. **Still we do not know if a Kružkov function which is not Lipschitz is necessarily Riemann monotone or not.** *For this reason, at some places it will be useful to distinguish between Lipschitz Kružkov and Kružkov.*

We start with the following result.

Lemma 13.3 (Conservative quasi-prefluxes: equivalence between Kružkov and σ_0-mononotonicity)
Let $N \geq 1$ and $\bar{\lambda} \in (0,+\infty)^N \cup \{+\infty\}^N$ a function $\hat{\gamma}_0 : [0,+\infty)^N \to \mathbb{R}^N$ with values in $[0,+\infty)^N$ if $\bar{\lambda} = (+\infty,\dots,+\infty)$. Assume that $\hat{\lambda}$ satisfies the third line of (13.1). Assume moreover that it satisfies the conservative relation (13.4) for an orientation $\sigma_0 \in \{\pm 1\}^N$.

i) *Then $\hat{\gamma}_0$ is a Kružkov quasi-preflux if and only if it satisfies the σ_0-monotone condition (13.3).*

ii) *In particular, any conservative quasi-preflux is Kružkov if and only if it is σ_0-monotone.*

Proof of Lemma 13.3

We assume that $\hat{\gamma}_0$ is σ_0-conservative (in the sense of (13.4)).

Step 1: quasi-preflux property

If $\hat{\gamma}_0$ satisfies moreover the σ_0-monotone condition (13.3), then Proposition 5.9 with $\sigma := (1,1,\dots,1)$ shows that $\hat{\gamma}_0 = \sigma \diamond \hat{\gamma}_0$ is Riemann monotone, and is then a quasi-preflux.

Step 2: equivalence

On the other hand, for any quasi-preflux $\hat{\gamma}_0$, Lemma 6.6 shows that $\hat{\gamma}_0$ is Kružkov if and only if it is σ_0-monotone. This ends the proof.

Theorem 13.4 (Riemann relaxation $\mathfrak{R}_{\bar{\lambda}}$ of $\bar{\lambda}$-quasi-prefluxes)
Let $N \geq 1$ and $\bar{\lambda} \in (0,+\infty)^N$, and consider a junction whose orientations of the branches are described by $\sigma_0 \in \{\pm 1\}^N$. Let us consider a $\bar{\lambda}$-quasi-preflux $\hat{\gamma}_0 : [0,+\infty)^N \to \mathbb{R}^N$ in the sense of Definition 13.1. We define the doubling set

$$\mathbb{D}_{\bar{\lambda}} := \left\{(\lambda^L, \lambda^R) \in [0,\bar{\lambda}]^2, \quad \max\left\{\lambda^{L,j}, \lambda^{R,j}\right\} = \bar{\lambda}^j \quad \textit{for all} \quad j = 1,\dots,N\right\} \tag{13.5}$$

For any $\bar{\gamma} \in [0,\bar{\lambda}]$, we consider $(\lambda^L,\lambda^R) \in \mathbb{D}_{\bar{\lambda}}$ solutions of

$$\min\left\{\bar{\gamma}^j, (\lambda^L)^j\right\} = \hat{\gamma}_0^j(\lambda^R), \quad j = 1,\dots,N. \tag{13.6}$$

Then the set

$$\mathcal{R}_{\bar{\gamma}} := \left\{(\lambda^L,\lambda^R) \in \mathbb{D}_{\bar{\lambda}} \quad \textit{solution of (13.6)}\right\}$$

is non-empty, and the set

$$\Lambda_{\bar{\gamma}} := \left\{\hat{\gamma}_0(\lambda^R), \quad (\lambda^L,\lambda^R) \in \mathcal{R}_{\bar{\gamma}}\right\} \tag{13.7}$$

is reduced to a singleton

$$\Lambda_{\bar{\gamma}} = \left\{\hat{\gamma}_{\bar{\lambda}}(\bar{\gamma})\right\}. \tag{13.8}$$

This defines a map $\hat{\gamma}_{\bar{\lambda}} : [0,\bar{\lambda}] \to [0,\bar{\lambda}]$, that we extend to $[0,+\infty)^N$, setting

$$\begin{array}{rccc} \hat{\gamma}_{\bar{\lambda}} : & [0,+\infty)^N & \to & [0,+\infty)^N \\ & \bar{\gamma} & \mapsto & (\hat{\gamma}_{\bar{\lambda}} \circ T_{\bar{\lambda}})(\bar{\gamma}) \end{array}$$

where $T_{\bar{\lambda}}$ is the truncation operator

$$T_{\bar{\lambda}}(\bar{\gamma}) := (\min\left\{\bar{\gamma}^1, \bar{\lambda}^1\right\}, \dots, \min\left\{\bar{\gamma}^N, \bar{\lambda}^N\right\}). \tag{13.9}$$

i) (Riemann relaxation operator $\mathfrak{R}_{\bar\lambda}$ "on the box $[0,\bar\lambda]$")
Then we set

$$\mathfrak{R}_{\bar\lambda}\hat\gamma_0 := \hat\gamma_{\bar\lambda} \tag{13.10}$$

with $\hat\gamma_{\bar\lambda}$ defined just above. Then the function $\hat\gamma_0 \circ T_{\bar\lambda}$ is a $\bar\lambda$-quasi-preflux, and we have

$$\mathfrak{R}_{\bar\lambda}(\hat\gamma_0 \circ T_{\bar\lambda}) = \mathfrak{R}_{\bar\lambda}\hat\gamma_0 \tag{13.11}$$

and $\hat\gamma_{\bar\lambda}$ is also a $\bar\lambda$-quasi-preflux.
ii) (Preflux property)
Moreover $\hat\gamma_{\bar\lambda} : [0,+\infty)^N \to [0,+\infty)^N$ is a preflux.
iii) (Projection property)
We also have $\mathfrak{R}_{\bar\lambda}\hat\gamma_{\bar\lambda} = \hat\gamma_{\bar\lambda}$. More generally, if $\hat\gamma$ is a preflux satisfying $\hat\gamma \circ T_{\bar\lambda} = \hat\gamma$, then $\mathfrak{R}_{\bar\lambda}\hat\gamma = \hat\gamma$.

Theorem 13.4 implies in particular Theorem 2.43 of the Introduction.

Remark 13.5 (Unfolding with the doubling set $\mathbb{D}_{\bar\lambda}$)
Recall that the preflux is a sort of folding of the Godunov flux, in the case of bell-shaped original fluxes. The situation is like for the square function, where every value in the image has two pre-images (except only one for the critical value of the function). For the prefluxes, one way to unfold the situation, is to introduce the doubling set $\mathbb{D}_{\bar\lambda}$.

Notice also that the notation (λ^L,λ^R) with L for left and R for right has to do with the positions of vectors in the couple (λ^L,λ^R), but nothing to do with the orientations of the branches of the junction.

Proof of Theorem 13.4
The proof is modeled on the proof of Proposition 7.1.
Step 1: preliminaries
We define a few maps and will use later on their properties. We define the injection

$$\begin{array}{rccc} \Phi: & [0,2\bar\lambda] & \to & [0,\bar\lambda]\times\{\pm1\}^N \\ & q & \mapsto & (g(q),\sigma_q) \quad \text{with} \end{array} \left\{ \begin{array}{l} g^j(q) := g^j(q^j) := \min\left\{q^j, 2\bar\lambda^j - q^j\right\}, \\ \\ \sigma_q^j := \left\{ \begin{array}{lll} +1 & \text{if} & q^j \in [0,\bar\lambda^j] \\ -1 & \text{if} & q^j \in (\bar\lambda^j, 2\bar\lambda^j] \end{array}\right. \end{array}\right. \tag{13.12}$$

Moreover for $q \in [0,2\bar\lambda]$, we define the functions

$$\left\{ \begin{array}{l} g^{j,+}(q^j) = g^j(\min\left\{q^j,\bar\lambda^j\right\}), \quad g^{j,-}(q^j) = g^j(\max\left\{q^j,\bar\lambda^j\right\}) \\ g^\sigma(q) := (g^{1,\sigma^1}(q^1),\dots,g^{N,\sigma^N}(q^N)) \end{array}\right.$$

Now for all $\lambda \in [0,\bar\lambda]$ and sign vector $\sigma \in \{\pm1\}^N$, we define $\lambda^\sigma \in [0,\bar\lambda]$ as

$$(\lambda^\sigma)^j := \left\{ \begin{array}{lll} \lambda^j & \text{if} & \sigma^j = 1 \\ \bar\lambda^j & \text{if} & \sigma^j = -1 \end{array}\right. \quad \Bigg| \quad j = 1,\dots,N \tag{13.13}$$

Then for any $q \in [0,2\bar\lambda]$, we get with notation (13.13)

$$\lambda^{\sigma\diamond\sigma_q} = g^\sigma(q) \quad \text{and} \quad \lambda^{-\sigma\diamond\sigma_q} = g^{-\sigma}(q) \quad \text{with} \quad (\lambda,\sigma_q) := \Phi(q) = (g(q),\sigma_q) \tag{13.14}$$

where the injection Φ is defined in (13.12).
It is also natural to introduce the map

$$\begin{array}{rccc} \Psi: & [0,\bar\lambda]\times\{\pm1\}^N & \to & \mathbb{D}_{\bar\lambda} \\ & (\lambda,\sigma) & \mapsto & (\lambda^{-\sigma},\lambda^\sigma) \end{array}$$

where $\mathbb{D}_{\bar\lambda}$ is defined in (13.5). It is easy to see that Ψ is injective on $\Phi([0,2\bar\lambda])$ (because for any (λ,σ) in the image of Φ, we have that $\lambda^j = \bar\lambda^j$ implies $\sigma^j = +1$). Moreover its is also easy to see that $\Psi_{|\Phi([0,2\bar\lambda])}$ is surjective onto $\mathbb{D}_{\bar\lambda}$, hence bijective.
Step 1: resolution on the box $[0,2\bar\lambda]$

We first notice that the function $g := (g^1, \ldots, g^N)$ is bell-shaped on $[a,b] := [0, 2\bar{\lambda}]$, and then for any choice of $\sigma \in \{\pm 1\}^N$, we see that

$$\hat{f} := \hat{\gamma}_0 \circ g^\sigma : [a,b] \to \mathbb{R}^N$$

is a Godunov quasi-flux for a junction oriented by σ. It is easy to check that $\hat{f}$ satisfies (7.1). In particular, notice that

$$\sigma^j \hat{f}^j(q)_{|q^j=0} \le 0, \quad \sigma^j \hat{f}^j(q)_{|q^j=2\bar{\lambda}^j} \ge 0$$

and $\sigma \diamond \hat{f}$ is Riemann monotone.

Hence we can consider its Riemann relaxation operator

$$\mathfrak{R} = \mathfrak{R}^\sigma$$

given in Proposition 7.1, where we make explicit the dependence on σ. Here σ is an artificial orientation of the branches, and is not necessarily equal to σ_0. This corresponds to solve the following equation

$$(\mathfrak{R}^\sigma \hat{f})^j(p) := G^{g^j}_{\sigma^j}(p^j, q^j) = \min \left\{ g^{j,\sigma^j}(p^j), g^{j,-\sigma_j}(q^j) \right\} = \hat{\gamma}_0^j((g^\sigma(q)), \quad j = 1, \ldots, N,$$

for some $q \in [0, 2\bar{\lambda}]$, and $(\mathfrak{R}^\sigma \hat{f})(p)$ is known to be independent of the choice of such q. Because the Godunov flux $\mathfrak{R}^\sigma \hat{f}$ is associated to bell-shaped flux g, we deduce from the polar decomposition, that there exists some preflux $\hat{\gamma}_{\bar{\lambda}}$ (unique on the image $[0, \bar{\lambda}]$ of the capacity g^σ) such that

$$\hat{\gamma}_{\bar{\lambda}}(\bar{\gamma}) := (\mathfrak{R}^\sigma \hat{f})(p) = \hat{\gamma}_0(g^\sigma(q)) \quad \text{for all} \quad \bar{\gamma} := g^\sigma(p) \in [0, \bar{\lambda}]. \tag{13.15}$$

This is well defined because $(\mathfrak{R}^\sigma \hat{f})(p)$ depends on p only through $g^\sigma(p) = \bar{\gamma}$, and q satisfies

$$\min \left\{ \bar{\gamma}^j, g^{j,-\sigma^j}(q^j) \right\} = \hat{\gamma}_0^j(g^\sigma(q)), \quad j = 1, \ldots, N \tag{13.16}$$

In particular, for such a solution $q \in [0, 2\bar{\lambda}]$, we deduce from (13.14) that

$$\lambda^R := \lambda^{\sigma \diamond \sigma_q} = g^\sigma(q) \quad \text{and} \quad \lambda^L := \lambda^{-\sigma \diamond \sigma_q} = g^{-\sigma}(q) \quad \text{with} \quad \lambda := g(q) \tag{13.17}$$

which shows that $(\lambda^L, \lambda^R) \in \mathcal{R}_{\bar{\gamma}}$, and then $\mathcal{R}_{\bar{\gamma}}$ is not empty.

Step 2: singleton property of $\Lambda_{\bar{\gamma}}$

Conversely, let us consider any $(\tilde{\lambda}^L, \tilde{\lambda}^R) \in R_{\bar{\gamma}}$, which then satisfies

$$\min \left\{ \bar{\gamma}^j, (\tilde{\lambda}^L)^j \right\} = \hat{\gamma}_0^j(\tilde{\lambda}^R), \quad j = 1, \ldots, N \tag{13.18}$$

Given $\sigma \in \{\pm 1\}^N$ as above, it is then easy to see (as for the bijection $\Psi \circ \Phi$) that there exists some $\tilde{q} \in [0, 2\bar{\lambda}]$ such that

$$(g^{-\sigma}(\tilde{q}), g^\sigma(\tilde{q})) = (\tilde{\lambda}^L, \tilde{\lambda}^R)$$

and then (13.18) shows that

$$\min \left\{ \bar{\gamma}^j, (g^{-\sigma}(\tilde{q}))^j \right\} = \hat{\gamma}_0^j(g^\sigma(\tilde{q})), \quad j = 1, \ldots, N$$

From (13.15), we deduce that

$$\hat{\gamma}_{\bar{\lambda}}(\bar{\gamma}) := (\mathfrak{R}^\sigma \hat{f})(p) = \hat{\gamma}_0(g^\sigma(\tilde{q})) = \hat{\gamma}_0(\tilde{\lambda}^R) \quad \text{for all} \quad (\tilde{\lambda}^L, \tilde{\lambda}^R) \in \mathcal{R}_{\bar{\gamma}}$$

which shows that $\Lambda_{\bar{\gamma}}$ is reduced to a singleton.

Step 3: proof of i) and ii)

We want in particular to show that $\bar{\gamma}_{\bar{\lambda}}$ satisfies (13.1). This follows from the fact that by construction $\hat{\gamma}_{\bar{\gamma}}$ is a preflux on the box $[0, \bar{\lambda}]$, and then on $[0, +\infty)^N$ by extension $\bar{\gamma}_{\bar{\lambda}} = \bar{\gamma}_{\bar{\lambda}} \circ T_{\bar{\lambda}}$, due to Lemma 11.12.

It remains to discuss the case of

$$\tilde{\gamma} := \hat{\gamma}_0 \circ T_{\bar{\lambda}}.$$

Let us show that the continuous function $\tilde{\gamma} : [0, +\infty)N \to [0, +\infty)^N$ is Riemann monotone. Assume that

$$(p - q) \diamond \{\tilde{\gamma}(p) - \tilde{\gamma}(q)\} \le 0$$

Let us define $\tilde{p} := T_{\bar{\lambda}}(p)$ and $\tilde{q} := T_{\bar{\lambda}}(q)$. Because we have

$$|\tilde{p}^j - \tilde{q}^j| \le |p^j - q^j| \quad \text{and} \quad (\tilde{p}^j - \tilde{q}^j)(p^j - q^j) \ge 0$$

we deduce

$$(\tilde{p} - \tilde{q}) \diamond \{\hat{\gamma}_0(\tilde{p}) - \hat{\gamma}_0(\tilde{q})\} \le 0$$

Because $\hat{\gamma}_0$ is Riemann monotone, we deduce that $\hat{\gamma}_0(\tilde{p}) - \hat{\gamma}_0(\tilde{q}) = 0$, i.e. $\tilde{\gamma}(p) - \tilde{\gamma}(q) = 0$, and this shows that $\tilde{\gamma}$ is also Riemann monotone.
Now notice that in (13.6) defining $\hat{\gamma}_{\bar{\lambda}}(\bar{\gamma})$, we have $\lambda^R \le \bar{\lambda}$, and then $\hat{\gamma}_0(\lambda^R) = (\hat{\gamma}_0 \circ T_{\bar{\lambda}})(\lambda^R)$. This shows that $\mathfrak{R}_{\bar{\lambda}}(\hat{\gamma}_0 \circ T_{\bar{\lambda}}) = \mathfrak{R}_{\bar{\lambda}}(\hat{\gamma}_0)$, and then proves (13.11).
Step 4: proof of iii)
Let $\hat{\gamma}$ be a preflux such that $\hat{\gamma} = \hat{\gamma} \circ T_{\bar{\lambda}}$. We set

$$\hat{\gamma}_{\bar{\lambda}} := \mathfrak{R}_{\bar{\lambda}}\hat{\gamma}, \quad \hat{f} := \hat{\gamma} \circ g^\sigma$$

where $\hat{f}$ is a Godunov flux on the box $[a,b] := [0, 2\bar{\lambda}]$ such that $\sigma \diamond \hat{f}$ is Riemann monotone. Then on the one hand, we have $\mathfrak{R}^\sigma \hat{f} = \hat{f}$ on $[a,b]$. On the other hand, for $\bar{\gamma} \in [0, \bar{\lambda}]$, we have by construction with $p \in [0, 2\bar{\lambda}]$

$$(\mathfrak{R}_{\bar{\lambda}}\hat{\gamma})(\bar{\gamma}) := (\mathfrak{R}^\sigma \hat{f})(p) = \hat{f}(p) = \hat{\gamma}(\bar{\gamma}) \quad \text{with} \quad \bar{\gamma} := g^\sigma(p)$$

i.e. $\mathfrak{R}_{\bar{\lambda}}\hat{\gamma} = \hat{\gamma}$ on $[0, 2\bar{\lambda}]$. This result then extends to $[0, +\infty)^N$. This shows point iii). This ends the proof of the theorem.

Remark 13.6 (An example for $N = 1$)
For $N = 1$, consider the following quasi-preflux $\hat{\gamma}_0(\bar{\gamma}^1) := \frac{1}{2}\bar{\gamma}^1$ and let $\bar{\lambda} := \bar{\lambda}^1 \in (0, +\infty)$. Then a direct computation shows that $(\mathfrak{R}_{\bar{\lambda}}\hat{\gamma}_0)(\bar{\gamma}^1) = \min\left\{\bar{\gamma}^1, \bar{\lambda}/2\right\}$. This shows that the values of the relaxation $\mathfrak{R}_{\bar{\lambda}}(\hat{\gamma}_0)_{|[0,\bar{\lambda}]}$ does depend on $\bar{\lambda}$.
Still in this case, we see that it is possible to define $\mathfrak{R}_\infty(\hat{\gamma}_0)$ as a limit of $\mathfrak{R}_{\bar{\lambda}}(\hat{\gamma}_0)$ as $\bar{\lambda} \to +\infty$, and this fact will be generalized in Proposition 13.8 for quasi-prefluxes which are uniformly locally constant at infinity.

Lemma 13.7 (Polar decomposition for Godunov quasi-fluxes and compatibility of relaxations)
Assume (2.2) for $N \ge 1$ for a junction (J, f) with compact box $[a,b] \subset \mathbb{R}^N$ and with orientations $\sigma_0 \in \{\pm 1\}^N$. Assume that $f = (f^1, \ldots, f^N)$ is bell-shaped in the sense of Definition 11.6, and let

$$\bar{\lambda}^j := f^j_{\max} = f^j(c^j), \quad j = 1, \ldots, N$$

For $p \in [a,b]$, let

$$f^{\sigma_0}(p) = (f^{1,\sigma_0^1}(p^1), \ldots, f^{N,\sigma_0^N}(p^N)) \quad \text{with} \quad \left\{ \begin{array}{l} f^{j,+}(p^j) := f^j(\min\left\{p^j, c^j\right\}), \\ f^{j,-}(p^j) := f^j(\max\left\{p^j, c^j\right\}) \end{array} \right| , \quad j = 1, \ldots, N. \tag{13.19}$$

Let $g_0 : [a,b] \to \mathbb{R}^N$ satisfying

$$\left\{ \begin{array}{l} g_0 : [a,b] \to [0, \bar{\lambda}] \text{ continuous} \\ \\ \sigma_0 \diamond g_0 : [a,b] \to \mathbb{R}^N \text{ Riemann monotone} \\ \\ 0 \le \sigma_0^j g_0^j(q)_{|q^j = b^j}, \quad 0 \ge \sigma_0^j g_0^j(q)_{|q^j = a^j} \end{array} \right. \tag{13.20}$$

which implies in particular (7.1). In particular g_0 is a a Godunov quasi-flux in the sense of Definition 2.22.
i)(Polar decomposition for Godunov quasi-fluxes and definition of $\hat{\gamma}_0$)
Then for $\bar{\lambda} := f(c) = (f^1(c^1), \ldots, f^N(c^N))$, there exists a uniquely defined function $\hat{\gamma}_0 : [0, +\infty)^N \to \mathbb{R}^N$ satisfying

$$\hat{\gamma}_0 = \hat{\gamma}_0 \circ T_{\bar{\lambda}} \tag{13.21}$$

where $T_{\bar{\lambda}}$ is the truncation operator defined in (13.9), with the following "polar decomposition"

$$g_0 = \hat{\gamma}_0 \circ f^{\sigma_0}$$

Moreover $\hat{\gamma}_0$ is a $\bar{\lambda}$-quasi-preflux.
ii)(Compatibility of relaxations for quasi-fluxes and quasi-prefluxes)
Then we have the following compatibility relation between Riemann relaxations for Godunov quasi-flux g_0 and $\bar{\lambda}$-quasi-preflux $\hat{\gamma}_0$:

$$\mathfrak{R}g_0 = \hat{\gamma}_{\bar{\lambda}} \circ f^{\sigma_0} \quad \textit{with} \quad \hat{\gamma}_{\bar{\lambda}} := \mathfrak{R}_{\bar{\lambda}}\hat{\gamma}_0$$

where $\mathfrak{R}$ is defined in (7.3) for $\sigma := \sigma_0$ and where $\mathfrak{R}_{\bar{\lambda}}$ is defined in (13.10).

Proof of Lemma 13.7
Point i) follows from the monotonicity of the functions f^{j,σ_0^j} and their strict monotonicity on $(f^{j,\sigma_0^j})^{-1}([0,\bar{\lambda}^j))$. This allows to define uniquely $(\hat{\gamma}_0)_{|[0,\bar{\lambda}]}$ and then to extend it through (13.21). Then the fact that $\hat{\gamma}_0$ satisfies (13.1) follows from (13.20). Point ii) follows from a variant of (13.15) in the special case $\sigma := \sigma_0$, with piecewise linear functions g replaced by bell-shaped functions f. This ends the proof of the lemma.

13.2 Riemann relaxation of quasi-prefluxes on the box $[0,+\infty]^N$

For an example of quasi-preflux illustrating this section, we refer the reader to Lemma 14.1.

Proposition 13.8 (Riemann relaxation of quasi-prefluxes on the box $[0,+\infty]^N$)
Assume $N \geq 1$. Let $\hat{\gamma}_0 : [0,+\infty)^N \to \mathbb{R}^N$ be a quasi-preflux which is **boundedly continuous on the box $[0,+\infty]^N$, and boundedly locally constant at infinity**, *in the sense of vi) and vi') in Definition 13.1.*

Given any $\bar{\lambda} \in [0,+\infty)^N$, let $\mathfrak{R}_{\bar{\lambda}}\hat{\gamma}_0$ be the preflux obtained by the Riemann relaxation of the quasi-preflux $\hat{\gamma}_0$, as defined in (13.10).
i) (Definition of $\mathfrak{R}_\infty\hat{\gamma}_0$)
Then there exists a preflux $\hat{\gamma} : [0,+\infty)^N \to [0,+\infty)^N$ such that for every $\bar{\gamma} \in [0,+\infty)^N$, we have

$$\hat{\gamma}(\bar{\gamma}) = \lim_{[0,+\infty)^N \ni \bar{\lambda} \to (+\infty,\ldots,+\infty)} (\mathfrak{R}_{\bar{\lambda}}\hat{\gamma}_0)(\bar{\gamma}) \tag{13.22}$$

We denote this preflux as

$$\mathfrak{R}_\infty\hat{\gamma}_0 := \hat{\gamma}$$

that we call the Riemann relaxation "on the box $[0,+\infty]^N$" of the quasi-preflux $\hat{\gamma}_0$.
ii) (A characterization of $\mathfrak{R}_\infty\hat{\gamma}_0$)
Let $\bar{\lambda}_\infty := (\bar{\lambda}_\infty^1,\ldots,\bar{\lambda}_\infty^N) := (+\infty,\ldots,+\infty)$. We define the doubling set

$$\mathbb{D}_{\bar{\lambda}_\infty} := \left\{(\lambda^L,\lambda^R) \in [0,\bar{\lambda}_\infty]^2, \quad \max\left\{\lambda^{L,j},\lambda^{R,j}\right\} = \bar{\lambda}_\infty^j \quad \textit{for all} \quad j = 1,\ldots,N\right\} \tag{13.23}$$

For any $\bar{\gamma} \in [0,+\infty)^N$, we consider $(\lambda^L,\lambda^R) \in \mathbb{D}_{\bar{\lambda}_\infty}$ solutions of

$$\min\left\{\bar{\gamma}^j,(\lambda^L)^j\right\} = \hat{\gamma}_0^j(\lambda^R), \quad j = 1,\ldots,N \tag{13.24}$$

where when λ^R contains infinite components, then $\hat{\gamma}_0(\lambda^R)$ is uniquely defined as any bounded limit. Then the set

$$\mathcal{R}_{\bar{\gamma}} := \left\{(\lambda^L,\lambda^R) \in \mathbb{D}_{\bar{\lambda}_\infty} \quad \textit{solution of (13.24)}\right\}$$

is non-empty, and the set

$$\Lambda_{\bar{\gamma}} := \left\{\hat{\gamma}_0(\lambda^R), \quad (\lambda^L,\lambda^R) \in \mathcal{R}_{\bar{\gamma}}\right\} \tag{13.25}$$

is reduced to a singleton

$$\Lambda_{\bar{\gamma}} = \{\hat{\gamma}(\bar{\gamma})\}. \tag{13.26}$$

iii) (Additional properties)
Moreover as a quasi-preflux, the function $\hat{\gamma}$ is boundedly continuous on the box $[0,+\infty]^N$ and boundedly locally constant at infinity.

Proof of Proposition 13.8
Step 1: limit along a subsequence
Fix some $\bar\gamma \in [0,+\infty)^N$, and consider some $\bar\lambda \in (0,+\infty)^N$. Then there exists $(\lambda^L_{\bar\lambda}, \lambda^R_{\bar\lambda}) \in \mathbb{D}_{\bar\lambda}$ solution of

$$\min\left\{\bar\gamma^j, \lambda^{jL}_{\bar\lambda}\right\} = \hat\gamma_0^j(\lambda^R_{\bar\lambda}), \quad j = 1,\dots,N$$

with

$$0 \le \hat\gamma_0^j(\lambda^R_{\bar\lambda}) \le \bar\gamma^j$$

As $\bar\lambda \to (+\infty,\dots,+\infty)$, we see that $\hat\gamma_0(\lambda^R_{\bar\lambda})$ stays bounded, and up to extract a subsequence (still denoted by $\bar\lambda$), we can assume that

$$\left\{\begin{array}{rl} (\lambda^L_{\bar\lambda}, \lambda^R_{\bar\lambda}) \to (\lambda^L_\infty, \lambda^R_\infty) & \in \mathbb{D}_{(+\infty,\dots,+\infty)},\\ \hat\gamma_0(\lambda^R_{\bar\lambda}) \to \hat\gamma_0(\lambda^R_\infty) & \in [0,+\infty)^N,\\ \min\left\{\bar\gamma^j, \lambda^{jL}_\infty\right\} = \hat\gamma_0^j(\lambda^R_\infty), & \quad j = 1,\dots,N \end{array}\right.$$

Moreover, because $\hat\gamma_0$ is boundedly continuous on the box $[0,+\infty]^N$ and boundedly locally constant at infinity, we see that there exists $\rho_0 > 0$ (large enough) such that for all $\rho = (\rho^1,\dots,\rho^N)$ with $\rho^j \ge \rho_0$ and

$$\lambda^{j\alpha}_\rho := \min\left\{\lambda^{j\alpha}_\infty, \rho^j\right\}, \quad \text{for} \quad j = 1,\dots,N, \quad \alpha = L, R$$

we have

$$\hat\gamma_0 = const = \hat\gamma_0(\lambda^R_\infty) \quad \text{on} \quad \lambda^R_\rho + \sum_{j\in I}[0,+\infty)e_j \quad \text{with} \quad I := \left\{j \in \{1,\dots,N\}, \quad \lambda^{jR} = +\infty\right\}$$

In particular, we can assume that

$$\left\{\begin{array}{lll} \rho_0 > \bar\gamma^j & \text{for all } j, \\ \rho_0 > \lambda^{jL} & \text{if} \quad \lambda^{jL} < +\infty, \\ \rho_0 > \lambda^{jR} & \text{if} \quad \lambda^{jR} < +\infty, \end{array}\right.$$

and then

$$\left\{\begin{array}{rl} (\lambda^L_\rho, \lambda^R_\rho) & \in \mathbb{D}_\rho,\\ \min\left\{\bar\gamma^j, \lambda^{jL}_\rho\right\} = \hat\gamma_0^j(\lambda^R_\rho) = \hat\gamma_0(\lambda^R_\infty), & \quad j = 1,\dots,N, \end{array}\right.$$

This shows that

$$\hat\gamma_0(\lambda^R_\infty) = (\mathfrak{R}_\rho\hat\gamma_0)(\bar\gamma) \quad \text{for all } \rho = (\rho^1,\dots,\rho^N) \text{ with} \quad \rho^j \ge \rho_0$$

Step 2: change of subsequence and definition of $\hat\gamma(\bar\gamma)$
For another choice of a subsequence, we may get some $(\tilde\lambda^L_\infty, \tilde\lambda^R_\infty) \in \mathbb{D}_{(+\infty,\dots,+\infty)}$ such that $\hat\gamma_0(\tilde\lambda^R_\infty)$ is the limit of $\hat\gamma_0(\lambda^R_{\bar\lambda})$. Up to increase ρ_0, the same reasoning as in Step 1 shows that

$$\hat\gamma_0(\tilde\lambda^R_\infty) = (\mathfrak{R}_\rho\hat\gamma_0)(\bar\gamma) \quad \text{for all } \rho = (\rho^1,\dots,\rho^N) \text{ with} \quad \rho^j \ge \rho_0$$

Hence the limit is independent on the choice of the subsequence, and then is unique. Therefore we can set

$$\hat\gamma(\bar\gamma) := \hat\gamma_0(\lambda^R_\infty)$$

Step 3: perturbation in $\bar\gamma$ and proof of i)
As usual, the uniqueness implies the continuity of $\hat\gamma$. Indeed, consider a sequence $\bar\gamma_n \to \bar\gamma_\infty \in [0,+\infty)^N$, and an associated sequence $(\lambda^L_n, \lambda^R_n) \in \mathbb{D}_{(+\infty,\dots,+\infty)}$ such that

$$\min\left\{\bar\gamma^j_n, (\lambda^L_n)^j\right\} = \hat\gamma_0^j(\lambda^R_n), \quad j = 1,\dots,N$$

Up to extract a convergence subsequence with

(13.27) $$(\lambda^L_n, \lambda^R_n) \to (\lambda^L_\infty, \lambda^R_\infty) \in \mathbb{D}_{(+\infty,\dots,+\infty)} \quad \text{and} \quad \hat\gamma_0(\lambda^R_n) \to \hat\gamma_0(\lambda^R_\infty)$$

we get

(13.28) $$\min\left\{\bar\gamma^j_\infty, (\lambda^L_\infty)^j\right\} = \hat\gamma_0^j(\lambda^R_\infty), \quad j = 1,\dots,N$$

Then

$$\hat{\gamma}(\bar{\gamma}_n) = \hat{\gamma}_0(\lambda_n^R) \to \hat{\gamma}(\bar{\gamma}_\infty) = \hat{\gamma}_0(\lambda_\infty^R)$$

where the last equality follows from arguments of Steps 1 and 2. This shows the continuity of $\hat{\gamma}$.

It remains to show that $\hat{\gamma}$ is locally constant on $\left\{\hat{\gamma} \neq id_{[0,+\infty)}\right\}$. But by construction, we know that for any $\bar{\gamma} \in [0,+\infty)^N$, we have

$$\mathfrak{R}_\rho(\hat{\gamma}_0)(\bar{\gamma}) \to \hat{\gamma}(\bar{\gamma}) \quad \text{as} \quad \rho \to (+\infty, \dots, +\infty)$$

as pointwise limit of functions $\mathfrak{R}_\rho(\hat{\gamma}_0)$ which are locally constant on $\left\{\mathfrak{R}_\rho(\hat{\gamma}_0) \neq id_{[0,+\infty)^N}\right\}$. This is then easy to see that this implies that the limit $\hat{\gamma}$ is also locally constant on $\left\{\hat{\gamma} \neq id_{[0,+\infty)}\right\}$. Therefore $\hat{\gamma}$ is a preflux.

Step 4: proof of ii)

We first notice that construction of Steps 1 and 2 shows that any bounded limit $\hat{\gamma}_0(\lambda^R)$ of $\hat{\gamma}_0(\lambda)$ as $\lambda \to \lambda^R \in [0,+\infty]^N$ is necessarily unique. Moreover, it shows that the set $\mathcal{R}_{\bar{\gamma}}$ is non empty. Because we also have

$$\hat{\gamma}(\bar{\gamma}) = (\mathfrak{R}_\rho \hat{\gamma}_0)(\bar{\gamma}) = \hat{\gamma}_0(\lambda^R)$$

this shows that $\Lambda_{\bar{\gamma}}$ is reduced to the singleton $\hat{\gamma}(\bar{\gamma})$.

Step 5: proof of iii)

Step 5.1: bounded continuity of $\hat{\gamma}$ on the box $[0,+\infty]$

Let us show that $\hat{\gamma}$ is boundedly continuous on the box $[0,+\infty]^N$. To this end, we consider a sequence $\bar{\gamma}_n \to \bar{\gamma}_\infty$ as in Step 3, but now with $\bar{\gamma}_\infty \in [0,+\infty]^N$, where we expect to have $\bar{\gamma}_\infty \in \hat{\gamma}^{-1}([0,\rho_0]^N)$ for some $\rho_0 > 0$. This gives again (13.27)-(13.28), i.e.

$$\left\{ \begin{array}{l} \min\left\{\bar{\gamma}^j_\infty, (\lambda^L_\infty)^j\right\} = \hat{\gamma}^j_0(\lambda^R_\infty) \in [0,\rho_0], \quad j = 1,\dots,N \\ (\lambda^L_\infty, \lambda^R_\infty) \in \mathbb{D}_{(+\infty,\dots,+\infty)} \end{array} \right. \tag{13.29}$$

and

$$\hat{\gamma}(\bar{\gamma}_n) = \hat{\gamma}_0(\lambda_n^R) \to \hat{\gamma}_0(\lambda_\infty^R)$$

where we recall that we assume $\hat{\gamma}_0(\lambda^R_\infty) \in [0,\rho_0]^N$. Hence this provides the candidate $\hat{\gamma}_0(\lambda^R_\infty)$ to be $\hat{\gamma}(\bar{\gamma}_\infty)$. We still have to show that this value is uniquely defined. But by assumption,

$$\min\left\{\tilde{\gamma}^j_{\rho_0}, (\lambda^L_\infty)^j\right\} = \hat{\gamma}^j_0(\lambda^R_\infty) \quad \text{with} \quad \tilde{\gamma}^j_{\rho_0} := \min\left\{\bar{\gamma}^j_\infty, \rho_0\right\}$$

Hence

$$\hat{\gamma}(\bar{\gamma}_\infty) = \hat{\gamma}_0(\lambda^R_\infty) = \hat{\gamma}(\tilde{\gamma}_{\rho_0})$$

which shows the uniqueness of the value, and then defines $\hat{\gamma}(\bar{\gamma}_\infty)$. This shows in particular that $\hat{\gamma}$ is continuous on $\overline{\hat{\gamma}^{-1}([0,\rho_0]^N)}^{[0,+\infty]^N}$. Therefore $\hat{\gamma}$ is boundedly continuous on the box $[0,+\infty]^N$ as a quasi-preflux in the sense of Definition 13.1.

Step 5.2: $\hat{\gamma}$ boundedly locally constant at infinity

Assume that there exists $\rho_0 > 0$ and $\bar{\gamma} \in [0,+\infty]^N$ such that $\hat{\gamma}(\bar{\gamma}) \in [0,\rho_0]^N$, and that

$$I := \left\{ j \in \{1,\dots,N\}, \quad \bar{\gamma}^j = +\infty \right\} \neq \emptyset$$

Let $(\lambda^L, \lambda^R) \in \mathbb{D}_{+\infty,\dots,+\infty)}$ be a solution of

$$\min\left\{\bar{\gamma}^j, (\lambda^L)^j\right\} = \hat{\gamma}^j_0(\lambda^R), \quad j = 1,\dots,N$$

This implies that $(\lambda^L)^j \leq \rho_0$ for all $j \in I$. Therefore

$$\hat{\gamma} = const = \hat{\gamma}(\bar{\gamma}) = \hat{\gamma}_0(\lambda^R) \quad \text{on} \quad \bar{\gamma}_{\rho_0} + \sum_{j\in I} [0,+\infty) e_j$$

This shows that $\hat{\gamma}$ is boundedly locally constant at infinity, as a quasi-preflux.

Step 5.3: conclusion

We conclude that Steps 5.1 and 5.2 show point iii).

This ends the proof of the proposition.

13.3 Transfer of properties by Riemann relaxation of quasi-prefluxes

Lemma 13.9 (Transfer of properties by Riemann relaxation of quasi-prefluxes)
Assume $N \geq 1$ with sign vector $\sigma_0 \in \{\pm 1\}^N$. Let $\bar{\lambda} \in (0,+\infty)^N$. Let $\hat{\gamma}_0 : [0,+\infty)^N \to \mathbb{R}^N$ be a $\bar{\lambda}$-quasi-preflux such that $\hat{\gamma}_0 = \hat{\gamma}_0 \circ T_{\bar{\lambda}}$, where $T_{\bar{\lambda}}$ is the truncation operator defined in (13.9). Let $\hat{\gamma}_{\bar{\lambda}} := \mathfrak{R}_{\bar{\lambda}}\hat{\gamma}_0$ be the preflux obtained by the Riemann relaxation of the quasi-preflux $\hat{\gamma}_0$, as defined in (13.10).
i) (Finite $\bar{\lambda}$)
If the quasi-preflux $\hat{\gamma}_0$ is HJ (resp. Kružkov, σ_0-monotone Kružkov, σ_0-monotone, conservative), then the preflux $\mathfrak{R}_{\bar{\lambda}}\hat{\gamma}_0$ is HJ (resp. Kružkov, σ_0-monotone Kružkov, σ_0-monotone, conservative).
In particular if the quasi-preflux $\hat{\gamma}_0$ is only Kružkov, then $\mathfrak{R}_{\bar{\lambda}}\hat{\gamma}_0$ is Lipschitz Kružkov (as a quasi-preflux).
ii) (Case of $\bar{\lambda} = (+\infty,\dots,+\infty)$)
Assume furthermore that the quasi-preflux $\hat{\gamma}_0 : [0,+\infty)^N \to [0,+\infty)^N$ is **boundedly continuous on the box $[0,+\infty]^N$, and boundedly locally constant at infinity**, *in the sense of vi) and vi') in Definition 13.1. Let $\mathfrak{R}_\infty\hat{\gamma}_0$ be the Riemann relaxation on the box $[0,+\infty]^N$ given by Proposition 13.8 for $\bar{\lambda} = (+\infty,\dots,+\infty)$. Then the result of point i) is still valid for $\mathfrak{R}_\infty\hat{\gamma}_0$.*

Proof of Lemma 13.9
Step 1: proof of i)
Recall that by construction, we have

$$(\mathfrak{R}_{\bar{\lambda}}\hat{\gamma}_0) \circ g^\sigma = \mathfrak{R}^\sigma \hat{f} \quad \text{with} \quad \hat{f} := \hat{\gamma}_0 \circ g^\sigma$$

where $\mathfrak{R}^\sigma := \mathfrak{R}$ is the Riemann relaxation operator for junctions oriented by σ (that may be chosen different or equal to σ_0). Then the result follows immediatetly from the polar decomposition i) of Lemma 13.7 and from Proposition 7.4, which is the analogue of Lemma 13.9 for relaxation of Godunov quasi-fluxes to Godunov fluxes, instead of relaxation of functions to prefluxes. Notice that for σ_0-monotonicity and conservativity, it is convenient to choose $\sigma := \sigma_0$.
Step 2: proof of ii)
It is easy to see that the result follows from formula (13.22) defining $\mathfrak{R}_\infty$ as a limit of $\mathfrak{R}_{\bar{\lambda}}$.
This ends the proof of the Lemma.

13.4 Gluing of quasi-prefluxes

We introduce the following definition (similar to Definition 5.11) which will be useful for gluing along index j.

Definition 13.10 (j-local quasi-constancy)
Let $N \geq 1$ and some quasi-preflux $\hat{\lambda} = (\hat{\lambda}^1,\dots,\hat{\lambda}^N) : [0,+\infty)^N \to \mathbb{R}^N$ in the sense of Definition 13.1.
Let $j \in \{1,\dots,N\}$. Then we say that $\hat{\lambda}$ is j-locally quasi-constant if it satisfies the following.
For any $\bar{\gamma} \in [0,+\infty)^N$, let

$$\Phi(x) := \hat{\lambda}(\bar{\gamma} + x \cdot e_j) \quad \textit{defined for } \bar{\gamma}^j + x \in [0,+\infty).$$

Then

$$\Phi = const = \Phi(0) \quad on \quad \left\{x \in [-\bar{\gamma}^j,+\infty), \quad \Phi^j(x) = \Phi^j(0)\right\}$$

Remark 13.11 *Notice that by definition, any preflux $\hat{\gamma}$ is a quasi-preflux which is j-locally quasi-constant in any index j.*

Example: (Lebacque quasi-preflux)
Consider the following Lebacque quasi-preflux for $\theta = (\theta^1,\theta^2) \in (0,1)^2$ with $\theta^1 + \theta^2 = 1$ and for $\bar{\gamma} = (\bar{\gamma}^0,\bar{\gamma}^1,\bar{\gamma}^2) \in [0,+\infty)^3$

$$\begin{cases} \hat{\lambda}^k(\bar{\gamma}) := \min\left\{\bar{\gamma}^k, \theta^k\bar{\gamma}^0\right\}, \quad k = 1,2 \\ \hat{\lambda}^0(\bar{\gamma}) := \hat{\lambda}^1(\bar{\gamma}) + \hat{\lambda}^2(\bar{\gamma}) \end{cases}$$

Then it is easy to check that the quasi-preflux $\hat{\lambda}$ is j-locally quasi-constant for all $j = 0,1,2$, but is not locally constant on $\left\{\hat{\gamma} \neq id_{[0,+\infty)^3}\right\}$.

Lemma 13.12 (Gluing of quasi-prefluxes)
*Let $\gamma := \alpha, \beta$, and $N_\gamma \geq 2$, and some quasi-prefluxes $\hat{\lambda}_\gamma : [0,+\infty)^{N_\gamma} \to \mathbb{R}^{N_\gamma}$, with notation $\hat{\lambda}_\gamma = (\hat{\lambda}^0_\gamma, \dots, \hat{\lambda}^{N_\gamma -1}_\gamma)$. Let $j_\gamma \in \{0,\dots,N_\gamma - 1\}$. Assume that each quasi-preflux $\hat{\lambda}_\gamma$ is j_γ-***locally quasi-constant** *in the sense of Definition 13.10. Let also consider the doubling set*

$$\mathbb{D}_{\bar{\lambda}^0} := \left\{(\lambda^L,\lambda^R) \in [0,\bar{\lambda}^0]^2, \quad \max\left\{\lambda^L,\lambda^R\right\} = \bar{\lambda}^0\right\} \quad \text{for some } \bar{\lambda}^0 \in [0,+\infty]$$

Then for any $\bar{\gamma} = (\bar{\gamma}_\alpha, \bar{\gamma}_\beta) \in [0,+\infty)^{N_\alpha -1} \times [0,+\infty)^{N_\beta - 1}$, consider the equation

$$\hat{\lambda}^{j_\alpha}_\alpha(\bar{\gamma}_\alpha, \lambda^L) = \hat{\lambda}^{j_\beta}_\beta(\bar{\gamma}_\beta, \lambda^R) \tag{13.30}$$

In order to simplify the presentation, let us assume that $j_\alpha = 0 = j_\beta$. Then the set

$$R := \left\{(\lambda^L,\lambda^R) \in \mathbb{D}_{\bar{\lambda}^0}, \quad \text{with } (\lambda^L,\lambda^R) \text{ solution of (13.30)}\right\}$$

is non empty and let us consider the set

$$\Lambda := \left\{\tilde{\lambda}(\bar{\gamma},\lambda^L,\lambda^R) \quad \text{with } (\lambda^L,\lambda^R) \in R\right\}$$

with

$$\tilde{\lambda}(\bar{\gamma},\lambda^L,\lambda^R) = (\hat{\lambda}^1_\alpha(\bar{\gamma}_\alpha,\lambda^L), \dots, \hat{\lambda}^{N_\alpha-1}_\alpha(\bar{\gamma}_\alpha,\lambda^L); \hat{\lambda}^1_\beta(\bar{\gamma}_\beta,\lambda^R), \dots, \hat{\lambda}^{N_\beta-1}_\beta(\bar{\gamma}_\beta,\lambda^R))$$

Then Λ is reduced to a singleton $\Lambda = \{\lambda\}$, **if $\bar{\lambda}^0$ is finite***. Then this defines the following map*

$$\begin{array}{llll} \underline{\hat{\lambda}}: & [0,+\infty)^{N_\alpha-1} \times [0,+\infty)^{N_\beta-1} & \to & \mathbb{R}^{N_\alpha+N_\beta-2} \\ & (\bar{\gamma}_\alpha,\bar{\gamma}_\beta) & \mapsto & \underline{\hat{\lambda}}(\bar{\gamma}_\alpha,\bar{\gamma}_\beta) := \lambda \end{array}$$

0) (Gluing quasi-prefluxes)
Then $\underline{\hat{\lambda}}$ is a quasi-preflux. We use the notation

$$\hat{\lambda}_\alpha \underset{(j_\alpha:j_\beta,\bar{\lambda}^0)}{\sharp} \hat{\lambda}_\beta := \underline{\hat{\lambda}} \tag{13.31}$$

which is defined here for $j_\alpha = 0 = j_\beta$ (and that can be easily generalized for indices $j_\alpha \in \{0,\dots,N_\alpha - 1\}$ and $j_\beta \in \{0,\dots,N_\beta-1\}$).
i) (Gluing Lipschitz Kružkov quasi-prefluxes)
Assume that $\hat{\lambda}_\gamma$ are Lipschitz Kružkov quasi-prefluxes for $\gamma = \alpha,\beta$. Then $\underline{\hat{\lambda}}$ is also a Lipschitz Kružkov quasi-preflux.
ii) (Gluing HJ quasi-prefluxes)
Assume that $\hat{\lambda}_\gamma$ are HJ quasi-prefluxes for $\gamma = \alpha,\beta$. Then $\underline{\hat{\lambda}}$ is also a HJ quasi-preflux.
iii) (Gluing monotone quasi-prefluxes)
Assume that $\hat{\lambda}_\gamma$ are monotone quasi-prefluxes for $\gamma = \alpha,\beta$ for orientations $\sigma_\gamma \in \{\pm 1\}^{N_\gamma}$. Then $\underline{\hat{\lambda}}$ is also a monotone quasi-preflux for orientation $\underline{\sigma}$ with

$$\underline{\sigma} = (\sigma^1_\alpha,\dots,\sigma^{N_\alpha-1}_\alpha;\sigma^1_\beta,\dots,\sigma^{N_\beta-1}_\beta). \tag{13.32}$$

iv) (Gluing conservative quasi-prefluxes)
Assume that $\hat{\lambda}_\gamma$ are conservative quasi-prefluxes for $\gamma = \alpha,\beta$ for orientations $\sigma_\gamma \in \{\pm 1\}^{N_\gamma}$. Then $\underline{\hat{\lambda}}$ is also a conservative quasi-preflux for orientation $\underline{\sigma}$ given by (13.32).
v) (Case $\bar{\lambda}^0 = +\infty$)
When $\bar{\lambda}^0 = +\infty$, we assume furthermore the following condition for both quasi-prefluxes $\hat{\lambda}_\gamma$ and in the sense of Definition 13.1:

$$\left\{\begin{array}{l} \hat{\lambda}_\gamma \text{ are boundedly continuous on the box } [0,+\infty]^{N_\gamma} \text{ and boundedly locally constant at infinity,} \\ \hat{\lambda}_\gamma \text{ are uniformly locally bounded,} \\ \hat{\lambda}_\gamma \text{ are } \sigma_\gamma\text{-conservative} \end{array}\right. \tag{13.33}$$

Then points 0)-iv) are still true. Moreover $\underline{\hat{\lambda}}$ does satisfy (13.33), and is also given by

$$\underline{\hat{\lambda}} = \lim_{\rho_0 \to +\infty} \quad \hat{\lambda}_\alpha \underset{(j_\alpha:j_\beta;\rho_0)}{\sharp} \hat{\lambda}_\beta$$

and we denote it as

$$\hat{\lambda}_\alpha \underset{(j_\alpha:j_\beta;+\infty)}{\sharp} \hat{\lambda}_\beta := \underline{\hat{\lambda}}.$$

Lemma 13.12 implies in particular Theorem 2.44 of the Introduction (because a preflux is in particular a quasi-preflux which is j-locally quasi-constant for all j).

Proof of Lemma 13.12
Step 1: case $\bar\lambda^0 < +\infty$
Step 1.1: general construction
For $\bar\lambda_\gamma = (\bar\lambda_\gamma^0, \dots, \bar\lambda_\gamma^{N_\gamma-1}) \in (0,+\infty)^{N_\gamma}$ with $\bar\lambda_\gamma^0 = \bar\lambda^0$ and $j \in \{0,\dots,N_\gamma - 1\}$, we define the function

$$g^j_{\bar\lambda_\gamma}(r) := \min\left\{r, 2\bar\lambda_\gamma^j - r\right\} \quad \text{for} \quad r \in [0, 2\bar\lambda^0]$$

Consider $\sigma_\gamma \in \{\pm 1\}^{N_\gamma}$ with $\sigma_\alpha^0 = -1$ and $\sigma_\beta^0 = 1$. Setting

$$\hat\gamma_\gamma := \hat\lambda_\gamma \circ g^{\sigma_\gamma}_{\bar\lambda_\gamma}$$

we see that $\hat\gamma_\gamma$ is a Godunov quasi-flux, which is moreover 0-locally quasi-constant, because the quasi-preflux $\hat\lambda_\gamma$ is 0-locally quasi-constant. Then it is easy to see that

$$\hat\gamma = \underline{\hat\lambda} \circ \underline{g}^{\underline{\sigma}} \quad \text{with} \quad \begin{cases} \hat\gamma := \hat\gamma_\alpha \underset{0:0}{\sharp} \hat\gamma_\beta \\ \underline{\hat\lambda} := \hat\lambda_\alpha \underset{(0:0,\bar\lambda^0)}{\sharp} \hat\lambda_\beta \end{cases}$$

with

$$\begin{cases} \underline{\sigma} := (\sigma_\alpha^1, \dots, \sigma_\alpha^{N_\alpha-1}; \sigma_\beta^1, \dots, \sigma_\beta^{N_\beta-1}) \\ \underline{g} := (g_\alpha^1, \dots, g_\alpha^{N_\alpha-1}; g_\beta^1, \dots, g_\beta^{N_\beta-1}) \end{cases}$$

where $\hat\gamma$ is a Godunov quasi-flux, because of Proposition 5.23. Hence polar decomposition for Godunov quasi-fluxes (see i) of Lemma 13.7) shows that $\underline{\hat\lambda}$ is a quasi-preflux, at least on the box $\prod_{j\neq 0}[0,\bar\lambda_\alpha^j] \times \prod_{j\neq 0}[0,\bar\lambda_\beta^j]$. This shows that $\underline{\hat\lambda}$ is a quasi-preflux in the limit $\bar\lambda_\gamma^j \to +\infty$ for $j \neq 0$.
Step 1.2: additional properties
When the quasi-prefluxes $\hat\lambda_\gamma$ are Lipschitz Kružkov (resp. HJ, σ_γ-monotone, conservative), then the quasi-fluxes $\hat\gamma_\gamma$ are also Kružkov (resp. HJ, monotone, conservative). Then from Proposition 5.23, we deduce that $\underline{\hat\gamma}$ is Lipschitz Kružkov (resp. HJ, monotone, conservative). From polar decomposition (see i) of Lemma 13.7) and the very special expression of our choice of $\underline{g}^{\underline{\sigma}}$, we deduce that $\underline{\hat\lambda}$ is then Lipschitz Kružkov (resp. HJ, $\underline{\sigma}$-monotone, conservative).
Step 2: case $\bar\lambda^0 = +\infty$
Step 2.1: definition of $\underline{\lambda}$ at infinity
Step 2.1.1: first value for a subsequence
We want to deduce the result from Step 1, in the limit $\bar\lambda^0 \to +\infty$. From (13.30), we have for $(\lambda^L, \lambda^R) = (\lambda^L_{\bar\lambda^0}, \lambda^R_{\bar\lambda^0})$

$$\hat\lambda^0_\alpha(\bar\gamma_\alpha, \lambda^L_{\bar\lambda^0}) = \hat\lambda^0_\beta(\bar\gamma_\beta, \lambda^R_{\bar\lambda^0})$$

Our assumptions (13.33) do ensure that

$$\begin{cases} \hat\lambda^j_\alpha(\bar\gamma_\alpha, \lambda^L_{\bar\lambda^0}) \le F(\bar\gamma^j_\alpha), \\ \hat\lambda^j_\beta(\bar\gamma_\beta, \lambda^R_{\bar\lambda^0}) \le F(\bar\gamma^j_\beta) \end{cases} \quad \text{for} \quad j \neq 0$$

The fact that each $\hat\lambda_\gamma$ is conservative implies

$$\begin{cases} \hat\lambda^0_\alpha(\bar\gamma_\alpha, \lambda^L_{\bar\lambda^0}) \le \displaystyle\sum_{j=1,\dots,N_\alpha-1} F(\bar\gamma^j_\alpha), \\ \hat\lambda^0_\beta(\bar\gamma_\beta, \lambda^R_{\bar\lambda^0}) \le \displaystyle\sum_{j=1,\dots,N_\beta-1} F(\bar\gamma^j_\beta) \end{cases}$$

We deduce that both $\hat\lambda_\alpha(\bar\gamma_\alpha, \lambda^L_{\bar\lambda^0})$ and $\hat\lambda_\beta(\bar\gamma_\beta, \lambda^R_{\bar\lambda^0})$ are bounded when $\bar\lambda^0 \to +\infty$. Because each quasi-preflux $\hat\lambda_\gamma$ is boundedly continuous on the box $[0,+\infty]^{N_\gamma}$ in the sense of Definition 13.1, we deduce (extracting a subsequence convergent in $[0,+\infty]^2$) that at the limit $(\lambda^L_{\bar\lambda^0}, \lambda^R_{\bar\lambda^0}) \to (\lambda^L_\infty, \lambda^R_\infty) \in [0,+\infty]^2$, we have

$$\hat\lambda^0_\alpha(\bar\gamma_\alpha, \lambda^L_\infty) = \hat\lambda^0_\beta(\bar\gamma_\beta, \lambda^R_\infty) \quad \text{with} \quad (\lambda^L_\infty, \lambda^R_\infty) \in \mathbb{D}_{+\infty} \tag{13.34}$$

with candidate for $\underline{\hat{\lambda}}(\bar{\gamma}_\alpha, \bar{\gamma}_\beta)$ the following quantity

$$\underline{\lambda}_\infty := ((\hat{\lambda}_\alpha^1, \dots, \hat{\lambda}_\alpha^{N_\alpha - 1})(\bar{\gamma}_\alpha, \lambda_\infty^L); (\hat{\lambda}_\beta^1, \dots, \hat{\lambda}_\beta^{N_\beta - 1})(\bar{\gamma}_\beta, \lambda_\infty^R)).$$

Step 2.1.2: value independent of the choice of the subsequence
At this point, let us consider another solution $(\tilde{\lambda}_\infty^L, \tilde{\lambda}_\infty^R) \in \mathbb{D}_{+\infty}$ of (13.34), and the quantity

$$\tilde{\underline{\lambda}}_\infty := ((\hat{\lambda}_\alpha^1, \dots, \hat{\lambda}_\alpha^{N_\alpha - 1})(\bar{\gamma}_\alpha, \tilde{\lambda}_\infty^L); (\hat{\lambda}_\beta^1, \dots, \hat{\lambda}_\beta^{N_\beta - 1})(\bar{\gamma}_\beta, \tilde{\lambda}_\infty^R)).$$

Then we have to show that

$$\tilde{\underline{\lambda}}_\infty = \underline{\lambda}_\infty \tag{13.35}$$

Indeed, because both quasi-prefluxes $\hat{\lambda}_\gamma$ are boundedly locally constant at infinity, we see that we can replace any infinite components of $(\lambda_\infty^L, \lambda_\infty^R)$ and $(\tilde{\lambda}_\infty^L, \tilde{\lambda}_\infty^R)$ by finite ones equal to $\rho_0 > 0$ for some $\rho_0 > 0$ large enough, and then assume that

$$(\lambda_\infty^L, \lambda_\infty^R), (\tilde{\lambda}_\infty^L, \tilde{\lambda}_\infty^R) \in \mathbb{D}_{\rho_0}$$

whithout changing the values of the quasi-prefluxes. This shows that

$$\underline{\lambda}_\infty = (\hat{\lambda}_\alpha \underset{(0:0;\rho_0)}{\sharp} \hat{\lambda}_\beta)(\bar{\gamma}) = \tilde{\underline{\lambda}}_\infty$$

which shows (13.35). This shows that $\underline{\hat{\lambda}} : [0, +\infty)^{N_\alpha + N_\beta - 2} \to \mathbb{R}^{N_\alpha + N_\beta - 2}$ is well defined, and satisfes

$$\underline{\hat{\lambda}} = \lim_{\rho_0 \to +\infty} \hat{\lambda}_\alpha \underset{(0:0;\rho_0)}{\sharp} \hat{\lambda}_\beta.$$

Step 2.2: first additional properties
We now have to show that $\underline{\hat{\lambda}}$ is a quasi-preflux. The continuity is easy, because this is a perturbation argument of Step 2.1, and follows (as it is usual) from the uniqueness of the limit value $\underline{\lambda}_\infty$. The bounds

$$\underline{\hat{\lambda}}^I_{|\bar{\gamma}^I = 0} = 0 \quad \text{for all index } I$$

follow immediately from the same property for each $\hat{\lambda}_\gamma$.
We still have to show that $\underline{\hat{\lambda}} : [0, +\infty)^{N_\alpha + N_\beta - 2} \to \mathbb{R}^{N_\alpha + N_\beta - 2}$ is Riemann monotone. To this end, consider $\bar{\gamma} = (\bar{\gamma}_\alpha, \bar{\gamma}_\beta)$ and $\bar{\gamma}' = (\bar{\gamma}'_\alpha, \bar{\gamma}'_\beta)$, and assume that

$$(\bar{\gamma}' - \bar{\gamma}) \diamond \left\{ \underline{\hat{\lambda}}(\bar{\gamma}') - \underline{\hat{\lambda}}(\bar{\gamma}) \right\} \leq 0$$

The point is that there exists ρ_0 large enough such that

$$\begin{cases} \underline{\hat{\lambda}}(\bar{\gamma}') = (\hat{\lambda}_\alpha \underset{(0:0;\rho_0)}{\sharp} \hat{\lambda}_\beta)(\bar{\gamma}') \\ \underline{\hat{\lambda}}(\bar{\gamma}) = (\hat{\lambda}_\alpha \underset{(0:0;\rho_0)}{\sharp} \hat{\lambda}_\beta)(\bar{\gamma}) \end{cases}$$

Hence, because $\hat{\lambda}_\alpha \underset{(0:0;\rho_0)}{\sharp} \hat{\lambda}_\beta$ is Riemann monotone, we see that this implies that $\underline{\hat{\lambda}}$ is also Riemann monotone.
Therefore $\underline{\hat{\lambda}}$ is a quasi-preflux.

In particular $\underline{\hat{\lambda}}$ is $\underline{\sigma}$-conservative and is uniformly locally bounded. It also satisfies all expected additional properties as in Step 1.2, for instance just by passage to the limit $\rho_0 \to +\infty$.
Step 2.3: checking that $\underline{\hat{\lambda}}$ is boundedly continuous on the box $[0, +\infty]^{N_\alpha + N_\beta - 2}$
Consider a sequence $\bar{\gamma}_n = (\bar{\gamma}_{\alpha,n}, \bar{\gamma}_{\beta,n}) \to \bar{\gamma}_\infty = (\bar{\gamma}_{\alpha,\infty}, \bar{\gamma}_{\beta,\infty}) \in [0, +\infty]^{N_\alpha + N_\beta - 2}$ such that $\underline{\hat{\lambda}}(\bar{\gamma}_n)$ stays bounded. Let $(\lambda_n^L, \lambda_n^R) \in \mathbb{D}_{+\infty}$ solution of

$$\hat{\lambda}_\alpha^0(\bar{\gamma}_{\alpha,n}, \lambda_n^L) = \hat{\lambda}_\beta^0(\bar{\gamma}_{\beta,n}, \lambda_n^R)$$

which stays bounded by the argument of Step 2.1. Hence, up to extract a subsequence (still denoted by (n)), we have

$$(\lambda_n^L, \lambda_n^R) \to (\lambda_\infty^L, \lambda_\infty^R) \in \mathbb{D}_{+\infty}$$

and

$$\hat{\lambda}^0_\alpha(\bar{\gamma}_{\alpha,\infty}, \lambda^L_\infty) = \hat{\lambda}^0_\beta(\bar{\gamma}_{\beta,\infty}, \lambda^R_\infty)$$

and

$$\underline{\hat{\lambda}}(\bar{\gamma}_n) \to ((\hat{\lambda}^1_\alpha, \dots, \hat{\lambda}^{N_\alpha-1}_\alpha)(\bar{\gamma}_{\alpha,\infty}, \lambda^L_\infty); (\hat{\lambda}^1_\beta, \dots, \hat{\lambda}^{N_\beta-1}_\beta)(\bar{\gamma}_{\beta,\infty}, \lambda^R_\infty)) =: \underline{\lambda}_\infty$$

Here the value $\underline{\lambda}_\infty$ is well-defined for our subsequence, because both quasi-prefluxes $\hat{\lambda}_\gamma$ are boundedly continuous on the boxes $[0,+\infty]^{N_\gamma}$. Moreover, up to replace every infinite components of $\bar{\gamma}_\infty$ and of $(\lambda^L_\infty, \lambda^R_\infty)$, we can apply the argument of Step 2.1.2, to deduce that the value $\underline{\lambda}_\infty$ is independent of the choice of the subsequence. Therefore the value $\underline{\hat{\lambda}}(\bar{\gamma}_\infty)$ is uniquely defined. Moreover, a variant of this argument also shows that this value is continuous as a function of $\bar{\gamma}_\infty$, i.e. that $\underline{\hat{\lambda}}(\bar{\gamma}'_\infty) \to \underline{\hat{\lambda}}(\bar{\gamma}_\infty)$ as $\underline{\hat{\lambda}}(\bar{\gamma}'_\infty)$ stays bounded and $[0,+\infty]^{N_\alpha+N_\beta-2} \ni \bar{\gamma}'_\infty \to \bar{\gamma}'_\infty \in [0,+\infty]^{N_\alpha+N_\beta-2}$. We conclude that $\underline{\hat{\lambda}}$ is boundedly continuous on the box $[0,+\infty]^{N_\alpha+N_\beta-2}$.

Step 2.4: checking that $\underline{\hat{\lambda}}$ is locally constant at infinity

Consider now some $\bar{\gamma} \in [0,+\infty]^{N_\alpha+N_\beta-2}$ with

$$\mathcal{I} := \left\{\text{index } K, \text{ with } \bar{\gamma}^K = +\infty \text{ with } \underline{\hat{\lambda}}(\bar{\gamma}) \text{ bounded}\right\} \neq \emptyset$$

Then there exists $(\lambda^L, \lambda^R) \in \mathbb{D}_{+\infty}$ such that

$$\hat{\lambda}^0(\bar{\gamma}_\alpha, \lambda^L) = \hat{\lambda}^0_\beta(\bar{\gamma}_\beta, \lambda^R) \in [0,+\infty)$$

and

$$\underline{\hat{\lambda}}(\bar{\gamma}) = ((\hat{\lambda}^1_\alpha, \dots, \hat{\lambda}^{N_\alpha-1}_\alpha)(\bar{\gamma}_\alpha, \lambda^L); (\hat{\lambda}^1_\beta, \dots, \hat{\lambda}^{N_\beta-1}_\beta)(\bar{\gamma}_\beta, \lambda^R))$$

Let $\rho > 0$ (large enough) such that for

$$\bar{\gamma}^j_{\gamma,\rho} := \begin{cases} \rho & \text{if} \quad (\gamma, j) \in \mathcal{I} \\ \bar{\gamma}^j_\gamma < \rho & \text{if} \quad (\gamma, j) \notin \mathcal{I} \end{cases}$$

and the sets

$$\mathcal{I}_\gamma := \{(\gamma, j) \in \mathcal{I}\}$$

and also

$$\lambda^C_\rho := \min\left\{\rho, \lambda^C\right\}, \quad C = L, R$$

with $\rho > \min\left\{\lambda^L, \lambda^R\right\}$ in case of $\min\left\{\lambda^L, \lambda^R\right\} < +\infty$. Then $(\lambda^L_\rho, \lambda^R_\rho) \in \mathbb{D}_\rho$. Because both $\hat{\lambda}_\gamma$ are boundedly locally constant at infinity, we deduce that

$$\begin{cases} \hat{\lambda}_\alpha = const \quad \text{on} \quad (\bar{\gamma}_{\alpha,\rho}, \lambda^L_\rho) + \left\{ \displaystyle\sum_{(\alpha,j)\in\mathcal{I}_\alpha} [0,+\infty)e_j \right\} + \delta_\alpha[0,+\infty)e_0 \\ \hat{\lambda}_\beta = const \quad \text{on} \quad (\bar{\gamma}_{\beta,\rho}, \lambda^R_\rho) + \left\{ \displaystyle\sum_{(\beta,j)\in\mathcal{I}_\beta} [0,+\infty)e_j \right\} + \delta_\beta[0,+\infty)e_0 \end{cases}$$

where

$$\delta_\alpha := \begin{cases} 1 & \text{if} \quad \lambda^L = +\infty \\ 0 & \text{if} \quad \lambda^L < +\infty \end{cases} \quad \text{and} \quad \delta_\beta := \begin{cases} 1 & \text{if} \quad \lambda^R = +\infty \\ 0 & \text{if} \quad \lambda^R < +\infty \end{cases}$$

In particular, it is easy to see that this implies

$$\underline{\hat{\lambda}} = const \quad \text{on} \quad \bar{\gamma}_\rho + \sum_{K\in\mathcal{I}} [0,+\infty)e_K \quad \text{with} \quad \bar{\gamma}_\rho := (\bar{\gamma}_{\alpha,\rho}, \bar{\gamma}_{\beta,\rho})$$

which shows that $\underline{\hat{\lambda}}$ is boundedly locally constant at infinity.

Step 2.5: conclusion

We conclude that $\underline{\hat{\lambda}}$ satisfies (13.33). This ends the proof of the lemma.

We refrain ourself to state and prove similar results on self-gluing and also on all possible associative properties, but the reader can easily get them, when necessary, following the lines of this work.

13.5 Commutation of relaxation and gluing of quasi-prefluxes

Lemma 13.13 (Commutation of Riemann relaxation and gluing of quasi-prefluxes)
*Let $\gamma := \alpha, \beta$, and $N_\gamma \ge 2$, and some quasi-prefluxes $\hat{\lambda}_\gamma : [0,+\infty)^{N_\gamma} \to [0,+\infty)^{N_\gamma}$, with notation $\hat{\lambda}_\gamma = (\hat{\lambda}^0_\gamma, \dots, \hat{\lambda}^{N_\gamma-1}_\gamma)$. Let $j_\gamma \in \{0,\dots,N_\gamma - 1\}$. Assume that each quasi-preflux $\hat{\lambda}_\gamma$ is j_γ-**locally constant** in the sense of Definition 13.10.*
Let $\bar{\lambda}_\gamma \in (0,+\infty)^{N_\gamma}$ be a flux-limiter with $\bar{\lambda}^0_\alpha = \bar{\lambda}^0_\beta =: \bar{\lambda}^0$. We set

$$\underline{\bar{\lambda}} := (\bar{\lambda}^1_\alpha, \dots, \bar{\lambda}^{N_\alpha-1}_\alpha; \bar{\lambda}^1_\beta, \dots, \bar{\lambda}^{N_\beta-1}_\beta) \in (0,+\infty)^{N_\alpha+N_\beta-2}$$

i) (Finite $\bar{\lambda}^0$)
Then we have the following commutation of gluing and Riemann relaxation

$$\mathfrak{R}_{\underline{\bar{\lambda}}}\underline{\hat{\lambda}} = (\mathfrak{R}_{\bar{\lambda}_\alpha}\hat{\lambda}_\alpha) \underset{(j_\alpha:j_\beta,\bar{\lambda}^0)}{\sharp} (\mathfrak{R}_{\bar{\lambda}_\beta}\hat{\lambda}_\beta) \quad \text{with} \quad \underline{\hat{\lambda}} := \hat{\lambda}_\alpha \underset{(j_\alpha:j_\beta,\bar{\lambda}^0)}{\sharp} \hat{\lambda}_\beta$$

ii) (Case $\bar{\lambda}^0 = +\infty$)
When $\bar{\lambda}^0 = +\infty$, we assume furthermore the following condition for both quasi-prefluxes $\hat{\lambda}_\gamma$ and in the sense of Definition 13.1:

$$(13.36)\quad \left\{ \begin{array}{l} \hat{\lambda}_\gamma \text{ are boundedly continuous on the box } [0,+\infty]^{N_\gamma} \text{ and boundedly locally constant at infinity,}\\ \hat{\lambda}_\gamma \text{ are uniformly locally bounded,}\\ \hat{\lambda}_\gamma \text{ are } \sigma_\gamma\text{-conservative} \end{array}\right.$$

Then point i) still holds true. Moreover $\mathfrak{R}_{\underline{\bar{\lambda}}}\underline{\hat{\lambda}}$ does satisfy (13.36).
iii) (Riemann relaxation $\mathfrak{R}_\infty$)
Under assumption (13.36), we also have

$$\mathfrak{R}_\infty\underline{\hat{\lambda}} = (\mathfrak{R}_\infty\hat{\lambda}_\alpha) \underset{(j_\alpha:j_\beta,+\infty)}{\sharp} (\mathfrak{R}_\infty\hat{\lambda}_\beta) \quad \text{with} \quad \underline{\hat{\lambda}} := \hat{\lambda}_\alpha \underset{(j_\alpha:j_\beta,+\infty)}{\sharp} \hat{\lambda}_\beta$$

where the operator $\mathfrak{R}_\infty$ is defined in point i) of Proposition 13.8.

Proof of Lemma 13.13
The result is a corollary of Proposition 7.6, using as usual polar decomposition, and translating the result on quasi-prefluxes/prefluxes. We skip the details. This ends the proof of the lemma.

14 Examples of conservative quasi-prefluxes and their relaxations

14.1 Lebacque 1:n conservative quasi-preflux and its relaxation

For divergent junctions $1:n$, Lebacque introduces a candidate in [37] (see there the last model presented in Subsection 6.2 "Modelling diverges"), which is also written in equation (5.3) in Lebacque, Khoshyaran [38]. We discuss this candidate below.

We will see that this Lebacque candidate is unfortunately not a preflux, which means that it is not associated to any Riemann solver. Nevertheless, it is a convenient quasi-preflux, and we show that it can be Riemann relaxed into a genuine preflux (useful for both $1:n$ and $n:1$ junctions).

Lemma 14.1 (Relaxation of Lebacque 1:n quasi-preflux)
Assume $N := n+1$ with $n \ge 1$ and a $1:n$ junction with $\theta = (\theta^1,\dots,\theta^n) \in (0,1]^n$ such that $\sum_{j=1,\dots,n} \theta^j = 1 =: \theta^0$. For $\bar{\gamma} = (\bar{\gamma}^0,\dots,\bar{\gamma}^n) \in [0,+\infty)^N$, we consider the following function

$$\hat{\lambda}^{Le} := \hat{\lambda} : [0,+\infty)^N \to [0,+\infty)^N$$

defined by

$$\left\{ \begin{array}{l} \hat{\lambda}^j(\bar{\gamma}) := \min\left\{\theta^j\bar{\gamma}^0, \bar{\gamma}^j\right\}, \quad j = 1,\dots,n\\ \hat{\lambda}^0(\bar{\gamma}) := \displaystyle\sum_{j=1,\dots,n} \hat{\lambda}^j(\bar{\gamma}) \end{array}\right.$$

i) (Lebacque quasi-preflux)
Then $\hat{\lambda}^{Le}$ is not a preflux for $n \geq 2$. For all $n \geq 1$, the function $\hat{\lambda}^{Le}$ is a $1:n$ conservative Kružkov quasi-preflux in the sense of Definition 13.1, which is moreover uniformly locally constant at infinity. We call it the Lebacque quasi-preflux.
ii) (Riemann relaxation of Lebacque quasi-preflux)
Let $\bar{\lambda} = (\bar{\lambda}^0, \ldots, \bar{\lambda}^n) \in (0,+\infty)^N$. We define

$$\hat{\gamma}^{Le}_{\bar{\lambda}} := \mathfrak{R}_{\bar{\lambda}} \hat{\lambda}^{Le}$$

where $\mathfrak{R}_{\bar{\lambda}}$ is the relaxation of quasi-prefluxes defined in (13.10) on the box $[0,\bar{\lambda}]$.
Then $\hat{\gamma}^{Le}_{\bar{\lambda}}$ is a $1:n$ conservative Kružkov preflux, called Lebacque preflux.
iii) (Explicit expression of Lebacque preflux)
Let $\hat{\gamma} := \hat{\gamma}^{Le}_{\bar{\lambda}}$ and let us define

$$\begin{cases} \tilde{\lambda}^j := \min\left\{\bar{\lambda}^j, \theta^j \bar{\lambda}^0\right\}, \quad j = 0, \ldots, n \\ \bar{\bar{\gamma}}^j := \min\left\{\bar{\gamma}^j, \tilde{\lambda}^j\right\}, \quad j = 0, \ldots, n \\ \\ \phi(\lambda^{0R}) := \displaystyle\sum_{j=1,\ldots,n} \min\left\{\bar{\bar{\gamma}}^j, \theta^j \lambda^{0R}\right\} \leq \phi_\infty := \sum_{j=1,\ldots,n} \bar{\bar{\gamma}}^j \end{cases}$$

Then we have

$$\begin{cases} \hat{\gamma}^0(\bar{\gamma}) := \min\left\{\bar{\bar{\gamma}}^0, \phi_\infty\right\}, \\ \hat{\gamma}^j(\bar{\gamma}) := \min\left\{\bar{\bar{\gamma}}^j, \theta^j \lambda^{0R}\right\}, \quad j = 1, \ldots, n \\ \lambda^{0R} := \begin{cases} \phi^{-1}(\bar{\bar{\gamma}}^0) & \text{if } \bar{\bar{\gamma}}^0 < \phi_\infty, \\ \bar{\bar{\gamma}}^0_\infty & \text{if } \bar{\bar{\gamma}}^0 \geq \phi_\infty, \end{cases} \end{cases} \tag{14.1}$$

with

$$\bar{\bar{\gamma}}^0_\infty := \sup_{j=1,\ldots,n} \frac{\bar{\bar{\gamma}}^j}{\theta^j}$$

In particular

$$\hat{\gamma}(\bar{\gamma}^0, \bar{\gamma}') \quad \text{is independent on } \bar{\gamma}^0 \text{ while } \quad \bar{\bar{\gamma}}^0 \geq \phi_\infty.$$

Moreover, we have

$$\boxed{\hat{\gamma}^{Le}_{\bar{\lambda}} = \hat{\gamma}^{Le}_{(+\infty,\ldots,+\infty)} \circ T_{\tilde{\lambda}}}$$

where $T_{\tilde{\lambda}} : [0,+\infty)^{n+1} \to [0,\tilde{\lambda}]$ is the truncation operator defined as

$$(T_{\tilde{\lambda}}(\bar{\gamma}))^j = \min\left\{\bar{\gamma}^j, \tilde{\lambda}^j\right\}, \quad j = 0, \ldots, n$$

and

$$\hat{\gamma}^{Le}_{(+\infty,\ldots,+\infty)} := \lim_{\bar{\lambda} \to (+\infty,\ldots,+\infty)} \hat{\gamma}^{Le}_{\bar{\lambda}} = \mathfrak{R}_\infty \hat{\lambda}^{Le}. \tag{14.2}$$

Lemma 14.2 (Special case $n = 2$ and Daganzo formulation)
As we will see later on in Corollary 14.9, in the special case $n = 2$, Lebacque preflux was first introduced in an equivalent formulation by Daganzo [17]. See in particular Remark 15.11 for the identification of Daganzo formulation with Data Network preflux, and then see Lemma 16.5 for the identification on the box $[0,+\infty]^3$ of Data Network preflux with Traffic Light preflux, whose special case is a Lebacque preflux for $n = 2$. The point is that Lebacque quasi-preflux is so natural and general, that it is very convenient to call Lebacque preflux its Riemann relaxation.

Remark 14.3 *Notice that the limit in (14.2) is well-defined on the explicit expression. This also follows from general result given in Proposition 13.8 applied to Lebacque quasi-preflux $\hat{\lambda}^{Le}$ (see for instance Lemma 14.10 below).*

Remark 14.4 (Application of the quasi-preflux to numerical schemes)
If the Lebacque quasi-preflux is implemented in a standard scheme, then it is possible to show that the numerical solution will converges at the limit to the solution to the continuous problem, whose associated preflux is $\hat{\gamma}^{Le}_{\bar{\lambda}}$.

We start with the following simple result, whose proof is left to the reader, but which will be used in the proof of Lemma 14.1.

Lemma 14.5 (A basic result)
Let $x, y \in [0,+\infty)$ *and* $\lambda^L, \lambda^R \in [0,+\infty]$ *satisfying*

$$\begin{cases} \min\left\{x, \lambda^L\right\} = \min\left\{y, \lambda^R\right\}, \\ \max\left\{\lambda^L, \lambda^R\right\} = +\infty \end{cases}$$

Then we have

$$\min\left\{x, \lambda^L\right\} = \min\left\{y, \lambda^R\right\} = \min\left\{x, y\right\} = \min\left\{\lambda^L, \lambda^R\right\}.$$

Proof of Lemma 14.1
Step 1: proof of i)
Step 1.1: quasi-preflux
By definition, we have $0 \le \hat{\lambda} \le id_{[0,+\infty)^N}$ and $\hat{\lambda}$ is continuous. Assume $n \ge 2$ and let us choose two indices $j_0, j_1 \in \{1, \ldots, n\}$ and $\bar{\gamma}_* \in (0,+\infty)^N$ such that

$$\begin{cases} \bar{\gamma}^{j_0}_* < \theta^{j_0} \bar{\gamma}^0_*, \\ \bar{\gamma}^{j_1}_* > \theta^{j_1} \bar{\gamma}^0_* \end{cases}$$

Then this implies that

$$\begin{cases} \hat{\lambda}^0(\bar{\gamma}_*) < \bar{\gamma}^0_*, \\ \hat{\lambda}^{j_1}(\bar{\gamma}_*) = \theta^{j_1} \bar{\gamma}^0_* \end{cases}$$

Hence there exists a neighborhood of $\bar{\gamma}_*$ such that for all $\bar{\gamma}$ in such a neighborhood, we have

$$\hat{\lambda}^{j_1}(\bar{\gamma}) = \theta^{j_1} \bar{\gamma}^0 \quad \text{and} \quad \hat{\lambda}^0(\bar{\gamma}) < \bar{\gamma}^0$$

which shows that $\hat{\lambda}^0$ is not locally constant on $\left\{\hat{\lambda}^0(\bar{\gamma}) < \bar{\gamma}^0\right\}$. Therefore $\hat{\lambda} = \hat{\lambda}^{Le}$ is not a preflux.
It is straightforward to check that $\hat{\lambda}^{Le}$ is a quasi-preflux.
Step 1.2: conservative Kružkov quasi-preflux
We first notice that $\hat{\lambda}$ is Lipschitz continuous and satisfies

$$\partial_j \hat{\lambda}^j \ge \sum_{k \in \{0,\ldots,n\} \setminus \{j\}} |\partial_j \hat{\lambda}^k| \quad \text{a.e. on} \quad [0,+\infty)^N, \quad \text{for all} \quad j = 0, \ldots, n$$

We deduce that $\hat{\lambda}$ is a Kružkov function in the following sense

$$D^{\hat{\lambda}}(p,q) := \sum_{j=0,\ldots,n} \text{sign}(p^j - q^j) \cdot \left\{\hat{\lambda}^j(p) - \hat{\lambda}^j(q)\right\} \ge 0 \quad \text{for all} \quad p, q \in [0,+\infty)^N$$

From Proposition 5.9, we deduce that $\hat{\lambda} : [0,+\infty)^N \to [0,+\infty)^N$ is Riemann monotone. Moreover $\hat{\lambda}$ satisfies

$$\hat{\lambda}^0 - \sum_{j=1,\ldots,n} \hat{\lambda}^j = 0$$

i.e. is conservative.
Step 2: proof of ii)
From Step 1 and Lemma 13.9, we deduce that the Riemann relaxation $\hat{\gamma}^{Le}_{\bar{\lambda}} := \mathfrak{R}_{\bar{\lambda}} \hat{\lambda}$ is a conservative Kružkov preflux.
Step 3: computation of the preflux and proof of iii)
Step 3.1: first computations
We set $\hat{\gamma} := \hat{\gamma}^{Le}_{\bar{\lambda}}$. By definition, we have

$$\hat{\gamma}^j(\bar{\gamma}) := \min\left\{\bar{\gamma}^j, \lambda^{jL}\right\} = \hat{\lambda}^j(\lambda^R), \quad j = 0, \ldots, n$$

with

$$(\lambda^L,\lambda^R)\in \mathbb{D}_{\bar\lambda} := \left\{(\tilde\lambda^L,\tilde\lambda^R)\in [0,\bar\lambda]^2, \quad \max\left\{\tilde\lambda^{jL},\tilde\lambda^{jR}\right\} = \bar\lambda^j \quad \text{for all} \quad j=1,\dots,N\right\}$$

Hence we get

$$\text{(14.3)} \qquad \begin{cases} \min\left\{\bar\gamma^j,\lambda^{jL}\right\} = \min\left\{\theta^j\lambda^{0R},\lambda^{jR}\right\}, & j=1,\dots,n\\ \min\left\{\bar\gamma^0,\lambda^{0L}\right\} = \displaystyle\sum_{j=1,\dots,n} \min\left\{\theta^j\lambda^{0R},\lambda^{jR}\right\}, & \\ \max\left\{\lambda^{kL},\lambda^{kR}\right\} = \bar\lambda^k, & k=0,\dots,n \end{cases}$$

The first line of (14.3) shows that (use for instance Lemma 14.5)

$$\hat\gamma^j(\bar\gamma) = \min\left\{\bar\gamma^j,\lambda^{jL}\right\} = \min\left\{\lambda^{jL},\lambda^{jR}\right\} = \min\left\{\bar\gamma^j,\theta^j\lambda^{0R}\right\}, \quad j=1,\dots,n$$

and the second line means

$$\min\left\{\bar\gamma^0,\lambda^{0L}\right\} = \sum_{j=1,\dots,n}\hat\gamma^j(\bar\gamma).$$

Now using the third line of (14.3) , we get

$$\min\left\{\bar\gamma^0,\bar\lambda^0,\lambda^{0L}\right\} = \sum_{j=1,\dots,n}\min\left\{\bar\gamma^j,\theta^j\lambda^{0R},\bar\lambda^j,\theta^j\bar\lambda^0\right\}$$

Setting

$$\bar{\bar\gamma}^k := \min\left\{\bar\gamma^k,\bar\lambda^k,\theta^k\bar\lambda^0\right\} \quad \text{for} \quad k=0,\dots,n$$

we get

$$\text{(14.4)} \qquad \min\left\{\bar{\bar\gamma}^0,\lambda^{0L}\right\} = \phi(\lambda^{0R}) \quad \text{with} \quad \phi(\lambda^{0R}) := \sum_{j=1,\dots,n}\min\left\{\bar{\bar\gamma}^j,\theta^j\lambda^{0R}\right\}$$

where ϕ is a concave nondecreasing function.

Step 3.2: study of the function $\phi:[0,+\infty)\to[0,+\infty)$

Then, up to relabel the indices $j=1,\dots,n$, we can assume that

$$0 \le \frac{\bar{\bar\gamma}^1}{\theta^1} \le \frac{\bar{\bar\gamma}^2}{\theta^2} \le \dots \le \frac{\bar{\bar\gamma}^n}{\theta^n} \le \bar\lambda^0$$

and then ϕ is increasing on $[0,\frac{\bar{\bar\gamma}^n}{\theta^n}]$ and then constant on $[\frac{\bar{\bar\gamma}^n}{\theta^n},+\infty)$, with value

$$\text{(14.5)} \qquad \phi_\infty := \sum_{j=1,\dots,n}\bar{\bar\gamma}^j \quad \text{which satisfies} \quad \phi_\infty = \max\ \phi = \phi(\bar\lambda^0) \le \bar\lambda^0$$

We define

$$\breve\gamma^j := \begin{cases} \dfrac{\bar{\bar\gamma}^j}{\theta^j} & \text{if} \quad j=1,\dots,n\\ 0 & \text{if} \quad j=0, \end{cases}$$

and get

$$\phi(\breve\gamma^n) = \phi_\infty.$$

Step 3.3: solving equation (14.4)

Then we set

$$(\lambda^{0L},\lambda^{0R}) = \begin{cases} (\bar\lambda^0,\phi^{-1}(\bar{\bar\gamma}^0)) & \text{if} \quad \bar{\bar\gamma}^0 < \phi_\infty\\ (\phi_\infty,\breve\gamma^n) & \text{if} \quad \bar{\bar\gamma}^0 \ge \phi_\infty \end{cases}$$

which is a solution of (14.4). This implies (14.1). The remaining results of point ii) are straightforward. This ends the proof of the lemma.

