Strictly convex Hamilton-Jacobi equations: strong trace of the derivatives in codimension ≥ 2

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Abstract

We consider Lipschitz continuous viscosity solutions to an evolutive Hamilton-Jacobi equation. The equation arises outside a closed set Γ. Under a condition of strict convexity of the Hamiltonian, we show that there exists a notion of strong trace of the derivatives of the solution on the Lipschitz boundary Γ of codimension $d \geq 2$. The very special case $d = 1$ is done in a separated work.

This result is based on a Liouville-type result of classification of global solutions with zero Dirichlet condition on the boundary Γ, where Γ is an affine subspace. We show in particular that such solutions only depend on the normal variable to Γ. As a consequence, we show more generally that the existence of a pointwise tangential gradient along Γ implies the existence of pointwise directional derivatives in all directions.

This result also holds true for Hamiltonians depending on the time-space variables, under an additional Dini condition involving certain moduli of continuity. We also give a counter-example for $d = 2$ in the stationary case, where the Hamiltonian is only continuous in the space variable, and where the solution has no directional derivatives in any directions normal to Γ. Such phenomenon does not hold for $d = 1$.

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1 Introduction

1.1 Main results

Let $m \geq 0$ and let us consider globally Lipschitz continuous solutions $u = u(t, x', x)$ of

(1.1) $$\begin{cases} u_t + H(Du) = 0 & \text{on} \quad \mathbb{R} \times \mathbb{R}^m \times \Omega \\ u = 0 & \text{on} \quad \mathbb{R} \times \mathbb{R}^m \times \partial\Omega \end{cases}$$

where the Dirichlet condition is satisfied in the strong sense (i.e. pointwisely). Here we consider the open set whose boundary has codimension bigger or equal to 2

$$\Omega := \mathbb{R}^d \setminus \{0\}, \quad \partial\Omega = \{0_{\mathbb{R}^d}\}, \quad \text{with} \quad d \geq 2.$$

The case of dimension $d = 1$ with a half line $\Omega = (0, +\infty)$ has been studied in [15]. Here for $d \geq 2$, we find a new method and get different results.
Assume

(1.2) $$H : \mathbb{R}^{m+d} \to \mathbb{R} \text{ is } C^1\text{, strictly convex, and superlinear (i.e. } \lim_{|P| \to +\infty} \frac{H(P)}{|P|} = +\infty)$$

where we recall that H strictly convex means

$$H(\lambda P + (1 - \lambda)Q) < \lambda H(P) + (1 - \lambda)H(Q) \quad \text{for all} \quad \lambda \in (0, 1), \quad P, Q \in \mathbb{R}^{m+d}, \quad P \neq Q.$$

We consider the coordinates $X := (t, \tilde{x})$ with $t \in \mathbb{R}$ and $\tilde{x} = (x', x) \in \mathbb{R}^m \times \mathbb{R}^d$.

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Theorem 1.1 (Liouville-type result)
Assume that the strictly convex function H satisfies (1.2). Then every globally Lipschitz continuous viscosity solution u of (1.1) only depends on the normal variable, i.e.

$$u(t,x',x) = u(0,0,x) \quad \text{for all} \quad (t,x',x) \in \mathbb{R}\times\mathbb{R}^m\times\overline{\Omega}.$$

For local problems, we will need to describe directional derivatives with a single object. To this end, we introduce the following definition.

Definition 1.2 (Bouligand derivative)
We say that a function $f : \mathbb{R}^n \supset B_1(0) \to \mathbb{R}$ has a Bouligand derivative at the origin if there exists a (positively) 1-homogeneous function $Bf(0) : \mathbb{R}^n \to \mathbb{R}$ (i.e. $Bf(0)(\lambda y) = \lambda Bf(0)(y)$ for all $\lambda \geq 0$) such that

$$f(y) - f(0) = Bf(0)(y) + o(|y|)$$

It is easy to see that any Lipschitz function has directional derivatives in any directions if and only if it has a Bouligand derivative. The formalism of Bouligand derivative has just the practical advantage to deal with all directional derivatives at the same time.

We now consider the following problem localized on a cylinder $Q_0 \subset \mathbb{R}^{1+m+d}$. We denote the boundary $\Gamma := \mathbb{R}^{1+m}\times\{0_{\mathbb{R}^d}\}$ and the open ball $B_r = B_r(0)$ of center the origin and radious $r > 0$. Then we consider the problem:

$$\left\{\begin{array}{llll} u_t + H(Du) = 0 & \text{on} & Q_0\backslash\Gamma & \subset \mathbb{R}^{1+m}\times\Omega, \\ u = g(t,x',0) & \text{on} & Q_0\cap\Gamma & \subset \mathbb{R}^{1+m}\times\partial\Omega. \end{array}\right. \tag{1.3}$$

We have the following result (at least surprising for the author).

Theorem 1.3 (Tangential gradient implies full directional derivatives)
Let $Q_0 := (-1,1)\times B_1 \subset \mathbb{R}\times\mathbb{R}^{m+d}$. Assume that $H : \mathbb{R}^{m+d} \to \mathbb{R}$ satisfies (1.2). Consider a Lipschitz continuous function $u : Q_0 \to \mathbb{R}$ which is a viscosity solution of (1.3) with $u(0) = 0$. Assume that u has a tangential gradient at the origin, i.e. there exists $(\lambda, p') \in \mathbb{R}\times\mathbb{R}^m$ such that

$$u(t,x',0) = \lambda t + p'\cdot x' + o(|(t,x')|) \quad \text{for all} \quad (t,x',0) \in \mathbb{R}\times\mathbb{R}^m\times\partial\Omega. \tag{1.4}$$

Then u has Bouligand time-space derivative at the origin, i.e. there exists a (positively) 1-homogeneous function $Bu(0) : \mathbb{R}^{1+m+d} \to \mathbb{R}$ such that for $X = (t,x',x) \in Q_0$

$$u(X) = Bu(0)(X) + o(|X|) \tag{1.5}$$

with the splitting $Bu(0)(X',x) = X'\cdot D'u(0) + Bu(0)(0,x)$ where $X'\cdot D'u(0) := \lambda t + p'\cdot x'$ for $X' = (t,x')$.

Theorem 1.4 (A notion of strong trace of directional derivatives)
Let $Q_0 := B_1\times B_1 \subset \mathbb{R}^{1+m}\times\mathbb{R}^d$. Assume that $H : \mathbb{R}^{m+d} \to \mathbb{R}$ satisfies (1.2). Consider a Lipschitz continuous function $u : Q_0 \to \mathbb{R}$ which is a viscosity solution of (1.3), with $g : Q_0\cap\Gamma \to \mathbb{R}$ Lipschitz continuous.
Then for $Y \in \mathbb{R}^{1+m+d}$ with $X + Y \in Q_0$, we have

$$u(X+Y) - u(X) = Bu(X)(Y) + o(|Y|) \quad \text{for a.e.} \quad X = (t,x',0) \in \Gamma \tag{1.6}$$

where $Bu(X) : \mathbb{R}^{1+m+d} \to \mathbb{R}$ is (positively) 1-homogeneous and Lipschitz. Moreover for a.e. $X \in \Gamma$, the quantity $Bu(X)$ satisfies the splitting $Bu(X)(Y',y) = Y'\cdot D'u(X) + Bu(X)(0,y)$ for all $Y' = (s,y') \in \mathbb{R}\times\mathbb{R}^m$ and $y \in \mathbb{R}^d$, with $Y'\cdot D'u(X) := \lambda s + p'\cdot y'$ where $\lambda := u_t(X)$, and the tangential space gradient is $p' := D(u_{|\Gamma})(X)$.

Moreover, we have the following limit for the time-space derivatives $\hat{D}u := (u_t, Du)$ with $X = (X',x)$

$$\lim_{\varepsilon\to 0^+}\int_{B_1\times B_1} |\hat{D}u(X',\varepsilon x) - \hat{D}(Bu(X',0))(0,x)|\; dX = 0. \tag{1.7}$$

This convergence is our definition of the strong trace of the time-space derivatives of the solution u on the boundary Γ of codimension $d \geq 2$.

Remark 1.5 *We see here that the strong trace of the gradient has to be a Bouligand derivative in general, and not a standard gradient. In other words, we see that taking the strong trace of the gradient, we may loose the linearity along the normal variable for the first order approximation map when the boundary* Γ *has codimension* $d \geq 2$. *In particular, for a.e. point* X *of the boundary* Γ, *we get that* u *has directional derivatives at* X.

Remark 1.6 *Notice that it is straightforward to adapt the proof of Theorem 1.4 to the case of Lipschitz continuous boundary* Γ.

Again for a cylinder $Q_0 \subset \mathbb{R}^{1+m+d}$, we consider now the X-dependence in the Hamiltonian H, for $X = (t, x', x)$, and the problem

$$\left\{ \begin{array}{lll} u_t + H(X, Du) = 0 & \text{on} & Q_0 \backslash \Gamma \\ u = g(X) & \text{on} & Q_0 \cap \Gamma. \end{array} \right. \tag{1.8}$$

Recall that we say that a function u is semiconcave on an open set Q_0 if there exists a modulus of continuity $\omega : (0, +\infty) \to (0, +\infty)$, with $\omega(0^+) = 0$ such that

$$\lambda u(X) + (1 - \lambda)u(Y) - u(\lambda X + (1 - \lambda)Y) \leq \lambda(1 - \lambda)|X - Y|\omega(|X - Y|) \quad \text{for all} \quad \lambda \in [0, 1], \ X, Y \in Q_0. \tag{1.9}$$

Recall also that semiconcavity implies the existence of directional derivatives everywhere (see [10]). Here we show that the lack of regularity in X can break semiconcavity property of the solution, even if the Hamiltonian is strictly convex in the gradient.

Proposition 1.7 (A counter-example to semiconcavity)
We assume that $m = 0$, $\partial_t = 0$ *and* $d = 2$. *Let* $h : [0, +\infty) \to (0, +\infty)$ *be* C^2 *with* $h(1) = 1$ *and* $h'(0) = 0$ *and* $h' > 0$ *on* $(0, +\infty)$ *and* $h'' > 0$ *on* $[0, +\infty)$. *Then there exists some continuous function* $a : \mathbb{R}^2 \supset B_1 \to \mathbb{R}$ *and* $H(x, P) = a(x)h(|P|)$ *such that there exists a Lipschitz continuous solution* $u = u(x)$ *of*

$$H(x, Du) = 1 \quad \text{on} \quad B_1 \subset \mathbb{R}^2.$$

We can choose the continuous function $a(\cdot)$ *such that our solution* u *has no directional derivatives at* $x = 0$. *In particular* u *is not semiconcave (i.e. does not satisfy (1.9) with* $Q_0 := B_{\frac{1}{2}}$*). Still, there exists a modulus of continuity* ω_0 *such that we have*

$$u(x + b) + u(x - b) - 2u(x) \leq |b|\omega_0(|b|) \quad \text{for all} \quad x + b, x - b \in B_{\frac{1}{2}} \tag{1.10}$$

but ω_0 *is not Dini integrable, i.e. that* $\int_0^r \frac{ds}{s}\omega_0(s) = +\infty$ *for all* $r > 0$.

We refer the reader to Theorem 2.1.10 on page 35 in Cannarsa, Sinestrari [10], which says that (1.10) implies semiconcavity estimate (1.9) with $\omega(r) := C \int_0^r \frac{ds}{s}\omega_0(s)$.

For our ptoblem, and under a certain Dini condition on the X-dependence of $H(X, P)$ which is continuous in both variables and C^1 strictly convex and superlinear in P, it is possible to recover the results of the homogeneous case. This is the result below.