Corollary 14.6 (Long explicit Lebacque 1:n preflux for $n \geq 1$)
We work under assumptions of Lemma 14.1 with $N = 1+n$ and consider Lebacque preflux $\hat{\gamma} = (\hat{\gamma}^0, \dots, \hat{\gamma}^n) : [0,+\infty)^{n+1} \to [0,+\infty)^{n+1}$ for $n \geq 1$, and $\bar{\gamma} = (\bar{\gamma}^0, \dots, \bar{\gamma}^n) \in [0,+\infty)^{n+1}$. Recall that $\theta^0 = 1 = \sum_{j=1,\dots,n} \theta^j$ with $\theta^j \in (0,1]$ and $\bar{\lambda} = (\bar{\lambda}^0, \dots, \bar{\lambda}^n) \in (0,+\infty)^{n+1}$. We also set for $\lambda^{0R} \in [0,+\infty)$

$$\left\{\begin{array}{l} \bar{\bar{\gamma}}^j := \min\left\{\bar{\gamma}^j, \bar{\lambda}^j, \theta^j \bar{\lambda}^0\right\}, \\ \breve{\gamma}^j := \dfrac{\bar{\bar{\gamma}}^j}{\theta^j}, \\ \phi(\lambda^{0R}) := \displaystyle\sum_{j=1,\dots,n} \min\left\{\bar{\bar{\gamma}}^j, \theta^j \lambda^{0R}\right\}, \qquad \text{for} \quad j = 0, \dots, n \\ \bar{\gamma}^j_* := \phi(\breve{\gamma}^j) \end{array}\right.$$

i) (Case $n = 1$)
Then $\theta^1 = 1$ and

$$\hat{\gamma}^0(\bar{\gamma}) = \hat{\gamma}^1(\bar{\gamma}) = \min\left\{\bar{\bar{\gamma}}^0, \bar{\bar{\gamma}}^1\right\} = \min\left\{\bar{\gamma}^0, \bar{\gamma}^1, \bar{\lambda}^{01}\right\} \quad \text{with} \quad \bar{\lambda}^{01} := \min\left\{\bar{\lambda}^0, \bar{\lambda}^1\right\}$$

where $\bar{\lambda}^{01}$ appears to be a classical flux limiter for some $1:1$ junction.
ii) (Case $n = 2$)
Then

$$\hat{\gamma}^0(\bar{\gamma}) = \min\left\{\bar{\bar{\gamma}}^0, \bar{\bar{\gamma}}^1 + \bar{\bar{\gamma}}^2\right\} = (\hat{\gamma}^1 + \hat{\gamma}^2)(\bar{\gamma})$$

and

$$(14.6) \qquad (\hat{\gamma}^1, \hat{\gamma}^2)(\bar{\gamma}) = \left\{\begin{array}{ll} (\theta^1 \bar{\bar{\gamma}}^0, \theta^2 \bar{\bar{\gamma}}^0) & \text{if} \quad \breve{\gamma}^0 < \bar{\gamma}^1_* \\ (\bar{\bar{\gamma}}^1, \bar{\bar{\gamma}}^0 - \bar{\bar{\gamma}}^1) & \text{if} \quad \bar{\gamma}^1_* \leq \breve{\gamma}^0 < \bar{\gamma}^2_* \\ (\bar{\bar{\gamma}}^1, \bar{\bar{\gamma}}^2) & \text{if} \quad \bar{\gamma}^2_* \leq \breve{\gamma}^0 \\ \text{assuming everywhere that} & \breve{\gamma}^1 \leq \breve{\gamma}^2 \\ \text{with} \quad \left\{\begin{array}{l} \bar{\gamma}^1_* := \breve{\gamma}^1 \\ \bar{\gamma}^2_* := \theta^1 \breve{\gamma}^1 + (1-\theta^1)\breve{\gamma}^2 \\ \bar{\gamma}^1_* \leq \bar{\gamma}^2_* \end{array}\right. & \end{array}\right.$$

and all other cases (not indicated in (14.6)) are then obtained by permutations on the indices.
iii) (Case $n = 3$)
Then

$$\hat{\gamma}^0(\bar{\gamma}) = \min\left\{\bar{\bar{\gamma}}^0, \bar{\bar{\gamma}}^1 + \bar{\bar{\gamma}}^2 + \bar{\bar{\gamma}}^3\right\} = (\hat{\gamma}^1 + \hat{\gamma}^2 + \hat{\gamma}^3)(\bar{\gamma})$$

and

$$(14.7) \qquad (\hat{\gamma}^1, \hat{\gamma}^2, \hat{\gamma}^3)(\bar{\gamma}) = \left\{\begin{array}{ll} (\theta^1 \bar{\bar{\gamma}}^0, \theta^2 \bar{\bar{\gamma}}^0, \theta^3 \bar{\bar{\gamma}}^0) & \text{if} \quad \breve{\gamma}^0 < \bar{\gamma}^1_* \\ (\bar{\bar{\gamma}}^1, \frac{\theta^2}{1-\theta^1} \cdot (\bar{\bar{\gamma}}^0 - \bar{\bar{\gamma}}^1), \frac{\theta^3}{1-\theta^1} \cdot (\bar{\bar{\gamma}}^0 - \bar{\bar{\gamma}}^1)) & \text{if} \quad \bar{\gamma}^1_* \leq \breve{\gamma}^0 < \bar{\gamma}^2_* \\ (\bar{\bar{\gamma}}^1, \bar{\bar{\gamma}}^2, \bar{\bar{\gamma}}^0 - (\bar{\bar{\gamma}}^1 + \bar{\bar{\gamma}}^2)) & \text{if} \quad \bar{\gamma}^2_* \leq \breve{\gamma}^0 < \bar{\gamma}^3_* \\ (\bar{\bar{\gamma}}^1, \bar{\bar{\gamma}}^2, \bar{\bar{\gamma}}^3) & \text{if} \quad \bar{\gamma}^3_* \leq \breve{\gamma}^0 \\ \text{assuming everywhere that} & \breve{\gamma}^1 \leq \breve{\gamma}^2 \leq \breve{\gamma}^3 \\ \text{with} \quad \left\{\begin{array}{l} \bar{\gamma}^1_* := \breve{\gamma}^1 \\ \bar{\gamma}^2_* := \theta^1 \breve{\gamma}^1 + (1-\theta^1)\breve{\gamma}^2 \\ \bar{\gamma}^3_* := \theta^1 \breve{\gamma}^1 + \theta^2 \breve{\gamma}^2 + \theta^3 \breve{\gamma}^3 \\ \bar{\gamma}^1_* \leq \bar{\gamma}^2_* \leq \bar{\gamma}^3_* \end{array}\right. & \end{array}\right.$$

and all other cases (not indicated in (14.7)) are then obtained by permutations on the indices.
iv) (Case $n \geq 4$)

Then

$$\hat{\gamma}^0(\bar{\gamma}) = \min\left\{\bar{\bar{\gamma}}^0, \bar{\bar{\gamma}}^1 + \cdots + \bar{\bar{\gamma}}^n\right\} = (\hat{\gamma}^1 + \cdots + \hat{\gamma}^n)(\bar{\gamma})$$

and for

$$\left\{\begin{array}{ll} \phi_\infty := \bar{\bar{\gamma}}^1 + \cdots + \bar{\bar{\gamma}}^n, & \\ \bar{\bar{\gamma}}^0_k := \bar{\bar{\gamma}}^0 - (\bar{\bar{\gamma}}^1 + \cdots + \bar{\bar{\gamma}}^k), & \\ \theta^j_k := \dfrac{\theta^j}{1-(\theta^1 + \cdots + \theta^k)}, & \quad j \geq k+1 \end{array}\right.$$

$$(14.8) \quad (\hat{\gamma}^1, \ldots, \hat{\gamma}^n)(\bar{\gamma}) = \left\{\begin{array}{ll} (\theta^1\bar{\bar{\gamma}}^0, \theta^2\bar{\bar{\gamma}}^0, \ldots, \theta^n\bar{\bar{\gamma}}^0) & \text{if} \quad \breve{\gamma}^0 < \bar{\gamma}^1_* \\ (\bar{\bar{\gamma}}^1, \theta^2_1\bar{\bar{\gamma}}^0_1, \ldots, \theta^n_1\bar{\bar{\gamma}}^0_1) & \text{if} \quad \bar{\gamma}^1_* \leq \breve{\gamma}^0 < \bar{\gamma}^2_* \\ (\bar{\bar{\gamma}}^1, \bar{\bar{\gamma}}^2, \theta^3_2\bar{\bar{\gamma}}^0_2, \ldots, \theta^{n-1}_2\bar{\bar{\gamma}}^0_2, \theta^n_2\bar{\bar{\gamma}}^0_2) & \text{if} \quad \bar{\gamma}^2_* \leq \breve{\gamma}^0 < \bar{\gamma}^3_* \\ \vdots & \\ (\bar{\bar{\gamma}}^1, \bar{\bar{\gamma}}^2, \ldots, \bar{\bar{\gamma}}^{n-2}, \theta^{n-1}_{n-2}\bar{\bar{\gamma}}^0_{n-2}, \theta^n_{n-2}\bar{\bar{\gamma}}^0_{n-2}) & \text{if} \quad \bar{\gamma}^{n-2}_* \leq \breve{\gamma}^0 < \bar{\gamma}^{n-1}_* \\ (\bar{\bar{\gamma}}^1, \bar{\bar{\gamma}}^2, \ldots, \bar{\bar{\gamma}}^{n-2}, \bar{\bar{\gamma}}^{n-1}, \theta^n_{n-1}\bar{\bar{\gamma}}^0_{n-1}) & \text{if} \quad \bar{\gamma}^{n-1}_* \leq \breve{\gamma}^0 < \bar{\gamma}^n_* \\ (\bar{\bar{\gamma}}^1, \bar{\bar{\gamma}}^2, \ldots, \bar{\bar{\gamma}}^n) & \text{if} \quad \bar{\gamma}^n_* \leq \breve{\gamma}^0 \\ \text{assuming everywhere that} & \breve{\gamma}^1 \leq \breve{\gamma}^2 \leq \cdots \leq \breve{\gamma}^n \\ \text{with} \quad \left\{\begin{array}{l} \bar{\gamma}^1_* := \breve{\gamma}^1 \\ \bar{\gamma}^2_* := \theta^1\breve{\gamma}^1 + (1-\theta^1)\breve{\gamma}^2 \\ \bar{\gamma}^3_* := \theta^1\breve{\gamma}^1 + \theta^2\breve{\gamma}^2 + (1-(\theta^1+\theta^2))\breve{\gamma}^3 \\ \vdots \\ \bar{\gamma}^n_* = \theta^n\breve{\gamma}^n + \displaystyle\sum_{j=1,\ldots,n-1} \theta^j\breve{\gamma}^j \\ \bar{\gamma}^1_* \leq \cdots \leq \bar{\gamma}^n_* \end{array}\right. & \end{array}\right.$$

and all other cases (not indicated in (14.8)) are then obtained by permutations on the indices $\{1, \ldots, n\}$*. Notice that* $\theta^n_{n-1} = 1$*.*

Remark 14.7 *Notice that in Corollary 14.6, the quantity* $\breve{\gamma}^j$ *for* $j = 0$ *has a different definition than in the proof of Lemma 14.1. Notice also that for* $\bar{\lambda} = (\bar{\lambda}^0, \ldots, \bar{\lambda}^n) = (+\infty, \ldots, +\infty)$*, then we have the simplification* $\bar{\bar{\gamma}}^j = \bar{\gamma}^j$*.*

Proof of Corollary 14.6
We simply make explicit the statement of Lemma 14.1. Again we can assume that the inequalities are strict

$$(14.9) \qquad 0 < \breve{\gamma}^1 < \cdots < \breve{\gamma}^n < \breve{\gamma}^{n+1} = +\infty,$$

because the general case with large inequalities can be recovered by continuity of the preflux.
Recall that $\phi(\breve{\gamma}^n) = \phi_\infty$ and $\phi < \phi_\infty$ on $[0, \breve{\gamma}^n)$. Hence

$$0 = \bar{\gamma}^0_* < \bar{\gamma}^1_* < \cdots < \bar{\gamma}^n_* = \bar{\gamma}^{n+1}_* := \phi_\infty$$

and

$$\left\{\begin{array}{l} \hat{\gamma}^0(\bar{\gamma}) = \min\left\{\bar{\bar{\gamma}}^0, \phi_\infty\right\}, \\ \hat{\gamma}^j(\bar{\gamma}) = \min\left\{\bar{\bar{\gamma}}^j, \theta^j\lambda^{0R}\right\} \end{array}\right. \qquad \text{with} \quad \lambda^{0R} = \left\{\begin{array}{ll} \phi^{-1}(\breve{\gamma}^0) & \text{if} \quad \breve{\gamma}^0 < \phi_\infty \\ \breve{\gamma}^n & \text{if} \quad \breve{\gamma}^0 \geq \phi_\infty \end{array}\right.$$

Recall that

$$\hat{\gamma}(\bar{\gamma}^0, \bar{\gamma}') \quad \text{is independent of } \bar{\gamma}^0 \text{ while } \breve{\gamma}^0 \geq \phi_\infty$$

and then we can assume that

$$\breve{\gamma}^0 \leq \phi_\infty$$

Now for $k = 0, \ldots, n-1$, we have for $\breve{\gamma}^0 \in [0, \phi_\infty]$

$$\begin{cases} \bar{\gamma}_*^k \leq \breve{\gamma}^0 < \bar{\gamma}_*^{k+1} & \iff & \breve{\gamma}^k \leq \lambda^{0R} < \breve{\gamma}^{k+1} \\ \breve{\gamma}^0 = \bar{\gamma}_*^n & \iff & \lambda^{0R} = \breve{\gamma}^n. \end{cases}$$

Case A: $\bar{\gamma}^0 < \bar{\gamma}_*^n = \phi_\infty$
We see that $\lambda^{0R} = \phi^{-1}(\bar{\gamma}^0)$ satisfies for some $k \in \{0, \ldots, n-1\}$

$$\bar{\gamma}^0 = \hat{\gamma}^0(\bar{\gamma}) = \left\{ \sum_{j \leq k} \bar{\gamma}^j \right\} + \sum_{j>k} \theta^j \lambda^{0R} \quad \text{if} \quad \breve{\gamma}^k \leq \lambda^{0R} < \breve{\gamma}^{k+1}$$

This means for $k = 0, \ldots, n-1$ that
(14.10)

$$\lambda^{0R} = \phi^{-1}(\bar{\gamma}^0) = \left(\sum_{j>k} \theta^j \right)^{-1} (\bar{\gamma}^0 - \sum_{j \leq k} \bar{\gamma}^j) = \left(1 - \sum_{j \leq k} \theta^j \right)^{-1} (\bar{\gamma}^0 - \sum_{j \leq k} \bar{\gamma}^j) \quad \text{when} \quad \breve{\gamma}^k \leq \lambda^{0R} < \breve{\gamma}^{k+1}$$

Hence we see that that the explicit shape of ϕ changes when $\bar{\gamma}^0$ takes values such that $\breve{\gamma}^k = \phi^{-1}(\bar{\gamma}^0)$, i.e. such that $\bar{\gamma}^0 := \bar{\gamma}_*^k$. Then, assuming (14.10), we get for $j = 1, \ldots, n$

$$\hat{\gamma}^j(\bar{\gamma}) = \theta^j \cdot \min\left\{ \breve{\gamma}^j, \lambda^{0R} \right\} = \begin{cases} \bar{\gamma}^j & \text{if} \quad j \leq k, \\ \theta^j \cdot \phi^{-1}(\bar{\gamma}^0) & \text{if} \quad j > k, \end{cases}$$

This means for $k = 0, \ldots, n-1$, that
(14.11)

$$\left\{ \begin{array}{l} (\hat{\gamma}^1, \ldots, \hat{\gamma}^n)(\bar{\gamma}) = (\bar{\gamma}^1, \ldots, \bar{\gamma}^k, \theta^{k+1} \lambda^{0R}, \ldots, \theta^n \lambda^{0R}) \\ \text{with} \quad \lambda^{0R} := \phi^{-1}(\bar{\gamma}^0) = \left(\sum_{\ell > k} \theta^\ell \right)^{-1} (\bar{\gamma}^0 - \sum_{\ell \leq k} \bar{\gamma}^\ell) = \left(1 - \sum_{\ell \leq k} \theta^\ell \right)^{-1} (\bar{\gamma}^0 - \sum_{\ell \leq k} \bar{\gamma}^\ell) \end{array} \right| \quad \text{if} \quad \bar{\gamma}_*^k \leq \bar{\gamma}^0 < \bar{\gamma}_*^{k+1}$$

where the second line is not defined for $k = n$ (but it then not used in the first line).
Case B: $\bar{\gamma}^0 \geq \phi_\infty$
Then $\bar{\gamma}^0 \geq \phi_\infty = \bar{\gamma}_*^{n+1}$, and

$$(\hat{\gamma}^1, \ldots, \hat{\gamma}^n)(\bar{\gamma}) = (\bar{\gamma}^1, \ldots, \bar{\gamma}^n). \tag{14.12}$$

This ends the proof of the corollary.

We will need the following technical lemma.

Lemma 14.8 (A specific inequality)
Let

$$\begin{cases} \hat{z} := \overline{z} - \theta^- y^- + \theta^+ y^+ \\ \hat{\theta} := \overline{\theta} - \theta^- + \theta^+ \end{cases}$$

assuming that we have

$$\begin{cases} y^- \leq \underline{y} \leq \overline{y} \leq y^+ \\ \theta^-, \theta^+ \geq 0, \\ \textit{with} \quad \overline{\theta}, \hat{\theta} < 1 \end{cases} \tag{14.13}$$

Let

$$\varphi(x) := \frac{x - \overline{z}}{1 - \overline{\theta}}, \quad \psi(x) := \frac{x - \hat{z}}{1 - \hat{\theta}}$$

and

$$\underline{y}_* := \varphi^{-1}(\underline{y}), \quad \overline{y}_* := \varphi^{-1}(\overline{y})$$

Then

$$\psi \leq \varphi \quad \textit{on} \quad [\underline{y}_*, \overline{y}_*] \quad \textit{with} \quad \underline{y}_* \leq \overline{y}_* \tag{14.14}$$

Proof of Lemma 14.8
We first notice that (14.14) is equivalent to

$$\frac{x-\overline{z}-(\theta^+y^+-\theta^-y^-)}{1-\overline{\theta}-(\theta^+-\theta^-)} \le \frac{x-\overline{z}}{1-\overline{\theta}} =: \varphi(x) \quad \text{for} \quad \varphi(x)\in[\underline{y},\overline{y}]$$

which itself is implied by the same inequality evaluated only for $y^- := \underline{y} \le y^+ := \overline{y}$, i.e.

$$\frac{x-\overline{z}-(\theta^+\overline{y}-\theta^-\underline{y})}{1-\overline{\theta}-(\theta^+-\theta^-)} \le \frac{x-\overline{z}}{1-\overline{\theta}} =: \varphi(x) \quad \text{for} \quad \varphi(x)\in[\underline{y},\overline{y}] \tag{14.15}$$

Now, let

$$\begin{cases} \hat{y} := \varphi(x), \\ \hat{\theta}^{\pm} := \dfrac{\theta^{\pm}}{1-\overline{\theta}} \end{cases}$$

Then (14.15) is equivalent to

$$\frac{\hat{y}-(\hat{\theta}^+\overline{y}-\hat{\theta}^-\underline{y})}{1-(\hat{\theta}^+-\hat{\theta}^-)} \le \hat{y} \quad \text{for} \quad \hat{y}\in[\underline{y},\overline{y}]$$

i.e. $\hat{\theta}^-(\underline{y}-\hat{y}) \le \hat{\theta}^+(\overline{y}-\hat{y})$, which is true because we even have

$$\hat{\theta}^-(\underline{y}-\hat{y}) \le 0 \le \hat{\theta}^+(\overline{y}-\hat{y})$$

This ends the proof of the lemma.

Corollary 14.9 (Short explicit Lebacque 1:n preflux for $n\ge 1$)
We work under assumptions of Lemma 14.1 with $N=1+n$ and consider Lebacque preflux $\hat{\gamma}=(\hat{\gamma}^0,\dots,\hat{\gamma}^n):[0,+\infty)^{n+1}\to[0,+\infty)^{n+1}$ for $n\ge 1$, and $\bar{\gamma}=(\bar{\gamma}^0,\dots,\bar{\gamma}^n)\in[0,+\infty)^{n+1}$. Recall that $\theta^0=1=\sum_{j=1,\dots,n}\theta^j$ with $\theta^j\in(0,1]$ and $\bar{\lambda}=(\bar{\lambda}^0,\dots,\bar{\lambda}^n)\in(0,+\infty)^{n+1}$. We also set

$$\begin{cases} \bar{\bar{\gamma}}^j := \min\left\{\bar{\gamma}^j, \bar{\lambda}^j, \theta^j\bar{\lambda}^0\right\}, \\ \breve{\gamma}^j := \dfrac{\bar{\bar{\gamma}}^j}{\theta^j}, \end{cases} \qquad \textit{for} \quad j=0,\dots,n$$

i) (Case $n=1$)
Then $\theta^1=1$ and

$$\hat{\gamma}^0(\bar{\gamma})=\hat{\gamma}^1(\bar{\gamma})=\min\left\{\bar{\bar{\gamma}}^0,\bar{\bar{\gamma}}^1\right\}$$

ii) (Case $n=2$)
Then

$$\hat{\gamma}^0(\bar{\gamma})=\min\left\{\bar{\bar{\gamma}}^0,\bar{\bar{\gamma}}^1+\bar{\bar{\gamma}}^2\right\}=(\hat{\gamma}^1+\hat{\gamma}^2)(\bar{\gamma})$$

and

$$\begin{cases} (\theta^1)^{-1}\hat{\gamma}^1(\bar{\gamma}) = \min\left\{\breve{\gamma}^1, \max\left\{\breve{\gamma}^0, \dfrac{\breve{\gamma}^0-\theta^2\breve{\gamma}^2}{1-\theta^2}\right\}\right\}, \\ (\theta^2)^{-1}\hat{\gamma}^2(\bar{\gamma}) = \min\left\{\breve{\gamma}^2, \max\left\{\breve{\gamma}^0, \dfrac{\breve{\gamma}^0-\theta^1\breve{\gamma}^1}{1-\theta^1}\right\}\right\}, \end{cases} \tag{14.16}$$

iii) (Case $n=3$)
Then

$$\hat{\gamma}^0(\bar{\gamma})=\min\left\{\bar{\bar{\gamma}}^0,\bar{\bar{\gamma}}^1+\bar{\bar{\gamma}}^2+\bar{\bar{\gamma}}^3\right\}=(\hat{\gamma}^1+\hat{\gamma}^2+\hat{\gamma}^3)(\bar{\gamma})$$

Let us consider

$$\begin{cases} X_j := \dfrac{\breve{\gamma}^0-\theta^j\breve{\gamma}^j}{1-\theta^j} & \textit{for} \quad j\in\{1,2,3\} \\ X_{jk} := \dfrac{\breve{\gamma}^0-(\theta^j\breve{\gamma}^j+\theta^k\breve{\gamma}^k)}{1-(\theta^j+\theta^k)} & \textit{for} \quad j,k\in\{1,2,3\} \quad \textit{with} \quad j\ne k \end{cases}$$

and

$$\begin{cases} S^3 := \max\left\{\breve{\gamma}^0, X_1, X_2, X_{12}\right\} \\ S^1 := \max\left\{\breve{\gamma}^0, X_2, X_3, X_{23}\right\} \\ S^2 := \max\left\{\breve{\gamma}^0, X_3, X_1, X_{31}\right\} \end{cases}$$

Then we have

$$(\theta^j)^{-1}\hat{\gamma}^j(\bar{\gamma}) = \min\left\{\breve{\gamma}^j, S^j\right\} \quad \text{for} \quad j = 1, 2, 3 \tag{14.17}$$

iv) (Case $n \geq 4$)
Then

$$\hat{\gamma}^0(\bar{\gamma}) = \min\left\{\bar{\bar{\gamma}}^0, \bar{\bar{\gamma}}^1 + \cdots + \bar{\bar{\gamma}}^n\right\} = (\hat{\gamma}^1 + \cdots + \hat{\gamma}^n)(\bar{\gamma})$$

For every set $I \subset_{\neq} \{1, \ldots, n\}$, we define (including $Y_\emptyset := \breve{\gamma}^0$)

$$Y_I := \frac{\breve{\gamma}^0 - \displaystyle\sum_{j\in I} \theta^j \breve{\gamma}^j}{1 - \displaystyle\sum_{j \in I} \theta^j}$$

We also set

$$S^j := \max_{I \subset \{1,\ldots,n\}\setminus\{j\}} Y_I, \quad \text{for all} \quad j = 1, \ldots, n$$

Then

$$\breve{\gamma}^j := (\theta^j)^{-1}\hat{\gamma}^j(\bar{\gamma}) = \min\left\{\breve{\gamma}^j, S^j\right\}, \quad \text{for all} \quad j = 1, \ldots, n \tag{14.18}$$

Proof of Corollary 14.9
Case i) is trivial. We focus on the general case $n \geq 2$. We only do the analysis in the special case where

$$0 < \breve{\gamma}^1 < \cdots < \breve{\gamma}^n. \tag{14.19}$$

All the other cases can be obtained by permutations on the indices, and then by continuity.
We define

$$\begin{cases} \bar{\gamma}^1_* := \breve{\gamma}^1 \\ \bar{\gamma}^2_* := \theta^1\breve{\gamma}^1 + (1-\theta^1)\breve{\gamma}^2 \\ \bar{\gamma}^3_* := \theta^1\breve{\gamma}^1 + \theta^2\breve{\gamma}^2 + (1-(\theta^1+\theta^2))\breve{\gamma}^3 \\ \vdots \\ \bar{\gamma}^{j+1}_* := (1 - \displaystyle\sum_{k=1,\ldots,j} \theta^k)\breve{\gamma}^{j+1} + \sum_{k=1,\ldots,j} \theta^k\breve{\gamma}^k \\ \vdots \\ \bar{\gamma}^n_* = \theta^n\breve{\gamma}^n + \displaystyle\sum_{j=1,\ldots,n-1} \theta^j\breve{\gamma}^j \\ \bar{\gamma}^1_* < \cdots < \bar{\gamma}^n_* \end{cases}$$

which satisfy

$$\bar{\gamma}^1_* = \breve{\gamma}^1, \quad \bar{\gamma}^j_* < \breve{\gamma}^j, \quad j = 2, \ldots, n$$

We also define (including $\beta^0(\breve{\gamma}^0) := \breve{\gamma}^0$)

$$\beta^j(\breve{\gamma}^0) := Y_{\{1,\ldots,j\}} = \frac{\breve{\gamma}^0 - \displaystyle\sum_{k=1,\ldots,j} \theta^k\breve{\gamma}^k}{1 - \displaystyle\sum_{k=1,\ldots,j} \theta^k}, \quad j = 0, \ldots, n-1$$

which satisfy

$$\begin{cases} & \beta^0(\bar{\gamma}^1_*) = \breve{\gamma}^1, & \\ \beta^1(\bar{\gamma}^1_*) = \breve{\gamma}^1, & \beta^1(\bar{\gamma}^2_*) = \breve{\gamma}^2, & \\ \beta^2(\bar{\gamma}^2_*) = \breve{\gamma}^2, & \beta^2(\bar{\gamma}^3_*) = \breve{\gamma}^3, & \\ \vdots & \vdots & \\ \beta^j(\bar{\gamma}^j_*) = \breve{\gamma}^j, & \beta^j(\bar{\gamma}^{j+1}_*) = \breve{\gamma}^{j+1}, & 1 \leq j \leq n-1 \\ \vdots & \vdots & \\ \beta^{n-1}(\bar{\gamma}^{n-1}_*) = \breve{\gamma}^{n-1}, & \beta^{n-1}(\bar{\gamma}^n_*) = \breve{\gamma}^n, & \end{cases}$$

Case 1: $\breve{\gamma}^0 < \bar{\gamma}^1_* < \breve{\gamma}^2 < \cdots < \breve{\gamma}^n$
Then (14.8) gives

$$(\breve{\gamma}^1, \ldots, \breve{\gamma}^n) = (\beta^0, \ldots, \beta^0) = (\breve{\gamma}^0, \ldots, \breve{\gamma}^0)$$

We also have

$$Y_I \le (Y_I)_{|(\breve{\gamma}^1,\ldots,\breve{\gamma}^n):=(\breve{\gamma}^0,\ldots,\breve{\gamma}^0)} = \breve{\gamma}^0 \le \breve{\gamma}^j, \quad j = 1, \ldots, n$$

which implies $S^k = \breve{\gamma}^0$ and then shows (14.18).
Case 2: $\bar{\gamma}^1_* < \cdots < \bar{\gamma}^j_* \le \breve{\gamma}^0 < \bar{\gamma}^{j+1}_* < \breve{\gamma}^{j+2} < \cdots < \breve{\gamma}^n$ **for** $1 \le j \le n$
Then (14.8) gives

$$(\breve{\gamma}^1, \ldots, \breve{\gamma}^n) = (\breve{\gamma}^1, \ldots, \breve{\gamma}^j, \beta^j, \ldots, \beta^j)$$

Subcase 2.1: $1 \le k \le j$
Then, by definition of S^k, we have

$$S^k \ge Y_{\{1,\ldots,k-1\}} = \beta^k(\breve{\gamma}^0) \ge \beta^k(\bar{\gamma}^k_*) = \breve{\gamma}^k \tag{14.20}$$

Subcase 2.2: $j+1 \le k \le n$
Then let

$$I \subset \{1, \ldots, n\} \setminus \{k\}$$

We write

$$I = I_- \cup I_+ \quad \text{with} \quad \begin{cases} I_- \subset \{1, \ldots, j\}, \\ I_+ \subset \{j+1, \ldots, n\}, \end{cases}$$

Hence

$$Y_I = \frac{\breve{\gamma}^0 - (\sum_{l \in I_-} \theta^l \breve{\gamma}^l + \sum_{l \in I_+} \theta^l \breve{\gamma}^l)}{1 - (\sum_{l \in I_-} \theta^l + \sum_{l \in I_+} \theta^l)}$$

We write

$$\{1, \ldots, j\} = I_- \cup J_-, \quad \text{with} \quad I_- \cap J_- = \emptyset$$

and

$$\begin{cases} \overline{\theta} := \sum_{l=1,\ldots,j} \theta^l, \\ \theta^+ := \sum_{l \in I_+} \theta^l \ge 0, \\ \theta^- := \sum_{l \in J_-} \theta^l \ge 0, \\ \\ \overline{z} := \sum_{l=1,\ldots,j} \theta^l \breve{\gamma}^l, \\ y^+ := (\theta^+)^{-1} \sum_{l \in I_+} \theta^l \breve{\gamma}^l \ge \breve{\gamma}^{j+1} =: \overline{y}, \\ y^- := (\theta^-)^{-1} \sum_{l \in J_-} \theta^l \breve{\gamma}^l \le \breve{\gamma}^j =: \underline{y}, \\ \\ \varphi(\breve{\gamma}^0) := \dfrac{\breve{\gamma}^0 - \overline{z}}{1 - \overline{\theta}} \\ \underline{y}_* := \bar{\gamma}^j_*, \\ \overline{y}_* := \bar{\gamma}^{j+1}_*, \end{cases}$$

Then we get

$$\begin{aligned} Y_I &= \frac{\breve{\gamma}^0 - (\sum_{l=1,\ldots,j} \theta^l \breve{\gamma}^l - \sum_{l \in J_-} \theta^l \breve{\gamma}^l + \sum_{l \in I_+} \theta^l \breve{\gamma}^l)}{1 - (\sum_{l=1,\ldots,j} \theta^l - \sum_{l \in J_-} \theta^l + \sum_{l \in I_+} \theta^l)} \\ &= \frac{\breve{\gamma}^0 - (\overline{z} - \theta^- y^- + \theta^+ y^+)}{1 - (\overline{\theta} - \theta^- + \theta^+)} \\ &\le \frac{\breve{\gamma}^0 - \overline{z}}{1 - \overline{\theta}} = \varphi(\breve{\gamma}^0) = \beta^j(\breve{\gamma}^0) \quad \text{for} \quad \breve{\gamma}^0 \in [\underline{\gamma}_*, \overline{\gamma}_*] := [\bar{\gamma}^j_*, \bar{\gamma}^{j+1}_*] \end{aligned}$$

where the inequality follows from Lemma 14.8 and the fact that

$$\beta^j(\bar{\gamma}^l_*) = \check{\gamma}^l \quad \text{for} \quad l = j, j+1.$$

Therefore

$$S^k = \beta^j(\check{\gamma}^0) \leq \beta^j(\bar{\gamma}^{j+1}_*) = \check{\gamma}^{j+1} \leq \check{\gamma}^k \tag{14.21}$$

Conclusion of Case 2
From (14.20) and (14.21), we deduce that (14.18) holds true. In particular this implies Points ii), iii) and iv) respectively for $n = 2$, $n = 3$ and $n \geq 4$.
This ends the proof of the lemma.

14.2 1:n conservative preflux (I): central case with variable coefficients

We start with the following result about quasi-prefluxes.

Lemma 14.10 (A family of central 1:n conservative quasi-prefluxes with variable coefficients $\hat{\theta}$)
Let $N := 1 + n$ with $n \geq 1$. We consider functions $\hat{\theta}^j$ for $j = 1, \ldots, n$, satisfying

$$\begin{cases} \hat{\theta}^j : [0,+\infty) \to [0,+\infty) \quad \text{continuous,} & \text{for} \quad j = 0, 1, \ldots, n \\ \displaystyle\sum_{j=1,\ldots,n} \hat{\theta}^j(\lambda) = \lambda =: \hat{\theta}^0(\lambda), & \text{for all} \quad \lambda \in [0,+\infty) \end{cases} \tag{14.22}$$

We also consider the function $\hat{\lambda} = (\hat{\lambda}^0, \ldots, \hat{\lambda}^n) : [0,+\infty)^N \to [0,+\infty)^N$ defined for $\bar{\gamma} = (\bar{\gamma}^0, \ldots, \bar{\gamma}^n) \in [0,+\infty)^N$ by

$$\begin{cases} \hat{\lambda}^j(\bar{\gamma}) := \min\left\{\bar{\gamma}^j, \hat{\theta}^j(\bar{\gamma}^0)\right\}, & \text{for} \quad j = 1, \ldots, n \\ \hat{\lambda}^0 := \displaystyle\sum_{j=1,\ldots,n} \hat{\lambda}^j \end{cases} \tag{14.23}$$

i) (Quasi-preflux characterization)
Then the function $\hat{\lambda} : [0,+\infty)^N \to [0,+\infty)^N$ is a quasi-preflux if and only if the following additional condition is satisfied

$$\hat{\theta}^j : [0,+\infty) \to [0,+\infty) \quad \text{is nondecreasing} \quad \text{for} \quad j = 0, 1, \ldots, n \tag{14.24}$$

ii) (Additional properties)
Assume now (14.24). Then $\hat{\lambda}$ is a quasi-preflux which is conservative and is also (in the sense of Definition 13.1):
1. uniformly locally bounded,
2. boundedly continuous on the box $[0,+\infty]^N$,
3. j-locally quasi-constant for all $j = 1, \ldots, n$ in the sense of Definition 13.10.
iii) (Further properties)
iii.a)
If

$$\hat{\theta}^j \text{ is increasing on } [0,+\infty), \quad \text{for all} \quad j = 1, \ldots, n \tag{14.25}$$

then $\hat{\lambda}$ is j-locally quasi-constant for $j = 0$.
iii.b)
If

$$\hat{\theta}^j(+\infty) = +\infty \quad \text{or} \quad \hat{\theta}^j \text{ is constant on some interval } [\rho_j, +\infty), \quad \text{for all} \quad j = 1, \ldots, n \tag{14.26}$$

then $\hat{\lambda}$ is boundedly locally constant at infinity.

Proof of Lemma 14.10
Step 1: proof of i)
We first notice that $\hat\lambda$ is continuous and satisfies $0 \le \hat\lambda \le id_{[0,+\infty)^N}$. Recall that $\hat\lambda$ is then a quasi-preflux if and only if it is Riemann monotone.
Step 1.1: Riemann monotonicity implies (14.24)
Assume by contradiction that (14.24) is false. Up to relabel the functions, we can assume that $\hat\theta^1$ is not nondecreasing. Then $n \ge 2$. Moreover there exists $\bar\gamma^{0\prime}, \bar\gamma^0 \in [0,+\infty)$ such that

$$\hat\theta^1(\bar\gamma^{0\prime}) < \hat\theta^1(\bar\gamma^0) \quad \text{and} \quad \bar\gamma^{0\prime} > \bar\gamma^0 \ge 0$$

We define

$$\begin{cases} \bar\gamma^{1\prime} = \bar\gamma^1 = \hat\theta^1(\bar\gamma^0) > \hat\theta^1(\bar\gamma^{0\prime}), \\ \bar\gamma^{j\prime} = \bar\gamma^j := \min\left\{\hat\theta^j(\bar\gamma^{0\prime}), \hat\theta^j(\bar\gamma^0)\right\}, \quad j = 2,\dots,n \end{cases}$$

Then we have

$$(\bar\gamma' - \bar\gamma) \diamond \left\{\hat\lambda(\bar\gamma') - \hat\lambda(\bar\gamma)\right\} \le 0$$

Indeed, we have $(\bar\gamma' - \bar\gamma)^j = 0$ for $j = 1,\dots,n$ and

$$\begin{cases} \hat\lambda^1(\bar\gamma') = \min\left\{\bar\gamma^{1\prime}, \hat\theta^1(\bar\gamma^{0\prime})\right\} = \hat\theta^1(\bar\gamma^{0\prime}) < \min\left\{\bar\gamma^1, \hat\theta^1(\bar\gamma^0)\right\} = \hat\lambda^1(\bar\gamma) \\ \\ \hat\lambda^j(\bar\gamma') = \min\left\{\bar\gamma^{j\prime}, \hat\theta^j(\bar\gamma^{0\prime})\right\} = \min\left\{\bar\gamma^j, \hat\theta^j(\bar\gamma^0)\right\} = \hat\lambda^j(\bar\gamma), \quad j = 2,\dots,n \\ \\ \hat\lambda^0(\bar\gamma') < \hat\lambda^0(\bar\gamma') \end{cases}$$

Because $\hat\lambda(\bar\gamma') \ne \hat\lambda(\bar\gamma)$, we deduce that $\hat\lambda$ is not Riemann monotone. Contradiction. Therefore (14.24) holds true.
Step 1.2: (14.24) implies Riemann monotonicity
Consider $\bar\gamma', \bar\gamma \in [0,+\infty)^N$, and assume that

$$(\bar\gamma' - \bar\gamma) \diamond \left\{\hat\lambda(\bar\gamma') - \hat\lambda(\bar\gamma)\right\} \le 0 \tag{14.27}$$

Up to exchange $\bar\gamma'$ and $\bar\gamma$, we can assume that

$$\bar\gamma^{0\prime} \ge \bar\gamma^0$$

Now for $j = 1,\dots,n$, we have $\hat\theta^j(\bar\gamma^{0\prime}) \ge \hat\theta^j(\bar\gamma^0)$ and

$$\bar\gamma^{j\prime} \ge \bar\gamma^j \quad \Longrightarrow \quad \hat\lambda^j(\bar\gamma') \ge \hat\lambda^j(\bar\gamma)$$

Hence using (14.27), we get

$$\bar\gamma^{j\prime} > \bar\gamma^j \quad \Longrightarrow \quad \hat\lambda^j(\bar\gamma') = \hat\lambda^j(\bar\gamma) \tag{14.28}$$

Let

$$I := \left\{j \in \{1,\dots,n\}, \quad \hat\lambda^j(\bar\gamma') \ne \hat\lambda^j(\bar\gamma)\right\}$$

If $j \in I$, then (14.28) implies $\bar\gamma^{j\prime} \le \bar\gamma^j$. Therefore

$$I = I_+ \cup I_0 \quad \text{with} \quad I_+ := \left\{j \in I, \quad \bar\gamma^j > \bar\gamma^{j\prime}\right\}, \quad I_0 := \left\{j \in I, \quad \bar\gamma^j = \bar\gamma^{j\prime}\right\}$$

Case A: $j \in I_+$
Then (14.27) implies $\hat\lambda^j(\bar\gamma') > \hat\lambda^j(\bar\gamma)$ and

$$\bar\gamma^j > \bar\gamma^{j\prime} \ge \min\left\{\bar\gamma^{j\prime}, \hat\theta^j(\bar\gamma^{0\prime})\right\} = \hat\lambda^j(\bar\gamma') > \hat\lambda^j(\bar\gamma) = \min\left\{\bar\gamma^j, \hat\theta^j(\bar\gamma^0)\right\}$$

This forces $\hat\lambda^j(\bar\gamma^0) = \hat\theta^j(\bar\gamma^0) < \hat\theta^j(\bar\gamma^{0\prime})$ and then

$$\hat\lambda^j(\bar\gamma') > \hat\lambda^j(\bar\gamma) \quad \text{with} \quad \bar\gamma^{0\prime} > \bar\gamma^0$$

Case B: $j \in I_0$
Then $\bar\gamma^j = \bar\gamma^{j\prime}$ and $\hat\lambda^j(\bar\gamma') \ge \hat\lambda^j(\bar\gamma)$. Moreover because $j \in I$, this forces again

$$\hat\lambda^j(\bar\gamma') > \hat\lambda^j(\bar\gamma) \quad \text{with} \quad \bar\gamma^{0\prime} > \bar\gamma^0$$

Conclusion
If $I \neq \emptyset$, then we get

$$\bar\gamma^{0\prime} > \bar\gamma^0 \quad \text{and} \quad \hat\lambda^0(\bar\gamma') = \sum_{j=1,\dots,n} \hat\lambda^j(\bar\gamma') > \sum_{j=1,\dots,n} \hat\lambda^j(\bar\gamma) = \hat\lambda^0(\bar\gamma)$$

which gives a contradiction with (14.27). Therefore $I = \emptyset$ and

$$\hat\lambda(\bar\gamma') = \hat\lambda(\bar\gamma)$$

which shows that $\hat\lambda$ is Riemann monotone.
Step 2: proof of ii)
Step 2.1: uniform local boundedness
Because $0 \le \hat\lambda \le id_{[0,+\infty)^N}$, we deduce that $\hat\lambda$ is uniformly locally bounded.
Step 2.2: bounded continuity on the box $[0,+\infty]^N$
Let $\rho > 0$ and a sequence $\bar\gamma_k \in [0,+\infty)^N$ such that

$$\hat\lambda(\bar\gamma_k) \in [0,\rho]^N$$

Up to extract a subsequence (still denoted $(\bar\gamma_k)_{k\in\mathbb{N}}$), we have

$$\bar\gamma_k^0 \to \bar\gamma_\infty^0 \in [0,+\infty] \quad \text{and} \quad \bar\gamma_k \to \bar\gamma_\infty \in \left\{\bar\gamma_\infty^0\right\} \times [0,+\infty]^n$$

Two cases may hold.
Case A: $\bar\gamma_\infty^0 = +\infty$
From the monotonicity of the functions $\hat\theta^j$, we deduce that

$$\hat\theta(\bar\gamma_k^0) \to \hat\theta_\infty \in \{+\infty\} \times [0,+\infty]^n$$

where $\hat\theta_\infty$ is independent on the sequence. Let us consider the set

$$I := \left\{ j \in \{1,\dots,n\}, \quad \hat\theta_\infty^j = +\infty \right\}$$

which is non empty because $\displaystyle\sum_{j=1,\dots,n} \hat\theta_\infty^j = \hat\theta_\infty^0 = \bar\gamma_\infty^0 = +\infty$. Then

$$\bar\gamma_\infty^j \in [0,\rho] \quad \text{for all} \quad j \in I$$

Again, it is easy to see that $\hat\lambda(\bar\gamma_k)$ converges to a limit which is continuous in $\bar\gamma_\infty^j \in [0,+\infty]$ for all $j \in \{1,\dots,n\}$.
Case B: $\bar\gamma_\infty^0 \in [0,+\infty)$
Then it is straightworward to see that $\hat\lambda(\bar\gamma_k)$ converges to a limit which is continuous in $\bar\gamma_\infty^0 \in [0,+\infty)$.
Conclusion
Cases A and B are not sufficient to conclude. But more generally if now $\bar\gamma_k \in [0,+\infty]^N$ such that $\hat\lambda(\bar\gamma_k) \in [0,\rho]^N$, then proceeding as in Cases A and B, we can easily conclude that $\hat\lambda$ extends to a continuous function on $\overline{(\hat\lambda)^{-1}([0,\rho]^N)}^{[0,+\infty]^N}$. This shows that $\hat\lambda$ is boundedly continuous on the box $[0,+\infty]^N$.
Step 2.3: j-local quasi-constancy for $j = 1,\dots,n$
Let $\bar\gamma \in [0,+\infty)^N$ and

$$\Phi(x) := \hat\lambda(\bar\gamma + x \cdot e_j)$$

Then $(\Phi^j)^{-1}(\Phi(0))$ is not reduced to a singleton if and only if $\bar\gamma^j > \hat\theta^j(\bar\gamma^0)$. In that case, we see that $\hat\lambda$ is locally independent on $\bar\gamma^j$, and then Φ is constant on $(\Phi^j)^{-1}(\Phi(0))$. This shows that $\hat\lambda$ is j-locally quasi constant for such j.

Step 3: proof of iii)
Step 3.1: proof of iii.a)
Case 1: $\bar{\gamma}^k > \hat{\theta}^k(\bar{\gamma}^0)$ **for some** $k \in \{1, \dots, n\}$
Then locally $\hat{\lambda}^k(\bar{\gamma}) = \hat{\theta}^k(\bar{\gamma}^0)$, and then $\hat{\lambda}^0 = \sum_{l=1,\dots,n} \hat{\lambda}^l$ does depend on $\hat{\theta}^k(\bar{\gamma}^0)$. Under assumption (14.25), we see that $\hat{\lambda}^0$ is not locally constant on $\bar{\gamma}^0$. Therefore $(\Phi^0)^{-1}(\Phi^0(0))$ is reduced to a singleton, and in particular, we get

$$\Phi = const \quad \text{on} \quad (\Phi^0)^{-1}(\Phi^0(0)) \tag{14.29}$$

Case B: $\bar{\gamma}^k = \hat{\theta}^k(\bar{\gamma}^0)$ **for some** $k \in \{1, \dots, n\}$
Similarly, we get that $[-\bar{\gamma}^0, 0] \cap (\Phi^0)^{-1}(\Phi^0(0))$ is reduced to a singleton. Notice that $[0, +\infty) \cap (\Phi^0)^{-1}(\Phi^0(0))$ may be non reduced to a singleton, but then $\Phi^k = \bar{\gamma}^k$ is constant on it. We conclude that

$$\Phi^k = const \quad \text{on} \quad (\Phi^0)^{-1}(\Phi^0(0)) \tag{14.30}$$

Case C: $\bar{\gamma}^j \le \hat{\theta}^j(\bar{\gamma}^0)$ **for all** $j \in \{1, \dots, n\}$
Assume that $\bar{\gamma}^k < \hat{\theta}^k(\bar{\gamma}^0)$ for some $k \in \{1, \dots, n\}$. Then (14.30) is still true. Therefore, using Case B, we see that (14.30) holds true for all $k = 1, \dots, n$ and this implies (14.29).
Conclusion
We conclude that under assumption (14.25), then $\hat{\lambda}$ is 0-locally quasi-constant.
Step 3.2: proof of iii.b)
Let $\bar{\gamma} \in [0, +\infty]^N$ such that $\hat{\lambda}(\bar{\gamma}) \in [0, \rho_0]^N$ for some $\rho_0 > 0$. Let

$$I := \left\{ j \in \{0, \dots, n\}, \quad \bar{\gamma}^j = +\infty \right\}$$

and for $\rho > 0$ large enough, consider

$$\bar{\gamma}^j_\rho := \begin{cases} \rho & \text{if} \quad j \in I, \\ \bar{\gamma}^j & \text{if} \quad j \notin I, \end{cases}$$

Notice that either $\bar{\gamma}^0 \in [0, +\infty)$ or $\bar{\gamma}^0 = +\infty$. In both cases (using (14.26) in case $\bar{\gamma}^0 = +\infty$), we deduce for $\rho > 0$ large enough that

$$\hat{\lambda} = const = \hat{\lambda}(\bar{\gamma}_\rho) \quad \text{on} \quad \bar{\gamma}_\rho + \sum_{j \in I} [0, +\infty) e_j$$

which shows that $\hat{\lambda}$ is boundedly locally constant at infinity.
This ends the proof of the lemma.

Lemma 14.11 (Central $1:n$ conservative Kružkov prefluxes)
Let $N := 1 + n$ with $n \ge 1$. We consider functions $\hat{\theta}^j$ for $j = 0, \dots, n$, satisfying

$$\begin{cases} \hat{\theta}^j : [0, +\infty) \to [0, +\infty) \quad \textit{continuous increasing surjective}, \quad \textit{for} \quad j = 0, 1, \dots, n \\ \displaystyle\sum_{j=1,\dots,n} \hat{\theta}^j(\lambda) = \lambda =: \hat{\theta}^0(\lambda) \quad \textit{for all} \quad \lambda \in [0, +\infty) \end{cases} \tag{14.31}$$

Given $\bar{\gamma} := (\bar{\gamma}^0, \bar{\gamma}^1, \dots, \bar{\gamma}^n) \in [0, +\infty)^N$, we consider

$$\begin{cases} \hat{\lambda}^j(\bar{\gamma}) := \min \left\{ \bar{\gamma}^j, \hat{\theta}^j(\bar{\gamma}^0) \right\}, \qquad \textit{for} \quad j = 1, \dots, n \\ \hat{\lambda}^0 := \displaystyle\sum_{j=1,\dots,n} \hat{\lambda}^j \end{cases} \tag{14.32}$$

Then $\hat{\lambda}$ is a $1:n$ conservative Kružkov quasi-preflux. Moreover its Riemann relaxation on the box $[0, +\infty]^N$

$$\hat{\gamma} := \mathfrak{R}_\infty \hat{\lambda}$$

is well-defined and is a $1:n$ *conservative Kružkov preflux.*
Setting for $\lambda^{0R} \in [0,+\infty)$

$$\left\{ \begin{array}{l} \phi_\infty := \displaystyle\sum_{j=1,\dots;n} \bar\gamma^j, \\ \phi(\lambda^{0R}) := \displaystyle\sum_{j=1,\dots,n} \min\left\{\bar\gamma^j, \hat\theta^j(\lambda^{0R})\right\} \end{array} \right.$$

we moreover have

$$\left\{ \begin{array}{l} \hat\gamma^0(\bar\gamma) = \min\left\{\bar\gamma^0, \phi_\infty\right\} \\ \hat\gamma^j(\bar\gamma) = \min\left\{\bar\gamma^j, \hat\theta^j(\lambda^{0R})\right\} \quad \textit{with} \quad \lambda^{0R} := \left\{ \begin{array}{ll} \phi^{-1}(\bar\gamma^0) & \textit{if} \quad \bar\gamma^0 < \phi_\infty \\ \bar\gamma^0_\infty & \textit{if} \quad \bar\gamma^0 \ge \phi_\infty \end{array} \right. \end{array} \right. \tag{14.33}$$

where

$$\bar\gamma^0_\infty := \sup_{j=1,\dots,n} \breve\gamma^j \quad \textit{with} \quad \breve\gamma^j := (\hat\theta^j)^{-1}(\bar\gamma^j), \quad j = 0,1,\dots,n.$$

Moreover the map

$$[0,+\infty)^{1+n} \ni \bar\gamma \mapsto \lambda^{0R} \in [0,+\infty) \quad \textit{is continuous.}$$

In particular, we also have

$$\hat\gamma(\min\left\{\bar\gamma^0, \phi_\infty\right\}, \bar\gamma') = \hat\gamma(\bar\gamma) \quad \textit{for} \quad \bar\gamma = (\bar\gamma^0, \bar\gamma') \quad \textit{with} \quad \bar\gamma' = (\bar\gamma^1, \dots, \bar\gamma^n).$$

Proof of Lemma 14.11
Step 1: preliminaries
Notice that $\hat\lambda$ is conservative for the orientation $\sigma := (\sigma^0, \dots, \sigma^n) = (1,-1,-1,\dots,-1)$ and is also σ-monotone, i.e. that

$$\sigma^j \sigma^k \partial_k \hat\lambda^j \le 0 \quad \text{for all} \quad k \ne j.$$

Then from ii) of Lemma 13.3, we easily deduce that $\hat\lambda$ is a Kružkov quasi-preflux. Moreover, from Lemma 14.10, we deduce from assumption (14.31) that $\hat\lambda$ is a quasi-preflux which is boundedly continuous on the box $[0,+\infty]^N$ and boundedly locally constant at infinity. From Proposition 13.8, we deduce that the Riemann relaxation $\hat\gamma := \mathfrak{R}_\infty \hat\lambda$ is a well-defined preflux. Moreover, from ii) of Lemma 13.9, we also deduce that $\hat\gamma$ is a σ-conservative Kružkov preflux.

Even if it is not useful here, let us also notice that $\hat\lambda$ is also uniformly locally bounded, and is j-locally quasi-constant for all $j = 0,\dots,n$, in the sense of Definition 13.10.
Step 2: computation of $\hat\gamma$
From ii) of Proposition 13.8, and for $\lambda_\infty := (+\infty,\dots,+\infty) \in \{+\infty\}^{1+n}$, we have the existence of some $(\lambda^L, \lambda^R) \in \mathbb{D}_{\lambda_\infty}$ such that

$$\left\{ \begin{array}{llll} \hat\gamma^j(\bar\gamma) & = \min\left\{\bar\gamma^j, \lambda^{jL}\right\} & = \min\left\{\lambda^{jR}, \hat\theta^j(\lambda^{0R})\right\}, & j = 1,\dots,n, \\ \hat\gamma^0(\bar\gamma) & = \min\left\{\bar\gamma^0, \lambda^{0L}\right\} & = \displaystyle\sum_{j=1,\dots,n} \min\left\{\lambda^{jR}, \hat\theta^j(\lambda^{0R})\right\} & \end{array} \right.$$

Hence

$$\left\{ \begin{array}{llll} \hat\gamma^j(\bar\gamma) & = \min\left\{\bar\gamma^j, \hat\theta^j(\lambda^{0R})\right\} & = \min\left\{\lambda^{jL}, \lambda^{jR}\right\}, & j = 1,\dots,n, \\ \hat\gamma^0(\bar\gamma) & = \min\left\{\bar\gamma^0, \lambda^{0L}\right\} & = \displaystyle\sum_{j=1,\dots,n} \min\left\{\bar\gamma^j, \hat\theta^j(\lambda^{0R})\right\} =: \phi(\lambda^{0R}) & \end{array} \right. \tag{14.34}$$

We then define (with $\hat\theta^0 = id_{[0,+\infty)} \ge \phi$)

$$\breve\gamma^j := (\hat\theta^j)^{-1}(\bar\gamma^j), \quad j = 1,\dots,n$$

Up to relabel the indices, let us assume that

$$\breve\gamma^0 := 0 \le \breve\gamma^1 \le \breve\gamma^2 \le \dots \le \breve\gamma^n < \breve\gamma^{n+1} := +\infty \tag{14.35}$$

Moreover, we can assume that the inequalities are strict

$$\breve\gamma^0 = 0 < \breve\gamma^1 < \dots < \breve\gamma^n < \breve\gamma^{n+1} = +\infty, \tag{14.36}$$

because the general case can be recovered by continuity of the preflux.
Then ϕ is increasing on $[0, \breve{\gamma}^n]$ and then constant on $[\breve{\gamma}^n, +\infty)$ with value

$$\phi_\infty := \sum_{j=1,\dots,n} \bar{\gamma}^j$$

Then

$$\bar{\gamma}^n_* = \phi(\breve{\gamma}^n) = \phi_\infty.$$

As a curiosity, let us notice that we get $\breve{\gamma}^n = \sum_{l=1,\dots,n} \hat{\theta}^l(\breve{\gamma}^n) \geq \sum_{l=1,\dots,n} \hat{\theta}^l(\breve{\gamma}^l) = \sum_{l=1,\dots,n} \bar{\gamma}^l = \phi_\infty$, i.e.

$$\breve{\gamma}^n \geq \phi_\infty.$$

Then it is easy to see that

$$(\lambda^{0L}, \lambda^{0R}) = \begin{cases} (+\infty, \phi^{-1}(\bar{\gamma}^0)) & \text{if } \ \bar{\gamma}^0 < \phi_\infty, \\ (\phi_\infty, \breve{\gamma}^n) & \text{if } \ \bar{\gamma}^0 \geq \phi_\infty, \end{cases} \tag{14.37}$$

is a solution. With (14.34), we see that this implies the first and second lines of (14.33). Moreover, under assumption (14.36), it is easy to check that

$$\lim_{\phi_\infty > \bar{\gamma}^0 \to \phi_\infty} \phi^{-1}(\bar{\gamma}^0) = \breve{\gamma}^n$$

It stays true under assumption (14.35), and then shows that more generally, we have

$$\lim_{\phi_\infty > \bar{\gamma}^0 \to \phi_\infty} \phi^{-1}(\bar{\gamma}^0) = \bar{\gamma}^0_\infty = \sup_{j=1,\dots,n} \breve{\gamma}^j$$

This shows that the map $\bar{\gamma} \mapsto \lambda^{0R}$ (as defined in (14.37)) is continuous. This ends the proof of the lemma.

Corollary 14.12 (Long explicit central 1:n conservative preflux)
We work under assumptions of Lemma 14.11 with $N = 1 + n$ and $n \geq 1$, and consider the $1:n$ conservative Kružkov preflux $\hat{\gamma} = (\hat{\gamma}^0, \dots, \hat{\gamma}^n) : [0, +\infty)^{n+1} \to [0, +\infty)^{n+1}$ as given in Lemma 14.11. Recall that for $\lambda^{0R} \in [0, +\infty)$

$$\begin{cases} \phi(\lambda^{0R}) := \displaystyle\sum_{j=1,\dots,n} \min\left\{\bar{\gamma}^j, \hat{\theta}^j(\lambda^{0R})\right\} \\ \bar{\gamma}^j_* := \phi(\breve{\gamma}^j) \quad \textit{with} \quad \breve{\gamma}^j := (\hat{\theta}^j)^{-1}(\bar{\gamma}^j), \quad j = 1, \dots, n \end{cases}$$

with $\phi : [0, +\infty) \to [0, +\infty)$ nondecreasing.
i) (Case $n = 1$)
Then $\hat{\theta}^1 = id_{[0,+\infty)}$ and

$$\hat{\gamma}^0(\bar{\gamma}) = \hat{\gamma}^1(\bar{\gamma}) = \min\left\{\bar{\gamma}^0, \bar{\gamma}^1\right\}.$$

ii) (Case $n = 2$)
Then

$$\hat{\gamma}^0(\bar{\gamma}) = \min\left\{\bar{\gamma}^0, \bar{\gamma}^1 + \bar{\gamma}^2\right\} = (\hat{\gamma}^1 + \hat{\gamma}^2)(\bar{\gamma})$$

and

$$(\hat{\gamma}^1, \hat{\gamma}^2)(\bar{\gamma}) = \begin{cases} (\hat{\theta}^1(\bar{\gamma}^0), \hat{\theta}^2(\bar{\gamma}^0)) & \textit{if } \ \bar{\gamma}^0 < \bar{\gamma}^1_* \\ (\bar{\gamma}^1, \bar{\gamma}^0 - \bar{\gamma}^1) & \textit{if } \ \bar{\gamma}^1_* \leq \bar{\gamma}^0 < \bar{\gamma}^2_* \\ (\bar{\gamma}^1, \bar{\gamma}^2) & \textit{if } \ \bar{\gamma}^2_* \leq \bar{\gamma}^0 \\ \textit{assuming everywhere that} & \breve{\gamma}^1 \leq \breve{\gamma}^2 \end{cases} \tag{14.38}$$

and all other cases (not indicated in (14.38)) are then obtained by permutations on the indices.
iii) (Case $n = 3$)
Then

$$\hat{\gamma}^0(\bar{\gamma}) = \min\left\{\bar{\gamma}^0, \bar{\gamma}^1 + \bar{\gamma}^2 + \bar{\gamma}^3\right\} = (\hat{\gamma}^1 + \hat{\gamma}^2 + \hat{\gamma}^3)(\bar{\gamma})$$

and
(14.39)

$$
(\hat\gamma^1,\hat\gamma^2,\hat\gamma^3)(\bar\gamma) = \begin{cases} (\hat\theta^1(\bar\gamma^0),\hat\theta^2(\bar\gamma^0),\hat\theta^3(\bar\gamma^0)) & \text{if} \quad \bar\gamma^0 < \bar\gamma^1_* \\ (\bar\gamma^1, \quad (\hat\theta^2\circ(id_{[0,+\infty)}-\hat\theta^1)^{-1})(\bar\gamma^0-\bar\gamma^1), \quad (\hat\theta^3\circ(id_{[0,+\infty)}-\hat\theta^1)^{-1})(\bar\gamma^0-\bar\gamma^1)) & \text{if} \quad \bar\gamma^1_*\le\bar\gamma^0<\bar\gamma^2_* \\ (\bar\gamma^1,\bar\gamma^2,\bar\gamma^0-(\bar\gamma^1+\bar\gamma^2)) & \text{if} \quad \bar\gamma^2_*\le\bar\gamma^0<\bar\gamma^3_* \\ (\bar\gamma^1,\bar\gamma^2,\bar\gamma^3) & \text{if} \quad \bar\gamma^3_*\le\bar\gamma^0 \\ \text{assuming everywhere that} & \breve\gamma^1\le\breve\gamma^2\le\breve\gamma^3 \end{cases}
$$

and all other cases (not indicated in (14.39)) are then obtained by permutations on the indices.
iv) (Case $n\ge 4$)
Then

$$\hat\gamma^0(\bar\gamma)=\min\left\{\bar\gamma^0,\bar\gamma^1+\cdots+\bar\gamma^n\right\}=(\hat\gamma^1+\cdots+\hat\gamma^n)(\bar\gamma)$$

and for

$$\begin{cases} \bar\gamma^0_k := \bar\gamma^0-(\bar\gamma^1+\cdots+\bar\gamma^k), \\ \hat\theta^j_k := \hat\theta^j\circ(id_{[0,+\infty)}-(\hat\theta^1+\cdots+\hat\theta^k))^{-1}, \qquad j\ge k+1 \end{cases}$$

$$
(14.40) \qquad (\hat\gamma^1,\dots,\hat\gamma^n)(\bar\gamma) = \begin{cases} (\hat\theta^1(\bar\gamma^0),\hat\theta^2(\bar\gamma^0),\dots,\hat\theta^n(\bar\gamma^0)) & \text{if} \quad \bar\gamma^0<\bar\gamma^1_* \\ (\bar\gamma^1,\hat\theta^2_1(\bar\gamma^0_1),\dots,\hat\theta^n_1(\bar\gamma^0_1)) & \text{if} \quad \bar\gamma^1_*\le\bar\gamma^0<\bar\gamma^2_* \\ (\bar\gamma^1,\bar\gamma^2,\hat\theta^3_2(\bar\gamma^0_2),\dots,\hat\theta^{n-1}_2(\bar\gamma^0_2),\hat\theta^n_2(\bar\gamma^0_2)) & \text{if} \quad \bar\gamma^2_*\le\bar\gamma^0<\bar\gamma^3_* \\ \vdots & \\ (\bar\gamma^1,\bar\gamma^2,\dots,\bar\gamma^{n-2},\hat\theta^{n-1}_{n-2}(\bar\gamma^0_{n-2}),\hat\theta^n_{n-2}(\bar\gamma^0_{n-2})) & \text{if} \quad \bar\gamma^{n-2}_*\le\bar\gamma^0<\bar\gamma^{n-1}_* \\ (\bar\gamma^1,\bar\gamma^2,\dots,\bar\gamma^{n-2},\bar\gamma^{n-1},\hat\theta^n_{n-1}(\bar\gamma^0_{n-1})) & \text{if} \quad \bar\gamma^{n-1}_*\le\bar\gamma^0<\bar\gamma^n_* \\ (\bar\gamma^1,\bar\gamma^2,\dots,\bar\gamma^n) & \text{if} \quad \bar\gamma^n_*\le\bar\gamma^0 \\ \text{assuming everywhere that} & \breve\gamma^1\le\breve\gamma^2\le\cdots\le\breve\gamma^n \end{cases}
$$

and all other cases (not indicated in (14.40)) are then obtained by permutations on the indices $\{1,\dots,n\}$*.*
Notice that $\hat\theta^n_{n-1}=id_{[0,+\infty)}$ *and* $\bar\gamma^n_*=\bar\gamma^1+\cdots+\bar\gamma^n$*.*

Proof of Corollary 14.12
We simply make explicit the statement of Lemma 14.11. Again we can assume that the inequalities are strict

$$(14.41) \qquad \breve\gamma^0=0<\breve\gamma^1<\cdots<\breve\gamma^n<\breve\gamma^{n+1}=+\infty,$$

because the general case with large inequalities can be recovered by continuity of the preflux.
Recall that $\phi(\breve\gamma^n)=\phi_\infty$ and $\phi<\phi_\infty$ on $[0,\breve\gamma^n)$. Hence

$$0=\bar\gamma^0_*<\bar\gamma^1_*<\cdots<\bar\gamma^n_*=\bar\gamma^{n+1}_*:=\phi_\infty$$

and

$$\begin{cases} \hat\gamma^0(\bar\gamma)=\min\left\{\bar\gamma^0,\phi_\infty\right\}, \\ \hat\gamma^j(\bar\gamma)=\min\left\{\bar\gamma^j,\hat\theta^j(\lambda^{0R})\right\} \end{cases} \quad \text{with} \quad \lambda^{0R}=\begin{cases} \phi^{-1}(\bar\gamma^0) & \text{if} \quad \bar\gamma^0<\phi_\infty \\ \breve\gamma^n & \text{if} \quad \bar\gamma^0\ge\phi_\infty \end{cases}$$

Recall that

$$\hat\gamma(\min\left\{\bar\gamma^0,\phi_\infty\right\},\bar\gamma')=\hat\gamma(\bar\gamma^0,\bar\gamma')$$

and then we can assume that

$$\bar\gamma^0\le\phi_\infty$$

Now for $k = 0, \ldots, n-1$, we have for $\bar\gamma^0 \in [0, \phi_\infty]$

$$\begin{cases} \bar\gamma_*^k \le \bar\gamma^0 < \bar\gamma_*^{k+1} & \iff \quad \breve\gamma^k \le \lambda^{0R} < \breve\gamma^{k+1} \\ \bar\gamma^0 = \bar\gamma_*^n & \iff \quad \lambda^{0R} = \breve\gamma^n. \end{cases}$$

Case A: $\bar\gamma^0 < \bar\gamma_*^n = \phi_\infty$
We see that $\lambda^{0R} = \phi^{-1}(\bar\gamma^0)$ satisfies for some $k \in \{0, \ldots, n-1\}$

$$\bar\gamma^0 = \hat\gamma^0(\bar\gamma) = \left\{ \sum_{j \le k} \bar\gamma^j \right\} + \sum_{j>k} \hat\theta^j(\lambda^{0R}) \quad \text{if} \quad \breve\gamma^k \le \lambda^{0R} < \breve\gamma^{k+1}$$

This means for $k = 0, \ldots, n-1$ that
(14.42)

$$\lambda^{0R} = \phi^{-1}(\bar\gamma^0) = \left(\sum_{j>k} \hat\theta^j \right)^{-1} (\bar\gamma^0 - \sum_{j \le k} \bar\gamma^j) = \left(id_{[0,+\infty)} - \sum_{j \le k} \hat\theta^j \right)^{-1} (\bar\gamma^0 - \sum_{j \le k} \bar\gamma^j) \quad \text{when} \quad \breve\gamma^k \le \lambda^{0R} < \breve\gamma^{k+1}$$

Hence we see that that the explicit shape of ϕ changes when $\bar\gamma^0$ takes values such that $\breve\gamma^k = \phi^{-1}(\bar\gamma^0)$, i.e. such that $\bar\gamma^0 := \bar\gamma_*^k$. Then, assuming (14.42), we get for $j = 1, \ldots, n$

$$\hat\gamma^j(\bar\gamma) = \hat\theta^j(\min\{\breve\gamma^j, \lambda^{0R}\}) = \begin{cases} \bar\gamma^j & \text{if} \quad j \le k, \\ (\hat\theta^j \circ \phi^{-1})(\bar\gamma^0) & \text{if} \quad j > k, \end{cases}$$

This means for $k = 0, \ldots, n-1$, that
(14.43)

$$\left\{ \begin{array}{l} (\hat\gamma^1, \ldots, \hat\gamma^n)(\bar\gamma) = (\bar\gamma^1, \ldots, \bar\gamma^k, \hat\theta^{k+1}, \ldots, \hat\theta^n)(\phi^{-1}(\bar\gamma^0)) \\ \text{with} \quad \phi^{-1}(\bar\gamma^0) = \left(\sum_{\ell>k} \hat\theta^\ell \right)^{-1} (\bar\gamma^0 - \sum_{\ell \le k} \bar\gamma^\ell) = \left(id_{[0,+\infty)} - \sum_{\ell \le k} \hat\theta^\ell \right)^{-1} (\bar\gamma^0 - \sum_{\ell \le k} \bar\gamma^\ell) \end{array} \right| \quad \text{if} \quad \bar\gamma_*^k \le \bar\gamma^0 < \bar\gamma_*^{k+1}$$

where the second line is not defined for $k = n$ (but it then not used in the first line).
Case B: $\bar\gamma^0 \ge \phi_\infty$
Then $\bar\gamma^0 \ge \phi_\infty = \bar\gamma_*^{n+1}$, and

$$(\hat\gamma^1, \ldots, \hat\gamma^n)(\bar\gamma) = (\bar\gamma^1, \ldots, \bar\gamma^n). \tag{14.44}$$

This ends the proof of the corollary.

We will need the following technical lemma (generalizing Lemma 14.8).

Lemma 14.13 (An other specific inequality)
Let $\hat\theta, \hat\theta^\pm : [0, +\infty) \to [0, +\infty)$ be continuous increasing surjective maps such that the map $\hat\theta - (\hat\theta^+ - \hat\theta^-)$: $[0, +\infty) \to [0, +\infty)$ is also continuous increasing surjective. Let us consider real numbers satisfying

$$0 \le y^- \le y^+$$

Then

$$\left\{ \hat\theta - (\hat\theta^+ - \hat\theta^-) \right\}^{-1} (\hat\theta(y) - (\hat\theta^+(y^+) - \hat\theta^-(y^-)) \le y \quad \textit{for all} \quad y \in [y^-, y^+] \tag{14.45}$$

Proof of Lemma 14.13
We notice that (14.45) is equivalent to

$$\hat\theta(y) - (\hat\theta^+(y^+) - \hat\theta^-(y^-)) \le \left\{ \hat\theta - (\hat\theta^+ - \hat\theta^-) \right\}(y)$$

i.e.

$$\hat\theta^+(y) - \hat\theta^-(y) \le \hat\theta^+(y^+) - \hat\theta^-(y^-)$$

which is obviously true for $y \in [y^-, y^+]$, because of the monotonicities of $\hat\theta^\pm$. This ends the proof of the lemma.

Corollary 14.14 (Short explicit central 1:n conservative preflux)
We work under assumptions of Lemma 14.11 with $N = 1+n$ and $n \geq 1$, and consider the $1:n$ conservative Kružkov preflux $\hat{\gamma} = (\hat{\gamma}^0, \ldots, \hat{\gamma}^n) : [0,+\infty)^{n+1} \to [0,+\infty)^{n+1}$ as given in Lemma 14.11. Recall that

$$\breve{\gamma}^j := (\hat{\theta}^j)^{-1}(\bar{\gamma}^j), \quad j = 1, \ldots, n.$$

Then the preflux $\hat{\gamma}$ is given explicitly as follows.
i) (Case $n = 1$)
Then $\hat{\theta}^1 = id_{[0,+\infty)}$ and

$$\hat{\gamma}^0(\bar{\gamma}) = \hat{\gamma}^1(\bar{\gamma}) = \min\left\{\bar{\gamma}^0, \bar{\gamma}^1\right\}$$

ii) (Case $n = 2$)
Then

$$\begin{cases} \hat{\gamma}^1(\bar{\gamma}) = \min\left\{\bar{\gamma}^1, \max\left\{\hat{\theta}^1(\bar{\gamma}^0), \bar{\gamma}^0 - \bar{\gamma}^2\right\}\right\}, \\ \hat{\gamma}^2(\bar{\gamma}) = \min\left\{\bar{\gamma}^2, \max\left\{\hat{\theta}^2(\bar{\gamma}^0), \bar{\gamma}^0 - \bar{\gamma}^1\right\}\right\}, \end{cases}$$

and

$$\hat{\gamma}^0(\bar{\gamma}) = \min\left\{\bar{\gamma}^0, \bar{\gamma}^1 + \bar{\gamma}^2\right\} = (\hat{\gamma}^1 + \hat{\gamma}^2)(\bar{\gamma})$$

iii) (Case $n = 3$)
Setting $x_+ := \max(x, 0)$, let us consider

$$\begin{cases} X_1 := (\hat{\theta}^2 + \hat{\theta}^3)^{-1}(\bar{\gamma}^0 - \bar{\gamma}^1)_+, \\ X_2 := (\hat{\theta}^1 + \hat{\theta}^3)^{-1}(\bar{\gamma}^0 - \bar{\gamma}^2)_+, \\ X_3 := (\hat{\theta}^1 + \hat{\theta}^2)^{-1}(\bar{\gamma}^0 - \bar{\gamma}^3)_+, \\ \\ X_{12} := (\hat{\theta}^3)^{-1}(\bar{\gamma}^0 - (\bar{\gamma}^1 + \bar{\gamma}^2))_+, \\ X_{23} := (\hat{\theta}^1)^{-1}(\bar{\gamma}^0 - (\bar{\gamma}^2 + \bar{\gamma}^3))_+, \\ X_{31} := (\hat{\theta}^2)^{-1}(\bar{\gamma}^0 - (\bar{\gamma}^3 + \bar{\gamma}^1))_+, \end{cases}$$

and

$$\begin{cases} S^3 := \max\left\{\breve{\gamma}^0, X_1, X_2, X_{12}\right\} \\ S^1 := \max\left\{\breve{\gamma}^0, X_2, X_3, X_{23}\right\} \\ S^2 := \max\left\{\breve{\gamma}^0, X_3, X_1, X_{31}\right\} \end{cases}$$

Then we have

$$(\hat{\theta}^j)^{-1}(\hat{\gamma}^j(\bar{\gamma})) = \min\left\{\breve{\gamma}^j, S^j\right\} \quad \text{for} \quad j = 1, 2, 3 \tag{14.46}$$

i.e.

$$\begin{cases} \hat{\gamma}^1(\bar{\gamma}) = \min\left\{\bar{\gamma}^1, \max\left\{\hat{\theta}^1(\bar{\gamma}^0), \hat{\theta}^1 \circ (\hat{\theta}^1 + \hat{\theta}^3)^{-1}(\bar{\gamma}^0 - \bar{\gamma}^2)_+, \hat{\theta}^1 \circ (\hat{\theta}^1 + \hat{\theta}^2)^{-1}(\bar{\gamma}^0 - \bar{\gamma}^3)_+, \bar{\gamma}^0 - (\bar{\gamma}^2 + \bar{\gamma}^3)\right\}\right\}, \\ \hat{\gamma}^2(\bar{\gamma}) = \min\left\{\bar{\gamma}^2, \max\left\{\hat{\theta}^2(\bar{\gamma}^0), \hat{\theta}^2 \circ (\hat{\theta}^2 + \hat{\theta}^3)^{-1}(\bar{\gamma}^0 - \bar{\gamma}^1)_+, \hat{\theta}^2 \circ (\hat{\theta}^2 + \hat{\theta}^1)^{-1}(\bar{\gamma}^0 - \bar{\gamma}^3)_+, \bar{\gamma}^0 - (\bar{\gamma}^1 + \bar{\gamma}^3)\right\}\right\}, \\ \hat{\gamma}^3(\bar{\gamma}) = \min\left\{\bar{\gamma}^3, \max\left\{\hat{\theta}^3(\bar{\gamma}^0), \hat{\theta}^3 \circ (\hat{\theta}^3 + \hat{\theta}^1)^{-1}(\bar{\gamma}^0 - \bar{\gamma}^2)_+, \hat{\theta}^3 \circ (\hat{\theta}^3 + \hat{\theta}^2)^{-1}(\bar{\gamma}^0 - \bar{\gamma}^1)_+, \bar{\gamma}^0 - (\bar{\gamma}^1 + \bar{\gamma}^2)\right\}\right\}, \end{cases}$$

and

$$\hat{\gamma}^0(\bar{\gamma}) = \min\left\{\bar{\gamma}^0, \bar{\gamma}^1 + \bar{\gamma}^2 + \bar{\gamma}^3\right\} = (\hat{\gamma}^1 + \hat{\gamma}^2 + \hat{\gamma}^3)(\bar{\gamma})$$

iv) (General case $n \geq 2$)
For every set $I \subset_{\neq} \{1, \ldots, n\}$, we define (including $Y_\emptyset := \bar{\gamma}^0$ for $I := \emptyset$)

$$Y_I := \left(id_{[0,+\infty)} - \sum_{j \in I} \hat{\theta}^j\right)^{-1} (\bar{\gamma}^0 - \sum_{j \in I} \bar{\gamma}^j)_+$$

We also set

$$S^j := \max_{I \subset \{1,\ldots,n\} \setminus \{j\}} Y_I, \quad \text{for all} \quad j = 1, \ldots, n$$

Then

$$(14.47)\qquad \breve{\gamma}^j := (\hat{\theta}^j)^{-1}(\hat{\gamma}^j(\bar{\gamma})) = \min\left\{\breve{\gamma}^j, S^j\right\}, \quad \text{for all} \quad j=1,\dots,n$$

i.e.

$$\hat{\gamma}^j(\bar{\gamma}) = \min\left\{\bar{\gamma}^j, \hat{\theta}^j(S^j)\right\}, \quad j=1,\dots,n$$

and

$$\hat{\gamma}^0(\bar{\gamma}) = \min\left\{\bar{\gamma}^0, \bar{\gamma}^1+\dots+\bar{\gamma}^n\right\} = (\hat{\gamma}^1+\dots+\hat{\gamma}^n)(\bar{\gamma})$$

Corollary 14.14 implies Theorem 2.54 of the Introduction.

Proof of Corollary 14.14
Starting from Corollary 14.11, the proof consists to reformulate the expression of $\hat{\gamma}$.
Case i) is trivial. We focus on the general case $n \geq 2$. We only do the analysis in the special case where (14.36) is assumed. All the other cases can be obtained by permutations on the indices, and then by continuity.
We define (including $\beta^0(\bar{\gamma}^0) := \bar{\gamma}^0$) for $\bar{\gamma}^0 \in [0,+\infty)$

$$\beta^j(\bar{\gamma}^0) := Y_{\{1,\dots,j\}} = \left\{id_{[0,+\infty)} - \sum_{k=1,\dots,j} \hat{\theta}^k\right\}^{-1} \left(\bar{\gamma}^0 - \sum_{k=1,\dots,j} \bar{\gamma}^k\right)_+, \quad \text{for all} \quad j=0,\dots,n-1$$

which satisfy

$$\beta^j = \phi^{-1} \quad \text{on} \quad [\bar{\gamma}^j_*, \bar{\gamma}^{j+1}_*], \quad \text{if} \quad \bar{\gamma}^j_* < \bar{\gamma}^{j+1}_*, \quad \text{for all} \quad j=0,\dots,n$$

and then

$$(14.48)\qquad \begin{cases} & \beta^0(\bar{\gamma}^1_*) = \breve{\gamma}^1, \\ \beta^1(\bar{\gamma}^1_*) = \breve{\gamma}^1, & \beta^1(\bar{\gamma}^2_*) = \breve{\gamma}^2, \\ \beta^2(\bar{\gamma}^2_*) = \breve{\gamma}^2, & \beta^2(\bar{\gamma}^3_*) = \breve{\gamma}^3, \\ \vdots & \vdots \\ \beta^j(\bar{\gamma}^j_*) = \breve{\gamma}^j, & \beta^j(\bar{\gamma}^{j+1}_*) = \breve{\gamma}^{j+1}, \qquad \text{if} \quad \bar{\gamma}^j_* < \bar{\gamma}^{j+1}_*, \quad 1 \leq j \leq n-1 \end{cases}$$

Notice also for $j=0,\dots,n-1$ that $\bar{\gamma}^j_* = \bar{\gamma}^{j+1}_*$ implies $\bar{\gamma}^j_* = \bar{\gamma}^{j+1}_* = \phi_\infty$ and $j \geq 1$.
Case 1: $\bar{\gamma}^0_* = 0 \leq \bar{\gamma}^0 < \bar{\gamma}^1_* \leq \breve{\gamma}^1 < \breve{\gamma}^2 < \dots < \breve{\gamma}^n$
Then (14.43) gives

$$(\breve{\gamma}^1,\dots,\breve{\gamma}^n) = (\beta^0,\dots,\beta^0)(\bar{\gamma}^0) = (\bar{\gamma}^0,\dots,\bar{\gamma}^0)$$

Because

$$Y_I := \left(id_{[0,+\infty)} - \sum_{j\in I} \hat{\theta}^j\right)^{-1} (\bar{\gamma}^0 - \sum_{j\in I} \hat{\theta}^j(\breve{\gamma}^j))_+$$

we also have

$$Y_I \leq (Y_I)_{|(\breve{\gamma}^1,\dots,\breve{\gamma}^n):=(\bar{\gamma}^0,\dots,\bar{\gamma}^0)} = \bar{\gamma}^0 < \bar{\gamma}^1_* \leq \bar{\gamma}^1 \leq \breve{\gamma}^j, \quad j=1,\dots,n.$$

which implies $S^k = \breve{\gamma}^0$ for $I = \emptyset$ and then shows (14.47).
Case 2: $\bar{\gamma}^0_* = 0 < \bar{\gamma}^1_* < \dots < \bar{\gamma}^j_* \leq \bar{\gamma}^0 < \bar{\gamma}^{j+1}_* \leq \breve{\gamma}^{j+1} < \breve{\gamma}^{j+2} < \dots$ **for** $1 \leq j \leq n-1$
This implies $\bar{\gamma}^0 < \bar{\gamma}^{j+1}_* \leq \bar{\gamma}^n_* = \phi_\infty$ and (14.43) gives

$$(\breve{\gamma}^1,\dots,\breve{\gamma}^n) = (\breve{\gamma}^1,\dots,\breve{\gamma}^j,\beta^j,\dots,\beta^j)(\bar{\gamma}^0)$$

Subcase 2.1: $1 \leq k \leq j$
Then, by definition of S^k, we have in particular

$$(14.49)\qquad S^k \geq Y_{\{1,\dots,k-1\}} = \beta^{k-1}(\bar{\gamma}^0) \geq \beta^{k-1}(\bar{\gamma}^k_*) = \breve{\gamma}^k.$$

Subcase 2.2: $j+1 \leq k \leq n$
Then consider any set I satisfying

$$I \subset \{1,\dots,n\} \setminus \{k\}$$

We write

$$I = I_- \cup I_+ \quad \text{with} \quad \begin{cases} I_- \subset \{1,\dots,j\}, \\ I_+ \subset \{j+1,\dots,n\}, \end{cases}$$

Hence

$$Y_I = \left\{ id_{[0,+\infty)} - \left(\sum_{l\in I_-} \hat{\theta}^l + \sum_{l\in I_+} \hat{\theta}^l \right) \right\}^{-1} \left(\bar{\gamma}^0 - \left(\sum_{l\in I_-} \hat{\theta}^l(\breve{\gamma}^l) + \sum_{l\in I_+} \hat{\theta}^l(\breve{\gamma}^l) \right) \right)$$

We write

$$\{1,\dots,j\} = I_- \cup J_-, \quad \text{with} \quad I_- \cap J_- = \emptyset$$

and

$$\begin{cases} \hat{\theta} := id_{[0,+\infty)} - \displaystyle\sum_{l=1,\dots,j} \hat{\theta}^l, \\ \hat{\theta}^+ := \displaystyle\sum_{l\in I_+} \hat{\theta}^l, \\ \hat{\theta}^- := \displaystyle\sum_{l\in J_-} \hat{\theta}^l, \\ \hat{\theta}(y) := \bar{\gamma}^0 - \displaystyle\sum_{l=1,\dots,j} \hat{\theta}^l(\breve{\gamma}^l) = \bar{\gamma}^0 - \displaystyle\sum_{l=1,\dots,j} \bar{\gamma}^l, \\ y^- := \breve{\gamma}^j, \quad y^+ := \breve{\gamma}^{j+1} \end{cases}$$

Then

$$id_{[0,+\infty)} - \sum_{l\in I_-} \hat{\theta}^l = id_{[0,+\infty)} - \Big(\sum_{l=1,\dots,j} \hat{\theta}^l - \sum_{l\in J_-} \hat{\theta}^l \Big) = \hat{\theta} + \hat{\theta}^-$$

and we get

$$\begin{aligned} Y_I &= \left\{ \hat{\theta} - (\hat{\theta}^+ - \hat{\theta}^-) \right\}^{-1} \left(\hat{\theta}(y) - \left(\sum_{l\in I_+} \hat{\theta}^l(\breve{\gamma}^l) - \sum_{l\in J_-} \hat{\theta}^l(\breve{\gamma}^l) \right) \right) \\ &\le \left\{ \hat{\theta} - (\hat{\theta}^+ - \hat{\theta}^-) \right\}^{-1} \left(\hat{\theta}(y) - \left(\sum_{l\in I_+} \hat{\theta}^l(\breve{\gamma}^{j+1}) - \sum_{l\in J_-} \hat{\theta}^l(\breve{\gamma}^j) \right) \right) \\ &\le \left\{ \hat{\theta} - (\hat{\theta}^+ - \hat{\theta}^-) \right\}^{-1} \left(\hat{\theta}(y) - \left(\hat{\theta}^+(\breve{\gamma}^{j+1}) - \hat{\theta}^-(\breve{\gamma}^j) \right) \right) \\ &\le y = \beta^j(\bar{\gamma}^0) \quad \text{if} \quad y \in [y^-, y^+] := [\breve{\gamma}^j, \breve{\gamma}^{j+1}] \end{aligned}$$

we have used the technical Lemma 14.13 in the last inequality. Because we know from (14.48) for $j \le n-1$ that

$$\beta^j(\bar{\gamma}_*^{j+1}) = \breve{\gamma}^{j+1}, \quad \beta^j(\bar{\gamma}_*^j) = \breve{\gamma}^j \quad \text{and} \quad \bar{\gamma}_*^j \le \bar{\gamma}^0 < \bar{\gamma}_*^{j+1}$$

we deduce that $y \in [y^-, y^+]$. Hence

$$Y_I \le \beta^j(\bar{\gamma}^0) = Y_{\{1,\dots,j\}}$$

Therefore

(14.50) $$S^k = \beta^j(\bar{\gamma}^0) \le \beta^j(\bar{\gamma}_*^{j+1}) = \breve{\gamma}^{j+1} \le \breve{\gamma}^k$$

Conclusion of Case 2
From (14.49) and (14.50), we deduce that (14.47) holds true.
Case 3: $\bar{\gamma}^0 \ge \bar{\gamma}_*^n = \phi_\infty$

If $\bar{\gamma}^0 \geq \phi_\infty$, then for $j = 0, \dots, n-1$, we have

$$\begin{aligned}
\beta^j(\bar{\gamma}^0) &\geq \beta^j(\phi_\infty) \\
&= \left\{ \sum_{l>j} \hat{\theta}^l \right\}^{-1} \left(\sum_{l>j} \bar{\gamma}^j \right)_+ \\
&= \left\{ \sum_{l>j} \hat{\theta}^l \right\}^{-1} \left(\sum_{l>j} \hat{\theta}^l(\breve{\gamma}^j) \right)_+ \\
&\geq \left\{ \sum_{l>j} \hat{\theta}^l \right\}^{-1} \left(\sum_{l>j} \hat{\theta}^l(\breve{\gamma}^{j+1}) \right)_+ \\
&= \breve{\gamma}^{j+1}
\end{aligned}$$

and then for $k = 1, \dots, n$, we have

$$S^k \geq Y_{\{1,\dots,k-1\}} = \beta^{k-1}(\bar{\gamma}^0) \geq \breve{\gamma}^k$$

Then we deduce that (14.44) implies (14.47) holds true.
Conclusion of Cases 1,2,3
In particular this implies Points ii), iii) and iv) respectively for $n = 2$, $n = 3$ and $n \geq 4$.
This ends the proof of the lemma.

14.3 1:n conservative preflux (II): fundamental limiters

Lemma 14.15 (1:n conservative preflux (II): fundamental limiters)
i) (General assumptions)
Let $N := 1 + n$ with $n \geq 2$. We consider functions $\hat{\theta}^j$ for $j = 0, \dots, n$, satisfying

$$(14.51) \qquad \begin{cases} \hat{\theta}^j : [0, +\infty) \to [0, +\infty) \quad \text{continuous increasing bijective}, \quad \text{for} \quad j = 0, 1, \dots, n \\ \hat{\theta}^0 = id_{[0,+\infty)} = \displaystyle\sum_{j=1,\dots,n} \hat{\theta}^j \end{cases}$$

Let $\hat{\lambda}_0 := \hat{\gamma}$ be the preflux given by Lemma 14.11, that we write for $\bar{\gamma} = (\bar{\gamma}^0, \dots, \bar{\gamma}^n) \in [0, +\infty)^{1+n}$

$$\hat{\lambda}_0^j(\bar{\gamma}) = \min\left\{\bar{\gamma}^j, B^j(\bar{\gamma})\right\}, \quad j = 0, \dots, n$$

with $B^j : [0, +\infty)^{1+n} \to [0, +\infty)$ independent on $\bar{\gamma}^j$, with moreover

$$B^0(\bar{\gamma}) = \bar{\gamma}^1 + \dots + \bar{\gamma}^n \quad \text{and} \quad \hat{\lambda}_0^0 = \hat{\lambda}_0^1 + \dots + \hat{\lambda}_0^n$$

ii) (Barrier functions)
For $j, k = 1, \dots, n$ with $k \neq j$, we introduce the following functions

$$\underline{A}_j^k(\bar{\gamma}^j) := \sup_{\hat{\theta}^j(p^0) \leq \bar{\gamma}^j} \hat{\theta}^k(p^0)$$

Then we have

$$(14.52) \qquad \underline{A}_j^k \circ \underline{A}_k^j \geq id_{[0,+\infty)} \quad \text{for all} \quad k \neq j$$

For $j \in \{1, \dots, n\}$, we set

$$\underline{\underline{A}}^j(\bar{\gamma}') := \sup_{k \in \{1,\dots,n\} \backslash \{j\}} \underline{A}_k^j(\bar{\gamma}^k)$$

with $\underline{\underline{A}}^j$ independent of $\bar{\gamma}^j$.
iii) (The result with limiters A^j)

For $j = 1, \ldots, n$, we consider continuous functions $A^j : [0, +\infty)^n \to [0, +\infty)$, with A^j independent on $\bar{\gamma}^j$, satisfying moreover the following generalized epigraph condition

$$A^j(\bar{\gamma}') \geq \underline{\underline{A}}^j(\bar{\gamma}') \quad \text{for all} \quad \bar{\gamma}' = (\bar{\gamma}^1, \ldots, \bar{\gamma}^n) \in [0, +\infty)^n \tag{14.53}$$

We define for $\bar{\gamma} = (\bar{\gamma}^0, \bar{\gamma}')$

$$\begin{cases} \hat{\lambda}^j(\bar{\gamma}) := \min\left\{A^j(\bar{\gamma}'), \hat{\lambda}_0^j(\bar{\gamma})\right\} = \min\left\{\bar{\gamma}^j, A^j(\bar{\gamma}'), B^j(\bar{\gamma})\right\}, \quad j = 1, \ldots, n, \\ \hat{\lambda}^0 := \hat{\lambda}^1 + \cdots + \hat{\lambda}^n \end{cases} \tag{14.54}$$

Then

(14.55)

$$\hat{\lambda}^0(\bar{\gamma}) = \min\left\{\bar{\gamma}^0, \bar{\gamma}^1 + \cdots + \bar{\gamma}^n, A_{\bar{1}}^0(\bar{\gamma}'), \ldots, A_{\bar{n}}^0(\bar{\gamma}')\right\} \quad \text{with} \quad A_{\bar{j}}^0(\bar{\gamma}') := A^j(\bar{\gamma}') + \sum_{\ell \in \{1,\ldots,n\} \setminus \{j\}} \bar{\gamma}^\ell \quad \text{for} \quad j = 1, \ldots, n$$

where $A_{\bar{j}}^0$ is independent on $\bar{\gamma}^j$, and $\hat{\lambda}$ is a 1:n conservative preflux (which is called limited by the limiters A^j).

Proof of Lemma 14.15

Notice that point i) is just a reformulation of Lemma 14.11.

Step 1: proof of ii)

We notice that (14.52) follows from

$$\underline{A}_j^k = \hat{\theta}^k \circ (\hat{\theta}^j)^{-1}.$$

Step 2: proof of iii)

Step 2.1: preliminaries

We recall that for $\lambda^{0R} \in [0, +\infty)$

$$\begin{cases} \phi(\lambda^{0R}) := \sum_{j=1,\ldots,n} \min\left\{\bar{\gamma}^j, \hat{\theta}^j(\lambda^{0R})\right\} \\ \phi_\infty := \bar{\gamma}^1 + \cdots + \bar{\gamma}^n \\ \bar{\gamma}_*^j := \phi(\breve{\gamma}^j), \quad j = 1, \ldots, n \end{cases}$$

with

$$\breve{\gamma}^j := (\hat{\theta}^j)^{-1}(\bar{\gamma}^j)$$

and $\phi : [0, +\infty) \to [0, +\infty)$ concave nondecreasing with $\phi(0) = 0$ and $\phi'(0) \leq 1$.

We also have

$$\hat{\lambda}_0^0(\bar{\gamma}) = \min\left\{\bar{\gamma}^0, \bar{\gamma}^1 + \cdots + \bar{\gamma}^n\right\} = (\hat{\lambda}_0^1 + \cdots + \hat{\lambda}_0^n)(\bar{\gamma})$$

and for

$$\begin{cases} \bar{\gamma}_k^0 := \bar{\gamma}^0 - (\bar{\gamma}^1 + \cdots + \bar{\gamma}^k), \\ \hat{\theta}_k^j := \hat{\theta}^j \circ (id_{[0,+\infty)} - (\hat{\theta}^1 + \cdots + \hat{\theta}^k))^{-1}, \qquad j \geq k+1 \end{cases}$$

$$\bar{\gamma}' \geq (\hat{\lambda}_0^1, \ldots, \hat{\lambda}_0^n)(\bar{\gamma}) = \begin{cases} (\hat{\theta}^1(\bar{\gamma}^0), \hat{\theta}^2(\bar{\gamma}^0), \ldots, \hat{\theta}^n(\bar{\gamma}^0)) & \text{if} \quad \bar{\gamma}^0 < \bar{\gamma}_*^1 \\ (\bar{\gamma}^1, \hat{\theta}_1^2(\bar{\gamma}_1^0), \ldots, \hat{\theta}_1^n(\bar{\gamma}_1^0)) & \text{if} \quad \bar{\gamma}_*^1 \leq \bar{\gamma}^0 < \bar{\gamma}_*^2 \\ (\bar{\gamma}^1, \bar{\gamma}^2, \hat{\theta}_2^3(\bar{\gamma}_2^0), \ldots, \hat{\theta}_2^{n-1}(\bar{\gamma}_2^0), \hat{\theta}_2^n(\bar{\gamma}_2^0)) & \text{if} \quad \bar{\gamma}_*^2 \leq \bar{\gamma}^0 < \bar{\gamma}_*^3 \\ \vdots & \\ (\bar{\gamma}^1, \bar{\gamma}^2, \ldots, \bar{\gamma}^{n-2}, \hat{\theta}_{n-2}^{n-1}(\bar{\gamma}_{n-2}^0), \hat{\theta}_{n-2}^n(\bar{\gamma}_{n-2}^0)) & \text{if} \quad \bar{\gamma}_*^{n-2} \leq \bar{\gamma}^0 < \bar{\gamma}_*^{n-1} \\ (\bar{\gamma}^1, \bar{\gamma}^2, \ldots, \bar{\gamma}^{n-2}, \bar{\gamma}^{n-1}, \hat{\theta}_{n-1}^n(\bar{\gamma}_{n-1}^0)) & \text{if} \quad \bar{\gamma}_*^{n-1} \leq \bar{\gamma}^0 < \bar{\gamma}_*^n \\ (\bar{\gamma}^1, \bar{\gamma}^2, \ldots, \bar{\gamma}^n) & \text{if} \quad \bar{\gamma}_*^n \leq \bar{\gamma}^0 \\ \text{assuming everywhere that} & \breve{\gamma}^1 \leq \breve{\gamma}^2 \leq \cdots \leq \breve{\gamma}^n \end{cases} \tag{14.56}$$

and all other cases (not indicated in (14.56)) are then obtained by permutations on the indices $\{1, \dots, n\}$. Notice that $\hat{\theta}^n_{n-1} = id_{[0,+\infty)}$.

Step 2.2: proof of (14.55)

We want to prove (14.55), namely

$$\hat{\lambda}^0(\bar{\gamma}) = \min\left\{\bar{\gamma}^0, \bar{\gamma}^1 + \dots + \bar{\gamma}^n, A^0_1(\bar{\gamma}'), \dots, A^0_n(\bar{\gamma}')\right\} \tag{14.57}$$

We assume that

$$0 < \breve{\gamma}^1 < \breve{\gamma}^2 < \dots < \breve{\gamma}^n \tag{14.58}$$

where the general case is obtained by continuity (and permutations).

Notice that

$$\bar{\gamma}^1_* = \phi(\breve{\gamma}^1) = \sum_{i=1,\dots,n} \hat{\theta}^i(\min\left\{\breve{\gamma}^i, \breve{\gamma}^1\right\}) = \breve{\gamma}^1$$

and

$$\bar{\gamma}^n_* = \phi(\breve{\gamma}^n) = \sum_{i=1,\dots,n} \hat{\theta}^i(\min\left\{\breve{\gamma}^i, \breve{\gamma}^n\right\}) = \phi_\infty$$

Step 2.2.1: preliminary part of the proof

Our goal is to show (14.60), which will be used in Case 2 of Step 2.2.2 below.

To this end, we first notice that for $m \geq j$ with $m, j \in \{1, \dots, n\}$, we get

$$\bar{\gamma}^m_* = \phi(\breve{\gamma}^m) = \sum_{i=1,\dots,n} \hat{\theta}^i(\min\left\{\breve{\gamma}^i, \breve{\gamma}^m\right\}) = (\bar{\gamma}^1 + \dots + \bar{\gamma}^j) + (id - (\hat{\theta}^1 + \dots + \hat{\theta}^j))(\breve{\gamma}^m)$$

and therefore, we get the remarquable relation (for $\ell \geq j+1$)

$$\hat{\theta}^\ell_j(\bar{\gamma}^m_* - (\bar{\gamma}^1 + \dots + \bar{\gamma}^j)) = \hat{\theta}^\ell(\breve{\gamma}^m) = \begin{cases} \hat{\theta}^\ell \circ (\hat{\theta}^m)^{-1}(\bar{\gamma}^m) = \underline{A}^\ell_m(\bar{\gamma}^m) & \text{if} \quad \ell \neq m \\ \bar{\gamma}^m & \text{if} \quad \ell = m \end{cases}$$

Consider condition

$$\bar{\gamma}^0 < \bar{\gamma}^{j+1}_* \leq \bar{\gamma}^{j+2}_* \quad \text{with} \quad 1 \leq j \leq n-2. \tag{14.59}$$

we get for $\ell \geq j+1$

$$\hat{\theta}^\ell_j(\bar{\gamma}^0_j) \leq \left\{ \begin{array}{lll} \hat{\theta}^\ell_j(\bar{\gamma}^{j+1}_* - (\bar{\gamma}^1 + \dots + \bar{\gamma}^j))_+ = \underline{A}^\ell_{j+1}(\bar{\gamma}^{j+1}) & \text{if} & \ell \geq j+2 \\ \hat{\theta}^\ell_j(\bar{\gamma}^{j+2}_* - (\bar{\gamma}^1 + \dots + \bar{\gamma}^j))_+ = \underline{A}^\ell_{j+2}(\bar{\gamma}^{j+2}) & \text{if} & \ell = j+1 \end{array} \right\} \leq \underline{\underline{A}}^\ell(\bar{\gamma}'). \tag{14.60}$$

Step 2.2.2: remaining part of the proof

Case 1: $\bar{\gamma}^0_* = 0 \leq \bar{\gamma}^0 < \bar{\gamma}^1_* = \breve{\gamma}^1 < \breve{\gamma}^2 < \dots < \breve{\gamma}^n$

Then (14.56) gives

$$(\hat{\lambda}^1_0, \dots, \hat{\lambda}^n_0)(\bar{\gamma}) = (\hat{\theta}^1, \dots, \hat{\theta}^n)(\bar{\gamma}^0) \leq \bar{\gamma}'$$

Then for $\ell = 1, \dots, n$, we have

$$\hat{\lambda}^\ell(\bar{\gamma}) = \min\left\{\hat{\theta}^\ell(\bar{\gamma}^0), A^\ell(\bar{\gamma}')\right\}$$

and for $n \geq 2$, we get

$$A^\ell(\bar{\gamma}') \geq \underline{\underline{A}}^\ell(\bar{\gamma}') \geq \underline{A}^\ell_m(\bar{\gamma}^m) = \sup_{\hat{\theta}^m(p^0) \leq \bar{\gamma}^m} \hat{\theta}^\ell(p^0) \geq \hat{\theta}^\ell(\bar{\gamma}^0) \quad \text{with} \quad m \neq \ell, \quad m \geq 1 \tag{14.61}$$

which implies

$$(\hat{\lambda}^0, \dots, \hat{\lambda}^n) = (\hat{\theta}^0, \hat{\theta}^1, \dots, \hat{\theta}^n)(\bar{\gamma}^0) = (\hat{\lambda}^1_0, \dots, \hat{\lambda}^n_0)$$

which implies (14.57).

Case 2: $\bar{\gamma}^0_* = 0 < \bar{\gamma}^1_* < \dots < \bar{\gamma}^j_* \leq \bar{\gamma}^0 < \bar{\gamma}^{j+1}_* \leq \breve{\gamma}^{j+1} < \breve{\gamma}^{j+2} < \dots$ **for** $1 \leq j \leq n-1$

This implies $\bar{\gamma}^0 < \bar{\gamma}^{j+1}_* \leq \bar{\gamma}^n_* = \phi_\infty$ and (14.56) gives

$$(\hat{\lambda}^1_0, \dots, \hat{\lambda}^n_0)(\bar{\gamma}) = (\bar{\gamma}^1, \dots, \bar{\gamma}^j, \hat{\theta}^{j+1}_j, \dots, \hat{\theta}^n_j)(\bar{\gamma}^0_j) \leq \bar{\gamma}' \quad \text{with} \quad \bar{\gamma}^0_j := \bar{\gamma}^0 - (\bar{\gamma}^1 + \dots + \bar{\gamma}^j)$$

Hence

$$\hat{\lambda}^k(\bar{\gamma}) = \begin{cases} \min\left\{\bar{\gamma}^k, A^k(\bar{\gamma}')\right\} & \text{if} \quad 1 \le k \le j, \\ \min\left\{\hat{\theta}^k_j(\bar{\gamma}^0_j), A^k(\bar{\gamma}')\right\} & \text{if} \quad n \ge k \ge j+1, \end{cases}$$

Assume by contradiction that for some $k \in \{1, \dots, n-1\}$, we have

$$\bar{\gamma}^k > A^k(\bar{\gamma}') \tag{14.62}$$

Then we get

$$\bar{\gamma}^k > A^k(\bar{\gamma}') \ge \underline{\underline{A}}^k(\bar{\gamma}') \ge \underline{A}^k_n(\bar{\gamma}^n) = \hat{\theta}^k \circ (\hat{\theta}^n)^{-1}(\bar{\gamma}^n) \tag{14.63}$$

which means

$$\breve{\gamma}^k \ge \breve{\gamma}^n$$

Contradiction with (14.58). Therefore (14.62) is false, and this shows in particular that

$$\hat{\lambda}^k(\bar{\gamma}) = \bar{\gamma}^k = \hat{\lambda}^k_0(\bar{\gamma}) \quad \text{for all} \quad k \le j.$$

Case 2.a: $j \le n-2$
For $\ell = j+1, \dots, n$, we get using (14.60)

$$A^\ell(\bar{\gamma}') \ge \underline{\underline{A}}^\ell(\bar{\gamma}') \ge \hat{\theta}^\ell_j(\bar{\gamma}^0_j)$$

and we deduce that $\hat{\lambda}^\ell(\bar{\gamma}) = \hat{\lambda}^\ell_0(\bar{\gamma})$ for $\ell = 1, \dots, n$, which implies $\hat{\lambda}(\bar{\gamma}) = \hat{\lambda}_0(\bar{\gamma})$.
Case 2.b: $j = n-1$
Then

$$(\hat{\lambda}^1_0, \dots, \hat{\lambda}^n_0)(\bar{\gamma}) = (\bar{\gamma}^1, \dots, \bar{\gamma}^{n-1}, \bar{\gamma}^0 - (\bar{\gamma}^1 + \dots + \bar{\gamma}^{n-1})) \le \bar{\gamma}'$$

Hence

$$(\hat{\lambda}^1, \dots, \hat{\lambda}^n)(\bar{\gamma}) = (\bar{\gamma}^1, \dots, \bar{\gamma}^{n-1}, \min\left\{\bar{\gamma}^0 - (\bar{\gamma}^1 + \dots + \bar{\gamma}^{n-1}), A^n\right\})$$

Therefore

$$\hat{\lambda}^0 = \hat{\lambda}^1 + \dots + \hat{\lambda}^n = \min\left\{\bar{\gamma}^0, A^0_{\bar{n}}\right\}$$

which also implies (14.57). Moreover

$$(\hat{\lambda}^1, \dots, \hat{\lambda}^n)(\bar{\gamma}) \in \begin{cases} (\bar{\gamma}^1, \dots, \bar{\gamma}^{n-1}, \bar{\gamma}^0 - (\bar{\gamma}^1 + \dots + \bar{\gamma}^{n-1})), \\ (\bar{\gamma}^1, \dots, \bar{\gamma}^{n-1}, A^n(\bar{\gamma}')) \end{cases} \tag{14.64}$$

Case 3: $\bar{\gamma}^0 \ge \bar{\gamma}^n_* = \phi_\infty$
Then

$$(\hat{\lambda}^1_0, \dots, \hat{\lambda}^n_0)(\bar{\gamma}) = (\bar{\gamma}^1, \dots, \bar{\gamma}^n)$$

and for $j = 1, \dots, n$, we have

$$\hat{\lambda}^j = \min\left\{\bar{\gamma}^j, A^j\right\}$$

We now claim that (in all cases) we have

$$\left\{\bar{\gamma}^1 > A^1(\bar{\gamma}'), \quad \bar{\gamma}^2 \ge A^2(\bar{\gamma}')\right\} = \emptyset \tag{14.65}$$

Using our epigraph condition (14.53), the left hand side of (14.65) implies

$$\begin{cases} \bar{\gamma}^1 > \underline{A}^1_2(\bar{\gamma}^2), \\ \bar{\gamma}^2 \ge \underline{A}^2_1(\bar{\gamma}^1), \end{cases}$$

and then (14.52) implies by composition

$$\bar{\gamma}^1 > \underline{A}^1_2(\bar{\gamma}^2) \ge (\underline{A}^1_2 \circ \underline{A}^2_1)(\bar{\gamma}^1) \ge \bar{\gamma}^1$$

which leads to a contradiction, which shows (14.65).
More generally replacing $\{1, 2\}$ by any pair of indices in $\{1, \dots, n\}$, we get a similar result

$$\left\{\bar{\gamma}^j > A^j(\bar{\gamma}'), \quad \bar{\gamma}^k \ge A^k(\bar{\gamma}')\right\} = \emptyset \quad \text{for} \quad j \ne k \tag{14.66}$$

Hence

$$\hat{\lambda}^0(\bar{\gamma}) = \min\left\{\bar{\gamma}^1 + \cdots + \bar{\gamma}^n, A^0_{\bar{1}}, \ldots, A^0_{\bar{n}}\right\} \leq \bar{\gamma}^0$$

which implies (14.57). Moreover, we have

$$(14.67) \qquad (\hat{\lambda}^1, \ldots, \hat{\lambda}^n)(\bar{\gamma}) \in \left\{\begin{array}{l} (\bar{\gamma}^1, \ldots, \bar{\gamma}^n), \\ (A^1(\bar{\gamma}'), \bar{\gamma}^2, \bar{\gamma}^3, \ldots, \bar{\gamma}^n), \\ (\bar{\gamma}^1, A^2(\bar{\gamma}'), \bar{\gamma}^3, \ldots, \bar{\gamma}^n), \\ \ldots \\ (\bar{\gamma}^1, \bar{\gamma}^2, \bar{\gamma}^3, \ldots, A^n(\bar{\gamma}')). \end{array}\right.$$

Step 2.3: proof that $\hat{\lambda}$ is a preflux
By definition, the map $\hat{\lambda} : [0, +\infty)^{1+n} \to [0, +\infty)^{1+n}$ is continuous and satisfies

$$\hat{\lambda}^j(\bar{\gamma}) = 0 \quad \text{if} \quad \bar{\gamma}^j = 0, \quad \text{if } j \in \{0, \ldots, n\}$$

It remains to show that $\hat{\lambda}$ is locally constant. We distinguish cases.
Case A: $\hat{\lambda}^0(\bar{\gamma}) < \bar{\gamma}^0$
From (14.55), we deduce that

$$(14.68) \qquad \hat{\lambda}^1 + \cdots + \hat{\lambda}^n = \hat{\lambda}^0 = \min\left\{\bar{\gamma}^1 + \cdots + \bar{\gamma}^n, A^0_{\bar{1}}, \ldots, A^0_{\bar{n}}\right\}$$

We know that for $j = 1, \ldots, n$, we have

$$\hat{\lambda}^j \leq \min\left\{\bar{\gamma}^j, A^j(\bar{\gamma}')\right\}$$

and introduce $\varepsilon_j \geq 0$ such that

$$\hat{\lambda}^j + \varepsilon_j = \min\left\{\bar{\gamma}^j, A^j\right\}$$

we deduce that

$$(14.69) \qquad \hat{\lambda}^0 + \varepsilon_1 + \cdots + \varepsilon_n = \min\left\{\bar{\gamma}^1, A^1\right\} + \cdots + \min\left\{\bar{\gamma}^n, A^n\right\}$$

Because of (14.66) and $A^0_{\bar{j}} := A^j + \sum_{k\neq j} \bar{\gamma}^k$, we deduce that

$$\min\left\{\bar{\gamma}^1, A^1\right\} + \cdots + \min\left\{\bar{\gamma}^n, A^n\right\} \leq \hat{\lambda}^0$$

Then (14.69) implies $\varepsilon_1 + \cdots + \varepsilon_n \leq 0$, i.e. $\varepsilon_j = 0$ for all $j = 1, \ldots, n$. Therefore for all $j = 1, \ldots, n$, we get

$$\hat{\lambda}^j(\bar{\gamma}) = \min\left\{\bar{\gamma}^j, A^j(\bar{\gamma}')\right\}$$

This shows that $\hat{\lambda}(\bar{\gamma})$ is locally constant in $\bar{\gamma}^0$ on $\left\{\hat{\lambda}^0(\bar{\gamma}) < \bar{\gamma}^0\right\}$.
Case B: other indices $j = 1, \ldots, n$
From Step 2.2, we know that when $\hat{\lambda}$ differs from $\hat{\lambda}_0$, then it satisfies either (14.64) or (14.67). In both cases if $\hat{\lambda}^j(\bar{\gamma}) < \bar{\gamma}^j$, then $\hat{\lambda}$ is independent on $\bar{\gamma}^j$ locally. Because this is also true for $\hat{\lambda}_0$, we deduce that in general $\hat{\lambda}$ is locally constant on $\left\{\hat{\lambda}^j(\bar{\gamma}) < \bar{\gamma}^j\right\}$.
Conclusion
We conclude that $\hat{\lambda}$ is locally constant on $\left\{\hat{\lambda} \neq id_{[0,+\infty)^{1+n}}\right\}$.
This ends the proof of the lemma.

14.4 1:n conservative preflux (III): truncation

Lemma 14.16 (1:n conservative preflux (III): truncation)
Under the assumptions of Lemma 14.15, let us consider the 1:n conservative preflux $\hat{\lambda}$ given in (14.54). Let $\bar{\lambda}^0 \in (0, +\infty)$ and $\bar{\lambda} := (\bar{\lambda}^0, +\infty, \ldots, +\infty) \in [0, +\infty]^{1+n}$ and the truncation $T_{\bar{\lambda}}$. Then

$$\hat{\lambda}_* := \hat{\lambda} \circ T_{\bar{\lambda}}$$

is a 1:n conservative preflux. Setting

$$\begin{cases} \bar{\bar{\gamma}}^0 := \min\left\{\bar{\gamma}^0, \bar{\lambda}^0\right\}, \\ B^j(\bar{\gamma}) := \hat{\theta}^j \left(\max_{I\subset\{1,\dots,n\}\setminus\{j\}} Y_I\right) \quad \text{with} \quad Y_I := \left(id_{[0,+\infty)} - \sum_{j\in I} \hat{\theta}^j\right)^{-1} (\bar{\gamma}^0 - \sum_{j\in I}\bar{\gamma}^j)_+, \\ \text{(including } Y_\emptyset := \bar{\gamma}^0 \text{ for } I := \emptyset\text{)}, \\ A^0_{\bar{j}}(\bar{\gamma}') := A^j(\bar{\gamma}') + \sum_{\ell\in\{1,\dots,n\}\setminus\{j\}} \bar{\gamma}^\ell \quad \text{for} \quad j = 1,\dots,n, \end{cases}$$

we have

$$\text{(14.70)} \qquad \begin{cases} \hat{\lambda}^j_*(\bar{\gamma}) = \min\left\{\bar{\gamma}^j, A^j(\bar{\gamma}'), B^j(\bar{\bar{\gamma}}^0, \bar{\gamma}')\right\}, \quad j = 1,\dots,n, \\ \hat{\lambda}^0_*(\bar{\gamma}) := (\hat{\lambda}^1_* + \cdots + \hat{\lambda}^n_*)(\bar{\gamma}) = \min\left\{\bar{\bar{\gamma}}^0, \bar{\gamma}^1 + \cdots + \bar{\gamma}^n, A^0_{\bar{1}}(\bar{\gamma}'), \dots, A^0_{\bar{n}}(\bar{\gamma}')\right\} \end{cases}$$

We define for $j = 1,\dots,n$

$$\begin{cases} \bar{A}^j(\bar{\gamma}') := \min\left\{A^j(\bar{\gamma}'), B^j(\bar{\lambda}^0, \bar{\gamma}')\right\} \\ \bar{A}^0_{\bar{j}}(\gamma') := \bar{A}^j(\bar{\gamma}') + \sum_{\ell\in\{1,\dots,n\}\setminus\{j\}} \bar{\gamma}^\ell \quad \text{for} \quad j = 1,\dots,n, \end{cases}$$

Then we have

$$\text{(14.71)} \qquad \begin{cases} \hat{\lambda}^j_*(\bar{\gamma}) = \min\left\{\bar{\gamma}^j, \bar{A}^j(\bar{\gamma}'), B^j(\bar{\gamma})\right\}, \quad j = 1,\dots,n, \\ \hat{\lambda}^0_*(\bar{\gamma}) := (\hat{\lambda}^1_0 + \cdots + \hat{\lambda}^n_0)(\bar{\gamma}) = \min\left\{\bar{\gamma}^0, \bar{\gamma}^1 + \cdots + \bar{\gamma}^n, \bar{\lambda}^0, \bar{A}^0_{\bar{1}}(\bar{\gamma}'), \dots, \bar{A}^0_{\bar{n}}(\bar{\gamma}')\right\} \end{cases}$$

Proof of Lemma 14.16
Step 1: first computations
Because the map $\bar{\gamma}^0 \mapsto B^j(\bar{\gamma}^0, \bar{\gamma}')$ is nondecreasing for $j = 1,\dots,n$, we deduce that

$$B^j(\bar{\bar{\gamma}}^0, \bar{\gamma}') = B^j(\min\left\{\bar{\lambda}^0, \bar{\gamma}^0\right\}, \bar{\gamma}') = \min\left\{B^j(\bar{\lambda}^0, \bar{\gamma}'), B^j(\bar{\gamma}^0, \bar{\gamma}')\right\}$$

Hence

$$\hat{\lambda}^j_*(\bar{\gamma}) = \min\left\{\bar{\gamma}^j, \bar{A}^j(\bar{\gamma}')\right\} \quad \text{with} \quad \bar{A}^j(\bar{\gamma}') := \min\left\{A^j(\bar{\gamma}'), B^j(\bar{\lambda}^0, \bar{\gamma}')\right\}$$

We also deduce that

$$\hat{\lambda}^0_*(\bar{\gamma}) = \min\left\{\bar{\gamma}^0, \bar{\gamma}^1 + \cdots + \bar{\gamma}^n, \bar{\lambda}^0, A^0_{\bar{1}}(\bar{\gamma}'), \dots, A^0_{\bar{n}}(\bar{\gamma}')\right\}$$

We can conclude if we show that for all $j = 1,\dots,n$

$$\text{(14.72)} \qquad \min\left\{\bar{\lambda}^0, A^0_{\bar{j}}(\bar{\gamma}')\right\} = \min\left\{\bar{\lambda}^0, \bar{A}^0_{\bar{j}}(\bar{\gamma}')\right\}$$

Step 2: proof of (14.72)
We first notice that

$$\text{(14.73)} \qquad B^j(\bar{\lambda}^0, \bar{\gamma}') + \Sigma^j \ge \bar{\lambda}^0 \quad \text{with} \quad \Sigma^j := \sum_{\ell\in\{1,\dots,n\}\setminus\{j\}} \bar{\gamma}^\ell$$

as shows the choice of $I = \{1,\dots,n\}\setminus\{j\}$ in the suppremum defining B^j. Hence

$$\text{(14.74)} \qquad \begin{aligned} \bar{A}^0_{\bar{j}} &:= \bar{A}^j(\bar{\gamma}') + \Sigma^j \\ &= \min\left\{A^j(\bar{\gamma}'), B^j(\bar{\lambda}^0, \bar{\gamma}')\right\} + \Sigma^j \\ &= \min\left\{A^j(\bar{\gamma}') + \Sigma^j, B^j(\bar{\lambda}^0, \bar{\gamma}') + \Sigma^j\right\} \\ &= \min\left\{A^0_{\bar{j}}(\bar{\gamma}'), B^j(\bar{\lambda}^0, \bar{\gamma}') + \Sigma^j\right\} \end{aligned}$$

Therefore

$$\begin{aligned} \min\left\{\bar{A}^0_{\bar{j}}, \bar{\lambda}^0\right\} &= \min\left\{A^0_{\bar{j}}(\bar{\gamma}'), B^j(\bar{\lambda}^0, \bar{\gamma}') + \Sigma^j, \bar{\lambda}^0\right\} \\ &= \min\left\{A^0_{\bar{j}}(\bar{\gamma}'), \bar{\lambda}^0\right\} \end{aligned}$$

where we have used (14.73) in the last equality. This establishes (14.72) and ends the proof of the lemma.

We get the following result.

Corollary 14.17 (Explicit 1:n conservative prefluxes of total limiter $\bar{\lambda}^0$)
i) (General assumptions)
Let $N := 1 + n$ with $n \geq 2$. We consider functions $\hat{\theta}^j$ for $j = 0, \dots, n$, satisfying

$$(14.75)\qquad \left\{ \begin{array}{l} \hat{\theta}^j : [0,+\infty) \to [0,+\infty) \quad \textit{continuous increasing bijective}, \quad \textit{for} \quad j = 0, 1, \dots, n \\ \hat{\theta}^0 = id_{[0,+\infty)} = \displaystyle\sum_{j=1,\dots,n} \hat{\theta}^j \end{array} \right.$$

Let for $\bar{\gamma} = (\bar{\gamma}^0, \bar{\gamma}')$ with $\bar{\gamma}' := (\bar{\gamma}^1, \dots, \bar{\gamma}^n)$ and for $j = 1, \dots, n$

$$\left\{ \begin{array}{l} B^j(\bar{\gamma}) := \hat{\theta}^j \left(\displaystyle\max_{I \subset \{1,\dots,n\} \backslash \{j\}} Y_I \right) \quad \textit{with} \quad \left\{ \begin{array}{l} Y_I := \left(id_{[0,+\infty)} - \displaystyle\sum_{j \in I} \hat{\theta}^j \right)^{-1} (\bar{\gamma}^0 - \displaystyle\sum_{j \in I} \bar{\gamma}^j)_+, \\ \textit{(including } Y_\emptyset := \bar{\gamma}^0 \textit{ for } I := \emptyset\textit{)}, \end{array} \right. \\ \underline{\underline{A}}^j(\bar{\gamma}') := \displaystyle\sup_{k \in \{1,\dots,n\} \backslash \{j\}} \underline{A}^j_k(\bar{\gamma}^k) \quad \textit{with} \quad \underline{A}^k_j(\bar{\gamma}^j) := \displaystyle\sup_{\hat{\theta}^j(p^0) \leq \bar{\gamma}^j} \hat{\theta}^k(p^0) \\ A^0_{\bar{j}}(\bar{\gamma}') := A^j(\bar{\gamma}') + \displaystyle\sum_{\ell \in \{1,\dots,n\} \backslash \{j\}} \bar{\gamma}^\ell, \end{array} \right.$$

where $A^j : [0,+\infty)^n \to [0,+\infty)$ is a continuous function, independent of the variable $\bar{\gamma}^j$, satisfying

$$(14.76)\qquad \min \left\{ B^j(\bar{\lambda}^0, \bar{\gamma}'), \underline{\underline{A}}^j(\bar{\gamma}') \right\} \leq A^j(\bar{\gamma}') \leq B^j(\bar{\lambda}^0, \bar{\gamma}')$$

for some fixed constant $\bar{\lambda}^0 \in [0,+\infty)$, which is a called the total limiter. Moreover, we have

$$(14.77)\qquad \left\{ \begin{array}{lll} \underline{\underline{A}}^j = A^j_* = B^j(\bar{\lambda}^0, \cdot) & \textit{on} & A'_* + [0,+\infty)^n, \\ \underline{\underline{A}}^j < A^j_* < B^j(\bar{\lambda}^0, \cdot) & \textit{on} & \displaystyle\prod_{k=1,\dots,n} [0, A^k_*) \end{array} \right| \quad \textit{for} \quad j = 1, \dots, n$$

with

$$A'_* := (\hat{\theta}^1, \dots, \hat{\theta}^n)(\bar{\lambda}^0).$$

ii) (The result with limiters A^j)
Then the function $\hat{\lambda}_* : [0,+\infty)^N \to [0,+\infty)^N$ defined by

$$(14.78)\qquad \left\{ \begin{array}{l} \hat{\lambda}^j_*(\bar{\gamma}) := \min \left\{ \bar{\gamma}^j, A^j(\bar{\gamma}'), B^j(\bar{\gamma}) \right\}, \quad j = 1, \dots, n, \\ \hat{\lambda}^0_* := \hat{\lambda}^1_* + \dots + \hat{\lambda}^n_* \end{array} \right.$$

is a 1:n conservative preflux, which satisfies moreover

$$(14.79)\qquad \hat{\lambda}^0_*(\bar{\gamma}) = \min \left\{ \bar{\gamma}^0, \bar{\gamma}^1 + \dots + \bar{\gamma}^n, \bar{\lambda}^0, A^0_{\bar{1}}(\bar{\gamma}'), \dots, A^0_{\bar{n}}(\bar{\gamma}') \right\}$$

iii) (Further property)
Let

$$M_0 := \max_{j=0,\dots,n} (\hat{\theta}^j)^{-1}(\bar{\lambda}^0)$$

Then the values of $\hat{\lambda}_*$ depend on the function $\hat{\theta} = (\hat{\theta}^j)_{j=0,\dots,n}$ only through the values of the restriction $\hat{\theta}_{|[0,M_0]}$, for some $\hat{\theta}$ satisfying (14.75).
iv) (Kružkov case)
Moreover, under the previous assumptions, the 1:n conservative preflux $\hat{\lambda}_*$ is Kružkov if and only if

$$(14.80)\qquad -1 \leq \partial_k A^j \leq 0 \quad \textit{for all} \quad j, k \in \{1, \dots, n\} \quad \textit{with} \quad j \neq k.$$

Proof of Corollary 14.17
This is a straightforward consequence of Lemma 14.16, except for (14.77) and point iv).
Step 1: proof of the first line of (14.77)

We notice that for $j = 1, \ldots, n$

$$B^j(\bar{\lambda}^0, \bar{\gamma}') \geq A_*^j$$

as it can be seen from the choice $I := \emptyset$ in the supremum defining B^j. By definition of B^j, it is easy to check that

$$B^j(\bar{\lambda}^0, A_*') = A_*^j$$

Because by definition of B^j, we know that $B^j(\bar{\lambda}^0, \cdot)$ is nonincreasing in each variable $\bar{\gamma}^k$ for $k \neq 0$, we deduce that

$$B^j(\bar{\lambda}^0, \cdot) = A_*^j \quad \text{on} \quad A_*' + [0, +\infty)^n$$

By definition of $\underline{\underline{A}}^j$, it is also easy to see that

$$\underline{\underline{A}}^j(\bar{\gamma}') \geq A_*^j \quad \text{if there exists } k \in \{1, \ldots, n\} \setminus \{j\} \text{ such that } \bar{\gamma}^k \geq A_*^k$$

Hence we deduce that

$$\underline{\underline{A}}^j \geq A_*^j \quad \text{on} \quad A_*' + [0, +\infty)^n$$

and then

$$\min \left\{ \underline{\underline{A}}^j, B^j(\bar{\lambda}^0, \cdot) \right\} = A_*^j \quad \text{on} \quad A_*' + [0, +\infty)^n$$

which shows the first line of (14.77).

Step 2: proof of the second line of (14.77)

Now notice that

$$B^j(\bar{\lambda}^0, \bar{\gamma}') \geq \bar{\lambda}^0 - \sum_{k \neq j} \bar{\gamma}^k$$

as it can be seen from the choice $I := \{1, \ldots, n\} \setminus \{j\}$ in the supremum defining B^j. Hence

$$B^j(\bar{\lambda}^0, \cdot) > A_*^j \quad \text{on} \quad [0, A_*'] \setminus \{A_*'\}$$

On the other hand, we have

$$\underline{\underline{A}}^j(\bar{\gamma}') = \max_{k \neq j} \ \hat{\theta}^j \circ (\hat{\theta}^k)^{-1}(\bar{\gamma}^k) \quad < A_*^j \quad \text{for all} \quad \bar{\gamma}' \in \prod_{k=1,\ldots,n} [0, A_*^k)$$

and this implies the second line of (14.77).

Step 3: proof of iii)

From the definition of the functions $B^j(\bar{\lambda}^0, \cdot)$, we see that they satisfy

$$B^j(\bar{\lambda}^0, \cdot) \leq \bar{\lambda}^0$$

and then only depend on the restrictions $\left(\hat{\theta}^j_{|(\hat{\theta}^j)^{-1}([0, \bar{\lambda}^0])} \right)_{j=0,\ldots,n}$. On the other hand the function $\underline{\underline{A}}^j$ plays a role in $\min \left\{ \underline{\underline{A}}^j, B^j(\bar{\lambda}^0, \cdot) \right\}$ only if $\underline{\underline{A}}^j \leq \bar{\lambda}^0$, which means

$$(\hat{\theta}^k)^{-1}(\bar{\gamma}^k) \leq (\hat{\theta}^j)^{-1}(\bar{\lambda}^0) \quad \text{for all} \quad k \neq j$$

which implies

$$\bar{\gamma}^k \leq \hat{\theta}^k(M_0)$$

which only involves $\hat{\theta}_{|[0, M_0]}$. This implies point iii).

Step 4: proof of iv)

We first notice that $(B^j(\bar{\lambda}^0, \cdot))_{j=1,\ldots,n}$ satisfies (14.80) and that $B^j(\cdot, \cdot)$ satisfies

$$\partial_0 B^j \geq 0 \quad \text{for all} \quad j \in \{1, \ldots, n\}. \tag{14.81}$$

Because we know that $\hat{\lambda}_*$ is 1:n conservative Kružkov if and only if it is σ-monotone, we deduce that

$$\left\{ \begin{array}{lll} \partial_k \hat{\lambda}_*^j \leq 0 & \text{for all} & k, j \in \{1, \ldots, n\} \quad \text{with} \quad k \neq j, \\ \partial_0 \hat{\lambda}_*^j \geq 0 & \text{for all} & j \in \{1, \ldots, n\}, \\ \partial_k \hat{\lambda}^0 \geq 0 & \text{for all} & k \in \{1, \ldots, n\}, \end{array} \right.$$

Using (14.81), we see that it is equivalent to

$$\begin{cases} \partial_k A^j \le 0 & \text{for all} \quad k,j \in \{1,\dots,n\} \quad \text{with} \quad k \neq j, \\ \text{nothing} & \\ \partial_k A^0_{\bar{j}} \ge 0 & \text{for all} \quad k \in \{1,\dots,n\}, \end{cases}$$

which is then equivalent to (14.80).
This ends the proof of the corollary.

14.5 n:m conservative preflux

In this subsection, by gluing, we construct a large family of n:m conservative prefluxes.

Lemma 14.18 (n:m conservative prefluxes, by gluing)
i) (General assumptions)
Let $N := n+m$ with $n,m \ge 2$. For $\alpha = L,R$, we write $n_L := n$ and $n_R := m$. We consider N functions satisfying
(14.82)

$$\begin{cases} \hat{\theta}^j_L : [0,+\infty) \to [0,+\infty) \quad \textit{continuous increasing bijective}, \quad \textit{for} \quad j \in I_L := \{1,\dots,n\} \\ \hat{\theta}^j_R : [0,+\infty) \to [0,+\infty) \quad \textit{continuous increasing bijective}, \quad \textit{for} \quad j \in I_R := \{1+n,\dots,m+n\} \\ \displaystyle\sum_{j=1,\dots,n} \hat{\theta}^j_L = id_{[0,+\infty)} = \sum_{j=1+n,\dots,m+n} \hat{\theta}^j_R \end{cases}$$

Let for $\bar{\gamma} = (\bar{\gamma}_L, \bar{\gamma}_R)$, with $\bar{\gamma}_L := (\bar{\gamma}^1,\dots,\bar{\gamma}^n)$, $\bar{\gamma}_R := (\bar{\gamma}^{1+n},\dots,\bar{\gamma}^{m+n})$, and $\bar{\gamma}^0 \in [0,+\infty)$ and for $\alpha = L,R$ and $j \in I_\alpha$

$$\begin{cases} B^j_\alpha(\bar{\gamma}^0, \bar{\gamma}_\alpha) := \hat{\theta}^j_\alpha\left(\displaystyle\max_{I \subset I_\alpha \setminus \{j\}} Y_I\right) \quad \textit{with} \quad \begin{cases} Y_I := \left(id_{[0,+\infty)} - \displaystyle\sum_{j\in I} \hat{\theta}^j_\alpha\right)^{-1} (\bar{\gamma}^0 - \displaystyle\sum_{j\in I} \bar{\gamma}^j)_+, \\ \textit{(including } Y_\emptyset := \bar{\gamma}^0 \textit{ for } I := \emptyset\textit{)}, \end{cases} \\ \underline{\underline{A}}^j_\alpha(\bar{\gamma}_\alpha) := \displaystyle\sup_{k \in I_\alpha \setminus \{j\}} \underline{A}^j_k(\bar{\gamma}^k) \quad \textit{with} \quad \underline{A}^k_j(\bar{\gamma}^j) := \sup_{\hat{\theta}^j(p^0) \le \bar{\gamma}^j} \hat{\theta}^k(p^0) \\ A^0_{\alpha,\bar{j}}(\bar{\gamma}_\alpha) := A^j_\alpha(\bar{\gamma}_\alpha) + \displaystyle\sum_{\ell \in I_\alpha \setminus \{j\}} \bar{\gamma}^\ell, \end{cases}$$

where $A^j_\alpha : [0,+\infty)^{n_\alpha} \to [0,+\infty)$ is a continuous function, independent of the variable $\bar{\gamma}^j$, satisfying

$$\min\left\{B^j_\alpha(\bar{\lambda}^0, \bar{\gamma}_\alpha), \underline{\underline{A}}^j_\alpha(\bar{\gamma}_\alpha)\right\} \le A^j_\alpha(\bar{\gamma}_\alpha) \le B^j_\alpha(\bar{\lambda}^0, \bar{\gamma}_\alpha) \tag{14.83}$$

where the A^j_α are called limiters. Here the constant $\bar{\lambda}^0 \in [0,+\infty)$ is fixed, and is a called the total limiter. Moreover, we have

$$\left\{ \begin{array}{lll} \underline{\underline{A}}^j_\alpha = A^j_{\alpha,*} = B^j_\alpha(\bar{\lambda}^0,\cdot) & \textit{on} & A_{\alpha,*} + [0,+\infty)^{n_\alpha}, \\ \underline{\underline{A}}^j_\alpha < A^j_{\alpha,*} < B^j_\alpha(\bar{\lambda}^0,\cdot) & \textit{on} & \displaystyle\prod_{k=1,\dots,n} [0, A^k_{\alpha,*}) \end{array} \right| \quad \textit{for} \quad j = 1,\dots,n \tag{14.84}$$

with

$$A_{\alpha,*} := \begin{cases} (\hat{\theta}^1,\dots,\hat{\theta}^n)(\bar{\lambda}^0) & \textit{if} \quad \alpha = L, \\ (\hat{\theta}^{1+n},\dots,\hat{\theta}^{m+n})(\bar{\lambda}^0) & \textit{if} \quad \alpha = R. \end{cases} \tag{14.85}$$

ii) (The result with limiters (A_L, A_R))

Then the function $\hat\lambda_* : [0,+\infty)^N \to [0,+\infty)^N$ *defined by*

$$(14.86)\qquad \left\{\begin{array}{l} \hat\lambda^j_*(\bar\gamma) := \min\left\{\bar\gamma^j, A^j_\alpha(\bar\gamma_\alpha), B^j_\alpha(\lambda^{0\alpha},\bar\gamma_\alpha)\right\}, \quad j\in I_\alpha, \quad \alpha = L,R\\ C^0_L := \min\left\{\bar\gamma^1+\dots+\bar\gamma^n, \bar\lambda^0, A^0_{L,\bar 1}(\bar\gamma_L),\dots,A^0_{L,\bar n}(\bar\gamma_L)\right\},\\ C^0_R := \min\left\{\bar\gamma^{1+n}+\dots+\bar\gamma^{m+n}, \bar\lambda^0, A^0_{R,\overline{1+n}}(\bar\gamma_R),\dots,A^0_{R,\overline{m+n}}(\bar\gamma_R)\right\},\\ \bar\gamma^0 := \min\left\{C^0_L, C^0_R\right\},\\ (\lambda^{0L},\lambda^{0R}) = \left\{\begin{array}{ll} (\bar\lambda^0,\bar\gamma^0) & \textit{if}\quad C^0_L\le C^0_R,\\ (\bar\gamma^0,\bar\lambda^0) & \textit{if}\quad C^0_L > C^0_R,\end{array}\right.\end{array}\right.$$

is a n:m conservative preflux, and satisfies moreover

$$\sum_{j\in I_L}\hat\lambda^j_* = \sum_{j\in I_R}\hat\lambda^j_* = \bar\gamma^0.$$

iii) (Further property)
Let

$$M_0 := \max_{j\in I_L\cup I_R} (\hat\theta^j)^{-1}(\bar\lambda^0).$$

Then the values of $\hat\lambda_*$ *depend on the function* $\hat\theta = (\hat\theta^j)_{j\in I_L\cup I_R}$ *only through the values of the restriction* $\hat\theta_{|[0,M_0]}$*, for some* $\hat\theta$ *satisfying (14.75).*
iv) (Kružkov case)
Moreover, under the previous assumptions, the n:m conservative preflux $\hat\lambda_*$ *is Kružkov if*

$$(14.87)\qquad -1\le \partial_k A^j_\alpha \le 0 \quad \textit{for all}\quad j,k\in I_\alpha \quad \textit{with}\quad j\ne k, \quad \alpha = L,R.$$

v) (Extension to the cases $n,m\ge 1$**)**
The result extends to the cases $n_L := n, n_R := m \ge 1$*, with the convention that*

$$\left\{\begin{array}{ll} B^j_\alpha \equiv +\infty, & \\ \underline{\underline{A}}^j_\alpha \equiv 0, & \textit{if}\quad n_\alpha = 1,\\ \overline{A}^j_\alpha = const = A^j_{\alpha,*}, & \end{array}\right.$$

where $A^j_{\alpha,*}$ *is prescribed in (14.85).*

Proof of Lemma 14.18
The proof follows easily from Corollary 14.17 by gluing of prefluxes using Theorem 11.17 in the limit case where $\bar\lambda_L = (\bar\lambda^0,+\infty,\dots,+\infty)\in[0,+\infty]^{1+n}$ and $\bar\lambda_R = (\bar\lambda^0,+\infty,\dots,+\infty)\in[0,+\infty]^{1+m}$. Notice that the $+\infty$ may be replaced by any sufficiently large constant here. We then notice that for gluing some n:1 preflux with some 1:m preflux as given in Corollary 14.17, we are reduced to solve the equation: find $(\lambda^{0L},\lambda^{0R})\in\mathbb{D}_{\bar\lambda^0}$ with $\bar\lambda^0\in[0,+\infty)$ solving (using (14.79) for the left and the right contributions)

$$\min\left\{\lambda^{0L},\bar\gamma^1+\dots+\bar\gamma^n,\bar\lambda^0,A^0_{L,\bar 1}(\bar\gamma_L),\dots,A^0_{L,\bar n}(\bar\gamma_L)\right\} = \min\left\{\lambda^{0R},\bar\gamma^{1+n}+\dots+\bar\gamma^{m+n},\bar\lambda^0,A^0_{R,\overline{1+n}}(\bar\gamma_R),\dots,A^0_{R,\overline{m+n}}(\bar\gamma_R)\right\}$$

Hence we have the constraint

$$\max\left\{\lambda^{0L},\lambda^{0R}\right\} = \bar\lambda^0$$

and find

$$\min\left\{\lambda^{0L},\lambda^{0R}\right\} := \bar\gamma^0$$

where $\bar\gamma^0$ is given in (14.86). The result easily follows. In point v), notice also that when $n=1$ or $m=1$, we have nothing to glue. This ends the proof of the lemma.

Remark 14.19 (Simplified n:m conservative Kružkov preflux with no limiters)
We work with notation of Lemma 14.18. In the special case where $\bar\lambda^0 := +\infty$ *and* $A^j_\alpha \equiv +\infty$*, we get the preflux*

$$\left\{\begin{array}{l} \hat\lambda^j_*(\bar\gamma) := \min\left\{\bar\gamma^j, B^j_\alpha(\lambda^{0\alpha},\bar\gamma_\alpha)\right\}, \quad j\in I_\alpha, \quad \alpha = L,R\\ C^0_L := \bar\gamma^1+\dots+\bar\gamma^n,\\ C^0_R := \bar\gamma^{1+n}+\dots+\bar\gamma^{m+n},\\ \bar\gamma^0 := \min\left\{C^0_L, C^0_R\right\},\\ (\lambda^{0L},\lambda^{0R}) = \left\{\begin{array}{ll} (+\infty,\bar\gamma^0) & \textit{if}\quad C^0_L\le C^0_R,\\ (\bar\gamma^0,+\infty) & \textit{if}\quad C^0_L > C^0_R,\end{array}\right.\end{array}\right.$$

Now using property (14.40), we see that

$$B_\alpha(\lambda^{0\alpha}, \bar\gamma_\alpha) = B_\alpha(C^0_\alpha, \bar\gamma_\alpha) \quad \textit{if} \quad \lambda^{0\alpha} \geq C^0_\alpha$$

and then the preflux simplifies in

$$(14.88) \qquad \left\{ \begin{array}{l} \hat\lambda^j_*(\bar\gamma) := \min\left\{\bar\gamma^j, B^j_\alpha(\bar\gamma^0, \bar\gamma_\alpha)\right\}, \quad j \in I_\alpha, \quad \alpha = L, R \\ \bar\gamma^0 := \min\left\{\bar\gamma^1 + \cdots + \bar\gamma^n, \bar\gamma^{1+n} + \cdots + \bar\gamma^{m+n}\right\}. \end{array}\right.$$

which is a n:m conservative Kružkov preflux.

15 Examples of conservative germs and prefluxes

15.1 Kružkov cases

15.1.1 Vanishing viscosity germ revisited

The germ $\mathcal{G}^{VV}$ obtained by vanishing viscosity on a junction has been studied in several works. We can cite Andreianov, Karlsen, Risebro [4] for 1 : 1 junctions, and for $n : m$ junctions Coclite, Garavello [13] for existence, Andreianov, Coclite, Donadello [2] for existence and uniqueness for bell-shaped fluxes, and more recently Musch, Fjordholm, Risebro [40] for monotone fluxes and Fjordholm, Musch, Risebro [20] for fluxes with finite number of extrema.

Here we do not try to justify that $\mathcal{G}^{VV}$ is obtained in the vanishing viscosity limit. On the contrary, we take $\mathcal{G}^{VV}$ (or more precisely its associated Godunov flux) as a definition, and show that it is a conservative Kružkov germ, removing in particular the technical condition of a finite number of extrema of the fluxes on each branch.

We have the following result.

Proposition 15.1 (Vanishing viscosity germ and its Godunov flux)
Assume (2.2) with $N \geq 1$ and junction (J, f). Assume that $[a^j, b^j] = [a^0, b^0]$ for all $j = 1, \dots, N$. Assume moreover that either

$$(15.1) \qquad \left\{ \begin{array}{lll} f^j \geq 0 & \textit{on} & [a^j, b^j] \subset \mathbb{R} \\ f^j = 0 & \textit{on} & \partial[a^j, b^j] \end{array}\right. \qquad \textit{for all indices} \quad j = 1, \dots, N$$

or

$$(15.2) \qquad [a^0, b^0] \cap \mathbb{R} = \mathbb{R} \quad \textit{and } N \geq 2 \textit{ with} \quad \theta^{j_+} = \sigma^{j_+}, \quad \theta^{j_-} = -\sigma^{j_-} \quad \textit{for two indices } j_-, j_+ \textit{ in (2.2).}$$

Now for any $p \in [a, b] \cap \mathbb{R}^N$, consider

$$R_p := \left\{ r \in [a^0, b^0], \quad \sum_{\sigma^j = 1} G^j(p^j, r) = \sum_{\sigma^j = -1} G^j(r, p^j) \right\}$$

and

$$\Lambda_p := \left\{ g(r) := (g^1, \dots, g^N)(r) \quad \textit{with} \quad g^j(r) := \left\{ \begin{array}{ll} G^j(p^j, r) & \textit{if} \quad \sigma^j = 1 \\ G^j(r, p^j) & \textit{if} \quad \sigma^j = -1 \end{array}\right. \middle| \quad \textit{for all} \quad r \in R_p \right\}$$

Then $\Lambda_p = \left\{\hat f(p)\right\}$ is reduced to a singleton. Moreover $\hat f : [a, b] \to \mathbb{R}^N$ defines a map such that the set

$$\mathcal{G}^{VV} := \left\{ p \in [a, b], \quad \hat f(p) = f(p) \right\}$$

is a conservative Kružkov germ. Moreover $\hat f$ is the associated Godunov flux $\hat f = \hat f_{\mathcal{G}^{VV}}$.

Proof of Proposition 15.1
Step 1: $R_p \neq \emptyset$

We recall that

$$G^j(\uparrow,\downarrow) = G^j(p,q) = \begin{cases} \min\limits_{[p,q]} f^j & \text{if} \quad p \le q \\ \max\limits_{[q,p]} f^j & \text{if} \quad p \ge q \end{cases}$$

Then we set

$$g(r) := \sum_{\sigma^j=1} G^j(p^j,r) - \sum_{\sigma^j=-1} G^j(r,p^j)$$

which is nonincreasing in r.

Step 1.1: under assumption (15.1)

Inequality $f^j \ge 0$ implies $G^j \ge 0$ and then

$$\begin{cases} g(b^0) = \sum\limits_{\sigma^j=1} 0 - \sum\limits_{\sigma^j=-1} (\ge 0) \le 0 \\ g(a^0) = \sum\limits_{\sigma^j=1} (\ge 0) - \sum\limits_{\sigma^j=-1} 0 \ge 0 \end{cases}$$

Because g is continuous, we deduce the existence of some $r \in [a^0, b^0]$ such that $g(r) = 0$, i.e. $r \in R_p$.

Step 1.2: under assumption (15.2)

Recall that the function $\theta^j f^j$ is coercive, i.e. $\liminf\limits_{|p^j|\to+\infty} \theta^j f^j(p^j) = +\infty$. Hence

$$G^j(p^j,r) \to \begin{cases} -\infty & \text{if} \quad r \to +\infty \quad \text{and} \quad \theta^j = -1 = -\sigma^j \\ +\infty & \text{if} \quad r \to -\infty \quad \text{and} \quad \theta^j = 1 = \sigma^j \\ & \\ \min\limits_{[p^j,+\infty)} f^j & \text{if} \quad r \to +\infty \quad \text{and} \quad \theta^j = 1 \\ \max\limits_{(-\infty,p^j]} f^j & \text{if} \quad r \to -\infty \quad \text{and} \quad \theta^j = -1 \end{cases}$$

while

$$G^j(r,p^j) \to \begin{cases} +\infty & \text{if} \quad r \to +\infty \quad \text{and} \quad \theta^j = 1 = -\sigma^j \\ -\infty & \text{if} \quad r \to -\infty \quad \text{and} \quad \theta^j = -1 = \sigma^j \\ & \\ \max\limits_{[p^j,+\infty)} f^j & \text{if} \quad r \to +\infty \quad \text{and} \quad \theta^j = -1 \\ \min\limits_{(-\infty,p^j]} f^j & \text{if} \quad r \to -\infty \quad \text{and} \quad \theta^j = 1 \end{cases}$$

Hence under assumption (15.2), we deduce that

$$\begin{cases} g(r) \to -\infty \quad \text{as} \quad r \to +\infty \quad \text{because of } j_- \\ g(r) \to +\infty \quad \text{as} \quad r \to -\infty \quad \text{because of } j_+ \end{cases}$$

By continuity of g, we deduce the existence of some $r \in \mathbb{R}$ such that $g(r) = 0$, i.e. $r \in R_p$.

Step 2: Λ_p is reduced to a singleton

Assume that $r, \bar{r} \in R_p$ with $r < \bar{r}$. Hence $g(r) = 0 = g(\bar{r})$. Because g is a sum of N nonincreasing functions, we deduce that each function is constant on $[r, \bar{r}]$, i.e. $G^j(p^j, r) = G^j(p^j, \bar{r})$ for $\sigma^j = 1$ and $G^j(r, p^j) = G^j(\bar{r}, p^j)$ for $\sigma^j = -1$. Up to redefine $r, \bar{r}$, we can choose such elements such that $[r, \bar{r}]$ is the maximal interval in $[a^0, b^0]$ where g vanishes. This implies that $g = const$ on $[r, \bar{r}]$, and then that $\Lambda_p = \{g(r)\}$ is a singleton. We set $\hat{f}(p) := g(r)$.

Step 3: continuity of $\hat{f}$

As usual the continuity of $\hat{f}$ follows from the singleton property of Λ_p.

Step 4: bounds on $\hat{f}$

We want to check that $f_- \le \hat{f} \le f_+$. We only do it for components j such that $\sigma^j = 1$ (the case $\sigma^j = -1$ is similar). This follows from $f^j_-(p^j) = G^j(p^j, b^j) \le \hat{f}^j(p) = G^j(p^j, r) \le G^j(p^j, a^j) = f^j_+(p^j)$.

Step 5: local constancy

Consider $p_* \in [a,b] \cap \mathbb{R}^N$ and assume that $K_{p_*} := \Big\{ j \in \{1, \dots, N\}\,,\ \hat{f}^j(p_*) \ne f^j(p_*) \Big\} \ne \emptyset$. Let $j \in K_{p_*}$. Assume that $\sigma^j = 1$ (the case $\sigma^j = -1$ is similar). Then we have with $r_* \in R_{p_*}$

$$f^j(p^j_*) \ne \hat{f}^j(p_*) = G^j(p^j_*, r_*) = \begin{cases} \min\limits_{[p^j_*, r_*]} f^j < f^j(p^j_*) & \text{if} \quad p^j_* < r_* \\ \max\limits_{[r_*, p^j_*]} f^j > f^j(p^j_*) & \text{if} \quad p^j_* > r_* \end{cases}$$

Then there exists $\varepsilon > 0$ such that for

$$Q_\varepsilon(p_*) := \left(p_* + \sum_{j\in K_{p_*}} (-\varepsilon,\varepsilon)e_j\right) \cap [a,b]$$

and all $p \in Q_\varepsilon(p_*)$, we have $G^j(p^j, r_*) = G^j(p^j_*, r_*)$ for $\sigma^j = 1$ and $G^j(r_*, p^j) = G^j(r_*, p^j_*)$ for $\sigma^j = -1$. Hence $r^* \in R_p$ and moreover $\hat{f}(p) = \hat{f}(p_*)$. This shows that $\hat{f}$ is locally constant on $Q_\varepsilon(p_*)$ and then on $\left\{\hat{f} \neq f\right\}$.

Step 6: basic monotonicities

Notice from Lemma 11.4 that basic monotonicities are automatically satisfied for prefluxes. Here we still give a proof, because arguments of the proof will be also used later on in Step 8.

Consider some $p \in [a,b] \cap \mathbb{R}^N$, and fix some index j_0. Assume also that $\sigma^{j_0} = 1$ (the case $\sigma^{j_0} = -1$ is similar). Then consider

$$[a,b] \ni \bar{p} = p + qe_{j_0} \quad \text{with} \quad q \in [0,+\infty).$$

Step 6.1: monotonicity in r

Consider the minimal $r \in \mathbb{R}$ such that $r \in R_p$ and any $\bar{r} \in R_{\bar{p}}$. Then we have

$$g(\bar{p},\bar{r}) := G^{j_0}(\bar{p}^{j_0}, \bar{r}) + \sum_{\sigma^j=1,\ j\neq j_0} G^j(p^j,\bar{r}) - \sum_{\sigma^j=-1} G^j(\bar{r}, p^j) = 0$$

Because of the monotonicities $G^j(\uparrow,\downarrow)$ and the fact that $\bar{p}^{j_0} \geq p^{j_0}$, we deduce that $0 = g(\bar{p},\bar{r}) \geq g(p,\bar{r})$. Then the monotonicity of $g(p,\cdot)$ implies that $r \leq \bar{r}$.

Step 6.2: monotonicity of $\hat{f}$

We have

$$\begin{aligned} \hat{f}^{j_0}(\bar{p}) &= G^{j_0}(\bar{p}^{j_0}, \bar{r}) \\ &= \sum_{\sigma^j=-1} G^j(\bar{r}, p^j) - \sum_{\sigma^j=1,\ j\neq j_0} G^j(p^j, \bar{r}) \\ &\geq \sum_{\sigma^j=-1} G^j(r, p^j) - \sum_{\sigma^j=1,\ j\neq j_0} G^j(p^j, r) \\ &= G^{j_0}(p^{j_0}, r) \\ &= \hat{f}^{j_0}(p) \end{aligned}$$

which shows the monotonicity of $\sigma^j \hat{f}^j$ in p^j for $j = j_0$, i.e. the basic monotonicities of $\hat{f}$.

Step 7: conservative Riemann germ

From the previous steps, we deduce that $\mathcal{G}^{VV}$ is a Riemann germ and $\hat{f} = \hat{f}_{\mathcal{G}^{VV}}$. Moreover by construction, the germ is conservative.

Step 8: monotone germ

We start as in Step 6 with index j_0 such that $\sigma^{j_0} = 1$ (the case $\sigma^{j_0} = -1$ is similar), and $\bar{p} = p + qe_j$ with $q \geq 0$. Using $\bar{r} \in R_{\bar{p}}$ and minimal $r \in R_p$, we get $\bar{r} \geq r$.

Case A: $\sigma^j = 1$ **with** $j \neq j_0$

We have

$$\hat{f}^j(\bar{p}) = G^j(p^j, \bar{r}) \leq G^j(p^j, r) = \hat{f}^j(p)$$

Case B: $\sigma^j = -1$

We have

$$\sigma^j \hat{f}^j(\bar{p}) = \sigma^j G^j(\bar{r}, p^j) \leq \sigma^j G^j(r, p^j) = \sigma^j \hat{f}^j(p)$$

Conclusion

In all cases, we get that the map $p \mapsto \sigma^j \hat{f}^j(p)$ is nonincreasing in p^{j_0} for all $j \neq j_0$. Because this is true for all indices j_0, this implies that the germ $\mathcal{G}^{VV}$ is monotone. But a conservative monotone germ is a conservative Kružkov germ. This ends the proof of the proposition.

15.1.2 Holden-Risebro theory revisited

We revisit the pioneering work of Holden and Riesebro [26].

Lemma 15.2 (Convex optimization)
Let $\Psi : \mathbb{R}^N \to \mathbb{R}$ be a strictly convex function and some fixed vector $\mathbb{R}^N \ni \bar{\gamma} = (\bar{\gamma}^1, \dots, \bar{\gamma}^N) \geq 0$ and $\sigma = (\sigma^1, \dots, \sigma^N) \in \{\pm 1\}^N$ with $\sigma \neq (1, \dots, 1)$ and $\sigma \neq (-1, \dots, -1)$. Let $L : \mathbb{R}^N \to \mathbb{R}$ be the linear map defined by $L(\gamma) := \sum_{j=1}^N \sigma^j \gamma^j$ for $\gamma \in \mathbb{R}^N$. Let

$$\hat{\gamma}(\bar{\gamma}) := \underset{K(\bar{\gamma})}{Argmin}\, \Psi \quad \text{with the convex} \quad K(\bar{\gamma}) := \{\gamma \in \Gamma(\bar{\gamma}), \quad L(\gamma) = 0\} \quad \text{and the box} \quad \Gamma(\bar{\gamma}) := [0, \bar{\gamma}] := \prod_{j=1}^N [0, \bar{\gamma}^j]$$

i) (Preflux properties)
Then $\hat{\gamma} : [0, +\infty)^N \to [0, +\infty)^N$ is a conservative preflux in the sense of Definition 11.1.
ii) (Holden-Risebro preflux: σ-monotonicity, when Ψ has separated variables)
Assume moreover that there exists N strictly convex functions $\Psi_j : \mathbb{R} \to \mathbb{R}$ such that $\Psi(\gamma) = \sum_{j=1}^N \Psi_j(\gamma^j)$ for $\gamma \in \mathbb{R}^N$. Then $\hat{\gamma}$ is σ-monotone in the sense of Definition 11.1, and we denote it $\hat{\gamma}^{HR}$ and call it a Holden-Risebro preflux. In particular $\hat{\gamma}^{HR}$ is a conservative Kružkov preflux.

Remark 15.3 *In Lemma 15.2, $\bar{\gamma}$ denotes a vector of $[0, +\infty)^N$ which must not be confused with the capacity of Definition 11.6.*

Proof of Lemma 15.2
Step 1: proof of continuity, basic monotonicity, conservation
Step 1.1: continuity
Notice that $0_{\mathbb{R}^N} \in K(\bar{\gamma})$ for all $\bar{\gamma} \geq 0$ and then $K(\bar{\gamma})$ is always non empty and compact convex. Then the strict convexity of Ψ implies the uniqueness of the minimizer $\hat{\gamma}(\bar{\gamma})$, which is then well-defined. Moreover this uniqueness also implies the continuity of the map $\bar{\gamma} \mapsto \hat{\gamma}(\bar{\gamma})$.
Step 1.2: basic monotonicitity
Notice from Lemma 11.4 that basic monotonicities are automatically satisfied for prefluxes. Here we still give a proof, because arguments of the proof will be also used later on in Step 3.1.
Consider $\bar{\gamma}, \bar{\gamma}_* \in [0, +\infty)^N$ such that $\bar{\gamma} - \bar{\gamma}_* = \varepsilon e_k$ with $\varepsilon > 0$ and $k = 1$ to fix the ideas. Assume moreover that $\sigma^1 = 1$ (the case $\sigma^1 = -1$ is similar).
Assume by contradiction that

$$\hat{\gamma}^1(\bar{\gamma}) \leq \hat{\gamma}^1(\bar{\gamma}_*) \quad \text{and} \quad \hat{\gamma}(\bar{\gamma}) \neq \hat{\gamma}(\bar{\gamma}_*) \tag{15.3}$$

This implies that $\hat{\gamma}(\bar{\gamma}) \in [0, \bar{\gamma}_*]$. Because $L(\hat{\gamma}(\bar{\gamma})) = 0$, we deduce that $\hat{\gamma}(\bar{\gamma}) \in K(\bar{\gamma}_*) \subset K(\bar{\gamma})$. This implies that

$$\inf_{K(\bar{\gamma})} \Psi = \Psi(\hat{\gamma}(\bar{\gamma})) \geq \inf_{K(\bar{\gamma}_*)} \Psi = \Psi(\hat{\gamma}(\bar{\gamma}_*)) \geq \inf_{K(\bar{\gamma})} \Psi$$

Hence $\Psi(\hat{\gamma}(\bar{\gamma})) = \Psi(\hat{\gamma}(\bar{\gamma}_*))$, and because Ψ is strictly convex, we deduce that $\hat{\gamma}(\bar{\gamma}) = \hat{\gamma}(\bar{\gamma}_*)$. Contradiction with (15.3). We conclude that $\hat{\gamma}^1(\bar{\gamma}) > \hat{\gamma}^1(\bar{\gamma}_*)$ or $\hat{\gamma}(\bar{\gamma}) = \hat{\gamma}(\bar{\gamma}_*)$. This shows in particular that the map $\bar{\gamma} \mapsto \hat{\gamma}^j(\bar{\gamma})$ is nondecreasing in $\bar{\gamma}^j$, for $j = 1$, and then similarly for all j.
Step 1.3: conservation
By construction, we have $L(\hat{\gamma}(\bar{\gamma})) = 0$, and then $\hat{\gamma}$ is conservative.
Step 2: proof of local constancy
To simplify the notation, let us introduce $\hat{\gamma}_* := \hat{\gamma}(\bar{\gamma}_*)$.
Step 2.1: preliminary
Now, given $\bar{\gamma}_* \in [0, +\infty)^N$, the function Ψ is minimal at $\hat{\gamma}_*$ on $K(\bar{\gamma}_*)$ with $L(\hat{\gamma}_*) = 0$. Looking at the subdifferential $\partial(\Psi_{|\Gamma(\bar{\gamma}_*)})(\hat{\gamma}_*)$ of the convex function Ψ, it is easy to see that there exists some $\lambda \in \mathbb{R}$ (that can be interpreted as a Lagrange multiplier), such that the function $\Phi := \{\Psi - \lambda L\}_{|\Gamma(\bar{\gamma}_*)}$ is minimal on $\Gamma(\bar{\gamma}_*)$ at $\hat{\gamma}_*$. Hence the convex function Φ satisfies

$$\inf_{\Gamma(\bar{\gamma}_*)} \Phi \geq \Phi(\hat{\gamma}_*) \qquad \text{and} \qquad \xi \cdot \partial\Phi(\hat{\gamma}_*) \geq 0 \quad \text{for all} \quad \xi \in T_{\hat{\gamma}_*}\Gamma(\bar{\gamma}_*) := \lim_{\delta \to 0^+} \delta^{-1}\left(\Gamma(\bar{\gamma}_*) - \hat{\gamma}_*\right) \tag{15.4}$$

where $\xi \cdot \partial\Phi(x_0) \geq 0$ means $\xi \cdot v \geq 0$ for all $v \in \partial\Phi(x_0)$.
Step 2.2: variations and local constancy

Now assume that $I := \left\{j \in \{1, \ldots, N\},\ \hat\gamma^j(\bar\gamma_*) \neq \bar\gamma_*^j\right\} \neq \emptyset$, and for $\varepsilon > 0$, consider $\bar\gamma \in [0, +\infty)^N$ such that

$$\bar\gamma^j \quad \left\{\begin{array}{lll} = \bar\gamma_*^j & & \text{if} \quad j \notin I \\ \in (\bar\gamma_*^j - \varepsilon, \bar\gamma_*^j + \varepsilon) \cap [0, +\infty) & & \text{if} \quad j \in I \end{array}\right. \tag{15.5}$$

Assume that $\sigma^j = 1$ (the case $\sigma^j = -1$ is similar). Then for ε small enough, we have $\left\{\begin{array}{llll} \bar\gamma^j & = \hat\gamma_*^j & \text{if} & \bar\gamma_*^j = \hat\gamma_*^j \\ \bar\gamma^j & > \hat\gamma_*^j & \text{if} & \bar\gamma_*^j > \hat\gamma_*^j \end{array}\right.$.
Hence $[0, \bar\gamma^j] \subset [0, \hat\gamma_*^j] + T_{\hat\gamma_*^j}[0, \bar\gamma_*^j]$, and we conclude that

$$\hat\gamma_*, \hat\gamma(\bar\gamma) \in \Gamma(\bar\gamma) \subset \Gamma(\bar\gamma_*) + T_{\hat\gamma_*}\Gamma(\bar\gamma_*) \tag{15.6}$$

Therefore

$$\Phi(\hat\gamma_* + \xi) \geq \Phi(\hat\gamma_*) + \xi \cdot v \geq \Phi(\hat\gamma_*) \quad \text{for all} \quad \xi \in T_{\hat\gamma_*}\Gamma(\bar\gamma_*), \quad v \in \partial\Phi(\hat\gamma_*)$$

where we have used the second part of (15.4) in the last inequality. Therefore, using now the first part of (15.4) and also (15.6), we get $\inf_{\Gamma(\bar\gamma)} \Phi \geq \Phi(\hat\gamma_*)$. Because $L = 0$ on $K(\bar\gamma)$, and $L(\hat\gamma_*) = 0$, we deduce $\inf_{K(\bar\gamma)} \Psi \geq \Psi(\hat\gamma_*)$. Because Ψ is strictly convex and $\hat\gamma_* \in K(\bar\gamma)$, we conclude that $\hat\gamma(\bar\gamma) = \hat\gamma_*$. Then Definition (15.5) of $\bar\gamma$ shows exactly that $\hat\gamma$ is locally constant on $\left\{\hat\gamma \neq id_{[0,+\infty)^N}\right\}$.

Step 3: proof of ii)

We assume that $\Psi(\gamma) = \sum_{j=1}^N \Psi_j(\gamma^j)$, with $\Psi_j : \mathbb{R} \to \mathbb{R}$ strictly convex. We fix the index $k = 1$ and assume that $\sigma^1 = 1$ (the other cases are similar for $\sigma^1 = -1$ or other indices). We want to show that $\bar\gamma^1 \mapsto \sigma^j \hat\gamma^j(\bar\gamma)$ is nonincreasing for $j \neq 1$.

Step 3.1: a simplified approximate situation

For $j \neq 1$, fix $\bar\gamma^j \geq 0$, and approximate each function Ψ_j, by a smooth convex coercive function $\Psi_j^\varepsilon : \mathbb{R} \to \mathbb{R}$ such that pointwisely, we have

$$\lim_{\varepsilon \to 0^+} \Psi_j^\varepsilon(x) = \quad \left\{\begin{array}{lll} \Psi_j(x) & \text{if} & x \in \left[0, \bar\gamma^j\right] \\ +\infty & \text{if} & x \notin \left[0, \bar\gamma^j\right] \end{array}\right.$$

We set $\Psi^\varepsilon(\gamma) := \Psi_1(\gamma^1) + \sum_{j=2}^N \Psi_j^\varepsilon(\gamma^j)$, $K^\varepsilon(\bar\gamma^1) := \left\{\gamma \in \Gamma^\varepsilon(\bar\gamma^1),\ L(\gamma) = 0\right\}$, $\Gamma^\varepsilon(\bar\gamma^1) = [0, \bar\gamma^1] \times \mathbb{R}^{N-1}$ and consider the approximate problem

$$\hat\gamma^\varepsilon(\bar\gamma^1) := \operatorname*{Argmin}_{K^\varepsilon(\bar\gamma^1)} \Psi^\varepsilon$$

Now it is easy to analyse the optimization problem for which we have suppressed the constraints for $j \neq 1$. We deduce that there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ of the constraint $L = 0$, such that for $\bar\gamma' := (\bar\gamma^2, \ldots, \bar\gamma^N)$, we have $D_{\bar\gamma'}(\Psi^\varepsilon - \lambda L)(\hat\gamma^\varepsilon(\bar\gamma^1)) = 0$, i.e.

$$\left(\begin{array}{l} (\Psi_2^\varepsilon)'(\hat\gamma^{\varepsilon,2}(\bar\gamma^1)) \\ \ldots \\ (\Psi_N^\varepsilon)'(\hat\gamma^{\varepsilon,N}(\bar\gamma^1)) \end{array}\right) = \lambda \left(\begin{array}{l} \sigma^2 \\ \ldots \\ \sigma^N \end{array}\right)$$

Similarly for some $\bar\gamma_*^1 > \bar\gamma^1$, we get the existence of some $\lambda_* \in \mathbb{R}$ such that

$$\left(\begin{array}{l} (\Psi_2^\varepsilon)'(\hat\gamma^{\varepsilon,2}(\bar\gamma_*^1)) \\ \ldots \\ (\Psi_N^\varepsilon)'(\hat\gamma^{\varepsilon,N}(\bar\gamma_*^1)) \end{array}\right) = \lambda_* \left(\begin{array}{l} \sigma^2 \\ \ldots \\ \sigma^N \end{array}\right)$$

Because each function $(\Psi_j^\varepsilon)'$ is nondecreasing, we deduce that

$$(\lambda_* - \lambda) \cdot \sigma^j \left\{\hat\gamma^{\varepsilon,j}(\bar\gamma_*^1) - \hat\gamma^{\varepsilon,j}(\bar\gamma^1)\right\} \geq 0 \quad \text{for all} \quad j \neq 1 \tag{15.7}$$

Moreover relation $L(\hat\gamma^\varepsilon(\bar\gamma_*^1) - \hat\gamma^\varepsilon(\bar\gamma^1)) = 0$ shows that

$$\left\{\hat\gamma^{\varepsilon,1}(\bar\gamma_*^1) - \hat\gamma^{\varepsilon,1}(\bar\gamma^1)\right\} + \sum_{j=2}^N \sigma^j \left\{\hat\gamma^{\varepsilon,j}(\bar\gamma_*^1) - \hat\gamma^{\varepsilon,j}(\bar\gamma^1)\right\} = 0.$$

Notice that (15.7) shows that all terms in the sum $\sum_{j=2}^N$ have the same sign. Moreover the raisoning of Step 1.2 shows that $\left\{\hat{\gamma}^{\varepsilon,1}(\bar{\gamma}_*^1) - \hat{\gamma}^{\varepsilon,1}(\bar{\gamma}^1)\right\} \ge 0$. We deduce that

$$\sigma^j \left\{\hat{\gamma}^{\varepsilon,j}(\bar{\gamma}_*^1) - \hat{\gamma}^{\varepsilon,j}(\bar{\gamma}^1)\right\} \le 0 \quad \text{for all} \quad j \neq 1 \tag{15.8}$$

Step 3.2: the limit $\varepsilon \to 0$
Using the strict convexity of Ψ, it is easy to show that $\lim\limits_{\varepsilon\to 0^+} \hat{\gamma}^\varepsilon(\bar{\gamma}^1) = \hat{\gamma}(\bar{\gamma})$. We deduce from (15.8) that

$$\sigma^j \left\{\hat{\gamma}^j(\bar{\gamma}_*^1, \bar{\gamma}') - \hat{\gamma}^j(\bar{\gamma}^1, \bar{\gamma}')\right\} \le 0 \quad \text{for all} \quad j \neq 1, \quad \text{with} \quad \bar{\gamma}_*^1 > \bar{\gamma}^1 \quad \text{and} \quad \sigma^1 = 1$$

which shows exactly that the maps $\bar{\gamma} \mapsto \sigma^j \hat{\gamma}^j(\bar{\gamma})$ are nonincreasing in $\sigma^1\bar{\gamma}^1$ for all $j \neq 1$. Similarly, doing the reasoning for all indices, this implies the σ-monotonicity of $\hat{\gamma}$.
Step 3.3: conservative Kružkov
The fact that the preflux $\hat{\gamma}$ is conservative Kružkov is equivalent to the fact that $\hat{\gamma}$ is conservative σ-monotone (see v) of Theorem 11.8). This ends the proof of the lemma.

For later use, we will also need the following adaptation of the previous lemma.

Lemma 15.4 (Convex optimization, local variation)
Let $\Psi : \mathbb{R}^N \to \mathbb{R}$ be a strictly convex function and some fixed vector $\mathbb{R}^N \ni \bar{\gamma}_* = (\bar{\gamma}_*^1, \dots, \bar{\gamma}_*^N) \ge 0$ and $\sigma = (\sigma^1, \dots, \sigma^N) \in \{\pm 1\}^N$. Let $L : \mathbb{R}^N \to \mathbb{R}$ be the linear map defined by $L(\gamma) := \sum\limits_{j=1} \sigma^j \gamma^j$ for $\gamma \in \mathbb{R}^N$. Assume that $L\bar{\gamma}_* \neq 0$, and let $\kappa \in (\min\{0, L\bar{\gamma}_*\}, \max\{0, L\bar{\gamma}_*\})$. Let

$$\hat{\gamma}(\bar{\gamma}) := \underset{K_\kappa(\bar{\gamma})}{\mathit{Argmin}}\, \Psi \quad \text{with the convex} \quad K_\kappa(\bar{\gamma}) := \{\gamma \in \Gamma(\bar{\gamma}), \quad L(\gamma) = \kappa\} \quad \text{and the box} \quad \Gamma(\bar{\gamma}) := [0, \bar{\gamma}] := \prod_{j=1}^N [0, \bar{\gamma}^j]$$

Then there exists $\varepsilon > 0$ small enough such that the following holds for

$$Q_\varepsilon := \left(\bar{\gamma}_* + \sum_{j=1,\dots,N} (-\varepsilon, \varepsilon) e_j\right) \cap [0, +\infty)^N.$$

i) (Preflux properties)
Then $\hat{\gamma} : Q_\varepsilon \to [0, +\infty)^N$ is well defined and is (locally on Q_ε) a preflux in the sense of Definition 11.1.
ii) (σ-monotonicity, when Ψ has separated variables)
Assume moreover that there exists N strictly convex functions $\Psi_j : \mathbb{R} \to \mathbb{R}$ such that $\Psi(\gamma) = \sum\limits_{j=1}^N \Psi_j(\gamma^j)$ for $\gamma \in \mathbb{R}^N$. Then $\hat{\gamma}$ is (locally on Q_ε) σ-monotone in the sense of Definition 11.1.

Proof of Lemma 15.4
The proof follows exactly the same lines of the proof of Lemma 15.2, but replacing the linear map L by the affine map $\gamma \mapsto L\gamma - \kappa$. This ends the proof of the lemma.

Proposition 15.5 (Holden-Risebro germ)
Assume (2.2) for some $n : m$ junction (J, f) with $N = n + m$ and $n, m \ge 1$. Recall that $\sigma \in \{\pm 1\}^N$ and σ^j denotes the orientation of the branch J^j. Assume that $\sigma \neq (1, \dots, 1)$ and $\sigma \neq (-1, \dots, -1)$. Assume also that f is bell-shaped, and call $\bar{\gamma} : [a, b] \to [0, +\infty)^N$ the capacity $\bar{\gamma}^j := f^{j,\sigma^j}$ given in Definition 11.6.
For $p \in [a, b]$, consider the convex set

$$K_0(p) := \{\gamma \in \Gamma_0(p), \quad L(\gamma) = 0\} \quad \text{with} \quad L(\gamma) := \sum_{k=1}^N \sigma^k \gamma^k \quad \text{and the box} \quad \Gamma_0(p) := [0, \bar{\gamma}(p)] = \prod_{k=1,\dots,N} [0, \bar{\gamma}^k(p)]$$

Let $\Psi : \mathbb{R}^N \to \mathbb{R}$ be a strictly convex function.
i) (Riemann germ for strictly convex Ψ)
Then the set

$$\mathcal{G} = \mathcal{G}_\Psi := \left\{p \in [a, b], \quad \hat{f}(p) = f(p)\right\} \quad \text{with} \quad \hat{f}(p) := \underset{K_0(p)}{\mathit{Argmin}}\, \Psi$$

is a conservative Riemann germ.

ii) (Conservative Kruškov germ when Ψ has separated variables)

If $\Psi(\gamma) = \sum_{j=1}^{N} \Psi_j(\gamma^j)$ *with each map* $\Psi_j : \mathbb{R} \to \mathbb{R}$ *is strictly convex, then* $\mathcal{G}_\Psi$ *is a conservative Kruškov germ, which is called a Holden-Risebro germ.*

Remark 15.6 *Notice that the original Holden-Risebro germ only concerns point ii) of Proposition 15.5. The reader can also consult* HOLLE *[28] for a different discussion of the conservative Kružkov property of Holden-Risebro germs. Here point i) is a natural generalization, with weaker properties.*

Proof of Proposition 15.5

We notice that the capacity $\bar{\gamma} : [a,b] \to [0,+\infty)^N$ given by Definition 11.6 is continuous and each map $p^j \mapsto \sigma^j \bar{\gamma}^j(p^j)$ is nondecreasing. Moreover with notation $\hat{\gamma}, K, \Gamma$ of Lemma 15.2, we have

$$\hat{f} = \hat{\gamma} \circ \bar{\gamma}, \quad \Gamma_0 = \Gamma \circ \bar{\gamma}, \quad K_0 = K \circ \bar{\gamma}.$$

Step 1: Riemann germ property

Then i) of Lemma 15.2 and Theorem 11.8 show that $\mathcal{G}_\Psi$ is a conservative Riemann germ.

Step 2: additional monotonicities for Ψ with separated variables

Assume that $\Psi(\gamma) = \sum_{j=1}^{N} \Psi_j(\gamma^j)$. Then ii) of Lemma 15.2 shows that the preflux $\hat{\gamma}$ is σ-monotone. Then iii) of Theorem 11.8 shows that $\mathcal{G}_\Psi$ is monotone. We know (see Theorem 2.29) that any conservative monotone germ is a Kružkov germ, and this ends the proof of the proposition.

15.1.3 Data Network solver $\mathcal{R}S_2$ (and convergent junctions)

The main result of this Subsection is Lemma 15.10 on Data Network prefluxes introduced by Garavello and Piccoli for $\mathcal{R}S_2$ in [25], and also in [24] for convergent junctions of type $n:1$ (see Remark 15.11).

We need to start with the following result.