Theorem 1.8 (Generalization to variable coefficients X**)**
Theorems 1.3 and 1.4 hold for equation (1.8), under the condition that $H : Q_0 \times \mathbb{R}^{m+d} \to \mathbb{R}$ *is continuous and that* $P \mapsto H(X, P)$ *satisfies (1.2) for all* $X \in Q_0$, *under the additional Dini condition (1.13) on a certain modulus given below.*

Assume that the solution u *to (1.8) has Lipschitz constant* $L > 0$. *Let* $\mathcal{L}$ *be the Legendre-Fenchel transform of* H *defined by*

$$\mathcal{L}(X, \tilde{\xi}) := \sup_{P \in \mathbb{R}^{m+d}} \tilde{\xi} \cdot P - H(X, P). \tag{1.11}$$

Assume also that there exists a map $\mathcal{L}_0 : \mathbb{R}^{m+d} \to \mathbb{R}$ *such that* $\mathcal{L}(X, \tilde{\xi}) \geq \mathcal{L}_0(\tilde{\xi})$ *for all* $X \in Q_0$, $\tilde{\xi} \in \mathbb{R}^{m+d}$, *and for* $L > 0$, *we consider* $R_L \geq 1$ *such that*

$$\left\{ \tilde{\xi} \in \mathbb{R}^{m+d}, \quad \mathcal{L}_0(\tilde{\xi}) \leq L|(1, \tilde{\xi})| \right\} \subset \overline{B}_{R_L}. \tag{1.12}$$

i) (modulus of strict convexity of the map $\tilde{\xi} \mapsto \mathcal{L}(X, \tilde{\xi})$)
For every $R > 0$, we assume that there exists a (monotone) modulus of continuity $\tilde{\omega}_R : (0, +\infty) \to (0, +\infty)$ with $\tilde{\omega}_R(0^+) = 0$ such that for $r > 0$, we set

$$\tilde{\Omega}_R(r) := \int_0^r \tilde{\omega}_R(s)\ ds \quad \text{with} \quad \tilde{\omega}_R(r) := \inf_{|\tilde{\xi}_1 - \tilde{\xi}_2| \geq r,\ \tilde{\xi}_i \in \overline{B}_R,\ X \in Q_0} \left(\frac{\tilde{\xi}_1 - \tilde{\xi}_2}{|\tilde{\xi}_1 - \tilde{\xi}_2|}, D\mathcal{L}(X, \tilde{\xi}_1) - D\mathcal{L}(X, \tilde{\xi}_2) \right)$$

where $\tilde{\Omega}_R$ is convex increasing.
ii) (modulus of continuity of the map $X \mapsto \mathcal{L}(X, \tilde{\xi})$)
We assume the existence of the following monotone modulus of continuity

$$\omega_R(r) := \sup_{X, Y \in Q_0,\ |X - Y| \leq r,\ \tilde{\xi} \in \overline{B}_R} \left| \mathcal{L}(X, \tilde{\xi}) - \mathcal{L}(Y, \tilde{\xi}) \right|.$$

Then we require the modulus $\tilde{\Omega}_R^{-1} \circ \omega_R$ to satisfy the following Dini condition

$$\int_0^1 \frac{ds}{s}\ (\tilde{\Omega}_R^{-1} \circ \omega_R)(s) < +\infty \quad \text{for} \quad R := 1 + R_L. \tag{1.13}$$

Remark 1.9 *It is easy to check that Dini condition (1.13) is satisfied for instance if $P \mapsto H(X, P)$ is C^2 with $\delta^{-1} \geq D^2_{PP} H \geq \delta > 0$ and $X \mapsto H(X, P)$ is β-Hölder continuous for some $\beta \in (0, 1]$. Such result can be compared to the precise interior semiconcavity results in Cannarsa, Cardaliaguet [9] for Hölder continuous Hamiltonians in the space variable, in the case of stationary solutions where the Hamiltonian is 1-homogeneous in the gradient.*

Remark 1.10 *It is straightforward to adapt Theorem 1.8 to the case of a Lipschitz continuous boundary Γ. It would be also interesting (and not quite difficult) to develop an interior theory (i.e. with $\Gamma = \emptyset$) along the same lines as in the present paper. For instance, the analogue of the Liouville-type result claims that Lipschitz global time-space solutions are concave. Here we refrain us to go further in that direction.*

1.2 Brief review of the literature

We refer to the pioneering work of Lions [14] on viscosity solutions of Hamilton-Jacobi equations and their properties. The reader can also consult the reference books Cannarsa, Sinestrari [10] on semiconcave functions and Bardi, Capuzo-Dolcetta [3] for Hamilton-Jacobi equations related to control problems. In Cannarsa, Soner [11] (see also Theorem 5.3.8 on page 118 in [10]), it is proven that for Hamiltonians $H(X, P)$ which are locally Lipschitz in all variables and strictly convex in P, the locally Lipschitz continuous solutions are semiconcave. This result has been extended to stationary solutions for 1-homogeneous Hamiltonians which are Hölder in the space variable in Cannarsa, Cardaliaguet [9].

In Bianchini, De Lellis, Robyr [7], the authors show that for a uniformly C^2 Hamiltonian H, the time-space gradient of the solution is in SBV_{loc}, which can be seen as a refinement of semiconcavity estimates. This result has been extended to the case of C^3 Hamiltonians depending also on (t, x) in Bianchini, Tonon [8]. We also refer the reader to Rifford [17] for further interesting regularity results of solutions.

In the context of homogeneous scalar conservation laws, a notion of strong trace on a Lipschitz boundary of a domain (i.e. for $d = 1$) has been introduced by Vasseur [21] under a condition of genuine nonlinearity of the C^3 flux function. This result has been generalized by Panov [16] to the case of C^0 homogeneous fluxes, and C^1 boundary (the case of Lipschitz boundary is also claimed to remain valid with the same proof).

In Monneau [15], the existence of strong traces of the gradient of the solution has been obtained for strictly convex Hamilton-Jacobi equations with boundary Γ of codimension $d = 1$. For $d = 1$, no Dini condition is required: only the continuity of the map $(X, P) \mapsto H(X, P)$ is sufficient.

Actual researches try to understand convex Hamilton-Jacobi equations either on stratified domains (see for instance Barles, Chasseigne [4, 5]), or with a point defect (see Achdou, Le Bris [1]). Here we hope that our present work will help to understand better these probems where codimensions play a key role.

1.3 Organization of the paper

In Section 2, we recall the representation formula of the solutions to convex Hamilton-Jacobi equations.

In Section 3, we construct a fundamental solution $\hat{u}$ associated to the Hamiltonian $H(P)$, which is the analogue of the distance to the origin for the eikonal equation. Then we show that $u_+ := \hat{u}$ and $u_- = -\hat{u}(-x)$ are both barriers, and that every global solution with zero value on the boundary Γ is sandwiched in between $u_+ \geq u_-$.

This is a key step towards the Liouville-type result (Theorem 1.1) whose proof is done in Section 4. This proof is in particular based on a key equality (Lemma 4.2) satisfied by the solution on a characteristic trajectory.

In Section 5, we study in details the characteristics of 1-homogeneous concave global solutions. This section is of independent interest and uses the notion of extreme points and exposed points of a compact convex set. This section is also a key step towards the proof of Theorem 1.3 on the existence of full directional derivatives for local solutions.

Section 6 is a technical result which localizes the result of Lemma 4.2, and which is necessary for the proof of Theorem 1.3.

Section 7 is fully devoted to the proof of Theorem 1.3 (on the existence of directional derivatives) using ingredients of Sections 5 and 6.

Section 8 is devoted to the proof of Theorem 1.4 on the strong traces of the directional derivatives. It starts with a building block result, which shows the strong L^1_{loc} convergence of the gradient of the blow-up of the solution (see Proposition 8.1). The remaining part of the section proves Theorem 1.4, using covering arguments and identification of the limits through Young measures.

Section 9 gives the proof of the counter-example to semiconcavity, namely Proposition 1.7.

Section 10 gives the proof of Theorem 1.8 which generalizes our results to Hamiltonians with time-space dependence.

Section 11 is an appendix where we collect results on exposed points of convex sets, which are used in the main part of the paper.

1.4 Main notations

$\Omega = \mathbb{R}^d \setminus \{0\}$	= reduced space domain
$\mathbb{R}^m \times \Omega$	= space domain
$\Gamma = \mathbb{R}^{1+m} \times \{0_{\mathbb{R}^d}\}$	= time-space boundary
$\tilde{\Gamma} = \mathbb{R}^m \times \{0_{\mathbb{R}^d}\}$	= space boundary
$x \in \mathbb{R}^d$	= normal coordinates
$X' = (t, x') \in \mathbb{R} \times \mathbb{R}^m$	= tangential coordinates
$\tilde{x} = (x', x) \in \mathbb{R}^m \times \mathbb{R}^d$	= space coordinates
$X = (t, x', x) = (t, \tilde{x}) = (X', x)$	= time-space coordinates
$Q_0 = (-1,1) \times B_1$ or $B_1 \times B_1$	= cylinder in $\mathbb{R} \times \mathbb{R}^{m+d}$ or $\mathbb{R}^{1+m} \times \mathbb{R}^d$
$Q_{\tau,\rho} = (-\tau, 0) \times B_\rho$	= local cylinder in $\mathbb{R} \times \mathbb{R}^{m+d}$
$\Gamma_{\tau,\rho} = \overline{Q}_{\tau,\rho} \cap \Gamma$	= (closed) tangential boundary of the local cylinder
$P, Q \in \mathbb{R}^{m+d}$	= space gradient
$\hat{D}u = (u_t, Du)$	= time-space gradient
Bu	= Bouligand derivative of u
H	= the Hamiltonian
$\mathcal{L} = H^*$	= the Legendre-Fenchel transform of H
$\tilde{\xi} = (\xi', \xi) \in \mathbb{R}^m \times \mathbb{R}^d$	= velocity
$\ell(\xi) = \inf_{\xi' \in \mathbb{R}^m} \mathcal{L}(\xi', \xi)$	= the Legendre-Fenchel transform of $H(0, \cdot)$
$\hat{u}$	= fundamental convex 1-homogeneous solution
$u_\pm$	= barriers.

2 Representation formula

In this section, we recall the representation formula of the solutions to convex Hamilton-Jacobi equations.

Given $X = (t, \tilde{x}) \in \mathbb{R} \times \mathbb{R}^{m+d}$ and $\xi(\cdot) \in L^1_{loc}((-\infty, t]; \mathbb{R}^{m+d})$, we consider the following backward trajectory

$$\frac{d}{d\sigma}\gamma_X^\xi(\sigma) = \xi(\sigma) \quad \text{for} \quad \sigma \leq t, \quad \text{with terminal data} \quad \gamma_X^\xi(t) = \tilde{x}$$

and call for all $t_0 < t$

$$\mathcal{E}_X^{t_0} := \left\{ \begin{array}{l|c|ll} (s,\xi) \in [t_0, t) \times L^1_{loc}((-\infty, t]; \mathbb{R}^{m+d}), & & \gamma_X^\xi(s) \in \mathbb{R}^m \times \partial\Omega & \text{if} \quad s \in (t_0, t) \\ & \text{with} & & \\ \gamma_X^\xi(\sigma) \in \mathbb{R}^m \times \Omega, \quad \text{for all} \quad \sigma \in (s,t), & & \gamma_X^\xi(s) \in \mathbb{R}^m \times \overline{\Omega} & \text{if} \quad s = t_0 \end{array} \right\}$$

which is the set of parameters such that the backward trajectory stays in the set $\mathbb{R}^m \times \Omega$ and in a time interval contained in $[t_0, t]$.

We recall the following standard result for convex Hamiltonians (which can be seen as a generalization of Lax-Hopf formula).

Lemma 2.1 (Representation formula)
Assume that $H : \mathbb{R}^{1+m+d} \times \mathbb{R}^{m+d} \to \mathbb{R}$ be continuous such that $P \mapsto H(X, P)$ satisfies (1.2) for all X, and let $\mathcal{L}$ be the Legendre-Fenchel transform of H given in (1.11). Assume that $u : \mathbb{R}^{1+m+d} \to \mathbb{R}$ is a globally Lipschitz continuous function satisfying $u = g$ on $\Gamma = \mathbb{R}^{1+m} \times \{0_{\mathbb{R}^d}\}$.

Then u satisfies for all $X = (t, \tilde{x}) \in \mathbb{R}^{1+m+d}$ and all $t_0 \in (-\infty, t)$

$$(2.1) \qquad \left\{ \begin{array}{l} u(X) = \displaystyle\inf_{(s,\xi)\in\mathcal{E}_X^{t_0}} G(s,t;\gamma_X^\xi) \\ \textit{with} \\ G(s,t;\gamma_X^\xi) - \displaystyle\int_s^t \mathcal{L}\left(\sigma, \gamma_X^\xi(\sigma), \frac{d}{d\sigma}\gamma_X^\xi(\sigma)\right) d\sigma := \left\{ \begin{array}{ll} u(s, \gamma_X^\xi(s)) & \textit{if} \quad \gamma_X^\xi(s) \in \mathbb{R}^m \times \Omega \\ g(s, \gamma_X^\xi(s)) & \textit{if} \quad \gamma_X^\xi(s) \in \mathbb{R}^m \times \partial\Omega \end{array} \right. \end{array} \right.$$

if and only if u solves (1.8) with $Q_0 = \mathbb{R}^{1+m+d}$.

Representation formula (2.1) means that $u(t,x)$ is the infimum of some cost function over all trajectories with terminal point $(t, \tilde{x})$ and initial point on the part of the boundary $([t_0, t) \times \mathbb{R}^m \times \partial\Omega) \cup (\{t_0\} \times \mathbb{R}^m \times \Omega)$.

Sketch of the proof
The standard proof first shows the dynamic programming principle which implies (by variations/comparison) the viscosity inequalities on the time interval $(t_0, +\infty)$ (see for instance [13], or [10] for results of the same flavour). Conversely, the comparison principle implies that every solution of (1.8) on the time interval $[t_0, +\infty)$ coincides with the unique solution given by the representation formula (2.1). Notice that the comparison principle is valid here because u is globally Lipschitz continuous. This ends the sketch of the proof.

3 Existence of barriers

In this section, we show that $\inf_{\mathbb{R}^d} H(0, \cdot) \leq 0$, and that when the inequality is strict, then we can describe the maximal solution u_+ and the minimal solution u_- to equation (1.1). We show that $u_+ = \hat{u}$ and $u_- = -\hat{u}(-x)$, where $\hat{u}$ is a convex 1-homogeneous solution, whose we study the rich properties. The existence of the two barriers $u_\pm$ is a key step towards the Liouville-type result which will be developed in the next section.

Lemma 3.1 (Dichotomy)
Assume that $H : \mathbb{R}^{m+d} \to \mathbb{R}$ satisfies (1.2) and that u is a globally Lipschitz continuous solution of (1.1). Then either $\inf_{\mathbb{R}^d} H(0, \cdot) = 0$ and there exists a unique $p_0 \in \mathbb{R}^d$ such that $H(0, p_0) = 0$ with $u(t, x', x) = p_0 \cdot x$, or

$$(3.1) \qquad \inf_{\mathbb{R}^d} H(0, \cdot) < 0.$$

Proof of Lemma 3.1
We define

$$\overline{u}(x) := \sup_{(t,x')\in\mathbb{R}\times\mathbb{R}^m} u(t,x',x), \quad \underline{u}(x) := \inf_{(t,x')\in\mathbb{R}\times\mathbb{R}^m} u(t,x',x).$$

Because u is globally Lipschitz continuous this is also the case of $\overline{u}$ and $\underline{u}$. Moreover $\overline{u}$ is a subsolution, and by classical Barron-Jensen argument (see [6]) for convex Hamiltonians and Lipschitz solutions, the minimum (here infimum) of solutions is still a solution. Hence $H(0,D\overline{u}) \le 0$ and $H(0,D\underline{u}) = 0$ on Ω. Using a test function from above, we see that $\mu := \inf_{\mathbb{R}^d} H(0,\cdot) \le 0$. In case $\mu = 0$, the strict convexity of H implies the uniqueness of some p_0 such that $H(0,p_0) = 0$. Then any test function φ touching from above either $\overline{u}$ or $\underline{u}$ at some point $x_0 \in \Omega$, satisfies $D\varphi(x_0) = p_0$. It is then easy to see that $D\overline{u} = p_0 = D\underline{u}$ which gives the result. This ends the proof of the lemma.

In the remaining part of the paper, we assume the negativity of the infimum of H as in (3.1) and define the convex function

$$\hat{u}(x) := \sup_{p\in K} p\cdot x \quad \text{for all} \quad x\in\mathbb{R}^d, \quad \text{with the convex set} \quad K := \left\{p\in\mathbb{R}^d, \quad H(0,p)\le 0\right\}. \tag{3.2}$$

Notice that K is a strictly convex set under assumption (1.2) on H. The function $\hat{u}$ plays a key role in the definition of barriers, and we start to study its nice (probably classical) properties.

Lemma 3.2 (Properties of $\hat{u}$)
Under assumptions (1.2) and (3.1) on H, the function $\hat{u}$ defined in (3.2) is convex Lipschitz continuous on $\mathbb{R}^d$, (positively) 1-homogeneous (i.e. $u(\lambda x) = \lambda u(x)$ for all $\lambda \ge 0$) and belongs to $C^1(\mathbb{R}^d\setminus\{0\})$.
Moreover, we have

$$\hat{u}(x) = x\cdot D\hat{u}(x) \quad \textit{with} \quad D\hat{u}(x)\in\partial K \quad \textit{for all} \quad x\in\Omega \tag{3.3}$$

where the convex K is defined in (3.2). In particular $\hat{u}$ is a (viscosity) solution of

$$\begin{cases} H(0,D\hat{u}) = 0 & \textit{on} \quad \Omega \\ \hat{u} = 0 & \textit{on} \quad \partial\Omega = \{0_{\mathbb{R}^d}\}. \end{cases} \tag{3.4}$$

We have also for all $x\in\mathbb{R}^d$

$$\hat{u}(x) = \inf_{\tau>0} \tau\ell(\frac{x}{\tau}) \quad \textit{where} \quad \ell(\xi) := \sup_{p\in\mathbb{R}^d}\{\xi\cdot p - h(p)\} \quad \textit{for all} \quad \xi\in\mathbb{R}^d, \quad \textit{with} \quad h(p) := H(0,p) \tag{3.5}$$

and for $x\neq 0$
(3.6)

$$\inf_{\tau>0} \tau\ell(\frac{x}{\tau}) = \tau_*\ell(\frac{x}{\tau_*}) \quad \textit{for some unique} \quad \tau_* = \tau_*(x) > 0 \quad \textit{and} \quad \begin{cases} \xi_* := \dfrac{x}{\tau_*(x)}, \quad p_* := D\hat{u}(x)\in\partial K, \\ \xi_* = D_x H(0,p_*), \\ p_* = D\ell(\xi_*). \end{cases}$$

Moreover the two Legendre-Fenchel transforms ℓ and $\mathcal{L}$ satisfy

$$\ell(\xi) = \inf_{\xi'\in\mathbb{R}^m} \mathcal{L}(\xi',\xi) \quad \textit{with} \quad \mathcal{L}(\xi',\xi) = \sup_{P\in\mathbb{R}^{m+d}} \{(\xi',\xi)\cdot P - H(P)\}, \quad (\xi',\xi)\in\mathbb{R}^m\times\mathbb{R}^d. \tag{3.7}$$

Finally, $\hat{u}$ enjoys the following strict convexity property (not along the radials):

$$\hat{u}(\lambda x + (1-\lambda)y) < \lambda\hat{u}(x) + (1-\lambda)\hat{u}(y) \quad \textit{for all} \quad \lambda\in(0,1), \quad \textit{if} \quad \begin{cases} [0,+\infty)\cdot x \neq [0,+\infty)\cdot y, \\ x,y\neq 0. \end{cases} \tag{3.8}$$

Proof of Lemma 3.2
Step 1: proof that $\hat{u}\in C^1(\mathbb{R}^d\setminus\{0\})$
We first notice that by construction, $\hat{u}$ is convex, (positively) homogeneous of degree one and Lipschitz continuous. We now check that the strict convexity of K implies $\hat{u}\in C^1(\mathbb{R}^d\setminus\{0\})$.
To this end, we introduce the lower semi-continuous convex function $v = (+\infty)\cdot 1_{\mathbb{R}^d\setminus K}$ which satisfies $v = 0$

on K and $v = +\infty$ outside K. Then the Legendre-Fenchel transform v^* of v satisfies $v^* = \hat{u}$, and by convex duality, we have $\hat{u}^* = v$. Now assume by contradiction that for some $x_0 \neq 0$

$$p_1 \neq p_2 \quad \text{with} \quad p_1, p_2 \in \partial\hat{u}(x_0) := \left\{p \in \mathbb{R}^d, \quad \hat{u}(x_0 + y) - \hat{u}(x_0) \geq p \cdot y, \quad \text{for all} \quad y \in \mathbb{R}^d\right\}$$

where $\partial\hat{u}(x_0)$ is the subdifferential of $\hat{u}$ at x_0. Then the full segment satisfies $[p_1, p_2] \subset \partial\hat{u}(x_0)$, which (by convex duality) implies $x_0 \in \partial\hat{u}^*(p) = \partial v(p)$ for all $p \in [p_1, p_2]$. In particular, we get $v(p) < +\infty$, and then $v(p) = 0$, i.e. $[p_1, p_2] \subset K$. If there exists $p \in [p_1, p_2] \cap \text{Int}(K)$, then we have $x_0 \in \partial v(p) = \{0\}$ which is impossible. Hence $[p_1, p_2] \subset \partial K$ which is also impossible because K is strictly convex. Hence we deduce that

$$\partial\hat{u}(x) = \text{singleton} = \{D\hat{u}(x)\} \quad \text{for all} \quad x \neq 0$$

which implies $\hat{u} \in C^1(\mathbb{R}^d \setminus \{0\})$.

Step 2: proof of (3.3)

We notice that $\hat{u}$ satisfies

$$\hat{u}(x) = \sup_{p \in \partial K} \; p \cdot x.$$

In particular for every $x \neq 0$, there exists some $p \in \partial K$ such that $\hat{u}(x) = p \cdot x$ and then $p \in \partial\hat{u}(x)$. Because $\hat{u} \in C^1(\mathbb{R}^d \setminus \{0\})$, we deduce that $p = D\hat{u}(x)$ is unique. This shows (3.3) and consequently (3.4).

Step 3: proof of (3.5)

We now define $\hat{u}_+(x) := \inf_{\tau > 0} \; \tau\ell(\frac{x}{\tau})$. By definition of ℓ, we have

$$\begin{aligned}
\hat{u}_+(x) \quad &= \inf_{\tau>0} \; \tau \sup_{p \in \mathbb{R}^d} \left\{\frac{x}{\tau} \cdot p - H(0,p)\right\} \\
&\geq \inf_{\tau>0} \; \tau \sup_{p \in K} \left\{\frac{x}{\tau} \cdot p - H(0,p)\right\} \\
&\geq \inf_{\tau>0} \; \tau \sup_{p \in K} \left\{\frac{x}{\tau} \cdot p\right\} \\
&= \sup_{p \in K} \; x \cdot p \\
&= \hat{u}(x).
\end{aligned} \tag{3.9}$$

From assumption (3.1), we have

$$\ell(0) = -\inf_{p \in \mathbb{R}^d} H(0,p) < 0. \tag{3.10}$$

Hence

$$\hat{u}_+(0) = 0. \tag{3.11}$$

Now for $x \neq 0$, (3.10) shows on the one hand that the infimum defining $\hat{u}_+(x)$ is not reached as $\tau \to +\infty$. On the other hand, we know that H satisfies (1.2), and then this is also the case of h, and by duality of ℓ. Therefore ℓ is superlinear and the infimum defining $\hat{u}_+(x)$ is not reached as $\tau \to 0$. Hence it is reached for some $\tau_* \in (0, +\infty)$. The first variation in τ of the map $\tau \mapsto \tau\ell(\frac{x}{\tau})$ shows that

$$\ell(\xi_*) = \xi_* \cdot p_* \quad \text{with} \quad \xi_* := \frac{x}{\tau_*} \quad \text{and} \quad p_* := D\ell(\xi_*). \tag{3.12}$$

Hence

$$h(p_*) = \sup_{\xi \in \mathbb{R}^d} \{p_* \cdot \xi - \ell(\xi)\} = p_* \cdot \xi_* - \ell(\xi_*) = 0 \quad \text{and} \quad p_* \in \partial K. \tag{3.13}$$

We get for $x \neq 0$

$$\begin{aligned}
\hat{u}_+(x) \quad &= \tau_*\ell(\xi_*) \\
&= \tau_*\xi_* \cdot D\ell(\xi_*) \\
&= x \cdot p_* \\
&\leq \hat{u}(x).
\end{aligned}$$

Together with (3.9) for $x \neq 0$ and (3.11) for $x = 0$, we deduce that $\hat{u}_+ = \hat{u}$, which shows (3.5).