Lemma 15.7 (Preflux by orthogonal projection in $\mathbb{R}^N$)

Let $N \geq 1$ *and* $\tilde{\sigma} = (\tilde{\sigma}^1, \dots, \tilde{\sigma}^{N+1}) = (1, \dots, 1, -1)$. *For* $y = (y^1, \dots, y^N) \in [0,+\infty)^N$, *define*

$$Ly := \sum_{j=1,\dots,N} y^j$$

Fix $\theta \in (0,1]^N$ *such that* $L\theta = 1$. *For* $\gamma = (\gamma^1, \dots, \gamma^N) \in [0,+\infty)^N$ *and* $\gamma^{N+1} \in [0,+\infty)$, *define the following function*

$$\Psi(\cdot, \gamma^{N+1}) : [0,+\infty)^N \to [0,+\infty) \quad \textit{with} \quad \Psi(y, \gamma^{N+1}) := |y - \gamma^{N+1}\theta|^2 \quad \textit{and the box} \quad \Gamma(\gamma) := [0,\gamma] := \prod_{j=1}^{N} [0, \gamma^j]$$

We set

$$\hat{\gamma}(\gamma, \gamma^{N+1}) := \underset{K(\gamma,\gamma^{N+1})}{\mathit{Argmin}} \ \Psi(\cdot, \gamma^{N+1}) \qquad \textit{with} \quad K(\gamma, \gamma^{N+1}) := \left\{ y \in \Gamma(\gamma), \quad Ly = \gamma^{N+1} \right\} \tag{15.9}$$

Fix some $\tilde{\gamma}_* = (\gamma_*, \gamma_*^{N+1}) = (\gamma_*^1, \dots, \gamma_*^{N+1}) \in [0,+\infty)^{N+1}$ *such that* $L\gamma_* > \gamma_*^{N+1}$. *Then there exists* $\varepsilon > 0$ *small enough such that the following holds for*

$$\tilde{Q}_\varepsilon(\tilde{\gamma}_*) := \left(\tilde{\gamma}_* + \sum_{j=1,\dots,N+1} (-\varepsilon, \varepsilon) e_j \right) \cap [0,+\infty)^{N+1}.$$

The function $\tilde{\hat{\gamma}} : \tilde{Q}_\varepsilon(\tilde{\gamma}_*) \to [0,+\infty)^{N+1}$ *given by* $\tilde{\hat{\gamma}}(\gamma, \gamma^{N+1}) := (\hat{\gamma}(\gamma, \gamma^{N+1}), \gamma^{N+1})$ *is well defined and is (locally on* $\tilde{Q}_\varepsilon(\tilde{\gamma}_*)$*) a* $\tilde{\sigma}$*-conservative* $\tilde{\sigma}$*-monotone preflux in the sense of Definition 11.1.*

Remark 15.8 *Notice that* $\hat{\gamma}$ *is the orthogonal projection of* $\gamma^{N+1}\theta$ *onto the closed convex set* $K(\gamma, \gamma^{N+1})$.

Proof of Lemma 15.7
Because $\Psi(\cdot,\gamma^{N+1})$ is strictly convex and $\left\{\dfrac{\gamma_*^{N+1}}{L\gamma_*}\right\}\cdot\gamma_*\in K(\gamma_*,\gamma_*^{N+1})\neq\emptyset$, we deduce that $\hat{\gamma}$ (and then $\tilde{\hat{\gamma}}$) is well-defined and is continuous.
Step 1: bounds
By construction, we have $\tilde{\hat{\gamma}}\in[0,\tilde{\gamma}]$ and then $0\le\tilde{\hat{\gamma}}^j(\tilde{\gamma})\le\tilde{\gamma}^j$ for all indices $j=1,\dots,N+1$.
Step 2: local constancy
By assumption, we have $L\gamma_*>\gamma_*^{N+1}$ and $L\hat{\gamma}(\gamma_*,\gamma_*^{N+1})=\gamma_*^{N+1}$, which forces to have $I:=\left\{j\in\{1,\dots,N\},\ \hat{\gamma}^j(\gamma_*,\gamma_*^{N+1})<\gamma_*^j\right\}\neq\emptyset$. Moreover for $\varepsilon>0$, consider

$$Q_\varepsilon(\gamma_*):=\left(\gamma_*+\sum_{j\in I}(-\varepsilon,\varepsilon)e_j\right)\cap[0,+\infty)^N$$

Then for $\varepsilon>0$ small enough and $\gamma\in Q_\varepsilon(\gamma_*)$, we still have $L\gamma>\gamma_*^{N+1}$. For fixed $\gamma_*^{N+1}\ge 0$, and using Lemma 15.4, and the fact that $\Psi(\cdot,\gamma_*^{N+1})$ is a sum of strictly convex functions in separated variables, we deduce that locally around γ_*, the map $\gamma\mapsto\hat{\gamma}(\gamma,\gamma_*^{N+1})$ is a preflux. Therefore the map $\gamma\mapsto\hat{\gamma}(\gamma,\gamma_*^{N+1})$ is locally constant in $\{\gamma,\ \hat{\gamma}(\gamma,\gamma_*^{N+1})\neq\gamma\}$ close to γ_*. This shows the desired local constancy of $\hat{\gamma}$, and then the local constancy of $\tilde{\hat{\gamma}}$ on $\left\{\tilde{\gamma},\ \tilde{\hat{\gamma}}(\tilde{\gamma})\neq\tilde{\gamma}\right\}$, close to $\tilde{\gamma}_*$.
Step 3: basic monotonicity
Notice from Lemma 11.4 that basic monotonicities are automatically satisfied for prefluxes. Here we still give a proof, because the proof is very short.
For fixed γ_* and γ^{N+1} close enough to γ_*^{N+1}, consider $\gamma\in Q_\varepsilon(\gamma_*)$. Then the argument of Step 2 shows that $\gamma\mapsto\hat{\gamma}(\gamma,\gamma^{N+1})$ is a preflux. In particular the j component of $\hat{\gamma}$ is nondecreasing in γ^j for each $j=1,\dots,N$. With $\tilde{\hat{\gamma}}^{N+1}(\gamma,\gamma^{N+1})=\gamma^{N+1}$, we conclude that $\tilde{\hat{\gamma}}$ satisfies the basic monotonicities, and $\tilde{\hat{\gamma}}$ is then a preflux on $\tilde{Q}_\varepsilon(\tilde{\gamma}_*)$. Moreover it is $\tilde{\sigma}$-conservative by construction.
Step 4: $\tilde{\sigma}$-monotonicity
Because we have seen that $\gamma\mapsto\hat{\gamma}(\gamma,\gamma^{N+1})$ is a preflux, we know that for fixed γ^{N+1} it is σ-monotone in γ. Because $\tilde{\hat{\gamma}}^{N+1}(\gamma,\gamma^{N+1})=\gamma^{N+1}$, it only remains to show that the maps $\gamma^{N+1}\mapsto\hat{\gamma}^j(\gamma,\gamma^{N+1})$ are nondecreasing for all $j=1,\dots,N$. Having (15.9) in mind, we see that the new difficulty here is that the convex function $\Psi(\cdot,\gamma^{N+1})$ itself does depend on γ^{N+1}.
Step 4.1: Generic case
Assume that $\tilde{\gamma}_*=(\gamma_*,\gamma_*^{N+1})\in(0,+\infty)^N\times(0,+\infty)$ satisfies $L\gamma_*>\gamma_*^{N+1}$. Notice that the unique minimizer $\hat{\gamma}_*\in K(\gamma_*,\gamma_*^{N+1})$ is the orthogonal projection of $\gamma_*^{N+1}\theta$ onto the convex set $K(\gamma_*,\gamma_*^{N+1})\ni\left\{\dfrac{\gamma_*^{N+1}}{L\gamma_*}\right\}\cdot\gamma_*$, which is characterized by the following variational inequality

$$(\gamma_*^{N+1}\theta-\hat{\gamma}_*,y-\hat{\gamma}_*)\le 0\quad\text{for all}\quad y\in K(\gamma_*,\gamma_*^{N+1})=\left\{y\in\Gamma(\gamma_*),\quad Ly=\gamma_*^{N+1}\right\}\tag{15.10}$$

Step 4.1.1: decomposition
Define

$$I^\alpha:=\left\{j\in\{1,\dots,N\},\ 0<\hat{\gamma}_*^j<\gamma_*^j\right\},\quad I^\beta:=\left\{j\in\{1,\dots,N\},\ \hat{\gamma}_*^j=\gamma_*^j\right\},\quad I^\delta:=\left\{j\in\{1,\dots,N\},\ \hat{\gamma}_*^j=0\right\}$$

We set $W:=V^\alpha\oplus V^\beta\oplus V^\delta$ with $V^c:=\bigoplus_{k\in I^c}\mathbb{R}e_k$ for $c=\alpha,\beta,\delta$, and for any $\gamma\in W$, we write $\gamma=\gamma^\alpha+\gamma^\beta+\gamma^\delta$ with $\gamma^c\in V^c$ for $c=\alpha,\beta,\delta$. Notice that the minimizer $\hat{\gamma}_*$ is uniquely determined by the existence of Lagrange multipliers $\lambda_*^\alpha\in\mathbb{R}$ and $\mu_*^\beta\in V^\beta\cap[0,+\infty)^N$ and $\mu_*^\delta\in V^\delta\cap[0,+\infty)^N$ such that

$$\hat{\gamma}_*-\gamma_*^{N+1}\theta=D_y\Psi(\hat{\gamma}_*,\gamma_*^{N+1})=\lambda_*^\alpha DL-\mu_*^\beta\diamond DL_{|V^\beta}+\mu_*^\delta\diamond DL_{|V^\delta}\tag{15.11}$$

Indeed for $\Phi(y):=\Psi(y,\gamma_*^{N+1})-\lambda_*^\alpha(Ly-\gamma_*^{N+1})$, we have $\Phi(y)\ge\Phi(\hat{\gamma}_*)+D_y\Phi(\hat{\gamma}_*)\cdot(y-\hat{\gamma}_*)=\Phi(\hat{\gamma}_*)+(\mu_*^\delta-\mu_*^\beta,y-\hat{\gamma}_*)\ge\Phi(\hat{\gamma}_*)$ for $y\in\Gamma(\gamma_*)$, and with $Ly=\gamma_*^{N+1}=L\hat{\gamma}_*$, we get

$$\Psi(y,\gamma_*^{N+1})\ge\Psi(\hat{\gamma}_*,\gamma_*^{N+1})\quad\text{for all}\quad y\in K(\gamma_*,\gamma_*^{N+1})$$

which shows that this characterizes the unique minimizer $\hat{\gamma}_*$ of the strictly convex function $\Psi(\cdot,\gamma_*^{N+1})$ on the convex set $K(\gamma_*,\gamma_*^{N+1})$ (this can also be seen as a special case of Karush-Kuhn-Tucker theorem). We deduce

$$(15.12)\qquad \begin{cases} \hat{\gamma}^\delta_* - \gamma^{N+1}_*\theta^\delta = \lambda^\alpha_* DL_{|V^\delta} + \mu^\delta_* \ge \lambda^\alpha_* DL_{|V^\delta} \\ \hat{\gamma}^\beta_* - \gamma^{N+1}_*\theta^\beta = \lambda^\alpha_* DL_{|V^\beta} - \mu^\beta_* \le \lambda^\alpha_* DL_{|V^\beta} \\ \hat{\gamma}^\alpha_* - \gamma^{N+1}_*\theta^\alpha = \lambda^\alpha_* DL_{|V^\alpha} \end{cases}$$

Step 4.1.2: proof that $I^\delta = \emptyset$ and $\lambda^\alpha_* \ge 0$
The fact that $\hat{\gamma}^\delta_* = 0$ implies with the first line of (15.12) that $I^\delta = \emptyset$ or $\lambda^\alpha_* < 0$. Assume by contradiction that $I^\delta \ne \emptyset$ and then that $\lambda^\alpha_* < 0$.
Case A: $I^\beta \ne \emptyset$
Then the second and third lines of (15.12) show that $\gamma^{N+1}_*\theta^\beta > \hat{\gamma}^\beta_*$ in V^β, and $\gamma^{N+1}_*\theta^\alpha \ge \hat{\gamma}^\alpha_*$ in V^α. Then $\gamma^{N+1}_*\theta \ge \hat{\gamma}_*$ with $\gamma^{N+1}_*\theta \ne \hat{\gamma}_*$. Hence $\gamma^{N+1}_* = L(\gamma^{N+1}_*\theta) \ge L\hat{\gamma}_* = \gamma^{N+1}_*$ with $L(\gamma^{N+1}_*\theta) \ne L\hat{\gamma}_*$. Contradiction.
Case B: $I^\beta = \emptyset$
Then the third line of (15.12) shows that

$$(15.13)\qquad \begin{aligned} \lambda^\alpha_*|I^\alpha| = \ & = L(\hat{\gamma}^\alpha_* - \gamma^{N+1}_*\theta^\alpha) \\ & = \left\{\gamma^{N+1}_* - L(\hat{\gamma}^\beta_* + \hat{\gamma}^\delta_*)\right\} - \gamma^{N+1}_* L\theta^\alpha \\ & = L(\gamma^{N+1}_*(\theta^\beta + \theta^\delta) - \gamma^\beta_*) \\ & = \gamma^{N+1}_* L\theta^\delta > 0 \end{aligned}$$

where in the second line we have used $L\hat{\gamma}_* = \gamma^{N+1}_*$, in the third line we have used $L\theta = 1$, $\hat{\gamma}^\delta_* = 0$, $\hat{\gamma}^\beta_* = \gamma^\beta_*$, and in the last line we have used $I^\beta = \emptyset$ and $I^\delta \ne \emptyset$. Contradiction with $\lambda^\alpha_* < 0$.
Conclusion
We conclude that $\lambda^\alpha_* \ge 0$ and $I^\delta = \emptyset$.
Step 4.1.3: proof that $I^\alpha \ne \emptyset$
Assume by contradiction that $I^\alpha = \emptyset$. Then this implies that $\gamma^{N+1}_*\theta \ge \hat{\gamma}_* = \gamma_*$ and then $\gamma^{N+1}_* = L\gamma^{N+1}_*\theta \ge L\gamma_*$. Contradiction with $L\gamma_* > \gamma^{N+1}_*$. Hence $I^\alpha \ne \emptyset$. Setting $e^\alpha := |I^\alpha|^{-1} \sum_{j \in I^\alpha} e_j$ which satisfies $Le^\alpha = 1$, we deduce from the third line of (15.12) that

$$(15.14)\qquad \hat{\gamma}^\alpha_* = \gamma^{N+1}_*\theta^\alpha + \lambda_* e^\alpha \quad \text{with} \quad 0 \le \lambda_* := \lambda^\alpha_*|I^\alpha| = L(\gamma^{N+1}_*\theta^\beta - \gamma^\beta_*)$$

where in the last equality, we have used the third line of (15.13) with $I^\delta = \emptyset$. Hence we get

$$(15.15)\qquad \hat{\gamma}(\gamma_*, \gamma^{N+1}_*) = \hat{\gamma}_* = \gamma^{N+1}_*(\theta^\alpha + (L\theta^\beta)e^\alpha) - (L\gamma^\beta_*)e^\alpha + \gamma^\beta_*$$

From (15.11), we also have

$$(15.16)\qquad \hat{\gamma}_* - \gamma^{N+1}_*\theta = \lambda^\alpha_* DL - \mu^\beta_* \quad \text{with} \quad \begin{cases} 0 < \hat{\gamma}^\alpha_* < \gamma^\alpha_* \quad \text{in} \quad V^\alpha \\ \hat{\gamma}^\beta_* = \gamma^\beta_* \quad \text{and} \quad \mu^\beta_* \ge 0 \end{cases}$$

which characterizes $\hat{\gamma}_* \in K(\gamma_*, \gamma^{N+1}_*)$.

We now discuss a complement of independent interest. Because $\hat{\gamma}^\alpha_* \le \gamma^\alpha_*$ and $\lambda_* \ge 0$, we deduce from (15.14) that $\gamma^{N+1}_*\theta^\alpha \in \Gamma(\gamma^\alpha_*)$. Now assume by contradiction that there exists $j \in I^\alpha$ such that $\gamma^{N+1}_*\theta^j = \gamma^j_*$. Then $\lambda_* \ge 0$ implies that $\hat{\gamma}^j_* = \gamma^j_*$. Contradiction with the definition of I^α. Hence we get moreover $\gamma^{N+1}_*\theta^\alpha < \gamma^\alpha_*$ in V^α.
Step 4.2: Generic case and variations
Now consider $\gamma^{N+1} := \gamma^{N+1}_* + q$ with $q \in (0, +\infty)$. Following (15.15), let us consider as a candidate for

$\hat{\gamma}(\gamma_*, \gamma_*^{N+1} + q)$, the vector $\xi := \hat{\gamma}_* + q\left\{\theta^\alpha + (L\theta^\beta)e^\alpha\right\}$. Following (15.16), we now compute

$$\begin{aligned}
\xi - \gamma^{N+1}\theta &= \left\{\hat{\gamma}_* - \gamma_*^{N+1}\theta\right\} + q\left\{(L\theta^\beta)e^\alpha - \theta^\beta\right\} \\
&= \left\{\lambda_*^\alpha DL - \mu_*^\beta\right\} + q\left\{(L\theta^\beta)e^\alpha - \theta^\beta\right\} \\
&= \left\{\lambda_*^\alpha DL - \mu_*^\beta\right\} + q\left\{|I^\alpha|^{-1}(L\theta^\beta)\sum_{j\in I^\alpha} e_j - \theta^\beta\right\} \\
&= \left\{\lambda_*^\alpha DL - \mu_*^\beta\right\} + q\left\{|I^\alpha|^{-1}(L\theta^\beta)\left\{DL - \sum_{j\in I^\beta} e_j\right\} - \theta^\beta\right\} \\
&= \lambda^\alpha DL - \mu^\beta \quad \text{with} \quad \left\{\begin{array}{l} \lambda^\alpha = \lambda_*^\alpha + q|I^\alpha|^{-1}L\theta^\beta \geq 0 \\ \mu^\beta = \mu_*^\beta + q\left\{\theta^\beta + |I^\alpha|^{-1}(L\theta^\beta)\displaystyle\sum_{j\in I^\beta} e_j\right\} \geq 0 \quad \text{and} \quad \xi^\beta = \gamma_*^\beta \end{array}\right.
\end{aligned}$$

which satisfies $\xi^\alpha < \gamma_*^\alpha$ for $q > 0$ small enough. This shows that

$$\xi - \gamma^{N+1}\theta = \lambda^\alpha DL - \mu^\beta \quad \text{with} \quad \left\{\begin{array}{l} 0 < \xi^\alpha < \gamma_*^\alpha \quad \text{in} \quad V^\alpha \\ \xi^\beta = \gamma_*^\beta \quad \text{and} \quad \mu^\beta \geq 0 \end{array}\right.$$

As in (15.11), this relation characterizes $\hat{\gamma}(\gamma_*, \gamma^{N+1}) = \xi$. Because $\xi \geq \hat{\gamma}_*$, we deduce that $\hat{\gamma}(\gamma_*, \gamma^{N+1}) \geq \hat{\gamma}(\gamma_*, \gamma_*^{N+1})$. Hence for each $j = 1, \ldots, N$, this shows that the map $\gamma^{N+1} \mapsto \hat{\gamma}^j(\gamma_*, \gamma^{N+1})$ is locally nondecreasing.

Step 4.3: General case

Let $\tilde{\gamma}_* = (\gamma_*, \gamma_*^{N+1}) \in [0, +\infty)^{N+1}$ such that $L\gamma_* > \gamma_*^{N+1}$. Using the continuity of $\hat{\gamma}$, we get (by perturbations) that $\hat{\gamma}^j$ is nondecreasing in γ^{N+1} on $\tilde{Q}_\varepsilon(\tilde{\gamma}_*)$ for all $j = 1, \ldots, N$. With the other monotonicities, we deduce that $\tilde{\hat{\gamma}}$ is $\tilde{\sigma}$-monotone on $\tilde{Q}_\varepsilon(\tilde{\gamma}_*)$.

This ends the proof of the lemma.

Remark 15.9 *Notice that Step 4 of the proof of Lemma 15.7 provides a proof of the monotonicity of the map $\gamma^{N+1} \mapsto \hat{\gamma}(\gamma_*, \gamma^{N+1})$ stated in Lemma 4 in [25], with more precise justifications.*

The next result presents the preflux used by Garavello and Piccoli for $\mathcal{RS}_2$ in [25].

Lemma 15.10 (Data Networks preflux)

Let $N \geq 1$ and $\sigma = (\sigma^1, \ldots, \sigma^N) \in \{\pm 1\}^N$ with $\sigma \neq (1, \ldots, 1)$ and $\sigma \neq (-1, \ldots, -1)$. For $\gamma = (\gamma^1, \ldots, \gamma^N) \in [0, +\infty)^N$, define

$$L^+(\gamma) := \sum_{\sigma^j = 1} \gamma^j, \quad L^-(\gamma) := \sum_{\sigma^j = -1} \gamma^j$$

Let $\theta \in (0,1)^N$ such that $L^+(\theta) = 1 = L^-(\theta)$, fix some $\bar{\gamma} = (\bar{\gamma}^1, \ldots, \bar{\gamma}^N) \in [0, +\infty)^N$, and define the passing flux

$$\bar{\gamma}_0(\bar{\gamma}) := \bar{\gamma}_0 := \min\left\{L^+(\bar{\gamma}), L^-(\bar{\gamma})\right\}$$

and

$$\left\{\begin{array}{ll} K^+(\bar{\gamma}) := \{\gamma \in \Gamma^+(\bar{\gamma}), \quad L^+(\gamma) = \bar{\gamma}_0\}, & \textit{with} \quad \Gamma^+(\bar{\gamma}) := \displaystyle\sum_{\sigma^j = 1} [0, \bar{\gamma}^j] e_j \\ K^-(\bar{\gamma}) := \{\gamma \in \Gamma^-(\bar{\gamma}), \quad L^-(\gamma) = \bar{\gamma}_0\}, & \textit{with} \quad \Gamma^-(\bar{\gamma}) := \displaystyle\sum_{\sigma^j = -1} [0, \bar{\gamma}^j] e_j \end{array}\right.$$

Let us consider the convex compact set

$$K(\bar{\gamma}) := \left\{\gamma \in [0, +\infty)^N, \quad \gamma = \gamma^+ + \gamma^-, \quad (\gamma^+, \gamma^-) \in K^+(\bar{\gamma}) \times K^-(\bar{\gamma})\right\}$$

and the orthogonal projection on it

$$\hat{\gamma}^{DN}(\bar{\gamma}) := Proj^{\perp}_{|K(\bar{\gamma})}(\bar{\gamma}_0(\bar{\gamma}) \cdot \theta)$$

Then $\hat{\gamma}^{DN} : [0, +\infty)^N \to [0, +\infty)^N$ is a conservative σ-monotone preflux in the sense of Definition 11.1.

Remark 15.11 *The convergent junction case in Section 5.2.2 of* GARAVELLO, PICCOLI *[24] is a special case of data networks (Riemann solver $\mathcal{RS}_2$ in [25]), which corresponds to $N = n+1$ with n ingoing branches $\sigma^j = 1$ for $j = 1, \dots, n$ and a single outgoing branch $\sigma^{n+1} = -1$.*
For $N = 1+n$ and the special case $n = 2$, such solver has originally been introduced by Daganzo [17] (with an equivalent formulation, see Fig. 4 and (7a)-(7b)-(7c) in [17]).

Proof of Lemma 15.10
Step 1: preflux
By construction, the function $\hat{\gamma} := \hat{\gamma}^{DN}$ is continuous and conservative.
Moreover, we see that

$$\hat{\gamma} = \hat{\gamma}^+ + \hat{\gamma}^- \quad \text{with} \quad \begin{cases} \hat{\gamma}^+ := \mathrm{Proj}^{\perp}_{|K^+(\bar{\gamma})}(\bar{\gamma}_0 \theta^+) & \text{with} \quad \theta^+ := \displaystyle\sum_{\sigma^j = 1} \theta^j e_j \\ \hat{\gamma}^- := \mathrm{Proj}^{\perp}_{|K^-(\bar{\gamma})}(\bar{\gamma}_0 \theta^-) & \text{with} \quad \theta^- := \displaystyle\sum_{\sigma^j = -1} \theta^j e_j \end{cases}$$

Step 1.1: bounds
By construction, we have $\hat{\gamma}^{\pm} \in K^{\pm}(\bar{\gamma}) \subset \Gamma^{\pm}(\bar{\gamma})$ and then $0 \le \hat{\gamma}^j(\bar{\gamma}) \le \bar{\gamma}^j$ for all indices j.
Step 1.2: local constancy
For $\gamma \in [0, +\infty)^N$, we consider the general splitting

$$\gamma = \gamma^+ + \gamma^- \quad \text{with} \quad (\gamma^+, \gamma^-) \in \Gamma^+(\gamma) \times \Gamma^-(\gamma)$$

For some $\bar{\gamma}_* \in [0, +\infty)^N$, assume that $I := \left\{ j \in \{1, \dots, N\}, \ \hat{\gamma}^j(\bar{\gamma}_*) < \bar{\gamma}_*^j \right\} \neq \emptyset$. Assume by contradiction that there exists $j, k \in I$ such that $\sigma^j = 1$ and $\sigma^k = -1$. Then this implies that $L^{\pm}\hat{\gamma}(\bar{\gamma}_*) < L^{\pm}\bar{\gamma}_*$, and then $L^{\pm}\hat{\gamma}(\bar{\gamma}_*) < \bar{\gamma}_0(\bar{\gamma}_*) := \min\{L^+\bar{\gamma}_*, L^-\bar{\gamma}_*\}$, i.e. $\hat{\gamma}^{\pm}(\bar{\gamma}_*) \notin K^{\pm}(\bar{\gamma}_*)$. Contradiction. Therefore we conclude that we have either $\sigma^j = 1$ for all indices $j \in I$, or $\sigma^j = -1$ for all indices $j \in I$.

Without loss of generality, let us assume that $\sigma^j = 1$ for all $j \in I$ (the case $\sigma^j = -1$ is similar). Then $L^+\bar{\gamma}_*^+ > L^+\hat{\gamma}_*^+ = L^-\bar{\gamma}_*^- = \bar{\gamma}_0(\bar{\gamma}_*)$. Therefore $K^-(\bar{\gamma}_*) = \{\bar{\gamma}_*^-\}$ and $\hat{\gamma}^-(\bar{\gamma}_*) = \bar{\gamma}_*^-$ Moreover for $\varepsilon > 0$, consider

$$Q_\varepsilon(\bar{\gamma}_*) := \left(\bar{\gamma}_* + \sum_{j \in I} (-\varepsilon, \varepsilon) e_j \right) \cap [0, +\infty)^N$$

Then for $\varepsilon > 0$ small enough and $\bar{\gamma} \in Q_\varepsilon(\bar{\gamma}_*)$, we have $\bar{\gamma}_0(\bar{\gamma}) = \bar{\gamma}_0(\bar{\gamma}_*)$. Therefore we still have $K^-(\bar{\gamma}) = \{\bar{\gamma}^-\} = \{\bar{\gamma}_*^-\}$, and then $\hat{\gamma}^-(\bar{\gamma}) = \bar{\gamma}_*^-$. Because $\hat{\gamma}^+$ is the orthogonal projection on $K^+(\bar{\gamma})$ of $\bar{\gamma}_0\theta^+$, we deduce that

$$\hat{\gamma}^+(\bar{\gamma}) := \underset{K^+(\bar{\gamma})}{\mathrm{Argmin}} \ \Psi(\cdot, \bar{\gamma}_0) \quad \text{with} \quad \Psi(y, \bar{\gamma}_0) := |y - \bar{\gamma}_0\theta^+|^2 \tag{15.17}$$

Because $\bar{\gamma}_0(\bar{\gamma}) = \bar{\gamma}_0(\bar{\gamma}_*) =: \bar{\gamma}_0^*$ is fixed, we can use Lemma 15.4, and the fact that $\Psi(\cdot, \bar{\gamma}_0^*)$ is a sum of strictly convex functions in separated variables, to deduce that for $\bar{\gamma}$ locally around $\bar{\gamma}_*$, the map $\bar{\gamma}^+ \mapsto \hat{\gamma}^+(\bar{\gamma}^+ + \bar{\gamma}_*^-) = \hat{\gamma}^+(\bar{\gamma})$ is a preflux. Therefore the map $\bar{\gamma} \mapsto \hat{\gamma}(\bar{\gamma}) = (\hat{\gamma}^+(\bar{\gamma}), \bar{\gamma}_*^-)$ is locally constant in $\{\bar{\gamma}, \ \hat{\gamma}(\bar{\gamma}) \neq \bar{\gamma}\}$ close to $\bar{\gamma}_*$. This is the desired local constancy of $\hat{\gamma}$.
Step 1.3: basic monotonicities and σ-monotonicity
Notice from Lemma 11.4 that basic monotonicities are automatically satisfied for prefluxes. Here we prove it directly and also prove more.
Consider $\bar{\gamma}_*, \bar{\gamma} \in [0, +\infty)^N$, such that for some index j, say with $\sigma^j = 1$ (the case $\sigma^j = -1$ is similar), we have

$$\bar{\gamma} - \bar{\gamma}_* = q e_j \quad \text{with} \quad q \in (0, +\infty) \tag{15.18}$$

Case A: $L^+\bar{\gamma}_* \ge L^-\bar{\gamma}_*$
Then $L^+\bar{\gamma}_* \ge L^-\bar{\gamma}_* = \bar{\gamma}_0(\bar{\gamma}_*)$. Hence $L^+\bar{\gamma} > L^-\bar{\gamma} = L^-\bar{\gamma}_* = \bar{\gamma}_0(\bar{\gamma})$. Therefore as in Step 1.2, we get (15.17), and Lemma 15.4 shows that $\bar{\gamma}^+ \mapsto \hat{\gamma}^+(\bar{\gamma}^+ + \bar{\gamma}_*^-)$ is a $(1, \dots, 1)$-monotone preflux. In particular, for $\sigma^k = 1$, the component $\sigma^k\hat{\gamma}^k$ is nonincreasing in $\sigma^j\bar{\gamma}^j$ for $k \neq j$, and nondecreasing in $\sigma^j\bar{\gamma}^j$ for $k = j$. Moreover, we have $K^-(\bar{\gamma}) = \{\bar{\gamma}^-\} = \{\bar{\gamma}_*^-\}$, and then $\hat{\gamma}^-(\bar{\gamma}) = \bar{\gamma}_*^-$. Hence for $\sigma^k = -1$, each component $\sigma^k\hat{\gamma}^k$ is constant in $\sigma^j\bar{\gamma}^j$, and it is in particular nonincreasing.

Case B: $L^+\bar{\gamma}_* < L^-\bar{\gamma}_*$
Then $\bar{\gamma}_0^* := \bar{\gamma}_0(\bar{\gamma}_*) = L^+\bar{\gamma}_* < L^-\bar{\gamma}_*$, and then $K^+(\bar{\gamma}_*) = \{\bar{\gamma}_*^+\}$. Then for $q > 0$ small enough such that $\bar{\gamma}_0^* < \bar{\gamma}_0(\bar{\gamma}) = L^+\bar{\gamma} < L^-\bar{\gamma} = L^-\bar{\gamma}_*$, we have $K^+(\bar{\gamma}) = \{\bar{\gamma}^+\}$ and $\hat{\gamma}^+(\bar{\gamma}) = \bar{\gamma}^+$ and in particular $\hat{\gamma}^k(\bar{\gamma}) = \bar{\gamma}^k = \bar{\gamma}_*^k$ for all k such that $\sigma^k = 1$ (for $k \neq j$ and for $k = j$). For $q > 0$ not small, notice that there exists some unique $q_* > 0$, such that we get equality $\bar{\gamma}_0(\bar{\gamma}_* + q_* e_j) = L^+(\bar{\gamma}_* + q_* e_j) = L^-\bar{\gamma}_*$, and then we are back to Case A with $\bar{\gamma}_*$ replaced by $\bar{\gamma}_* + q_* e_j$.

Now notice that $\hat{\gamma}^-$ is the orthogonal projection on $K^-(\bar{\gamma})$ of $\bar{\gamma}_0\theta^-$, and we deduce that
(15.19)
$$\hat{\gamma}^-(\bar{\gamma}) := \underset{K^-(\bar{\gamma})}{\text{Argmin}}\, \Psi(\cdot, \bar{\gamma}_0(\bar{\gamma})) \quad \text{with} \quad \Psi(y, \bar{\gamma}_0) := |y - \bar{\gamma}_0\theta^-|^2 \quad \text{with} \quad K^-(\bar{\gamma}) = \left\{\gamma \in \Gamma^-(\bar{\gamma}), \quad L^-\gamma = L^-(\bar{\gamma}_0\theta^-)\right\}$$

with $L^-(\bar{\gamma}_0\theta^-) = \bar{\gamma}_0 = L^+\bar{\gamma} = L^+\bar{\gamma}^+$. Then Lemma 15.7 shows that the map $(\bar{\gamma}^-, \bar{\gamma}_0) \mapsto (\hat{\gamma}^-(\bar{\gamma}), \bar{\gamma}_0)$ is a $(1, 1, \ldots, 1, -1)$-monotone preflux. In particular each component of $\hat{\gamma}^-$ is nondecreasing in the agglomerated variable $\bar{\gamma}_0(\bar{\gamma}) = L^+\bar{\gamma}^+$. Therefore $\sigma^k\hat{\gamma}^k$ is nonincreasing in $\sigma^j\bar{\gamma}^j$ for all k such that $\sigma^k = -1$. This shows that $\hat{\gamma}$ is σ-monotone, and has the basic monotonicities ($\sigma^j\hat{\gamma}^j$ is nondecreasing in $\sigma^j\bar{\gamma}^j$). This ends the proof of the lemma.

Corollary 15.12 (Data Networks germ; $\mathcal{R}S_2$ in [25])
Assume (2.2) for some $n : m$ junction (J, f) with $N = n + m$ and $n, m \geq 1$. Recall that $\sigma \in \{\pm 1\}^N$ and σ^j denotes the orientation of the branch J^j. Assume that $\sigma \neq (1, \ldots, 1)$ and $\sigma \neq (-1, \ldots, -1)$. Assume also that f is bell-shaped, and call $\bar{\gamma} : [a, b] \to [0, +\infty)^N$ the capacity $\bar{\gamma}^j := f^{j,\sigma^j}$ given in Definition 11.6.
Then the set

$$\mathcal{G} = \mathcal{G}_{\hat{f}} := \left\{p \in [a, b], \quad \hat{f}(p) = f(p)\right\} \quad \text{with} \quad \hat{f}(p) := \hat{\gamma}^{DN} \circ \bar{\gamma} \tag{15.20}$$

is a conservative Kružkov germ, where $\hat{\gamma}^{DN}$ is the data network preflux given in Lemma 15.10.

Proof of Corollary 15.12
From Lemma 15.10, we know that the preflux $\hat{\gamma}^{DN}$ is conservative and σ-monotone. This implies that $\mathcal{G}$ is conservative monotone germ, hence conservative Kružkov. This ends the proof of the corollary.

In [25], the authors provide an important result that gives us an enlighting heuristic for the proof of Corollary 15.12. We now state this result.

Lemma 15.13 (A contraction result, Lemma 6 in [25])
We work under the assumptions of Corollary 15.12, with furthermore the specific case of $f^j = f^0$ for $j = 1, \ldots, N$. For $\mathcal{G}$ defined in (15.20), and for all fixed $j \in \{1, \ldots, N\}$, all $\hat{p} \in \mathcal{G}$, and $q \in [a, b]$ such that $q - \hat{p} \in \mathbb{R}^* e_j$ satisfies

$$\sigma^j c^j \geq 0 \quad \text{with} \quad c^j := \frac{f^j(q^j) - f^j(\hat{p}^j)}{q^j - \hat{p}^j} \quad \text{and} \quad \sigma^j = \begin{cases} 1 & \text{for ingoing road} \\ -1 & \text{for outgoing road} \end{cases}$$

then we have (recalling that $\hat{f} = f \circ \pi_{\mathcal{G}}$ and $\pi_{\mathcal{G}} \circ \pi_{\mathcal{G}} = \pi_{\mathcal{G}}$)

$$|\hat{f}^j(q) - f^j(q^j)| + \sum_{k \in \{1,\ldots,N\}\backslash\{j\}} |\hat{f}^k(q) - \hat{f}^k(\hat{p})| = |\hat{f}^j(\hat{p}) - f^j(q^j)|$$

Heuristic motivating the proof of Corollary 15.12
We now propose an heuristic to motivate the proof of Corollary 15.12 in the special case $f^j = f^0$ for $j = 1, \ldots, N$, using Lemma 15.13. Using the fact that the map $x \mapsto |x - c|$ is 1-Lipschitz, we deduce that $|x - c| - |y - c| \leq |x - y|$ and get from Lemma 15.13 that $\displaystyle\sum_{k \in \{1,\ldots,N\}\backslash\{j\}} |\hat{f}^k(q) - \hat{f}^k(\hat{p})| \leq |\hat{f}^j(q) - \hat{f}^j(\hat{p})|$.
In the limit $q \to \hat{p}$, this gives formally $\displaystyle\sum_{k \in \{1,\ldots,N\}\backslash\{j\}} |\partial_j \hat{f}^k| \leq |\partial_j \hat{f}^j|$ at $\hat{p}$ when $\sigma^j (f^j)'(\hat{p}^j) \geq 0$. Using vii) of Proposition 2.14, if we know that $\hat{f}$ is a Godunov flux associated to a Riemann germ $\mathcal{G}$, then we expect $\sigma^j \partial_j \hat{f}^j(\hat{p}) \in \left\{0, \max\left\{0, \sigma^j (f^j)'(\hat{p}^j)\right\}\right\}$. Then, in the very best case, we may expect to have on $\mathcal{G}$

$$\sum_{k \in \{1,\ldots,N\}\backslash\{j\}} |\partial_j \hat{f}^k| \leq \sigma^j \partial_j \hat{f}^j$$

Because any Godunov flux $\hat{f}$ is locally constant on $\left\{\hat{f} \neq f\right\}$, we may expect that this inequality is indeed true not only on $\mathcal{G}$, but a.e. on $[a, b]$. If it is the case, then i) of Theorem 2.26 implies that $\mathcal{G}$ is a Kružkov germ. Because by construction $\mathcal{G}$ is conservative, we get that $\mathcal{G}$ is a conservative Kružkov germ. This ends the heuristic.

Remark 15.14 (Comments on other Riemann solvers in [25])
In [25], Riemann solver $\mathcal{RS}_3$ is not associated to a (conservative) Kružkov germ, because $\Gamma_i := \min(\gamma_i^{max}, \gamma_{i+n}^{max}) = \min(\bar{\gamma}^i, \bar{\gamma}^{i+n})$ is symmetric in ingoing index i and outgoing index $i+n$, which is not compatible with σ-monotonicity of the associated preflux for $\sigma^i = 1$ and $\sigma^{i+n} = -1$.

Similarly in [25], Riemann solver $\mathcal{RS}_1$ is not associated to a (conservative) Kružkov germ for some $n : m$ junction with $1 \leq n \leq m$ and $m \geq 2$. Indeed if the ingoing branches $j = 2, \ldots, n$ are empty with $\bar{\gamma}^j = 0$, then the problem is simply described by some $1 : m$ junction with effective passing flux equal to $\min\limits_{k=n+1,\ldots,n+m} \left\{\bar{\gamma}^1, \frac{\bar{\gamma}^k}{\theta^k}\right\}$ where $\theta^k := \alpha_{1k}$ and where $\bar{\gamma}$ is the capacity. As explained in Section 2.3 of [30], this corresponds to a (quasi) HJ problem (hence (quasi) HJ germ) for functions $f^1 := \frac{f^0}{1}$ and $f^k = \frac{f^0}{\theta^k}$. But monotonicities of the Godunov flux for (quasi) HJ germs (with $\hat{f}^k = h(\uparrow, \downarrow, \ldots, \downarrow) = h(p^1, p^{n+1}, \ldots, p^{n+m})$) and for monotone germs (with $\partial_{p^{k'}}(\sigma^k \hat{f}^k) \leq 0$ for $k' \neq k$, and $\partial_{p^k}(\sigma^k \hat{f}^k) \geq 0$) are indeed incompatible for $m \geq 2$.

15.1.4 Traffic Lights germs revisited

We revisit the traffic light germs discovered in Cardaliaguet, Forcadel, Monneau [11] (see also Towers [46] for a special case), and give as a new result the explicit expression of their associated Godunov flux.

Consider a divergent junction $1 : 2$ with bell-shaped function $f = (f^0; f^1, f^2)$ in the sense of Definition 11.6 (with definitions of $f^{j,\pm}$), with ingoing flux f^0 and outgoing fluxes f^1, f^2. We consider the following assumptions

$$(15.21) \qquad \left\{ \begin{array}{lll} \bar{\lambda}^j \in [0, f^j_{max}] & & \text{for} \quad j = 0, 1, 2 \\ \bar{\lambda}^0 = \bar{\lambda}^1 + \bar{\lambda}^2 & & \\ \text{the maps} \quad \hat{\theta}^k : [0, f^0_{max}] \to [0, \bar{\lambda}^k] & \text{are continuous nondecreasing} & \text{for} \quad k = 1, 2 \\ \hat{\theta}^k(0) = 0, \quad \hat{\theta}^k(\bar{\lambda}^0) = \bar{\lambda}^k & & \text{for} \quad k = 1, 2 \\ \hat{\theta}^1(\lambda) + \hat{\theta}^2(\lambda) = \min(\lambda, \bar{\lambda}^0) & & \text{for} \quad \lambda \in [0, f^0_{max}] \end{array} \right.$$

For a divergent junction $1 : 2$, and for $\Lambda := (\bar{\lambda}, \hat{\theta}) = (\bar{\lambda}^0, \bar{\lambda}^1, \bar{\lambda}^2, \hat{\theta}^1, \hat{\theta}^2)$, we consider the following Traffic Light germ, which is known to be a Kružkov germ (see [11])

$$(15.22) \qquad \mathcal{G}^{1:2}_{\Lambda} := \left\{ U = (u^0, u^1, u^2) \in \mathbb{R}^3, \quad \left| \begin{array}{ll} a^j \leq u^j \leq b^j, & j = 0, 1, 2 \\ 0 \leq f^j(u^j) \leq \bar{\lambda}^j, & j = 0, 1, 2 \\ f^0(u^0) = f^1(u^1) + f^2(u^2) & \\ f^{k,+}(u^k) \geq \hat{\theta}^k(f^{0,+}(u^0)), & k = 1, 2 \end{array} \right. \right\}$$

Lemma 15.15 (Godunov flux of Traffic Light germ $\mathcal{G}^{1:2}_{\Lambda}$)
For $N = 3$, assume (2.2) for some $1 : 2$ junction (J, f) with $f = (f^0, f^1, f^2)$, with incoming branch denoted by J^0 ($\sigma^0 = 1$) and outgoing branches J^k ($\sigma^k = -1$) for indices $k = 1, 2$. Assume also that f is bell-shaped in the sense of Definition 11.6, and call $\bar{\gamma} : [a, b] \to [0, +\infty)^N$ the capacity $\bar{\gamma}^j := f^{j,\sigma^j}$ given in Definition 11.6. We set $f^j_{max} := f^j(c^j)$ for $j = 0, 1, 2$ and assume (15.21) with $\bar{\lambda} = (\bar{\lambda}^0, \bar{\lambda}^1, \bar{\lambda}^2) \in [0, +\infty)^N$ and

$$T_{\bar{\lambda}} : [0, +\infty)^N \to [0, +\infty)^N, \quad \text{with} \quad T_{\bar{\lambda}}(\gamma) = \left(\min\left\{\gamma^0, \bar{\lambda}^0\right\}, \min\left\{\gamma^1, \bar{\lambda}^1\right\}, \min\left\{\gamma^2, \bar{\lambda}^2\right\}\right)$$

We define $\hat{\gamma}_0 = (\hat{\gamma}^0_0, \hat{\gamma}^1_0, \hat{\gamma}^2_0)$ for $\gamma = (\gamma^0, \gamma^1, \gamma^2) \in [0, +\infty)^N$

$$(15.23) \qquad \left\{ \begin{array}{ll} \hat{\gamma}^1_0(\gamma) & = \min\left\{\gamma^1, \max\left\{\hat{\theta}^1(\gamma^0), \gamma^0 - \gamma^2\right\}\right\}, \\ \hat{\gamma}^2_0(\gamma) & = \min\left\{\gamma^2, \max\left\{\hat{\theta}^2(\gamma^0), \gamma^0 - \gamma^1\right\}\right\}, \\ \hat{\gamma}^0_0 & = \hat{\gamma}^1_0 + \hat{\gamma}^2_0 \end{array} \right.$$

Then

$$\mathcal{G} = \mathcal{G}_{\hat{f}} := \left\{ p \in [a,b], \quad \hat{f}(p) = f(p) \right\} \quad \text{with} \quad \hat{f} := \hat{\gamma} \circ \bar{\gamma} : [a,b] \to [0,+\infty)^N \quad \text{with} \quad \hat{\gamma} := \hat{\gamma}_0 \circ T_{\bar{\lambda}}$$

is a conservative Kružkov germ $\mathcal{G} \subset [a,b]$ *with respect to* (J,f)*, and* $\hat{\gamma}_0$ *and* $\hat{\gamma}$ *are prefluxes. Moreover we have* $\mathcal{G} = \mathcal{G}^{1:2}_{\Lambda}$.
We furthermore have the simple expression

$$\hat{\gamma}^0_0(\gamma) = \min\left\{\gamma^0, \gamma^1 + \gamma^2\right\} \quad \text{for} \quad \gamma^0 \in [0, \bar{\lambda}^0] \tag{15.24}$$

Proof of Lemma 15.15
Step 1: properties of $\hat{\gamma}$
Step 1.1: $\hat{\gamma}_0$ is a conservative preflux
We first notice that by construction, we have the function $\hat{\gamma}_0 : [0,+\infty)^N \to [0,+\infty)^N$ is continuous, and is conservative because $\hat{\gamma}^0_0 = \hat{\gamma}^1_0 + \hat{\gamma}^2_0$.
Step 1.2: preliminaries
We consider the following four sets ($\hat{T}$ like triangle, $\hat{Q}$ like sQuare)

$$\begin{cases} \hat{T}_0 & := \left\{\gamma \in [0,+\infty)^N, \quad \gamma^1 + \gamma^2 \le \gamma^0\right\} \\ \hat{T}_1 & := \left\{\gamma \in [0,+\infty)^N, \quad \gamma^1 + \gamma^2 \ge \gamma^0, \quad \gamma^1 \le \hat{\theta}^1(\gamma^0)\right\} \\ \hat{T}_2 & := \left\{\gamma \in [0,+\infty)^N, \quad \gamma^1 + \gamma^2 \ge \gamma^0, \quad \gamma^2 \le \hat{\theta}^2(\gamma^0)\right\} \\ \hat{Q} & := \left\{\gamma \in [0,+\infty)^N, \quad \gamma^k \ge \hat{\theta}^k(\gamma^0), \quad k = 1,2\right\} \end{cases}$$

and we have (using the fact that $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for $x \vee y := \max\{x,y\}$ and $x \wedge y = \min\{x,y\}$)

$$(\hat{\gamma}^1_0, \hat{\gamma}^2_0) = \begin{cases} (\gamma^1, & \gamma^2) & \text{if} \quad \gamma \in \hat{T}_0 \\ (\gamma^1, & \gamma^0 - \gamma^1) & \text{if} \quad \gamma \in \hat{T}_1 \\ (\gamma^0 - \gamma^2, & \gamma^2) & \text{if} \quad \gamma \in \hat{T}_2 \\ (\hat{\theta}^1(\gamma^0), & \hat{\theta}^2(\gamma^0)) & \text{if} \quad \gamma \in \hat{Q} \quad \text{and} \quad \gamma^0 \le \bar{\lambda}^0 \\ & & \\ (\gamma^1, & \gamma^2) & \text{if} \quad \gamma \in \hat{Q} \quad \text{and} \quad \gamma^0 > \bar{\lambda}^0 \quad \text{and} \quad \gamma^1 + \gamma^2 \le \gamma^0 \\ (\gamma^0 - \gamma^2, & \gamma^0 - \gamma^1) & \text{if} \quad \gamma \in \hat{Q} \quad \text{and} \quad \gamma^0 > \bar{\lambda}^0 \quad \text{and} \quad \gamma^1 + \gamma^2 \ge \gamma^0 \end{cases} \tag{15.25}$$

Step 1.3: bounds
By construction, we have $\hat{\gamma}_0 \ge 0$ and $\hat{\gamma}^k_0(\gamma) \le \gamma^k$ for $k = 1,2$. Moreover from (15.25), we get $\hat{\gamma}^0_0(\gamma) = \hat{\gamma}^1_0 + \hat{\gamma}^2_0 \le \gamma^0$.
Step 1.4: local constancy
We see that $\hat{\gamma}_0$ is not locally constant on $\left\{\hat{\gamma}_0 \ne id_{[0,+\infty)^N}\right\}$ for $(\hat{\gamma}^1_0, \hat{\gamma}^2_0) = (\gamma^0 - \gamma^2, \gamma^0 - \gamma^1)$. On the contrary the restriction of $\hat{\gamma}_0$ to the box $K := [0, \bar{\lambda}]$ avoids this behaviour, and we see easily from the four first lines of (15.25), and from the possible combinations, that $(\hat{\gamma}_0)_{|K}$ is locally constant on $\left\{(\hat{\gamma}_0)_{|K} \ne id_K\right\}$.
Step 1.5: basic monotonicity
It is easy to check that $\gamma \mapsto \hat{\gamma}^j_0(\gamma)$ is nondecreasing in γ^j for $j = 0,1,2$ (even if we know already from Lemma 11.4 that basic monotonicities are automatically satisfied for prefluxes).
Step 1.6: σ-monotonicity (for $1:2$ junctions)
In terms of $\gamma = (\gamma^0, \gamma^1, \gamma^2)$, we now want to check the σ-monotonicity of $\hat{\gamma}_0$, i.e. that $\sigma^j \hat{\gamma}^j_0$ is nonincreasing in $\sigma^k \gamma^k$ for $j \ne k$, i.e. the following monotonities (with indeed $* = \uparrow$)

$$\begin{cases} \hat{\gamma}^0_0(*, \uparrow, \uparrow) \\ \hat{\gamma}^1_0(\uparrow, *, \downarrow) \\ \hat{\gamma}^2_0(\uparrow, \downarrow, *) \end{cases}$$

which is the case.
Step 1.7: conclusion
We now conclude by composition (similarly to the proof of Lemma 11.13) that $\hat{\gamma} = (\hat{\gamma}_0)_{|K} \circ T_{\bar{\lambda}}$ is a conservative σ-monotone preflux.
Step 2: properties of $\hat{f}$

From Theorem 11.8, we deduce that $\mathcal{G}_{\hat{f}}$ is a monotone conservative Riemann germ, hence a conservative Kružkov germ (as follows from Theorem 2.29).

Step 3: identification of the germ

We now want to show that $\mathcal{G} := \mathcal{G}_{\hat{f}} = \mathcal{G}^{1:2}_\Lambda$. It is known from Lemma 1.5 in [11] that there is a generating set $E_\Lambda \subset \mathcal{G}^{1:2}_\Lambda$ such that $E_\Lambda \subset \mathcal{G}$ implies $\mathcal{G} = \mathcal{G}^{1:2}_\Lambda$. Here the generating set is given by $E_\Lambda := \Gamma \cup \{P_1, P_2, P_3\}$, with

$$\begin{cases} \Gamma := \left\{P(\gamma^0), \quad \gamma^0 \in [0,\bar\lambda^0]\right\} \quad \text{with} \quad P(\gamma^0) := (u^0_+(\gamma^0), u^1_+(\hat\theta^1(\gamma^0)), u^2_+(\hat\theta^2(\gamma^0))) \\ \\ P_1 := (u^0_-(\bar\lambda^1), u^1_+(\bar\lambda^1), u_-(0)) \\ P_2 := (u^0_-(\bar\lambda^2), u^1_-(0), u^2_+(\bar\lambda^2)) \\ P_3 := (u^0_-(0), u^1_-(0), u^2_-(0)) \end{cases}$$

and

$$f^j(u^j_\pm(\lambda)) = \lambda, \quad u^j_+(\lambda) \in [a^j, c^j], \quad u^j_-(\lambda) \in [c^j, b^j], \quad \lambda \in [0, f^j(c^j)]$$

Step 3.1: checking Γ

Notice that $\bar\gamma(P(\gamma^0)) = (\gamma^0, f^1_{max}, f^2_{max})$, and then for $\gamma^0 \in [0,\bar\lambda^0]$, we get $\hat f(P(\gamma^0)) = (\gamma^0, \hat\theta^1(\gamma^0), \hat\theta^2(\gamma^0)) = f(P(\gamma^0))$.

Step 3.2: checking P_1

We have $\bar\gamma(P_1) = (f^0_{max}, f^1_{max}, 0)$, and then $(T_K \circ \bar\gamma)(P_1) = (\bar\lambda^0, \bar\gamma^1, 0)$. Therefore $\hat f(P_1) = (\bar\lambda^1, \bar\lambda^1, 0) = f(P_1)$.

Step 3.3: checking P_2

Symmetrically, we get $\hat f(P_2) f(P_2)$.

Step 3.4: checking P_3

We have $\bar\gamma(P_3) = (f^0_{max}, 0, 0)$, and $(T_K \circ \bar\gamma)(P_3) = (\bar\lambda^0, 0, 0)$. Therefore $\hat f(P_3) = (0,0,0) = f(P_3)$.

We conclude that $E_\Lambda \subset \mathcal{G}$, and then $\mathcal{G} = \mathcal{G}^{1:2}_\Lambda$.

Step 4: checking (15.24)

Relation (15.24) follows from lemma 15.16 just below with $K := \gamma^0$, $\alpha := \gamma^1$, $\beta := \gamma^2$ and $\mu_\alpha := \frac{\hat\theta^1(\gamma^0)}{\gamma^0}$ and $\mu_\beta := \frac{\hat\theta^2(\gamma^0)}{\gamma^0}$, using the fact that $\mu_\alpha + \mu_\beta = 1$ for $\gamma^0 \le \bar\lambda^0$, as a consequence of (15.21).

This ends the proof of the lemma.

Lemma 15.16 (A remarkable equality)

For $K, \alpha, \beta \in \mathbb{R}$ and $\mu_\alpha, \mu_\beta \in [0,1]$ with $\mu_\alpha + \mu_\beta = 1$, we set

$$S := S(K,\alpha,\beta,\mu_\alpha,\mu_\beta) := \min\{\alpha, \max\{\mu_\alpha K, K-\beta\}\} + \min\{\beta, \max\{\mu_\beta K, K-\alpha\}\}$$

Then

$$S = \min(K, \alpha+\beta)$$

Proof of Lemma 15.16

We use notation $x \vee y = \max(x,y)$. Then

$$S = \min\{\alpha, \mu_\alpha K + 0 \vee (\mu_\beta K - \beta)\} + \min\{\beta, \mu_\beta K + 0 \vee (\mu_\alpha K - \alpha)\}$$

Case A: $\alpha \le \mu_\alpha K$ **and** $\beta \le \mu_\beta K$

Then

$$S = \alpha + \beta = \min\{\alpha+\beta, K\}$$

Case B: $\alpha \ge \mu_\alpha K$ **and** $\beta \ge \mu_\beta K$

Then

$$S = \mu_\alpha K + \mu_\beta K = K = \min\{\alpha+\beta, K\}$$

Case C: $\alpha \le \mu_\alpha K$ **and** $\beta \ge \mu_\beta K$

Then

$$S = \alpha + \min\{\beta, K-\alpha\} = \min\{\alpha+\beta, K\}$$

Case D: $\alpha \ge \mu_\alpha K$ **and** $\beta \le \mu_\beta K$

This case is symmetric to Case C, and find again $S = \min\{\alpha+\beta, K\}$.

This ends the proof of the lemma.

Remark 15.17 (Convergent junctions $2:1$**)**
Notice that for convergent junctions $2:1$, the germ $\mathcal{G}^{2:1}_{\Lambda}$ has the same expression as $\mathcal{G}^{1:2}_{\Lambda}$ in (15.22) with only $f^{j,+}$ naturally changed in $f^{j,-}$ for $j=0,1,2$. Similarly the expression of Godunov flux is still given by $\hat{f}=\hat{\gamma}\circ\bar{\gamma}$, where the preflux $\hat{\gamma}=\hat{\gamma}_0\circ T_{\bar{\lambda}}$ is unchanged, but only the capacity $\bar{\gamma}^j=f^{j,\sigma^j}$ is naturally changed under the transformation $\sigma\to-\sigma$, for $\sigma=(\sigma^0,\sigma^1,\sigma^2)$.

15.1.5 Newell's preflux (and on-ramp junction)

In [41] (paragraph 4. on a single bottleneck), Newell considers an on-ramp on a highway with full priority given to the on-ramp traffic (see also the presentation in paragraph 2.4 in [42]). This corresponds to the following preflux given in (15.23) for $\bar{\lambda}^0=+\infty$ and

$$\hat{\theta}^1(\bar{\gamma}^0)=\theta^1_*\bar{\gamma}^0,\quad \hat{\theta}^2(\bar{\gamma}^0)=\theta^2_*\bar{\gamma}^0\quad\text{with}\quad \theta^1_*=1,\quad \theta^2_*=0$$

i.e. to the following Newell's preflux $\hat{\gamma}^N=\hat{\gamma}=(\hat{\gamma}^0,\hat{\gamma}^1,\hat{\gamma}^2):[0,+\infty)^3\to[0,+\infty)^3$ given for $\bar{\gamma}=(\bar{\gamma}^0,\bar{\gamma}^1,\bar{\gamma}^2)\in[0,+\infty)^3$ as

$$\left\{\begin{array}{ll}\hat{\gamma}^1(\bar{\gamma}) &=\min\left\{\bar{\gamma}^1,\bar{\gamma}^0\right\},\\ \hat{\gamma}^2(\bar{\gamma}) &=\min\left\{\bar{\gamma}^2,\max\left\{0,\bar{\gamma}^0-\bar{\gamma}^1\right\}\right\},\\ \hat{\gamma}^0 &=\hat{\gamma}^1_0+\hat{\gamma}^2_0\end{array}\right.\tag{15.26}$$

Here the full priority is given to the flux $\bar{\gamma}^1$, and the flux $\bar{\gamma}^2$ occupies the remaining allowed part of traffic, if any.

Remark 15.18 *Notice that Newell's preflux $\hat{\gamma}$ can be obtained as the limit as $\theta:=(\theta^1,\theta^2)\to\theta_*:=(1,0)$ in the relaxation $\mathfrak{R}_\infty\hat{\lambda}_\theta$ for $n=2$ with Lebacque quasi-preflux $\hat{\lambda}_\theta$ for $N=1+n$ given by*

$$\left\{\begin{array}{l}\hat{\lambda}^j_\theta(\bar{\gamma})=\min\left\{\bar{\gamma}^j,\theta^j\bar{\gamma}^0\right\},\quad j=1,\dots,n\\ \hat{\lambda}^0_\theta=\displaystyle\sum_{j=1,\dots,n}\hat{\lambda}^j_\theta\end{array}\right.$$

Nevertheless, the reader may notice that in this case θ_* is so degenerate (with $\theta^2_*=0$) that

$$\mathfrak{R}_\infty\hat{\lambda}_\theta\to\hat{\gamma}\neq\mathfrak{R}_\infty\hat{\lambda}_{\theta_*}=\hat{\lambda}_{\theta_*}\quad as\quad \theta\to\theta_*$$

with $\hat{\gamma}$ given in (15.26). In particular $\hat{\gamma}$ is a conservative Kružkov preflux, as limit of conservative Kružkov prefluxes.

In the same spirit, notice that for instance for $N=1+n$ with $n=3$ and some small enough $\varepsilon>0$

$$\left\{\begin{array}{l}\theta^1_\varepsilon:=1-\varepsilon(\bar{\theta}^2+\bar{\theta}^3),\\ \theta^2_\varepsilon=\varepsilon\bar{\theta}^2,\\ \theta^3_\varepsilon=\varepsilon\bar{\theta}^3,\\ \\ \bar{\theta}:=(\bar{\theta}^2,\bar{\theta}^3)\in(0,+\infty)^2\end{array}\right.$$

using Lebacque preflux given in Corollary 14.9, we get

$$\mathfrak{R}_\infty\hat{\lambda}_{\theta_\varepsilon}\to\hat{\gamma}_{\bar{\theta}}$$

with

$$\left\{\begin{array}{l}\hat{\gamma}^1_{\bar{\theta}}(\bar{\gamma})=\min\left\{\bar{\gamma}^1,\bar{\gamma}^0\right\},\\ \hat{\gamma}^2_{\bar{\theta}}(\bar{\gamma})=\min\left\{\bar{\gamma}^2,\max\left\{0,\dfrac{\bar{\theta}^2}{\bar{\theta}^2+\bar{\theta}^3}(\bar{\gamma}^0-\bar{\gamma}^1),\bar{\gamma}^0-(\bar{\gamma}^1+\bar{\gamma}^3)\right\}\right\},\\ \hat{\gamma}^3_{\bar{\theta}}(\bar{\gamma})=\min\left\{\bar{\gamma}^3,\max\left\{0,\dfrac{\bar{\theta}^3}{\bar{\theta}^2+\bar{\theta}^3}(\bar{\gamma}^0-\bar{\gamma}^2),\bar{\gamma}^0-(\bar{\gamma}^1+\bar{\gamma}^2)\right\}\right\},\\ \hat{\gamma}^0_{\bar{\theta}}=\displaystyle\sum_{j=1,2,3}\hat{\gamma}^j_{\bar{\theta}}\end{array}\right.$$

In the limit $\bar\theta \to \bar\theta_* = (1,0)$, we get

$$\begin{cases} \hat\gamma^1_{\bar\theta_*}(\bar\gamma) = \min\left\{\bar\gamma^1, \bar\gamma^0\right\}, \\ \hat\gamma^2_{\bar\theta_*}(\bar\gamma) = \min\left\{\bar\gamma^2, \max\left\{0, \bar\gamma^0 - \bar\gamma^1\right\}\right\}, \\ \hat\gamma^3_{\bar\theta_*}(\bar\gamma) = \min\left\{\bar\gamma^3, \max\left\{0, \bar\gamma^0 - (\bar\gamma^1 + \bar\gamma^2)\right\}\right\}, \\ \hat\gamma^0_{\bar\theta_*} = \displaystyle\sum_{j=1,2,3} \hat\gamma^j_{\bar\theta} \end{cases}$$

which is an example of a preflux where the full priority is given to road 1. If there is still some remaining capacity to welcome further traffic (on road 0), then the next full priority is then given to road 2 (with respect to road 3). This is just a natural generalization of Newell's preflux.

15.1.6 Jin-Zhang quasi-preflux and its relaxation (and fair merging)

We consider the following Jin-Zhang function $\hat\lambda := (\hat\lambda^0, \hat\lambda^1, \hat\lambda^2) : [0,+\infty)^3 \to [0,+\infty)^3$, defined for $(\bar\gamma := (\bar\gamma^0, \bar\gamma^1, \bar\gamma^2) \in [0,+\infty)^3$ by

$$\begin{cases} \hat\lambda^1(\bar\gamma) = \min\left\{\bar\gamma^1, \dfrac{\bar\gamma^1}{\bar\gamma^1+\bar\gamma^2}\cdot\bar\gamma^0\right\} \\ \hat\lambda^2(\bar\gamma) = \min\left\{\bar\gamma^2, \dfrac{\bar\gamma^2}{\bar\gamma^1+\bar\gamma^2}\cdot\bar\gamma^0\right\} \\ \hat\lambda^0 = \hat\lambda^1 + \hat\lambda^2 \end{cases} \tag{15.27}$$

with the convention that $\hat\lambda = (0,0,0)$ if $\bar\gamma^1 = \bar\gamma^2 = 0$. This is called the fair merging (for merging junctions $2:1$), and we refer the reader to formulas (11) on page 526 and (13) on page 530 in [33].

Lemma 15.19 (Jin-Zhang quasi-preflux and its relaxation)
Let $\hat\lambda$ as defined in (15.27).
i) (Jin-Zhang quasi-preflux)
Then the function $\hat\lambda : [0,+\infty)^3 \to [0,+\infty)^3$ is a conservative Kružkov quasi-preflux in the sense of Definition 13.1, and is called the Jin-Zhang quasi-preflux. Moreover, we have

$$\hat\lambda^0(\bar\gamma) = \min\left\{\bar\gamma^0, \bar\gamma^1 + \bar\gamma^2\right\}$$

ii) (Relaxation of Jin-Zhang quasi-preflux)
Let $\bar\lambda = (\bar\lambda^0, \bar\lambda^1, \bar\lambda^2) \in (0,+\infty)^3$, and let

$$\hat\gamma := \mathfrak{R}_{\bar\lambda}\hat\lambda$$

be its associated preflux obtained by relaxation (which is a conservative Kružkov preflux). Then all for $\bar\gamma \in [0,\bar\lambda]$, we have

$$\begin{cases} \hat\gamma^1(\bar\gamma) = \min\left\{\bar\gamma^1, \max\left\{\theta^1\bar\gamma^0, \bar\gamma^0 - \bar\gamma^2\right\}\right\}, \\ \hat\gamma^2(\bar\gamma) = \min\left\{\bar\gamma^2, \max\left\{\theta^2\bar\gamma^0, \bar\gamma^0 - \bar\gamma^1\right\}\right\}, \\ \hat\gamma^0 = \hat\gamma^1 + \hat\gamma^2 \end{cases} \quad \textit{for} \quad \theta^j := \frac{\bar\lambda^j}{\bar\lambda^1 + \bar\lambda^2}, \quad j = 1,2$$

and

$$\hat\gamma = \hat\gamma \circ T_{\bar\lambda}.$$

Proof of Lemma 15.19
For the proof, we partially follow Appendix C in Jin [31].
Step 1: proof of i)
Assume that $\bar\gamma^1 + \bar\gamma^2 > 0$. Because $\bar\gamma^j < \frac{\bar\gamma^j}{\bar\gamma^1+\bar\gamma^2}\cdot\bar\gamma^0$ for $j = 1$ or 2 is equivalent to $\bar\gamma^1 + \bar\gamma^2 < \bar\gamma^0$, we deduce that we have

$$\hat\lambda^0(\bar\gamma) = \min\left\{\bar\gamma^0, \bar\gamma^1 + \bar\gamma^2\right\}$$

even for $\bar\gamma^1 + \bar\gamma^2 \geq 0$.
Step 1.1: checking that $\hat\lambda$ is a conservative Kružkov quasi-preflux

We first notice that $\hat{\lambda} : [0,+\infty)^3 \to [0,+\infty)^3$ is continuous, satisfies $0 \le \hat{\gamma}_0 \le id_{[0,+\infty)^3}$. Moreover the function

$$\varphi(\bar{\gamma}) := \frac{\bar{\gamma}^1}{\bar{\gamma}^1+\bar{\gamma}^2}\cdot \bar{\gamma}^0$$

satisfies $\varphi(\bar{\gamma}) = \bar{\gamma}^0 - \dfrac{\bar{\gamma}^2}{\bar{\gamma}^1+\bar{\gamma}^2}\cdot\bar{\gamma}^0$, which shows that $\partial_0\varphi \ge 0$, $\partial_1\varphi \ge 0$ and $\partial_2\varphi \le 0$. Hence for $\sigma := (\sigma^0,\sigma^1,\sigma^2) = (-1,1,1)$, we deduce that $\partial_k\hat{\lambda}^k \ge 0$ for $k=0,1,2$, and

$$\sigma^\ell\sigma^k\partial_k\hat{\lambda}^\ell \le 0 \quad \text{for all} \quad \ell \ne k$$

We conclude from Lemma 13.3 that $\hat{\lambda}$ is a Kružkov quasi-preflux.
Step 1.2: further properties at infinity
Notice that $\hat{\gamma}_0$ is uniformly locally bounded. Notice also that $\hat{\gamma}_0$ is not boundedly continuous on $[0,+\infty]^3$, because for bounded $\bar{\gamma}^0$, the quasi-preflux is bounded, but has no unique limit as $(\bar{\gamma}^1,\bar{\gamma}^2)\to(+\infty,+\infty)$. Hence $\mathfrak{R}_\infty\hat{\lambda}$ is not defined.
Step 2: Computation of its relaxation
Let $\bar{\lambda} = (\bar{\lambda}^0,\bar{\lambda}^1,\bar{\lambda}^2)\in(0,+\infty)^3$, and $\hat{\gamma} := \mathfrak{R}_{\bar{\lambda}}\hat{\lambda}$. We have to solve for $(\lambda^L,\lambda^R)\in\mathbb{D}_{\bar{\lambda}}$

$$\begin{cases} \hat{\gamma}^1(\bar{\gamma}) = \min\left\{\bar{\gamma}^1,\lambda^{1L}\right\} = \min\left\{\lambda^{1R},\dfrac{\lambda^{1R}}{\lambda^{1R}+\lambda^{2R}}\cdot\lambda^{0R}\right\}\\ \hat{\gamma}^2(\bar{\gamma}) = \min\left\{\bar{\gamma}^2,\lambda^{2L}\right\} = \min\left\{\lambda^{2R},\dfrac{\lambda^{2R}}{\lambda^{1R}+\lambda^{2R}}\cdot\lambda^{0R}\right\}\\ \hat{\gamma}^0(\bar{\gamma}) = \min\left\{\bar{\gamma}^0,\lambda^{0L}\right\} = \min\left\{\lambda^{1R},\dfrac{\lambda^{1R}}{\lambda^{1R}+\lambda^{2R}}\cdot\lambda^{0R}\right\} + \min\left\{\lambda^{2R},\dfrac{\lambda^{2R}}{\lambda^{1R}+\lambda^{2R}}\cdot\lambda^{0R}\right\} \end{cases}$$

Let us assume that

$$\bar{\gamma}\in[0,\bar{\lambda}).$$

Step 2.1: the passing flux
We first show that

$$\hat{\gamma}^0 = \min\left\{\bar{\gamma}^0,\bar{\gamma}^1+\bar{\gamma}^2\right\} \tag{15.28}$$

Indeed, we already have $\hat{\gamma}^0 \le \min\left\{\bar{\gamma}^0,\bar{\gamma}^1+\bar{\gamma}^2\right\}$. Assume by contradiction that

$$\hat{\gamma}^0 < \min\left\{\bar{\gamma}^0,\bar{\gamma}^1+\bar{\gamma}^2\right\}$$

Then $\lambda^{0R} = \bar{\lambda}^0$ and at least $\lambda^{jL} < \bar{\gamma}^j$ for some index $j\in\{1,2\}$. Assume for instance this is $j=1$ (the case $j=2$ is similar) and $\lambda^{1R} = \bar{\lambda}^1$. Then $\hat{\gamma}^1 = \lambda^{1L} < \bar{\gamma}^1$ and

$$\bar{\lambda}^1 > \bar{\gamma}^1 > \hat{\gamma}^1 = \min\left\{\lambda^{1R},\frac{\lambda^{1R}}{\lambda^{1R}+\lambda^{2R}}\cdot\lambda^{0R}\right\} = \min\left\{\bar{\lambda}^1,\frac{\bar{\lambda}^1}{\bar{\lambda}^1+\lambda^{2R}}\cdot\bar{\lambda}^0\right\}$$

Hence $\lambda^{0R} = \bar{\lambda}^0 < \bar{\lambda}^1+\lambda^{2R} = \lambda^{1R}+\lambda^{2R}$, i.e. $\lambda^{0R} < \lambda^{1R}+\lambda^{2R}$. Hence

$$\bar{\lambda}^0 > \bar{\gamma}^0 > \hat{\gamma}^0 = \hat{\gamma}^1+\hat{\gamma}^2 = \lambda^{0R} = \bar{\lambda}^0$$

Contradiction. We conclude to (15.28).
Step 2.2: the other fluxes
Case A: $\bar{\gamma}^1+\bar{\gamma}^2 < \bar{\gamma}^0$
We want to show that

$$(\hat{\gamma}^1,\hat{\gamma}^2,\hat{\gamma}^0) = (\bar{\gamma}^1,\bar{\gamma}^2,\bar{\gamma}^1+\bar{\gamma}^2).$$

We choose

$$\begin{cases} \lambda^{1L} = \bar{\lambda}^1, & \lambda^{1R} = \bar{\gamma}^1\\ \lambda^{2L} = \bar{\lambda}^2, & \lambda^{2R} = \bar{\gamma}^2\\ \lambda^{0L} = \bar{\gamma}^1+\bar{\gamma}^2, & \lambda^{0R} = \bar{\lambda}^0 \end{cases}$$

which is allowed because $\lambda^{0L}\le\bar{\lambda}^0$ follows from $\lambda^{0L} = \bar{\gamma}^1+\bar{\gamma}^2 < \bar{\gamma}^0 < \bar{\lambda}^0$.
Case B: $\bar{\gamma}^1+\bar{\gamma}^2 > \bar{\gamma}^0$

Case B.1: $\bar\gamma^j > \theta^j\bar\gamma^0$ **with** $\theta^j := \dfrac{\bar\lambda^j}{\bar\lambda^1+\bar\lambda^2}$ **and** $j=1,2$

We want to show that
$$(\hat\gamma^1,\hat\gamma^2,\hat\gamma^0) = (\theta^1\bar\gamma^0,\theta^2\bar\gamma^0,\bar\gamma^0).$$
We choose
$$\begin{cases} \lambda^{1L} = \theta^1\bar\gamma^0, & \lambda^{1R} = \bar\lambda^1\\ \lambda^{2L} = \theta^2\bar\gamma^0, & \lambda^{2R} = \bar\lambda^2\\ \lambda^{0L} = \bar\lambda^0, & \lambda^{0R} = \bar\gamma^0 \end{cases}$$
which is allowed because $\begin{cases} \lambda^{1L} \le \bar\lambda^1,\\ \lambda^{2L} \le \bar\lambda^2 \end{cases}$ follows from $\begin{cases} \lambda^{1L} = \theta^1\bar\gamma^0 < \bar\gamma^1 < \bar\lambda^1,\\ \lambda^{2L} = \theta^2\bar\gamma^0 < \bar\gamma^2 < \bar\lambda^2 \end{cases}$.

Case B.2: $\bar\gamma^1 < \theta^1\bar\gamma^0$ **and** $\bar\gamma^2 > \theta^2\bar\gamma^0$

We want to show that
$$(\hat\gamma^1,\hat\gamma^2,\hat\gamma^0) = (\bar\gamma^1,\bar\gamma^0-\bar\gamma^1,\bar\gamma^0).$$
We choose
$$\begin{cases} \lambda^{1L} = \bar\lambda^1, & \lambda^{1R} = \dfrac{\bar\lambda^2}{\bar\gamma^0-\bar\gamma^1}\cdot\bar\gamma^1\\ \lambda^{2L} = \bar\gamma^0-\bar\gamma^1 < \bar\lambda^2, & \lambda^{2R} = \bar\lambda^2\\ \lambda^{0L} = \bar\lambda^0, & \lambda^{0R} = \bar\gamma^0 \end{cases}$$
which is allowed because $\begin{cases} \lambda^{1R} \le \bar\lambda^1,\\ \lambda^{2L} \le \bar\lambda^2 \end{cases}$ follows from the following. We first notice that
$$\lambda^{2L} = \bar\gamma^0-\bar\gamma^1 < \bar\gamma^2 < \bar\lambda^2.$$
We also have (because $\theta^1\bar\gamma^0 > \bar\gamma^1 \ge 0$)
$$\begin{aligned} \lambda^{1R} < \bar\lambda^1 \quad &\iff \quad \frac{\bar\lambda^2}{\bar\gamma^0-\bar\gamma^1}\cdot\bar\gamma^1 < \bar\lambda^1\\ &\iff \quad (\bar\lambda^1+\bar\lambda^2)\bar\gamma^1 < \bar\lambda^1\bar\gamma^0\\ &\iff \quad \bar\gamma^1 < \theta^1\bar\gamma^0 \end{aligned}$$
which shows the expected inequality.

Case B.3: $\bar\gamma^1 > \theta^1\bar\gamma^0$ **and** $\bar\gamma^2 < \theta^2\bar\gamma^0$

Symmetric to Case B.2.

Case B.4: $\bar\gamma^1 < \theta^1\bar\gamma^0$ **and** $\bar\gamma^2 < \theta^2\bar\gamma^0$

Impossible because $\bar\gamma^0 < \bar\gamma^1+\bar\gamma^2$.

Conclusion

By continuity, we conclude to point ii) and this ends the proof of the lemma.

15.1.7 Relaxation of Lebacque quasi-preflux with unnormalized coefficients

As an example considered in Lebacque [37] (Section 6.3), we may consider for $N=1+n$ and $n=2$, the following function $\hat\lambda = (\hat\lambda^0,\hat\lambda^1,\hat\lambda^2) : [0,+\infty)^3 \to [0,+\infty)^3$ defined for $\bar\gamma = (\bar\gamma^0,\bar\gamma^1,\bar\gamma^2) \in [0,+\infty)^3$ as

(15.29)
$$\begin{cases} \hat\lambda^j(\bar\gamma) = \min\left\{\bar\gamma^j, \tilde\theta^1\bar\gamma^0\right\}, & j=1,2,\\ \hat\lambda^0 = \hat\lambda^1+\hat\lambda^2 \end{cases}$$
with
$$\tilde\theta^1,\tilde\theta^2 > 0$$
where we may have $\tilde\theta^1+\tilde\theta^2 > 1$ (or also $\tilde\theta^1+\tilde\theta^2 < 1$).

Lemma 15.20 (Relaxation of Lebacque quasi-preflux with unnormalized coefficients, $n=2$**)**
Let $N=1+n$ with $n=2$ and $\hat\lambda : [0,+\infty)^N \to [0,+\infty)^N$ be the function defined in (15.29).

i) (Quasi-preflux)

Then $\hat\lambda$ is a conservative Kružkov quasi-preflux.

ii) (Relaxation)

Let $\hat{\gamma} := \mathfrak{R}_\infty \hat{\lambda}$ be the Riemann relaxation of $\hat{\lambda}$ on the box $[0,+\infty]^N$. Then $\hat{\gamma}$ is a "standard" Lebacque preflux

$$\left\{\begin{array}{l} \hat{\gamma}^1(\bar{\gamma}) = \min\left\{\bar{\gamma}^1, \max\left\{\theta^1\bar{\gamma}^0, \bar{\gamma}^0 - \bar{\gamma}^2\right\}\right\}, \\ \hat{\gamma}^2(\bar{\gamma}) = \min\left\{\bar{\gamma}^2, \max\left\{\theta^2\bar{\gamma}^0, \bar{\gamma}^0 - \bar{\gamma}^1\right\}\right\}, \\ \hat{\gamma}^0 = \hat{\gamma}^1 + \hat{\gamma}^2 \end{array}\right. \tag{15.30}$$

with normalized coefficients:

$$\theta^j := \frac{\tilde{\theta}^j}{\tilde{\theta}^1 + \tilde{\theta}^2}, \quad j = 1, 2.$$

Proof of Lemma 15.20
Step 1: proof of i)
The proof of i) is similar to the proof of i) and ii) of Lemma 14.1.
Step 2: proof of ii)
The proof follows the same lines as the proof of Lemma 14.1 and Corollaries 14.6 and 14.9. We give a few details below. Notice in particular that $\mathfrak{R}_\infty \hat{\lambda}$ is well-defined, due to lemma 14.10.
We start to relax $\hat{\lambda}$ finding $(\lambda^L, \lambda^R) \in \mathbb{D}_{\lambda_\infty}$ with $\lambda_\infty = (+\infty, +\infty, +\infty)$, i.e. solution of

$$\left\{\begin{array}{l} \min\left\{\bar{\gamma}^1, \lambda^{1L}\right\} = \min\left\{\lambda^{1R}, \tilde{\theta}^1\lambda^{0R}\right\} =: \hat{\gamma}^1(\bar{\gamma}), \\ \min\left\{\bar{\gamma}^2, \lambda^{2L}\right\} = \min\left\{\lambda^{2R}, \tilde{\theta}^2\lambda^{0R}\right\} =: \hat{\gamma}^2(\bar{\gamma}), \\ \min\left\{\bar{\gamma}^0, \lambda^{0L}\right\} = \hat{\gamma}^1(\bar{\gamma}) + \hat{\gamma}^2(\bar{\gamma}) \end{array}\right.$$

which gives (using Lemma 14.5)

$$\min\left\{\bar{\gamma}^0, \lambda^{0L}\right\} = \min\left\{\bar{\gamma}^1, \tilde{\theta}^1\lambda^{0R}\right\} + \min\left\{\bar{\gamma}^2, \tilde{\theta}^2\lambda^{0R}\right\}$$

We set

$$\left\{\begin{array}{l} \tilde{\theta}^0 := \tilde{\theta}^1 + \tilde{\theta}^2, \\ \breve{\gamma}^j := \dfrac{\bar{\gamma}^j}{\tilde{\theta}^j}, \quad j = 0, 1, 2, \\ (\tilde{\lambda}^L, \tilde{\lambda}^R) := (\dfrac{\lambda^{0L}}{\tilde{\theta}^0}, \lambda^{0R}) \in \mathbb{D}_\infty \end{array}\right.$$

and get

$$\min\left\{\breve{\gamma}^0, \tilde{\lambda}^{0L}\right\} = \phi(\tilde{\lambda}^{0R}) \quad \text{with} \quad \phi(\tilde{\lambda}^{0R}) := \theta^1 \min\left\{\breve{\gamma}^1, \tilde{\lambda}^{0R}\right\} + \theta^2 \min\left\{\breve{\gamma}^2, \tilde{\lambda}^{0R}\right\} \tag{15.31}$$

From the solution to (14.4), given for instance in (14.16), we deduce that

$$\left\{\begin{array}{l} (\tilde{\theta}^1)^{-1}\hat{\gamma}^1(\bar{\gamma}) = \min\left\{\breve{\gamma}^1, \max\left\{\breve{\gamma}^0, \dfrac{\breve{\gamma}^0 - \theta^2\breve{\gamma}^2}{1 - \theta^2}\right\}\right\}, \\ (\tilde{\theta}^2)^{-1}\hat{\gamma}^2(\bar{\gamma}) = \min\left\{\breve{\gamma}^2, \max\left\{\breve{\gamma}^0, \dfrac{\breve{\gamma}^0 - \theta^1\breve{\gamma}^1}{1 - \theta^1}\right\}\right\}, \end{array}\right.$$

which implies (15.30). This ends the proof of the lemma.

Remark 15.21 *Notice that we may also consider and compute (if necessary) Riemann relaxation of $\hat{\lambda} \circ T_{\bar{\lambda}}$ with a truncation operator $T_{\bar{\lambda}}$ for some $\bar{\lambda} \in [0,+\infty)^3$.*

15.1.8 Kedem-Katchalsky quasi-preflux and its zero relaxation

This subsection is inspired by [3] and [12]. For $N \geq 1$ and $\bar{\lambda} = (\bar{\lambda}^1, \ldots, \bar{\lambda}^N) \in (0,+\infty)^N$, and indices $j, k \in \{1, \ldots, N\}$. Following Example 2.2 in [3], we consider the kernel functions

$$\left\{\begin{array}{l} K_{jk} : [0, \bar{\lambda}^j] \times [0, \bar{\lambda}] \to \mathbb{R} \quad \text{continuous map}, \\ K_{jk}(\uparrow, \downarrow) \quad \text{monotonicities}, \\ K_{jk}(0,0) = 0 = K_{jk}(\bar{\lambda}^j, \bar{\lambda}^k), \\ K_{jk}(\bar{\gamma}^j, \bar{\gamma}^k) = -K_{kj}(\bar{\gamma}^k, \bar{\gamma}^j), \\ K_{jj} = 0 \end{array}\right. \tag{15.32}$$

where $\uparrow$ means non-decreasing and $\downarrow$ means non-increasing.

Remark 15.22 (Explicit example of a Kedem-Katchalsky quasi-preflux)
A typical example is

$$\left\{\begin{array}{l} K_{jk}(\bar\gamma^j,\bar\gamma^k) := c_{jk}(\bar\gamma^j-\bar\gamma^k),\\ c_{jk}=c_{kj}>0 \quad \textit{for} \quad j\neq k,\\ c_{jj}=0\\ \bar\lambda^j=\bar\lambda^k>0 \quad \textit{for all} \quad j,k \end{array}\right.$$

Lemma 15.23 (Kedem-Katchalsky quasi-preflux and its zero relaxation)
For $N\geq 1$ and $\bar\lambda=(\bar\lambda^1,\dots,\bar\lambda^N)\in(0,+\infty)^N$, assume (15.32). Consider the function $\hat\lambda:[0,+\infty)^N\to\mathbb{R}^N$, first defined on $[0,\bar\lambda]$ by

$$\hat\lambda^j(\bar\gamma) := \sum_{k=1,\dots,N} K_{jk}(\bar\gamma^j,\bar\gamma^k),\quad j=1,\dots,N,\quad \textit{for all}\quad \bar\gamma\in[0,\bar\lambda]$$

and extended to $[0,+\infty)^N$ by

$$\hat\lambda := \hat\lambda\circ T_{\bar\lambda}$$

where $T_{\bar\lambda}$ is the standard truncation operator defined in (13.9).
Then $\hat\lambda:[0,+\infty)^N\to\mathbb{R}^N$ is a Kružkov $\bar\lambda$-quasi-preflux in the sense of Definition 13.1. Moreover, its relaxation on the box $[0,\bar\lambda]$ satisfies

$$\mathfrak{R}_{\bar\lambda}\hat\lambda = 0 \tag{15.33}$$

i.e. is the zero Godunov flux.

Proof of Lemma 15.23
Step 1: cheking Kružkov quasi-preflux
By definition, $\hat\lambda$ is continuous and it is easy to see that it satisfies

$$\hat\lambda^j_{|\{\bar\gamma^j=0\}}\leq 0 \quad\text{and}\quad \hat\lambda^j_{|\left\{\bar\gamma^j\geq\bar\lambda^j\right\}}\geq 0.$$

It is also easy to see that

$$\sum_{j=1,\dots,N}\hat\lambda^j=0$$

i.e. that $\hat\lambda$ is conservative for $\sigma=(1,\dots,1)\in\mathbb{R}^N$. It is also easy to see that it is σ-monotone, and then Lemma 13.3 implies that $\hat\lambda$ is a Kružkov quasi-preflux. By construction it is then a $\bar\lambda$-quasi-preflux.
Step 2: its Riemann relaxation is zero
Let us consider the Riemann relaxation on the box $[0,\bar\lambda]$ given by

$$\hat\gamma := \mathfrak{R}_{\bar\lambda}\hat\lambda$$

which is by construction a Kružkov preflux associated to the Kedem-Katchalsky quasi-preflux. By definition, we have

$$\hat\gamma^j(\bar\gamma) := \min\left\{\bar\gamma^j,\lambda^{jL}\right\} = \hat\lambda^j(\lambda^R)\quad\text{for}\quad(\lambda^L,\lambda^R)\in\mathbb{D}_{\bar\lambda} \tag{15.34}$$

Then

$$(\lambda^L,\lambda^R)=(0_{\mathbb{R}^N},\bar\lambda)\in\mathbb{D}_{\bar\lambda}$$

is a solution of (15.34), and then

$$\hat\gamma^j(\bar\gamma)=\min\left\{\bar\gamma^j,0\right\}=0$$

which shows (15.33). This ends the proof of the lemma.

15.1.9 Construction of some $2:2$ conservative Kružkov prefluxes and germs by gluing

In this subsection, we give explicitly the Godunov flux of some $2:2$ conservative Kružkov germ, that is obtained by gluing of two traffic lights germs of type $2:1$ and of type $1:2$.