Step 4: proof of (3.6)

Let us consider the function $f(\tau,x):=\tau\ell(\frac{x}{\tau})$, for all $(\tau,x)\in(0,+\infty)\times\mathbb{R}^d$. For $\tau_1,\tau_2>0$ and $\frac{x_1}{\tau_1}\neq\frac{x_2}{\tau_2}$ and $\lambda\in(0,1)$ and $\tau:=\lambda\tau_1+(1-\lambda)\tau_2$, we get

$$\begin{aligned}\tau^{-1}\left\{\lambda f(\tau_1,x_1)+(1-\lambda)f(\tau_2,x_2)\right\} &= \tau^{-1}\left\{\lambda\tau_1\ell(\frac{x_1}{\tau_1})+(1-\lambda)\tau_2\ell(\frac{x_2}{\tau_2})\right\}\\ &> \ell\left(\tau^{-1}\left\{\lambda x_1+(1-\lambda)x_2\right\}\right)\\ &= \tau^{-1}f(\tau,\lambda x_1+(1-\lambda)x_2).\end{aligned}$$

In particular for $x_1=x_2=x\neq 0$ and $\tau_1\neq\tau_2$, we see that the map $\tau\mapsto f(\tau,x)$ is strictly convex on $(0,+\infty)$. This shows the uniqueness of the minimizer $\tau_*=\tau_*(x)$ which is the first part of (3.6).
On the other hand, we have $\hat{u}(x)=\tau_*\ell(\xi_*)$ with $\xi_*=\frac{x}{\tau_*}$, and (3.12), (3.13) show that

$$\hat{u}(x)=x\cdot p_*\quad\text{with}\quad D\ell(\xi_*)=p_*\in\partial K$$

and Step 2 shows that $p_*=D\hat{u}(x)$. Therefore

$$\frac{x}{\tau_*}=\xi_*=Dh(p_*)=D_xH(0,p_*)=D_xH(0,D_x\hat{u}(x))\quad\text{and also}\quad D\hat{u}(x)=D\ell\left(\xi_*\right)\quad\text{with}\quad \xi:=\frac{x}{\tau_*(x)}$$

which shows the last part of (3.6).
Step 5: proof of (3.7)
From convex duality, we have

$$\sup_q\left\{p\cdot q-\ell(q)\right\}=H(0,p)=\sup_{(q',q)}\left\{(0,p)\cdot(q',q)-\mathcal{L}(q',q)\right\}=\sup_q\left\{p\cdot q-\inf_{q'}\mathcal{L}(q',q)\right\}$$

which shows by reverse convex duality that $\ell(q)=\inf_{q'}\mathcal{L}(q',q)$ which is (3.7).
Step 6: proof of (3.8)
Assume by contradiction that $\hat{u}$ is affine on the segment $[x_1,x_2]$ with $[0,+\infty)\cdot x_1\neq[0,+\infty)\cdot x_2$ and $x_1,x_2\neq 0$. From the positive 1-homogeneity of $\hat{u}$, we deduce that $\hat{u}$ is linear on the cone $\Lambda:=[0,+\infty)x_1+[0,+\infty)x_2$. Moreover, by property (3.6) of $\hat{u}$, we have

$$\hat{u}(x_k)=\tau_k\ell(\xi_k),\quad \xi_k:=\frac{x_k}{\tau_k},\quad \tau_k=\tau_*(x_k),\quad k=1,2$$

i.e. $\hat{u}(\xi_k)=\ell(\xi_k)$. Hence for $\lambda\in(0,1)$ and $\xi_\lambda:=\lambda\xi_1+(1-\lambda)\xi_2$, we get

$$\ell(\xi_\lambda)\geq\inf_{\tau>0}\ \tau\ell(\frac{\xi_\lambda}{\tau})=\hat{u}(\xi_\lambda)=\lambda\hat{u}(\xi_1)+(1-\lambda)\hat{u}(\xi_2)=\lambda\ell(\xi_1)+(1-\lambda)\ell(\xi_2)>\ell(\xi_\lambda)$$

where the last inequality follows from the strict convexity of ℓ. Contradiction. Hence we deduce that $\hat{u}$ is convex not affine on $[x_1,x_2]$ as desired, which proves (3.8).
This ends the proof of the lemma.

Lemma 3.3 (Barrier u_+)
Under assumptions (1.2) and (3.1) on H, for $\hat{u}$ defined in (3.2), we set

$$u_+(t,x',x):=\hat{u}(x)\quad\textit{for all}\quad(t,x',x)\in\mathbb{R}\times\mathbb{R}^m\times\mathbb{R}^d.$$

Then u_+ is a globally Lipschitz continuous viscosity solution of (1.1). Moreover any globally Lipschitz continuous viscosity solution u of (1.1) satisfies $u_+\geq u$.

Proof of Lemma 3.3
Recall that $\inf H(0,\cdot)=H(0,p_0)<0$ with $p_0\in\mathrm{Int}(K)$. Up to redefine H, we can assume that $p_0=0$. Then we have $u_+(0_{\mathbb{R}^{1+m}},\cdot)>0$ on Ω. The graph of the solution u_+ is a cone that we will shrink like an umbrella λu_+ for $\lambda>1$ to get a strict supersolution above u. Precisely, for any $\varepsilon>0$, we have for large $\lambda>1$

$$u_+^{\varepsilon,\lambda}:=\varepsilon+\lambda u_+>u\quad\text{on}\quad\mathbb{R}^{1+m}\times\overline{\Omega}.$$

This is always possible, because u is globally Lipschitz. Then we now continuously unfold the umbrella (i.e. decrease λ). Either we reach the value $\lambda = 1$, or we have to stop at a heigher level $\lambda_* > 1$ and get

$$u_+^{\varepsilon,\lambda_*} \geq u \quad \text{on} \quad \mathbb{R}^{1+m} \times \overline{\Omega}$$

and for any $\delta > 0$, there exists $X_\delta = (X'_\delta, x_\delta) \in \mathbb{R}^{1+m} \times \mathbb{R}^d$ such that $(u_+^{\varepsilon,\lambda_*-\delta} - u)(X_\delta) < 0$.
Case 1: x_δ stays bounded
Then we set $u_\delta := u(\cdot + (X'_\delta, 0))$ and up to extract a subsequence, we have $x_\delta \to x_0$, $u_\delta \to u_0$, and we get

$$u_+^{\varepsilon,\lambda_*} \geq u_0 \quad \text{with equality at} \quad (0, x_0) \in \mathbb{R}^{1+m} \times \overline{\Omega}.$$

Because $u_+^{\varepsilon,\lambda_*} - u_0 = \varepsilon$ on $\mathbb{R}^{1+m} \times \partial\Omega$, and because u_0 is globally Lipschitz continuous, we deduce that $(0, x_0) \in \mathbb{R}^{1+m} \times \Omega$. Then $u_+^{\varepsilon,\lambda_*}$ is a test function for u at $(0, x_0)$, and this gives a contradiction because $\lambda_* > 1$.
Case 2: $|x_\delta| \to +\infty$
Then we set $u_\delta := u(\cdot + X_\delta) - u(X_\delta)$ and $u_{+,\delta}^{\varepsilon,\lambda_*} = u_+^{\varepsilon,\lambda_*}(\cdot + X_\delta) - u(X_\delta)$, and up to extract a subsequence, we have $u_\delta \to u_0$ and $u_{+,\delta}^{\varepsilon,\lambda_*} \to u_{+,0}^{\varepsilon,\lambda_*}$, which gives

$$u_{+,0}^{\varepsilon,\lambda_*} \geq u_0 \quad \text{with equality at} \quad 0.$$

Here $u_{+,0}^{\varepsilon,\lambda_*}$ is a strict supersolution of $u_t + H(Du) = 0$ on $\mathbb{R}^{1+m+d}$, because $\lambda_* > 1$. Again this leads to a contradiction because $u_{+,0}^{\varepsilon,\lambda_*}$ is affine.
Conclusion
We conclude that $\lambda_* > 1$ is impossible, and then we always reach the value $\lambda = 1$. We get $\varepsilon + u_+ \geq u$. Because this is true for every $\varepsilon > 0$, we can pass to the limit $\varepsilon \to 0$ and get $u_+ \geq u$. This ends the proof of the lemma.

Lemma 3.4 (Barrier u_-)
Under assumptions (1.2) and (3.1) on H, for $\hat{u}$ defined in (3.2), we set

$$u_-(t, x', x) := -\hat{u}(-x) \quad \textit{for all} \quad (t, x', x) \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^d.$$

Then u_- is a globally Lipschitz continuous viscosity solution of (1.1). Moreover any globally Lipschitz continuous viscosity solution u of (1.1) satisfies $u \geq u_-$.

Proof of Lemma 3.4
The fact that $\hat{u} \in C^1(\Omega)$ and that $H(0, D\hat{u}) = 0$ implies that u_- is also a viscosity solution of (1.1). We now want to show that

$$u \geq u_- \quad \text{on} \quad \mathbb{R}^{1+m} \times \overline{\Omega}. \tag{3.14}$$

We start with short preliminaries and then proceed to a proof by contradiction.
Step 1: preliminaries
Step 1.1: representation formula for any solution of (1.1)
From Lemma 2.1, we have for all $t > s$
(3.15)

$$u(t, \tilde{x}) = \min \left[\inf_{\bar{s} \in [s,t),\ \tilde{y} \in \mathbb{R}^m \times \partial\Omega} \left\{ 0 + \int_{\bar{s}}^t \mathcal{L}\left(\frac{\tilde{x} - \tilde{y}}{t - \bar{s}}\right) d\sigma \right\}, \quad \inf_{\tilde{y} \in \mathbb{R}^m \times \Omega} \left\{ u(s, \tilde{y}) + \int_0^t \mathcal{L}\left(\frac{\tilde{x} - \tilde{y}}{t - s}\right) d\sigma \right\} \right].$$

Step 1.2: reformulation of $\hat{u}$
From Lemma 3.2, we have for $x \neq 0$

$$\begin{aligned} \hat{u}(x) &= \tau_* \ell(\xi_*) && \text{with} \quad \xi_* := \frac{x}{\tau_*} \\ &= \tau_* \mathcal{L}(\xi'_*, \xi_*) && \text{with} \quad \mathcal{L}(\xi'_*, \xi_*) = \inf_{\xi' \in \mathbb{R}^m} \mathcal{L}(\xi', \xi_*) \end{aligned}$$

where ξ'_* is uniquely defined from the strict convexity of $\mathcal{L}$, inherited from (1.2) satisfied by H.
Step 2: comparison with u_-

We proceed by contradiction.
Step 2.1: first consequences (of the contrapositive)
If (3.14) is false, then there exists some $\varepsilon > 0$ and some $\bar{X}_0 \in \mathbb{R}^{1+m} \times \overline{\Omega}$ such that

$$\varepsilon + u(\bar{X}_0) < u_-(\bar{X}_0). \tag{3.16}$$

From the global Lipschitz continuity of u and u_- and their common zero value on $\mathbb{R}^{1+m} \times \partial\Omega$, we deduce that there exists $r > 0$ and some ball $\bar{B}_r = \bar{B}_r(0) \subset \mathbb{R}^d$ such that

$$\varepsilon + u \geq u_- \quad \text{on} \quad \mathbb{R}^{1+m} \times \bar{B}_r. \tag{3.17}$$

Because of (3.16), we can increase the size of the ball up to some finite $r_* > 0$ such that (3.17) holds true for $r = r_*$, but not for any $r > r_*$. This implies that for any $\delta > 0$, there exists $X_\delta \in \mathbb{R}^{1+m} \times (\bar{B}_{r_*+\delta} \backslash \bar{B}_{r_*})$ such that $\varepsilon + u(X_\delta) < u_-(X_\delta)$. For $X_\delta = (t_\delta, x'_\delta, x_\delta)$ with $x_\delta \in \bar{B}_{r_*+\delta} \backslash \bar{B}_{r_*}$, we define the points $\bar{X}_\delta := (t_\delta, x'_\delta, 0)$ and $Y_\delta := (0, 0, x_\delta)$ and get

$$\varepsilon + u_\delta(Y_\delta) < u_-(Y_\delta) \quad \text{with} \quad u_\delta(X) := u(X + \bar{X}_\delta). \tag{3.18}$$