We consider a function g satisfying

$$(15.35)\qquad g:[0,1]\to[0,+\infty)\quad \text{stricly concave with}\quad \left\{\begin{array}{l} g(0)=0=g(1)\\ \text{and maximum at } c\in(0,1) \text{ with } g_{\max}:=g(c)\end{array}\right.$$

and its monotone envelopes

$$g^+(u):=\left\{\begin{array}{lll} g(u) & \text{for} & u\in[0,c]\\ g(c) & \text{for} & u\in[c,1]\end{array}\right. \quad \text{and}\quad g^-(u):=\left\{\begin{array}{lll} g(c) & \text{for} & u\in[0,c]\\ g(u) & \text{for} & u\in[c,1]\end{array}\right.$$

We also consider functions $\hat\lambda^k$ satisfying the following

$$(15.36)\qquad \left\{\begin{array}{lll} \bar\lambda^j:=g_{\max} & \text{for} & j=1,\dots,4\\ \bar\lambda^0:=2g_{\max} & & \\ \text{the maps}\quad \hat\lambda^k:[0,+\infty)\to[0,+\infty)\quad \text{continuous nondecreasing} & \text{for} & k=1,\dots,4\\ \hat\lambda^k(0)=0,\quad \hat\lambda^k(\bar\lambda^0)=g_{\max} & \text{for} & k=1,\dots,4\\ \hat\lambda^1(\lambda)+\hat\lambda^2(\lambda)=\lambda=\hat\lambda^3(\lambda)+\hat\lambda^4(\lambda) & \text{for all} & \lambda\in[0,+\infty)\end{array}\right.$$

Lemma 15.24 (Example of some $2:2$ conservative Kružkov preflux and germ)
Assume (15.35) and (15.36). For $U:=(u^1,u^2,u^3,u^4)\in[0,1]^4$, we define

$$\lambda^0:=\min\left\{g^+(u^1)+g^+(u^2),g^-(u^3)+g^-(u^4)\right\}$$

and the flux $\hat f(U):=(\hat f^1,\hat f^2,\hat f^3,\hat f^4)(U)$ as follows

(15.37)

$$\left\{\begin{array}{l} \hat f^1(U)=g^+(u^1)\\ \hat f^2(U)=g^+(u^2)\\ (\hat f^3(U),\hat f^4(U))=\left\{\begin{array}{lll} (\hat\lambda^3(\lambda^0),\hat\lambda^4(\lambda^0)) & \textit{if} & \left\{\begin{array}{l} g^-(u^3)\ge\hat\lambda^3(\lambda^0)\\ g^-(u^4)\ge\hat\lambda^4(\lambda^0)\end{array}\right.\\ (\lambda^0-g^-(u^4),g^-(u^4)) & \textit{if} & \left\{\begin{array}{l} g^-(u^3)\ge\hat\lambda^3(\lambda^0)\\ \boxed{g^-(u^4)\le}\hat\lambda^4(\lambda^0)\end{array}\right.\\ (g^-(u^3),\lambda^0-g^-(u^3)) & \textit{if} & \left\{\begin{array}{l} \boxed{g^-(u^3)\le}\hat\lambda^3(\lambda^0)\\ g^-(u^4)\ge\hat\lambda^4(\lambda^0)\end{array}\right.\end{array}\right.\end{array}\right. \quad\Bigg|\quad \textit{if}\quad \lambda^0<g^-(u^3)+g^-(u^4)$$

(where $g^-(u^j)\ge\hat\lambda^j(\lambda^0)$ never happens for $j=3,4$ at the same time) and

(15.38)

$$\left\{\begin{array}{l} (\hat f^1(U),\hat f^2(U))=\left\{\begin{array}{lll} (\hat\lambda^1(\lambda^0),\hat\lambda^2(\lambda^0)) & \textit{if} & \left\{\begin{array}{l} g^+(u^1)\ge\hat\lambda^1(\lambda^0)\\ g^+(u^2)\ge\hat\lambda^2(\lambda^0)\end{array}\right.\\ (\lambda^0-g^+(u^2),g^+(u^2)) & \textit{if} & \left\{\begin{array}{l} g^+(u^1)\ge\hat\lambda^1(\lambda^0)\\ \boxed{g^+(u^2)\le}\hat\lambda^2(\lambda^0)\end{array}\right.\\ (g^+(u^1),\lambda^0-g^+(u^1)) & \textit{if} & \left\{\begin{array}{l} \boxed{g^+(u^1)\le}\hat\lambda^1(\lambda^0)\\ g^+(u^2)\ge\hat\lambda^2(\lambda^0)\end{array}\right.\end{array}\right.\\ \hat f^3(U)=g^-(u^3)\\ \hat f^4(U)=g^-(u^4)\end{array}\right. \quad\Bigg|\quad \textit{if}\quad \lambda^0<g^+(u^1)+g^+(u^2)$$

(where $g^+(u^j)\ge\hat\lambda^j(\lambda^0)$ never happens for $j=1,2$ at the same time), and

$$(15.39)\qquad \left\{\begin{array}{l} \hat f^1(U)=g^+(u^1)\\ \hat f^2(U)=g^+(u^2)\\ \hat f^3(U)=g^-(u^3)\\ \hat f^4(U)=g^-(u^4)\end{array}\right. \quad\Bigg|\quad \textit{if}\quad \lambda^0=g^+(u^1)+g^+(u^2)=g^-(u^3)+g^-(u^4)$$

i) (The germ)
Then the set $\mathcal{G}$ defined by

$$\mathcal{G} := \left\{ U = (u^1, u^2, u^3, u^4) \in [0,1]^4, \quad \hat{f}(U) = f(U) \right\}$$

with $f = (g, g, g, g)$, is a conservative Kružkov germ for junction of type $2:2$.
ii) (The associated 2:2 conservative Kružkov preflux)
Moreover for

$$(\bar{\gamma}^1, \bar{\gamma}^2, \bar{\gamma}^3, \bar{\gamma}^4) := (g^+(u^1), g^+(u^2), g^-(u^3) g^-(u^4))$$

we have

$$\hat{f}(U) = \hat{\gamma}(\bar{\gamma}) \quad \text{with 2:2 conservative Kružkov preflux} \quad \hat{\gamma} = \hat{\gamma}_0 \circ T_{\bar{\lambda}}, \quad \bar{\lambda} := (g_{\max}, g_{\max}, g_{\max}, g_{\max})$$

where $\hat{\gamma}_0 : [0,+\infty)^4 \to [0,+\infty)^4$ is also a 2:2 conservative Kružkov preflux given by

$$(15.40) \qquad \begin{cases} \lambda^0 := \min\left\{\bar{\gamma}^1 + \bar{\gamma}^2, \bar{\gamma}^3 + \bar{\gamma}^4\right\}, \\ \\ \hat{\gamma}_0^1(\bar{\gamma}) = \min\left\{\bar{\gamma}^1, \max\left\{\hat{\lambda}^1(\lambda^0), \lambda^0 - \bar{\gamma}^2\right\}\right\}, \\ \hat{\gamma}_0^2(\bar{\gamma}) = \min\left\{\bar{\gamma}^2, \max\left\{\hat{\lambda}^2(\lambda^0), \lambda^0 - \bar{\gamma}^1\right\}\right\}, \\ \\ \hat{\gamma}_0^3(\bar{\gamma}) = \min\left\{\bar{\gamma}^3, \max\left\{\hat{\lambda}^3(\lambda^0), \lambda^0 - \bar{\gamma}^4\right\}\right\}, \\ \hat{\gamma}_0^4(\bar{\gamma}) = \min\left\{\bar{\gamma}^4, \max\left\{\hat{\lambda}^4(\lambda^0), \lambda^0 - \bar{\gamma}^3\right\}\right\}, \end{cases}$$

Proof of Lemma 15.24
Step 1: proof of i)
The idea if that the set $\mathcal{G}$ is obtained by gluing of two conservative Kružkov germs $\mathcal{G}_\alpha$ and $\mathcal{G}_\beta$, i.e. that $\mathcal{G} = \mathcal{G}_\alpha \sharp \mathcal{G}_\beta$. Then by Proposition 5.13, the set $\mathcal{G}$ is therefore also a conservative Kružkov germ. Here $\mathcal{G}_\alpha$ is of type $2:1$ with fluxes $f_\alpha := (f_\alpha^1, f_\alpha^2, f_\alpha^0) = (g, g, 2g)$ (where $2g$ is the flux on the outgoing branch) and $\mathcal{G}_\beta$ of type $1:2$ with fluxes $f_\beta := (f_\beta^0, f_\beta^3, f_\beta^4) = (2g, g, g)$ (where $2g$ is the flux on the ingoing branch). Precisely $\mathcal{G}_\alpha$ and $\mathcal{G}_\beta$ are two particular traffic lights germs. For $u := (u^1, u^2, u^0) \in [0,1]^3$, we consider the flux

$$\begin{cases} \hat{f}_\alpha^1(u) := \min\left\{g^+(u^1), \max\left\{\hat{\lambda}^1(2g^-(u^0)), 2g^-(u^0) - g^+(u^2)\right\}\right\} \\ \hat{f}_\alpha^2(u) := \min\left\{g^+(u^2), \max\left\{\hat{\lambda}^2(2g^-(u^0)), 2g^-(u^0) - g^+(u^1)\right\}\right\} \\ \hat{f}_\alpha^0(u) := \min\left\{2g^-(u^0), g^+(u^1) + g^+(u^2)\right\} \end{cases}$$

and for $\tilde{u} := (u^0, u^3, u^4) \in [0,1]^3$, the flux

$$\begin{cases} \hat{f}_\beta^0(\tilde{u}) := \min\left\{2g^+(u^0), g^-(u^3) + g^-(u^4)\right\} \\ \hat{f}_\beta^3(\tilde{u}) := \min\left\{g^-(u^3), \max\left\{\hat{\lambda}^3(2g^+(u^0)), 2g^+(u^0) - g^-(u^4)\right\}\right\} \\ \hat{f}_\beta^4(\tilde{u}) := \min\left\{g^-(u^4), \max\left\{\hat{\lambda}^4(2g^+(u^0)), 2g^+(u^0) - g^-(u^3)\right\}\right\} \end{cases}$$

We set

$$\mathcal{G}_\alpha := \left\{ u = (u^1, u^2, u^0) \in [0,1]^3, \quad \hat{f}_\alpha(u) = f_\alpha(u) \right\}$$

and

$$\mathcal{G}_\beta := \left\{ \tilde{u} = (u^0, u^3, u^4) \in [0,1]^3, \quad \hat{f}_\beta(\tilde{u}) = f_\beta(\tilde{u}) \right\}$$

Given $U := (u^1, u^2, u^3, u^4) \in [0,1]^4$, we then look for some $u^0 \in [0,1]$ such that

$$\hat{f}_\alpha^0(u^1, u^2, u^0) = \hat{f}_\beta^0(u^0, u^3, u^4)$$

Then it is easy to see that

$$\begin{cases} \lambda^0 := 2g(u^0) = \min\left\{g^+(u^1) + g^+(u^2), g^-(u^3) + g^-(u^4)\right\} & \\ g^+(u^0) = g(u^0) & \text{if} \quad 2g(u^0) < g^-(u^3) + g^-(u^4) \\ g^-(u^0) = g(u^0) & \text{if} \quad 2g(u^0) < g^+(u^1) + g^+(u^2) \end{cases}$$

and it is easy to see that for $(\hat{f}^1, \hat{f}^2, \hat{f}^3, \hat{f}^4)(U) = (\hat{f}^1_\alpha(u), \hat{f}^2_\alpha(u), \hat{f}^3_\beta(\tilde{u}), \hat{f}^4_\beta(\tilde{u}))$ with $u := (u^1, u^2, u^0)$ and $\tilde{u} := (u^0, u^3, u^4)$, we have (15.37)-(15.38), and (15.39) then follows by continuity.
Finally, $\mathcal{G}$ is of type $2:2$ with ingoing branches labelled $1, 2$ and outgoing branches labelled $3, 4$.
Step 2: proof of ii)
By construction we also see that we have point ii) with (15.40). In particular when $\lambda^0 < \bar{\gamma}^1 + \bar{\gamma}^2$, notice that $\lambda^0 = \bar{\gamma}^3 + \bar{\gamma}^4$, and then

$$\begin{cases} \max\left\{\hat{\lambda}^3(\lambda^0), \lambda^0 - \bar{\gamma}^4\right\} \geq \bar{\gamma}^3, \\ \max\left\{\hat{\lambda}^4(\lambda^0), \lambda^0 - \bar{\gamma}^3\right\} \geq \bar{\gamma}^4, \end{cases}$$

which implies (15.38), and then (15.40). The raisoning is similar when $\lambda^0 < \bar{\gamma}^3 + \bar{\gamma}^4$, and obtained by continuity when $\lambda^0 = \bar{\gamma}^1 + \bar{\gamma}^2 = \bar{\gamma}^3 + \bar{\gamma}^4$. This ends the proof of the lemma.

15.2 Non Kružkov cases

15.2.1 Daganzo FIFO preflux (and diverging junctions)

Let $N = 1 + n$ with $n \geq 2$ and

$$\theta^0 = 1 = \sum_{j=1,\ldots,n} \theta^j \quad \text{with} \quad \theta^j \in (0,1) \quad \text{for} \quad j = 1, \ldots, n. \tag{15.41}$$

We consider the following First In First Out (FIFO) function $\hat{\gamma} := (\hat{\gamma}^0, \ldots, \hat{\gamma}^n) : [0, +\infty)^N \to [0, +\infty)^N$

$$\hat{\gamma}^k(\bar{\gamma}) = \theta^k \min\left\{\frac{\bar{\gamma}^0}{\theta^0}, \frac{\bar{\gamma}^1}{\theta^1}, \ldots, \frac{\bar{\gamma}^n}{\theta^n}\right\}, \quad k = 0, 1, \ldots, n \tag{15.42}$$

which has been introduced for $n = 2$ by Daganzo [17] (equations (9a)-(9b) on page 88) originally for a diverging junction $1:n$ (useful for instance to describe emergency evacuation). See also [30] for the associated Hamilton-Jacobi theory for $n \geq 1$.

Lemma 15.25 (Daganzo FIFO preflux)
Assume (15.41). Then the function $\hat{\gamma} : [0, +\infty)^N \to [0, +\infty)^N$ given in (15.42) is a conservative quasi Hamilton-Jacobi preflux in the sense of Definition 11.1.

Proof of Lemma 15.25
We first notice that $0 \leq \hat{\gamma}^{qHJ} \leq id_{[0,+\infty)^N}$ is continuous and locally constant on $\left\{\hat{\gamma}^{qHJ} \neq id_{[0,+\infty)^N}\right\}$, hence is a preflux. Moreover its N components take the same value up to the prefactor $\theta^k > 0$, hence it is a quasi HJ preflux. For $\sigma := (\sigma^0, \sigma^1, \quad, \sigma^n) = (1, -1, \ldots, -1)$ we have

$$\sigma^j \sigma^l \partial_j (\hat{\gamma}^{qHJ,l}) > 0$$

when $\bar{\gamma}^j < \bar{\gamma}^0, \bar{\gamma}^l$. This shows that the preflux $\hat{\gamma}^{qHJ}$ is not σ-monotone. Moreover, we have

$$\sum_{j=0,1,\ldots,n} \sigma^j (\hat{\gamma}^{qHJ,j})(\bar{\gamma}) = 0$$

which shows that the preflux $\hat{\gamma}^{qHJ}$ is conservative. This ends the proof of the lemma.

15.2.2 Coclite-Garavello-Piccoli preflux and case study of $2:2$ junctions

In this subsection, we consider the Coclite-Garavello-Piccoli preflux for traffic on $n:m$ junctions introduced in [14] (see also the book [24], and germ $\mathcal{RS}_1$ in [25]). We will apply it in particular for $1:2$ junctions.

For $n:m$ junctions with $n, m \geq 1$, we consider indices orienting the branches (n ingoing branches and m outgoing branches) such that

$$\begin{cases} \sigma^j := 1, \quad j = 1, \ldots, n \\ \sigma^i := -1, \quad i = 1+n, \ldots, m+n \end{cases} \tag{15.43}$$

We assume that the preferred distribution of the traffic is given by a real matrix $A \in \mathbb{R}^{m \times n}$ satisfying

$$A := (a_{ij})_{(i,j)\in\{1+n,\ldots,m+n\}\times\{1,\ldots,n\}}, \quad 0 < a_{ji} < 1, \qquad \sum_{i=1+n,\ldots,m+n} a_{ij} = 1 \quad \text{for all} \quad j = 1, \ldots, n \tag{15.44}$$

such that we expect to get the outgoing fluxes in term of the ingoing fluxes as follows

$$\bar{\gamma}^i = \sum_{j=1,\dots,n} a_{ij}\bar{\gamma}^j, \quad i = 1+n,\dots,m+n \tag{15.45}$$

We call A a traffic distribution matrix. This matrix encodes the statistical preferences of the drivers depending on their incoming roads. The Coclite-Garavello-Piccoli preflux is particularly designed in order to satisfy relations (15.45) and also to maximize the total flux passing through the junction point.

We have the following result.

Lemma 15.26 (An approximate preflux)
Let $N := m+n$ with $n, m \geq 1$ and a matrix A satisfying (15.44). Let

$$K(\bar{\gamma}) := \{\gamma \in \Gamma(\bar{\gamma}), \quad L(\gamma) = 0\} \quad \textit{with} \quad \Gamma(\bar{\gamma}) := \prod_{k=1,\dots,N} [0, \bar{\gamma}^k] \quad \textit{and} \quad L(\gamma) := \sum_{k=1,\dots,N} \sigma^k \gamma^k$$

with σ defined in (15.43).
Let $\varepsilon, \delta > 0$ and the strictly convex function $\Psi : [0,+\infty)^N \to [0+\infty)$ defined as

$$\Psi_{\varepsilon,\delta}(\gamma) := \varepsilon|\gamma|^2 + \sum_{j=1,\dots,n} \left\{ \gamma^j + \delta^{-1} \left| \sum_{i=1+n,\dots,m+n} a_{ij}\gamma^j - \gamma^i \right|^2 \right\}$$

For every $\bar{\gamma} \in [0,+\infty)^N$, we define

$$\hat{\gamma}_{\varepsilon,\delta}(\bar{\gamma}) := \underset{K(\bar{\gamma})}{Argmin} \quad \Psi_{\varepsilon,\delta}$$

Then $\hat{\gamma}_{\varepsilon,\delta} : [0,+\infty)^N \to [0,+\infty)^N$ is a conservative preflux in the sense of Definition 11.1.

Proof of Lemma 15.26
Point i) follows immediately from Lemma 15.2.

Corollary 15.27 (Coclite-Garavello-Piccoli preflux for $n : m$ junctions with $n \leq m$)
Let $N := m+n$ with $n, m \geq 1$ and a matrix A satisfying (15.44). Let

$$K(\bar{\gamma}) := \{\gamma \in \Gamma(\bar{\gamma}), \quad L(\gamma) = 0\} \quad \textit{with} \quad \Gamma(\bar{\gamma}) := \prod_{k=1,\dots,N} [0, \bar{\gamma}^k] \quad \textit{and} \quad L(\gamma) := \sum_{k=1,\dots,N} \sigma^k \gamma^k$$

with σ defined in (15.43).
Let the convex function $\Psi : [0,+\infty)^N \to [0+\infty)$ defined as

$$\Psi(\gamma) := -\sum_{j=1,\dots,n} \gamma^j$$

and

$$K_A(\bar{\gamma}) := \left\{\gamma = (\gamma^1,\dots,\gamma^N) \in \Gamma(\bar{\gamma}) \quad \textit{with} \quad (\gamma^{1+n},\dots,\gamma^{m+n})^T = A\cdot(\gamma^1,\dots,\gamma^n)^T\right\} \quad \subset K(\bar{\gamma})$$

i) (Definition of Coclite-Garavello-Piccoli preflux)
Assume that for every $\bar{\gamma} \in [0,+\infty)^N$,

$$\textit{the value} \quad \hat{\gamma}^{CGP}(\bar{\gamma}) := \underset{K_A(\bar{\gamma})}{Argmin} \quad \Psi \quad \textit{is uniquely defined} \tag{15.46}$$

with $\hat{\gamma}^{CGP}(\bar{\gamma}) \in K_A(\bar{\gamma})$. Then the map $\hat{\gamma}^{CGP} : [0,+\infty)^N \to [0,+\infty)^N$ is a conservative preflux in the sense of Definition 11.1, which is called the Coclite-Garavello-Piccoli preflux.
ii) (A characterization when $\hat{\gamma}^{CGP}$ is uniquely defined)
For $\mathbb{R}^n$ with standard orthonormal basis $(e_1,\dots,e_n)$, we define the vectors b_i for $i = 1,\dots,n+m$ as

$$b_i := \begin{cases} e_i & \textit{if} \quad i = 1,\dots,n, \\ (a_{i1},\dots,a_{i,n}) & \textit{if} \quad i = 1+n,\dots,m+n, \end{cases}$$

and set furthermore the vector $e := (1, \dots, 1) \in \mathbb{R}^n$. *We also define for*

$$\sigma^I \in \{\pm 1\}^I \quad \text{for} \quad \emptyset \neq I \subset \{1, \dots, n+m\} \tag{15.47}$$

the convex cone

$$K_{I,\sigma^I} := \bigcap_{i \in I} \{\gamma' \in \mathbb{R}^n, \quad \sigma^i(b_i \cdot \gamma') \geq 0\}$$

Then $\hat{\gamma}^{CGP}$ *is uniquely defined if and only if the following geometric condition is satisfied*

$$\text{for all } (I, \sigma^I) \text{ as in (15.47)}, \quad K_{I,\sigma^I} \subset H^e_{\leq} \quad \text{implies} \quad K_{I,\sigma^I} \cap e^\perp = \{0\}\,. \tag{15.48}$$

iii) (A sufficient condition)
The following condition

$$\text{for every subset } \emptyset \neq I \subset \{1, \dots, m+n\}, \text{ we have:} \quad \left(\{0\} \neq V_I := \bigcap_{i \in I} b_i^\perp\right) \quad \Longrightarrow \quad V_I \not\subset e^\perp \tag{15.49}$$

implies the geometric condition (15.48).
iv (A further property)
Moreover condition (15.49) is equivalent to the following condition

$$\text{for every subset } \emptyset \neq I \subset \{1, \dots, m+n\}, \text{ we have:} \quad e \in \sum_{i \in I} \mathbb{R} b_i \quad \Longrightarrow \quad \sum_{i \in I} \mathbb{R} b_i = \mathbb{R}^n \tag{15.50}$$

Remark 15.28 *Condition (15.49) can easily be seen to be a reformulation of condition (C) in Coclite, Garavello, Piccoli introduced in [14].*

Proof of Corollary 15.27
Step 1: proof of i)
Point i) follows from Lemma 15.26, in the limit $\delta \to 0$ and then $\varepsilon \to 0$.
Step 2: proof of ii)
We want to maximize $E(\gamma') := \gamma^1 + \dots + \gamma^n$ for $\gamma' := (\gamma^1, \dots, \gamma^n)$ in the compact convex set

$$K_*(\bar{\gamma}) := K_1 \cap K_2 \quad \text{with} \quad \left\{ \begin{array}{l} K_1 := [0, (\bar{\gamma}^1, \dots, \bar{\gamma}^n)] := \prod_{i=1,\dots,n} [0, \bar{\gamma}^i], \\ K_2 := A^{-1}[0, (\bar{\gamma}^{1+n}, \dots, \bar{\gamma}^{m+n})] \end{array} \right.$$

Assume to simplify that $\bar{\gamma}^k > 0$ for all $k = 1, \dots, n+m$ (the other cases being limit cases).
Step 2.1: preliminaries
Let us consider a point $p \in \partial K_*$. Then locally around p, there exists I, σ^I as above such that we can write $K_*(\bar{\gamma}) - p$ as

$$\tilde{K}_{I,\sigma^I} := \bigcap_{i \in I} \left\{ \gamma' \in \mathbb{R}^n, \quad \left| \begin{array}{lll} \sigma^i(e_i, \gamma')_{\mathbb{R}^n} \geq 0 & \text{if} & i \in \{1, \dots, n\}\,, \\ \sigma^i(A\gamma', e_i)_{\mathbb{R}^m} \geq 0 & \text{if} & i \in \{1+n, \dots, m+n\}\,, \end{array} \right. \right\}$$

where $(e_{1+n}, \dots, e_{m+n})$ is an orthonormal basis of $\mathbb{R}^m$. Hence

$$\tilde{K}_{I,\sigma^I} := \bigcap_{i \in I} \left\{ \gamma' \in \mathbb{R}^n, \quad \left| \begin{array}{lll} \sigma^i(e_i, \gamma')_{\mathbb{R}^n} \geq 0 & \text{if} & i \in \{1, \dots, n\}\,, \\ \sigma^i(A^T e_i, \gamma')_{\mathbb{R}^n} \geq 0 & \text{if} & i \in \{1+n, \dots, m+n\}\,, \end{array} \right. \right\}$$

i.e.

$$\tilde{K}_{I,\sigma^I} := \bigcap_{i \in I} \{\gamma' \in \mathbb{R}^n, \quad \sigma^i(b_i, \gamma')_{\mathbb{R}^n} \geq 0\} = K_{I,\sigma^I}$$

Moreover, if p is an optimizer of E on $K_*(\bar{\gamma})$, then

$$K_*(\bar{\gamma}) - p \subset H^e_{\leq} := \{\gamma' \in \mathbb{R}^n, \quad (e, \gamma') \leq 0\}$$

which implies

$$K_{I,\sigma^I} \subset H^e_{\leq}.$$

We now want to show that the following condition

(15.51) there exists (I,σ^I) as in (15.47) and a point $\bar q$ such that $\quad K_{I,\sigma^I}\subset H^e_\le \quad$ and $\quad 0\neq \bar q\in K_{I,\sigma^I}\cap e^\perp$

is equivalent to the fact that there exists $\bar\gamma>0$ such that the minimizers of E over $K_*(\bar\gamma)$ are not unique.

Step 2.2: sufficient condition

Assume (15.51) and let us show that there exists $\bar\gamma\in(0,+\infty)^{n+m}$ and $p\in K_*(\bar\gamma)$ a maximizer of E on $K_*(\bar\gamma)$ such that $K_*(\bar\gamma)-p$ is locally equal to K_{I,σ^I} in a neighborhood of 0. We define

$$I^\pm := \left\{i\in I,\quad \sigma^i=\pm 1\right\},\quad \bar I := \{1,\dots,n+m\}\setminus I$$

and set

$$\hat K_*(\bar\gamma) := \left\{ \gamma'\in\mathbb{R}^n,\quad \left| \begin{array}{lll} \gamma^i\le\bar\gamma^i & \text{if} & i\in I^-\cap\{1,\dots,n\},\\ \gamma^i\ge 0 & \text{if} & i\in I^+\cap\{1,\dots,n\},\\ (A\gamma')^i\le\bar\gamma^i & \text{if} & i\in I^-\cap\{1+n,\dots,m+n\},\\ (A\gamma')^i\ge 0 & \text{if} & i\in I^+\cap\{1+n,\dots,m+n\}, \end{array}\right.\right\}$$

which satisfies

$$\hat K_*(0)=K_{I,\sigma^I}$$

Hence given $\bar\gamma\in(0,+\infty)^{n+m}$, we have

$$\hat K_*(\bar\gamma)=\hat K_*(\bar\gamma+\sum_{j\in\bar I}[0,+\infty)e_j)$$

and we can consider a maximizer p of $E(\gamma')=\gamma^1+\dots+\gamma^n$ on $\hat K_*(\bar\gamma)$ which is then independent on the components $\bar\gamma^j$ for $j\in\bar I$. We have

$$p^i\ge 0 \quad \text{for all} \quad i\in I^+\cap\{1,\dots,n\}$$

but only

$$p^i\le\bar\gamma^i \quad \text{for all} \quad i\in I^-\cap\{1,\dots,n\}$$

Up to add to p some

$$\bar p\in \sum_{j\in(\bar I\cup I^-)\cap\{1,\dots,n\}}[0,+\infty)e_j$$

and to redefine $\bar\gamma$, we can assume that

$$0\le p\in\hat K_*(\bar\gamma) \quad \text{which implies} \quad Ap\ge 0.$$

Moreover up to increase only the components $\bar\gamma^j$ for $j\in(I^+\cup\bar I)\cap\{1+n,\dots,m+n\}$, we can insure that

$$Ap\le\bar\gamma'':=(\bar\gamma^{1+n},\dots,\bar\gamma^{m+n})$$

Then $p\in K_*(\bar\gamma)$ and $K_*(\bar\gamma)$ is locally equal to $\hat K_*(\bar\gamma)$ around p.

Hence using (15.51) and the convexity of the considered sets, we see that $p+\varepsilon q\in\hat K_*(\bar\gamma)$ for all $\varepsilon\in[0,1]$ small enough. Because $\hat K_*(\bar\gamma)$ is equal to $K_*(\bar\gamma)$ in a neighborhood of p, we deduce that $p+\varepsilon\bar q\in K_*(\bar\gamma)$ for $\varepsilon>0$ small enough. Finally $E(p)=E(p+\varepsilon\bar q)$ shows the non uniqueness of the optimizer.

Step 2.3: necessary condition

Assume now that there exists two distinct minimizers $p,p+\bar q\in K_*(\bar\gamma)$ of E. From Step 2.1, there exists (I,σ^I) such that locally around p, we can write $K_*(\bar\gamma)-p$ as K_{I,σ^I} and

$$0\neq\bar q\in K_*(\bar\gamma)-p\subset H^e_\le \quad \text{with} \quad K_{I,\sigma^I}\subset H^e_\le$$

Because $E(p)=E(p+\bar q)$, we deduce that

$$0\neq\bar q\in K_{I,\sigma^I}\cap e^\perp$$

which shows condition (15.51).

Step 2.4: consequences

By contraposition of condition (15.51), we get that E has a unique minimizer on $K_*(\bar\gamma)$ if and only (15.48) holds.

Step 3: proof of iii)
We assume (15.49). We then show the result by contradiction. Assume that (15.48) is not true, i.e. that (15.51), holds. This means that there exists some (I,σ^I) as in (15.47) and a point $\bar{q}$ such that

$$K_{I,\sigma^I} \subset H^e_{\leq} \quad \text{and} \quad 0 \neq \bar{q} \in K_{I,\sigma^I} \cap e^{\perp}.$$

We know that $K_{I,\sigma^I} \cap e^{\perp}$ is a closed convex set of dimension $1 \leq k \leq n-1$ (the dimension k being the maximal integer such that we can find $k+1$ points barycentrically independent in the closed convex set). Up to redefine $\bar{q}$ and choose it to be the barycenter of $k+1$ points in the set

$$K := K_{I,\sigma^I} \cap e^{\perp}$$

we can now consider the tangential cone to the closed convex set K at $\bar{q}$ given by

$$T_{\bar{q}}K := \{v \in \mathbb{R}^n, \quad \bar{q} + \varepsilon v_\varepsilon \in K, \quad v_\varepsilon \to v \quad \text{as} \quad \varepsilon \to 0\}$$

Because K is polyedral, we also have

$$T_{\bar{q}}K := \{v \in \mathbb{R}^n, \quad \bar{q} + \varepsilon v \in K, \quad \text{for } \varepsilon > 0 \text{ small enough}\}$$

(and moreover ε can be chosen independently of v, because K is polyedral). By construction, we have

$$\mathbb{R}^k \simeq V_0 := T_{\bar{q}}K \subset T_{\bar{q}}K_{I,\sigma^I} \subset T_{\bar{q}}H^e_{\leq} = H^e_{\leq} \quad \text{with} \quad V_0 \subset e^{\perp}$$

Hence there exists $I' \subset I$ such that locally around $\bar{q}$, the set $K_{I,\sigma^I} - \bar{q}$ is equal to the cone $K_{I',\sigma^{I'}}$. Hence the the closed convex cone

$$K' := K_{I',\sigma^{I'}} \cap e^{\perp}$$

satisfies (because it is a cone)

$$K' = T_0K' = T_{\bar{q}}K = V_0 \quad \text{and} \quad K_{I',\sigma^{I'}} \subset H^e_{\leq}$$

i.e.

$$V_0 = K' \subset K_{I',\sigma^{I'}} \subset H^e_{\leq} \tag{15.52}$$

Notice that for $I' \subset I'' \subset I$, we may have $K_{I',\sigma^{I'}} = K_{I'',\sigma^{I''}}$, so that we can always choose I' to be minimal. Then each constraint defining $K_{I',\sigma^{I'}}$ is active on the boundary of $K_{I',\sigma^{I'}}$ in $\text{Span}_{\mathbb{R}}(K_{I',\sigma^{I'}})$. This implies that

$$V_{I'} \subset \mathbb{R}K_{I',\sigma^{I'}} \subset H^e_{\leq}$$

We deduce that

$$V_{I'} \subset e^{\perp}$$

On the other hand (15.52) implies that

$$V_0 \subset V_{I'}$$

because V_0 is a vector space, while constaints defining K_{I,σ^I} are constraint on half spaces. Therefore $1 \leq k := \dim_{\mathbb{R}} V_0$ implies

$$\{0\} \neq V_0 \subset V_{I'} \subset e^{\perp}$$

which leads to a contradiction with assumption (15.49). Therefore (15.48) holds true.

Step 4: proof of iv)
Recall that $V_I := \bigcap_{i\in I} b_i^{\perp}$, and set

$$W_I := \left\{ \sum_{i\in I} z_i b_i, \quad z^I \in \mathbb{R}^I \right\}$$

Then we definition, we have

$$W_I^{\perp} = V_I$$

Hence condition (15.49), namely

$$\{0\} \neq V_I \quad \Longrightarrow \quad V_I \not\subset e^{\perp}$$

means

$$e \in V_I^\perp \quad \Longrightarrow \quad V_I = \{0\}$$

which is equivalent to

$$e \in W_I \quad \Longrightarrow \quad W_I = \mathbb{R}^n$$

which is exactly condition (15.50). This ends the proof of the corollary.

Lemma 15.29 (A case study $2:2$ preflux)
Let $N := m+n$ with $n = m = 2$ for a $2:2$ junction (hence with $\sigma = (1,1,-1,-1)$) and a matrix

$$A = \begin{pmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix} \quad \text{with} \quad a_{j1} \neq a_{j2}, \quad j = 3,4$$

satisfying (15.44), i.e.

$$A = \begin{pmatrix} \alpha & \beta \\ 1-\alpha & 1-\beta \end{pmatrix} \quad \text{with} \quad \alpha, \beta \in (0,1), \quad \alpha > \beta.$$

Let

$$K(\bar\gamma) := \left\{ \gamma \in \Gamma(\bar\gamma), \quad \gamma^j = L^j_{12}(\gamma), \quad j = 3,4 \right\} \quad \text{with} \quad \Gamma(\bar\gamma) := \prod_{k=1,\dots,N} [0, \bar\gamma^k] \quad \text{and} \quad L^j_{12}(\gamma) := \sum_{i=1,2} a_{ji}\gamma^i$$

Let us consider the convex function $\Psi : [0,+\infty)^N \to [0+\infty)$ defined for $\gamma = (\gamma^1, \gamma^2, \gamma^3, \gamma^4)$ as

$$\Psi(\gamma) := \gamma^1 + \gamma^2$$

Then for every $\bar\gamma \in [0,+\infty)^N$, the following quantity

$$\hat\gamma(\bar\gamma) := \underset{K(\bar\gamma)}{Argmin} \quad \Psi$$

is well defined.

i) (Conservative preflux)
Then $\hat\gamma : [0,+\infty)^N \to [0,+\infty)^N$ is a conservative preflux in the sense of Definition 11.1. In particular it satisfies

$$\hat\gamma^1 + \hat\gamma^2 = \hat\gamma^3 + \hat\gamma^4$$

ii) (Particular monotonicities)
Moreover, we have $\alpha > \beta$ and we have the following monotonicities

$$\left\{ \begin{array}{l} \hat\gamma^1(\uparrow, \boxed{\downarrow}; \quad \uparrow, \boxed{\downarrow}), \\ \hat\gamma^2(\boxed{\downarrow}, \uparrow; \quad \boxed{\downarrow}, \uparrow), \\ \hat\gamma^3(\uparrow, \uparrow; \quad \uparrow, \uparrow), \\ \hat\gamma^4(\uparrow, \uparrow; \quad \uparrow, \uparrow), \end{array} \right.$$

iii) (Explicit preflux)
We set

$$\left\{ \begin{array}{lrr} L^3_{12}(\gamma) := & \alpha\gamma^1 & +\beta\gamma^2, \\ L^4_{12}(\gamma) := & (1-\alpha)\gamma^1 & +(1-\beta)\gamma^2, \end{array} \right. \tag{15.53}$$

Then we consider the set $\Lambda \subset H := \left\{ \gamma = (\gamma^1, \gamma^2, \gamma^3, \gamma^4) \in \mathbb{R}^4, \quad \gamma^1 + \gamma^2 = \gamma^3 + \gamma^4 \right\}$ and four of its different parametrizations

$$\left\{ \begin{array}{l} \Lambda := \left\{ \gamma \in \mathbb{R}^4, \quad (\gamma^3, \gamma^4) = (L^3_{12}, L^4_{12})(\gamma) \right\}, \\ \Lambda := \left\{ \gamma \in \mathbb{R}^4, \quad (\gamma^1, \gamma^2) = (L^1_{34}, L^2_{34})(\gamma) \right\}, \\ \Lambda := \left\{ \gamma \in \mathbb{R}^4, \quad (\gamma^1, \gamma^4) = (L^1_{23}, L^4_{23})(\gamma) \right\}, \\ \Lambda := \left\{ \gamma \in \mathbb{R}^4, \quad (\gamma^2, \gamma^3) = (L^2_{14}, L^3_{14})(\gamma) \right\}, \\ \Lambda := \left\{ \gamma \in \mathbb{R}^4, \quad (\gamma^1, \gamma^3) = (L^1_{24}, L^3_{24})(\gamma) \right\}, \\ \Lambda := \left\{ \gamma \in \mathbb{R}^4, \quad (\gamma^2, \gamma^4) = (L^2_{13}, L^4_{13})(\gamma) \right\}, \end{array} \right. \tag{15.54}$$

defining L^i_{jk} are defined explicitely in (15.53),(15.58),(15.59),(15.60),(15.61),(15.62) with the convention that

$$\partial_l L^i_{jk} = 0 \quad \text{if} \quad l \notin \{j,k\}$$

With some abuse of notation, we also use the notation

$$L^i_{jk}(\gamma) = L^i_{jk}(\gamma^j, \gamma^k).$$

Then $\hat{\gamma}(\bar{\gamma}) \in \Lambda$ and the following holds true.

iii.a) (Preflux by zone)

We have for $\hat{\gamma}(\bar{\gamma}) = (\hat{\gamma}^1, \hat{\gamma}^2, \hat{\gamma}^3, \hat{\gamma}^4)(\bar{\gamma})$

(15.55)

$$\hat{\gamma}(\bar{\gamma}) = \begin{cases} (\; L^1_{34}(\bar{\gamma}), \; L^2_{34}(\bar{\gamma}), \; \bar{\gamma}^3, \; \bar{\gamma}^4 \;) & \text{if} \; \begin{cases} 0 < L^1_{34}(\bar{\gamma}) < \bar{\gamma}^1, \\ 0 < L^2_{34}(\bar{\gamma}) < \bar{\gamma}^2, \end{cases} & \textbf{(Case A)} \\ (\; L^1_{23}(\bar{\gamma}), \; \bar{\gamma}^2, \; \bar{\gamma}^3, \; L^4_{23}(\bar{\gamma}) \;) & \text{if} \; \begin{cases} L^3_{12}(0,\bar{\gamma}^2) < \bar{\gamma}^3 < L^3_{12}(\bar{\gamma}), \\ L^2_{34}(\bar{\gamma}) > \bar{\gamma}^2, \end{cases} & \textbf{(Case B)} \\ (\; \bar{\gamma}^1, \; L^2_{14}(\bar{\gamma}), \; L^3_{14}(\bar{\gamma}), \; \bar{\gamma}^4 \;) & \text{if} \; \begin{cases} L^1_{34}(\bar{\gamma}) > \bar{\gamma}^1, \\ L^4_{12}(\bar{\gamma}^1,0) < \bar{\gamma}^4 < L^4_{12}(\bar{\gamma}), \end{cases} & \textbf{(Case C)} \\ (\; \bar{\gamma}^1, \; \bar{\gamma}^2, \; L^3_{12}(\bar{\gamma}), \; L^4_{12}(\bar{\gamma}) \;) & \text{if} \; \begin{cases} \bar{\gamma}^3 > L^3_{12}(\bar{\gamma}), \\ \bar{\gamma}^4 > L^4_{12}(\bar{\gamma}), \end{cases} & \textbf{(Case D)} \\ (\; 0, \; L^2_{13}(0,\bar{\gamma}^3), \; \bar{\gamma}^3, \; L^4_{13}(0,\bar{\gamma}^3) \;) & \text{if} \; \begin{cases} 0 < \bar{\gamma}^3 < L^3_{12}(0,\bar{\gamma}^2), \\ L^1_{34}(\bar{\gamma}) < 0, \end{cases} & \textbf{(Case B')} \\ (\; L^1_{24}(0,\bar{\gamma}^4), \; 0, \; L^3_{24}(0,\bar{\gamma}^4), \; \bar{\gamma}^4 \;) & \text{if} \; \begin{cases} 0 < \bar{\gamma}^4 < L^4_{12}(\bar{\gamma}^1,0), \\ L^2_{34}(\bar{\gamma}) < 0, \end{cases} & \textbf{(Case C')} \end{cases}$$

iii.b) (Preflux based on a minimum)

We have

$$(15.56) \qquad \begin{cases} \hat{\gamma}^3 = \min\left\{\bar{\gamma}^3, L^3_{12}(\bar{\gamma}), L^3_{14}(\bar{\gamma}), L^3_{24}(0,\bar{\gamma}^4)\right\}, \\ \hat{\gamma}^4 = \min\left\{\bar{\gamma}^4, L^4_{12}(\bar{\gamma}), L^4_{23}(\bar{\gamma}), L^4_{13}(0,\bar{\gamma}^3)\right\}, \\ \\ \hat{\gamma}^1 = L^1_{34}(\hat{\gamma}^3, \hat{\gamma}^4), \\ \hat{\gamma}^2 = L^2_{34}(\hat{\gamma}^3, \hat{\gamma}^4), \end{cases}$$

Moreover, we have

$$(15.57) \qquad (\hat{\gamma}^1 + \hat{\gamma}^2)(\bar{\gamma}) = (\hat{\gamma}^3 + \hat{\gamma}^4)(\bar{\gamma}) \leq \min\left\{\bar{\gamma}^1 + \bar{\gamma}^2, \bar{\gamma}^3 + \bar{\gamma}^4\right\}$$

where the inequality is strict in Cases B,C,B',C'.

Remark 15.30 *In Figure 10, we set $\bar{P} := (\bar{\gamma}^1, \bar{\gamma}^2)$ and $Q := (L^1_{34}, L^2_{34})(\bar{\gamma})$, and define $P := (\hat{\gamma}^1, \hat{\gamma}^2)(\bar{\gamma})$, in the six different cases (A),(B),(C),(D),(B'),(C').*

Proof of Lemma 15.29

Step 1: proof of i)

Point i) follows from Corollary 15.27, using the fact that $\mathrm{Ker}(A) = \{0\}$ when $\alpha \neq \beta$.

Step 2: proof of ii)

Point ii) follows from point iii) and the explicit expressions of L^i_{jk} given in the proof below.

Step 3: proof of iii)

Step 3.1: proof of iii.a)

From the definition of L^3_{12} and L^4_{12}, we deduce by inversion

$$(15.58) \qquad \begin{cases} L^1_{34}(\gamma) := \quad \left(\dfrac{1-\beta}{\alpha-\beta}\right)\gamma^3 - \left(\dfrac{\beta}{\alpha-\beta}\right)\gamma^4, \\ \\ L^2_{34}(\gamma) := -\left(\dfrac{1-\alpha}{\alpha-\beta}\right)\gamma^3 + \left(\dfrac{\alpha}{\alpha-\beta}\right)\gamma^4, \end{cases}$$

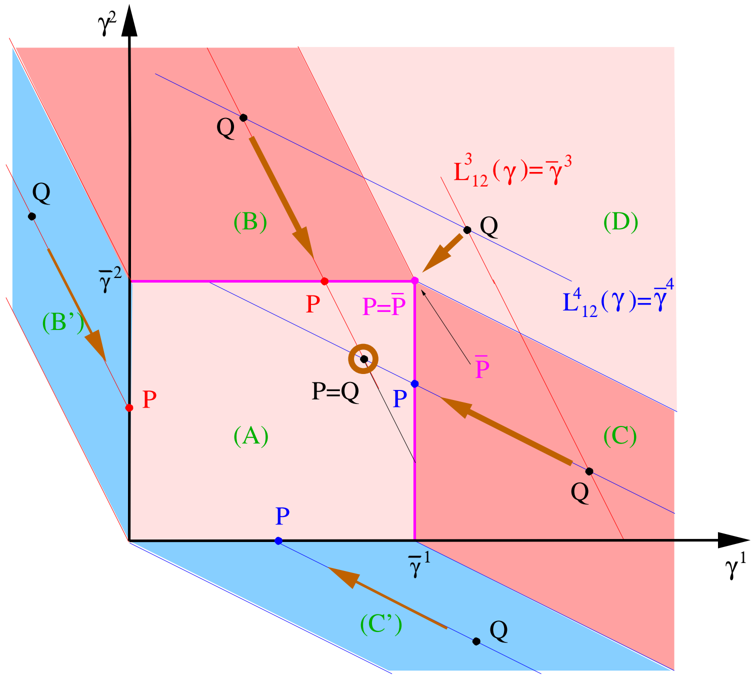

Figure 10: Coclite, Garavello, Piccoli preflux for 2 : 2 junctions, with $a_{31} = \alpha > \beta = a_{41}$: projection $Q \mapsto P$.

and

(15.59)
$$\begin{cases} L^1_{23}(\gamma) := & -\left(\dfrac{\beta}{\alpha}\right)\gamma^2 & +\left(\dfrac{1}{\alpha}\right)\gamma^3, \\ L^4_{23}(\gamma) := & \left(\dfrac{\alpha-\beta}{\alpha}\right)\gamma^2 & +\left(\dfrac{1-\alpha}{\alpha}\right)\gamma^3, \end{cases}$$

and

(15.60)
$$\begin{cases} L^2_{14}(\gamma) := & -\left(\dfrac{1-\alpha}{1-\beta}\right)\gamma^1 & +\left(\dfrac{1}{1-\beta}\right)\gamma^4, \\ L^3_{14}(\gamma) := & \left(\dfrac{\alpha-\beta}{1-\beta}\right)\gamma^1 & +\left(\dfrac{\beta}{1-\beta}\right)\gamma^4. \end{cases}$$

and

(15.61)
$$\begin{cases} L^1_{24}(\gamma) := & -\left(\dfrac{1-\beta}{1-\alpha}\right)\gamma^2 & +\left(\dfrac{1}{1-\alpha}\right)\gamma^4, \\ L^3_{24}(\gamma) := & -\left(\dfrac{\alpha-\beta}{1-\alpha}\right)\gamma^2 & +\left(\dfrac{\alpha}{1-\alpha}\right)\gamma^4, \end{cases}$$

and also

(15.62)
$$\begin{cases} L^2_{13}(\gamma) := & -\left(\dfrac{\alpha}{\beta}\right)\gamma^1 & +\left(\dfrac{1}{\beta}\right)\gamma^3, \\ L^4_{13}(\gamma) := & -\left(\dfrac{\alpha-\beta}{\beta}\right)\gamma^1 & +\left(\dfrac{1-\beta}{\beta}\right)\gamma^3. \end{cases}$$

We now use $\alpha > \beta$, and (see for instance Figure 10) we define (for $P = (P^1, P^2) \in [0, +\infty)^2$ and

$\bar{P} := (\bar{\gamma}^1, \bar{\gamma}^2)$) the sets

$$\begin{cases} (A) := \left\{(\gamma^1,\gamma^2) \in \mathbb{R}^2, \quad 0<\gamma^1<\bar{\gamma}^1, \quad 0<\gamma^2<\bar{\gamma}^2\right\}, \\ (B) := \left\{(\gamma^1,\gamma^2) \in \mathbb{R}^2, \quad L^3_{12}(0,\bar{\gamma}^2) < L^3_{12}(\gamma) < L^3_{12}(\bar{P}), \quad \gamma^2 > \bar{\gamma}^2\right\}, \\ (C) := \left\{(\gamma^1,\gamma^2) \in \mathbb{R}^2, \quad \gamma^1 > \bar{\gamma}^1, \quad L^4_{12}(\bar{\gamma}^1,0) < L^4_{12}(\gamma) < L^4_{12}(\bar{P})\right\}, \\ (D) := \left\{(\gamma^1,\gamma^2) \in \mathbb{R}^2, \quad L^3_{12}(\gamma) > L^3_{12}(\bar{P}), \quad L^4_{12}(\gamma) > L^4_{12}(\bar{P})\right\}, \\ (B') := \left\{(\gamma^1,\gamma^2) \in \mathbb{R}^2, \quad \gamma^1 < 0, \quad 0 < L^3_{12}(\gamma) < L^3_{12}(0,\bar{\gamma}^2)\right\}, \\ (C') := \left\{(\gamma^1,\gamma^2) \in \mathbb{R}^2, \quad \gamma^2 < 0, \quad 0 < L^4_{12}(\gamma) < L^4_{12}(\bar{\gamma}^1,0)\right\}, \end{cases}$$

Notice that the Argmax defining $\hat{\gamma}(\bar{\gamma})$ looks at the maximum of $\gamma^1+\gamma^2$ on the intersection of the box $[0,\bar{\gamma}^1]\times[0,\bar{\gamma}^2]$ with the parallogram $(L^1_{34},L^2_{34})([0,\bar{\gamma}^3]\times[0,\bar{\gamma}^4])$ of slopes in the (γ^1,γ^2)-plane which are $-\dfrac{\alpha}{\beta} < -1 < -\left(\dfrac{1-\alpha}{1-\beta}\right)$, like on Figure 10.

Using a geometrical reasoning in the (γ^1,γ^2)-plane for the , we get in particular for $Q=(Q^1,Q^2) := (L^1_{34},L^2_{34})(\bar{\gamma}) \in (A),(B),(C),(D),(B'),(C')$ the following

(15.63)

$$\hat{\gamma}(\bar{\gamma}) \in \Lambda \quad \text{with} \quad \begin{cases} (\hat{\gamma}^3,\hat{\gamma}^4) = (\bar{\gamma}^3,\bar{\gamma}^4) & \text{if } 0<Q^1<\bar{\gamma}^1 \text{ and } 0<Q^2<\bar{\gamma}^2, & \textbf{(A)} \\ (\hat{\gamma}^2,\hat{\gamma}^3) = (\bar{\gamma}^2,\bar{\gamma}^3) & \text{if } L^3_{12}(0,\bar{\gamma}^2) < L^3_{12}(Q) < L^3_{12}(\bar{P}) \text{ and } Q^2>\bar{\gamma}^2, & \textbf{(B)} \\ (\hat{\gamma}^1,\hat{\gamma}^4) = (\bar{\gamma}^1,\bar{\gamma}^4) & \text{if } Q^1>\bar{\gamma}^1 \text{ and } L^4_{12}(\bar{\gamma}^1,0) < L^4_{12}(Q) < L^4_{12}(\bar{P}), & \textbf{(C)} \\ (\hat{\gamma}^1,\hat{\gamma}^2) = (\bar{\gamma}^1,\bar{\gamma}^2) & \text{if } L^3_{12}(Q) > L^3_{12}(\bar{P}) \text{ and } L^4_{12}(Q) > L^4_{12}(\bar{P}), & \textbf{(D)} \\ (\hat{\gamma}^1,\hat{\gamma}^3) = (0,\bar{\gamma}^3) & \text{if } Q^1<0 \text{ and } 0<L^3_{12}(Q)<L^3_{12}(0,\bar{\gamma}^2), & \textbf{(B')} \\ (\hat{\gamma}^2,\hat{\gamma}^4) = (0,\bar{\gamma}^4) & \text{if } Q^2<0 \text{ and } 0<L^4_{12}(Q)<L^4_{12}(\bar{\gamma}^1,0), & \textbf{(C')} \end{cases}$$

We also notice that algebraically, we get for $\gamma=(\gamma^1,\gamma^2,\gamma^3,\gamma^4)$, we have

$$(L^3_{12},L^4_{12}) \circ (L^1_{34},L^2_{34}) = id_{[0,+\infty)^2}$$

Hence, we get

$$\hat{\gamma}(\bar{\gamma}) \in \Lambda \quad \text{with} \quad \begin{cases} (\hat{\gamma}^3,\hat{\gamma}^4) = (\bar{\gamma}^3,\bar{\gamma}^4) & \text{if } 0<L^1_{34}(\bar{\gamma})<\bar{\gamma}^1 \text{ and } 0<L^2_{34}(\bar{\gamma})<\bar{\gamma}^2, & \textbf{(A)} \\ (\hat{\gamma}^2,\hat{\gamma}^3) = (\bar{\gamma}^2,\bar{\gamma}^3) & \text{if } L^3_{12}(0,\bar{\gamma}^2) < \bar{\gamma}^3 < L^3_{12}(\bar{\gamma}) \text{ and } L^2_{34}(\bar{\gamma})>\bar{\gamma}^2, & \textbf{(B)} \\ (\hat{\gamma}^1,\hat{\gamma}^4) = (\bar{\gamma}^1,\bar{\gamma}^4) & \text{if } L^1_{34}(\bar{\gamma})>\bar{\gamma}^1 \text{ and } L^4_{12}(\bar{\gamma}^1,0) < \bar{\gamma}^4 < L^4_{12}(\bar{\gamma}), & \textbf{(C)} \\ (\hat{\gamma}^1,\hat{\gamma}^2) = (\bar{\gamma}^1,\bar{\gamma}^2) & \text{if } \bar{\gamma}^3 > L^3_{12}(\bar{\gamma}) \text{ and } \bar{\gamma}^4 > L^4_{12}(\bar{\gamma}), & \textbf{(D)} \\ (\hat{\gamma}^1,\hat{\gamma}^3) = (0,\bar{\gamma}^3) & \text{if } L^1_{34}(\bar{\gamma})<0 \text{ and } 0<\bar{\gamma}^3<L^3_{12}(0,\bar{\gamma}^2), & \textbf{(B')} \\ (\hat{\gamma}^2,\hat{\gamma}^4) = (0,\bar{\gamma}^4) & \text{if } L^2_{34}(\bar{\gamma})<0 \text{ and } 0<\bar{\gamma}^4<L^4_{12}(\bar{\gamma}^1,0), & \textbf{(C')} \end{cases}$$

Because $\hat{\gamma}(\bar{\gamma}) \in \Lambda$, this implies more precisely that

(15.64)

$$\boxed{\hat{\gamma}(\bar{\gamma}) = (\hat{\gamma}^1,\hat{\gamma}^2,\hat{\gamma}^3,\hat{\gamma}^4)(\bar{\gamma}) = \begin{cases} (\; L^1_{34}(\bar{\gamma}), & L^2_{34}(\bar{\gamma}), & \bar{\gamma}^3, & \bar{\gamma}^4 &) & \text{in Case } \textbf{(A)} \\ (\; L^1_{23}(\bar{\gamma}), & \bar{\gamma}^2, & \bar{\gamma}^3, & L^4_{23}(\bar{\gamma}) &) & \text{in Case } \textbf{(B)} \\ (\; \bar{\gamma}^1, & L^2_{14}(\bar{\gamma}), & L^3_{14}(\bar{\gamma}), & \bar{\gamma}^4 &) & \text{in Case } \textbf{(C)} \\ (\; \bar{\gamma}^1, & \bar{\gamma}^2, & L^3_{12}(\bar{\gamma}), & L^4_{12}(\bar{\gamma}) &) & \text{in Case } \textbf{(D)} \\ (\; 0, & L^2_{13}(0,\bar{\gamma}^3), & \bar{\gamma}^3, & L^4_{13}(0,\bar{\gamma}^3) &) & \text{in Case } \textbf{(B')} \\ (\; L^1_{24}(0,\bar{\gamma}^4), & 0, & L^3_{24}(0,\bar{\gamma}^4), & \bar{\gamma}^4 &) & \text{in Case } \textbf{(C')} \end{cases}}$$

and by construction, we have $\bar{\gamma} \geq \hat{\gamma}(\bar{\gamma})$, but it is also easy to check it directly.

Step 3.2: explicit expressions

We start with

Cases $\backslash\hat{\gamma}$	$\hat{\gamma}^1$	$\hat{\gamma}^2$	$\hat{\gamma}^3$	$\hat{\gamma}^4$	comparison 1	comparison 2
A	$\bar{\gamma}^1 > L^1_{34}$	$\bar{\gamma}^2 > L^2_{34}$	0	0	0	0
B	0	$L^2_{34} > \bar{\gamma}^2$	$L^3_{12} > \bar{\gamma}^3$	0	0	0
C	$L^1_{34} > \bar{\gamma}^1$	0	0	$L^4_{12} > \bar{\gamma}^4$	0	0
D	0	0	$\bar{\gamma}^3 > L^3_{12}$	$\bar{\gamma}^4 > L^4_{12}$	0	0
B'	$L^1_{34} < 0$	0	$L^3_{12}(0,\bar{\gamma}^2) > \bar{\gamma}^3$	0	0	0
C'	0	$L^2_{34} < 0$	0	$L^4_{12}(\bar{\gamma}^1,0) > \bar{\gamma}^4$	0	0

Notice that for $\alpha > \beta$, we get algebraically for $L^k_{ij} = L^k_{ij}(\bar{\gamma})$

(15.65)

$$\left\{\begin{array}{lllllllllll}
L^1_{34} > L^1_{23} & \Leftrightarrow & \bar{\gamma}^4 < L^4_{23} & \Leftrightarrow & \bar{\gamma}^2 > L^2_{34} & \Leftrightarrow & L^1_{34} > L^1_{24} & \Leftrightarrow & L^1_{23} > L^1_{24} & \Leftrightarrow & \bar{\gamma}^3 > L^3_{24} \\
L^2_{34} > L^2_{14} & \Leftrightarrow & \bar{\gamma}^3 < L^3_{14} & \Leftrightarrow & \bar{\gamma}^1 > L^1_{34} & \Leftrightarrow & L^2_{34} > L^2_{13} & \Leftrightarrow & L^2_{14} > L^2_{13} & \Leftrightarrow & \bar{\gamma}^4 > L^4_{13} \\
L^3_{14} > L^3_{12} & \Leftrightarrow & \bar{\gamma}^4 > L^4_{12} & \Leftrightarrow & \bar{\gamma}^2 < L^2_{14} & \Leftrightarrow & L^3_{14} < L^3_{24} & \Leftrightarrow & L^3_{12} < L^3_{24} & \Leftrightarrow & \bar{\gamma}^1 < L^1_{24} \\
L^4_{23} > L^4_{12} & \Leftrightarrow & \bar{\gamma}^3 > L^3_{12} & \Leftrightarrow & \bar{\gamma}^1 < L^1_{23} & \Leftrightarrow & L^4_{23} < L^4_{13} & \Leftrightarrow & L^4_{12} < L^4_{13} & \Leftrightarrow & \bar{\gamma}^2 < L^2_{13}
\end{array}\right.$$

From (15.65) we get (forgetting temporarily the Cases B' and C')

Cases $\backslash\hat{\gamma}$	$\hat{\gamma}^1$	$\hat{\gamma}^2$	$\hat{\gamma}^3$	$\hat{\gamma}^4$	comparison 1	comparison 2
A	$\bar{\gamma}^1 > L^1_{34} > L^1_{23}$	$\bar{\gamma}^2 > L^2_{34} > L^2_{14}$	$\boxed{L^3_{14} > \bar{\gamma}^3}$	$\boxed{L^4_{23} > \bar{\gamma}^4}$	0	0
B	$\bar{\gamma}^1 > L^1_{23} > L^1_{34}$	$L^2_{34} > \bar{\gamma}^2$	$\boxed{L^3_{12} > \bar{\gamma}^3}$	$\left\{\begin{array}{l} L^4_{12} > L^4_{23}, \\ \bar{\gamma}^4 > L^4_{23} \end{array}\right.$	0	0
C	$L^1_{34} > \bar{\gamma}^1$	$\bar{\gamma}^2 > L^2_{14} > L^2_{34}$	$\left\{\begin{array}{l} \bar{\gamma}^3 > L^3_{14} \\ L^3_{12} > L^3_{14} \end{array}\right.$	$\boxed{L^4_{12} > \bar{\gamma}^4}$	0	0
D	$L^1_{23} > \bar{\gamma}^1$	$L^2_{14} > \bar{\gamma}^2$	$\left\{\begin{array}{l} \bar{\gamma}^3 > L^3_{12} \\ L^3_{14} > L^3_{12} \end{array}\right.$	$\left\{\begin{array}{l} \bar{\gamma}^4 > L^4_{12} \\ L^4_{23} > L^4_{12} \end{array}\right.$	0	0

where the boxed inequalities are those which are not sufficient to identify the effective components $\hat{\gamma}^j(\bar{\gamma})$. Therefore, from the transitivity of the order ">", we get the additional comparisons

Cases $\backslash\hat{\gamma}$	$\hat{\gamma}^1$	$\hat{\gamma}^2$	$\hat{\gamma}^3$	$\hat{\gamma}^4$	comparison 1	comparison 2
A	$\bar{\gamma}^1 > L^1_{34} > L^1_{23}$	$\bar{\gamma}^2 > L^2_{34} > L^2_{14}$	$\boxed{L^3_{14} > \bar{\gamma}^3}$	$\boxed{L^4_{23} > \bar{\gamma}^4}$	$\bar{\gamma}^1 > L^1_{23}$	$\bar{\gamma}^2 > L^2_{14}$
B	$\bar{\gamma}^1 > L^1_{23} > L^1_{34}$	$L^2_{34} > \bar{\gamma}^2$	$\boxed{L^3_{12} > \bar{\gamma}^3}$	$\left\{\begin{array}{l} L^4_{12} > L^4_{23}, \\ \bar{\gamma}^4 > L^4_{23} \end{array}\right.$	$\bar{\gamma}^1 > L^1_{34}$	0
C	$L^1_{34} > \bar{\gamma}^1$	$\bar{\gamma}^2 > L^2_{14} > L^2_{34}$	$\left\{\begin{array}{l} \bar{\gamma}^3 > L^3_{14} \\ L^3_{12} > L^3_{14} \end{array}\right.$	$\boxed{L^4_{12} > \bar{\gamma}^4}$	0	$\bar{\gamma}^2 > L^2_{34}$
D	$L^1_{23} > \bar{\gamma}^1$	$L^2_{14} > \bar{\gamma}^2$	$\left\{\begin{array}{l} \bar{\gamma}^3 > L^3_{12} \\ L^3_{14} > L^3_{12} \end{array}\right.$	$\left\{\begin{array}{l} \bar{\gamma}^4 > L^4_{12} \\ L^4_{23} > L^4_{12} \end{array}\right.$	0	0

Hence from the new comparisons and (15.65), we deduce

(15.66)

Cases $\backslash\hat{\gamma}$	$\hat{\gamma}^1$	$\hat{\gamma}^2$	$\hat{\gamma}^3$	$\hat{\gamma}^4$	comparison 1	comparison 2
A	$\bar{\gamma}^1 > L^1_{34} > L^1_{23}$	$\bar{\gamma}^2 > L^2_{34} > L^2_{14}$	$L^3_{12} > L^3_{14} > \bar{\gamma}^3$	$L^4_{12} > L^4_{23} > \bar{\gamma}^4$	0	0
B	$\bar{\gamma}^1 > L^1_{23} > L^1_{34}$	$\left\{\begin{array}{l} L^2_{34} > \bar{\gamma}^2 \\ L^2_{34} > L^2_{14} \end{array}\right.$	$\left\{\begin{array}{l} L^3_{12} > \bar{\gamma}^3 \\ L^3_{14} > \bar{\gamma}^3 \end{array}\right.$	$\left\{\begin{array}{l} L^4_{12} > L^4_{23}, \\ \bar{\gamma}^4 > L^4_{23} \end{array}\right.$	0	0
C	$L^1_{34} > \bar{\gamma}^1$	$\bar{\gamma}^2 > L^2_{14} > L^2_{34}$	$\left\{\begin{array}{l} \bar{\gamma}^3 > L^3_{14} \\ L^3_{12} > L^3_{14} \end{array}\right.$	$\left\{\begin{array}{l} L^4_{12} > \bar{\gamma}^4 \\ L^4_{23} > \bar{\gamma}^4 \end{array}\right.$	0	0
D	$L^1_{23} > \bar{\gamma}^1$	$L^2_{14} > \bar{\gamma}^2$	$\left\{\begin{array}{l} \bar{\gamma}^3 > L^3_{12} \\ L^3_{14} > L^3_{12} \end{array}\right.$	$\left\{\begin{array}{l} \bar{\gamma}^4 > L^4_{12} \\ L^4_{23} > L^4_{12} \end{array}\right.$	0	0

In the two substeps below, we will now reintroduce the Cases B' and C'.

Step 3.2.a: study of $\hat{\gamma}^3$

We can now use the faithful Figure 10, to deduce geometrically, that

$$\begin{cases} L^3_{24}(0,\bar\gamma^4) > \bar\gamma^3 & \text{in} \quad \left\{\gamma^2 > 0\right\} \quad \supset (A)\cup(B)\cup(B'),\\ L^3_{24}(0,\bar\gamma^4) < \bar\gamma^3 & \text{in} \quad \left\{\gamma^2 < 0\right\} \quad \supset (C'),\\ L^3_{24}(0,\bar\gamma^4) > L^3_{12}(\bar\gamma) & \text{in} \quad (D),\\ L^3_{24}(0,\bar\gamma^4) > L^3_{14}(\bar\gamma) & \iff \quad \bar\gamma^1 < L^1_{24}(0,\bar\gamma^4) \quad \text{which is true in} \quad (C),\end{cases}$$

where in the fourth line, the equivalence follows from (15.65). Hence

$$\hat\gamma^3(\bar\gamma) = \min\left\{\bar\gamma^3, L^3_{14}(\bar\gamma), L^3_{12}(\bar\gamma), L^3_{24}(0,\bar\gamma^4)\right\} \quad \text{in} \quad (A)\cup(B)\cup(C)\cup(D)$$

Similarly, we get

$$\begin{cases} L^3_{24}(0,\bar\gamma^4) < L^3_{14}(\bar\gamma) & \iff \quad \bar\gamma^1 > L^1_{24}(0,\bar\gamma^4) \quad \text{which is true in} \quad (C'),\\ L^3_{24}(0,\bar\gamma^4) < L^3_{12}(\bar\gamma) & \text{in} \quad (C'),\\ \\ \bar\gamma^3 < L^3_{14}(\bar\gamma) & \iff \quad \bar\gamma^1 > L^1_{34}(\bar\gamma) \quad \text{which is true in} \quad (B'),\\ \bar\gamma^3 < L^3_{12}(\bar\gamma) & \text{in} \quad (B'),\end{cases}$$

which also implies

$$\hat\gamma^3(\bar\gamma) = \min\left\{\bar\gamma^3, L^3_{14}(\bar\gamma), L^3_{12}(\bar\gamma), L^3_{24}(0,\bar\gamma^4)\right\} \quad \text{in} \quad (B')\cup(C')$$

Step 3.2.b: study of $\hat\gamma^4$
Similarly to Step 3.2.a, we get

$$\begin{cases} L^4_{13}(0,\bar\gamma^3) > \bar\gamma^4 & \text{in} \quad \left\{\gamma^1 > 0\right\} \supset (A)\cup(C)\cup(C'),\\ L^4_{13}(0,\bar\gamma^3) < \bar\gamma^4 & \text{in} \quad \left\{\gamma^1 < 0\right\} \supset (B'),\\ L^4_{13}(0,\bar\gamma^3) > L^4_{12}(\bar\gamma) & \text{in} \quad (D),\\ L^4_{13}(0,\bar\gamma^3) > L^4_{23}(\bar\gamma) & \iff \quad \bar\gamma^2 < L^2_{13}(0,\bar\gamma^3) \quad \text{which is true in} \quad (B)\end{cases}$$

where in the fourth line, the equivalence follows from (15.65). Hence

$$\hat\gamma^4(\bar\gamma) = \min\left\{\bar\gamma^4, L^4_{23}(\bar\gamma), L^4_{12}(\bar\gamma), L^4_{13}(0,\bar\gamma^3)\right\} \quad \text{in} \quad (A)\cup(B)\cup(C)\cup(D)$$

Similarly, we get

$$\begin{cases} L^4_{13}(0,\bar\gamma^3) < L^4_{23}(\bar\gamma) & \iff \quad \bar\gamma^2 > L^2_{13}(0,\bar\gamma^3) \quad \text{which is true in} \quad (B')\\ L^4_{13}(0,\bar\gamma^3) < L^4_{12}(\bar\gamma) & \text{in} \quad (B'),\\ \\ \bar\gamma^4 < L^4_{23}(\bar\gamma) & \iff \quad \bar\gamma^2 > L^2_{34}(\bar\gamma) \quad \text{which is true in} \quad (C'),\\ \bar\gamma^4 < L^4_{12}(\bar\gamma) & \text{in} \quad (C'),\end{cases}$$

which also implies

$$\hat\gamma^4(\bar\gamma) = \min\left\{\bar\gamma^4, L^4_{23}(\bar\gamma), L^4_{12}(\bar\gamma), L^4_{13}(0,\bar\gamma^3)\right\} \quad \text{in} \quad (B')\cup(C')$$

Step 3.2.c: conclusion
We deduce (15.56).

Step 3.3: deduction of explicit expression (15.57)
Moreover, we also get

Cases $\backslash\hat\gamma$	$\hat\gamma^0 := \hat\gamma^1 + \hat\gamma^2$	$= \hat\gamma^3 + \hat\gamma^4$	$\bar\gamma^1 + \bar\gamma^2$	$\bar\gamma^3 + \bar\gamma^4$
A	$(L^1_{34} + L^2_{34})(\bar\gamma)$	$\bar\gamma^3 + \bar\gamma^4$	$<$	$=$
B	$L^1_{23}(\bar\gamma) + \bar\gamma^2$	$\bar\gamma^3 + L^4_{23}(\bar\gamma)$	$<$	$<$
C	$\bar\gamma^1 + L^2_{14}(\bar\gamma)$	$L^3_{14}(\bar\gamma) + \bar\gamma^4$	$<$	$<$
D	$\bar\gamma^1 + \bar\gamma^2$	$(L^3_{12} + L^4_{12})(\bar\gamma)$	$=$	$<$
B'	$0 + L^2_{13}(0,\bar\gamma^3)$	$\bar\gamma^3 + L^4_{13}(0,\bar\gamma^3)$	$(*)$	$<$
C'	$L^1_{24}(0,\bar\gamma^4) + 0$	$L^3_{24}(0,\bar\gamma^4) + \bar\gamma^4$	$(**)$	$<$

where comparisons in two Cases B' and C' are obtained geometrically, using the faithful Figure 10.
a) Study of (*)

We have

$$L^2_{13}(0,\bar\gamma^3) < \bar\gamma^2 \quad \text{which is true in (B')}$$

Therefore $0 + L^2_{13}(0,\bar\gamma^3) < \bar\gamma^1 + \bar\gamma^2$ and $(*)$ means $<$.

b) Study of (**)

We have

$$L^1_{24}(0,\bar\gamma^4) < \bar\gamma^1 \quad \text{which is true in (C')}$$

Therefore $L^1_{24}(0,\bar\gamma^4) + 0 < \bar\gamma^1 + \bar\gamma^2$ and $(**)$ means $<$.

c) conclusion

This shows (15.57) and ends the proof of the lemma.

16 Comparison of several germs for a $1:2$ junctions

The main result of this section is Theorem 16.6 which shows that in certain circonstances, and for $1:2$ junctions, the three following germs may coincide: Vanishing Viscosity, Data Network and Holden-Risebro. Moreover the Traffic Light germ can be seen as a flux limitation of the previous three identical germs.

We consider a $1:2$ junction with

$$\begin{cases} J^0 \simeq (-\infty,0), \quad J^1 \simeq J^2 \simeq (0,+\infty) \\ f^j = g, \quad j=0,1,2, \end{cases}$$

modeled on the following bell-shaped flux function with its monotone envelopes

$$\text{(16.1)} \qquad \begin{cases} g:[0,2]\to\mathbb{R}, \\ g(x)=\min\{x,2-x\}, \end{cases} \quad \text{with} \quad \begin{cases} g^+(x) := g(\min\{1,x\}), \\ g^-(x) := g(\max\{1,x\}). \end{cases}$$

We will split half of the traffic on road 1 and half of the traffic on road 2, as much as possible. We will consider several models of germs and will compare them.

16.1 Data Network

Lemma 16.1 (An explicit Data Network germ for $1:2$ junction)
Consider Lemma 15.10 with $N=3$ describing a $1:2$ junction with $\sigma_0 = (\sigma_0^0,\sigma_0^1,\sigma_0^2) = (1,-1,-1)$. Recall that for $\bar\gamma := (\bar\gamma^0,\bar\gamma^1,\bar\gamma^2) \in [0,+\infty)^3$, the passing flux is

$$m := \min\left\{\bar\gamma^0, \bar\gamma^1+\bar\gamma^2\right\}$$

and let

$$K^+(\bar\gamma) = \{m\}, \quad K^-(\bar\gamma) := \left\{(\gamma^1,\gamma^2)\in[0,\bar\gamma^1]\times[0,\bar\gamma^2], \quad \gamma^1+\gamma^2 = m\right\}$$

Recall that the Data Network preflux is defined by

$$\hat\gamma^{DN}(\bar\gamma) := Proj^{\perp}_{K^+(\bar\gamma)\times K^-(\bar\gamma)}(m\tilde\theta) \quad \text{for the choice} \quad \tilde\theta = (\theta^0,\theta^1,\theta^2) \quad \text{with} \quad \theta^0 = 1 = \theta^1+\theta^2 \quad \text{and} \quad \theta^1,\theta^2\in(0,1)$$

i) (Explicit preflux)
Then the preflux $\hat\gamma := \hat\gamma^{DN}$ is explicitly given by

$$\text{(16.2)} \qquad \begin{pmatrix} \hat\gamma^0 \\ \hat\gamma^1 \\ \hat\gamma^2 \end{pmatrix}(\bar\gamma) = \begin{cases} \begin{pmatrix} \bar\gamma^1+\bar\gamma^2 \\ \bar\gamma^1 \\ \bar\gamma^2 \end{pmatrix} & \text{if} \quad \bar\gamma^1+\bar\gamma^2 \le \bar\gamma^0 & (T_0) \\ \begin{pmatrix} \bar\gamma^0 \\ \theta^1\bar\gamma^0 \\ \theta^2\bar\gamma^0 \end{pmatrix} & \text{if} \quad \theta^1\bar\gamma^0 < \bar\gamma^1 \quad \text{and} \quad \theta^2\bar\gamma^0 < \bar\gamma^2 & (Q_0) \\ \begin{pmatrix} \bar\gamma^0 \\ \bar\gamma^1 \\ \bar\gamma^0-\bar\gamma^1 \end{pmatrix} & \text{if} \quad \begin{cases} \bar\gamma^0 < \bar\gamma^1+\bar\gamma^2 \\ \bar\gamma^1 \le \theta^1\bar\gamma^0 \end{cases} & (T_1) \\ \begin{pmatrix} \bar\gamma^0 \\ \bar\gamma^0-\bar\gamma^2 \\ \bar\gamma^2 \end{pmatrix} & \text{if} \quad \begin{cases} \bar\gamma^0 < \bar\gamma^1+\bar\gamma^2 \\ \bar\gamma^2 \le \theta^2\bar\gamma^0 \end{cases} & (T_2) \end{cases}$$

which satisfies

$$\hat\gamma^0(\bar\gamma) = \min\left\{\bar\gamma^0, \bar\gamma^1+\bar\gamma^2\right\}$$

ii) (The three characteristic curves of the conservative preflux)
The conservative preflux $\hat\gamma_{\bar\lambda} := \hat\gamma^{DN}\circ T_{\bar\lambda}$ *for* $1:2$ *junctions, is characterized by the following three curves*

$$\text{(16.3)}\qquad \left\{\begin{array}{l} \tilde\Gamma_0 := \left\{(\lambda^0, \theta^1\lambda^0, \theta^2\lambda^0)\quad \lambda^0\in[0,1]\right\}\\ \tilde\Gamma_1 := \left\{(1, \lambda^1, 1-\lambda^1)\quad \lambda^1\in[0,\theta^1]\right\}\\ \tilde\Gamma_2 := \left\{(1, 1-\lambda^2, \lambda^2)\quad \lambda^2\in[0,\theta^2]\right\}\end{array}\right.$$

whose projection onto the plane $\mathbb{R}e_1+\mathbb{R}e_2$ *are the curves* $\Gamma_0,\Gamma_1,\Gamma_2$ *defined in ii) of Lemma 12.5. Here* $T_{\bar\lambda}$ *is the truncation operator given by*

$$T_{\bar\lambda}(p) = \left(\min\left\{p^0,\bar\lambda^0\right\}, \min\left\{p^1,\bar\lambda^1\right\}, \min\left\{p^2,\bar\lambda^2\right\}\right)\quad \textit{with}\quad \bar\lambda = (\bar\lambda^0,\bar\lambda^1,\bar\lambda^2) := (1,+\infty,+\infty).$$

iii) (The Data Network germ)
Let g, g^+, g^- *be the functions defined in (16.1). Then the set*

$$\text{(16.4)}\qquad \mathcal{G}^{DN} := \left\{p=(p^0,p^1,p^2)\in[0,2]^3,\quad \hat\gamma^{DN}(g^+(p^0), g^-(p^1), g^-(p^2)) = (g(p^0), g(p^1), g(p^2))\right\}$$

is the Data Network germ. We set for $j=0,1,2$*, the reflection maps*

$$\begin{array}{llll}\tau_j: & [0,2]^3 & \to & [0,2]^3\\ & p & \mapsto & \tau_j(p) := p + 2(1-p^j)e_j\end{array}$$

and for $\sigma = (\sigma^0,\sigma^1,\sigma^2)\in\{\pm1\}^3$ *the reflection*

$$\tau^\sigma := \tau_0^{n_0}\tau_1^{n_1}\tau_2^{n_2}\quad \textit{with}\quad n_j = \left\{\begin{array}{ll} 0 & \textit{if}\quad \sigma^j = 1,\\ 1 & \textit{if}\quad \sigma^j=-1,\end{array}\right.$$

Then the conservative germ $\mathcal{G}^{DN}$ *is characterized by the following three curves defined for* $Q := [0,1]^3$

$$Q\cap(\tau^\sigma)^{-1}(\mathcal{G}^{DN}) = \left\{\begin{array}{ll}\tilde\Gamma_0 & \textit{if}\quad \sigma = (+,+,+)\\ \tilde\Gamma_1 & \textit{if}\quad \sigma=(-,-,+)\\ \tilde\Gamma_2 & \textit{if}\quad \sigma = (-,+,-)\end{array}\right.$$

where each σ *differs from* $-\sigma_0 = (-,+,+)$ *from only a single sign.*

Remark 16.2 *See also Lemma 16.5, where an explicit compact formula is given for* $\hat\gamma^{DN} = \hat\gamma_0$.

Proof of Lemma 16.1
Step 1: proof of i)
Notice that by definition, we have

$$\hat\gamma^{DN}(\bar\gamma) := \text{Proj}^{\perp}_{K^+(\bar\gamma)\times K^-(\bar\gamma)}(m\tilde\theta) = (m,P)\quad\text{with}\quad P := \text{Proj}^{\perp}_{K^+(\bar\gamma)}(m\theta)\quad\text{with}\quad \theta := (\theta^1,\theta^2)$$

Notice that

$$\left\{\begin{array}{llll}\left\{(\bar\gamma^1,\bar\gamma^2)\right\} & = K^-(\bar\gamma) & \text{if} & \bar\gamma^1+\bar\gamma^2\le\bar\gamma^0\\ (\bar\gamma^1,\bar\gamma^2) & \notin K^-(\bar\gamma) & \text{if} & \bar\gamma^1+\bar\gamma^2>\bar\gamma^0\end{array}\right.$$

In the second line, we can distinguish (three) cases: either we have both $m\theta^j<\bar\gamma^j$ for both $j=1,2$, or we only have one of the strict reverse inequalities (only for $j=1$ or only for $j=2$). We get

$$P = \left\{\begin{array}{llll}(\bar\gamma^1,\bar\gamma^2) & & \text{if} & \bar\gamma^1+\bar\gamma^2\le\bar\gamma^0\\ m(\theta^1,\theta^2) & \text{if}\ \left\{\begin{array}{l} m\theta^1<\bar\gamma^1,\\ m\theta^2<\bar\gamma^2\end{array}\right. & \text{with} & \bar\gamma^1+\bar\gamma^2>\bar\gamma^0\\ (\bar\gamma^1, m-\bar\gamma^1) & \text{if}\ \left\{\begin{array}{l} m\theta^1\ge\bar\gamma^1,\\ m\theta^2<\bar\gamma^2\end{array}\right. & \text{with} & \bar\gamma^1+\bar\gamma^2>\bar\gamma^0\\ (m-\bar\gamma^2,\bar\gamma^2) & \text{if}\ \left\{\begin{array}{l} m\theta^1<\bar\gamma^1,\\ m\theta^2\ge\bar\gamma^2\end{array}\right. & \text{with} & \bar\gamma^1+\bar\gamma^2>\bar\gamma^0\end{array}\right.$$

Hence

$$\begin{pmatrix} \hat{\gamma}^0 \\ \hat{\gamma}^1 \\ \hat{\gamma}^2 \end{pmatrix} = \begin{cases} \begin{pmatrix} \bar{\gamma}^1 + \bar{\gamma}^2 \\ \bar{\gamma}^1 \\ \bar{\gamma}^2 \end{pmatrix} & \text{if} \quad \bar{\gamma}^1 + \bar{\gamma}^2 \leq \bar{\gamma}^0 \\[2ex] \begin{pmatrix} m \\ m\theta^1 \\ m\theta^2 \end{pmatrix} & \text{if} \quad \begin{cases} m\theta^1 < \bar{\gamma}^1 \\ m\theta^2 < \bar{\gamma}^2 \end{cases} \\[2ex] \begin{pmatrix} m \\ \bar{\gamma}^1 \\ m - \bar{\gamma}^1 \end{pmatrix} & \text{if} \quad \begin{cases} m < \bar{\gamma}^1 + \bar{\gamma}^2 \\ m\theta^1 \geq \bar{\gamma}^1 \end{cases} \\[2ex] \begin{pmatrix} m \\ m - \bar{\gamma}^2 \\ \bar{\gamma}^2 \end{pmatrix} & \text{if} \quad \begin{cases} m < \bar{\gamma}^1 + \bar{\gamma}^2 \\ m\theta^2 \geq \bar{\gamma}^2 \end{cases} \end{cases}$$

which gives (16.2) with $m = \bar{\gamma}^0 < \bar{\gamma}^1 + \bar{\gamma}^2$.

Step 2: proof of ii)

From Lemma 12.5, we have for $\hat{\gamma}_{\bar{\lambda}} = \hat{\gamma}_{\bar{\lambda}}(\bar{\gamma})$

$$\begin{cases} \tilde{\Gamma}_0 = \hat{\gamma}_{\bar{\lambda}}\left(\Omega_{12}\right) & \text{with} \quad \Omega_{12} := \left\{\hat{\gamma}^0_{\bar{\lambda}} = \bar{\gamma}^0, \quad \hat{\gamma}^1_{\bar{\lambda}} < \bar{\gamma}^1, \quad \hat{\gamma}^2_{\bar{\lambda}} < \bar{\gamma}^2\right\}) \\ \tilde{\Gamma}_1 = \hat{\gamma}_{\bar{\lambda}}\left(\Omega_{02}\right) & \text{with} \quad \Omega_{02} := \left\{\hat{\gamma}^0_{\bar{\lambda}} < \bar{\gamma}^0, \quad \hat{\gamma}^1_{\bar{\lambda}} = \bar{\gamma}^1, \quad \hat{\gamma}^2_{\bar{\lambda}} < \bar{\gamma}^2\right\}) \\ \tilde{\Gamma}_2 = \hat{\gamma}_{\bar{\lambda}}\left(\Omega_{01}\right) & \text{with} \quad \Omega_{01} := \left\{\hat{\gamma}^0_{\bar{\lambda}} < \bar{\gamma}^0, \quad \hat{\gamma}^1_{\bar{\lambda}} < \bar{\gamma}^1, \quad \hat{\gamma}^2_{\bar{\lambda}} = \bar{\gamma}^2\right\}) \end{cases}$$

and Γ_j is the projection of $\tilde{\Gamma}_j$ onto the plane $\mathbb{R}e_1 + \mathbb{R}e_2$.

Case A: $j = 0$

For $j = 0$, we deduce that $\Omega_{12} \subset Q_0 \cap \left\{\bar{\gamma}^0 \leq 1\right\}$ with Q_0 defined in (16.2). Hence

$$\tilde{\Gamma}_0 = \left\{(\bar{\gamma}^0, \theta^1\bar{\gamma}^0, \theta^2\bar{\gamma}^0), \quad \bar{\gamma}^0 \in [0,1]\right\}$$

and

$$\Gamma_0 = \left\{(\theta^1\bar{\gamma}^0, \theta^2\bar{\gamma}^0), \quad \bar{\gamma}^0 \in [0,1]\right\}$$

Case B: $j = 1$

For $j = 1$, if $\bar{\gamma}^0 \in [0,1]$, then $\hat{\gamma}_{\bar{\lambda}}(\bar{\gamma}) = \hat{\gamma}(\bar{\gamma})$. Assume by contradiction that $\bar{\gamma} \in \Omega_{02}$. Then

$$\hat{\gamma}^0 = \min\left\{\bar{\gamma}^0, \bar{\gamma}^1 + \bar{\gamma}^2\right\} = \hat{\gamma}^0_{\bar{\lambda}} < \bar{\gamma}^0.$$

Hence

$$\hat{\gamma}^0 = \bar{\gamma}^1 + \bar{\gamma}^2 > \hat{\gamma}^1_{\bar{\lambda}} + \hat{\gamma}^2_{\bar{\lambda}} = \hat{\gamma}^1 + \hat{\gamma}^2 = \hat{\gamma}^0.$$

Contradiction.

Therefore if $\bar{\gamma} \in \Omega_{02}$, then $\bar{\gamma}^0 > 1$. We deduce that $\Omega_{02} \subset T_1$ with T_1 defined in (16.2). Hence $T_{\bar{\lambda}}(\bar{\gamma}) = (1, \bar{\gamma}^1, \bar{\gamma}^2)$ and

$$\tilde{\Gamma}_1 = \left\{(1, \bar{\gamma}^1, 1 - \bar{\gamma}^1), \quad \bar{\gamma}^1 \in [0, \theta^1]\right\}$$

and

$$\Gamma_1 = \left\{(\bar{\gamma}^1, 1 - \bar{\gamma}^1), \quad \bar{\gamma}^1 \in [0, \theta^1]\right\}$$

Case C: $j = 2$

This case is the symmetric of Case B. We get $\Omega_{01} \subset T_2$ and

$$\tilde{\Gamma}_2 = \left\{(1, 1 - \bar{\gamma}^2, \bar{\gamma}^2), \quad \bar{\gamma}^2 \in [0, \theta^2]\right\}$$

and

$$\Gamma_2 = \left\{(1 - \bar{\gamma}^2, \bar{\gamma}^2), \quad \bar{\gamma}^2 \in [0, \theta^2]\right\}$$

Step 3: proof of iii)

We first notice that in the definition of $\mathcal{G}^{DN}$, we can replace the preflux $\hat{\gamma}^{DN}$ by the preflux $\hat{\gamma}_{\bar{\lambda}}$. Then we simply apply iii) of Lemma 12.10 for $R := R'$.

This ends the proof of the lemma.

16.2 Vanishing Viscosity

Lemma 16.3 (Vanishing viscosity germ: identification)
Let

$$G(p,q) := G^g(p,q) = \min\left\{g^+(p), g^-(q)\right\}$$

be the Godunov flux associated to the bell-shaped function g, with g and its monotone envelopes g^+, g^- defined in (16.1). For $p = (p^0, p^1, p^2) \in [0,2]^3$, then the following set

$$R_p := \left\{r \in [0,2], \quad G(p^0, r) = G(r, p^1) + G(r, p^2)\right\}$$

is not empty. Moreover the function

$$\hat{f}^{VV}(p) := (G(p^0, r), G(r, p^1), G(r, p^2)) \quad \text{for any} \quad r \in R_p$$

is well-defined, and the set

$$\mathcal{G}^{VV} := \left\{p \in [0,2]^3, \quad \hat{f}^{VV}(p) = f(p)\right\} \quad \text{with} \quad f(p) := (g(p^0), g(p^1), g(p^2)) \tag{16.5}$$

is the Vanishing Viscosity germ for a $1:2$ junction.
Then

$$\mathcal{G}^{VV} = \mathcal{G}^{DN}$$

where $\mathcal{G}^{DN}$ is the Data Network germ defined in (16.4) for $\theta^1 = \theta^2 = \dfrac{1}{2}$.

Proof of Lemma 16.3
Step 1: preliminaries
We apply iii) of Lemma 12.10 for $R' = R$ and the structure of conservative prefluxes given in Lemma 12.5, we know that $\mathcal{G}^{VV}$ is characterized by the following three curves

$$\mathcal{G}^{VV} \cap Q_1^\sigma =: \begin{cases} \tilde{\Gamma}_0^\sigma & \text{for} \quad \sigma = (+;+,+) \\ \tilde{\Gamma}_1^\sigma & \text{for} \quad \sigma = (-;-,+) \\ \tilde{\Gamma}_2^\sigma & \text{for} \quad \sigma = (-;+,-) \end{cases}$$

where Γ_j^σ are defined in Lemma 12.10 for $j = 0,1,2$. We call $\hat{\gamma}^{VV}$ the preflux associated to the germ $\mathcal{G}^{VV}$ from the polar decomposition. For $p \in \mathcal{G}^{VV} \cap Q_1^\sigma$, we set

$$\begin{cases} \bar{\gamma} := (g^+(p^0), g^-(p^1), g^-(p^2)) \\ \lambda = (\lambda^0, \lambda^1, \lambda^2) := (g(p^0), g(p^1), g(p^2)) \end{cases}$$

We have

$$\lambda = \hat{\gamma}^{VV}(\bar{\gamma}) = \hat{f}^{VV}(p) \tag{16.6}$$

Step 2: Cases
We distinguish cases.
Case A: $\sigma = (+,+,+)$
Then $\bar{\gamma} = (\lambda^0, 1, 1)$ and (16.6) implies

$$\begin{cases} \lambda^0 = \min\left\{\lambda^0, g^-(r)\right\}, \\ \lambda^1 = \min\left\{g^+(r), 1\right\}, \\ \lambda^2 = \min\left\{g^+(r), 1\right\}, \\ \lambda^0 = \lambda^1 + \lambda^2 \end{cases}$$

Hence $r \leq 1$, and

$$\lambda^1 = \lambda^2 = \lambda^0/2$$

Using the expression of g, we deduce that

$$\tilde{\Gamma}_0 = \left\{(\lambda^0, \lambda^0/2, \lambda^0/2), \quad \lambda^0 \in [0,1]\right\}$$

Case B: $\sigma = (-,-,+)$

Then we have $\bar{\lambda} = (1, \lambda^1, 1)$ and (16.6) implies

$$\begin{cases} \lambda^0 = \min\{1, g^-(r)\}, \\ \lambda^1 = \min\{g^+(r), \lambda^1\}, \\ \lambda^2 = \min\{g^+(r), 1\}, \\ \lambda^0 = \lambda^1 + \lambda^2 \end{cases}$$

Hence

$$\lambda^1 \le \lambda^2 = g^+(r) \le \lambda^0 \le g^-(r)$$

Hence $r \le 1$ and $\lambda^0 = g^-(r) = 1$ and

$$\lambda^0 = 1 = \lambda^1 + \lambda^2, \quad \lambda^1 \le \lambda^2$$

We deduce that

$$\tilde{\Gamma}_1 = \left\{(1, \lambda^1, 1 - \lambda^1), \quad \lambda^1 \in [0, 1/2]\right\}$$

Case C: $\sigma = (-, +, -)$
This case is the symmetric of Case B. We then get

$$\tilde{\Gamma}_2 = \left\{(1, 1 - \lambda^2, \lambda^2), \quad \lambda^2 \in [0, 1/2]\right\}$$

Step 3: Conclusion
The fact that the characteristic curves $\tilde{\Gamma}_j$ for $j = 0, 1, 2$ are the same as the ones defined in Lemma 16.1 for $\theta^1 = \theta^2 = \dfrac{1}{2}$ implies the equality of the conservatives germs $\mathcal{G}^{VV} = \mathcal{G}^{DN}$ for a $1:2$ junction. This ends the proof of the lemma.

16.3 Holden-Risebro

Lemma 16.4 (A Holden-Risebro preflux: identification)
Consider a $1:2$ junction with $\sigma = (\sigma^0, \sigma^1, \sigma^2) = (1, -1, -1)$. For $\bar{\gamma} \in [0, +\infty)^3$, we set

$$\begin{cases} K_0(\bar{\gamma}) := \{\gamma \in [0, \bar{\gamma}], \quad L(\gamma) = 0\} & \textit{with} \quad L(\gamma) := \gamma^0 - (\gamma^1 + \gamma^2) \\ \Psi(\gamma) := \dfrac{1}{2}|\gamma - \tilde{\theta}|^2 & \textit{with} \quad \tilde{\theta} = (\theta^0, \theta^1, \theta^2) \quad \textit{with} \quad \theta^0 = 1 = \theta^1 + \theta^2 \quad \textit{and} \quad \theta^1, \theta^2 \in (0, 1) \end{cases}$$

and define the following Holden-Risebro preflux

$$\hat{\gamma}^{HR}(\bar{\gamma}) := \underset{K_0(\bar{\gamma})}{Argmin}\ \Psi$$

In the special case

$$\theta^1 = \theta^2 = \frac{1}{2}$$

the following holds true.
i) (Identification)
Then

$$\hat{\gamma}^{HR} \circ T_{\bar{\lambda}} = \hat{\gamma}^{DN} \circ T_{\bar{\lambda}} \tag{16.7}$$

where $\hat{\gamma}^{DN}$ and $T_{\bar{\lambda}}$ are defined in Lemma 16.1 with $\bar{\lambda} := (1, +\infty, +\infty)$.
ii) (Definition)
Let g, g^+, g^- be the functions defined in (16.1). Then the set

$$\mathcal{G}^{HR} := \left\{p = (p^0, p^1, p^2) \in [0, 2]^3, \quad \hat{\gamma}^{HR}(g^+(p^0), g^-(p^1), g^-(p^2)) = (g(p^0), g(p^1), g(p^2))\right\} \tag{16.8}$$

is a Holden-Risebro germ.

Proof of Lemma 16.4
We first notice that

$$\hat{\gamma}^{HR}(\bar{\gamma}) := \mathrm{Proj}^{\perp}_{|K_0(\bar{\gamma})}(\tilde{\theta})$$

and set

$$\hat{\gamma}_{\bar{\lambda}} := \hat{\gamma}^{HR}(\bar{\gamma}) \circ T_{\bar{\lambda}} \quad \text{with} \quad T_{\bar{\lambda}}(\bar{\gamma}) = \left(\min\left\{\bar{\gamma}^0, \bar{\lambda}^0\right\}, \min\left\{\bar{\gamma}^1, \bar{\lambda}^1\right\}, \min\left\{\bar{\gamma}^2, \bar{\lambda}^2\right\}\right)$$

For $\hat{\gamma}_{\bar{\lambda}} := \hat{\gamma}_{\bar{\lambda}}(\bar{\gamma})$, let

$$\left\{ \begin{array}{lll} \tilde{\Gamma}_0 = \hat{\gamma}_{\bar{\lambda}}\left(\Omega_{12}\right) & \text{with} & \Omega_{12} := \left\{\hat{\gamma}^0_{\bar{\lambda}} = \bar{\gamma}^0, \quad \hat{\gamma}^1_{\bar{\lambda}} < \bar{\gamma}^1, \quad \hat{\gamma}^2_{\bar{\lambda}} < \bar{\gamma}^2\right\}) \\ \tilde{\Gamma}_1 = \hat{\gamma}_{\bar{\lambda}}\left(\Omega_{02}\right) & \text{with} & \Omega_{02} := \left\{\hat{\gamma}^0_{\bar{\lambda}} < \bar{\gamma}^0, \quad \hat{\gamma}^1_{\bar{\lambda}} = \bar{\gamma}^1, \quad \hat{\gamma}^2_{\bar{\lambda}} < \bar{\gamma}^2\right\}) \\ \tilde{\Gamma}_2 = \hat{\gamma}_{\bar{\lambda}}\left(\Omega_{01}\right) & \text{with} & \Omega_{01} := \left\{\hat{\gamma}^0_{\bar{\lambda}} < \bar{\gamma}^0, \quad \hat{\gamma}^1_{\bar{\lambda}} < \bar{\gamma}^1, \quad \hat{\gamma}^2_{\bar{\lambda}} = \bar{\gamma}^2\right\}) \end{array} \right.$$

We will prove (16.7) through the identification of the three characteristic curves $\Gamma_0, \Gamma_1, \Gamma_2$ and Theorem 12.7.

Case A: $\tilde{\Gamma}_0$
Let $\bar{\gamma} \in \Omega_{12}$. Then $\bar{\gamma}^0 \le 1$, and we can choose $\bar{\gamma}^1, \bar{\gamma}^2 > 1$. Hence

$$K_0(\bar{\gamma}) = \left\{(\lambda^1 + \lambda^2, \lambda^1, \lambda^2) \in [0, +\infty)^3, \quad \lambda^1 + \lambda^2 \le \bar{\gamma}^0\right\}$$

and geometrically get in the special case $\theta^1 = \theta^2 = \dfrac{1}{2}$

$$\hat{\gamma}_{\bar{\lambda}}(\bar{\gamma}) = \mathrm{Proj}^{\perp}_{|K_0(\bar{\gamma})}(\tilde{\theta}) = (\bar{\gamma}^0, \bar{\gamma}^0/2, \bar{\gamma}^0/2)$$

and then

$$\tilde{\Gamma}_0 = \left\{(\bar{\gamma}^0, \bar{\gamma}^0/2, \bar{\gamma}^0/2), \quad \bar{\gamma}^0 \in [0,1]\right\}$$

Case B: $\tilde{\Gamma}_1$
Let $\bar{\gamma} \in \Omega_{12}$. Then $\bar{\gamma}^0 > 1$, and we can choose $\bar{\gamma}^2 > 1$. Hence

$$T_{\bar{\gamma}}(\bar{\gamma}) = (1, \bar{\gamma}^1, \bar{\gamma}^2)$$

$$K_0(T_{\bar{\gamma}}(\bar{\gamma})) = \left\{(\lambda^1 + \lambda^2, \lambda^1, \lambda^2) \in [0, +\infty)^3, \quad 0 \le \lambda^1 + \lambda^2 \le 1, \quad 0 \le \lambda^1 \le \bar{\gamma}^1\right\}$$

For $\bar{\gamma}^1 > 1/2 = \theta^1$, we get

$$\hat{\gamma}_{\bar{\lambda}}(\bar{\gamma}) = \mathrm{Proj}^{\perp}_{|K_0(T_{\bar{\gamma}}(\bar{\gamma}))}(\tilde{\theta}) = \tilde{\theta}$$

and then $\bar{\gamma} \notin \Omega_{02}$. On the contrary, for $\bar{\gamma}^1 \in [0, 1/2]$, we get geometrically

$$\hat{\gamma}_{\bar{\lambda}}(\bar{\gamma}) = \mathrm{Proj}^{\perp}_{|K_0(T_{\bar{\gamma}}(\bar{\gamma}))}(\tilde{\theta}) = (1, \bar{\gamma}^1, 1 - \bar{\gamma}^1)$$

and then

$$\tilde{\Gamma}_1 = \left\{(1, \bar{\gamma}^1, 1 - \bar{\gamma}^1), \quad \bar{\gamma}^1 \in [0, 1/2]\right\}$$

Case C: $\tilde{\Gamma}_2$
This case is symmetric of Case B, and we get

$$\tilde{\Gamma}_2 = \left\{(1, 1 - \bar{\gamma}^2, \bar{\gamma}^2), \quad \bar{\gamma}^2 \in [0, 1/2]\right\}$$

Conclusion
We recognize the three characteristic curves given in Lemma 16.1 for the conservative preflux $\hat{\gamma}^{DN} \circ T_{\bar{\gamma}}$. From 1.ii) of Lemma 12.5, we deduce (16.7). This ends the proof of the lemma.

16.4 Traffic light

Lemma 16.5 (Traffic light preflux: identification)
For a 1 : 2 junction, we consider the function $\hat{\gamma}_0 = (\hat{\gamma}^0_0, \hat{\gamma}^1_0, \hat{\gamma}^2_0) : [0, +\infty)^3 \to [0, +\infty)^3$ *for* $\gamma = (\gamma^0, \gamma^1, \gamma^2) \in [0, +\infty)^3$ *with* $\theta^1 + \theta^2 = 1$, $\theta^1, \theta^2 \in (0, 1)$

$$\left\{ \begin{array}{ll} \hat{\gamma}^1_0(\gamma) & = \min\left\{\gamma^1, \max\left\{\theta^1\gamma^0, \gamma^0 - \gamma^2\right\}\right\}, \\ \hat{\gamma}^2_0(\gamma) & = \min\left\{\gamma^2, \max\left\{\theta^2\gamma^0, \gamma^0 - \gamma^1\right\}\right\}, \\ \hat{\gamma}^0_0 & = \hat{\gamma}^1_0 + \hat{\gamma}^2_0 \end{array} \right.$$

Then $\hat{\gamma}_0$ is a conservative preflux which satisfies moreover

$$\hat{\gamma}_0^0(\gamma) = \min\left\{\gamma^0, \gamma^1 + \gamma^2\right\} \tag{16.9}$$

i) (Identification of $\hat{\gamma}_0$)
Then

$$\hat{\gamma}_0 = \hat{\gamma}^{DN} \tag{16.10}$$

where $\hat{\gamma}^{DN}$ is the Data Network preflux defined in (16.2).
ii) (Definition of $\hat{\gamma}^{TL}$)
A Traffic Light preflux is given by

$$\hat{\gamma}^{TL} := \hat{\gamma}_0 \circ T_{\bar{\lambda}_0} \quad \textit{with} \quad T_{\bar{\lambda}_0}(\bar{\gamma}) = \left(\min\left\{\bar{\gamma}^0, \bar{\lambda}_0^0\right\}, \min\left\{\bar{\gamma}^1, \bar{\lambda}_0^1\right\}, \min\left\{\bar{\gamma}^2, \bar{\lambda}_0^2\right\}\right) \quad \textit{where} \quad \bar{\lambda}_0^0 = \bar{\lambda}_0^1 + \bar{\lambda}_0^2$$

with

$$\bar{\lambda}_0 = (\bar{\lambda}_0^0, \bar{\lambda}_0^1, \bar{\lambda}_0^2) := (1, \theta^1, \theta^2)$$

iii) (The three characteristic curves of the conservative preflux)
Moreover, the conservative preflux $\hat{\gamma}^{TL}$ for $1:2$ junctions, is characterized by the following three curves

$$\left\{\begin{array}{l} \tilde{\Gamma}_0 := \left\{(\lambda^0, \theta^1\lambda^0, \theta^2\lambda^0) \quad \lambda^0 \in [0,1]\right\} \\ \tilde{\Gamma}_1 := \left\{(\lambda^1 + \theta^2, \lambda^1, \theta^2) \quad \lambda^1 \in [0,\theta^1]\right\} \\ \tilde{\Gamma}_2 := \left\{(\lambda^2 + \theta^1, \theta^1, \lambda^2) \quad \lambda^2 \in [0,\theta^2]\right\} \end{array}\right. \tag{16.11}$$

whose projection onto the plane $\mathbb{R}e_1 + \mathbb{R}e_2$ are the curves $\Gamma_0, \Gamma_1, \Gamma_2$ defined in ii) of Lemma 12.5.

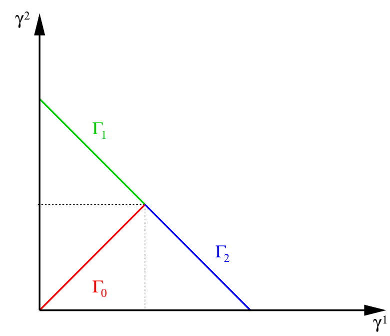

Figure 11: Curves Γ_j for $\hat{\gamma}^{DN} \circ T_{(1,1,1)}$ with $\bar{\gamma} = (\bar{\gamma}^0, \bar{\gamma}^1, \bar{\gamma}^2)$.

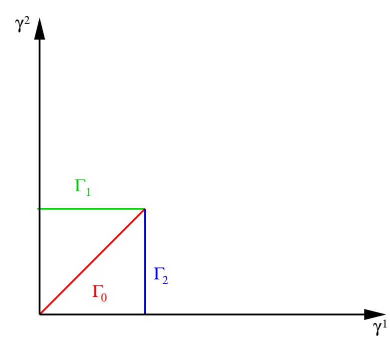

Figure 12: Curves Γ_j for $\hat{\gamma}^{TL} = \hat{\gamma}^{DN} \circ T_{(1,\theta^1,\theta^2)}$ with $\bar{\gamma} = (\bar{\gamma}^0, \bar{\gamma}^1, \bar{\gamma}^2)$.

Proof of Lemma 16.5
Step 1: proof of i) and ii)

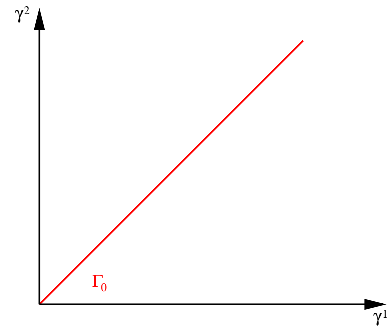

Figure 13: Curves Γ_j for $\hat{\gamma}^{DN}$ with $\bar{\gamma} = (\bar{\gamma}^0, \bar{\gamma}^1, \bar{\gamma}^2)$, with Γ_1, Γ_2 at infinity.

The fact that $\hat{\gamma}_0$ is a conservative preflux and that $\hat{\gamma}_0 \circ T_{\bar{\lambda}_0}$ is a Traffic Light preflux follows from Lemma 15.15. We now check (16.10) using expression (16.2) of $\hat{\gamma}^{DN}$ and distinguishing several cases.

Case A: T_0

Let $\bar{\gamma} \in T_0$. Then $\bar{\gamma}^1 + \bar{\gamma}^2 \leq \bar{\gamma}^0$, which implies $\bar{\gamma}^0 - \bar{\gamma}^2 \geq \bar{\gamma}^1$ and then $\max\left\{\theta^1\bar{\gamma}^0, \bar{\gamma}^0 - \bar{\gamma}^2\right\} \geq \bar{\gamma}^1$, and then $\hat{\gamma}_0^1(\bar{\gamma}) = \bar{\gamma}^1$. Similarly, we get $\hat{\gamma}_0^2(\bar{\gamma}) = \bar{\gamma}^2$, and conclude that

$$\hat{\gamma}_0 = \hat{\gamma}^{DN} \quad \text{on} \quad T_0$$

Case B: Q_0

Let $\bar{\gamma} \in Q_0$. Then $\theta^1\bar{\gamma}^0 < \bar{\gamma}^1$ and $\theta^2\bar{\gamma}^0 < \bar{\gamma}^2$. Hence $\theta^1\bar{\gamma}^0 > \bar{\gamma}^0 - \bar{\gamma}^2$, and $\max\left\{\theta^1\bar{\gamma}^0, \bar{\gamma}^0 - \bar{\gamma}^2\right\} = \theta^1\bar{\gamma}^0 < \bar{\gamma}^1$, which implies $\hat{\gamma}_0^1(\bar{\gamma}) = \theta^1\bar{\gamma}^0$. Similarly, we get $\hat{\gamma}_0^2(\bar{\gamma}) = \theta^2\bar{\gamma}^0$. Therefore

$$\hat{\gamma}_0 = \hat{\gamma}^{DN} \quad \text{on} \quad Q_0$$

Case C: T_1

Let $\bar{\gamma} \in T_1$. Then $\bar{\gamma}^0 < \bar{\gamma}^1 + \bar{\gamma}^2$ and $\bar{\gamma}^1 \leq \theta^1\bar{\gamma}^0$. Hence $\bar{\gamma}^0 - \bar{\gamma}^1 \geq \theta^2\bar{\gamma}^0$. Hence $\max\left\{\theta^2\bar{\gamma}^0, \bar{\gamma}^0 - \bar{\gamma}^1\right\} = \bar{\gamma}^0 - \bar{\gamma}^1 < \bar{\gamma}^2$, which shows that $\hat{\gamma}_0^2(\bar{\gamma}) = \bar{\gamma}^0 - \bar{\gamma}^1$. From (16.9), we also deduce that $\hat{\gamma}_0^0(\bar{\gamma}) = \bar{\gamma}^0$. Hence

$$\hat{\gamma}_0 = \hat{\gamma}^{DN} \quad \text{on} \quad T_1$$

Case D: T_2

This case is symmetric of Case D. Hence we get

$$\hat{\gamma}_0 = \hat{\gamma}^{DN} \quad \text{on} \quad T_2$$

Conclusion

Because we have a partition $[0, +\infty)^3 = T_0 \cup Q_0 \cup T_1 \cup T_2$, we deduce that $\hat{\gamma}_0 = \hat{\gamma}^{DN}$.

Step 2: proof of iii)

Recall from (12.16) that the curves are characterized by

$$\hat{\gamma}^{TL}(\hat{\Sigma}_j^*) = \tilde{\Gamma}_j, \quad j = 0, 1, 2$$

Let $T_{\bar{\lambda}_0}$ be a truncation satisfying

$$\bar{\lambda}_0 \in \tilde{\Gamma}_0^{DN}$$

where $\tilde{\Gamma}_0^{DN}$ is the curve associated to the conservative preflux $\hat{\gamma}^{DN}$, as defined in (16.3). Then it is easy (at least from the geometric interpretation of the problem on Figures 7 and 8) to check that constant A_* and $\tilde{\Gamma}_0$ are unchanged. Moreover, it is easy to see that only $\tilde{\Gamma}_j$ for $j = 1, 2$ are modified and are given as in (16.11). This ends the proof of the lemma.

Theorem 16.6 (Traffic Light germ: identification)

Let us consider the bell-shaped function $g : [0,2] \to [0,1]$ and its monotone envelopes g^+, g^- defined in (16.1). We consider the following set

$$\mathcal{G}^{TL} := \left\{ p = (p^0, p^1, p^2) \in [0,2]^3, \quad \left| \begin{array}{l} g(p^0) = g(p^1) + g(p^2), \\ g^+(p^1) \geq \theta^1 g^+(p^0) \\ g^+(p^2) \geq \theta^2 g^+(p^0) \\ g(p^0) \leq \bar{\lambda}_0^0 \\ g(p^1) \leq \bar{\lambda}_0^1 \\ g(p^2) \leq \bar{\lambda}_0^2 \end{array} \right. \right\}$$

with $\bar{\lambda}_0^0 = \bar{\lambda}_0^1 + \bar{\lambda}_0^2$ and $\bar{\lambda}_0 = (\bar{\lambda}_0^0, \bar{\lambda}_0^1, \bar{\lambda}_0^2) := (1, \theta^1, \theta^2)$ with $\theta^1 + \theta^2 = 1$ and $\theta^1, \theta^2 \in (0,1)$.

i) (Definition)

Then $\mathcal{G}^{TL}$ is a Traffic light germ.

ii) (Further identification in the special case $\theta^1 = \theta^2 = \dfrac{1}{2}$)

For $\mu \in [0,1]$, let us consider

$$\mathcal{G}_\mu := \left\{ (p^L, p^R) \in [0,2]^2, \quad \min\left\{\mu, g^+(p^L), g^-(p^R)\right\} = g(p^L) = g(p^R) \right\}$$

which a conservative germ for a $1:1$ junction with flux g on each branch, and flux limited to the value μ. Then we have the following identification by gluing

$$\mathcal{G}^{TL} = (\mathcal{G}^{VV} \underset{1:L}{\sharp} \mathcal{G}_{\bar{\lambda}_0^1}) \underset{2:L}{\sharp} \mathcal{G}_{\bar{\lambda}_0^2} \quad \text{with} \quad \mathcal{G}^{VV} = \mathcal{G}^{DN} = \mathcal{G}^{HR} \tag{16.12}$$

where $\mathcal{G}^{VV}$ is the vanishing viscosity germ defined in (16.5), $\mathcal{G}^{DN}$ is the data network germ defined in (16.2) and $\mathcal{G}^{HR}$ is the Holden-Risebro germ defined in (16.8).

In words, (16.12) means that the traffic light germ $\mathcal{G}^{TL}$ is obtained from the vanishing viscosity germ $\mathcal{G}^{VV}$ by a limitation of the flux to the maximal value $\bar{\lambda}_0^j$ on each outgoing branch j, for $j = 1$ and $j = 2$.

Proof of Theorem 16.6

Point i) follows from Lemma 15.15. Moreover it shows that the preflux associated to $\mathcal{G}^{TL}$ is

$$\hat{\gamma}^{TL} = \hat{\gamma}_0 \circ T_{\bar{\lambda}_0}$$

Now Lemma 11.16 gives the interpretation in terms of gluing which leads to (16.12). This ends the proof of the Theorem.

16.5 A fat germ

Lemma 16.7 (The fat germ)

Let us consider the bell-shaped function $g : [0,2] \to [0,1]$ and its monotone envelopes g^+, g^- defined in (16.1). For a $1:2$ junction, let us consider the identical preflux

$$\hat{\gamma}^{Fat} = id_{[0,+\infty)^3}$$

We consider the following set

$$\mathcal{G}^{Fat} := \left\{ p = (p^0, p^1, p^2) \in [0,2]^3, \quad \hat{f}^{Fat}(p) = f(p) \right\} \quad \text{with} \quad \left\{ \begin{array}{l} \hat{f}^{Fat}(p) := \hat{\gamma}^{Fat}(g^+(p^0), g^-(p^1), g^-(p^2)) \\ f(p) := (g(p^0), g(p^1), g(p^2)) \end{array} \right.$$

Then $\mathcal{G}^{Fat} \subset [0,2]^3$ is a Riemann germ which is monotone Kružkov and non-conservative.

Proof of Lemma 16.7

Let $\sigma = (\sigma^0, \sigma^1, \sigma^2) = (1, -1, -1)$. Because $\hat{\gamma}^{Fat} = id_{[0,+\infty)^3}$, we see that $\hat{\gamma}^{Fat,j}(\bar{\gamma})$ only depends on $\bar{\gamma}^j$, and then is σ-monotone. From Theorem 11.8, we deduce that the germ $\mathcal{G}^{Fat}$ is monotone.

Notice now that

$$D^{\hat{f}^{Fat}}(p,q) = \text{sign}(p^0 - q^0) \cdot \{g^+(p^0) - g^+(q^0)\} - \{\text{sign}(p^1 - q^1) \cdot \{g^-(p^1) - g^-(q^1)\} + \text{sign}(p^2 - q^2) \cdot \{g^-(p^2) - g^-(q^2)\}\}$$

Because g^+ is nondecreasing and g^- is nonincreasing, we deduce that

$$D^{\hat{f}^{Fat}}(p,q) \geq 0$$

and then we conclude that $\mathcal{G}^{Fat}$ is a Kružkov germ. Finally it is obvious that this germ is not conservative (see Figure 16). This ends the proof of the lemma.

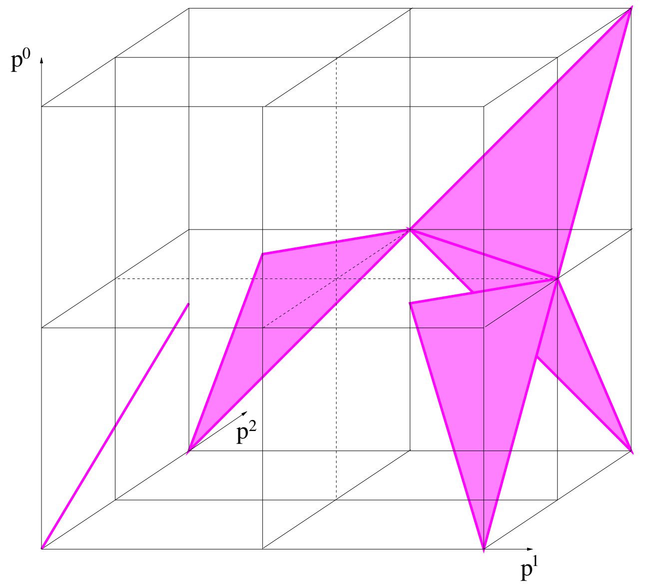

Figure 14: The germ $\mathcal{G}^{VV} = \mathcal{G}^{DN} = \mathcal{G}^{HR}$ for $\theta^1 = \theta^2 = \dfrac{1}{2}$

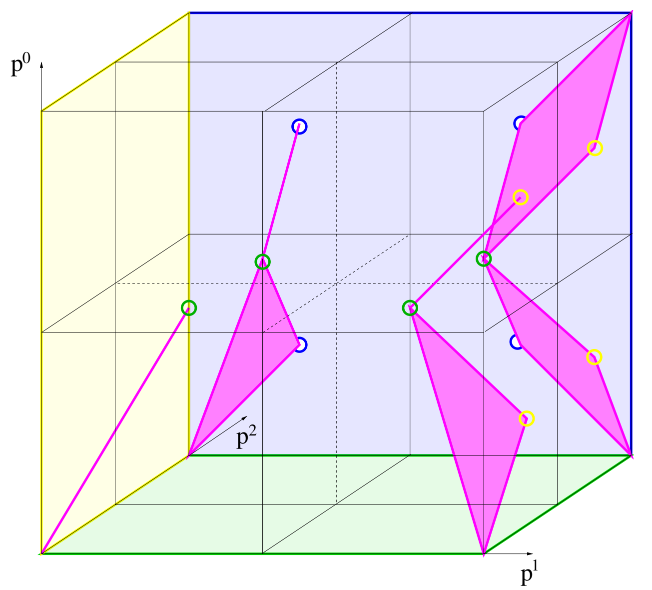

Figure 15: The Traffic Light germ $\mathcal{G}^{TL}$ for $\theta^1 = \theta^2 = \dfrac{1}{2}$

16.6 A conservative quasi Hamilton-Jacobi germ as a thin germ

Lemma 16.8 (A conservative quasi Hamilton-Jacobi germ)
Let us consider the bell-shaped function $g : [0,2] \to [0,1]$ and its monotone envelopes g^+, g^- defined in (16.1). For a $1:2$ junction, let us consider the identical preflux $\hat{\gamma}^{qHJ} : [0,+\infty)^3 \to [0,+\infty)^3$ for $\theta^0 = 1 = \theta^1 + \theta^2$ with $\theta^1, \theta^2 \in (0,1)$, defined by

$$(\hat{\gamma}^{qHJ})^k(\bar{\gamma}) := \theta^k \min\left\{\frac{\bar{\gamma}^0}{\theta^0}, \frac{\bar{\gamma}^1}{\theta^1}, \frac{\bar{\gamma}^2}{\theta^2}\right\}, \quad k = 0,1,2$$

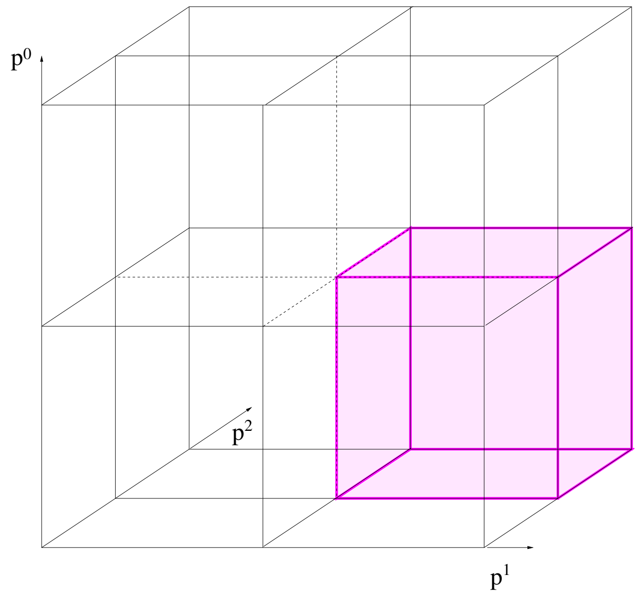

Figure 16: A fat germ $\mathcal{G}^{Fat}$

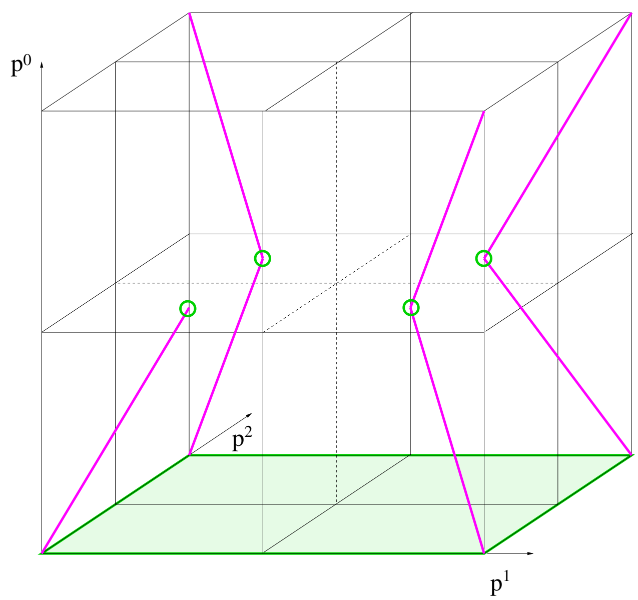

Figure 17: A thin germ: Hamilton-Jacobi germ $\mathcal{G}^{qHJ}$

We consider the following set

$$\mathcal{G}^{qHJ} := \left\{ p = (p^0, p^1, p^2) \in [0,2]^3, \quad \hat{f}^{qHJ}(p) = f(p) \right\} \quad \text{with} \quad \left\{ \begin{array}{l} \hat{f}^{qHJ}(p) := \hat{\gamma}^{qHJ}(g^+(p^0), g^-(p^1), g^-(p^2)) \\ f(p) := (g(p^0), g(p^1), g(p^2)) \end{array} \right.$$

For a $1:2$ *junction with flux* f*, then* $\mathcal{G}^{qHJ} \subset [0,2]^3$ *is a Riemann germ which is conservative quasi HJ, but not monotone (neither Kružkov). Precisely, we have*

$$\hat{f}^{qHJ,0} = \hat{f}^{qHJ,1} + \hat{f}^{qHJ,2}$$

Proof of Lemma 16.8
We use (the proof of) the Lemma 15.25 in the spacial case $n = 2$ for the preflux.
Then Theorem 11.8 implies that the Riemann germ $\mathcal{G}^{qHJ}$ is conservative quasi HJ, but not monotone (and then not Kružkov). This ends the proof of the lemma.

Part III

Existence and uniqueness theory for Kružkov germs

17 Properties of semisolutions

17.1 Stability

Lemma 17.1 (Solutions versus sub/supersolutions)
Assume (2.2) with $N \geq 1$, and let $\mathcal{G} \subset [a,b]$ be a generalized Riemann germ. Then u is a $\mathcal{G}$-entropy solution of (2.4) if and only if it is a $\mathcal{G}$-entropy subsolution and supersolution of (2.4).

Proof of Lemma 17.1
We know that $\mathcal{G}^{SUB} \cap \mathcal{G}^{SUP} = \mathcal{G}$. Hence the desired property is true at the junction point for $\mathcal{G}$-entropy. We now want to check (it is probably very classical) that on each branch a function is an entropy solution if and only if it is an entropy subsolutions and supersolution. To this end, we consider the case $N = 1$ and drop the index j. We get with notation ψ^j in (2.53) and $\psi^j_\pm$ in (2.54)

$$\left\{\begin{array}{l} |u-k| = |u-k|_+ + |u-k|_- \quad \text{and} \quad |u-k|_- = |k-u|_+ \\ \psi(u,k) = \psi_+(u,k) + \psi_-(u,k) \end{array}\right.$$

and this implies that if a function u is both an entropy subsolution and supersolution, then it is an entropy solution. Conversely for an entropy solution (hence bounded), we have for all $k \in \mathbb{R}$

$$(17.1) \qquad \left\{\begin{array}{l} |u-k| + |u-a| = 2|u-k|_+ + |k-a| + 2|u-a|_- \\ \psi(u,k) + \psi(u,a) = 2\psi_+(u,k) + \{f(k) - f(a))\} + 2\psi_-(u,a) \end{array}\right.$$

and

$$(17.2) \qquad \left\{\begin{array}{l} |u-k| + |u-b| = 2|u-k|_- + |k-b| + 2|u-b|_+ \\ \psi(u,k) + \psi(u,b) = 2\psi_-(u,k) - \{f(k) - f(b))\} + 2\psi_+(u,b) \end{array}\right.$$

Notice that $u \in [a,b]$ implies $0 = 2|u-a|_- = 2\psi_-(u,a) = 2|u-b|_+ = 2\psi_+(u,b)$. Hence this shows in the integral formulation that every standard entropy solution u satisfying $a \leq u \leq b$ also satisfies both conditions of standard entropy subsolution and supersolution for $k \in [a,b]$, and then for all $k \in \mathbb{R}$.
This ends the proof of the lemma.

In general, we do not expect to have stability of $\mathcal{G}$-solutions for all Riemann germs $\mathcal{G}$. Here we present stability for certain subclasses.

Lemma 17.2 (Stability of solutions and of sub/supersolutions)
Assume (2.2) with $N \geq 1$, and let $\mathcal{G} \subset [a,b]$ be a set.
We consider a sequence of functions $(u_n)_{n\in\mathbb{N}}$ with $u_n : [0,+\infty) \times J \to \mathbb{R}$, such that

$$u_n \to u_\infty \quad \text{in} \quad L^1_{loc}([0,+\infty) \times J)$$

i) (Solutions for Kružkov germs)
If $\mathcal{G}$ is a Kružkov germ and if each function u_n is a $\mathcal{G}$-entropy solution of (2.4), then the limit u is also a $\mathcal{G}$-entropy solution of (2.4).
ii) (Subsolutions/supersolutions for monotone Kružkov germs)
If $\mathcal{G}$ is a monotone Kružkov germ and if each function u_n is a $\mathcal{G}$-entropy subsolution (resp. supersolution) of (2.4), then the limit u is also a $\mathcal{G}$-entropy subsolution (resp. supersolution) of (2.4).

Proof of Lemma 17.2
First recall that the stability of Kružkov entropy solutions/subsolutions/supersolutions is classical outside the junction point, and follows from the very definition of Kružkov entropy solutions. Hence it remains to show the stability of the boundary condition at the junction point.
Step 1: proof of i)

We simply use the integral formulation of (2.4), which is recalled in Proposition 2.65. Forgetting the initial data, and focusing on the junction point, this means the following for u_n. For all test functions $0 \le \varphi^j \in C^1_c((0,+\infty)\times \bar{J}^j)$, and with $\bar{J}^j = \{0\}\cup J^j \simeq [0,+\infty)$ or $(-\infty,0]$ with $\varphi^j(t,0)=\varphi^k(t,0)$ for all $t\in[0,+\infty)$ and all index j,k, we have

$$(17.3)\quad \sum_k \left\{ \int_{(0,+\infty)\times J^k} \left\{\eta^k(u_n,c)\varphi^k_t + \psi^k(u_n,c)\varphi^k_x\right\}\ dtdx \right\} \ge 0 \quad \text{for all elements} \quad c=(c^1,\dots,c^N)\in\mathcal{G}$$

At the limit $n\to+\infty$, the function u_∞ still satisfies (17.3). Choosing

$$\varphi^j(t,x)=\beta(t)\alpha^j_\varepsilon(x) \quad \text{with} \quad \alpha^j_\varepsilon(x)=\alpha^j(\varepsilon^{-1}x) \quad \text{with} \quad 0\le\beta\in C^1_c(0,+\infty), \quad 0\le\alpha^j\in C^1_c(\bar{J}^j), \quad \alpha^j(0)=1$$

and using the existence of strong Panov's traces, we get at the limit $\varepsilon\to 0$:

$$\int_{(0,+\infty)\times\{0\}} \beta D(u_\infty,c)\ge 0 \quad \text{with} \quad D(u_\infty,c):=\sum_{J^k\simeq(-\infty,0)}\psi^k(u_\infty,c)-\sum_{J^k\simeq(0,+\infty)}\psi^k(u_\infty,c)$$

We deduce that for a.e. time $t\in(0,+\infty)$, we have $D(u_\infty(t,0),c)\ge 0$ for all $c\in\mathcal{G}$. Because $\mathcal{G}$ is a D-germ, it is D-maximal, and this implies that $u_\infty(t,0)\in\mathcal{G}$ for a.e. time $t\in(0,+\infty)$, which shows that u_∞ is a $\mathcal{G}$-entropy solution of (2.4).
Step 2: proof of ii)
The proof follows the same lines as in Step 1. Recall that at the junction point, $\mathcal{G}$-entropy subsolutions (resp supersolutions) u_n of (2.4) satisfy the following. For all test functions $0\le\varphi^j\in C^1_c((0,+\infty)\times\bar{J}^j)$ with $\bar{J}^j=\{0\}\cup J^j\simeq[0,+\infty)$ or $(-\infty,0]$, and with $\varphi^j(t,0)=\varphi^k(t,0)$ for all $t\in[0,+\infty)$ and all index j,k, we have the following inequality

$$(17.4)\qquad \sum_k \left\{ \int_{(0,+\infty)\times J^k} \left\{\eta^k_+(u_n,c)\varphi^k_t + \psi^k_+(u_n,c)\varphi^k_x\right\}\ dtdx \right\} \ge 0 \quad \text{for all} \quad c\in\mathcal{G}^{SUB}$$

(respectively the same relation with $(\eta^k_-,\psi^k_-,\mathcal{G}^{SUP})$ instead of $(\eta^k_+,\psi^k_+,\mathcal{G}^{SUB})$, with $\psi^k_-(u_n,c)=\psi^k_+(c,u_n)$). Then we deduce $D_+(u_\infty(t,0),c)\ge 0$ for all $c\in\mathcal{G}^{SUB}$. Using the fact that the left-dual satisfies ${}^*\mathcal{G}^{SUB}=\mathcal{G}^{SUB}$, we deduce that $u_\infty(t,0)\in\mathcal{G}^{SUB}$ for a.e. time $t\in(0,+\infty)$, which shows that u_∞ is a $\mathcal{G}$-entropy subsolution (resp. supersolution) of (2.4). This ends the proof of the lemma.

17.2 L^1-contraction, uniqueness and comparison

Lemma 17.3 **(L^1-contraction, uniqueness and comparison)**
Assume (2.2) with $N\ge 1$, and let $\mathcal{G}\subset[a,b]$ be a set. Let us consider two initial data $u_0,v_0\in p_0+L^1(J)$ for some constant $p_0\in\mathbb{R}^N$.
i) (L^1-contraction and uniqueness)
Assume that $\mathcal{G}$ is a Kružkov germ. Let u,v be two $\mathcal{G}$-entropy solutions of (2.4) with respective initial data u_0,v_0. Then we have

$$\int_{\{t\}\times J}|u-v|\le\int_{\{0\}\times J}|u-v| \quad \textit{for all} \quad t\ge 0$$

In particular, if $u_0=v_0$, we get $u=v$, i.e. we have uniqueness of the solution.
ii) (Comparison)
Assume that $\mathcal{G}$ is a monotone Kružkov germ. Let u (resp. v) be a $\mathcal{G}$-entropy subsolution (resp. supersolution) of (2.4), with respective initial data u_0,v_0. Then we have

$$\int_{\{t\}\times J}|u-v|_+\le\int_{\{0\}\times J}|u-v|_+ \quad \textit{for all} \quad t\ge 0$$

Proof of Lemma 17.3
Step 1: proof of i)
Recall that the doubling of variable method introduced by Kružkov, allows to claim for inequalities on $\mathcal{D}'=\mathcal{D}'((0,+\infty)\times J^j)$ that

$$(17.5)\qquad \left.\begin{array}{l}\partial_t\eta^j(u,k)+\partial_x\psi^j(u,k)\le 0\\ \partial_t\eta^j(k,v)+\partial_x\psi^j(k,v)\le 0\end{array}\right\} \text{for all } k\in\mathbb{R}^N, \quad \text{implies} \quad \partial_t\eta^j(u,v)_t+\partial_x\psi^j(u,v)\le 0$$

Notice that in the original paper [35], Kružkov uses the Lipschitz continuity of the fluxes f^j (that we also assume), even if it also works for continuous fluxes (using the Lebesgue Dominated Convergence theorem). Inspired by the integral formulation of (2.4) given in Proposition 2.65 at the junction point, we consider the following. For all test functions $0 \le \varphi^j \in C^1_c((0,+\infty) \times \bar{J}^j)$ with $\bar{J}^j = \{0\} \cup J^j \simeq [0,+\infty)$ or $(-\infty, 0]$ with

$$\varphi^j(t,0) = \varphi^k(t,0) =: \varphi(t,0) \quad \text{for all } t \in [0,+\infty) \text{ and all index } j,k \tag{17.6}$$

we set

$$I(\varphi) := \sum_k \left\{ \int_{(0,+\infty)\times J^k} \left\{ \eta^k(u,v)\varphi^k_t + \psi^k(u,v)\varphi^k_x \right\} \, dtdx \right\} \tag{17.7}$$

Now set

$$\tilde{\varphi}^j_\varepsilon(t,x) := \varphi(t,0)\alpha^j_\varepsilon(x) \quad \text{with} \quad \alpha^j_\varepsilon(x) := \alpha^j(\varepsilon^{-1}(x), \quad 0 \le \alpha^j \in C^1_c(\bar{J}^j), \quad \alpha^j(0) = 1$$

Then we write $I(\varphi) = I(\tilde{\varphi}_\varepsilon) + I(\varphi - \tilde{\varphi}_\varepsilon)$. Because $\varphi - \tilde{\varphi}_\varepsilon$ is the limit of functions in $C^1_c((0,+\infty) \times (J\backslash \{0\}))$, we deduce from (17.5) that $I(\varphi - \tilde{\varphi}_\varepsilon) \ge 0$. In the other hand, we have using Panov's traces that

$$I(\tilde{\varphi}_\varepsilon) \to I_0 \quad \text{as} \quad \varepsilon \to 0, \quad \text{with} \quad I_0 := \int_{(0,+\infty)\times\{0\}} \varphi D(u,v)$$

Because $\varphi \ge 0$ and $D(\mathcal{G},\mathcal{G}) \ge 0$, we deduce that $I_0 \ge 0$. Hence $I(\varphi) \ge 0$. Using Panov's traces at time $t = 0^+$, and the same argument as before but in time instead of space, we get for all test functions $0 \le \varphi^j \in C^1_c([0,+\infty) \times \bar{J}^j)$ satisfying (17.6) that

$$\sum_k \left\{ \int_{(0,+\infty)\times J^k} \left\{ \eta^k(u,v)\varphi^k_t + \psi^k(u,v)\varphi^k_x \right\} \, dtdx + \int_{\{0\}\times J^k} \eta^k(u,v)\varphi^k \, dx \right\} \ge 0$$

Using the fact that the fluxes f^j are locally Lipschitz continuous (say with constant L), we know that we have finite propagation. Now, given any compact set $K_0 \subset J$, and for $t \ge 0$, let $K_t := J\backslash\Omega_t$ with $\Omega_t := B_{tL} + \Omega_0$ and $\Omega_0 := J\backslash K_0$. Then we can find a sequence of test functions $\varphi_n(t,x)$ approximating $1_{K_t}(x)1_{[0,s]}(t)$ for any $s > 0$. Hence we get

$$\int_{\{s\}\times K_s} |u - v| \le \int_{\{0\}\times K_0} |u - v| = \int_{\{0\}\times K_0} |u_0 - v_0|$$

This property implies that $u, v \in C^0([0,+\infty); L^1_{loc}(J))$ (for instance testing u with a function v which is locally constant on $K_0 \not\ni 0$, we can repeat it for all such K_0 and all translations in time). Now assuming that $u_0, v_0 \in p_0 + L^1(J)$, we deduce from monotone convergence theorem, that we can pass to the limit where K_0 tends to the whole junction J, which gives

$$\int_{\{s\}\times J} |u - v| \le \int_J |u_0 - v_0| = \int_{\{0\}\times J} |u - v|$$

Step 2: proof of ii)
The proof follows the lines of Step 1. The method of doubling of variables of Kružkov also works for Kružkov semi-entropies, and gives that

$$\left.\begin{array}{l} \partial_t\eta^j_+(u,k) + \partial_x\psi^j_+(u,k) \le 0 \\ \partial_t\eta^j_+(k,v) + \partial_x\psi^j_+(k,v) \le 0 \end{array}\right\} \text{ for all } k \in \mathbb{R}^N, \quad \text{implies} \quad \partial_t(\eta^j_+(u,v) + \partial_x\psi^j_+(u^j,v^j) \le 0$$

Then for all test functions $0 \le \varphi^j \in C^1_c((0,+\infty) \times \bar{J}^j)$ with $\bar{J}^j = \{0\} \cup J^j \simeq [0,+\infty)$ or $(-\infty, 0]$ satisfying (17.6), we set

$$I_+(\varphi) := \sum_k \left\{ \int_{(0,+\infty)\times J^k} \left\{ \eta^k_+(u,v)\varphi^k_t + \psi^k_+(u,v)\varphi^k_x \right\} \, dtdx \right\} \tag{17.8}$$

and using the fact that $D_+(\mathcal{G}^{SUB},\mathcal{G}^{SUP})\ge 0$, we get that $I_+(\varphi)\ge 0$ and also deduce that

$$\sum_k \left\{ \int_{(0,+\infty)\times J^k} \left\{\eta_+^k(u,v)\varphi_t^k+\psi_+^k(u,v)\varphi_x^k\right\}\ dtdx+\int_{\{0\}\times J^k}\eta_+^k(u,v)\varphi^k\ dx\right\}\ge 0$$

Panov's traces also work for subsolutions and supersolutions, and this is also the case for the finite propagation behaviour. We get in particular that $|u-v|_+\in C^0([0,+\infty);L^1(J))$, and that

$$\int_{\{s\}\times J}|u-v|_+\le \int_J |u_0-v_0|_+=\int_{\{0\}\times J}|u-v|_+$$

This ends the proof of the lemma.

17.3 Maximum and minimum of semisolutions

Lemma 17.4 (Maximum/minimum of sub/supersolutions)
Assume (2.2) with $N\ge 1$, and let $\mathcal{G}\subset[a,b]$ be a monotone Kružkov germ.
i) (Maximum of two subsolutions)
Let u,w be two $\mathcal{G}$-entropy subsolutions of (2.4). Then $\max(u,w)$ *is a $\mathcal{G}$-entropy subsolutions of (2.4).*
ii) (Minimum of two supersolutions)
Let u,w be two $\mathcal{G}$-entropy supersolutions of (2.4). Then $\min(u,w)$ *is a $\mathcal{G}$-entropy supersolutions of (2.4).*

Proof of Lemma 17.4
We prove point i) (the proof of ii) is similar).
Step 1: checking that $\max(u,w)$ is an entropy subsolution, outside the junction point
The result of point i) should be standard, but we are not aware of a direct proof (see nevertheles Bianca, Dogbe [8] for an indirect proof). Because we only want to check that each component $\max(u^j,w^j)$ is an entropy solution on each brancj J^j, we can consider the case $N=1$ and drop the index j in all expressions. We define the entropy $\eta_0(\alpha,\beta;\gamma):=\eta_+(\alpha\vee\beta,\gamma)=|\alpha\vee\beta-\gamma|_+$, which is clearly symmetric in α,β, and satisfies (using $\alpha\vee\beta=\beta+|\alpha-\beta|_+$)

$$\eta_0(\alpha,\beta;\gamma)=|\beta+|\alpha-\beta|_+-\gamma|_+=\left\{\begin{array}{lll}|\alpha-\beta|_++\beta-\gamma & \text{if} & \beta\ge\gamma\\ |\alpha-\gamma|_+ & \text{if} & \beta\le\gamma\end{array}\right\}=|\alpha-\beta\vee\gamma|_++\text{sign}^+(\beta-\gamma)\cdot(\beta-\gamma)$$

For $\psi_+(\beta,\gamma)=\text{sign}^+(\beta-\gamma)\cdot\{f(\beta)-f(\gamma)\}$, we now define $q_0(\alpha,\beta;\gamma):=\psi_+(\alpha,\beta\vee\gamma)+\psi_+(\beta,\gamma)$. At this stage, it is not clear if q_0 is symmetric or not in α,β, and we set

$$\delta q_0(\alpha,\beta;\gamma):=q_0(\alpha,\beta;\gamma)-q_0(\beta,\alpha;\gamma)\quad=\psi_+(\alpha,\beta\vee\gamma)+\psi_+(\beta,\gamma)-\psi_+(\beta,\alpha\vee\gamma)-\psi_+(\alpha,\gamma)$$

which is antisymmetric in α,β. We now only consider the case $\alpha\le\beta$, (because the other case is symmetric).

$\alpha\le\beta$	$\psi_+(\alpha,\beta\vee\gamma)$	$\psi_+(\beta,\alpha\vee\gamma)$	$\psi_+(\alpha,\gamma)$	$\psi_+(\beta,\gamma)$	$2\psi_+(\alpha\vee\beta,\gamma)$	$\delta q_0(\alpha,\beta;\gamma)$
$\alpha\le\beta\le\gamma$	$\psi_+(\alpha,\gamma)=0$	$\psi_+(\beta,\gamma)=0$	$\psi_+(\alpha,\gamma)=0$	$\psi_+(\beta,\gamma)=0$	$2\psi_+(\beta,\gamma)=0$	0
$\alpha\le\gamma\le\beta$	$\psi_+(\alpha,\beta)=0$	$\psi_+(\beta,\gamma)$	$\psi_+(\alpha,\gamma)=0$	$\psi_+(\beta,\gamma)$	$2\psi_+(\beta,\gamma)$	0
$\gamma\le\alpha\le\beta$	$\psi_+(\alpha,\beta)=0$	$\psi_+(\beta,\alpha)$	$\psi_+(\alpha,\gamma)$	$\psi_+(\beta,\gamma)$	$2\psi_+(\beta,\gamma)$	0

We deduce from the table that q_0 is symmetric in α,β and that $2\psi_+(\alpha\vee\beta;\gamma)=q_0(\alpha,\beta,\gamma)+q_0(\beta,\alpha,\gamma)$, which shows that $q_0(\alpha,\beta;\gamma)=\psi_+(\alpha\vee\beta;\gamma)$. Hence we can apply the method of doubling of variables of Kružkov, which gives that

$$\left.\begin{array}{l}\partial_t\eta_0(u,k;c)+\partial_x q_0(u,k;c)\le 0\\ \partial_t\eta_0(w,\ell;c)+\partial_x q_0(w,\ell;c)\le 0\end{array}\right\}\text{ for all } k,\ell\in\mathbb{R}^N,\quad\text{implies}\quad \partial_t\eta_0(u,w;c)+\partial_x q_0(u,w;c)\le 0$$

i.e. $\partial_t\eta_+(u\vee w,c)+\partial_x\psi_+(u\vee w,c)\le 0$ for all $c\in\mathbb{R}$. This shows that $\max(u,w)$ is a Kružkov entropy subsolution (outside the origin).
Step 2: checking that $\max(u,w)$ is a $\mathcal{G}$-entropy subsolution, at the junction point
Here this is the simplest part. We just have to check that $\max(u,w)(t,0)\in\mathcal{G}^{SUB}$ for a.e. time $t>0$, which follows from Lemma 8.16. This ends the proof of the lemma.

17.4 Proof of Theorem 2.64: properties of semisolutions for monotone Kružkov germs

Proof of Theorem 2.64
For the proof we refer to the table of Subsection 2.5. The result follows from Lemmata 17.2 (stability), 17.3 (Max/min) and 17.4 (L^1-comparison).

18 Existence via vanishing viscosity for Kružkov germs

In this section, our goal is to get existence (and indeed uniqueness) of $\mathcal{G}$-entropy solutions to problem (2.52), which describes scalar conservation laws on a junction. We will be able to reach the end of this program only in the special case of Kružkov germs $\mathcal{G}$.

18.1 General strategy for the proof of existence

Here we present very briefly the general strategy of the proof of existence. We only give the heuristics, without justification. The key idea is to get a priori bounds on the solution, and then to justify them using some method of approximation. To simplify, we assume that J is a junction of type $0 : N$, i.e. that $J^j \simeq (0,+\infty)$ for all j. We consider solutions $u = (u^1,\dots,u^N)$ of

$$\begin{cases} u_t + (f(u))_x = 0 & \text{on} \quad (0,+\infty)\times J \\ u(t,0)\in\mathcal{G} & \text{a.e. on} \quad (0,+\infty)\times\{0\} \\ u = u_0 & \text{on} \quad \{0\}\times J \end{cases}$$

0. L^∞ estimate
Assuming u_0 bounded, we first construct a bounded box $[\bar a,\bar b]\supset u_0(J)$ such that $\mathcal{G}\cap[\bar a,\bar b]$ is still a Riemann germ on $[\bar a,\bar b]$, which is not a straightforward result. We then show by the maximum principle (up to the boundary) that $u(t,\cdot)\in[\bar a,\bar b]$ for all $t>0$.
1. u_t estimate
Then for $J^* := J\setminus\{0\}$, for two solutions u,v we have $|u-v|_t + (\psi^f(u,v))_x \le 0$ with $\psi^{f^j}(u,v) := \text{sign}(u^j - v^j)\cdot\left\{f^j(u^j)-f^j(v^j)\right\}$. The integration by parts on $(0,t)\times J^*$ gives the contraction estimate

$$\int_{(0,t)\times\{0\}} D^f(u,v) + \int_{\{t\}\times J}|u-v| \le \int_{\{0\}\times J}|u-v| \qquad \text{where} \quad D^f(u,v) := \sum_{j=1,\dots,N} -\psi^{f^j}(u,v)\ge 0$$

for $u(t,0), v(t,0)\in\mathcal{G}$, because $\mathcal{G}$ is a Kružkov germ. In particular for $v(t,x) := u(t+h,x)$ and dividing by $h\to 0^+$, we get formally

$$\int_{\{t\}\times J}|u_t| \le \int_{\{0\}\times J}|u_t|$$

Up to a boundary layer correction term when $u_0(0)\notin\mathcal{G}$, we can show that such inequality holds true.
2. u_x estimate
We write $(u_x)_t + (f'(u)u_x)_x = 0$. Multiplying by $\text{sign}(u_x)$, we get $|u_x|_t + (f'(u)|u_x|)_x \le 0$. Integrating on $(0,t)\times(\delta,+\infty)$, we get

$$\begin{aligned}\int_{\{t\}\times(\delta,+\infty)}|u_x| &\le \int_{\{0\}\times(\delta,+\infty)}|u_x| + \int_{(0,t)\times\{\delta\}}|(f(u))_x| \\ &\le \int_{\{0\}\times(\delta,+\infty)}|u_x| + \int_{(0,t)\times\{\delta\}}|u_t|\end{aligned}$$

which is the boundary BLN estimate. Integrating on δ, and using the u_t estimate, we get what we call the interior BLN estimate on u_x.
3. Recovering the junction condition
The previous bounds give some a priori BV_{loc} estimates, which are sufficient for any reasonable approximation process. Still in such approximation process, the key point is to recover the junction condition $u(t,0)\in\mathcal{G}$, at the limit. This is done using a weak version of the boundary condition. Precisely, this is the following for $\varphi = (\varphi^1,\dots,\varphi^N)$ with $0\le\varphi^j\in C^1((0,+\infty)\times\bar J^j)$, and for all stationary constant solution $c\in\mathcal{G}$

$$0 \le \int_{(0,+\infty)\times J^*}\left\{|u-c|\varphi_t + \psi^f(u,c)\varphi_x\right\}$$

Focusing φ on the junction point $x = 0$, we get for all $0 \le \phi \in C^1_c(0,+\infty)$

$$0 \le \int_{(0,+\infty)\times\{0\}} D^f(u,c)\phi(t), \qquad \text{i.e.} \quad 0 \le D^f(u,c) \quad \text{for all} \quad c \in \mathcal{G}$$

which implies $u(t,0) \in \mathcal{G}$, because $\mathcal{G}$ is maximal, due to the fact that $\mathcal{G}$ is Kružkov.

18.2 Strategy of the proof by vanishing viscosity

Here we present briefly the strategy of the proof of existence by vanishing viscosity. Given a Kružkov germ $\mathcal{G}$, we call $\hat{f} := \hat{f}_{\mathcal{G}}$ its associated Godunov flux. In a first step, the existence is reached using several approximations of the problem, and in a second step we relax all those approximations. We list below our set of approximations:

1) (Boundedness)
We assume that the box $[a,b]$ is bounded. We also work on a truncated junction $J_R := J \cap B_R(0)$ with bounded branches of length R, with suitable boundary condition for $x = \pm R$, encoded in some suitable germ $\mathcal{G}_R$ and some associated flux $\hat{f}_R$.
2) (Regularized fluxes)
We consider smooth flux functions $f_\eta, \hat{f}_\eta, \hat{f}_{R,\eta}$, instead of functions $f, \hat{f}, \hat{f}_R$ which are only locally Lipschitz continuous.
3) (Regularization of the solution by vanishing viscosity)
We introduce some viscosity $\varepsilon > 0$. We also assume that initial data is smooth, bounded and satisfies some compatibility conditions.

With those approximations in hands, we will be able to justify the following estimates on the solution u^ε
(18.1)

$$\begin{cases} u^{\varepsilon,j}(t,x) \in [a^j,b^j] \quad \text{for all} \quad (t,x) \in [0,+\infty)\times J^j_R, \quad j = 1,\dots,N & \textbf{(Box estimate)} \\ \displaystyle\int_{\{t\}\times J_R} |u^\varepsilon_t| \le \int_{\{0\}\times J_R} |u^\varepsilon_t| & \textbf{(Contraction estimate)} \\ \displaystyle\int_{\{t\}\times J_{\delta,R-\delta}} |u^\varepsilon_x| \le \int_{\{0\}\times J_{\delta,R-\delta}} |u^\varepsilon_x| + \int_{(0,t)\times\partial J_{\delta,R-\delta}} |u^\varepsilon_t| & \textbf{(Boundary BLN estimate)} \end{cases}$$

for all small $\delta > 0$ with

$$J_{\delta,R-\delta} := (J \cap B_{R-\delta}) \backslash \bar{B}_\delta$$

Once we have estimates (18.1), it is sufficient to get BV bounds on the solution u^ε, locally outside the origin $x = 0$ and outside $x \in \partial J_R$. We can then remove fluxes approximations 2). Then notice that all $c \in \mathcal{G}$ are solutions for $x \notin \partial J_R$, while all $c_R \in \mathcal{G}_R$ are solutions for all $x \ne 0$. This allows us show the following property with $\varphi \in \mathrm{Lip}([0,+\infty)\times \bar{J}_R;\mathbb{R}_+)$

$$(18.2) \qquad \begin{cases} \displaystyle\int_{\{0\}\times J_R} |u^\varepsilon - c|\varphi + \int_{(0,+\infty)\times J_R} \left\{ |u^\varepsilon - c|\varphi_t + \psi^f(u^\varepsilon,c)\varphi_x \right\} \ge -\varepsilon C_\varphi \\ \quad \text{for all} \quad \varphi \in C_c([0,+\infty)\times J_R) \quad \text{and all} \quad c \in \mathcal{G} \\ \\ \displaystyle\int_{\{0\}\times J_R} |u^\varepsilon - c_R|\varphi + \int_{(0,+\infty)\times J_R} \left\{ |u^\varepsilon - c_R|\varphi_t + \psi^f(u^\varepsilon,c_R)\varphi_x \right\} \ge -\varepsilon C_\varphi \\ \quad \text{for all} \quad \varphi \in C_c([0,+\infty)\times (\bar{J}_R \backslash \{0\})) \quad \text{and all} \quad c_R \in \mathcal{G}_R \end{cases}$$

We can then remove the viscosity approximation 3) with $\varepsilon \to 0$. We end up with entropy solutions with zero viscosity on the bounded junction J_R. The boundary conditions at $x = 0$ and $x \in \partial J_R$ then follow from (18.2).

In a final step, we can consider the limit $R \to +\infty$, and recover the desired solution on the full junction J.

As a guide for the remaining part of this section, let us indicate the flux $\hat{f}_R$ that we choose. For $\sigma = (-1, \dots, -1)$, and given some $p_0 \in [a, b]$, and $p = (p^1, \dots, p^N) \in [a, b]$, we set for all $j = 1, \dots, N$

$$\hat{f}^j_R(p) := G^{f^j}(p^j, p^j_0) \quad \text{and} \quad J^j_R \simeq (0, R) \tag{18.3}$$

where G^{f^j} is the standard Godunov flux associated to f^j. Then the associated germ is $\mathcal{G}_R := \Big\{p \in [a, b],\ \hat{f}_R(p) = f(p)\Big\}$.

In order to simplify the presentation, we will also use extensively the following result. It is a reduction result for a problem with viscosity $\varepsilon \geq 0$ on the full junction J. Its proof is straightforward.

Lemma 18.1 (Reduction from $n : m$ junction to $0 : n + m$ junction)
Assume (2.2) with $N \geq 1$, and let $\mathcal{G} \subset [a, b]$ be a Riemann germ with respect to (J, f) of orientation $\sigma \in \{\pm 1\}^N$ with $J^j \simeq \sigma^j \cdot (-\infty, 0)$, with Godunov flux $\hat{f} := \hat{f}_{\mathcal{G}} : [a, b] \to \mathbb{R}^N$. Let u^ε be a solution of

$$\begin{cases} u^{\varepsilon,j}_t + (f^j(u^{\varepsilon,j}))_x & = \varepsilon u^{\varepsilon,j}_{xx} \quad \text{on} \quad (0, +\infty) \times J^j \\ f^j(u^{\varepsilon,j}) - \varepsilon u^{\varepsilon,j}_x & = \hat{f}^j(u^\varepsilon) \quad \text{on} \quad (0, +\infty) \times \{0\} \\ u^{\varepsilon,j} & = u^j_0 \quad \text{on} \quad \{0\} \times J \end{cases} \tag{18.4}$$

Define the following type of reversion transform

$$(\tilde{u}^j_0(x), \tilde{u}^{\varepsilon,j}(t, x), \tilde{J}^j, \tilde{f}^j(p^j), \hat{\tilde{f}}^j(p)) := \begin{cases} (\tilde{u}^j_0(x), & u^{\varepsilon,j}(t, x), & J^j, & f^j(p^j), & \hat{f}^j(p)) & \text{if} \quad \sigma^j = -1 \\ (\tilde{u}^j_0(-x), & u^{\varepsilon,j}(t, -x), & -J^j, & -f^j(p^j), & -\hat{f}^j(p)) & \text{if} \quad \sigma^j = 1 \end{cases}$$

Then u^ε solves (18.4) if and only if $\tilde{u}^\varepsilon := (\tilde{u}^{\varepsilon,1}, \dots, \tilde{u}^{\varepsilon,N})$ solves

$$\begin{cases} \tilde{u}^{\varepsilon,j}_t + (\tilde{f}^j(\tilde{u}^{\varepsilon,j}))_x & = \varepsilon \tilde{u}^{\varepsilon,j}_{xx} \quad \text{on} \quad (0, +\infty) \times \tilde{J}^j \\ \tilde{f}^j(\tilde{u}^{\varepsilon,j}) - \varepsilon \tilde{u}^{\varepsilon,j}_x & = \hat{\tilde{f}}^j(\tilde{u}^\varepsilon) \quad \text{on} \quad (0, +\infty) \times \{0\} \\ \tilde{u}^{\varepsilon,j} & = \tilde{u}^j_0 \quad \text{on} \quad \{0\} \times \tilde{J} \end{cases}$$

and junction $\tilde{J}$ is of type $0 : N$ with $\tilde{J}^j \simeq (0, +\infty)$ for all j.

18.3 Viscous regime for truncated junction with regularized fluxes

Lemma 18.2 (Existence in the viscous regime for truncated junction with regularized fluxes)
Assume (2.2) with $N \geq 1$ with junction J of type $0 : N$ and bounded box $[a, b] \subset \mathbb{R}^N$. Let $R > 0$, $J_R := J \cap B_R$ and $\sigma = (-1, \dots, -1) = -\sigma_R \in \mathbb{R}^N$ and let us consider functions for all $j = 1, \dots, N$ and $p \in \mathbb{R}^N$

$$\begin{cases} \hat{f}_\eta, \hat{f}_{R,\eta} : \mathbb{R}^N \to \mathbb{R}^N, \\ f_\eta, \hat{f}_\eta, \hat{f}_{R,\eta} \in W^{1,\infty}(\mathbb{R}^N; \mathbb{R}^N) \\ \\ f_\eta = (f^1_\eta, \dots, f^N_\eta) \quad \text{with} \quad f^j_\eta : \mathbb{R} \to \mathbb{R} \\ p^j \mapsto \hat{f}^j_\eta(p) \quad \text{is nonincreasing} \\ p^j \mapsto \hat{f}^j_{R,\eta}(p) \quad \text{is nondecreasing} \\ \\ \left.\begin{array}{ll} \sigma^j \cdot (\hat{f}^j_\eta(p))_{|p^j = a^j} & \leq \sigma^j \cdot f^j_\eta(a^j) \\ \sigma^j_R \cdot (\hat{f}^j_{R,\eta}(p))_{|p^j = a^j} & \leq \sigma^j_R \cdot f^j_\eta(a^j) \\ \\ \sigma^j \cdot (\hat{f}^j_\eta(p))_{|p^j = b^j} & \geq \sigma^j \cdot f^j_\eta(b^j) \\ \sigma^j_R \cdot (\hat{f}^j_{R,\eta}(p))_{|p^j = b^j} & \geq \sigma^j_R \cdot f^j_\eta(b^j) \end{array}\right| \quad \text{for all} \quad p \in [a, b] \end{cases} \tag{18.5}$$

Assume that the initial data $u_0 = (u^1_0, \dots, u^N_0)$ satisfies $u^j_0 \in C^\infty(\bar{J}^j_R; [a^j, b^j])$ with $J^j_R \simeq (0, R)$. Let $\varepsilon > 0$.
i) (Existence)
Then there exists $u^\varepsilon = (u^{\varepsilon,j})_{j=1,\dots,N}$ with

$$u^{\varepsilon,j} : [0, +\infty) \times \bar{J}^j_R \to [a^j, b^j] \tag{18.6}$$

solution of

$$
(18.7)\qquad \left\{\begin{array}{rll} u_t^{\varepsilon,j}+(f_\eta^j(u^{\varepsilon,j}))_x & =\varepsilon u_{xx}^{\varepsilon,j} & on\quad (0,+\infty)\times J_R^j\\ f_\eta^j(u^{\varepsilon,j})-\varepsilon u_x^{\varepsilon,j} & =\hat f_\eta^j(u^\varepsilon) & on\quad (0,+\infty)\times\{0\}\\ f_\eta^j(u^{\varepsilon,j})-\varepsilon u_x^{\varepsilon,j} & =\hat f_{R,\eta}^j(u^\varepsilon) & on\quad (0,+\infty)\times\{R\}\end{array}\right.
$$

with initial condition

$$
(18.8)\qquad u^{\varepsilon,j}=u_0^j\quad on\quad \{0\}\times\bar J_R^j
$$

In particular we have $u^{\varepsilon,j}\in C_{t,x}^{\frac{\alpha}{2},\alpha}([0,+\infty)\times\bar J_R^j)$ *for all* $\alpha\in(0,1)$.

ii) (Further regularity)

Assume moreover that the initial data u_0 *satisfies the following compatibility conditions for all* $j=1,\dots,N$

$$
(18.9)\qquad \left\{\begin{array}{ll} f_\eta^j(u_0^j)-\varepsilon(u_0^j)_x=\hat f_\eta^j(u_0) & for\quad x=0\\ f_\eta^j(u_0^j)-\varepsilon(u_0^j)_x=\hat f_{R,\eta}^j(u_0) & for\quad x=R.\end{array}\right.
$$

and that we have the following additional regularity

$$
(18.10)\qquad f_\eta,\hat f_\eta,\hat f_{R,\eta}\in W_{loc}^{2,\infty}(\mathbb{R}^N;\mathbb{R}^N)
$$

Then the solution has regularity $u^{\varepsilon,j}\in C_{t,x}^{\frac{2+\alpha}{2},2+\alpha}([0,+\infty)\times\bar J_R^j)$ *for all* $\alpha\in(0,1)$.

Proof of Lemma 18.2

First notice that, up to rescale the PDE in space and redefine properly the functions $f_\eta,\hat f_\eta,\hat f_{R,\eta}$, we can assume that $\varepsilon=1$.

Step 1: first global existence result

We can apply Lemma 21.7 to $h(u):=-f_\eta(u)$, $g_0(u):=\hat f_\eta(u)-f_\eta(u)$ and $g_R(u):=\hat f_{R,\eta}(u)-f_\eta(u)$. We get the existence of a global solution $u=(u^1,\dots,u^N)$ with $u^j\in C_{t,x}^{\frac{\alpha}{2},\alpha}([0,+\infty)\times\bar J_R^j)$ for any $\alpha\in(0,1)$, of

$$
(18.11)\qquad \left\{\begin{array}{rlll} u_t+(f_\eta(u))_x & =u_{xx} & \text{on} & (0,+\infty)\times J_R^*\\ f_\eta(u)-u_x & =\hat f_\eta(u) & \text{on} & (0,+\infty)\times\{0\}\\ f_\eta(u)-u_x & =\hat f_{R,\eta}(u) & \text{on} & (0,+\infty)\times(J\cap\partial B_R)\\ u & =u_0 & \text{on} & \{0\}\times J_R^*\end{array}\right.
$$

Step 2: Box bounds (18.6)

In order to show that the solution u stays in the box $[a,b]$ if the initial data is, we want to use the maximum principle. Recall that $\hat f_\eta,\hat f_{R,\eta}$ satisfy the barrier bounds (the two last lines of (18.5)). Given some index j, we want to show that the solutions satisfy for instance

$$
(18.12)\qquad u^j\le b^j
$$

Given the PDE on the branch J_R^j, i.e. $(u^j-b^j)_t+(f_\eta^j(u^j)-f_\eta^j(b^j))_x=(u^j-b^j)_{xx}$ and multiplying it by $\text{sign}^+(u^j-b^j)$ (or an approximation of it), we get (with $\psi_+^{f_\eta^j}$ defined in (2.54))

$$
(|u^j-b^j|_+)_t+(\psi_+^{f_\eta^j}(u^j,b^j))_x\le(|u^j-b^j|_+)_{xx}
$$

and integrating over $J_R^j\simeq(0,R)$, we get

$$
\frac{d}{dt}\int_{\{t\}\times J_R^j}|u^j-b^j|_++\left[\psi_+^{f_\eta^j}(u^j,b^j)-\text{sign}^+(u^j-b^j)u_x^j\right]_{x=0}^{x=R}(t)\le 0,\qquad\text{i.e.}\quad \frac{d}{dt}\int_{\{t\}\times J_R^j}|u^j-b^j|_++\mathcal{D}^j\le 0
$$

with

$$
\mathcal{D}^j:=\text{sign}^+(u^j-b^j)\cdot\left\{\hat f_{R,\eta}^j(u^j)-f_\eta^j(b^j)\right\}_{|x=R}-\text{sign}^+(u^j-b^j)\cdot\left\{\hat f_\eta^j(u)-f_\eta^j(b^j)\right\}_{|x=0}=:\mathcal{D}_R^j+\mathcal{D}_0^j
$$

Recall that at $x=0$ with $J^j\simeq(0,+\infty)$, we have $\hat f_\eta^j(p)_{|p^j=b^j}\le f_\eta^j(b^j)$, where we have used the two last lines of (18.5) for $\hat f_\eta$ and $\sigma^j=-1$. Recall also that the map $p^j\mapsto\hat f^j(p)$ is nonincreasing. This property implies that $\mathcal{D}_0^j \geq 0$. Similarly, at $x = R$ with $J_R^j \simeq (0, R)$ (and $\sigma_R^j = 1$), we have $\hat{f}_{R,\eta}^j(p)_{|p^j=b^j} \geq f_\eta^j(b^j)$, where we have used the two last lines of (18.5) for $\hat{f}_{R,\eta}$ and $\sigma_R^j = 1$. This shows that $\mathcal{D}_R^j \geq 0$. Hence

$$\frac{d}{dt}\int_{\{t\}\times J_R^j} |u^j - b^j|_+ \leq 0$$

Because the initial data satisfies $u_0^j \leq b^j$, this shows that $u^j \leq b^j$, i.e. the maximum principle. All those arguments can be made rigorous, as it is classical. Similarly, we show the other bounds, and get that $u(t,x) \in [a,b]$ for all $(t,x) \in [0,+\infty) \times \bar{J}_R$. Notice that regularity $C_{t,x}^{\frac{\alpha}{2},\alpha,}$ of the solution u^ε follows from i) of Lemma 21.7.

Step 3: further regularity

Notice that point ii) of the lemma follows immediately from assumption (18.9), additional regularity (18.10) and from regularity results ii) and iii) of Lemma 21.7. Finally, changing the variables back from $\varepsilon = 1$ to the original ε, we get the results for the solution u^ε of (18.7). This ends the proof of the lemma.

Lemma 18.3 (Contraction-dissipation in the viscous regime for $0 : N$ junction J)

We work under assumptions of Lemma 18.2 i), and assume that $u^\varepsilon, v^\varepsilon$ are two solutions of (18.7) with respective initial data u_0, v_0. Then we have the following Kato inequality

$$(18.13)\qquad \int_{\{t\}\times J_R} |u^\varepsilon - v^\varepsilon| + \int_{(0,t)\times\{0\}} D^{\hat{f}_\eta}(u^\varepsilon, v^\varepsilon) + \int_{(0,t)\times\{R\}} D^{\hat{f}_{R,\eta}}(u^\varepsilon, v^\varepsilon) \leq \int_{\{0\}\times J_R} |u^\varepsilon - v^\varepsilon|$$

with

$$D^{\hat{f}_\eta}(u^\varepsilon, v^\varepsilon) := \sum_{j=1}^N \sigma^j\ \psi^{\hat{f}_\eta^j}(u^\varepsilon, v^\varepsilon) \quad \textit{and} \quad \psi^{\hat{f}_\eta^j}(u^\varepsilon, v^\varepsilon) := sign(u^{\varepsilon,j} - v^{\varepsilon,j}) \cdot \left\{\hat{f}_\eta^j(u^\varepsilon) - \hat{f}_\eta^j(v^\varepsilon)\right\} \quad \textit{with} \quad \sigma^j = -1$$

and

$$D^{\hat{f}_{R,\eta}}(u^\varepsilon, v^\varepsilon) := \sum_{j=1}^N \sigma_R^j\ \psi^{\hat{f}_{R,\eta}^j}(u^\varepsilon, v^\varepsilon) \quad \textit{with} \quad \sigma_R^j = 1.$$

Moreover, for every test function $0 \leq \varphi^j \in C_c^1([0,+\infty) \times \bar{J}^j)$ such that

$$(18.14)\qquad \left\{\begin{array}{l} \varphi^j(t,0) = \varphi^k(t,0) =: \varphi(t,0) \quad \textit{for all indices } j,k \\ \varphi^j(t,R) = \varphi^k(t,R) =: \varphi(t,R) \quad \textit{for all indices } j,k \end{array}\right.$$

we have

$$(18.15)\qquad \begin{array}{l} \displaystyle\int_{(0,+\infty)\times\{0\}} D^{\hat{f}_\eta}(u^\varepsilon, v^\varepsilon)\varphi + \int_{(0,+\infty)\times\{R\}} D^{\hat{f}_{R,\eta}}(u^\varepsilon, v^\varepsilon)\varphi \\ \displaystyle\leq \quad \varepsilon C_{R,\varphi} + \int_{\{0\}\times J_R} |u^\varepsilon - v^\varepsilon|\varphi + \int_{(0,+\infty)\times J_R} \left\{|u^\varepsilon - v^\varepsilon|\varphi_t + \psi^{f_\eta}(u^\varepsilon, v^\varepsilon)\varphi_x\right\} \end{array}$$

where $C_{R,\varphi}$ depends on R and φ and is independent on ε.

Moreover for $N = 1$, inequality (18.13) reads

$$(18.16)\quad \int_{\{t\}\times J_R} |u^\varepsilon - v^\varepsilon| + \int_{(0,t)\times\{0\}} |\hat{f}_\eta(u^\varepsilon) - \hat{f}_\eta(v^\varepsilon)| + \int_{(0,t)\times\{R\}} |\hat{f}_{\eta,R}(u^\varepsilon) - \hat{f}_{\eta,R}(v^\varepsilon)| \leq \int_{\{0\}\times J_R} |u^\varepsilon - v^\varepsilon|$$

Proof of Lemma 18.3

Step 1: preparation

Consider $\theta_\nu : \mathbb{R} \to \mathbb{R}$ a (symmetric) approximation of the absolute value with

$$\theta'_\nu(y) := \left\{\begin{array}{ccc} 1 & \text{if} & y > \nu \\ \nu^{-1}y & \text{if} & |y| \leq \nu \\ -1 & \text{if} & y < -\nu \end{array}\right.$$

Now multiplying by $\theta'_\nu(u^{\varepsilon,j}-v^{\varepsilon,j})$ the difference of the two PDEs satisfied by $u^{\varepsilon,j}$ and $v^{\varepsilon,j}$, we get the very classical estimate (at least in the sense of distributions)

$$\partial_t\theta_\nu(u^{\varepsilon,j}-v^{\varepsilon,j})+\partial_x\psi_\nu^{f_\eta^j}(u^\varepsilon,v^\varepsilon)+A_\nu^j=\varepsilon(\theta_\nu(u^{\varepsilon,j}-v^{\varepsilon,j}))_{xx}-\varepsilon\theta''_\nu(u^{\varepsilon,j}-v^{\varepsilon,j})|(u^{\varepsilon,j}-v^{\varepsilon,j})_x|^2 \tag{18.17}$$

where $\psi_\nu^{f_\eta^j}(u^\varepsilon,v^\varepsilon):=\theta'_\nu(u^{\varepsilon,j}-v^{\varepsilon,j})\cdot\left\{f_\eta^j(u^\varepsilon)-f_\eta^j(v^\varepsilon)\right\}$ with $f_\eta^j(u^\varepsilon):=f_\eta^j(u^{\varepsilon,j})$ and

$$A_\nu^j:=A_\nu^j[u^\varepsilon,v^\varepsilon]:=-\theta''_\nu(u^{\varepsilon,j}-v^{\varepsilon,j})\cdot\left\{f_\eta^j(u^\varepsilon)-f_\eta^j(v^\varepsilon)\right\}\cdot(u^{\varepsilon,j}-v^{\varepsilon,j})_x$$

Now, following the ideas of [7], we have the monotone convergence for any radius $R>0$

$$\int_{[0,R]\cap\{|w|\le\nu\}}|w_x|\;dx\to\int_{[0,R]\cap\{w=0\}}|w_x|\;dx=0\quad\text{as}\quad\nu\to 0 \tag{18.18}$$

where the equality to zero is classical for $w\in W^{1,2}([0,R])$. This implies that $A_\nu^j\to 0$ in the sense of distributions as $\nu\to 0$, and using $\theta''_\nu\ge 0$, as $\nu\to 0$, we recover the standard result in the sense of distributions

$$\partial_t|u^{\varepsilon,j}-v^{\varepsilon,j}|+\partial_x\psi^{f_\eta^j}(u^\varepsilon,v^\varepsilon)\le\varepsilon|u^{\varepsilon,j}-v^{\varepsilon,j}|_{xx}$$

where $\psi^{f_\eta^j}(u^\varepsilon,v^\varepsilon):=\mathrm{sign}(u^{\varepsilon,j}-v^{\varepsilon,j})\cdot\left\{f_\eta^j(u^\varepsilon)-f_\eta^j(v^\varepsilon)\right\}$. To simplify the notation, we set

$$\left\{\begin{array}{l}\theta_\nu^j(u^\varepsilon-v^\varepsilon):=\theta_\nu(u^{\varepsilon,j}-v^{\varepsilon,j})\\ \theta_\nu^{j\,\prime}(u^\varepsilon-v^\varepsilon):=\theta'_\nu(u^{\varepsilon,j}-v^{\varepsilon,j})\\ \theta_\nu^{j\,\prime\prime}(u^\varepsilon-v^\varepsilon):=\theta''_\nu(u^{\varepsilon,j}-v^{\varepsilon,j})\end{array}\right.$$

Multiplying first (18.17) by a smooth nonnegative test function $0\le\varphi^j\in C_c^\infty([0,+\infty)\times\bar J_R^j)$, we get by integration by parts on $J_R^j\simeq(0,R)$

$$\begin{aligned}&-\int_{\{0\}\times J_R^j}\theta_\nu^j(u^\varepsilon-v^\varepsilon)\varphi^j-\int_{(0,+\infty)\times\{0\}}\psi_\nu^{f_\eta^j}(u^\varepsilon,v^\varepsilon)\varphi^j+\int_{(0,+\infty)\times\{R\}}\psi_\nu^{f_\eta^j}(u^\varepsilon,v^\varepsilon)\varphi^j\\
&-\int_{(0,+\infty)\times J_R^j}\left\{\theta_\nu^j(u^\varepsilon-v^\varepsilon)\varphi_t^j+\psi_\nu^{f_\eta^j}(u^\varepsilon,v^\varepsilon)\varphi_x^j\right\}+\int_{(0,+\infty)\times J_R^j}A_\nu^j\varphi^j\\
&=\int_{(0,+\infty)\times\{R\}}\left\{\varepsilon(\theta_\nu^j(u^\varepsilon-v^\varepsilon))_x\varphi^j-\varepsilon\theta_\nu^j(u^\varepsilon-v^\varepsilon)\varphi_x^j\right\}-\int_{(0,+\infty)\times\{0\}}\left\{\varepsilon(\theta_\nu^j(u^\varepsilon-v^\varepsilon))_x\varphi^j-\varepsilon\theta_\nu^j(u^\varepsilon-v^\varepsilon)\varphi_x^j\right\}\\
&+\int_{(0,+\infty)\times J_R^j}\varepsilon\theta_\nu^j(u^\varepsilon-v^\varepsilon)\varphi_{xx}^j-\int_{(0,+\infty)\times J_R^j}\varepsilon\theta_\nu^{j\,\prime\prime}(u^\varepsilon-v^\varepsilon)|(u^{\varepsilon,j}-v^{\varepsilon,j})_x|^2\varphi^j\end{aligned}$$

Using the fact that the solutions satisfy

$$\left\{\begin{array}{ll}\psi_\nu^{f_\eta^j}(u^\varepsilon,v^\varepsilon)-\varepsilon(\theta_\nu^j(u^\varepsilon-v^\varepsilon))_x=\psi_\nu^{\hat f^j}(u^\varepsilon,v^\varepsilon)&\text{on}\quad x=0\\[2ex] \psi_\nu^{f_\eta^j}(u^\varepsilon,v^\varepsilon)-\varepsilon(\theta_\nu^j(u^\varepsilon-v^\varepsilon))_x=\psi_\nu^{\hat f^j_{R,\eta}}(u^\varepsilon,v^\varepsilon)&\text{on}\quad x=R\end{array}\right.$$

(which is already satisfied by the initial data), we get

$$\begin{aligned}&\int_{(0,+\infty)\times\{R\}}\psi_\nu^{\hat f^j_{R,\eta}}(u^\varepsilon,v^\varepsilon)\varphi^j-\int_{(0,+\infty)\times\{0\}}\psi_\nu^{\hat f^j_\eta}(u^\varepsilon,v^\varepsilon)\varphi^j\\
\le\;&\int_{(0,+\infty)\times\{0\}}\varepsilon\theta_\nu^j(u^\varepsilon-v^\varepsilon)\varphi_x^j-\int_{(0,+\infty)\times\{R\}}\varepsilon\theta_\nu^j(u^\varepsilon-v^\varepsilon)\varphi_x^j+\int_{(0,+\infty)\times J_R^j}\varepsilon\theta_\nu^j(u^\varepsilon-v^\varepsilon)\varphi_{xx}^j\\
&+\int_{\{0\}\times J_R^j}\theta_\nu^j(u^\varepsilon-v^\varepsilon)\varphi^j+\int_{(0,+\infty)\times J_R^j}\left\{\theta_\nu^j(u^\varepsilon-v^\varepsilon)\varphi_t^j+\psi_\nu^{f_\eta^j}(u^\varepsilon,v^\varepsilon)\varphi_x^j\right\}\\
&-\int_{(0,+\infty)\times J_R^j}A_\nu^j\varphi^j\end{aligned}$$

In the limit $\nu \to 0$, this gives with the special convention $|u^\varepsilon - v^\varepsilon|^j := |u^{\varepsilon,j} - v^{\varepsilon,j}|$

(18.19)
$$\left\{\int_{(0,+\infty)\times\{R\}} \psi^{\hat{f}^j_{R,\eta}}(u^\varepsilon,v^\varepsilon)\varphi^j + \varepsilon|u^\varepsilon - v^\varepsilon|^j\varphi^j_x\right\} - \left\{\int_{(0,+\infty)\times\{0\}} \psi^{\hat{f}^j_\eta}(u^\varepsilon,v^\varepsilon)\varphi^j + \varepsilon|u^\varepsilon - v^\varepsilon|^j\varphi^j_x\right\}$$
$$\leq \int_{(0,+\infty)\times J^j_R} \varepsilon|u^\varepsilon - v^\varepsilon|^j\varphi^j_{xx} + \int_{\{0\}\times J^j_R} |u^\varepsilon - v^\varepsilon|^j\varphi^j + \int_{(0,+\infty)\times J^j_R} \left\{|u^\varepsilon - v^\varepsilon|^j\varphi^j_t + \psi^{f^j_\eta}(u^\varepsilon,v^\varepsilon)\varphi^j_x\right\}$$

Step 2: proof of (18.13)

For $s > 0$ and $\varphi^j(t,x) := 1_{[0,s]}(t)$, and summing on j in (18.19), we get (18.13) with t replaced by s.