Up to extract a subsequence, we have

$$u_\delta \to u_0, \quad Y_\delta = (0,0,x_\delta) \to (0,0,y_0) = Y_0 \in \{0_{\mathbb{R}^{1+m}}\} \times \partial B_{r_*}.$$

Passing to the limit in (3.18), we get $\varepsilon + u_0(Y_0) \leq u_-(Y_0)$, and then passing to the limit in (3.17) for $r = r_*$, we get

$$\begin{cases} \varepsilon + u_0 & \geq u_- \quad \text{on} \quad & \mathbb{R}^{1+m} \times \bar{B}_{r_*} \\ \varepsilon + u_0 & = u_- \quad \text{at} \quad Y_0 = (0, y_0) \in & \mathbb{R}^{1+m} \times \partial B_{r_*}. \end{cases} \tag{3.19}$$

Step 2.2: getting a contradiction
From Step 2.1, we have
(3.20)

$$u_-(Y_0) = -\hat{u}(-y_0) = -\tau_* \mathcal{L}(\tilde{\xi}_*) \quad \text{with} \quad \tilde{\xi}_* = (\xi'_*, \xi_*), \quad \xi_* = \frac{-y_0}{\tau_*}, \quad \tau_* = \tau_*(-y_0), \quad \ell(\xi_*) = \mathcal{L}(\xi'_*, \xi_*).$$

We also have

$$\begin{aligned} 0 &= u_0(\tau_*, \tau_* \xi'_*, 0) \\ &= u_0(Y_0 + \tau_*(1, \tilde{\xi}_*)) \\ &\leq u_0(Y_0) + \int_{-\tau_*}^0 \mathcal{L}(\tilde{\xi}_*)\, d\sigma \\ &\underset{(3.20)}{=} u_0(Y_0) - u_-(Y_0) \\ &\underset{(3.19)}{=} -\varepsilon \end{aligned}$$

where we have used the representation formula (3.15) in the third line. This gives a contradiction and ends the proof of the lemma.

4 Proof of Theorem 1.1: a Liouville-type result

The proof of the Liouville-type result is based on the barriers identified in the previous section. We make full use of the representation formula along characteristics trajectories, for which we show a key equality (Lemma 4.2), that is probably standard in other contexts. From this equality we deduce that the global solutions u are concave on $\{u < u_+\}$, which will very soon imply the Liouville-type result.

Lemma 4.1 (Solution along an optimal trajectory)
Assume that H satisfies (1.2) and (3.1). Let u be a globally Lipschitz continuous solution of (1.1). Let $X_0 := (t_0, \tilde{x}_0) \in \mathbb{R}^{1+m} \times \Omega$ be such that

$$u(X_0) < u_+(X_0).$$

Then for any $\tau > 0$, there exists $X_1 := (t_0 - \tau, \tilde{y}^\tau) \in \mathbb{R}^{1+m} \times \Omega$ such that

$$u(X_0) = u(X_1) + \tau \mathcal{L}(\tilde{\xi}^\tau) \quad \textit{with} \quad \tilde{\xi} := \frac{\tilde{x}_0 - \tilde{y}^\tau}{\tau} \quad \textit{and} \quad u(X_1) < u_+(X_1).$$

Proof of Lemma 4.1

Step 1: splitting the representation formula in two parts

Recall that from the representation formula, we have for $X_0 = (t_0, \tilde{x}_0)$ and $t < t_0$

$$u(X_0) = \min(u_b^t(X_0), u_d^t(X_0)) \tag{4.1}$$

with the boundary term

$$u_b^t(X_0) := \inf_{s_0 \in [t,t_0)} \bar{u}_b^{s_0}(X_0) \quad \text{with} \quad \bar{u}_b^{s_0}(X_0) := \inf_{\tilde{y} \in \mathbb{R}^m \times \partial\Omega} \left\{ 0 + \int_{\bar{s}}^{t} \mathcal{L}\left(\frac{\tilde{x}_0 - \tilde{y}}{t_0 - s_0}\right) d\sigma \right\}$$

and the domain term

$$u_d^t(X_0) := \inf_{\tilde{y} \in \mathbb{R}^m \times \Omega} \left\{ u(t, \tilde{y}) + \int_0^t \mathcal{L}\left(\frac{\tilde{x}_0 - \tilde{y}}{t_0 - t}\right) d\sigma \right\}. \tag{4.2}$$

Step 2: The boundary contribution

We write $\tilde{x}_0 = (x'_0, x_0)$ and define $(\tau_0, \tilde{\xi}_+) \in (0, +\infty) \times \mathbb{R}^{m+d}$ such that

$$u_+(X_0) = \hat{u}(x_0) = \inf_{\tau > 0} \tau \ell(\frac{x_0}{\tau}) = \tau_0 \ell(\frac{x_0}{\tau_0}) = \tau_0 \mathcal{L}(\tilde{\xi}_+), \quad \tilde{\xi}_+ = (\xi'_+, \xi_+), \quad \xi_+ := \frac{x_0}{\tau_0}$$

where we have used Lemma 3.2.

Now for $\tau > 0$ and $s_0 := t_0 - \tau$, we get

$$\bar{u}_b^{s_0}(X_0) = \inf_{\tilde{y} \in \mathbb{R}^m \times \partial\Omega} \tau \mathcal{L}\left(\frac{\tilde{x}_0 - \tilde{y}}{\tau}\right) = \tau \mathcal{L}(\tilde{\xi}^\tau) \quad \text{for} \quad \tilde{\xi}^\tau := \frac{\tilde{x}_0 - \tilde{y}^\tau}{\tau} \quad \text{for some} \quad \tilde{y}^\tau \in \mathbb{R}^m \times \partial\Omega.$$

Recall that by convexity, we have for any $\tilde{\xi} \in \mathbb{R}^{m+d}$ with $P_+ := D\mathcal{L}(\tilde{\xi}_+) = (0, p_+)$ with $p_+ \in \partial K$

$$\begin{aligned} \mathcal{L}(\tilde{\xi}) \quad &\geq \mathcal{L}(\tilde{\xi}_+) + (\tilde{\xi} - \tilde{\xi}_+) \cdot D\mathcal{L}(\tilde{\xi}_+) \\ &\geq \mathcal{L}(\tilde{\xi}_+) + (\tilde{\xi} - \tilde{\xi}_+) \cdot P_+ \\ &= \tilde{\xi} \cdot P_+ - H(P_+) \\ &= \tilde{\xi} \cdot P_+ \end{aligned} \tag{4.3}$$

where we have used in the third line the fact that $P_+ = D\mathcal{L}(\tilde{\xi}_+)$ is equivalent to $\mathcal{L}(\tilde{\xi}_+) = \tilde{\xi}_+ \cdot P_+ - H(P_+)$. Hence

$$\tau^{-1} \bar{u}_b^{s_0}(X_0) = \mathcal{L}(\tilde{\xi}^\tau) \geq \tilde{\xi}^\tau \cdot P_+ = \frac{\tilde{x}_0}{\tau} \cdot P_+ = \tau^{-1} u_+(X_0)$$

and then $\bar{u}_b^{s_0}(X_0) \geq u_+(X_0)$, which implies

$$u_b^t(X_0) \geq u_+(X_0). \tag{4.4}$$

Step 3: The domain contribution

Because $u(X_0) < u_+(X_0)$, we deduce from (4.1) and (4.4) that

$$u(X_0) = u_d^t(X_0) \quad \text{for all} \quad t < t_0. \tag{4.5}$$

Setting $\tau := t_0 - t > 0$ and $G(\tilde{y}) := u(t, \tilde{y}) + \tau \mathcal{L}\left(\frac{\tilde{x}_0 - \tilde{y}}{\tau}\right)$, we get

$$u_+(X_0) > u(X_0) = u_d^t(X_0) = \inf_{\tilde{y} \in \mathbb{R}^m \times \Omega} G(\tilde{y}) = \inf_{\tilde{y} \in \mathbb{R}^m \times \overline{\Omega}} G(\tilde{y})$$

where the infimum is reached for some $\tilde{y}^\tau \in \mathbb{R}^m \times \overline{\Omega}$. We have used the superlinearity of $\mathcal{L}$ and the global Lipschitz regularity of u. Notice that (4.2) implies that $\tilde{y}^\tau \notin \mathbb{R}^m \times \partial\Omega$, i.e. $\tilde{y}^\tau \in \mathbb{R}^m \times \Omega$. We get

$$P_+ \cdot \tilde{x}_0 = u_+(x_0) > u(X_0) = u(t, \tilde{y}^\tau) + \tau \mathcal{L}(\tilde{\xi}^\tau) \geq u(t, \tilde{y}^\tau) + \tau P_+ \cdot \tilde{\xi}^\tau \quad \text{with} \quad \tilde{\xi}^\tau := \frac{\tilde{x}_0 - \tilde{y}}{\tau}$$

where we have used (4.3) in the last inequality. For $X_1 := (t, \tilde{y}^\tau)$, this implies $u_+(X_1) > u(X_1)$, which is the desired inequality. This ends the proof of the lemma.

Lemma 4.2 (Key equality along a characteristic $\tilde{\xi}_-$)
Assume that H satisfies (1.2) and (3.1). Let u be a globally Lipschitz continuous solution of (1.1). Let $X_0 := (t_0, \tilde{x}_0) \in \mathbb{R}^{1+m} \times \Omega$ be such that $u(X_0) < u_+(X_0)$.
Then there exists $\tilde{\xi}_- \in \mathbb{R}^{m+d}$ such that

$$P_- := D\mathcal{L}(\tilde{\xi}_-) = (0, p_-) \quad \text{with} \quad p_- \in \partial K \tag{4.6}$$

such that for all $\tau > 0$, we have

$$u(X_0) = u(X_0 - \tau(1, \tilde{\xi}_-)) + \tau\mathcal{L}(\tilde{\xi}_-).$$

Remark 4.3 *Notice that Lemma 4.2 does not exclude the existence of possibly different admissible values of $\tilde{\xi}_-$, in particular at points X_0 where u is not C^1.*

Proof of Lemma 4.2
Step 1: the direction $\tilde{\xi}_2$ remains fixed for $\tau_2 > \tau_0$
We apply two times Lemma 4.1. For any τ_0, the first time applied from the point X_0 gives the existence of some direction $\tilde{\xi}_{X_0}^{\tau_0}$ such that

$$u(X_0) = u(X_1) + \tau_0\mathcal{L}(\tilde{\xi}_{X_0}^{\tau_0}), \quad X_1 := X_0 - \tau_0(1, \tilde{\xi}_{X_0}^{\tau_0}), \quad u(X_1) < u_+(X_1).$$

For any $\tau_1 > 0$, the second time applied from the point X_1, gives the direction $\tilde{\xi}_{X_1}^{\tau_1}$ such that

$$u(X_1) = u(X_2) + \tau_1\mathcal{L}(\tilde{\xi}_{X_1}^{\tau_1}), \quad X_2 := X_1 - \tau_1(1, \tilde{\xi}_{X_1}^{\tau_1}), \quad u(X_2) < u_+(X_2).$$