For $N = 1$, notice that the monotonicity of $\hat{f}_\eta$ implies $D^{\hat{f}_\eta}(u^\varepsilon,v^\varepsilon) = |\hat{f}_\eta(u^\varepsilon) - \hat{f}_\eta(v^\varepsilon)|$ and similarly for $D^{\hat{f}_{\eta,R}}(u^\varepsilon,v^\varepsilon) = |\hat{f}_{\eta,R}(u^\varepsilon) - \hat{f}_{\eta,R}(v^\varepsilon)|$ which yields to (18.16).

Step 3: proof of (18.15)

We come back to (18.19) with general test function φ^j satisfying (18.14). We now consider a constant $M \geq \max_j \max(|a^j|,|b^j|)$. Summing over indices j, we get

$$\int_{(0,+\infty)\times\{0\}} D^{\hat{f}_\eta}(u^\varepsilon,v^\varepsilon)\varphi + \int_{(0,+\infty)\times\{R\}} D^{\hat{f}_{R,\eta}}(u^\varepsilon,v^\varepsilon)\varphi$$
$$\leq 2\varepsilon M C'_{R,\varphi} + \sum_{j=1}^N \int_{\{0\}\times J^j} |u^\varepsilon - v^\varepsilon|^j\varphi^j + \sum_{j=1}^N \int_{(0,+\infty)\times J^j} \left\{|u^\varepsilon - v^\varepsilon|^j\varphi^j_t + \psi^{f^j_\eta}(u^\varepsilon,v^\varepsilon)\varphi^j_x\right\}$$

where $C'_{R,\varphi}$ is a constant only depending on R and φ, and we set $C_{R,\varphi} := 2MC'_{R,\varphi}$. This is (18.15). This ends the proof of the lemma.

Corollary 18.4 (True contraction in the viscous regime)
We work under assumptions of Lemma 18.2 ii). We assume that u^ε is a solution of (18.7) with initial data u_0 satisfying in particular compatibility conditions (18.9). We assume moreover that $\hat{f}_\eta$ and $\hat{f}_{R,\eta}$ satisfy for all $j = 1,\ldots,N$

(18.20)
$$\left\{\begin{array}{l} -\partial_j \hat{f}^j_\eta \geq \displaystyle\sum_{k\in\{1,\ldots,N\}\setminus\{j\}} |\partial_j \hat{f}^k_\eta| \\ \partial_j \hat{f}^j_{R,\eta} \geq \displaystyle\sum_{k\in\{1,\ldots,N\}\setminus\{j\}} |\partial_j \hat{f}^k_{R,\eta}| \end{array}\right| \quad on \quad [a,b]$$

Then we have

(18.21)
$$\int_{\{t\}\times J_R} |u^\varepsilon - v^\varepsilon| \leq \int_{\{0\}\times J_R} |u^\varepsilon - v^\varepsilon|$$

Moreover we have

(18.22)
$$\int_{\{t\}\times J_R} |u^\varepsilon_t| \leq \int_{\{0\}\times J_R} |u^\varepsilon_t|$$

and for $N = 1$, we get

(18.23)
$$\int_{\{t\}\times J_R} |u^\varepsilon_t| + \int_{(0,t)\times\{0\}} |\hat{f}_\eta(u^\varepsilon)_t| + \int_{(0,t)\times\{R\}} |\hat{f}_{\eta,R}(u^\varepsilon)_t| \leq \int_{\{0\}\times J_R} |u^\varepsilon_t|$$

Proof of Corollary 18.4

Notice that assumption (18.20) implies $D^{\hat{f}_\eta} \geq 0$, $D^{\hat{f}_{R,\eta}} \geq 0$ on $[a,b]^2$. Together with (18.13), this implies (18.21). Second inequality (18.22) follows from the choice of $v^\varepsilon(t,x) := u^\varepsilon(t+\tau,x)$, and dividing the difference by τ and passing to the limit. Here, we use in particular regularity of the solution up to the time $t = 0$ with $u^{\varepsilon,j} \in C^{1+\frac{\beta}{2},2+\beta}_{t,x}([0,+\infty)\times \bar{J}^j_R)$, where we have $(u^{\varepsilon,j})_t(0,\cdot) = \varepsilon(u^j_0)_{xx} - (f^j_\eta)'(u^j_0)\cdot(u^j_0)_x$. This ends the proof of the corollary.

Lemma 18.5 (Boundary and interior BLN estimates for a single truncated branch)
Assume (2.2) with $N = 1$, and denote $J^1 = (0, +\infty)$, and the truncated branch $J^1_R := (0, R)$ for some $R > 0$ and a bounded interval $[a, b] \subset \mathbb{R}$. Consider $f_\eta = f^1_\eta \in (W^{1,\infty} \cap W^{2,\infty}_{loc})(\mathbb{R}; \mathbb{R})$. For $\varepsilon > 0$, we consider a solution $u^\varepsilon \in C^{1+\frac{\beta}{2}, 2+\beta}_{t,x}([0, +\infty) \times \bar{J}^1_R; [a, b])$ for some $\beta \in (0, 1)$ of

$$u^\varepsilon_t + (f_\eta(u^\varepsilon))_x = \varepsilon u^\varepsilon_{xx} \quad on \quad (0, +\infty) \times J^1_R$$

Then for any $\delta \in (0, R/4)$ and $t > s \geq 0$, we have

$$\int_{\{t\}\times(\delta,R-\delta)} |u^\varepsilon_x| \leq \int_{\{s\}\times(\delta,R-\delta)} |u^\varepsilon_x| + \left\{ \int_{(s,t)\times\{\delta\}} |u^\varepsilon_t| + \int_{(s,t)\times\{R-\delta\}} |u^\varepsilon_t| \right\} \quad \textbf{(Boundary BLN estimate)}$$

where the lateral integral lies in the bracket.
Moreover, we have
(18.24)

$$\int_{\{t\}\times(2\delta,R-2\delta)} |u^\varepsilon_x| \leq \int_{\{s\}\times(\delta,R-\delta)} |u^\varepsilon_x| + \delta^{-1} \left\{ \int_{(s,t)\times(\delta,2\delta)} |u^\varepsilon_t| + \int_{(s,t)\times(R-2\delta,R-\delta)} |u^\varepsilon_t| \right\} \quad \textbf{(Interior BLN estimate)}$$

Proof of Lemma 18.5
Step 1: Boundary BLN estimate
Recall that the classical BLN estimate (14) in Bardos, Leroux, Nedelec [7] holds for zero Dirichlet boundary conditions. Here we do not assume any boundary conditions, but get a sort of BLN estimate up to the boundary, that we simply call boundary BLN estimate. We follow quite closely the ideas of the proof in [7]. Precisely, we adapt the estimate on u_x. We take the x-derivative of the PDE for $u := u^\varepsilon$

$$\partial_t u_x + \partial_x F = \varepsilon (u_x)_{xx} \quad \text{with} \quad F := (f_\eta(u))_x$$

and multiply by $\theta'_\nu(u_x)$, we get

$$\partial_t \theta_\nu(u_x) + \partial_x \Phi_\nu + B_\nu = \varepsilon(\theta_\nu(u_x))_{xx} - \varepsilon\theta''_\nu(u_x)|u_{xx}|^2 \tag{18.25}$$

with $\Phi_\nu := F \cdot \theta'_\nu(u_x)$, $B_\nu := -F \cdot \theta''_\nu(u_x) u_{xx}$ and $|F| \leq L|u_x|$, where L is the Lipschitz constant of f_η. Hence we have $B_\nu \to 0$ as $\nu \to 0$ (like in (18.18)). Therefore, as $\nu \to 0$, it gives $\partial_t |u_x| + \partial_x \{F \cdot \text{sign}(u_x)\} - 0 \leq \varepsilon(|u_x|)_{xx}$, which is not good enough. Indeed, we must use the PDE, in some sense on the artificial boundary $x \in \partial(\delta, R - \delta)$.
We now do the proof formally and explain later how to make it rigorous. Integrating (18.25), we get

$$\int_{J_{R,\delta}} \partial_t \theta_\nu(u_x) + [\Phi_\nu]_{\partial J_{R,\delta}} + \int_{J_{R,\delta}} B_\nu = \varepsilon\left[(\theta_\nu(u_x))_x\right]_{\partial J_{R,\delta}} - \int_{J_{R,\delta}} \varepsilon\theta''_\nu(u_x)|u_{xx}|^2 \quad \text{with} \quad J_{R,\delta} := (\delta, R - \delta)$$

Now the PDE is valid up on the boundary, i.e. $u_t + F = \varepsilon u_{xx}$ on $(0, +\infty) \times \partial J_{R,\delta}$. Hence, multiplying by $\theta'_\nu(u_x)$, we get

$$\begin{aligned} [\Phi_\nu]_{\partial J_{R,\delta}} &= [\{\varepsilon u_{xx} - u_t\} \cdot \theta'_\nu(u_x)]_{\partial J_{R,\delta}} \\ &= [\varepsilon(\theta_\nu(u_x))_x - u_t \cdot \theta'_\nu(u_x)]_{\partial J_{R,\delta}} \end{aligned}$$

Hence this gives

$$\int_{J_{R,\delta}} \partial_t \theta_\nu(u_x) + [-u_t \cdot \theta'_\nu(u_x)]_{\partial J_{R,\delta}} + \int_{J_{R,\delta}} B_\nu = -\int_{J_{R,\delta}} \varepsilon\theta''_\nu(u_x)|u_{xx}|^2$$

Integrating in time on (s, t), and using $|\theta'_\nu| \leq 1$, we get

$$\int_{\{t\}\times J_{R,\delta}} \theta_\nu(u_x) + \int_{(s,t)\times J_{R,\delta}} B_\nu \leq \int_{\{s\}\times J_{R,\delta}} \theta_\nu(u_x) + \int_{(s,t)\times\partial J_{R,\delta}} |u_t|$$

Passing to the limit $\nu \to 0$, we have $B_\nu \to 0$ and get the boundary BLN estimate

$$\int_{\{t\}\times J_{R,\delta}} |u_x| \leq \int_{\{s\}\times J_{R,\delta}} |u_x| + \int_{(s,t)\times\partial J_{R,\delta}} |u_t| \tag{18.26}$$

Now this formal proof of (18.26) can easily be justified using test functions as in the proof of Lemma 18.3.
Step 2: interior BLN estimate
From the boundary BLN estimate, we can always deduce an interior BLN estimate as follows. We follow an idea introduced in Lemma 4.2 in [10]. Replacing δ by $h > 0$ in (18.26), and integrating (18.26) for $h \in (\delta, 2\delta)$, we get (dividing by δ)

$$\delta^{-1}\int_{(\delta,2\delta)} dh \int_{\{t\}\times J_{R,h}} |u_x| \le \delta^{-1}\int_{(\delta,2\delta)} dh \int_{\{s\}\times J_{R,h}} |u_x| + \delta^{-1}\int_{(\delta,2\delta)} dh \int_{(s,t)\times \partial J_{R,h}} |u_t|$$

and then

$$\int_{\{t\}\times J_{R,2\delta}} |u_x| \le \int_{\{s\}\times J_{R,\delta}} |u_x| + \delta^{-1}\left\{\int_{(s,t)\times(\delta,2\delta)} |u_t| + \int_{(s,t)\times(R-2\delta,R-\delta)} |u_t|\right\}$$

which is (18.24). This ends the proof of the lemma.

18.4 Removing regularization of the fluxes

Proposition 18.6 (Existence in the viscous regime for truncated junctions)
Assume (2.2) with $N \ge 1$ with junction J of type $0 : N$ and bounded box $[a,b] \subset \mathbb{R}^N$. Let $R > 0$ and $\sigma = (-1,\dots,-1) = -\sigma_R \in \mathbb{R}^N$ and let us consider functions for all $j = 1,\dots,N$ and all $p \in [a,b]$

$$(18.27)\qquad \begin{cases} \hat f, \hat f_R : [a,b] \to \mathbb{R}^N, \\ f, \hat f, \hat f_R \in W^{1,\infty}([a,b];\mathbb{R}^N) \\ f = (f^1,\dots,f^N) \quad \text{with} \quad f^j : [a^j,b^j] \to \mathbb{R} \\ \sigma^j \cdot (\hat f^j(p))_{|p^j=a^j} \le \sigma^j \cdot f^j(a^j) \\ \sigma^j_R \cdot (\hat f^j_R(p))_{|p^j=a^j} \le \sigma^j_R \cdot f^j(a^j) \\ \sigma^j \cdot (\hat f^j(p))_{|p^j=b^j} \ge \sigma^j \cdot f^j(b^j) \\ \sigma^j_R \cdot (\hat f^j_R(p))_{|p^j=b^j} \ge \sigma^j_R \cdot f^j(b^j) \end{cases}$$

We assume moreover that $\hat f$ and $\hat f_R$ satisfy for all $j = 1,\dots,N$ and a.e. on $[a,b]$

$$(18.28)\qquad \begin{cases} -\partial_j \hat f^j \ge \sum_{k\in\{1,\dots,N\}\setminus\{j\}} |\partial_j \hat f^k| \\ \partial_j \hat f^j_R \ge \sum_{k\in\{1,\dots,N\}\setminus\{j\}} |\partial_j \hat f^k_R| \end{cases}$$

Assume that the initial data $u_0 = (u_0^1,\dots,u_0^N)$ satisfies

$$(18.29)\qquad u_0^j \in C^\infty(\bar J^j_R;[a^j,b^j]) \quad \text{with} \quad J^j_R \simeq (0,R).$$

Let $\varepsilon > 0$.
i) (Existence)
Then there exists $u^\varepsilon = (u^{\varepsilon,j})_{j=1,\dots,N}$ with

$$(18.30)\qquad u^{\varepsilon,j} : [0,+\infty) \times \bar J^j_R \to [a^j,b^j]$$

solution of

$$(18.31)\qquad \begin{cases} u^{\varepsilon,j}_t + (f^j(u^{\varepsilon,j}))_x = \varepsilon u^{\varepsilon,j}_{xx} & \text{on} \quad (0,+\infty)\times J^j_R \\ f^j(u^{\varepsilon,j}) - \varepsilon u^{\varepsilon,j}_x = \hat f^j(u^\varepsilon) & \text{on} \quad (0,+\infty)\times\{0\} \\ f^j(u^{\varepsilon,j}) - \varepsilon u^{\varepsilon,j}_x = \hat f^j_R(u^\varepsilon) & \text{on} \quad (0,+\infty)\times\{R\} \end{cases}$$

where the boundary conditions are satisfied in a weak sense (i.e. against test functions) and with initial condition

$$(18.32)\qquad u^{\varepsilon,j} = u_0^j \quad \text{on} \quad \{0\}\times \bar J^j_R$$

In particular we have $u^{\varepsilon,j} \in C^{\frac{\alpha}{2},\alpha}_{t,x}([0,+\infty) \times \bar{J}^j_R)$ *for all* $\alpha \in (0,1)$.

ii) (Additional bounds)

Assume moreover that the initial data u_0 *satisfies the following compatibility conditions for all* $j = 1,\dots,N$

$$\left\{ \begin{array}{ll} f^j(u^j_0) - \varepsilon(u^j_0)_x = \hat{f}^j(u_0) & \text{for} \quad x = 0 \\ f^j(u^j_0) - \varepsilon(u^j_0)_x = \hat{f}^j_R(u_0) & \text{for} \quad x = R. \end{array} \right. \tag{18.33}$$

and set

$$\left\{ \begin{array}{l} u^\varepsilon_t := \varepsilon(u_0)_{xx} - (f(u_0))_x \\ u^\varepsilon_x := (u_0)_x \end{array} \right. \Bigg| \quad \text{on} \quad \{0\} \times J^*_R \quad \text{with} \quad J^*_R := J_R \setminus \{0\} \tag{18.34}$$

Then $u^\varepsilon_t \in L^\infty((0,t);\mathcal{M}(J^*_R))$ *and we have the following bounds for a.e.* $t > 0$ *in the sense of measures*

$$|u^\varepsilon_t|(\{t\} \times J^*_R) \le |u^\varepsilon_t|(\{0\} \times J^*_R) \le \{\varepsilon|(u_0)_{xx}| + |(f(u_0))_x|\}(\{0\} \times J^*_R) \tag{18.35}$$

and for all $\delta \in (0, R/4)$

(18.36)

$$|u^\varepsilon_x|(\{t\} \times (2\delta, R-2\delta)) \le |u^\varepsilon_x|(\{0\} \times (\delta, R-\delta)) + \delta^{-1}\{|u^\varepsilon_t|((0,t) \times (\delta, 2\delta)) + |u^\varepsilon_t|((0,t) \times (R-2\delta, R-\delta))\}$$

Moreover for $N = 1$, *inequality (18.35) can be improved in (still in the sense of measures for a.e.* $t > 0$*)*

$$|u^\varepsilon_t|(\{t\} \times J^*_R) + TV(\hat{f}(u^\varepsilon);(0,t) \times \{0\}) + TV(\hat{f}_R(u^\varepsilon);(0,t) \times \{R\}) \le |u^\varepsilon_t|(\{0\} \times J^*_R) \tag{18.37}$$

where $TV(\hat{f}(u^\varepsilon);(0,t) \times \{0\})$ *is the time Total Variation of the trace of* $\hat{f}(u)$ *for* $x = 0$ *on the time interval* $(0,t)$.

Proof of Proposition 18.6

We first notice that (18.28) implies on $[a,b]$ and for all j, the map $p^j \mapsto \hat{f}^j(p)$ is nonincreasing, while the map $p^j \mapsto \hat{f}^j_R(p)$ is nondecreasing.

Step 1: Extension of the fluxes to the whole space

Let us simply extend $f^j : [a^j, b^j] \to \mathbb{R}$ by continuity to a function still denoted by $f^j : \mathbb{R} \to \mathbb{R}$, with constant value $f^j(a^j)$ on the left, and constant value $f^j(b^j)$ on the right of the interval $[a^j, b^j]$. This automatically extends to a bounded and Lipschitz continuous function $f : \mathbb{R}^N \to \mathbb{R}^N$ (with our standard abuse of notation). We now want to extend the bounded and Lipschitz vectorial functions $\hat{f}, \hat{f}_R : [a,b] \to \mathbb{R}^N$ to the whole $\mathbb{R}^N$. We only do it for $\hat{f}$ (this is similar for $\hat{f}_R$). We first notice that the first line of (18.28) for j means

$$-\partial_j \hat{f}^j \ge \sum_{k \in \{1,\dots,N\}\setminus\{j\}} \omega_{jk} \partial_j \hat{f}^k \quad \text{for all} \quad \omega_{jk} \in \{\pm 1\} \tag{18.38}$$

which is equivalent to the fact that the map

$$p^j \mapsto h^j_\omega(p) := \hat{f}^j(p) + \sum_{k \in \{1,\dots,N\}\setminus\{j\}} \omega_{jk} \hat{f}^k(p) \quad \text{is nonincreasing.}$$

Step 1.1: extension along p^1

We now first extend $\hat{f}$ along the first coordinate p^1 as follows for $p' = (p^2,\dots,p^N) \in \Pi'_2 := \prod_{j=2,\dots,N}[a^j, b^j]$

$$\hat{f}(p^1, p') := \left\{ \begin{array}{ll} \hat{f}(a^1, p') & \text{for} \quad p^1 < a^1 \\ \hat{f}(p^1, p') & \text{for} \quad p^1 \in [a^1, b^1] \\ \hat{f}(b^1, p') & \text{for} \quad p^1 > b^1 \end{array} \right.$$

Notice in particular that $p^1 \mapsto h^1_\omega(p)$ is also nonincreasing on $\Pi_1 := \mathbb{R} \times \Pi'_2$, because $\hat{f}$ is constant in p^1 on $\{p^1 < a^1\}$ and also on $\{p^1 > b^1\}$. For $j \neq 1$, the maps $p^j \mapsto h^j_\omega(p)$ are also nonincreasing on Π_1, because this is already the case for any fixed $p^1 \in [a^1, b^1]$, and in particular for $p^1 \in \partial[a^1, b^1]$. Moreover, for all j, the maps $\hat{f}^j$ are also bounded and Lipschitz continuous on Π_1.

Step 1.2: extension along p^2

Now for $p^1 \in \mathbb{R}$ and $p'' := (p^3, \dots, p^N) \in \Pi_3' := \prod_{j=3,\dots,N}[a^j, b^j]$, we set

$$\hat{f}(p^1,p^2,p'') := \begin{cases} \hat{f}(p^1,a^2,p'') & \text{for} \quad p^2 < a^2 \\ \hat{f}(p^1,p^2,p'') & \text{for} \quad p^2 \in [a^2,b^2] \\ \hat{f}(p^1,b^2,p'') & \text{for} \quad p^2 > b^2 \end{cases}$$

which extends $\hat{f}$ to $\Pi_2 := \mathbb{R}^2 \times \Pi_3'$. Exactly the same reasoning as in Step 1.1 applies and shows that $p^j \mapsto h^j_\omega(p)$ is nonincreasing on Π_2 for all j, and that $\hat{f}$ is bounded and Lipschitz continuous on Π_2.

Step 1.3: induction and conclusion

We do the proof by induction on j, and finally get that $p^j \mapsto h^j_\omega(p)$ is nonincreasing on $\Pi_N := \mathbb{R}^N$ for all ω and for all j. For all indices j, this shows that $-\partial_j \hat{f}^j \geq \sum_{k\in\{1,\dots,N\}\setminus\{j\}} |\partial_j \hat{f}^k|$ a.e. on $\mathbb{R}^N$. This implies in particular that $p^j \mapsto \hat{f}^j(p)$ is nonincreasing for all j. Moreover the map $\hat{f}$ is bounded and Lipschitz continuous over $\mathbb{R}^N$. We conclude that the extensions $f, \hat{f}, \hat{f}_R : \mathbb{R}^N \to \mathbb{R}^N$ are bounded and globally Lipschitz continuous. Moreover $\hat{f}, \hat{f}_R$ satisfy (18.28) a.e. on $\mathbb{R}^N$, and

$$\begin{cases} \mathrm{Lip}(f^j;\mathbb{R}) & = \mathrm{Lip}(f^j;[a^j,b^j]) \\ \mathrm{Lip}(\hat{f};\mathbb{R}^N) & = \mathrm{Lip}(\hat{f};[a,b]) \\ \mathrm{Lip}(\hat{f}_R;\mathbb{R}^N) & = \mathrm{Lip}(\hat{f}_R;[a,b]) \end{cases}$$

Step 2: Regularizations

Step 2.1: Regularization of the fluxes

We now consider some nonnegative function $\rho \in C^\infty_c(\mathbb{R})$ with $\int_\mathbb{R} \rho = 1$. For some $\eta \in (0,1]$ and for $p = (p^1,\dots,p^N)$, we set $\bar{\rho}_\eta(p) := \varepsilon^{-N}\bar{\rho}(\eta^{-1}p)$ with the product $\bar{\rho}(p) := \rho(p^1)\dots\rho(p^N)$. We set the mollifications $\hat{g}_\eta := \hat{f} \star_{\mathbb{R}^N} \bar{\rho}_\eta$ and $\hat{g}_{R,\eta} := \hat{f}_R \star_{\mathbb{R}^N} \bar{\rho}_\eta$. Applying mollification in (18.38), we get

$$-\partial_j \hat{g}^j_\eta \geq \sum_{k\in\{1,\dots,N\}\setminus\{j\}} \omega_{jk}\partial_j \hat{g}^k_\eta \quad \text{for all} \quad \omega_{jk} \in \{\pm 1\}$$

which shows that $-\partial_j \hat{g}^j_\eta \geq \sum_{k\in\{1,\dots,N\}\setminus\{j\}} |\partial_j \hat{g}^k_\eta|$ on $\mathbb{R}^N$. This implies that $\hat{g}_\eta, \hat{g}_{R,\eta}$ satisfy (18.28).

We now set the regularized fluxes $\hat{f}_\eta(p) := \hat{g}_\eta(p) - \mu\{p-m\}$ and $\hat{f}_{R,\eta}(p) := \hat{g}_\eta(p) + \mu\{p-m\}$ with $m := \dfrac{a+b}{2}$ and $\mu := \sqrt{\eta}$. Then $\hat{f}_\eta, \hat{f}_{R,\eta}$ still satisfy (18.28). They satisfy moreover the four last lines of (18.27) for our choice of μ and for $\eta > 0$ small enough. We also consider $f_\eta := f \star_{\mathbb{R}^N} \bar{\rho}_\eta$, which satisfies $f^j_\eta = f^j \star_\mathbb{R} \rho_\eta$ with $\rho_\eta(x) := \eta^{-1}\rho(\eta^{-1}x)$. Moreover we have

$$\begin{cases} \mathrm{Lip}(f^j_\eta;\mathbb{R}) & \leq \mathrm{Lip}(f^j;[a^j,b^j]) \\ \mathrm{Lip}(\hat{f}_\eta;\mathbb{R}^N) & \leq \mathrm{Lip}(\hat{f};[a,b]) \\ \mathrm{Lip}(\hat{f}_{R,\eta};\mathbb{R}^N) & \leq \mathrm{Lip}(\hat{f}_R;[a,b]) \end{cases}$$

Step 2.2: contraction-regularization of the initial data

For $\nu \in (0,1)$, we contract u_0, defining $u_{0,\nu} := m + (1-\nu)(u_0 - m)$ with $m := \frac{a+b}{2}$. Let $0 < \beta << \nu$. Then for $\nu > 0$ small enough, we have $u_{0,\nu}(x) \in [a,b]_{-\beta} := \prod_{j=1,\dots,N} [a^j + \beta, b^j - \beta]$ for all $x \in [0,R]$. Then we extend $u_{0,\nu}$ outside the interval $[0,R]$ in a C^∞ function such that $u_{0,\nu}(x) \in [a,b]_{-\beta/2}$ for all $x \in \mathbb{R}$, and such that $u_{0,\nu}$ takes a constant value $u_{0,\nu}(-1)$ for $x < -1$ and the constant value $u_{0,\nu}(R+1)$ for $x > R+1$. Now, consider the mollification $v_{0,\nu,\eta} := u_{0,\nu} \star_\mathbb{R} \rho_\eta$. It satisfies

$$v_{0,\nu,\eta}(x) \subset [a,b]_{-\beta/2} \quad \text{for all} \quad x \in \mathbb{R}. \tag{18.39}$$

We want our approximation to satisfy compatibility conditions (18.33). To this end, we now consider a function $\psi \in C^\infty_c([0,R_*))$ with $R_* := \min\{1,R\}$ and $\eta \in (0,1]$ such that $\psi_\eta(x) = \eta\psi(\eta^{-1}x)$, $\psi(0) = 0$ and $\psi'(0) = 1$. Setting $\lambda := \varepsilon^{-1}\left\{f_\eta - \hat{f}_\eta\right\}(v_{0,\nu,\eta}(0)) - v'_{0,\nu,\eta}(0)$ and $\lambda_R := \varepsilon^{-1}\left\{f_\eta - \hat{f}_{R,\eta}\right\}(v_{0,\nu,\eta}(R)) - v'_{0,\nu,\eta}(R)$, we see that the function

$$w_0(x) := v_{0,\nu,\eta}(x) + \lambda\psi_\eta(x) - \lambda_R\psi_\eta(R-x) \quad \text{for} \quad x \in [0,R]$$

satisfies

$$\begin{cases} f^j_\eta(w^j_0) - \varepsilon(w^j_0)_x = \hat f^j_\eta(w_0) & \text{for} \quad x = 0 \\ f^j_\eta(w^j_0) - \varepsilon(w^j_0)_x = \hat f^j_{R,\eta}(w_0) & \text{for} \quad x = R. \end{cases}$$

From (18.39), we deduce that for η small enough, we have $w_0(x) \subset [a,b]_{-\beta/4}$ for all $x \in \mathbb{R}$. Hence $f_\eta, \hat f_\eta, \hat f_{R,\eta} \in W^{2,\infty}_{loc}(\mathbb{R}^N, \mathbb{R}^N)$ satisfy (18.27)-(18.28) and w_0 satisfies (18.29), (18.33).

Step 3: A priori estimates

Let us call $U = u^\varepsilon_{\nu,\eta}$ the solution given by Lemma 18.2 for fluxes $(f_\eta, \hat f_\eta, \hat f_{R,\eta})$ and initial data $u_{0,\nu,\eta} := w_0$. We have in particular $U^j \in C^{\frac{2+\alpha}{2}, 2+\alpha}_{t,x}([0,+\infty) \times \bar J^j_R)$ for all $\alpha \in (0,1)$. From (18.22) and (18.24) , we get (18.40)

$$\begin{cases} U^j(t,x) \in [a^j, b^j] \quad \text{for all} \quad (t,x) \in [0,+\infty) \times \bar J^j_R \\ \\ \displaystyle\int_{\{t\}\times J_R} |U_t| \le \int_{\{0\}\times J_R} |U_t| = \int_{\{0\}\times J_R} |\varepsilon U_{xx} - (f_\eta(U))_x| \\ \\ \displaystyle\int_{\{t\}\times(2\delta, R-2\delta)} |U_x| \le \int_{\{0\}\times(\delta,R-\delta)} |U_x| + \delta^{-1}\left\{\int_{(0,t)\times(\delta,2\delta)} |U_t| + \int_{(0,t)\times(R-2\delta,R-\delta)} |U_t|\right\} \quad \text{for all} \quad \delta \in (0, R/4) \end{cases}$$

where

$$\left.\begin{cases} U_t(0,\cdot) = \varepsilon(w_0)_{xx} - (f_\eta(w_0))_x \\ U_x(0,\cdot) = (w_0)_x \end{cases}\right| \quad \text{with} \quad w_0 = u_{0,\nu,\eta}$$

For $N = 1$, we have moreover from (18.23)

$$\int_{\{t\}\times J_R} |U_t| + \int_{(0,t)\times\{0\}} |(\hat f_\eta(U))_t| + \int_{(0,t)\times\{R\}} |(\hat f_{\eta,R}(U))_t| \le \int_{\{0\}\times J_R} |U_t|$$

i.e. in the sense of measures

$$|U_t|(\{t\} \times J_R) + TV(\hat f_\eta(U); (0,t)\times\{0\}) + TV(\hat f_{\eta,R}(U); (0,t)\times\{R\}) \le |U_t|(\{0\}\times J_R) \tag{18.41}$$

Step 4: Removing regularization of the fluxes

Step 4.1: The initial value

We make a discussion for $U_t(0,\cdot)$ (the discussion is similar and simpler for $U_x(0,\cdot)$, and we skip it). Notice that we have $U_t(0,x) = A(x) + B(x)$ with $A(x) := \varepsilon(v_{0,\nu,\eta})_{xx} - (f_\eta(v_{0,\nu,\eta}))_x$ and $B(x) := U_t(0,x) - A(x)$ where B contains in particular boundary terms created by $\lambda\psi_\eta$ and $\lambda_R\psi_\eta(R-\cdot)$. Moreover, we have

$$\limsup_{(\nu,\eta)\to(0,0)} |A^j(x)| \le \varepsilon|(u^j_0)_{xx}(x)| + \mathrm{Lip}(f^j; [a,b]) \cdot |(u^j_0)_x(x)| \quad \text{for all} \quad x \in (0,R)$$

and

$$|A^j|_{L^\infty(0,R)} \le 1 + \varepsilon|(u^j_0)_{xx}|_{L^\infty(0,R)} + \mathrm{Lip}(f^j;[a,b]) \cdot |(u^j_0)_x|_{L^\infty(0,R)} \qquad \text{for } (\nu,\eta) \text{ close enough to } (0,0)$$

The limit contribution to B only arises through the functions $\psi_\eta(x)$ and $\psi_\eta(R-x)$. Because $|\psi''_\eta|_{L^1(0,R)} \le C$ and $|\mathrm{supp}(\psi_\eta)| \to 0$ and $\lambda, \lambda_R \to 0$ as $(\nu,\eta) \to (0,0)$, we deduce that $|B|_{L^1(0,R)} \to 0$ as $(\nu,\eta) \to (0,0)$.

Step 4.2: Positive times and conclusion

Setting $K[u_0] := \displaystyle\sum_{j=1,\dots,N} \left\{\varepsilon|(u^j_0)_{xx}|_{L^1(0,R)} + \mathrm{Lip}(f^j;[a,b]) \cdot |(u^j_0)_x|_{L^1(0,R)}\right\}$ and $K'[u_0] = 1 + K[u_0]$, and for (ν,η) close enough to $(0,0)$, from Step 3 we get

$$\begin{cases} U^j(t,x) \in [a^j, b^j] \quad \text{for all} \quad (t,x) \in [0,+\infty) \times \bar J^j_R \\ \\ \displaystyle\int_{\{t\}\times J_R} |U_t| \le K'[u_0] \\ \\ \displaystyle\int_{\{t\}\times(2\delta,R-2\delta)} |U_x| \le \int_{\{0\}\times(\delta,R-\delta)} |(u_0)_x| + \delta^{-1}K'[u_0] \quad \text{for all} \quad \delta \in (0,R/4) \end{cases} \tag{18.42}$$

We conclude from the compactness of the inclusion $BV_{loc} \subset L^1_{loc}$, that up to extract a subsequence (and using a diagonal extraction argument to cover $[0,+\infty) \times \bar{J}_R$), that we have

$$U = u^{\varepsilon}_{\nu,\eta} \to u^{\varepsilon} \quad \text{in} \quad L^1_{loc}([0,+\infty) \times \bar{J}_R) \quad \text{as} \quad (\nu,\eta) \to (0,0)$$

From (18.40), we deduce in the sense of measures that

$$\begin{cases} u^{\varepsilon,j}(t,x) \in [a^j, b^j] \quad \text{for all} \quad (t,x) \in [0,+\infty) \times J^j_R \\ \\ |u^{\varepsilon}_t|(\{t\} \times J^*_R) \le |u^{\varepsilon}_t|(\{0\} \times J^*_R) \le \{\varepsilon|(u_0)_{xx}| + |(f(u_0))_x|\}(\{0\} \times J^*_R) \\ \\ |u^{\varepsilon}_x|(\{t\} \times (2\delta, R-2\delta)) \\ \le |u^{\varepsilon}_x|(\{0\} \times (\delta, R-\delta)) + \delta^{-1}\{|u^{\varepsilon}_t|((0,t) \times (\delta,2\delta)) + |u^{\varepsilon}_t|((0,t) \times (R-2\delta, R-\delta))\} \quad \text{for all} \quad \delta \in (0,R/4) \end{cases}$$

where u^{ε} satisfies (18.34). Moreover we can pass to the limit in the PDE (and its weak formulation). Therefore u^{ε} solves (18.31) with initial data as in (18.32) with u_0 solving (18.33). Finally Lemma 18.2 shows that $u^{\varepsilon,j} \in C^{\frac{\alpha}{2},\alpha}_{t,x}([0,+\infty) \times \bar{J}^j_R)$ for all $\alpha \in (0,1)$.
Moreover for $N = 1$, we get from (18.41) that

(18.43) $$|u^{\varepsilon}_t|(\{t\} \times J^*_R) + TV(\hat{f}(u^{\varepsilon}); (0,t) \times \{0\}) + TV(\hat{f}_R(u^{\varepsilon}); (0,t) \times \{R\}) \le |u^{\varepsilon}_t|(\{0\} \times J^*_R)$$

This ends the proof of the proposition.

18.5 Removing the viscosity

For $w = (w^1, \dots, w^N)$, where each w^j denotes a finite measure on J^j_R, the associated norm is defined as $|w|_{\mathcal{M}(J^*_R)} := \sum_{j=1,\dots,N} |w^j|_{\mathcal{M}(J^j_R)}$, where $\mathcal{M}(J^j_R)$ is the set of (real valued) measures on the open interval $J^j_R \simeq (0,R)$ and $|\cdot|_{\mathcal{M}(J^j_R)}$ is the total variation of the measure.

Proposition 18.7 (Existence for truncated junctions)
Assume (2.2) with $N \ge 1$, nondegeneracy condition (2.17), for a junction J of type $0:N$ and bounded box $[a,b] \subset \mathbb{R}^N$. Let $R > 0$, $J_R := J \cap B_R$ with $J^j_R \simeq (0,R)$ for all j and $\sigma = (-1,\dots,-1) = -\sigma_R \in \mathbb{R}^N$. Assume that $f, \hat{f}, \hat{f}_R$ satisfy (18.27) and dissipative conditions (18.28). Then define the relaxations of the fluxes (given by Proposition 7.1)

$$\hat{f}_1 := \mathfrak{R}\hat{f} \quad \text{and} \quad \hat{f}_{R,1} := \mathfrak{R}\hat{f}_R$$

Then $\mathcal{G} := \left\{p \in [a,b], \quad \hat{f}_1(p) = f(p)\right\}$ and $\mathcal{G}_R := \left\{p \in [a,b], \quad \hat{f}_{R,1}(p) = f(p)\right\}$ are two Kružkov germs respectively at the junction point $x = 0$, and at the junction point $x = R$.
Assume that the initial data $u_0 = (u^1_0, \dots, u^N_0)$ is of bounded variations, i.e. satisfies $u^j_0 \in BV(J^j_R; [a^j, b^j])$.
i) (Existence and uniqueness)
Then there exists a unique $u = (u^j)_{j=1,\dots,N}$ with

(18.44) $$u^j : [0,+\infty) \times J^j_R \to [a^j, b^j]$$

entropy solution of

(18.45) $$\begin{cases} u^j_t + (f^j(u^j))_x & = 0 & \text{on} & (0,+\infty) \times J^j_R \\ f^j(u^j) & = \hat{f}^j_1(u) & a.e.\ \text{on} & (0,+\infty) \times \{0\} \\ f^j(u^j) & = \hat{f}^j_{R,1}(u) & a.e.\ \text{on} & (0,+\infty) \times \{R\} \end{cases}$$

where the boundary conditions are satisfied in the sense of strong traces and with initial condition

(18.46) $$u^j = u^j_0 \quad \text{on} \quad \{0\} \times J^j_R$$

This solution is constructed as a **vanishing viscosity limit of (18.31) with boundary fluxes $\hat{f}$ and $\hat{f}_R$.**

Moreover we have

$$u \in Lip([0,+\infty); \mathcal{M}(J_R^*))$$

where $\mathcal{M}(J_R^*)$ *denotes the set of measures on* $J_R^* := J_R \backslash \{0\}$.

ii) (Additional bounds)

Then we have the following bounds for all $t > 0$

(18.47)

$$TV(f(u)(t,\cdot); J_R^*) \le K_0 \quad \textit{with} \quad K_0 := \sum_{j=1,\dots,N} \left\{ TV(f^j(u_0^j); J_R^j) + |(f^j - \hat{f}^j)(u_0(0))| + |(f^j - \hat{f}_R^j)(u_0(R))| \right\}$$

and we have

$$TV(u(t,\cdot); J_{R-2\delta}\backslash \bar{B}_{2\delta})) \le TV(u_0; J_{R-\delta}\backslash \bar{B}_\delta) + \delta^{-1} t K_0 \quad \textit{for all} \quad \delta \in (0, R/4) \tag{18.48}$$

Moreover for $N = 1$, *we have for all* $t > 0$

$$TV(f(u)(t,\cdot); J_R^*) + TV(f(u); (0,t)\times\{0\}) + TV(f(u); (0,t)\times\{R\}) \le K_0 \tag{18.49}$$

Proof of Proposition 18.7

First notice that dissipative conditions (18.28) imply that $-\hat{f}$ and $\hat{f}_R$ are both Riemann monotone. From Proposition 7.4, we then deduce that both $\mathcal{G}$ and $\mathcal{G}_R$ are Kružkov germs.

Now consider the solution u^ε given in Proposition 18.6 for the fluxes $f, \hat{f}, \hat{f}_R$ and some initial data u_0^ε which approximates the function u_0 and which is described below.

Step 1: definition of the smooth initial data u_0^ε

Precisely, because $u_0 \in BV(J_R^*)$, we deduce that u_0^j has a two limits $u_0^j(0)$ and $u_0^j(R)$. We then extend u_0^j by the value $u_0^j(0)$ for $x \le 0$ and by the value $u_0^j(R)$ for $x \ge R$. We get the BV semi-norms $[u_0^j]_{BV(\mathbb{R})} = [u_0^j]_{BV(J_R^j)} := |(u_0^j)_x|_{\mathcal{M}(J_R^j)}$. For $\nu \in (0,1)$, we then contract u_0, defining $u_{0,\nu} := m + (1-\nu)(u_0 - m)$ with $m := \frac{a+b}{2}$. Let

$$0 < \beta << \nu \tag{18.50}$$

Then for $\nu > 0$ small, we have $u_{0,\nu}(x) \in [a,b]_{-\beta} := \prod_{j=1,\dots,N} [a^j + \beta, b^j - \beta]$ for all $x \in \mathbb{R}$. For $\eta, \mu > 0$, using the notation $\rho_\eta = \eta^{-1}\rho(\eta^{-1}\cdot)$ and $\psi_\mu = \mu\psi(\mu^{-1}\cdot)$ of Step 2 of the proof of Proposition 18.6, we set $v_{0,\nu,\eta} := u_{0,\nu} \star_{\mathbb{R}} \rho_\eta : \mathbb{R} \to [a,b]_{-\beta}$. Setting $\lambda := \varepsilon^{-1}\left\{f - \hat{f}\right\}(v_{0,\nu,\eta}(0)) - v'_{0,\nu,\eta}(0)$ and $\lambda_R := \varepsilon^{-1}\left\{f - \hat{f}_R\right\}(v_{0,\nu,\eta}(R)) - v'_{0,\nu,\eta}(R)$, we see for $\psi \in C_c^\infty([0,R_*))$, with $\psi_\mu(x) = \mu\psi(\mu^{-1}x)$, $\psi(0) = 0$ and $\psi'(0) = 1$, that the following function

$$u_0^\varepsilon(x) := v_{0,\nu,\eta}(x) + \lambda\psi_\mu(x) - \lambda_R\psi_\mu(R-x) \quad \text{for} \quad x \in [0,R]$$

satisfies

$$\left\{ \begin{array}{ll} f(u_0^\varepsilon) - \varepsilon(u_0^\varepsilon)_x = \hat{f}(u_0^\varepsilon) & \text{for} \quad x = 0 \\ f(u_0^\varepsilon) - \varepsilon(u_0^\varepsilon)_x = \hat{f}_R(u_0^\varepsilon) & \text{for} \quad x = R. \end{array} \right.$$

Notice that for

$$\mu << \eta << \varepsilon \tag{18.51}$$

we have in particular $u_{0,\nu,\eta,\mu}(x) \in [a,b]_{-\beta/2}$ for all $x \in [0,R]$.

Step 2: rought a priori bounds

Using $(v_{0,\nu,\eta})_{xx} = (u_{0,\nu})_x \star_{\mathbb{R}} (\rho_\eta)_x$ and (18.50)-(18.51), we get

$$\varepsilon|(v_{0,\nu,\eta}^j)_{xx}|_{L^1(J_R^j)} \le \varepsilon|(u_0^j)_x|_{\mathcal{M}(\mathbb{R})}|(\rho_\eta)_x|_{L^1(\mathbb{R})} = \varepsilon\eta^{-1}[u_0^j]_{BV(J_R^j)}|\rho_x|_{L^1(\mathbb{R})} \to 0 \quad \text{as} \quad \varepsilon \to 0$$

and

$$\varepsilon|(\lambda^j\psi_\mu)_{xx}|_{L^1(J_R^j)} = \varepsilon|\lambda^j||\psi''|_{L^1(0,+\infty)} \to |(f^j - \hat{f}^j)(u_0(0))| \cdot |\psi''|_{L^1(0,+\infty)} \quad \text{as} \quad \varepsilon \to 0$$

and similarly

$$|(v_{0,\nu,\eta}^j)_x|_{L^1(J_R^j)} \le |(u_0^j)_x|_{\mathcal{M}(\mathbb{R})}|\rho_\eta|_{L^1(\mathbb{R})} = [u_0^j]_{BV(J_R^j)}$$

and

$$|(\lambda^j\psi_\mu)_x|_{L^1(J^j_R)} = |\lambda^j\mu|\cdot|\psi'|_{L^1(0,+\infty)} \to 0 \quad \text{as} \quad \varepsilon\to 0$$

Notice that $\psi'(0)=1$ and $\psi\in C^\infty_c([0,R_*))$. Therefore $|\psi''|_{L^1(0,+\infty)} = [\psi']_{BV(0,+\infty)} \geq |\psi'(0)| = 1$, and we can choose ψ such that we have equality in the inequality. Hence

$$(18.52)\qquad \left\{\begin{array}{l} \varepsilon|(u_0^{\varepsilon,j})_{xx}|_{L^1(J^j_R)} \to \left\{|(f^j-\hat f^j)(u_0(0))| + |(f^j-\hat f^j_R)(u_0(R))|\right\}\\ |(f^j(u_0^{\varepsilon,j}))_x|_{L^1(J^j_R)} \to |(f^j(u_0^j))_x|_{L^1(J^j_R)} = TV(f^j(u^j_0);\{0\}\times(J^j_R))\end{array}\right| \quad \text{as} \quad \varepsilon\to 0$$

Setting

$$\left.\begin{array}{rl} K_0 & := \displaystyle\sum_{j=1,\dots,N}\left\{|(f^j-\hat f^j)(u_0(0))| + |(f^j-\hat f^j_R)(u_0(R))| + TV(f^j(u^j_0);J^j_R)\right\}\\ & \leq \displaystyle\sum_{j=1,\dots,N}\left\{|(f^j-\hat f^j)(u_0(0))| + |(f^j-\hat f^j_R)(u_0(R))| + \mathrm{Lip}(f^j;[a^j,b^j])\cdot TV(u^j_0;J^j_R)\right\}\end{array}\right| \quad K_0' = 1+K_0$$

and for ε close enough to 0, we deduce from (18.42) that for a.e. time $t>0$ we have

$$\left\{\begin{array}{l} u^{\varepsilon,j}(t,x)\in[a^j,b^j] \quad \text{for all} \quad (t,x)\in[0,+\infty)\times J^j_R\\ \\ |u^\varepsilon_t(t,\cdot)|(J^*_R)\leq K_0'\\ \\ |u^\varepsilon_x(t,\cdot)|(J_{R-2\delta}\backslash\bar B_{2\delta})) \leq TV(u_0;J_{R-\delta}\backslash\bar B_\delta) + \delta^{-1}K_0' \quad \text{for all} \quad \delta\in(0,R/4)\end{array}\right.$$

From compactness of $BV_{loc}\subset L^1_{loc}$, we get that, up to extract a subsequence, we have

$$\left\{\begin{array}{l} u^\varepsilon\to u \quad \text{in} \quad L^1_{loc}([0,+\infty)\times J_R)\\ \\ u_0^\varepsilon \to u_0 \quad \text{in} \quad L^1(J_R)\end{array}\right| \quad \text{as} \quad (\varepsilon,\nu)\to(0,0)$$

Step 3: test function formulations

Recall that (18.15) holds true for two solutions $u^\varepsilon, v^\varepsilon$ and fluxes satisfying assumption (18.5).

Step 3.1: interior formulations

In the special case where the test function satisfies $\varphi=(0,\dots,0,\varphi^j,0,\dots,0)$ with $0\leq\varphi^j\in C^\infty_c([0,+\infty)\times J^j_R)$, relation (18.15) still holds true if the solutions are only solutions on $(0,+\infty)\times J^j_R$. In particular, we can choose $v^\varepsilon = c := (0,\dots,0,c^j,0,\dots,0)$ for any $c^j\in[a^j,b^j]$. This gives

$$0\leq \varepsilon C_{R,\varphi} + \int_{\{0\}\times J^j_R}|u_0^{\varepsilon,j}-c^j|\varphi^j + \int_{(0,+\infty)\times J^j_R}\left\{|u^\varepsilon-c^j|\varphi^j_t + \psi^{f^j}(u^\varepsilon,c)\varphi^j_x\right\}$$

In the limit $\varepsilon\to 0$, we recover for all $c^j\in[a^j,b^j]$

$$0\leq \int_{\{0\}\times J^j_R}|u_0^j-c^j|\varphi^j + \int_{(0,+\infty)\times J^j_R}\left\{|u-c^j|\varphi^j_t + \psi^{f^j}(u,c)\varphi^j_x\right\}$$

which means exactly that u^j is an entropy solution of (18.45) only on $(0,+\infty)\times J^j_R$ and with initial data u^j_0.

Step 3.2: boundary formulation at $x=0$

In the special case where the test function satisfies $\varphi=(\varphi^1,\dots,\varphi^N)$ with $0\leq\varphi^j\in C^\infty_c([0,+\infty)\times(J^j_R\cup\{0\}))$ with $\varphi^j(t,0)=\varphi^k(t,0)$ for all k,j and all times $t\geq 0$, then relation (18.15) still holds true if the solutions are only solutions on $(0,+\infty)\times J_R$, where we recall that $0\in J_R$. Now let $\hat p\in\mathcal G$. From Lemma 7.3, we know that there exists $q\in[a,b]$ and $\hat r\in\mathcal G$ such that

$$(18.53)\qquad \left.\begin{array}{l} w\in BA(\hat p)\\ S:=\left\{j\in\{1,\dots,N\}, \quad w^j\neq\hat p^j\right\}\\ 0\leq \max\limits_{j\in S} D^{f^j}(w,\hat r)\end{array}\right\} \quad \text{implies} \quad S=\emptyset$$

and there exists $U: J^*\to\mathbb R$ with trace $U(0)\in\mathbb R^N$ such that

$$(18.54)\qquad U(0):=q \quad \text{and} \quad U' = f(U)-f(\hat r) \quad \text{on} \quad J^* \quad \text{with} \quad U(+\infty)=\hat r$$

In particular, we can check that the following function is an explicit solution

$$v^\varepsilon(t,x) := U\left(\frac{x}{\varepsilon}\right) \quad \text{with} \quad U(+\infty) = c := \hat{r} \in \mathcal{G}$$

and then

$$0 \le \int_{(0,+\infty)\times\{0\}} D^{\hat{f}}(u^\varepsilon, v^\varepsilon)\varphi \le \quad \varepsilon C_{R,\varphi} + \int_{\{0\}\times J_R} |u_0^\varepsilon - v^\varepsilon|\varphi + \int_{(0,+\infty)\times J_R} \left\{|u^\varepsilon - v^\varepsilon|\varphi_t + \psi^f(u^\varepsilon, v^\varepsilon)\varphi_x\right\}$$

In the limit $\varepsilon \to 0$, we recover $0 \le \int_{\{0\}\times J_R} |u_0 - c|\varphi + \int_{(0,+\infty)\times J_R} \left\{|u-c|\varphi_t + \psi^f(u,c)\varphi_x\right\}$. Now we can find a sequence of test functions approximating the functions $\varphi^j_\gamma(t,x) = \zeta(t)\max\left\{0, 1-\gamma^{-1}x\right\}$ for all j and $\gamma > 0$. Using nondegeneracy condition (2.17), we deduce from Theorem 2.58 that u has a strong trace at $x = 0$ for a.e. time $t > 0$. Therefore, in the limit $\gamma \to 0$ focusing on the junction point $x = 0$, we get

$$0 \le \int_{(0,+\infty)\times\{0\}} \left\{\sum_{j=1,\dots,N} -\psi^{f^j}(u,c)\right\}\zeta = \int_{(0,+\infty)\times\{0\}} D^f(u,c)\zeta$$

Because this is true for any function $0 \le \zeta \in C_c^\infty([0,+\infty))$, we deduce that for a.e. time $t > 0$, we have

$$0 \le D^f(u(t,0),c) = \sum_{j=1,\dots,N} D^{f^j}(u(t,0),c) \quad \text{for} \quad c := \hat{r}$$

This is in particular true for the choice of $\hat{p} \in \mathcal{G}$ such that $u(t_0,0) \in BA(\hat{p})$ where t_0 is a Lebesgue point of $u(\cdot,0)$. Therefore (18.53) implies for $t = t_0$

$$u(t_0,0) = \hat{p} \in \mathcal{G}$$

We deduce that

$$u(\cdot,0) \in \mathcal{G} \quad \text{a.e. on} \quad (0,+\infty)$$

This shows the second line of (18.45).

Step 3.3: boundary formulation at $x = R$

The proof is similar to Step 3.2, and shows the third line of (18.45).

Step 4: uniqueness

Finally the uniqueness of the solution u follows as usual, from the L^1-contraction as in i) of Lemma 17.3. We do not repeat the details here.

Step 5: fine a priori bounds

Using (18.52), and from the semi-continuity of BV norms/total variations of measures, we deduce from (18.35) that for a.e. $t > 0$

$$|u_t|(\{t\}\times J_R^*) \le K_0$$

Now from the PDE satisfied by u, we deduce that for a.e. $t > 0$

$$|u_t|(\{t\}\times J_R^*) = |(f(u))_x|(\{t\}\times J_R^*) = TV(f(u)(t,0); J_R^*)$$

Now from the continuity of the map $t \mapsto f(u)(t,\cdot) \in L^1(J_R)$ and the semi-continuity of the Total Variation, we deduce that for all $t > 0$ we have

$$TV(f(u)(t,0); J_R^*) \le K_0$$

which shows (18.47). Similarly for $N = 1$, we deduce (18.49) from (18.37) for all $t > 0$. Finally notice that (18.36) implies

$$\begin{aligned} |u_x^\varepsilon|(\{t\}\times(2\delta, R-2\delta)) &\le |u_x^\varepsilon|(\{0\}\times(\delta,R-\delta)) + \delta^{-1}|u_t^\varepsilon|((0,t)\times J_R^*) \\ &\le |u_x^\varepsilon|(\{0\}\times(\delta,R-\delta)) + \delta^{-1}t|u_t^\varepsilon|(\{0\}\times J_R^*) \end{aligned}$$

Therefore in the limit $\varepsilon \to 0$, we get for a.e. $t > 0$

$$\begin{aligned} |u_x|(\{t\}\times(2\delta, R-2\delta)) &\le |u_x|(\{0\}\times(\delta,R-\delta)) + \delta^{-1}tK_0 \\ &= TV(u_0; J_{R-\delta}\backslash\bar{B}_\delta) + \delta^{-1}tK_0 \end{aligned}$$

Finally this implies (18.48) for all $t > 0$. This ends the proof of the proposition.

18.6 Proof of Theorem 2.61: removing the truncations

Proposition 18.8 (Existence on the infinite junction)
Assume (2.2) with $N \geq 1$, nondegeneracy condition (2.17), and let $\mathcal{G} \subset [a,b]$ be a Kružkov germ. Consider an initial data u_0 with $u_0(J) \subset [a,b]$ such that $u_0 - p_0 \in (BV \cap L^1)(J)$ for some $p_0 \in [a,b] \cap \mathbb{R}^N$. Then there exists a unique $\mathcal{G}$-entropy solution u of (2.4) with initial data u_0.

Proof of Proposition 18.8 and of Theorem 2.61
Step 1: reduction to a bounded box
Because $u_0 \in BV(J)$, we deduce that u_0 is bounded. From Proposition 6.1, there exists a bounded box $[\bar{a}, \bar{b}] \subset [a,b]$ such that $u_0(J) \subset [\bar{a}, \bar{b}]$ and $\mathcal{G} \cap [\bar{a}, \bar{b}]$ is a Riemann germ on the bounded box $[\bar{a}, \bar{b}]$.
Step 2: construction of the germ $\mathcal{G}_R$
Let $R > 0$ and $J_R := J \cap B_R$ and $J_R^j := J^j \cap B_R$. We set

$$\hat{f}_R^j(u^j) := \begin{cases} G^{f^j}(u^j, p_0^j) & \text{if} \quad J_R^j \simeq (0,R) \\ G^{f^j}(p_0^j, u^j) & \text{if} \quad J^j \simeq (-R,0) \end{cases}$$

where $G^{f^j}(\uparrow, \downarrow)$ is the standard Godunov flux associated to f^j. Indeed, using Lemma 18.1 it is simpler to work with junctions J of type $0 : N$, and at the very end of the proof to use Lemma 18.1 to come back to the original problem. Hence in the remaining of the proof, we assume that J is of type $0 : N$, and then $J_R^j \simeq (0,R)$ for all j.
Here we have

$$\mathcal{G}_R := \left\{ p \in [a,b], \quad \hat{f}_R(p) = f(p) \right\} = \coprod_{j=1,\dots,N} \mathcal{G}_R^j \quad \text{with} \quad \mathcal{G}_R^j := \left\{ p^j \in [a^j, b^j], \quad \hat{f}_R^j(p) = f^j(p^j) \right\}$$

Notice that each $\mathcal{G}_R^j$ is a Kružkov germ (as any germ for a single branch), and then it is easy to check that the product $\mathcal{G}_R$ is also a Kružkov germ with $p_0 \in \mathcal{G}_R$.
Step 3: properties of the Godunov fluxes
From now on, and to simplify the presentation, let is assume without loss of generality that $[\bar{a}, \bar{b}] = [a,b]$ is a bounded box. Notice now that the four last lines of (18.5) are automatically satisfied for the Godunov fluxes $\hat{f} = \hat{f}_{\mathcal{G}}$ and $\hat{f}_R = \hat{f}_{\mathcal{G}_R}$, because of the monotone bounds (second line of (2.14)) satisfied by the Godunov fluxes.
Step 4: construction of a solution on J_R and the limit $R \to +\infty$
Therefore Proposition 18.7 applies with $\hat{f}_1 := \hat{f}$ and $\hat{f}_{R,1} := \hat{f}_R$ and $p_0 := u_0(R^-)$ and gives the existence of a solution u_R of (18.45) with initial data $(u_0)_{|J_R}$. In particular we have $\hat{f}_R(u_0) = f(u_0)$ at $x = R$. From the a priori bounds given in Proposition 18.7, we can pass to the limit with $u_R \to u$ as $R \to +\infty$. This leads to the existence of a solution u with bounds.
Step 5: uniqueness and L^1-contraction
The uniqueness and L^1-contraction follow from i) of Lemma 17.3.
Step 6: recovering a priori bounds
Finally, in the special case $p_0 = 0$, we recover the results of Theorem 2.61, and for general p_0, this proves Proposition 18.8.
This ends the proof.

19 Existence via semi-discretized schemes for Kružkov germs

19.1 The semi-discretized scheme

Assume (2.2) with $N \geq 1$ and junction (J, f) with $\sigma \in \{\pm 1\}^N$. We set the disjoint union

$$J_{\mathbb{N}} := \coprod_{\alpha=1,\dots,N} J_{\mathbb{N}}^\alpha \quad \text{with} \quad J_{\mathbb{N}}^\alpha := -\sigma^\alpha \mathbb{N}, \quad \text{for} \quad \alpha = 1, \dots, N \tag{19.1}$$

For $R \in \mathbb{N}$ with $R \geq 3$, we can also consider truncated branches

$$J_R := \coprod_{\alpha=1,\dots,N} J_R^\alpha \quad \text{with} \quad J_R^\alpha := \{k \in -\sigma^\alpha \mathbb{N}, \quad |k| \leq R\}, \quad \text{for} \quad \alpha = 1, \dots, N$$

We set $R^\alpha := \sigma^\alpha R$. Given some $p_0 \in [a,b]$, we also consider the following flux

$$\hat{f}^\alpha_R(p) := \begin{cases} G^\alpha(p^\alpha, p^\alpha_0) & \text{if} \quad \sigma^\alpha = -1 \\ G^\alpha(p^\alpha_0, p^\alpha,) & \text{if} \quad \sigma^\alpha = 1 \end{cases}$$

where $G^\alpha(\uparrow,\downarrow)$ is the standard Godunov flux associated to the function f^α. Recall that $\mathcal{G}_R := \left\{p \in [a,b],\ \hat{f}_R(p) = f(p)\right\}$ is a Kružkov germ with respect to junction $(-J,f)$.

We consider functions $u : [0,+\infty) \times J_R \to \mathbb{R}$ with $u = (u^\alpha_k(t))$ for $t \geq 0$, $k \in J^\alpha_R$ and $\alpha = 1,\dots,N$ solution of the following semi-discretized scheme for $u_k := (u^1_k,\dots,u^N_k)$ and $\Delta x > 0$

$$(19.2)\qquad \begin{cases} \partial_t u^\alpha_k + \dfrac{G^\alpha(u^\alpha_k,u^\alpha_{k+1}) - G^\alpha(u^\alpha_{k-1},u^\alpha_k)}{\Delta x} = 0 & \text{for} \quad k \in J^\alpha_{\mathbb{N}} \setminus \{0, R^\alpha\} \\ \partial_t u^\alpha_k + \dfrac{G^\alpha(u^\alpha_k,u^\alpha_{k+1}) - \hat{f}^\alpha(u_k)}{\Delta x} = 0 & \text{for} \quad k = 0 \qquad \text{with} \quad \sigma^\alpha = -1 \\ \partial_t u^\alpha_k + \dfrac{\hat{f}^\alpha(u_k) - G^\alpha(u^\alpha_{k-1},u^\alpha_k)}{\Delta x} = 0 & \text{for} \quad k = 0 \qquad \text{with} \quad \sigma^\alpha = 1 \\ \partial_t u^\alpha_k + \dfrac{\hat{f}^\alpha_R(u_k) - G^\alpha(u^\alpha_{k-1},u^\alpha_k)}{\Delta x} = 0 & \text{for} \quad k = R^\alpha \qquad \text{with} \quad \sigma^\alpha = -1 \\ \partial_t u^\alpha_k + \dfrac{G^\alpha(u^\alpha_k,u^\alpha_{k+1}) - \hat{f}^\alpha_R(u_k)}{\Delta x} = 0 & \text{for} \quad k = R^\alpha \qquad \text{with} \quad \sigma^\alpha = 1 \end{cases}$$

19.2 Preparation

Then we have the following result.

Lemma 19.1 (Existence of a semi-discrete solution)
Assume (2.2) with $N \geq 1$ and junction (J,f) with $\sigma \in \{\pm 1\}^N$ and $[a,b]$ bounded. Assume that $\mathcal{G} \subset [a,b]$ is a Kružkov germ, whose associated flux if $\hat{f} := \hat{f}_{\mathcal{G}}$. With notation (19.1), assume that the initial data $u(0)$ satisfies

$$(19.3)\qquad u^{0,\alpha}_k(t) \in [a^\alpha, b^\alpha] \quad \textit{for all} \quad k \in J^\alpha_R, \quad \alpha = 1,\dots,N$$

at the initial time $t = 0$. For any $\Delta x > 0$, there exists a unique solution $u : [0,+\infty) \times J_R \to \mathbb{R}$ of (19.2) with initial data $u(0)$. Moreover $u(t)$ satisfies (19.3) for all $t \geq 0$ and $u \in C^{1,1}([0,+\infty); L^\infty(J_R))$.

Proof of Lemma 19.1
We first extend the fluxes $f, \hat{f}, \hat{f}_R$ from the box $[a,b]$ to the whole space $\mathbb{R}^N$ as in Step 1 of the proof of Proposition 18.6. We also define the extended Godunov flux $G^\alpha = G^{f^\alpha}$ from the extended flux f^α.

Then the result follows easily from the fact that $\hat{f}, \hat{f}_R$ and f and $(G^\alpha)_\alpha$ are globally Lipschitz continuous functions. Then the classical Cauchy-Lipschitz theorem applies to the Banach space $L^\infty(J_R)$, and gives the existence and uniqueness of a solution in $C^{1,1}$. Moreover, the fluxes $\hat{f}, \hat{f}_R$ satisfy the second line of (2.14) which implies that $f, \hat{f}, \hat{f}_R$ satisfy (18.27). Let us now show that $u^\alpha_k(t)$ remains in the box $[a^\alpha, b^\alpha]$. Let us show for instance that $u^\alpha_k(t) \leq b^\alpha$. Formally, if the supremum of $u^\alpha_k(t_0)$ on $k \in J^\alpha_R$ is reached at some index

k_0 for some time t_0 with value $u^\alpha_{k_0}(t_0) = b^\alpha$, then we get for $k = k_0$

$$\partial_t u^\alpha_{k_0}(t_0) = \begin{cases} \dfrac{G^\alpha(u^\alpha_{k-1}, b^\alpha) - G^\alpha(b^\alpha, u^\alpha_{k+1})}{\Delta x} \le 0 & \text{if} \quad k_0 \in J^\alpha_R \backslash \{0, R^\alpha\} \\[2ex] \dfrac{\hat{f}^\alpha(u_k) - G^\alpha(b^\alpha, u^\alpha_{k+1})}{\Delta x} \le 0 & \text{if} \quad k_0 = 0 \qquad \text{and} \quad \sigma^\alpha = -1 \\[2ex] \dfrac{G^\alpha(u^\alpha_{k-1}, b^\alpha) - \hat{f}^\alpha(u_k)}{\Delta x} \le 0 & \text{if} \quad k_0 = 0 \qquad \text{and} \quad \sigma^\alpha = 1 \\[2ex] \dfrac{G^\alpha(u^\alpha_{k-1}, b^\alpha) - \hat{f}^\alpha_R(u_k)}{\Delta x} \le 0 & \text{if} \quad k_0 = R^\alpha \qquad \text{and} \quad \sigma^\alpha = -1 \\[2ex] \dfrac{\hat{f}^\alpha_R(u_k) - G^\alpha(b^\alpha, u^\alpha_{k+1})}{\Delta x} \le 0 & \text{if} \quad k_0 = R^\alpha \qquad \text{and} \quad \sigma^\alpha = 1 \end{cases}$$

where in the first line we have used the monotonicities of the Godunov flux $G^\alpha(\uparrow, \downarrow)$, while in the four last lines, we have also used the four last lines of (18.27). This shows formally that $u^\alpha_k(t) \le b^\alpha$ for all times $t \ge 0$ and $k \in J^\alpha_R$. As it is usual, this can be made rigorous, using for instance classical technics proving the comparison principle for viscosity solutions. This ends the proof of the lemma.

We will need the following easy result.

Lemma 19.2 (Comparison of entropy fluxes)
We define the entropy flux for $x', y', x, y \in [a^\alpha, b^\alpha]$ as

$$\Psi^\alpha(x', y'; x, y) := G^\alpha(x' \vee x, y' \vee y) - G^\alpha(x' \wedge x, y' \wedge y)$$

where $G^\alpha : [a^\alpha, b^\alpha]^2 \to \mathbb{R}$ is the standard Godunov flux associated to the function f^α. Then we have

$$\text{(19.4)} \qquad sign(y - y') \cdot \{G^\alpha(x, y) - G^\alpha(x', y')\} \le \Psi^\alpha(x', y'; x, y) \le sign(x - x') \cdot \{G^\alpha(x, y) - G^\alpha(x', y')\}$$

Proof of Lemma 19.2
We only prove the right inequality, i.e.

$$\text{(19.5)} \qquad G^\alpha(x' \vee x, y' \vee y) - G^\alpha(x' \wedge x, y' \wedge y) \le \text{sign}(x - x') \cdot \{G^\alpha(x, y) - G^\alpha(x', y')\}$$

Notice that the proof of the left inequality is similar.
Case A: $x - x' \ge 0$
Subcase A.1: $y \ge y'$
Then (19.5) means $G(x, y) - G(x', y') \le G(x, y) - G(x', y')$, which is indeed an equality.
Subcase A.2: $y \le y'$
Then (19.5) means $G(x, y') - G(x', y) \le G(x, y) - G(x', y')$, which is true because of the monotonicity of $G(x'', \downarrow)$.
Case B: $x - x' \le 0$
Subcase B.1: $y \ge y'$
Then (19.5) means $G(x', y) - G(x, y') \le -\{G(x, y) - G(x', y')\}$, which is true because of the monotonicity of $G(x'', \downarrow)$.
Subcase B.2: $y \le y'$
Then (19.5) means $G(x', y') - G(x, y) \le -\{G(x, y) - G(x', y')\}$, which is again an equality.
This ends the proof of the lemma.

For later use, we will need the following.

Definition 19.3 (The approximate solution u^ε given by the semi-discrete scheme)
We set $\varepsilon := \Delta x > 0$. Given a solution u of the semi-discretized scheme (19.2), we define $u^\varepsilon = (u^{\varepsilon,1}, \ldots, u^{\varepsilon,N})$ as
(19.6)

$$u^{\varepsilon,\alpha}(t, x) := \sum_{k \in J^\alpha_R} u^\alpha_k(t) \chi^{\varepsilon,\alpha}_k(x) \quad \textit{with} \quad \chi^{\varepsilon,\alpha}_k(x) := \left\{ \begin{array}{ll} 1_{[k\varepsilon,(k+1)\varepsilon)}(x) & \textit{if} \quad \sigma^\alpha = -1 \\ 1_{((k-1)\varepsilon,k\varepsilon]}(x) & \textit{if} \quad \sigma^\alpha = 1 \end{array} \right| \quad \textit{for} \quad \alpha = 1, \ldots, N$$

Lemma 19.4 (Entropy inequalities)
We work under assumptions of Lemma 19.1 and recall that $\sigma \in \{\pm\}^N$. Let $u(t), v(t)$ two solutions of the semi-discretized scheme (19.2) respectively with initial data $u(0), v(0)$. Then we have the following
i) (Pointwise entropy inequality)

$$\partial_t W_k^\alpha + \frac{\Psi^\alpha_{k+\frac{1}{2}} - \Psi^\alpha_{k-\frac{1}{2}}}{\Delta x} \le 0 \quad \textit{for all} \quad k \in J_R^\alpha, \quad \alpha = 1, \dots, N \tag{19.7}$$

with for $k \in \mathbb{Z}$

$$\begin{cases} W_k^\alpha := |u_k^\alpha - v_k^\alpha| \quad \textit{for} \quad k \in J_R^\alpha \\ \Psi^\alpha_{k+\frac{1}{2}} := \begin{cases} \left.\begin{array}{lll} \Psi^\alpha(u_k^\alpha, u_{k+1}^\alpha; v_k^\alpha, v_{k+1}^\alpha) & \textit{if} & k \in J_R^\alpha \backslash \{R\} \\ \Psi^\alpha_R(u_k, v_k) & \textit{if} & k = R \\ \Psi^\alpha_0(u_{k+1}, v_{k+1}) & \textit{if} & k = -1 \end{array}\right| & \textit{if} \quad \sigma^\alpha = -1 \\ \left.\begin{array}{lll} \Psi^\alpha(u_k^\alpha, u_{k+1}^\alpha; v_k^\alpha, v_{k+1}^\alpha) & \textit{if} & k \in J_R^\alpha \backslash \{0\} \\ \Psi^\alpha_0(u_k, v_k) & \textit{if} & k = 0 \\ \Psi^\alpha_R(u_{k+1}, v_{k+1}) & \textit{if} & k = -R-1 \end{array}\right| & \textit{if} \quad \sigma^\alpha = 1 \end{cases} \end{cases}$$

and

$$\begin{cases} \Psi^\alpha(x', y'; x, y) := G^\alpha(x' \vee x, y' \vee y) - G^\alpha(x' \wedge x, y' \wedge y) \\ \Psi^\alpha_0(z', z) = sign(z'^\alpha - z^\alpha) \cdot \left\{ \hat{f}^\alpha(z') - \hat{f}^\alpha(z) \right\} \\ \Psi^\alpha_R(z', z) = sign(z'^\alpha - z^\alpha) \cdot \left\{ \hat{f}^\alpha_R(z') - \hat{f}^\alpha_R(z) \right\} \end{cases}$$

ii) (Integral entropy inequality)
Let $u^\varepsilon, v^\varepsilon$ as in Definition 19.3 with $\varepsilon := \Delta x$. Define also
(19.8)

$$\Psi^{\varepsilon,\alpha}(t,x) := \sum_{k \in J_R^\alpha \backslash \{R^\alpha\}} \Psi^\alpha_{k+\frac{1}{2}}(t) \chi_k^{\varepsilon,\alpha}(x) \quad \textit{with} \quad \chi_k^{\varepsilon,\alpha}(x) := \left\{ \begin{array}{lll} 1_{[k\varepsilon,(k+1)\varepsilon)}(x) & \textit{if} & \sigma^\alpha = -1 \\ 1_{((k-1)\varepsilon, k\varepsilon]}(x) & \textit{if} & \sigma^\alpha = 1 \end{array} \right| \quad \textit{for} \quad \alpha = 1, \dots, N$$

Let us consider a smooth test function $0 \le \varphi_k^\alpha(t)$ with compact support in $t \in [0, +\infty)$ for $k \in J_R^\alpha$ which satisfies $\varphi_0^\alpha(t) = \varphi_0^\beta(t)$ for all α, β. Then we have the following integral entropy inequality

$$\int_{\{0\} \times J} |u^\varepsilon - v^\varepsilon| \varphi^\varepsilon + \int_{(0,+\infty) \times J} \left\{ |u^\varepsilon - v^\varepsilon| \varphi_t^\varepsilon - \sigma \Psi^\varepsilon \cdot \left\{ \frac{\varphi^\varepsilon(t, \cdot - \sigma\varepsilon) - \varphi^\varepsilon(t, \cdot)}{\varepsilon} \right\} \right\} \ge 0 \tag{19.9}$$

where φ^ε is defined from φ by the same formula defining u^ε from u.

Proof of Lemma 19.4
Step 1: proof of i)
We start with the semi-discretized scheme satisfied by u and v, and multiply the difference of the schemes by $\text{sign}(u_k^\alpha - v_k^\alpha)$ (starting with an approximation of it). We get

$$\partial_t |u_k^\alpha - v_k^\alpha| + \frac{\Psi_k^{\alpha,+} - \Psi_k^{\alpha,-}}{\Delta x} = 0 \quad \text{for} \quad k \in J_R^\alpha$$

with

$$\Psi_k^{\alpha,+} := \begin{cases} \left.\begin{array}{lll} \text{sign}(u_k^\alpha - v_k^\alpha) \cdot \left\{ G^\alpha(u_k^\alpha, u_{k+1}^\alpha) - G^\alpha(v_k^\alpha, v_{k+1}^\alpha) \right\} & \text{if} & k \in J_R^\alpha \backslash \{R\} \\ \Psi^\alpha_R(u_k, v_k) & \text{if} & k = R \end{array}\right| & \text{if} \quad \sigma^\alpha = -1 \\ \left.\begin{array}{lll} \text{sign}(u_k^\alpha - v_k^\alpha) \cdot \left\{ G^\alpha(u_k^\alpha, u_{k+1}^\alpha) - G^\alpha(v_k^\alpha, v_{k+1}^\alpha) \right\} & \text{if} & k \in J_R^\alpha \backslash \{0\} \\ \Psi^\alpha_0(u_k, v_k) & \text{if} & k = 0 \end{array}\right| & \text{if} \quad \sigma^\alpha = 1 \end{cases}$$

and

$$\Psi_k^{\alpha,-} := \begin{cases} \left. \begin{array}{ll} \mathrm{sign}(u_k^\alpha - v_k^\alpha)\cdot\left\{G^\alpha(u_{k-1}^\alpha, u_k^\alpha) - G^\alpha(v_{k-1}^\alpha, v_k^\alpha)\right\} & \text{if} \quad k \in J_R^\alpha\backslash\{0\} \\ \Psi_0^\alpha(u_k, v_k) & \text{if} \quad k = 0 \end{array} \right| & \text{if} \quad \sigma^\alpha = -1 \\ \\ \left. \begin{array}{ll} \mathrm{sign}(u_k^\alpha - v_k^\alpha)\cdot\left\{G^\alpha(u_{k-1}^\alpha, u_k^\alpha) - G^\alpha(v_{k-1}^\alpha, v_k^\alpha)\right\} & \text{if} \quad k \in J_R^\alpha\backslash\{-R\} \\ \Psi_R^\alpha(u_k, v_k) & \text{if} \quad k = -R \end{array} \right| & \text{if} \quad \sigma^\alpha = 1 \end{cases}$$

Then Lemma 19.2 means $\Psi^\alpha_{k+\frac12} \le \Psi_k^{\alpha,+}$ and $\Psi^\alpha_{k-\frac12} \ge \Psi_k^{\alpha,-}$ for all $k \in J_R^\alpha$, which implies (19.7).

Step 2: proof of ii)

Now, in order to simplify the presentation, let us assume that $\sigma^\alpha = -1$, say for all α (which can be done using a suitable change of variables like in Lemma 18.1). Consider any test function $0 \le \varphi_k^\alpha(t)$ which is with compact support for $t \in [0,+\infty)$. Multiplying (19.7) by φ_k^α for $k = 0,\dots,R$, and summing over k and integrating in time over $(0,+\infty)$, we get

$$\begin{aligned} 0 \quad &\ge \varepsilon \sum_{k=0,\dots,R} \int_{(0,+\infty)} \left\{ \{\partial_t W_k^\alpha\}\,\varphi_k^\alpha + \left\{ \frac{\Psi^\alpha_{k+\frac12} - \Psi^\alpha_{k-\frac12}}{\varepsilon} \right\} \cdot \varphi_k^\alpha \right\} \\ &= \int_{(0,+\infty)} \left\{ \varepsilon \sum_{k=0,\dots,R} \{\partial_t W_k^\alpha\}\,\varphi_k^\alpha + \left\{ \Psi_R^\alpha(u_R,v_R)\varphi_R^\alpha - \Psi_0^\alpha(u_0,v_0)\varphi_0^\alpha + \sum_{k=0,\dots,R-1} \Psi^\alpha_{k+\frac12}\varphi_k^\alpha - \sum_{k=1,\dots,R} \Psi^\alpha_{k-\frac12}\varphi_k^\alpha \right\}\right\} \\ &= \int_{(0,+\infty)} \{\Psi_R^\alpha(u_R,v_R)\varphi_R^\alpha - \Psi_0^\alpha(u_0,v_0)\varphi_0^\alpha\} + \int_{(0,+\infty)} \left\{ \varepsilon \sum_{k=0,\dots,R} \{\partial_t W_k^\alpha\}\,\varphi_k^\alpha + \left\{ \sum_{k=0,\dots,R-1} \Psi^\alpha_{k+\frac12}\left\{\varphi_k^\alpha - \varphi_{k+1}^\alpha\right\} \right\}\right\} \\ &= \int_{(0,+\infty)} \{\Psi_R^\alpha(u_R,v_R)\varphi_R^\alpha - \Psi_0^\alpha(u_0,v_0)\varphi_0^\alpha\} + \int_{(0,+\infty)\times J^\alpha} \left\{ \{\partial_t |u^{\varepsilon,\alpha} - v^{\varepsilon,\alpha}\}\,\varphi^{\varepsilon,\alpha} - \Psi^{\varepsilon,\alpha} \left\{ \frac{\varphi^{\varepsilon,\alpha}(\cdot,\cdot+\varepsilon) - \varphi^{\varepsilon,\alpha}}{\varepsilon} \right\}\right\} \end{aligned}$$

An integration by parts in time gives

$$\begin{aligned} &\int_{\{0\}\times J^\alpha} |u^{\varepsilon,\alpha} - v^{\varepsilon,\alpha}|\varphi^{\varepsilon,\alpha} + \int_{(0,+\infty)\times J^\alpha} \left\{ |u^{\varepsilon,\alpha} - v^{\varepsilon,\alpha}|\varphi_t^{\varepsilon,\alpha} + \Psi^{\varepsilon,\alpha} \left\{ \frac{\varphi^{\varepsilon,\alpha}(\cdot,\cdot+\varepsilon) - \varphi^{\varepsilon,\alpha}}{\varepsilon} \right\}\right\} \\ &\ge \int_{(0,+\infty)} \{\Psi_R^\alpha(u_R,v_R)\varphi_R^\alpha - \Psi_0^\alpha(u_0,v_0)\varphi_0^\alpha\} \end{aligned} \tag{19.10}$$

Summing over α and using $\varphi^\varepsilon \ge 0$, we get

$$\begin{aligned} &\int_{\{0\}\times J} |u^\varepsilon - v^\varepsilon|\varphi^\varepsilon + \int_{(0,+\infty)\times J} \left\{ |u^\varepsilon - v^\varepsilon|\varphi_t^\varepsilon + \Psi^\varepsilon \left\{ \frac{\varphi^\varepsilon(\cdot,\cdot+\varepsilon) - \varphi^\varepsilon}{\varepsilon} \right\}\right\} \\ &\ge \int_{(0,+\infty)\times(J\cap\partial B_{\varepsilon R})} D^{\hat f_R}(u^\varepsilon, v^\varepsilon)\varphi^\varepsilon + \int_{(0,+\infty)\times\{0\}} D^{\hat f}(u^\varepsilon, v^\varepsilon)\varphi^\varepsilon \end{aligned}$$

with

$$D^{\hat f}(u^\varepsilon, v^\varepsilon) := \sum_{\alpha=1}^N \sigma^\alpha\ \psi^{\hat f^\alpha}(u^\varepsilon, v^\varepsilon) \quad \text{and} \quad \psi^{\hat f^\alpha}(u^\varepsilon, v^\varepsilon) := \mathrm{sign}(u^{\varepsilon,\alpha} - v^{\varepsilon,\alpha})\cdot\left\{\hat f^\alpha(u^\varepsilon) - \hat f^\alpha(v^\varepsilon)\right\}$$

and

$$D^{\hat f_R}(u^\varepsilon, v^\varepsilon) := \sum_{\alpha=1}^N \sigma_R^\alpha\ \psi^{\hat f_R^\alpha}(u^\varepsilon, v^\varepsilon) \quad \text{with} \quad \sigma_R^\alpha = -\sigma^\alpha.$$

Because both $\mathcal G$ and $\mathcal G_R$ are Kružkov germs respectively to (J, f) and $(-J, f)$, we deduce that $D^{\hat f}, D^{\hat f_R} \ge 0$. Joint to the fact that $\varphi^\varepsilon \ge 0$, we see that (19.10) implies

$$\int_{\{0\}\times J} |u^\varepsilon - v^\varepsilon|\varphi^\varepsilon + \int_{(0,+\infty)\times J} \left\{ |u^\varepsilon - v^\varepsilon|\varphi_t^\varepsilon + \Psi^\varepsilon \left\{ \frac{\varphi^\varepsilon(\cdot,\cdot+\varepsilon) - \varphi^\varepsilon}{\varepsilon} \right\}\right\} \ge 0$$

More generally, for arbitrary signs $\sigma^\alpha \in \{\pm 1\}$ with $\sigma_R := -\sigma$, we get (19.9). This ends the proof of the lemma.

Corollary 19.5 (Contraction estimates)
We work under assumptions of Lemma 19.1. Let u, v be two solutions of the semi-discretized scheme (19.2). We use notation of Definition 19.3 for $u^\varepsilon, v^\varepsilon$. Then for all $t > 0$, we have for $R_\varepsilon := (R+1)\varepsilon$

$$\int_{\{t\}\times(J\cap B_{R_\varepsilon})} |u^\varepsilon - v^\varepsilon| \le \int_{\{0\}\times(J\cap B_{R_\varepsilon})} |u^\varepsilon - v^\varepsilon| \tag{19.11}$$

We also have

$$\int_{\{t\}\times(J\cap B_{R_\varepsilon})} |\partial_t u^\varepsilon| \le \int_{\{0\}\times(J\cap B_{R_\varepsilon})} |\partial_t u^\varepsilon| \tag{19.12}$$

Proof of Corollary 19.5
On the one hand, inequality (19.11) follows from the integral entropy inequality (19.9) for $\varphi^\varepsilon \equiv 1$. On the other hand, inequality (19.12) follows from (19.11) choosing $v^\varepsilon = u^\varepsilon(\cdot + \tau)$, dividing by τ and passing to the limit $\tau \to 0^+$. This ends the proof of the lemma.