Hence for $\tau_2 := \tau_0 + \tau_1$, we get

$$u(X_0) = u(X_2) + \tau_1\mathcal{L}(\tilde{\xi}_{X_1}^{\tau_1}) + \tau_0\mathcal{L}(\tilde{\xi}_{X_0}^{\tau_0}) \geq u(X_2) + \tau_2(\tilde{\xi}_2) \quad \text{with} \quad \tilde{\xi}_2 := \tau_2^{-1}(\tau_0\tilde{\xi}_{X_0}^{\tau_0} + \tau_1\tilde{\xi}_{X_1}^{\tau_1}) \tag{4.7}$$

where the convex inequality remains strict if $\tilde{\xi}_{X_1}^{\tau_1} \neq \tilde{\xi}_{X_0}^{\tau_0}$. Now for $X_2 := (t_2, \tilde{x}_2)$, we have $t_2 = t_0 - \tau_2$ and $\tilde{\xi}_2 = \dfrac{\tilde{x}_0 - \tilde{x}_2}{\tau_2}$ with $\tilde{x}_2 \in \mathbb{R}^m \times \Omega$, and we get

$$u(X_2) + \tau_2\mathcal{L}(\tilde{\xi}_2) \leq u(X_0) = u_d^{t_2}(X_0) = \inf_{\tilde{y} \in \mathbb{R}^m \times \Omega} \left\{ u(t_0 - \tau_2, \tilde{y}) + \tau_2\mathcal{L}(\tilde{\xi}) \right\} \quad \text{with} \quad \tilde{\xi} := \frac{x_0 - \tilde{y}}{\tau_2}$$

where we have used (4.5) for the first equality. Hence the infimum is reached for $\tilde{\xi} = \tilde{\xi}_2$ and we have equality in (4.7). This implies $\tilde{\xi}_{X_1}^{\tau_1} = \tilde{\xi}_{X_0}^{\tau_0} = \tilde{\xi}_2$. This also shows that we can choose $\tilde{\xi}_{X_0}^{\tau_2} = \tilde{\xi}_2$ and $\tilde{y}^{\tau_2} := \tilde{x}_2$, i.e. for all $\tau_2 > \tau_0$, there exists $\tilde{x}_2 \in \mathbb{R}^m \times \Omega$ such that $X_2 = (t_0 - \tau_2, \tilde{x}_2)$ satisfies

$$u(X_0) = u(X_2) + \tau_2\mathcal{L}(\tilde{\xi}_2) \quad \text{with} \quad \tilde{\xi}_2 = \frac{\tilde{x}_2 - x_2}{\tau_2} = \tilde{\xi}_{X_0}^{\tau_2} = \tilde{\xi}_{X_0}^{\tau_0}. \tag{4.8}$$

Because this is true for arbitrary $\tau_2 > \tau_0$, this shows that we can find a map $\tau \mapsto \tilde{\xi}_{X_0}^{\tau}$ which is constant equal to ξ_2.
Step 2: proof that $\tilde{\xi}_2 = \tilde{\xi}_-$ satisfies (4.6)
We now want to show that $\tilde{\xi}_2 = \tilde{\xi}_-$ is indeed specific, i.e. satisfies (4.6). By assumption, we have with $X_2 = X_0 - \tau_2(1, \tilde{\xi}_2)$

$$u_+(X_0) > u(X_0) = u(X_2) + \tau_2\mathcal{L}(\tilde{\xi}_2) \geq u_-(X_2) + \tau_2\mathcal{L}(\tilde{\xi}_2)$$

and then $u_+(X_0) - u_-(X_0) > u_-(X_2) - u_-(X_0) + \tau_2\mathcal{L}(\tilde{\xi}_2)$, i.e. for $P_-^{\tau_2} := Du_-(X_2)$ with $H(P_-^{\tau_2}) = 0$

$$\begin{aligned} \mathcal{L}(\tilde{\xi}_2) &\leq \frac{A}{\tau_2} + \tau_2^{-1}\{u_-(X_0) - u_-(X_2)\} \qquad \text{with} \quad A := u_+(X_0) - u_-(X_0) > 0 \\ &\leq \frac{A}{\tau_2} + Du_-(X_2) \cdot \left\{ \frac{X_0 - X_2}{\tau_2} \right\} \\ &\leq \frac{A}{\tau_2} + P_-^{\tau_2} \cdot \tilde{\xi}_2 \end{aligned}$$

where in the second line we have used the concavity of u_-. For any $P_- \in \mathbb{R}^{m+d}$ such that $H(P_-) = 0$, we set

$$S_\tau^A(P_-) := \left\{ \tilde{\xi} \in \mathbb{R}^{m+d}, \quad \mathcal{L}(\tilde{\xi}) < P_- \cdot \xi + \frac{A}{\tau} \right\}.$$

Setting $\tilde{\xi}_- := DH(P_-)$, we have $P_- = D\mathcal{L}(\tilde{\xi}_-)$ and $\mathcal{L}(\tilde{\xi}_-) = \tilde{\xi}_- \cdot P_- - H(P_-) = \tilde{\xi}_- \cdot P_-$. Therefore $\mathcal{L}(\tilde{\xi}) \geq \mathcal{L}(\tilde{\xi}_-) + (\tilde{\xi} - \tilde{\xi}_-) \cdot D\mathcal{L}(\tilde{\xi}_-) = \tilde{\xi} \cdot P_-$, i.e.

$$\mathcal{L}(\tilde{\xi}) \geq P_- \cdot \tilde{\xi} \quad \text{with equality only at} \quad \tilde{\xi} = \tilde{\xi}_- := DH(P_-) \quad \text{with} \quad H(P_-) = 0.$$

Hence from the strict convexity of $\mathcal{L}$, we deduce that for a fixed $P_- \in \mathbb{R}^{m+d}$, we have

$$\text{dist}(\left\{\tilde{\xi}_-\right\}, S^A_\tau(P_-)) \to 0 \quad \text{as} \quad \tau \to +\infty, \quad \text{if} \quad \left\{ \begin{array}{l} \tilde{\xi}_- := DH(P_-), \\ H(P_-) = 0. \end{array} \right.$$

Indeed this convergence is event true and uniform if P_- stays in some compact set. Hence we have also

$$\text{dist}(\left\{\tilde{\xi}^{\tau_2}_-\right\}, S^A_\tau(P^{\tau_2}_-)) \to 0 \quad \text{as} \quad \tau_2 \to +\infty, \quad \text{with} \quad \left\{ \begin{array}{l} \tilde{\xi}^{\tau_2}_- := DH(P^{\tau_2}_-), \\ H(P^{\tau_2}_-) = 0, \\ P^{\tau_2}_- = D\mathcal{L}(\tilde{\xi}^{\tau_2}_-), \\ P^{\tau_2}_- := Du_-(X_2) = Du_-(X_0 - \tau_2(1, \tilde{\xi}_2)). \end{array} \right.$$

Because $\tilde{\xi}_2 \in S^A_{\tau_2}(P^{\tau_2}_-)$ with $\tilde{\xi}_2$ fixed, we deduce that

$$|\tilde{\xi}_2 - \tilde{\xi}^{\tau_2}_-| \to 0, \quad P^{\tau_2}_- \to P_- = D\mathcal{L}(\tilde{\xi}_2) = Du_-(-(1, \tilde{\xi}_2)) = (0, p_-), \quad p_- \in \partial K.$$

This shows that $\tilde{\xi}^{\tau_2}_- \to \tilde{\xi}_- = \tilde{\xi}_2$ satisfies (4.6).
This ends the proof of the lemma.

Lemma 4.4 (Property of global solutions)
Assume that H satisfies (1.2) and (3.1). Let u be a globally Lipschitz continuous solution of (1.1) and let $X_0 := (t_0, \tilde{x}_0) \in \mathbb{R}^{1+m} \times \Omega$.
If $u(X_0) < u_+(X_0)$, then there exists $P_- = (0, p_-)$ with $p_- \in \partial K$ such that we have

$$u(t, \tilde{x}) \leq u(t_0, \tilde{x}_0) + P_- \cdot (\tilde{x} - \tilde{x}_0) \quad \textit{for all} \quad X := (t, \tilde{x}) \in \mathbb{R} \times \mathbb{R}^{m+d}.$$

Proof of Lemma 4.4
From Lemma 4.2 applied to the point X_0, we get the existence of some characteristic $\tilde{\xi}_-$, such that $P_- = D\mathcal{L}(\tilde{\xi}_-) = (0, p_-)$ with $p_- \in \partial K$, and for all $\tau > 0$

$$u(X_0) = u(Y_1) + \tau\mathcal{L}(\tilde{\xi}_-) \quad \text{with} \quad Y_1 := (t_1, \tilde{y}_1) = (t_0 - \tau, \tilde{x}_0 - \tau\tilde{\xi}_-).$$

From the representation formula, we also have for $X \in \mathbb{R}^{1+m+d}$ with $t - t_1 = (t - t_0) + \tau$ and $\tau > 0$ large enough

$$u(X) \leq u(Y_1) + (t - t_1)\mathcal{L}(\tilde{\xi}_1) \quad \text{with} \quad \tilde{\xi}_1 := \frac{\tilde{x} - \tilde{y}_1}{t - t_1} =: \tilde{\xi}_- + \tilde{\tilde{\xi}}, \quad \text{with} \quad \tilde{\tilde{\xi}} = \tau^{-1}A + o(\tau^{-1}), \quad A := \tilde{x} - \tilde{x}_0 - (t - t_0)\tilde{\xi}_-.$$

Hence we get

$$\begin{array}{lll} u(X) - u(X_0) & = u(X) - u(Y_1) - \tau\mathcal{L}(\tilde{\xi}_-) & \\ & \leq (t - t_0)\mathcal{L}(\tilde{\xi}_1) + \tau\left\{\mathcal{L}(\tilde{\xi}_1) - \tau\mathcal{L}(\tilde{\xi}_-)\right\} & \\ & \leq (t - t_0)\mathcal{L}(\tilde{\xi}_1) + \tau\displaystyle\int_0^1 d\sigma\, \tilde{\tilde{\xi}} \cdot D\mathcal{L}(\tilde{\xi}_- + \sigma\tilde{\tilde{\xi}}) & \\ & \to (t - t_0)\mathcal{L}(\tilde{\xi}_-) + A \cdot D\mathcal{L}(\tilde{\xi}_-) & \text{as} \quad \tau \to +\infty \\ & = (t - t_0)\left\{\mathcal{L}(\tilde{\xi}_-) - \tilde{\xi}_- \cdot D\mathcal{L}(\tilde{\xi}_-)\right\} + P_- \cdot (\tilde{x} - \tilde{x}_0) & \text{with} \quad P_- = D\mathcal{L}(\tilde{\xi}_-) \\ & = (t - t_0)H(P_-) + P_- \cdot (\tilde{x} - \tilde{x}_0) & \\ & = P_- \cdot (\tilde{x} - \tilde{x}_0) & \end{array}$$

which gives the desired result. This ends the proof of the lemma.

Corollary 4.5 (Characterization of global solutions)
Assume that H satisfies (1.2) and (3.1). Let u be a globally Lipschitz continuous solution of (1.1). Then $u(t,x',x) = u(0,0,x)$. Moreover either u is concave, or u is the minimum of the convex function $\hat{u}$ and of a concave function and $K_u := \{\hat{u} = u\}$ is strictly convex, possibly unbounded, and contains some small ball $B_r(0)$. Moreover we have

$$\hat{u}(y) \geq u(0,0,x+y) - u(0,0,x) \geq -\hat{u}(-y) \quad \textit{for all} \quad x,y \in \mathbb{R}^d. \tag{4.9}$$

Proof of Corollary 4.5
Step 1: main result
Because u is Lipschitz continuous in time-space, its time-space gradient $\hat{D}u := (u_t, Du)$ is defined almost everywhere on $\mathbb{R}^{1+m} \times \Omega$. If X_0 is such a point where $\hat{D}u$ is defined, then we have two cases.
Case 1: $u(X_0) = u_+(X_0)$
Because we know that $u \leq u_+$, and $\partial_t u_+ = 0 = D_{x'} u_+$, we deduce also that $\partial_t u(X_0) = 0 = D_{x'} u(X_0)$.
Case 2: $u(X_0) < u_+(X_0)$
Then from Lemma 4.4, we have for all $X = (t, \tilde{x}) \in \mathbb{R}^{1+m} \times \overline{\Omega}$

$$u(t,\tilde{x}) \leq u(t_0, \tilde{x}_0) + P_- \cdot (\tilde{x} - \tilde{x}_0), \quad P_- = (0, p_-), \quad p_- \in \partial K.$$

This implies that $\hat{D}u(X_0) = (0,0,p_-)$.
Conclusion
From both cases, we then conclude that $\partial_t u = 0 = D_{x'} u$ for a.e. $X_0 \in \mathbb{R}^{1+m} \times \Omega$. Hence the Lipschitz continuity of u implies

$$u(t,x',x) = u(0,0,x) \quad \text{for all} \quad (t,x',x) \in \mathbb{R} \times \mathbb{R}^m \times \overline{\Omega}.$$

Step 2: further properties of the solution
Step 2.1: first properties
In this step, we make abuse of notation, and simply note $u(x) = u(t,x',x) = u(0,0,x)$ and $u_+(x) = u_+(t,x',x) = u_+(0,0,x)$. From Lemma 4.4, if $x_0 \in \{u < u_+\} \cap \Omega$, then there exists $p_- = p_-(x_0) \in \partial K$ such that $u(x) \leq u(x_0) + (x - x_0) \cdot p_-(x_0)$ for all $x \in \overline{\Omega}$. We define for $x \in \overline{\Omega}$

$$w(x) := \inf_{x_0 \in \{u<u_+\} \cap \Omega} \{u(x_0) + (x - x_0) \cdot p_-(x_0)\}$$

which is concave by construction, and also viscosity solution of $H(0, Dw) = 0$ on $\mathbb{R}^d$. By construction of w, we also have for all $x, y \in \overline{\Omega}$

$$\hat{u}(y) := \sup_{p \in \partial K} p \cdot y \geq w(x+y) - w(x) \geq \inf_{p \in \partial K} p \cdot y \geq -\hat{u}(-y). \tag{4.10}$$

By construction of w, we also have $u \leq w$ and $u = w$ on $\{u < \hat{u}\}$, i.e.

$$u = \min(w, \hat{u}) \tag{4.11}$$

and then $w(0) \geq u(0) = 0$. Notice that (4.10) implies (4.9).
We now distinguish two cases.
Case 1: $w(0) = 0$
Then $w \leq \hat{u}$ and $u = w$ and u is concave.
Case 2: $w(0) > 0$
Then $K_u := \{u = \hat{u}\} = \{w \geq \hat{u}\} = \{\hat{u} - w \leq 0\} \subset \overline{\Omega}$. Because $\hat{u} - w$ is convex, we deduce that K_u is also convex. Moreover the condition $w(0) > 0 = \hat{u}(0)$ implies that the origin 0 is in the interior of K_u.
Step 2.2: strict convexity of K_u when $0 \in \mathbf{Int}(K_u)$
Assume by contradiction that K_u is not strictly concave. Hence there exists $[x_1, x_2] \subset \partial K_u$ with $x_1 \neq x_2$. Because $\partial K_u \subset \{\hat{u} = w\}$, we see that the equality of a convex function $\hat{u}$ and a concave function w, implies that $\hat{u} = w =$ affine on $[x_1, x_2]$. Because $\hat{u}$ is known to be strictly convex (see (3.8)), except along lines $\mathbb{R}e$ for some $e \in \mathbb{S}^{d-1}$, we deduce that there exists such e such that $[x_1, x_2] \subset \mathbb{R}e$. Moreover, the case $0 \in (x_1, x_2)$ is impossible, because we know that $p_0 = 0 \in \text{Int}(K)$, and then $\hat{u}$ is not linear on $(x_1, x_2) \ni 0$. We conclude that $[x_1, x_2] \subset (0, +\infty)e$ for such e. Then there exists an hyperplan Π tangential to the convex K_u at $x_2 \in \partial K_u$ which contains the vector e. Hence $0 \in \Pi$, which is impossible because $0 \in \text{Int}(K_u)$. We

conclude that K_u is strictly convex.
Step 2.3: conclusion
Hence u is convex on the the strictly convex set $K_u \supset B_r(0)$ for some small $r > 0$, and concave outside K_u. This ends the proof of the corollary.

Proof of Theorem 1.1
From Dichotomy Lemma 3.1, we have either $u(t, x', x) = p_0 \cdot x$ with $0 = H(0, p_0) = \inf H(0, \cdot)$, or $\inf H(0, \cdot) < 0$, and we get $u(t, x', x) = u(0, 0, x)$ from Corollary 4.5. This ends the proof of the theorem.

5 Explicit characteristics for concave 1-homogeneous solutions

We now provide a refinement of Lemma 4.2, where we characterize explicitely the set of characteristics ending at a point x_0 where the key equality holds. This result is of independent interest and is done for positively 1-homogeneous concave solutions. A large part of this section will be also used in the proof of Theorem 1.3 later in Section 7.

Lemma 5.1 (Explicit characteristics for concave 1-homogeneous solutions)
Assume that H satisfies (1.2) and (3.1), and let $K := \{H(0, \cdot) = 0\}$. Let w be a globally Lipschitz continuous solution of (1.1) which is (positively) 1-homogeneous. Let $x_0 \in \Omega$ be such that $w(x_0) < \hat{u}(x_0) = u_+(0, 0, x_0)$.
i) (General result)
Then w is concave.
Let us consider the following set

$$\Xi_-^{x_0} := \left\{ \xi_- \in \mathbb{R}^d, \quad w(x_0) = w(x_0 - \tau\xi_-) + \tau\ell(\xi_-) \quad \text{for all} \quad \tau > 0 \right\}$$

with ℓ defined in (3.5). Then we have

$$\Xi_-^{x_0} = D_x H(0, A_0) \quad \text{with} \quad A_0 := D^+ w(x_0) \cap \partial K \neq \emptyset, \quad D^+ w(x_0) \subset K_w \subset K$$

where there exists a unique compact convex set K_w (only depending on w) such that

$$w(x) = \inf_{p \in K_w} p \cdot x \quad \text{and} \quad D^+ w(x_0) = \{p \in K_w, \quad p \cdot x_0 = w(x_0)\}.$$

ii) (Properties of E_w)
Let

$$E_w := K_w \cap \partial K.$$

Then $E_w \neq \emptyset$ and we also have the convex hull reconstruction

$$K_w = co(E_w) \quad \text{and} \quad w(x) = \inf_{p \in E_w} p \cdot x. \tag{5.1}$$

Moreover, for any $p_- \in E_w$, and $\xi_- := D_x H(0, p_-)$, we have

$$\Xi_-^{x_0} = \{\xi_-\} \quad \text{for all} \quad x_0 = -\lambda\xi_- \quad \text{with} \quad \lambda > 0, \quad \text{and} \quad w(-\xi_-) = -\ell(\xi_-). \tag{5.2}$$

We also have

$$E_w = \{p_- \in \partial K, \quad w(-\xi_-) = -\hat{u}(\xi_-) \quad \text{for} \quad \xi_- := D_x H(0, p_-)\}. \tag{5.3}$$

Remark 5.2 *The notation E_w is chosen to think to it as the set of exposed points of K_w.*

Proof of Lemma 5.1
Step 1: preliminaries on w
Step 1.1: general preliminaries
Notice that w is in particular a solution to the Liouville-type problem. Hence by Corollary 4.5, either w is

concave, or the set $\{w = \hat{u}\}$ has the origin in its interior, which implies $w = \hat{u}$. This is impossible because $w(x_0) < \hat{u}(x_0)$. We conclude that w is concave, and then $w \geq -\hat{u}(-\cdot)$. Therefore

$$v(x) := -w(-x) \leq \hat{u}(x) = \sup_{p\in K} \; p\cdot x \tag{5.4}$$

is convex, and its the Legendre-Fenchel conjugate is with $K_v := D^-v(0)$

$$v^*(p) := \sup_{x\in\mathbb{R}^d} \{p\cdot x - v(x)\} = \left\{ \begin{array}{ll} 0 & \text{if} \quad p\in K_v \\ +\infty & \text{otherwise} \end{array} \right\} = (+\infty)\cdot 1_{\mathbb{R}^d\setminus K_v}(p)$$

where (5.4) implies $v^* \geq \hat{u}^* = (+\infty)\cdot 1_{\mathbb{R}^d\setminus K}$ and then $K_v \subset K$. This means that $v(x) = \sup_{p\in\mathbb{R}^d} \{p\cdot x - v^*(p)\} = \sup_{p\in K_v} \; p\cdot x$, i.e.

$$w(x) = \inf_{p\in K_w} \; p\cdot x \quad \text{with the compact convex set} \quad K_w := D^+w(0) = K_v \subset K. \tag{5.5}$$

Step 2: properties of every characteristics ending at x_0
We now choose $x_0 \neq 0$ and consider some $\xi_- \in \mathbb{R}^d$ such that

$$w(x_0) = w(x_0 - \tau\xi_-) + \tau\ell(\xi_-) \quad \text{for all} \quad \tau > 0. \tag{5.6}$$

The fact that $\ell(0) = -\inf H(0,\cdot) > 0$ excludes the case $\xi_- = 0$.
Step 2.1: far away consequences
From (5.6), we have $\ell(\xi_-) = \dfrac{w(x_0) - w(x_0 - \tau\xi_-)}{\tau} \to -w(-\xi_-)$. Recall that $w(x) \geq -\hat{u}(-x)$, and then

$$\ell(\xi_-) = -w(-\xi_-) \leq \hat{u}(\xi_-) = \inf_{\tau>0} \; \tau\ell(\frac{\xi_-}{\tau}) \leq \ell(\xi_-)$$

which shows that we have equality in each inequality. Hence

$$w(x) \geq -\hat{u}(-x) \quad \text{with equality at} \quad x := -\xi_-, \quad \text{and} \quad \tau_*(\xi_-) = 1.$$

The fact that $\hat{u} \in C^1(\Omega)$ implies that w has a derivative at $-\xi_- \neq 0$, which satisfies

$$p_- := Dw(-\xi_-) \quad \text{with} \quad K_w \ni Dw(-\xi_-) = D\hat{u}(\xi_-) \in \partial K \quad \text{and} \quad p_- = D\ell(\xi_-) \tag{5.7}$$

where we have identified the gradient in K_w using (5.5), and in ∂K using (3.3). Finally, we have used (3.6) in the last equality. Hence

$$\xi_- = D_xH(0,p_-) \quad \text{with} \quad p_- \in K_w \cap \partial K. \tag{5.8}$$

Step 2.2: close consequences
We also have $-w(-\xi_-) = \hat{u}(\xi_-) = \ell(\xi_-) = \dfrac{w(x_0) - w(x_0 - \tau\xi_-)}{\tau} = w(\dfrac{x_0}{\tau}) - w(\dfrac{x_0}{\tau} - \xi_-)$, i.e.

$$w(\frac{x_0}{\tau}) = w(\frac{x_0}{\tau} - \xi_-) - w(-\xi_-) = \frac{x_0}{\tau}\cdot p_- + o\left(\frac{x_0}{\tau}\right)$$

where we have used (5.7) to identify p_-. This gives $w(x_0) = x_0\cdot p_- + \tau\cdot o\left(\dfrac{x_0}{\tau}\right)$, which implies in the limit $\tau \to +\infty$

$$w(x_0) = x_0\cdot p_- \quad \text{with} \quad p_- \in K_w \cap \partial K.$$

From (5.5), we also deduce that $p_- \in D^+w(x_0)$, and then

$$p_- \in A_0 := D^+w(x_0) \cap \partial K. \tag{5.9}$$

Step 2.3: a property of $D^+w(x_0)$
We first recall that for $p \in D^+w(x_0)$, we have $w(x_0 + \tau h) - w(x_0) \leq p\cdot(\tau h)$, and in the limit $\tau \to +\infty$, we get $w(h) \leq p\cdot h$ which shows that $p \in D^+w(0) = K_w$. Hence $D^+w(x_0) \subset K_w$. We want to show that

$$D^+w(x_0) = K_w^{x_0} \quad \text{with} \quad K_w^{x_0} := \{p\in K_w, \quad p\cdot x_0 = w(x_0)\}. \tag{5.10}$$