Lemma 19.6 (BLN estimates)
We work under assumptions of Lemma 19.1. Given a solution u of the semi-discretized scheme (19.2), we consider u^ε as in Definition 19.3. Fix α such that $\sigma^\alpha = -1$. Let $k_i \in J_R^\alpha$ and $x_i = k_i\varepsilon$ for $i = 1,2$ with $x_1 < x_2$ and $2 \le k_i \le R-2$. Then we have boundary BLN estimate

$$\int_{\{t\}\times(x_1,x_2)} \left|\frac{u^{\varepsilon,\alpha}(\cdot,\cdot+\varepsilon) - u^{\varepsilon,\alpha}}{\varepsilon}\right| - \int_{\{0\}\times(x_1,x_2)} \left|\frac{u^{\varepsilon,\alpha}(\cdot,\cdot+\varepsilon) - u^{\varepsilon,\alpha}}{\varepsilon}\right| \le \sum_{i=1,2} \varepsilon^{-1} \int_{(0,t)\times(x_i,x_i+\varepsilon)} |\partial_t u^{\varepsilon,\alpha}| \tag{19.13}$$

Moreover for $R_\varepsilon := (R+1)\varepsilon$ and $x_1 = 2\delta = R_\varepsilon - x_2 + \varepsilon$ with $\delta \in \varepsilon(\mathbb{N}\setminus\{0,1\})$, we have the interior BLN estimate

$$\begin{aligned} &\int_{\{t\}\times(2\delta,R_\varepsilon-2\delta)} \left|\frac{u^{\varepsilon,\alpha}(\cdot,\cdot+\varepsilon) - u^{\varepsilon,\alpha}}{\varepsilon}\right| \\ &\le \int_{\{0\}\times(\delta,R_\varepsilon-\delta)} \left|\frac{u^{\varepsilon,\alpha}(\cdot,\cdot+\varepsilon) - u^{\varepsilon,\alpha}}{\varepsilon}\right| + \delta^{-1}\left\{\int_{(0,t)\times(\delta,2\delta)} |\partial_t u^{\varepsilon,\alpha}| + \int_{(0,t)\times(R_\varepsilon-2\delta,R_\varepsilon-\delta)} |\partial_t u^{\varepsilon,\alpha}|\right\} \end{aligned} \tag{19.14}$$

Proof of Lemma 19.6
Recall that $\sigma^\alpha = -1$, and define $\tilde{u}^{\varepsilon,\alpha}(t,x) := u^{\varepsilon,\alpha}(t,x+\varepsilon)$. For $i = 1,2$, assume $x_i = k_i\varepsilon \in J^\alpha \simeq (0,+\infty)$ for $i = 1,2$ with $k_i \in J_R^\alpha\setminus\{0,R\}$ with $2 \le k_i \le R-2$ and $x_1 < x_2$. Integrating (19.7) over (x_1,x_2), we get

$$\int_{\{t\}\times(x_1,x_2)} \left\{\partial_t|\tilde{u}^{\varepsilon,\alpha} - u^{\varepsilon,\alpha}| + \sum_{k\in J_R^\alpha} \left\{\frac{\Psi^\alpha_{k+\frac12} - \Psi^\alpha_{k-\frac12}}{\varepsilon}\right\} \chi_k^{\varepsilon,\alpha}\right\} \le 0$$

with $\Psi^\alpha_{k+\frac12} := \Psi^\alpha(\tilde{u}_k^\alpha, \tilde{u}_{k+1}^\alpha; u_k^\alpha, u_{k+1}^\alpha) = \Psi^\alpha(u_{k+1}^\alpha, u_{k+2}^\alpha; u_k^\alpha, u_{k+1}^\alpha)$. From (19.4), it satisfies

$$|\Psi^\alpha_{k+\frac12}| \le |G^\alpha(u_{k+1}^\alpha, u_{k+2}^\alpha) - G^\alpha(u_k^\alpha, u_{k+1}^\alpha)| = \varepsilon|\partial_t u_{k+1}^\alpha|$$

Hence

$$\begin{aligned} \partial_t \int_{\{t\}\times(x_1,x_2)} |\tilde{u}^{\varepsilon,\alpha} - u^{\varepsilon,\alpha}| &\le -\left\{\Psi^\alpha_{k_2-\frac12} - \Psi^\alpha_{k_1-\frac12}\right\} \\ &\le |\Psi^\alpha_{k_2-\frac12}| + |\Psi^\alpha_{k_1-\frac12}| \\ &= \varepsilon\left\{|\partial_t u_{k_2}^\alpha| + |\partial_t u_{k_1}^\alpha|\right\} \\ &= \sum_{i=1,2} \int_{\{t\}\times(x_1,x_1+\varepsilon)} |\partial_t u^{\varepsilon,\alpha}| \end{aligned}$$

Integrating on $(0,t)$, we get

$$\int_{\{t\}\times(x_1,x_2)} |\tilde{u}^{\varepsilon,\alpha} - u^{\varepsilon,\alpha}| - \int_{\{0\}\times(x_1,x_2)} |\tilde{u}^{\varepsilon,\alpha} - u^{\varepsilon,\alpha}| \le \sum_{i=1,2} \int_{(0,t)\times(x_i,x_i+\varepsilon)} |\partial_t u^{\varepsilon,\alpha}|$$

which implies (19.13). Alternatively, this inequality can also be obtained from the integral entropy inequality for a suitable test function, which is a characteristic function of the rectangle $(0,t)\times(x_1,x_2)$.
In particular, for $x_1 = h$ and $x_2 = R_\varepsilon - h - \varepsilon$, we get that (19.13) means

$$\int_{\{t\}\times(h,R_\varepsilon-h)} \left| \frac{u^{\varepsilon,\alpha}(\cdot,\cdot+\varepsilon) - u^{\varepsilon,\alpha}}{\varepsilon} \right|$$

$$\le \int_{\{0\}\times(h,R_\varepsilon-h)} \left| \frac{u^{\varepsilon,\alpha}(\cdot,\cdot+\varepsilon) - u^{\varepsilon,\alpha}}{\varepsilon} \right| + \varepsilon^{-1} \left\{ \int_{(0,t)\times(h,h+\varepsilon)} |\partial_t u^{\varepsilon,\alpha}| + \int_{(0,t)\times(R_\varepsilon-h-\varepsilon,R_\varepsilon-h)} |\partial_t u^{\varepsilon,\alpha}| \right\}$$

Now for $\Delta \in \mathbb{N}\backslash\{0,1\}$ and $\delta := \varepsilon\Delta$, summing over $\varepsilon^{-1}h = \Delta,\dots,2\Delta-1$, we get

$$\Delta \int_{\{t\}\times(2\delta,R_\varepsilon-2\delta)} \left| \frac{u^{\varepsilon,\alpha}(\cdot,\cdot+\varepsilon) - u^{\varepsilon,\alpha}}{\varepsilon} \right|$$

$$\le \Delta \int_{\{0\}\times(\delta,R_\varepsilon-\delta)} \left| \frac{u^{\varepsilon,\alpha}(\cdot,\cdot+\varepsilon) - u^{\varepsilon,\alpha}}{\varepsilon} \right| + \varepsilon^{-1} \left\{ \int_{(0,t)\times(\delta,2\delta)} |\partial_t u^{\varepsilon,\alpha}| + \int_{(0,t)\times(R_\varepsilon-2\delta,R_\varepsilon-\delta)} |\partial_t u^{\varepsilon,\alpha}| \right\}$$

which implies (19.14). This ends the proof of the lemma.

19.3 Convergence

Then we have the following result.

Proposition 19.7 (Convergence of the semi-discretized solution on the truncated junction)
Assume (2.2) with $N \ge 1$, nondegeneracy condition (2.17), and let $\mathcal{G} \subset [a,b]$ be a Kružkov germ with $[a,b]$ bounded. We assume that the junction is characterized by (J,f) with $\sigma \in \{\pm 1\}^N$ and associated flux $\hat{f} := \hat{f}_{\mathcal{G}}$.
Assume $\bar{U}_0 = (\bar{U}_0^1,\dots,\bar{U}_0^N)$ with $\bar{U}_0^\alpha \in (BV\cap L^1)(J^\alpha;[a^\alpha,b^\alpha])$. Define for $\alpha = 1,\dots,N$

$$u_k^\alpha(0) := \varepsilon^{-1}\int_{J^\alpha} \bar{U}_0^\alpha \chi_k^{\varepsilon,\alpha} \quad \text{with } \chi_k^{\varepsilon,\alpha} \text{ given in Definition 19.3}$$

Then the initial data $u(0)$ satisfies (19.3). For any $\varepsilon := \Delta x > 0$ and $R \ge 3$, let u be the unique solution $u : [0,+\infty)\times J_R \to \mathbb{R}$ of of the scheme (19.2) with initial data $u(0)$, and let $u^\varepsilon : [0,+\infty)\times(J\cap B_{R_\varepsilon}) \to \mathbb{R}$ with $R_\varepsilon := (R+1)\varepsilon$ given by Definition 19.3.
Then we have

$$u^\varepsilon \to \bar{U} \quad \text{in} \quad L^1_{loc}([0,+\infty)\times(J\cap B_{\bar{R}})) \quad \text{as} \quad (\varepsilon,R)\to(0,+\infty) \quad \text{with} \quad R_\varepsilon \to \bar{R} \in (0,+\infty)$$

where $\bar{U}$ is the unique solution of (18.45) (where R is replaced by $\bar{R}$) and with initial data $\bar{U}(0,\cdot) = \bar{U}_0$.

Remark 19.8 (When the scheme relaxes the boundary condition)
It is important to notice that Proposition 19.7 is still true when the flux $\hat{f}_{\mathcal{G}}$ is replaced by a function $\hat{f}$ satisfying (7.1), i.e. satisfying

$$(19.15) \qquad \begin{cases} \hat{f} : [a,b] \to \mathbb{R}^N \ \text{continuous} \\ \sigma \diamond \hat{f} : [a,b] \to \mathbb{R}^N \ \text{Riemann monotone} \\ \sigma^j f^j(b^j) \le \sigma^j \hat{f}^j(q)_{|q^j=b^j} \quad \text{and} \quad \sigma^j \hat{f}^j(q)_{|q^j=a^j} \le \sigma^j f^j(a^j) \end{cases}$$

Then setting $\hat{f}_1 := \mathfrak{R}\hat{f}$ where $\mathfrak{R}$ is defined in (7.3), there exists a Riemann germ $\mathcal{G}_1$ such that $\hat{f}_1 = \hat{f}_{\mathcal{G}_1}$. For instance in the special case where $\sigma = (1,\dots,1) \in \mathbb{R}^N$, for any $\hat{p} \in \mathcal{G}_1$, we know from the construction of $\mathfrak{R}\hat{f}$, that there exists $q \in [a,b]$ such that

$$G^g(\hat{p},\hat{p}) = f(\hat{p}) = f_1(\hat{p}) = G^f(\hat{p},q) = \hat{f}(q)$$

Then it is enough to notice that the function

$$u_k^\alpha := \begin{cases} \hat{p}^\alpha & \text{if} \quad k<0 \\ q^\alpha & \text{if} \quad k=0 \end{cases}$$

is a solution of the scheme (19.2). Then this solution can be used to show that the limit of the sequence u^ε still has a trace in $\mathcal{G}_1$, i.e. the result holds with $\mathcal{G} := \mathcal{G}_1$.

Proof of Proposition 19.7

Recall that $u^\varepsilon(t,x)$ belongs to the box $[a,b]$ which gives an L^∞ estimate, and we have both estimate (19.12) on u_t^ε, and interior BLN estimate (19.6) on $\dfrac{u^\varepsilon(\cdot,\cdot+\varepsilon)-u^\varepsilon}{\varepsilon}$, which approximates the x-derivative of u^ε. This shows that we have uniform bounds on u^ε in $BV_{loc;t,x}$. The compactness of $BV_{loc}\subset L^1_{loc}$ for bounded functions implies the convergence of u^ε in L^1_{loc} to some function $\bar{U}$, up to extraction of a subsequence. Now our integral entropy inequalities (19.9) also passes to the limit and gives (for any constant V both solution of the limit problem and of the discretized problem)

$$\int_{\{0\}\times(J\cap B_{\bar{R}})} |\bar{U}-V|\varphi + \int_{(0,+\infty)\times(J\cap B_{\bar{R}})} \left\{|\bar{U}-V|\varphi_t + \bar{\Psi}\cdot\varphi_x\right\} \geq 0 \tag{19.16}$$

with

$$\Psi^{\varepsilon,\alpha} \to \bar{\Psi}^\alpha = \Psi^\alpha(\bar{U}^\alpha,\bar{U}^\alpha,V^\alpha,V^\alpha)$$

where we recall that $\Psi^\alpha(x',y';x,y) := G^\alpha(x'\vee x, y'\vee y) - G^\alpha(x'\wedge x, y'\wedge y)$. This gives

$$\bar{\Psi}^\alpha = f^\alpha(\bar{U}^\alpha\vee V^\alpha) - f^\alpha(\bar{U}^\alpha\wedge V^\alpha) = \text{sign}(\bar{U}^\alpha - V^\alpha)\left\{f^\alpha(\bar{U}^\alpha)-f^\alpha(V^\alpha)\right\} = \psi^{f^\alpha}(\bar{U}^\alpha,V^\alpha)$$

which is exactly the flux of Kružkov entropy. Choosing the test function φ^α with compact support in the interior of each branch $J^\alpha\cap B_{\bar{R}}$ and any constant $V^\alpha\in[a^\alpha,b^\alpha]$, we recover that $\bar{U}$ is an entropy solution on each (truncated) branch. Now choosing $V\in\mathcal{G}$ and a test function φ with compact support in $J\cap B_{\bar{R}}$, with moreover φ focusing on the origin, we recover that

$$D^f(\bar{U}(t,0),V)\geq 0 \quad \text{for a.e. } t>0 \text{ and all} \quad V\in\mathcal{G}$$

Because $\mathcal{G}$ is Kružkov, we recover that $\bar{U}(t,0)\in\mathcal{G}$ for a.e. $t>0$.
Similarly, we show that $(\bar{U}(t,\cdot))_{J\cap\partial B_{\bar{R}}}\in\mathcal{G}_R$. This shows that $\bar{U}$ is solution of (18.45) (where R is replaced by $\bar{R}$) and with initial data $U(0,\cdot)=U_0$. The uniqueness of the solution shows the uniqueness of any accumulation point of the sequence u^ε. This implies the convergence of the full sequence u^ε towards $\bar{U}$. This ends the proof of the proposition.

20 Existence via fully discretized schemes for monotone Kružkov germs

20.1 The fully discretized scheme

Assume (2.2) with $N\geq 1$ and junction (J,f) with $\sigma\in\{\pm1\}^N$. We recall notation (19.1), namely

$$J_{\mathbb{N}} := \coprod_{\alpha=1,\ldots,N} J^\alpha_{\mathbb{N}} \quad \text{with} \quad J^\alpha_{\mathbb{N}} := -\sigma^\alpha\mathbb{N}, \quad \text{for} \quad \alpha=1,\ldots,N$$

We consider functions $u:\mathbb{N}\times J_{\mathbb{N}}\to\mathbb{R}$ with $u=(u_k^{n,\alpha})$ for $n\in\mathbb{N}$, $k\in J^\alpha_{\mathbb{N}}$ and $\alpha=1,\ldots,N$ solution of the following fully discretized explicit (finite volume) scheme for $u_0^n := (u_0^{n,1},\ldots,u_0^{n,N})$ and $\Delta x>0$, $\Delta t>0$ and $n\in\mathbb{N}$

(20.1)

$$\begin{cases} \dfrac{u_k^{n+1,\alpha}-u_k^{n,\alpha}}{\Delta t} + \dfrac{G^\alpha(u_k^{n,\alpha},u_{k+1}^{n,\alpha})-G^\alpha(u_{k-1}^{n,\alpha},u_k^{n,\alpha})}{\Delta x} = 0 & \text{for} \quad k\in J^\alpha_{\mathbb{N}}\setminus\{0\} \\ \dfrac{u_k^{n+1,\alpha}-u_k^{n,\alpha}}{\Delta t} + \dfrac{G^\alpha(u_k^{n,\alpha},u_{k+1}^{n,\alpha})-\hat{f}^\alpha(u_k^n)}{\Delta x} = 0 & \text{for} \quad k=0 \quad \text{with} \quad \sigma^\alpha=-1 \\ \dfrac{u_k^{n+1,\alpha}-u_k^{n,\alpha}}{\Delta t} + \dfrac{\hat{f}^\alpha(u_k^n)-G^\alpha(u_{k-1}^{n,\alpha},u_k^{n,\alpha})}{\Delta x} = 0 & \text{for} \quad k=0 \quad \text{with} \quad \sigma^\alpha=1 \end{cases}$$

where $G^\alpha = G^{f^\alpha}$ is the standard Godunov flux associated to the function $f^\alpha : [a^\alpha, b^\alpha] \to \mathbb{R}$. We also assume that $\hat{f} = \hat{f}_{\mathcal{G}} : [a,b] \to \mathbb{R}^N$ is the Godunov flux associated to a Kružkov germ $\mathcal{G}$. When it will be useful, we will furthermore assume that $\mathcal{G}$ is monotone.

20.2 Preparation

Then we have the following result.

Lemma 20.1 (Existence of a fully discrete solution)
Assume (2.2) with $N \geq 1$ and junction (J,f) with $\sigma \in \{\pm 1\}^N$ and $[a,b]$ bounded. We also assume that $\hat{f} = \hat{f}_{\mathcal{G}} : [a,b] \to \mathbb{R}^N$ is the Godunov flux associated to a Kružkov germ $\mathcal{G}$. With notation (19.1), assume that the initial data $u^0 : J_{\mathbb{N}} \to \mathbb{R}$ (with $u^0 = (u_k^{0,\alpha})$ with $k \in J_{\mathbb{N}}^\alpha$) satisfies

$$u_k^{n,\alpha} \in [a^\alpha, b^\alpha] \quad \text{for all} \quad k \in J_{\mathbb{N}}^\alpha, \quad \alpha = 1, \dots, N \tag{20.2}$$

at the initial time $n = 0$. For any $\Delta x > 0$, $\Delta t > 0$, satisfying the following CFL condition

$$L \frac{\Delta t}{\Delta x} \leq \frac{1}{2} \quad \text{with} \quad L := \sup_{\alpha = 1,\dots,N} Lip(f^\alpha; [a^\alpha, b^\alpha]) \tag{20.3}$$

there exists a unique solution $u : \mathbb{N} \times J_{\mathbb{N}} \to \mathbb{R}$ of (20.1) with initial data u^0. Moreover u satisfies (??) for all $n \geq 0$.

Furthermore, if the germ $\mathcal{G}$ is monotone, then the scheme is also monotone.

Proof of Lemma 20.1
Recall that $\hat{f} = \hat{f}_{\mathcal{G}}$ satisfies on $[a,b]$

$$\sigma^j \partial_j \hat{f}^j \geq \sum_{\alpha \in \{1,\dots,N\} \setminus \{j\}} |\partial_j \hat{f}^\alpha|, \quad \text{for all} \quad j = 1, \dots, N$$

because $\mathcal{G}$ is Kružkov. Furthermore, when $\mathcal{G}$ is monotone, we also have on $[a,b]$

$$\partial_j(\sigma^j \hat{f}^j) \geq 0, \quad \partial_k(\sigma^j \hat{f}^j) \leq 0 \quad \text{for all} \quad k \neq j \tag{20.4}$$

Step 1: extension, existence and uniqueness
We first extend the fluxes $f, \hat{f}$ from the box $[a,b]$ to the whole space $\mathbb{R}^N$ as in Step 1 of the proof of Proposition 18.6. Furthermore, when $\mathcal{G}$ is monotone, it is important to notice that monotonicities (20.4) are still true for the extension $\hat{f}$ on $\mathbb{R}^N$, as it follows easily from the method of extension. We also define the extended Godunov flux $G^\alpha = G^{f^\alpha}$ from the extended flux f^α.

Then the functions $\hat{f}, f$ and $(G^\alpha)_\alpha$ are defined on the whole $\mathbb{R}^N$, and are globally Lipschitz continuous satisfying in particular for all indices j, α

$$0 \leq \sigma^j \partial_j \hat{f}^j(p) \leq |(f^j)'(p^j)| \leq L, \quad |\partial_{u^L} G^\alpha(u^L, u^R)|, \ |\partial_{u^R} G^\alpha(u^L, u^R)| \leq L. \tag{20.5}$$

Therefore there is existence and uniqueness of a solution u to the scheme (20.1).
Step 2: monotonicity of the scheme when $\mathcal{G}$ is monotone
We have

$$u_k^{n+1,\alpha} = H_k^\alpha[u^n] := \begin{cases} H_k^\alpha(u_{k-1}^{n,\alpha}, u_k^{n,\alpha}, u_{k+1}^{n,\alpha}) & \text{if} \quad k \in J_{\mathbb{N}}^\alpha \setminus \{0\} \\ H_k^\alpha(u_k^n, u_{k+1}^{n,\alpha}) & \text{if} \quad k = 0 \qquad \text{and} \quad \sigma^\alpha = -1 \\ H_k^\alpha(u_{k-1}^{n,\alpha}, u_k^n) & \text{if} \quad k = 0 \qquad \text{and} \quad \sigma^\alpha = 1 \end{cases}$$

with

$$H_k^\alpha[u^n] := \begin{cases} u_k^{n,\alpha} + \dfrac{\Delta t}{\Delta x} \cdot \left\{ G^\alpha(u_{k-1}^{n,\alpha}, u_k^{n,\alpha}) - G^\alpha(u_k^{n,\alpha}, u_{k+1}^{n,\alpha}) \right\} & \text{if} \quad k \in J_{\mathbb{N}}^\alpha \setminus \{0\} \\ u_k^{n,\alpha} + \dfrac{\Delta t}{\Delta x} \cdot \left\{ \hat{f}^\alpha(u_k^n) - G^\alpha(u_k^{n,\alpha}, u_{k+1}^{n,\alpha}) \right\} & \text{if} \quad k = 0 \qquad \text{and} \quad \sigma^\alpha = -1 \\ u_k^{n,\alpha} + \dfrac{\Delta t}{\Delta x} \cdot \left\{ G^\alpha(u_{k-1}^{n,\alpha}, u_k^{n,\alpha}) - \hat{f}^\alpha(u_k^n) \right\} & \text{if} \quad k = 0 \qquad \text{and} \quad \sigma^\alpha = 1 \end{cases}$$

Under the CFL condition (20.3), and from estimates (20.5) and the monotonicities (20.4) on $\mathbb{R}^N$ and the natural monotonicities of the standard Godunov fluxes G^α's, we deduce that H_k^α is monotonous in its arguments.

Step 3: bounds in the box (even when $\mathcal{G}$ is not monotone)

Moreover, the flux $\hat{f}$ satisfies the second line of (2.14) which implies that $f, \hat{f}$ satisfy (18.27). Let us now show that $u_k^{n,\alpha}$ remains in the box $[a^\alpha, b^\alpha]$. Let us show for instance that $u_k^{n,\alpha} \le b^\alpha$. Formally, if the supremum of $(u_k^{n,\alpha})$'s on $k \in J_{\mathbb{N}}^\alpha$ is reached at some index k_0 for some time index n_0 with value $u_{k_0}^{n_0,\alpha} = b^\alpha$, then we get $(n,k) = (n_0, k_0)$

$$\frac{u_k^{n+1,\alpha} - u_k^{n,\alpha}}{\Delta t} = \begin{cases} \dfrac{G^\alpha(u_{k-1}^{n,\alpha}, b^\alpha) - G^\alpha(b^\alpha, u_{k+1}^{n,\alpha})}{\Delta x} \le 0 & \text{if} \quad k_0 \in J_{\mathbb{N}}^\alpha \setminus \{0\} \\[2ex] \dfrac{\hat{f}^\alpha(u_k^n) - G^\alpha(b^\alpha, u_{k+1}^{n,\alpha})}{\Delta x} \le 0 & \text{if} \quad k_0 = 0 \quad \text{and} \quad \sigma^\alpha = -1 \\[2ex] \dfrac{G^\alpha(u_{k-1}^{n,\alpha}, b^\alpha) - \hat{f}^\alpha(u_k^n)}{\Delta x} \le 0 & \text{if} \quad k_0 = 0 \quad \text{and} \quad \sigma^\alpha = 1 \end{cases}$$

where in the first line we have used the monotonicities of the Godunov flux $G^\alpha(\uparrow, \downarrow)$, while in the two last lines, we have also used the four last lines of (18.27). This shows formally that $u_k^{n,\alpha} \le b^\alpha$ for all times $n \ge 0$ and $k \in J_{\mathbb{N}}^\alpha$. As it is usual, this can be made rigorous, using for instance classical technics proving the comparison principle for viscosity solutions. This ends the proof of the lemma.

For later use, we will need the following.

Definition 20.2 (The approximate solution u^ε given by the fully discrete scheme)
We set $\tau := \Delta t > 0$, $\varepsilon := \Delta x > 0$ and $\tilde{\varepsilon} := (\tau, \varepsilon)$. Given a solution u of the semi-discretized scheme (19.2), we define $u^{\tilde{\varepsilon}} = (u^{\tilde{\varepsilon},1}, \ldots, u^{\tilde{\varepsilon},N})$ for $\alpha = 1, \ldots, N$

(20.6)

$$u^{\tilde{\varepsilon},\alpha}(t,x) := \sum_{n\in\mathbb{N},\ k\in J_R^\alpha} u_k^{n,\alpha}(t)\chi_k^{\tilde{\varepsilon},n,\alpha}(x) \quad \text{with} \quad \chi_k^{\tilde{\varepsilon},n,\alpha}(t,x) := \begin{cases} 1_{[n\tau,(n+1)\tau)}(t) \cdot 1_{[k\varepsilon,(k+1)\varepsilon)}(x) & \text{if} \quad \sigma^\alpha = -1 \\ 1_{[n\tau,(n+1)\tau)}(t) \cdot 1_{((k-1)\varepsilon,k\varepsilon]}(x) & \text{if} \quad \sigma^\alpha = 1 \end{cases}$$

Lemma 20.3 (Entropy inequalities for monotone Kružkov germs)
We work under assumptions of Lemma 20.1 and recall that $\sigma \in \{\pm\}^N$. Let u, v two solutions of the fully discretized scheme (20.1) respectively with initial data u^0, v^0. We assume that $\mathcal{G}$ is **monotone** *Kružkov. Then we have the following*

i) (Pointwise entropy inequality)

$$\frac{W_k^{n+1,\alpha} - W_k^{n,\alpha}}{\Delta t} + \frac{\Psi_{k+\frac12}^{n,\alpha} - \Psi_{k-\frac12}^{n,\alpha}}{\Delta x} \le 0 \quad \text{for all} \quad k \in J_{\mathbb{N}}^\alpha, \quad \alpha = 1, \ldots, N \tag{20.7}$$

with for $k \in \mathbb{Z}$, $n \in \mathbb{N}$

$$\begin{cases} W_k^{n,\alpha} := |u_k^{n,\alpha} - v_k^{n,\alpha}| \quad \text{for} \quad k \in J_{\mathbb{N}}^\alpha \\[2ex] \Psi_{k+\frac12}^{n,\alpha} := \begin{cases} \left.\begin{array}{ll} \Psi^\alpha(u_k^{n,\alpha}, u_{k+1}^{n,\alpha}; v_k^{n,\alpha}, v_{k+1}^{n,\alpha}) & \text{if} \quad k \in J_{\mathbb{N}}^\alpha \\ \Psi_0^\alpha(u_{k+1}^n, v_{k+1}^n) & \text{if} \quad k = -1 \end{array}\right| & \text{if} \quad \sigma^\alpha = -1 \\[2ex] \left.\begin{array}{ll} \Psi^\alpha(u_k^{n,\alpha}, u_{k+1}^{n,\alpha}; v_k^{n,\alpha}, v_{k+1}^{n,\alpha}) & \text{if} \quad k \in J_{\mathbb{N}}^\alpha \setminus \{0\} \\ \Psi_0^\alpha(u_k^n, v_k^n) & \text{if} \quad k = 0 \end{array}\right| & \text{if} \quad \sigma^\alpha = 1 \end{cases} \end{cases}$$

and

$$\begin{cases} \Psi^\alpha(x', y'; x, y) := G^\alpha(x' \vee x, y' \vee y) - G^\alpha(x' \wedge x, y' \wedge y) \\ \Psi_0^\alpha(z', z) = \hat{f}^\alpha(z' \vee z) - \hat{f}^\alpha(z' \wedge z) \\ \text{with} \quad (z' \vee z)^\alpha := z'^\alpha \vee z^\alpha \quad \text{and} \quad (z' \wedge z)^\alpha := z'^\alpha \wedge z^\alpha \end{cases}$$

ii) (Integral entropy inequality)
Let $u^{\tilde{\varepsilon}}, v^{\tilde{\varepsilon}}$ as in Definition 20.2. Define also

$$\Psi^{\tilde{\varepsilon},\alpha}(t,x) := \sum_{n\in\mathbb{N},\ k\in J_{\mathbb{N}}^\alpha} \Psi_{k+\frac12}^{n,\alpha}\chi_k^{\tilde{\varepsilon},n,\alpha}(t,x) \qquad \text{for} \quad \alpha = 1, \ldots, N \tag{20.8}$$

with $\chi^{\tilde{\varepsilon}}$ defined in (20.6). Let us consider a test function $0 \le \varphi_k^{n,\alpha}$ for $n \in \mathbb{N}$ for $k \in J_{\mathbb{N}}^\alpha$ which satisfies $\varphi_0^{n,\alpha} = \varphi_0^{n,\beta}$ for all α, β and $n \in \mathbb{N}$. For $\varphi^{\tilde{\varepsilon}}$ is defined from φ by the same formula defining $u^{\tilde{\varepsilon}}$ from u, and assuming that $\varphi^{\tilde{\varepsilon}}$ has compact support on $[0,+\infty) \times J$, we have the following integral entropy inequality (20.9)

$$\int_{\{0\}\times J} |u^{\tilde{\varepsilon}} - v^{\tilde{\varepsilon}}|\varphi^{\tilde{\varepsilon}} + \int_{(0,+\infty)\times J} \left\{ |u^{\tilde{\varepsilon}} - v^{\tilde{\varepsilon}}|(\cdot+\tau,\cdot) \cdot \left\{ \frac{\varphi^{\tilde{\varepsilon}}(\cdot+\tau,\cdot) - \varphi^{\tilde{\varepsilon}}}{\tau} \right\} - \sigma \Psi^{\tilde{\varepsilon}} \left\{ \frac{\varphi^{\tilde{\varepsilon}}(\cdot,\cdot-\sigma\varepsilon) - \varphi^{\tilde{\varepsilon}}}{\varepsilon} \right\} \right\} \ge 0$$

Proof of Lemma 20.3

Step 1: proof of i)

We first notice that for $x', x \in \mathbb{R}$, we have $|x'-x| := x' \vee x - x' \wedge x$. Because we assume that $\mathcal{G}$ is monotone, we know that the scheme is monotone. Hence

$$\left\{ \begin{array}{l} H_k^\alpha[u^n] \vee H_k^\alpha[v^n] \le H_k^\alpha[u^n \vee v^n] \\ H_k^\alpha[u^n] \wedge H_k^\alpha[v^n] \ge H_k^\alpha[u^n \wedge v^n] \\ \text{with} \quad (u^n \vee v^n)_k^\alpha := u_k^{n,\alpha} \vee v_k^{n,\alpha} \quad \text{and} \quad (u^n \wedge v^n)_k^\alpha := u_k^{n,\alpha} \wedge v_k^{n,\alpha} \end{array} \right.$$

Therefore, starting with the fully discretized scheme satisfied by u and v, and get (with obvious notation)

$$u^{n+1} \vee v^{n+1} - u^{n+1} \wedge v^{n+1} \le H[u^n \vee v^n] - H[u^n \wedge v^n]$$

Hence $W_k^{n,\alpha} := |u_k^{n,\alpha} - v_k^{n,\alpha}|$ satisfies

$$\frac{W_k^{n+1,\alpha} - W_k^{n,\alpha}}{\Delta t} + \frac{\Psi_{k+\frac12}^{n,\alpha} - \Psi_{k-\frac12}^{n,\alpha}}{\Delta x} \le 0 \quad \text{for all} \quad k \in J_{\mathbb{N}}^\alpha, \quad \alpha = 1, \dots, N$$

with $\Psi_{k\pm\frac12}^{n,\alpha}$ as in the statement of i) of Lemma 20.3. This shows (20.7).

Step 2: proof of ii)

Now, in order to simplify the presentation, let us assume that $\sigma^\alpha = -1$, say for all α (which can be done using a suitable change of variables like in Lemma 18.1). Consider any test function $0 \le \varphi_k^{n,\alpha}$. Multiplying (20.7) by $\varphi_k^{n,\alpha}$ for $k \in J_{\mathbb{N}}^\alpha$, and summing over k and integrating in time over $n \in \mathbb{N}$, we get for $\varphi^{\tilde{\varepsilon}}$ with compact support in $[0,+\infty) \times J$

$$0 \ge \varepsilon\tau \sum_{n\in\mathbb{N},\ k \in J_{\mathbb{N}}^\alpha} \left\{ \left\{ \frac{W_k^{n+1,\alpha} - W_k^{n,\alpha}}{\tau} \right\} \varphi_k^{n,\alpha} + \left\{ \frac{\Psi_{k+\frac12}^{n,\alpha} - \Psi_{k-\frac12}^{n,\alpha}}{\varepsilon} \right\} \cdot \varphi_k^{n,\alpha} \right\}$$

$$= \left\{ \begin{array}{l} \varepsilon \sum_{k\in J_{\mathbb{N}}^\alpha} \left\{ -W_k^{0,\alpha}\varphi_k^{0,\alpha} + \sum_{n\in\mathbb{N}} W_k^{n+1,\alpha}\varphi_k^{n,\alpha} - \sum_{n\in\mathbb{N}\setminus\{0\}} W_k^{n,\alpha}\varphi_k^{n,\alpha} \right\} \\ \\ \tau \sum_{n\in\mathbb{N}} \left\{ -\Psi_0^\alpha(u_0^n, v_0^n)\varphi_0^{n,\alpha} + \sum_{k\in J_{\mathbb{N}}^\alpha} \Psi_{k+\frac12}^{n,\alpha}\varphi_k^{n,\alpha} - \sum_{k\in J_{\mathbb{N}}^\alpha\setminus\{0\}} \Psi_{k-\frac12}^{n,\alpha}\varphi_k^{n,\alpha} \right\} \end{array} \right\}$$

$$= \left\{ \begin{array}{l} \varepsilon \sum_{k\in J_{\mathbb{N}}^\alpha} \left\{ -W_k^{0,\alpha}\varphi_k^{0,\alpha} - \sum_{n\in\mathbb{N}} W_k^{n+1,\alpha} \left\{ \varphi_k^{n+1,\alpha} - \varphi_k^{n,\alpha} \right\} \right\} \\ \\ \tau \sum_{n\in\mathbb{N}} \left\{ -\Psi_0^\alpha(u_0^n, v_0^n)\varphi_0^{n,\alpha} + \sum_{k\in J_{\mathbb{N}}^\alpha} \Psi_{k+\frac12}^{n,\alpha} \left\{ \varphi_k^{n,\alpha} - \varphi_{k+1}^{n,\alpha} \right\} \right\} \end{array} \right\}$$

$$= \left\{ \begin{array}{l} -\int_{\{0\}\times J^\alpha} W^{\tilde{\varepsilon},\alpha}\varphi^{\tilde{\varepsilon},\alpha} - \int_{(0,+\infty)\times J^\alpha} W^{\tilde{\varepsilon},\alpha}(\cdot+\tau,\cdot) \cdot \left\{ \frac{\varphi^{\tilde{\varepsilon},\alpha}(\cdot+\tau,\cdot) - \varphi^{\tilde{\varepsilon},\alpha}}{\tau} \right\} \\ \\ -\int_{(0,+\infty)\times\{0\}} \Psi_0^\alpha(u^{\tilde{\varepsilon}}, v^{\tilde{\varepsilon}})\varphi^{\tilde{\varepsilon},\alpha} - \int_{(0,+\infty)\times J^\alpha} \Psi^{\tilde{\varepsilon},\alpha} \left\{ \frac{\varphi^{\tilde{\varepsilon},\alpha}(\cdot,\cdot+\varepsilon) - \varphi^{\tilde{\varepsilon},\alpha}}{\varepsilon} \right\} \end{array} \right\}$$

i.e.
(20.10)
$$\int_{\{0\}\times J^\alpha} |u^{\tilde\varepsilon}-v^{\tilde\varepsilon}|\varphi^{\tilde\varepsilon} + \int_{(0,+\infty)\times J^\alpha} \left\{ |u^{\tilde\varepsilon}-v^{\tilde\varepsilon}|(\cdot+\tau,\cdot)\cdot \left\{ \frac{\varphi^{\tilde\varepsilon}(\cdot+\tau,\cdot)-\varphi^{\tilde\varepsilon}}{\tau} \right\} + \Psi^{\tilde\varepsilon} \left\{ \frac{\varphi^{\tilde\varepsilon}(\cdot,\cdot+\varepsilon)-\varphi^{\tilde\varepsilon}}{\varepsilon} \right\} \right\}$$

$$\geq -\int_{(0,+\infty)\times\{0\}} \Psi_0^\alpha(u^{\tilde\varepsilon},v^{\tilde\varepsilon})\varphi^{\tilde\varepsilon}$$

Summing over α and using $\varphi^{\tilde\varepsilon}\geq 0$, we get

$$\int_{\{0\}\times J} |u^{\tilde\varepsilon}-v^{\tilde\varepsilon}|\varphi^{\tilde\varepsilon} + \int_{(0,+\infty)\times J} \left\{ |u^{\tilde\varepsilon}-v^{\tilde\varepsilon}|(\cdot+\tau,\cdot)\cdot \left\{ \frac{\varphi^{\tilde\varepsilon}(\cdot+\tau,\cdot)-\varphi^{\tilde\varepsilon}}{\tau} \right\} + \Psi^{\tilde\varepsilon} \left\{ \frac{\varphi^{\tilde\varepsilon}(\cdot,\cdot+\varepsilon)-\varphi^{\tilde\varepsilon}}{\varepsilon} \right\} \right\}$$

$$\geq \int_{(0,+\infty)\times\{0\}} D^{\hat f}(u^{\tilde\varepsilon}\vee v^{\tilde\varepsilon}, u^{\tilde\varepsilon}\wedge v^{\tilde\varepsilon})\varphi^{\tilde\varepsilon}$$

with

$$D^{\hat f}(p,q) := \sum_{\alpha=1}^N \sigma^\alpha \ \psi^{\hat f^\alpha}(p,q) \quad \text{and} \quad \psi^{\hat f^\alpha}(p,q) := \operatorname{sign}(p^\alpha-q^\alpha)\cdot\left\{\hat f^\alpha(p)-\hat f^\alpha(q)\right\}$$

Because $\mathcal{G}$ is a Kružkov germ with respect to (J,f), we deduce that $D^{\hat f}\geq 0$. Joint to the fact that $\varphi^{\tilde\varepsilon}\geq 0$, we see that (20.10) implies

$$\int_{\{0\}\times J} |u^{\tilde\varepsilon}-v^{\tilde\varepsilon}|\varphi^{\tilde\varepsilon} + \int_{(0,+\infty)\times J} \left\{ |u^{\tilde\varepsilon}-v^{\tilde\varepsilon}|(\cdot+\tau,\cdot)\cdot \left\{ \frac{\varphi^{\tilde\varepsilon}(\cdot+\tau,\cdot)-\varphi^{\tilde\varepsilon}}{\tau} \right\} + \Psi^{\tilde\varepsilon} \left\{ \frac{\varphi^{\tilde\varepsilon}(\cdot,\cdot+\varepsilon)-\varphi^{\tilde\varepsilon}}{\varepsilon} \right\} \right\} \geq 0$$

More generally, for arbitrary signs $\sigma^\alpha\in\{\pm1\}$, we get (20.9). This ends the proof of the lemma.

Corollary 20.4 (Contraction estimates)
We work under assumptions of Lemma 20.1 with $\mathcal{G}$ monotone Kružkov. Let u,v be two solutions of the fully discretized scheme (20.1). We use notation of Definition 20.2 for $u^{\tilde\varepsilon},v^{\tilde\varepsilon}$. Then for all $t>0$, we have

$$\int_{\{t\}\times J} |u^{\tilde\varepsilon}-v^{\tilde\varepsilon}| \leq \int_{\{0\}\times J} |u^{\tilde\varepsilon}-v^{\tilde\varepsilon}| \tag{20.11}$$

where the right hand side may be infinite.
We also have

$$\int_{\{t\}\times J} \left| \frac{u^{\tilde\varepsilon}(\cdot+\tau,\cdot)-u^{\tilde\varepsilon}}{\varepsilon} \right| \leq \int_{\{0\}\times J} \left| \frac{u^{\tilde\varepsilon}(\cdot+\tau,\cdot)-u^{\tilde\varepsilon}}{\varepsilon} \right| \tag{20.12}$$

Proof of Corollary 20.4
On the one hand, inequality (20.11) follows from the integral entropy inequality (20.9) for $\varphi^{\tilde\varepsilon}\equiv 1$. On the other hand, inequality (20.12) follows from (20.11) choosing $v^\varepsilon = u^\varepsilon(\cdot+\tau,\cdot)$, dividing by τ. This ends the proof of the lemma.

Lemma 20.5 (BLN estimates)
We work under assumptions of Lemma 20.1. Given a solution u of the fully discretized scheme (20.1), we consider $u^{\tilde\varepsilon}$ as in Definition 20.2. Fix α such that $\sigma^\alpha=-1$. Let $k_i\in J^\alpha_{\mathbb{N}}$ and $x_i=k_i\varepsilon$ for $i=1,2$ with $x_1<x_2$ and $k_i\geq 2$. Then we have boundary BLN estimate for $t\in\tau\mathbb{N}\backslash\{0,1\}$
(20.13)
$$\int_{\{t\}\times(x_1,x_2)} \left| \frac{u^{\tilde\varepsilon,\alpha}(\cdot,\cdot+\varepsilon)-u^{\tilde\varepsilon,\alpha}}{\varepsilon} \right| - \int_{\{0\}\times(x_1,x_2)} \left| \frac{u^{\tilde\varepsilon,\alpha}(\cdot,\cdot+\varepsilon)-u^{\tilde\varepsilon,\alpha}}{\varepsilon} \right| \leq \sum_{i=1,2} \varepsilon^{-1} \int_{(0,t)\times(x_i,x_i+\varepsilon)} \left| \frac{u^{\tilde\varepsilon,\alpha}(\cdot+\tau,\cdot)-u^{\tilde\varepsilon,\alpha}}{\tau} \right|$$

Moreover for $x_1=2\delta$ with $\delta\in\varepsilon(\mathbb{N}\backslash\{0,1\})$, we have the interior BLN estimate for $t\in\tau\mathbb{N}\backslash\{0,1\}$
(20.14)
$$\int_{\{t\}\times(2\delta,+\infty)} \left| \frac{u^{\tilde\varepsilon,\alpha}(\cdot,\cdot+\varepsilon)-u^{\tilde\varepsilon,\alpha}}{\varepsilon} \right| \leq \int_{\{0\}\times(\delta,+\infty)} \left| \frac{u^{\tilde\varepsilon,\alpha}(\cdot,\cdot+\varepsilon)-u^{\tilde\varepsilon,\alpha}}{\varepsilon} \right| + \delta^{-1} \int_{(0,t)\times(\delta,2\delta)} \left| \frac{u^{\tilde\varepsilon,\alpha}(\cdot+\tau,\cdot)-u^{\tilde\varepsilon,\alpha}}{\tau} \right|$$

Proof of Lemma 20.5
Recall that $\sigma^\alpha = -1$, and define $v^{\tilde{\varepsilon},\alpha}(t,x) := u^{\tilde{\varepsilon},\alpha}(t,x+\varepsilon)$. For $i=1,2$, assume $x_i = k_i\varepsilon \in J^\alpha \simeq (0,+\infty)$ for $i=1,2$ with $k_i \in J^\alpha_{\mathbb{N}} \backslash \{0\}$ with $k_i \geq 2$ and $x_1 < x_2$. For $t \in \tau\mathbb{N}\backslash\{0,1\}$, consider the (limit) test function $\varphi^{\tilde{\varepsilon},\alpha}(s,x) = 1_{[0,t)}(s)\cdot 1_{[x_1,x_2)}(x)$, with $\varphi^{\tilde{\varepsilon},\beta} = 0$ for $\beta \neq \alpha$. Plugging this test function in the integral entropy inequality (20.9), we get

$$\int_{\{0\}\times(x_1,x_2)} |u^{\tilde{\varepsilon},\alpha} - v^{\tilde{\varepsilon},\alpha}| - \int_{\{t\}\times(x_1,x_2)} |u^{\tilde{\varepsilon},\alpha} - v^{\tilde{\varepsilon},\alpha}| + \varepsilon^{-1} \sum_{i=1,2} \int_{(0,t)\times(x_i-\varepsilon,x_i)} -(-1)^i \Psi^{\tilde{\varepsilon},\alpha} \geq 0$$

Recall that $\Psi^{n,\alpha}_{k+\frac{1}{2}} := \Psi^\alpha(v^{n,\alpha}_k, v^{n,\alpha}_{k+1}; u^{n,\alpha}_k, u^{n,\alpha}_{k+1}) = \Psi^\alpha(u^{n,\alpha}_{k+1}, u^{n,\alpha}_{k+2}; u^{n,\alpha}_k, u^{n,\alpha}_{k+1})$. From (19.4), it satisfies

$$|\Psi^{n,\alpha}_{k+\frac{1}{2}}| \leq |G^\alpha(u^{n,\alpha}_{k+1}, u^{n,\alpha}_{k+2}) - G^\alpha(u^{n,\alpha}_k, u^{n,\alpha}_{k+1})| = \varepsilon \left| \frac{u^{n+1,\alpha}_{k+1} - u^{n,\alpha}_{k+1}}{\tau} \right|$$

Hence for $t = n\tau$

$$\begin{aligned}
\int_{\{t\}\times(x_1,x_2)} |u^{\tilde{\varepsilon},\alpha} - v^{\tilde{\varepsilon},\alpha}| - \int_{\{0\}\times(x_1,x_2)} |u^{\tilde{\varepsilon},\alpha} - v^{\tilde{\varepsilon},\alpha}| \quad &\leq \sum_{i=1,2} \tau \sum_{m=0,\dots,n-1} |\Psi^{m,\alpha}_{k_i-\frac{1}{2}}| \\
&\leq \varepsilon \sum_{i=1,2} \tau \sum_{m=0,\dots,n-1} \left| \frac{u^{m+1,\alpha}_{k_i} - u^{m,\alpha}_{k_i}}{\tau} \right| \\
&= \sum_{i=1,2} \int_{(0,t)\times(x_i,x_i+\varepsilon)} \left| \frac{u^{\tilde{\varepsilon},\alpha}(\cdot+\tau,\cdot) - u^{\tilde{\varepsilon},\alpha}}{\tau} \right|
\end{aligned}$$

Hence we get

$$\int_{\{t\}\times(x_1,x_2)} \left| \frac{u^{\tilde{\varepsilon},\alpha}(\cdot,\cdot+\varepsilon) - u^{\tilde{\varepsilon},\alpha}}{\varepsilon} \right| - \int_{\{0\}\times(x_1,x_2)} \left| \frac{u^{\tilde{\varepsilon},\alpha}(\cdot,\cdot+\varepsilon) - u^{\tilde{\varepsilon},\alpha}}{\varepsilon} \right| \leq \varepsilon^{-1} \sum_{i=1,2} \int_{(0,t)\times(x_i,x_i+\varepsilon)} \left| \frac{u^{\tilde{\varepsilon},\alpha}(\cdot+\tau,\cdot) - u^{\tilde{\varepsilon},\alpha}}{\tau} \right|$$

which is (20.13).

In particular, for $x_1 = h$ and $x_2 = +\infty$, we get that (20.13) means

$$\begin{aligned}
&\int_{\{t\}\times(h,+\infty)} \left| \frac{u^{\varepsilon,\alpha}(\cdot,\cdot+\varepsilon) - u^{\varepsilon,\alpha}}{\varepsilon} \right| \\
&\leq \int_{\{0\}\times(h,+\infty)} \left| \frac{u^{\varepsilon,\alpha}(\cdot,\cdot+\varepsilon) - u^{\varepsilon,\alpha}}{\varepsilon} \right| + \varepsilon^{-1} \int_{(0,t)\times(h,h+\varepsilon)} \left| \frac{u^{\tilde{\varepsilon},\alpha}(\cdot+\tau,\cdot) - u^{\tilde{\varepsilon},\alpha}}{\tau} \right|
\end{aligned}$$

Now for $\Delta \in \mathbb{N}\backslash\{0,1\}$ and $\delta := \varepsilon\Delta$, summing over $\varepsilon^{-1}h = \Delta, \dots, 2\Delta - 1$, we get

$$\begin{aligned}
&\Delta \int_{\{t\}\times(2\delta,+\infty)} \left| \frac{u^{\varepsilon,\alpha}(\cdot,\cdot+\varepsilon) - u^{\varepsilon,\alpha}}{\varepsilon} \right| \\
&\leq \Delta \int_{\{0\}\times(\delta,+\infty)} \left| \frac{u^{\varepsilon,\alpha}(\cdot,\cdot+\varepsilon) - u^{\varepsilon,\alpha}}{\varepsilon} \right| + \varepsilon^{-1} \int_{(0,t)\times(\delta,2\delta)} \left| \frac{u^{\tilde{\varepsilon},\alpha}(\cdot+\tau,\cdot) - u^{\tilde{\varepsilon},\alpha}}{\tau} \right|
\end{aligned}$$

which implies (20.14). This ends the proof of the lemma.

20.3 Convergence

Then we have the following result.

Proposition 20.6 (Convergence of the fully discretized solution for monotone Kružkov germs)
Assume (2.2) with $N \geq 1$, nondegeneracy condition (2.17), and let $\mathcal{G} \subset [a,b]$ be a **monotone** *Kružkov germ with bounded $[a,b)$. We assume that the junction is characterized by (J,f) with $\sigma \in \{\pm 1\}^N$ and associated flux $\hat{f} := \hat{f}_{\mathcal{G}}$.*

Assume $\bar U_0 = (\bar U_0^1, \dots, \bar U_0^N)$ with $\bar U_0^\alpha \in (BV \cap L^1)(J^\alpha; [a^\alpha, b^\alpha])$. Define for $\alpha = 1, \dots, N$

$$u_k^{0,\alpha} := \varepsilon^{-1} \int_{J^\alpha} \bar U_0^\alpha \chi_k^{\varepsilon,\alpha} \quad \text{with } \chi_k^{\varepsilon,\alpha} \text{ given in Definition 20.2}$$

Then the initial data u^0 satisfies (20.2). Assume that $\tau := \Delta t > 0$ and $\varepsilon := \Delta x > 0$ satisfy the CFL condition (20.3), and let $\tilde\varepsilon := (\tau, \varepsilon)$. Let u be the unique solution $u : \mathbb{N} \times J_{\mathbb{N}} \to \mathbb{R}$ of the scheme (20.1) with initial data u^0. Let $u^{\tilde\varepsilon} : [0, +\infty) \times J \to \mathbb{R}$ given by Definition 20.2.

Then we have

$$u^{\tilde\varepsilon} \to \bar U \quad \text{in} \quad L^1_{loc}([0, +\infty) \times J) \quad \text{as} \quad \tilde\varepsilon \to (0, 0)$$

where $\bar U$ is the unique solution of (2.52) and with initial data $\bar U(0, \cdot) = \bar U_0$.

Remark 20.7 (When the scheme relaxes the boundary condition)
Notice that, as in Remark 19.8, we can claim that Proposition 20.6 is still true when the flux $\hat f_{\mathcal{G}}$ is replaced by a function $\hat f$ satisfying (7.1). Then setting $f_1 := \mathfrak{R} \hat f$ where $\mathfrak{R}$ is defined in (7.3), there exists a Riemann germ $\mathcal{G}_1$ such that $f_1 = \hat f_{\mathcal{G}_1}$. Again, with the same argument as in Remark 19.8, we see that the trace of the limit of $u^{\tilde\varepsilon}$ still belongs to the germ $\mathcal{G}_1$, i.e. the result holds with $\mathcal{G} := \mathcal{G}_1$.

Proof of Proposition 20.6
Recall that $u^{\tilde\varepsilon}(t, x)$ belongs to the box $[a, b]$ which gives an L^∞ estimate, and we have both contraction estimate (20.12) on $\dfrac{u^{\tilde\varepsilon}(\cdot + \tau, \cdot) - u^{\tilde\varepsilon}}{\tau}$ which approximates the t-derivative of $u^{\tilde\varepsilon}$, and interior BLN estimate (20.14) on $\dfrac{u^{\tilde\varepsilon}(\cdot, \cdot + \varepsilon) - u^{\tilde\varepsilon}}{\varepsilon}$, which approximates the x-derivative of $u^{\tilde\varepsilon}$. This shows that we have uniform bounds on $u^{\tilde\varepsilon}$ in $BV_{loc;t,x}$. Again, the compactness of $BV_{loc} \subset L^1_{loc}$ for bounded functions implies the convergence of $u^{\tilde\varepsilon}$ in L^1_{loc} to some function $\bar U$, up to extraction of a subsequence. The remaining part of the proof is similar to the one of the proof of Proposition 19.7. This ends the proof of the proposition.

21 Appendix of Part III

21.1 Tool box: some standard parabolic estimates

In this subsection, we recall some standard parabolic estimates. We consider the following problem for $T > 0$

$$(21.1) \qquad \begin{cases} v_t - v_{xx} = f & \text{on} \quad (0, T) \times I_R =: Q_{T,R}, \\ v = \varphi & \text{on} \quad (0, T) \times \partial I_R \\ v = v_0 & \text{on} \quad \{0\} \times \bar I_R \end{cases}$$

where for $R > 0$, we assume either that

$$I_R := (0, R) \quad \text{with} \quad \partial I_R = \{0, R\} \quad \text{and} \quad \bar I_R = [0, R]$$

or

$$I_R := \mathbb{R}/R\mathbb{Z} \quad \text{with} \quad \partial I_R = \emptyset \quad \text{and} \quad \bar I_R = I_R$$

In the following parabolic estimates, when it will be useful, we will make explicit the dependence of the constants in terms of T, R. We have the following results.

Lemma 21.1 (Calderon-Zygmund estimates)
Assume $T > 0$ and $I_R = \mathbb{R}/R\mathbb{Z}$ for some $R > 0$ and let $p \in (1, +\infty)$. Let v be a solution of (21.1) with $v_0 = 0$. Then we have

$$|v_t, v_{xx}|_{p, Q_{T,R}} \le c |v_t - v_{xx}|_{p, Q_{T,R}} \quad \text{with} \quad |\cdot|_{p, Q_{T,R}} := |\cdot|_{L^p(Q_{T,R})}$$

for $c = c(p)$ independent on T, R.

Proof of Lemma 21.1
First notice that the solution v can be extended by periodicity to the whole line $\mathbb{R}$. We first rescale the problem from (T, R) to $(1, R/\sqrt{T})$. Then we can consider $\tilde v_n = v \phi_n$ with $\phi_n = 1$ on $[0, n]$ and $\phi_n = 0$ on $\mathbb{R} \backslash [-1, (n+1)]$. Applying standard Calderon-Zygmund estimate to $\tilde v_n$ (see Theorem 9.1 in [36]), and taking the limit $n \to +\infty$, we recover the result with some constant c only depending on p. Scaling back the problem to (T, R), we get the result. This ends the proof of the lemma.

Lemma 21.2 (Precised Calderon-Zygmund estimates)
Assume $T > 0$ and $I_R = (0, R)$ for some $R > 0$ and let $p \in (1, +\infty)$. Let v be a solution of (21.1) with $v_0 = 0 = \varphi$. Then we have

$$\frac{|v|_{p,Q_{T,R}}}{T} + \frac{|v_x|_{p,Q_{T,R}}}{\sqrt{T}} + |v_t, v_{xx}|_{p,Q_{T,R}} \le c|v_t - v_{xx}|_{p,Q_{T,R}} \tag{21.2}$$

for $c = c(p)$ independent on T, R.

Proof of Lemma 21.2
Up to extend v by antisymmetry $v(t, -x) = -v(t, x)$, and then by $2R$-periodicity in x, we come back to problem covered by Lemma 21.1 with R replaced by $2R$. Now from Lemma 4.5 in [36] (on page 305), we have

$$|v|_{p,Q_{T,R}} \le \tilde{c}T|v_t, v_{xx}|_{p,Q_{T,R}}$$

and

$$|v_x|_{p,Q_{T,R}} \le \tilde{c}\sqrt{T}|v_t, v_{xx}|_{p,Q_{T,R}}$$

which implies (21.2) from Lemma 21.1. Notice that (21.2) also follows directly from the standard parabolic $W^{2,1;p}_{x,t}$ estimate, and by scaling in time. This ends the proof of the lemma.

We also have the following result.

Lemma 21.3 (Sobolev embedding in Hölder spaces)
Let $p > 3$ and $\alpha := 1 - 3/p \in (0, 1)$. For any $v \in W^{2,1;p}_{x,t}(Q_{T,R})$ with $I_R := (0, R)$. Then $v \in C^{1+\alpha,\frac{1+\alpha}{2}}_{x,t}(\bar{Q}_{T,R})$ with $v_x \in C^{\alpha,\frac{\alpha}{2}}_{x,t}(\bar{Q}_{T,R})$. Moreover we have

$$\frac{|v_x|_{\infty,Q_{T,R}}}{T^{\frac{\alpha}{2}}} + \langle v_x\rangle^{(\alpha)}_{Q_{T,R}} \le c\left\{|v_t, v_{xx}|_{p,Q_{T,R}} + \frac{|v|_{p,Q_{T,R}}}{T}\right\} \quad \text{with} \quad \left\{\begin{array}{l} |\cdot|_{\infty,Q_{T,R}} := |\cdot|_{L^\infty(Q_{T,R})} \\ \langle\cdot\rangle^{(\alpha)}_{Q_{T,R}} := [\cdot]_{C^{\alpha,\frac{\alpha}{2}}_{x,t}(Q_{T,R})} \end{array}\right. \tag{21.3}$$

with $c = c(p)$ independent on $T, R > 0$, if T/R^2 is small enough. Here $\langle\cdot\rangle^{(\alpha)}$ is the standard parabolic Hölder semi-norm of parameter $\alpha \in (0, 1)$.

Proof of Lemma 21.3
Up to a scaling argument, we can change (T, R) into $(T', R') := (T/R^2, 1)$. Then Lemma 3.3 in [36] implies that there exists $c = c(p)$ such that

$$|v_x|_{\infty,Q_{T',R'}} \le \frac{1}{2}c\left\{\delta^\alpha|v_t, v_{xx}|_{p,Q_{T',R'}} + \delta^{\alpha-2}|v|_{p,Q_{T',R'}}\right\}$$

and

$$\langle v_x\rangle^{(\alpha)}_{Q_{T',R'}} \le \frac{1}{2}c\left\{|v_t, v_{xx}|_{p,Q_{T',R'}} + \delta^{-2}|v|_{p,Q_{T',R'}}\right\}$$

with $\delta := \min\left\{\frac{1}{2}, \sqrt{T}'\right\}$, $R' = 1$ and $c = c(p)$ independent on T', R'. Now for T' small enough, we get

$$\frac{|v_x|_{\infty,Q_{T',R'}}}{T'^{\frac{\alpha}{2}}} + \langle v_x\rangle^{(\alpha)}_{Q_{T',R'}} \le c\left\{|v_t, v_{xx}|_{p,Q_{T',R'}} + \frac{|v|_{p,Q_{T',R'}}}{T'}\right\}$$

Then scaling back to (T, R), this gives (21.3). This ends the proof of the lemma.

Corollary 21.4 (Sobolev parabolic estimates)
Assume $T > 0$ and $I_R = (0, R)$ for some $R > 0$ and let $p \in (3, +\infty)$ with $\alpha := 1 - 3/p \in (0, 1)$. Let v be a solution of (21.1) with $v_0 = 0 = \varphi$. Then, with notation of Lemma 21.3, we have

$$\frac{|v_x|_{p,Q_{T,R}}}{\sqrt{T}} + \frac{|v_x|_{\infty,Q_{T,R}}}{T^{\frac{\alpha}{2}}} + \langle v_x\rangle^{(\alpha)}_{Q_{T,R}} + |v_{xx}|_{p,Q_{T,R}} \le c|v_t - v_{xx}|_{p,Q_{T,R}} \tag{21.4}$$

with $c = c(p)$ independent on $T, R > 0$, if T/R^2 is small enough.

Proof of Corollary 21.4
The result follows from Lemmata 21.3 and 21.2. This ends the proof of the corollary.

Lemma 21.5 (Schauder parabolic estimates)
Assume $T > 0$ and $I_R = (0, R)$ for some $R > 0$ and let $\alpha \in (0,1)$. Let v be a solution of (21.1) with $v_0 = 0 = \varphi$. Assume that

$$f = 0 \quad on \quad \{0\} \times \partial I_R \tag{21.5}$$

If $f \in C^{\alpha,\frac{\alpha}{2}}_{x,t}(\bar{Q}_{T,R})$, then $v \in C^{2+\alpha,1+\frac{\alpha}{2}}_{x,t}(\bar{Q}_{T,R})$.
If $f \in C^{1+\alpha,\frac{1+\alpha}{2}}_{x,t}(\bar{Q}_{T,R})$, then $v \in C^{3+\alpha,1+\frac{1+\alpha}{2}}_{x,t}(\bar{Q}_{T,R})$.

Proof of Lemma 21.5
The result follows from Theorem 5.2 in [36] (on page 320). Notice that on the singular boundary $\Gamma_0 := \{0\} \times \partial I_R$, the two compatibility conditions (of order zero and of order one) are satisfied. Recall that the conditions are the following

$$\left\{ \begin{array}{rll} v_0 &= \varphi \quad \text{on} \quad \Gamma_0, & \textbf{(order zero)} \\ \varphi_t - (v_0)_{xx} &= f \quad \text{on} \quad \Gamma_0, & \textbf{(order one)} \end{array} \right.$$

The first condition is satisfied because $v_0 = 0 = \varphi$, and the second condition is also satisfied because of (21.5). This ends the proof of the lemma.

21.2 Existence of a PDE solution by fixed point

The goal of this subsection is to find a fixed point solution to a semi-linear problem (21.13) on

$$Q_{T,R} := (0,T) \times (0,R) \quad \text{for} \quad T, R > 0.$$

As it is usual, we first start with a linear problem and insists on a priori estimates in short time, with sufficiently explicit dependence of the constants.

Precisely, we consider the following scalar linear problem

$$\left\{ \begin{array}{rll} u_t - u_{xx} &= h_x & \text{on} \quad (0,T) \times (0,R) \\ -u_x &= g_0(t) & \text{on} \quad (0,T) \times \{0\} \\ -u_x &= g_R(t) & \text{on} \quad (0,T) \times \{R\} \\ u &= u_0 & \text{on} \quad \{0\} \times (0,R) \end{array} \right. \tag{21.6}$$

with

$$\left\{ \begin{array}{rll} h &= 0 & \text{on} \quad (0,T) \times \{0\} \\ -u_0'(0) &= g_0(0) & \\ -u_0'(R) &= g_R(0) & \end{array} \right. \tag{21.7}$$

For problem (21.6), we will need several types of parabolic estimates on $Q_{T,R}$. To apply later on a fixed point method (with some contraction map say in $L^\infty(Q_{T,R})$), we will need estimates i) with explicit dependence in T, with constants small when T is small.

For later use, we will need to get controlled estimates as $R \to +\infty$. Hence for fixed $\rho > 0$, we will need local in space estimates on $Q_{T,\rho} := (0,T) \times (0,\rho)$ with constants depending only on T, ρ, but not on R.
For estimates ii) on the solution u in $C^{\frac{\alpha}{2},\alpha}_{t,x}(\overline{Q}_{T,\rho})$ and on u_x in $L^p(Q_{T,\rho})$, we will take advantage of a priori bounds in L^∞ on the fixed point solution, and then of u, h, g_0, g_R.
For estimates iii) on the solution u in $W^{1,2;p}_{t,x}(Q_{T,\rho})$, we will take advantage of a priori L^p estimates on h_x.

Lemma 21.6 (A priori parabolic estimates)
Assume that u solves (21.6) with the first line of (21.7). Let $p \in (3, +\infty)$ and $\alpha := 1 - 3/p \in (0,1)$. Assume that $u_0' \in L^p(0,R)$.
i) (Global space estimates on $Q_{T,R}$)

Then there exists a constant $c = c(p)$ independent on T, R, such that for T/R^2 small enough, we have

$$(21.8) \qquad \frac{|U|_p}{\sqrt{T}} + \frac{|U|_\infty}{T^{\frac{\alpha}{2}}} + \langle U\rangle^{(\alpha)}_{Q_{T,R}} + |U_x|_p \le c|W|_p \quad \textit{with} \quad \left\{ \begin{array}{l} W := (u_0', h, g_0, g_R, h(\cdot, R)) \\ \\ |\cdot|_p := |\cdot|_{L^p(Q_{T,R})} \\ \\ \langle \cdot \rangle^{(\alpha)}_{Q_{T,R}} := [\cdot]_{C^{\alpha,\frac{\alpha}{2}}_{x,t}(\overline{Q}_{T,R})} \end{array} \right.$$

and with

$$U := u - u_0 - R^{-1}G_R(t) \quad \textit{where} \quad G_R'(t) := g_0(t) - g_R(t) + h(t,R) \quad \textit{with} \quad G_R(0) = 0$$

ii) (Higher regularity)
Assume (21.7). Let $g := (g_0, g_R)$.
Moreover, if $u_0 \in C^{1+\alpha}_x([0,R])$ and $h, g \in C^{\alpha,\frac{\alpha}{2}}_{x,t}(\bar{Q}_{T,R})$, then $u \in C^{1+\alpha,\frac{1+\alpha}{2}}_{x,t}(\bar{Q}_{T,R})$.
iii) (Further regularity)
Assume furthermore that $u_0 \in C^{2+\alpha}_x([0,R])$ and $h, g \in C^{1+\alpha,\frac{1+\alpha}{2}}_{x,t}(\bar{Q}_{T,R})$; then $u \in C^{2+\alpha,\frac{2+\alpha}{2}}_{x,t}(\bar{Q}_{T,R})$.

Proof of Lemma 21.6
Step 0: preliminaries
We first symmetrize u in x and set (with the same notation) $u(t,x) := u(t,-x)$ for $x \in (-R,0)$, and also symmetrize similarly $u_0(x) = u_0(-x)$. From the first line of (21.7), we have $h(t,0) = 0$, and we can antisymmetrize h, setting $\check{h}(t,x) := -h(t,-x)$ for $x \in (-R,0)$ such that $\check{h}_x(t,x) = (h_x)(t,-x)$. We also set $g_{-R} := -g_R$. Then u satisfies

$$\left\{ \begin{array}{rll} u_t - u_{xx} & = 2g_0(t)\delta_0(x) + \check{h}_x & \text{on} \quad (0,T) \times (-R,R) \\ -u_x & = g_{\pm R}(t) & \text{on} \quad (0,T) \times \{\pm R\} \\ u & = u_0 & \text{on} \quad \{0\} \times (-R,R) \end{array} \right.$$

Now we set $u = v_x$ and $u_0 = (v_0)_x$ with $v_0(0) = 0$, and setting $v_0(x) = -v_0(-x)$, we consider $v(t,x) = -v(t,-x)$ solution of

$$(21.9) \qquad \left\{ \begin{array}{rll} v_t - v_{xx} & = g_0(t)\text{sign}(x) + \check{h} & \text{on} \quad (0,T) \times (-R,R) \\ v & = 0 & \text{on} \quad (0,T) \times \{0\} \\ v_t & = g_0(t)\text{sign}(x) + \check{h} - g_{\pm R}(t) & \text{on} \quad (0,T) \times \{\pm R\} \\ v & = v_0 & \text{on} \quad \{0\} \times (-R,R) \end{array} \right.$$

where the boundary condition in the third line comes from condition on $u_x = v_{xx}$, using the PDE satisfied by v up to the boundary (for instance in the case where all functions would be smooth, to fix the ideas). Setting (with $\check{h}(t,R) = h(t,R)$)

$$G_R'(t) := g_0(t) - g_R(t) + h(t,R) \quad \text{with} \quad G_R(0) = 0$$

we see that $\tilde{v}(t,x) := v(t,x) - G_R(t) \cdot R^{-1}x$ satisfies

$$\left\{ \begin{array}{rll} \tilde{v}_t - \tilde{v}_{xx} & = \tilde{h} & \text{on} \quad (0,T) \times (0,R) \\ \tilde{v} & = 0 & \text{on} \quad (0,T) \times \{0\} \\ \tilde{v} & = c_0 & \text{on} \quad (0,T) \times \{R\} \\ \tilde{v} & = v_0 & \text{on} \quad \{0\} \times (0,R) \end{array} \right.$$

with $\tilde{h}(t,x) := g_0(t) + h(t,x) - G_R'(t) \cdot R^{-1}x$ and the constant $c_0 := v_0(R)$.
Step 1: global space results i)
Then setting $\bar{v} := \tilde{v} - v_0$, we see that

$$\bar{v}(t,x) = v(t,x) - v_0(x) - G_R(t) \cdot R^{-1}x$$

solves

$$(21.10) \qquad \left\{ \begin{array}{rll} \bar{v}_t - \bar{v}_{xx} & = \bar{h} & \text{on} \quad (0,T) \times (0,R) \\ \bar{v} & = 0 & \text{on} \quad (0,T) \times \{0\} \\ \bar{v} & = 0 & \text{on} \quad (0,T) \times \{R\} \\ \bar{v} & = 0 & \text{on} \quad \{0\} \times (0,R) \end{array} \right.$$

with $\bar{h} = \tilde{h} + v_0''$, i.e.

$$\bar{h}(t,x) = g_0(t) + h(t,x) - G_R'(t) \cdot R^{-1}x + u_0'(x).$$

From Corollary 21.4, we deduce that $U := \bar{v}_x = u(t,x) - u_0(x) - R^{-1}G_R(t)$ satisfies

$$\text{(21.11)} \qquad \begin{array}{ll} \frac{|U|_p}{\sqrt{T}} + \frac{|U|_\infty}{T^{\frac{\alpha}{2}}} + \langle U\rangle^{(\alpha)} + |U_x|_p & \le c|\bar{h}|_p \\ & \le cK|W|_p \quad \text{with} \quad W := (u_0', h, g_0, g_R, h(\cdot, R)) \end{array}$$

where K is a numerical constant. This shows (21.8).

Step 2: complementar regularity results

Notice that for parabolic problem (21.10), compatibility conditions on the singular boundary $\{0\}\times\partial(0,R)$ are satisfied at order zero and order one. Indeed the condition at order zero is trivial, because its corresponds to the continuity of the solution $\bar{v}$ on the singular boundary. The condition at order one means that $\bar{v}_t = \bar{v}_{xx} + \bar{h}$ with $\bar{v}_{xx} = 0 = \bar{v}_t$, i.e. $\bar{h} = 0$ on $\{0\} \times \partial(0,R)$. It is then easy to check that this is true when (21.7) holds true.

Step 2.1: higher regularity ii)

Hence, if $u_0 \in C^{1+\alpha}([0,R])$ and $h, g \in C^{\alpha,\frac{\alpha}{2}}_{x,t}(\bar{Q}_{T,R})$, then using Schauder parabolic estimate (Lemma 21.5 ii)), we deduce that $\bar{v} \in C^{2+\alpha,1+\frac{\alpha}{2}}_{x,t}(\bar{Q}_{T,R})$. Then we have $U = \bar{v}_x \in C^{1+\alpha,\frac{1+\alpha}{2}}_{x,t}(\bar{Q}_{T,R})$. Because $U = u(t,x) - u_0(x) - R^{-1}G_R(t)$, we deduce that $u \in C^{1+\alpha,\frac{1+\alpha}{2}}_{x,t}(\bar{Q}_{T,R})$.

Step 2.2: further regularity iii)

This step is similar to Step 2.1. If $u_0 \in C^{2+\alpha}([0,R])$ and $h, g \in C^{1+\alpha,\frac{1+\alpha}{2}}_{x,t}(\bar{Q}_{T,R})$, then using Schauder parabolic estimate (Lemma 21.5 iii)), we deduce that $u \in C^{2+\alpha,\frac{2+\alpha}{2}}_{x,t}(\bar{Q}_{T,R})$.
This end the proof of the lemma.

Given $w = (w^1, \dots, w^N)$, and functions $g = (g^1, \dots, g^N) : \mathbb{R}^N \to \mathbb{R}^N$ and $h = (h^j)_{j=1,\dots,N}$ with $h^j : \mathbb{R} \to \mathbb{R}$, we now consider solutions $u = (u^1, \dots, u^N)$ to the system for $J_R := J \cap B_R$ and $J_R^* := J_R \setminus \{0\}$, with u^j defined on $[0,T) \times \bar{J}_R^j$ and $J_R^j := J^j \cap B_R \simeq -\sigma^j(0,R)$ for $j = 1, \dots, N$, and $\sigma^j \in \{\pm 1\}$.

$$\text{(21.12)} \qquad \left\{ \begin{array}{rll} u_t - u_{xx} & = (h(w) - h(w)(t,0))_x & \text{on} \quad (0,T) \times J_R^* \\ -u_x & = g_0(w) & \text{on} \quad (0,T) \times \{0\} \\ -u_x & = g_R(w) & \text{on} \quad (0,T) \times (J \cap \partial B_R) \\ u & = u_0 & \text{on} \quad \{0\} \times J_R^* \end{array} \right.$$

We also consider solutions to the fixed point problem $u \equiv w$ for $T = +\infty$

$$\text{(21.13)} \qquad \left\{ \begin{array}{rll} u_t - u_{xx} & = (h(u) - h(u)(t,0))_x & \text{on} \quad (0,+\infty) \times J_R^* \\ -u_x & = g_0(u) & \text{on} \quad (0,+\infty) \times \{0\} \\ -u_x & = g_R(u) & \text{on} \quad (0,+\infty) \times (J \cap \partial B_R) \\ u & = u_0 & \text{on} \quad \{0\} \times J_R^* \end{array} \right.$$

Consider also the following initial compatibility condition

$$\text{(21.14)} \qquad \left\{ \begin{array}{ll} -(u_0)_x = g_0(u_0) & \text{on} \quad \{0\} \times \{0\} \\ -(u_0)_x = g_R(u_0) & \text{on} \quad \{0\} \times (J \cap \partial B_R) \end{array} \right.$$

Lemma 21.7 (Existence by fixed point)

Let $R > 0$ be fixed. Assume that $h, g_0, g_R : \mathbb{R}^N \to \mathbb{R}^N$ are bounded functions which are globally Lipschitz continuous of Lipschitz constant $L_0 > 0$. Assume also that each component of the initial data satisfies $u_0^j \in C^{1+\alpha}_x(\bar{J}_R^j)$ for $j = 1, \dots, N$ and every $\alpha \in (0,1)$.

i) (Existence)

Then there exists a unique global solution u with $u^j \in L^\infty_{loc}([0,+\infty); L^\infty(\bar{J}_R^j)) \cap C^{\frac{\alpha}{2},\alpha}_{t,x}([0,+\infty) \times \bar{J}_R^j)$ of (21.13) for all $\alpha \in (0,1)$ (where the boundary conditions have to be understood in a weak sense).

ii) (Higher regularity)

Moreover, if the initial compatibility condition (21.14) is satisfied, then $u^j \in C^{\frac{1+\alpha}{2},1+\alpha}_{t,x}([0,+\infty) \times \bar{J}^j_R)$ *for any* $\alpha \in (0,1)$.

iii) (Further regularity)

Furthermore, if

$$\left\{ \begin{array}{ll} h, g_0, g_R \in W^{2,\infty}_{loc}(\mathbb{R}^N;\mathbb{R}^N) & \\ u^j_0 \in C^{2+\beta}_x(\bar{J}^j_R) & \text{for} \quad j=1,\dots,N, \end{array} \right| \quad \text{for some} \quad \beta \in (0,1)$$

then $u^j \in C^{\frac{2+\beta}{2},2+\beta}_{t,x}([0,+\infty) \times \bar{J}^j_R)$.

Proof of Lemma 21.7

Step 1: contraction for short time

Because u_0 is in particular Lipschitz continuous, we have

$$u_0' \in L^p(J_R) \tag{21.15}$$

and then $u_0' \in L^p(Q_{T,R})$ with $Q_{T,R} := (0,T) \times J_R$. Let u_a (resp. u_b) be the solution to system (21.12) for $w = w_a$ (resp. $w = w_b$). We set $\bar{u} := u_a - u_b$, and $\bar{w} := w_a - w_b$ and

$$|u|_p := |u|_{L^p(Q_{T,R})} := \sum_{j=1}^N |u^j|_{L^p(Q^j_{T,R})} \quad \text{with} \quad Q^j_{T,R} := (0,T) \times J^j_R$$

From (21.8) applied to $\bar{u}$, we get in particular for $(\bar{w}(t,\sigma R))^j := \bar{w}^j(t,\sigma^j R)$ (by linearity on the initial data which is zero for $\bar{u}$) the following bound on a quantity $\bar{U}$

$$\begin{array}{rcl} \dfrac{|\bar{U}|_\infty}{T^{\frac{\alpha}{2}}} & \leq & c' L_0 \left\{ |\bar{w}|_p + |\bar{w}(\cdot,0)|_p + |\bar{w}(\cdot,\sigma R)|_p \right\} \\ & \leq & 3c' L_0 N (TR)^{\frac{1}{p}} |\bar{w}|_\infty \end{array}$$

where

$$\bar{U} := \bar{u} - R^{-1}\bar{G}_R(t), \quad \bar{G}_R(0) = 0$$

with

$$\bar{G}_R'(t) = F[w_a](t) - F[w_b](t) \quad \text{and} \quad F[w_a](t) := g_0(w_a(t,0)) - g_R(w_a(t,\sigma R)) + h(w_a(t,\sigma R)) - h(w_a(t,0)).$$

Hence

$$|R^{-1}\bar{G}_R|_\infty \leq 4L_0 T R^{-1} |\bar{w}|_\infty$$

and then

$$|\bar{u}|_\infty \leq \beta |\bar{w}|_\infty \quad \text{for some } \beta \in (0,1)$$

for any $T \in (0,T_0]$ for some $T_0 > 0$ small enough (depending on R). This shows that the map $L^\infty(Q_{T,R}) \ni w \mapsto \Phi(w) := u \in L^\infty(Q_{T,R})$ is a contraction. Hence this map has a unique fixed point u, which is then solution of (21.12). Moreover we have $u = u_0 + \{\Phi(u_0 - u_0\} + \left\{\Phi^2(u_0) - \Phi(u_0)\right\} + \dots$, and then

$$|u|_\infty \leq |u_0|_\infty + \frac{1}{1-\beta}|\Phi(u_0) - u_0|_\infty \tag{21.16}$$

Notice that T_0 (and also β) are independent on the Lipschitz norm of the initial data u_0, nor on the $L^\infty(Q_{T,R})$ norm of the fixed point solution u.

Step 2: how to restart the time interval

Now if we want to restart the problem from a time $t_1 \in (T/2, T)$, as in (21.15), we then need to insure that $u_x(t_1,\cdot) \in L^p(J^*_R)$. Indeed, estimate (21.8) shows that

(21.17)

$$\frac{|U|_p}{\sqrt{T}} + \frac{|U|_\infty}{T^{\frac{\alpha}{2}}} + [U]_{C^{\alpha,\frac{\alpha}{2}}_{x,t}(\overline{Q}_{T,R})} + |U_x|_p \leq c''|W|_p \quad \text{with} \quad W := (u_0', h(u), h(u)(\cdot,0), g_0(u)(\cdot,0), g_R(u)(\cdot,R), h(u)(\cdot,R))$$

with

$$U := u - u_0 - R^{-1}G_R(t) \quad \text{with} \quad G_R'(t) = F[u](t) \quad \text{and} \quad G_R(0) = 0$$

and then $U_x = u_x - u_0'$. From (21.16), we know that $u \in L^\infty(Q_{T,R})$, and then the RHS of (21.17) is finite, and this implies the existence of some time t_1 such that $u_x(t_1,\cdot) \in L^p(J_R^*)$.

We can now iterate the procedure, and get the existence of a global solution u with $u^j \in L^\infty_{loc}([0,+\infty); L^\infty(\bar J_R^j))$. Moreover we see that $u^j \in C^{\frac{\alpha}{2},\alpha}_{t,x}([0,+\infty)\times \bar J_R^j)$ for all $\alpha \in (0,1)$.

Step 3: complementar regularity results

Step 3.1: proof of ii)

Moreover, notice that (21.17) implies that $u^j \in C^{\alpha,\frac{\alpha}{2}}_{x,t}(\overline{Q}^j_{T,R})$. Then, using the fact that h, g_0, g_R are globally Lipschitz, we deduce that $h^j(u), g_0^j(u)(\cdot,0), g_R^j(u)(\cdot,\sigma^j R) \in C^{\alpha,\frac{\alpha}{2}}_{x,t}(\overline{Q}^j_{T,R})$. If u_0 satisfies relations (21.14), then Lemma 21.6 implies that $u^j \in C^{1+\alpha,\frac{1+\alpha}{2}}_{x,t}(\overline{Q}^j_{T,R})$. In particular, relations (21.14) propagate in time. Precisely, this means that we can replace u_0 by $u(t_1,\cdot)$ in relations (21.14). Therefore regularity also propagates and we get that $u^j \in C^{1+\alpha,\frac{1+\alpha}{2}}_{x,t}([0,+\infty)\times \bar J_R^j)$.

Step 3.2: proof of iii)

The result follows similarly to Step 3.1. Notice in particular that $h \in W^{2,\infty}_{loc}(\mathbb{R}^N;\mathbb{R}^N)$ and $u \in C^{1+\beta,\frac{1+\beta}{2}}_{x,t}$ imply $h(u) \in C^{1+\beta,\frac{1+\beta}{2}}_{x,t}$ for $\beta \in (0,1)$. We get similar regularity for $g_0(u)$ and $g_R(u)$. Therefore further regularity in iii) of Lemma 21.6 implies that $u^j \in C^{2+\alpha,\frac{2+\alpha}{2}}_{x,t}([0,+\infty)\times \bar J_R^j)$.

This ends the proof of the lemma.

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References

[1] Adimurthi, Ghoshal, S.S., Dutta,R., Veerappa Gowda,G.D., *Existence and nonexistence of TV bounds for scalar conservation laws with discontinuous flux*. Comm. Pure Appl. Math. 64 (1) (2011), 84-115.

[2] B. Andreianov, G. M. Coclite, C. Donadello, *Well-posedness for vanishing viscosity solutions of scalar conservation laws on a network*, Discrete & Continuous Dynamical Systems 37 (11) (2017), 5913.

[3] B. Andreianov, G. M. Coclite, C. Donadello, *Provisional title: Characterization of vanishing viscosity solutions on a star-shaped graph under generalized Kedem-Katchalsky transmission conditions at the node*, work in progress.

[4] B. Andreianov, K.H. Karlsen, N.H. Risebro, *A Theory of L1-Dissipative Solvers for Scalar Conservation Laws with Discontinuous Flux*, Arch. Rational Mech. Anal. 201 (2011) 27-86.

[5] B. Andreianov, K. Sbihi, *Well-posedness of general boundary-value problems for scalar conservation laws*, Trans. Amer. Math. Soc. 367 (6) (2015), 3763-3806.

[6] M. Bardi, L.C. Evans, *On Hopf's formulas for solutions of Hamilton-Jacobi equations*, Nonlin. Anal. T.M.A. 8 (11) (1984), 1373-1381.

[7] C. Bardos, A.Y. Leroux and J.C. Nedelec, *First Order Quasilinear Equations with Boundary Conditions*. Communications in Partial Differential Equations, 4 (1979), 1017-1034.

[8] C. Bianca, C. Dogbe, *Regularity of entropy solutions to a class of conservation laws*, Nonlinear Studies 26 (1) (2019), 129-157.

[9] A. Bressan, Fang Yu, *Continuous Riemann solvers for traffic flow at a junction*. Discrete & Continuous Dynamical Systems 35 (9) (2015), 4149-4171.

[10] R. Bürger, A. Garcia, K.H. Karlsen, J.D. Towers, *A family of numerical schemes for kinematic flows with discontinuous flux*. J. Eng. Math. 60 (2008), 387-425.

[11] P. Cardaliaguet, N. Forcadel, R. Monneau, *A class of germs arising from homogenization in traffic flow on junctions*, preprint HAL hal-04289965, version 1, to appear in J. of Hyperbolic Differ. Eq.

[12] G.M. Coclite, C. Donadello, *Vanishing viscosity on a star-shaped graph under general transmission conditions at the node*, Networks and heterogeneous media 15 (2) (2020), 197-213.

[13] G.M. Coclite, M. Garavello, *Vanishing viscosity for traffic on networks.* SIAM J. Math. Anal. 42, 4 (2010), 1761-1783.

[14] G.M. Coclite, M. Garavello, B. Piccoli, *Traffic Flow on a Road Network*, SIAM J. Math. Anal. 36 (6) (2005), 1862-1886.

[15] M. G. Crandall and L. Tartar, *Some relations between nonexpansive and order preserving mappings*, Proc. Amer. Math. Soc., 78 (1980), 385-390.

[16] C.M. Dafermos, *Hyperbolic conservation laws in continuum physics*, 3rd edition, Springer, (2010).

[17] C.F. Daganzo, C.F., *The cell transmission model II: network traffic*. Transportation Research Part B 29 (2) (1995), 79-93.

[18] F. Facchinei, J.-S. Pang, *Finite-Dimensional Variational Inequalities and Complementarity Problems Volume I*, Springer Series in Operation Research (2003).

[19] M. Fiedler and V. Pták, *Some generalizations of positive definiteness and monotonicity*, Numerische Mathematik, 9 (1966), 163-172.

[20] U.S. Fjordholm, M. Musch, N.H. Risebro, *Well-posedness and convergence of finite volume method for conservation laws on networks*, SIAM J. Numer. Anal. 60 (2) (2022), 606-630.

[21] N. Forcadel, C. Imbert, R. Monneau, *Non-convex coercive Hamilton-Jacobi equations: Guerand's relaxation revisited*, preprint Sep 2023, https://hal.science/hal-04201310.

[22] N. Forcadel, C. Imbert, R. Monneau, *Germs for scalar conservation laws: the Hamilton-Jacobi equation point of view*, preprint hal-04635094 (2024).

[23] N. Forcadel, R. Monneau, *Beyong uniqueness: relaxation calculus of junction conditions for coercive Hamilton-Jacobi equations*, work in preparation.

[24] M. Garavello and B. Piccoli, *Traffic flow on networks*, AIMS Series on Applied Mathematics, Vol. 1, (2006).

[25] M. Garavello, B. Piccoli, *Conservation laws on complex networks*, Ann. I. H. Poincaré AN 26 (2009), 1925-1951.

[26] H. Holden, N.H. Risebro *A Mathematical model of traffic flow on a network of unidirectional roads* SIAM J. Math. Anal. 26 (4) (1995), 999-1017.

[27] H. Holden, N. H. Risebro, *Front Tracking for Hyperbolic Conservation Laws*, Applied Mathematical Sciences 152, 2nd edition, Springer, (2015).

[28] Y. Holle, *Entropy Dissipation at the Junction for Macroscopic Traffic Flow Models.* SIAM J. Math. Anal. 54 (1) (2022), 954-985.

[29] C. Imbert and R. Monneau, *Flux-limited solutions for quasi-convex Hamilton-Jacobi equations on networks.* Annales scientifiques de l'ENS 50 (2) (2017), 357-448.

[30] C. Imbert, R. Monneau, H. Zidani, *A Hamilton-Jacobi approach to junction problems and application to traffic flows*, ESAIM-COCV 19 (2013), 129-166.

[31] W.-L. Jin, *Analysis of Kinematic Waves Arising in Diverging Traffic Flow Models*, Transportation Science 49 (1) (2015), 28-45.

[32] W.-L. Jin, *Continuous kinematic wave models of merging traffic flow.* Transportation Research Part B 44 (8-9) (2010), 1084-1103.

[33] W.-L. Jin, H.M. Zhang, *On the distribution schemes for determining flows through a merge*. Transportation Research Part B 37 (6) (2003), 521-540.

[34] C.R. Johnson, R.L. Smith, M.J. Tsatsomeros, *Matrix Positivity*, Cambridge University Press, (2020).

[35] S.M. Kružkov, *First order quasi-linear equations in several independent variables*, Math. USSR Sbornik 10(2), 217-243 (1970).

[36] O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Ural'tseva, *Linear and Quasilinear Equations of Parabolic Type*, Izdat. "Nauka", Moscow, 1967 (Russian). English transl. : American Mathematical Society, Providence R. I., 1968.

[37] J.P. Lebacque, *The Godunov scheme and what it means for first order flow models*, in Transportation and Traffic Theory, Proceedings of the 13th ISTTT (J.B. Lesort ed.) (1996), Pergamon, Oxford, 647-677.

[38] J.P. Lebacque, M. M. Khoshyaran, *First Order Macroscopic Traffic Flow Models for Networks in the Context of Dynamic Assignment*, In Transportation Planning, Vol. 64 of Applied Optimization, (edited by M. Patriksson and M. Labbé). New York: Springer US, (2002), 119-140.

[39] J. Mawhin, *Simple Proofs of the Hadamard and Poincaré-Miranda Theorems Using the Brouwer Fixed Point Theorem*, The American Mathematical Monthly 126 (3) (2019), 260-263.

[40] M. Musch, U.S. Fjordholm, N.H. Risebro, *Well-posedness theory for nonlinear scalar conservation laws on networks*, Networks and heterogeneous media 17 (1) (2022), 101-128.

[41] G.F. Newell, *A simplified theory on kinematic wave in highway traffic, part II: queuing at freeway bottlenecks*, Transport. Res. B 27 (4) (1993), 289-303.

[42] D. Ni, J. Leonard, *A simplified kinematic wave model at a merge bottleneck*. Applied Mathematical Modelling 29 (11) (2005), 1054-1072.

[43] E. Panov, *Existence of strong traces for quasi-solutions of multidimensional scalar conservation laws*, J. Hyperbolic Differ. Equ. 4 (2007), 729-770.

[44] T. Parthasarathy, *On Global Univalence Theorems*, Lecture Notes in Mathematics 977, Springer-Verlag, Berlin (1983).

[45] D. Serre, *Systems of Conservation Laws 1. Hyperbolicity, Entropies, Shock Waves*, Cambridge University Press, (1999).

[46] J.D. Towers, *Well-posedness of a model of merging and branching traffic flow*, preprint (2020).

[47] A. Vasseur, *Strong traces for solutions of multidimensional scalar conservation laws*, Arch. Ration. Mech. Anal. 160 (3) (2001) 181-193.