Let $p \in D^+w(x_0)$. Then we have $w(x_0+h) - w(x_0) \le p \cdot h$. For $h = \pm \varepsilon x_0$ with $\varepsilon > 0$, we get in the limit $\varepsilon \to 0^+$ the two inequalities $\pm w(x_0) \le \pm p \cdot x_0$, which shows that

$$D^+w(x_0) \subset K_w^{x_0}.$$

Conversely, for any $p_0 \in K_w^{x_0}$, we have

$$w(x_0+h) - w(x_0) = \inf_{p \in K_w} \; p \cdot (x_0+h) - p_0 \cdot x_0 \le p_0 \cdot (x_0+h) - p_0 \cdot x_0 = p_0 \cdot h$$

and then $p_0 \in D^+w(x_0)$, i.e. $K_w^{x_0} \subset D^+w(x_0)$ which is the reverse inequality. We conclude to (5.10).
Step 3: reaching the set $DH(0, A_0)$ with $A_0 = D^+w(x_0) \cap \partial K$
First notice that Lemma 4.2 applies to w and shows that $\emptyset \neq \Xi_-^{x_0}$, while we know from Step 2 that $\Xi_-^{x_0} \subset D_x H(0, A_0)$. Therefore $A_0 \neq \emptyset$.
Because $D^+w(0) = K_w \subset K$, we also know that w is a subsolution at the origin. Moreover if $D^-w(0) \neq \emptyset$, then $w(x) = p \cdot x$ with $p \in K_w$ has to be a solution for $x \neq 0$, and then is also a supersolution at $x = 0$. Hence w is a solution on the whole space of

$$H(0, Dw) = 0 \quad \text{on} \quad \mathbb{R}^d.$$

From the representation formula, we have for any $\tau > 0$

$$\begin{aligned}
w(x_0) &= \inf_{\xi \in \mathbb{R}^d} \{w(x_0 - \tau\xi) + \tau\ell(\xi)\} \\
&= \inf_{\xi \in \mathbb{R}^d} \left\{ \left\{ \inf_{p \in K_w} \; p \cdot (x_0 - \tau\xi) \right\} + \tau\ell(\xi) \right\} \\
&= \inf_{p \in K_w} \left\{ p \cdot x_0 - \tau \sup_{\xi \in \mathbb{R}^d} \{p \cdot \xi - \ell(\xi)\} \right\} \\
&= \inf_{p \in K_w} \{p \cdot x_0 - \tau H(0,p)\} \\
&\le \inf_{p \in A_0} \{p \cdot x_0 - \tau H(0,p)\} \\
&= w(x_0)
\end{aligned}$$

where using (5.10), we have

$$A_0 = K_w^{x_0} \cap \partial K \neq \emptyset.$$

This shows that the infimum in p is reached for any $p_- \in A_0$. Then in the third line the suppremum in ξ is reached for $\xi = \xi_- = D\ell(p_-)$ uniquely associated to each p_-. Hence the infimum in the first line is reached for such $\xi = \xi_-$. This shows that for all $\tau > 0$, we have

$$w(x_0) = w(x_0 - \tau\xi_-) + \tau\ell(\xi_-) \quad \text{for all} \quad \xi_- := D_x H(0, p_-) \quad \text{with} \quad p_- \in A_0.$$

Step 4: proof of (5.2)
For any $x_1 \neq 0$, we know from Step 3, that $A_1 := D^+w(x_1) \cap \partial K \neq \emptyset$, and $A_1 \subset E_w := K_w \cap \partial K$. Hence

$$E_w \neq \emptyset.$$

and we can consider any $p_- \in E_w$ and $\xi_- := D_x H(0, p_-)$. Then $p_- = D\ell(\xi_-)$ and $\ell(\xi_-) + H(0, p_-) = p_- \cdot \xi_-$. Because $\inf H(0, \cdot) < 0$ and H is strictly convex, we first deduce that $\xi_- \neq 0$. We then notice that for all $\lambda > 0$ and $x_0 := -\lambda\xi_-$, we have

$$w(x_0) \ge -\hat{u}(-x_0) = -\lambda\hat{u}(\xi_-) = -\lambda\ell(\xi_-) = -\lambda p_- \cdot \xi_- = p_- \cdot x_0 \ge \inf_{p \in K_w} \; p \cdot x_0 = w(x_0) \tag{5.11}$$

which shows that we have equality in all inequalities. Because $\hat{u} \in C^1(\Omega)$, we deduce that w is C^1 at x_0 and we get

$$D^+w(x_0) = \{p_*\}, \quad p_* = D\hat{u}(-x_0) = D\hat{u}(\xi_-).$$

Moreover, we have $\ell(\xi_-) = \hat{u}(\xi_-) = \inf_{\tau > 0} \; \tau\ell(\frac{\xi_-}{\tau})$, where we have used (3.6) which also shows that $\tau_*(\xi_-) = 1$ and then that $p_* = D\hat{u}(\xi_-) = D\ell(\xi_-) = p_-$. Therefore $\Xi_-^{x_0} = \{\xi_-\}$. Moreover we have $w(x_0) = w(x_0 - \tau\xi_-) + \tau\ell(\xi_-)$, and then in the limit $\tau \to +\infty$, we get $w(-\xi_-) = -\ell(\xi_-)$.

Step 5: proof of (5.1)
Consider now some exposed point $p_0 \in \exp\, K_w$. Hence there exists a unit vector $n \in \mathbb{S}^{d-1}$ such that for $\Pi^{\pm}_{p_0,n} := \left\{p \in \mathbb{R}^d, \quad (p-p_0, \pm n) \ge 0\right\}$, we have

$$K_w \subset \Pi^-_{p_0,n}, \quad K_w \cap \Pi^+_{p_0,n} = \{p_0\}\,.$$

Then Lemma 11.3 shows that $v(x) = -w(-x)$ satisfies

$$\partial v(n) = \{p_0\}$$

i.e. that v is C^1 at n. This means that w is C^1 at $-n$ with $Dw(-n) = p_0$. Because the PDE is satisfied at $-n$, this shows that $p_0 \in \partial K$, and therefore, all exposed points of K_w are on ∂K. From ii) of Lemma 11.2 in the Appendix, we deduce that we have the following closure of the convex hull property

$$K_w = \overline{\text{co}}(\exp\, K_w) \quad \text{with} \quad \exp\, K_w \subset \partial K$$

which also implies (using the boundedness of A_2 to assert that $\overline{\text{co}}(A_2) = \text{co}(\overline{A_2})$, see for instance Theorem 1.4.3 on page 31 in [12])

$$K_w = \text{co}(\overline{\exp\, K_w}) \quad \text{with} \quad \overline{\exp\, K_w} \subset (K_w \cap \partial K) = E_w. \tag{5.12}$$

Hence $K_w = \text{co}(E_w)$. Furthermore point iii) of Lemma 11.2 shows that

$$w(x) = \inf_{p \in \overline{\exp\, K_w}} p \cdot x.$$

From the inclusion in (5.12), we deduce that

$$w(x) = \inf_{p \in E_w} p \cdot x.$$

Step 6: proof of (5.3)
We set

$$E'_w := \{p_- \in \partial K, \quad w(-\xi_-) = -\hat{u}(\xi_-) \quad \text{for} \quad \xi_- := D_x H(0, p_-)\}\,.$$

Notice that relation (5.11) for $\lambda = 1$ shows that

$$K_w \cap \partial K = E_w \subset E'_w. \tag{5.13}$$

Conversely, choose any $p_- \in \partial K$ and $\xi_- := D_x H(0, p_-)$. Because K is strictly convex and ξ_- is orthogonal to K at p_-, we get

$$\xi_- \cdot (p - p_-) < 0 \quad \text{for all} \quad p \in K \backslash \{p_-\}\,.$$

Now if $p_- \notin K_w \subset K$, then we have on the compact set K_w, the strict inequality $\sup_{p \in K_w} \xi_- \cdot (p - p_-) < 0$, i.e.

$$w(-\xi_-) = \inf_{p \in K_w} p \cdot (-\xi_-) > p_- \cdot (-\xi_-) = -\hat{u}(\xi_-)$$

where we have used (3.3) and (3.6) for the last inequality with $p_- = D\hat{u}(\xi_-)$ and $\tau_*(\xi_-) = 1$. This shows that $p_- \notin E'_w$, and then $(\partial K \backslash K_w) \cap E'_w = \emptyset$, i.e.

$$E_w \supset E'_w.$$

With (5.13), this shows that $E_w = E'_w$, which is (5.3).
This ends the proof of the lemma.

6 Key equality for localized solutions

In this section we present a localization of Lemma 4.2. For a cylinder $Q_0 \subset \mathbb{R}^{1+m+d}$, we consider solutions $u(t, x', x)$ of

$$\begin{cases} u_t + H(Du) &= 0 \qquad \text{on} \quad Q_0 \backslash \Gamma \\ u(t, x', 0) &= g(t, x', 0) \qquad \text{on} \quad \overline{Q}_0 \cap \Gamma. \end{cases} \tag{6.1}$$

We also set for $\tau, \rho > 0$

$$Q_{\tau,2\rho} := (-\tau, 0) \times B_{2\rho}, \quad \Gamma_{\tau,2\rho} := \overline{Q}_{\tau,2\rho} \cap \Gamma. \tag{6.2}$$

Lemma 6.1 (Key equality for localized solutions)
We use notation introduced in (6.2). Assume that H satisfies (1.2) and (3.1). For $\tau, \rho > 0$, let

$$u : \overline{Q}_0 \to \mathbb{R} \quad \text{with} \quad Q_0 := (-\tau, 0) \times B_{2\rho} \subset \mathbb{R} \times \mathbb{R}^{m+d}$$

be a Lipschitz continuous solution of (6.1), of Lipschitz constant $L > 0$. Define $R_L \geq 1$ such that

$$\left\{\tilde{\xi} \in \mathbb{R}^{m+d}, \quad \mathcal{L}(\tilde{\xi}) \leq L|(1,\tilde{\xi})|\right\} \subset \overline{B}_{R_L}$$

and assume that

$$\rho \geq \tau R_L \quad \text{and set} \quad \kappa := \sup_{x \in \partial B_1} \tau_*(x) > 0, \quad \alpha_0 := \inf_{n \in \mathbb{S}^{d-1},\ \xi_0 = (2\kappa)^{-1} n} \{\ell(\xi_0) - \xi_0 \cdot D\ell(\xi_0)\} > 0 \tag{6.3}$$

for τ_* defined in (3.6). For this local problem, we define the Dirichlet boundary (including the initial data)

$$\Gamma^D_{\tau,2\rho} := \Gamma_{\tau,2\rho} \cup \Gamma^{init}_{\tau,2\rho} \quad \text{with} \quad \Gamma^{init}_{\tau,2\rho} := \{-\tau\} \times \overline{B}_{2\rho}.$$

Then we have the representation formula

$$u(X) = \inf_{s \in (0,\tau],\ \tilde{\xi} \in \mathbb{R}^{m+d},\quad Y := X - s(1,\tilde{\xi}) \in \Gamma^D_{\tau,2\rho}} \left\{u(Y) + s\mathcal{L}(\tilde{\xi})\right\}, \quad \text{for all} \quad X = (0,\tilde{x}) \in \{0\} \times \overline{B}_\rho. \tag{6.4}$$

Moreover for $r \in (0,\rho]$, if $X_0 = (0,\tilde{x}_0) \notin \Gamma$ with $\tilde{x}_0 = (x_0', x_0) \in \overline{B}_r$, and

$$u(X_0) < u_+(X_0) + \inf_{s \in (0,\tau]} \left\{ \inf_{\Gamma_{s,r+sR_L}} u + \alpha_0 \max\{0, s - 2\kappa|x_0|\} \right\} \tag{6.5}$$

then there exists $\tilde{\xi}_- \in \overline{B}_{R_L}$ such that for all $s \in [0,\tau]$, we have

$$u(X_0) = u(X_0 - s(1,\tilde{\xi}_-)) + s\mathcal{L}(\tilde{\xi}_-). \tag{6.6}$$

Remark 6.2 *Lemma 6.1 allows to connect the behaviour of the solution at short distances (for small s) to large distances (large s). This result is crucial to show the uniqueness of the blow-up limit of the solutions at the boundary Γ (in Theorem 1.3).*

Proof of Lemma 6.1
We first notice that because the solution u is Lipschitz, and DH is continuous, we have finite propagation of information. Then it is possible to justify representation formula (6.4), which itself implies that minimizers $\tilde{\xi}^s$ have to satisfy $\tilde{\xi}^s \in \overline{B}_{R_L}$. Moreover for $X_0 = (0,\tilde{x}_0) \notin \Gamma$ with $\tilde{x}_0 = (x_0', x_0) \in \overline{B}_r$ and $r \in (0,\rho]$, if $Y = (-s, y', 0) \in \Gamma_{\tau,2\rho}$ is such that

$$u(X_0) = u(Y) + s\mathcal{L}(\tilde{\xi}^s), \quad Y = X_0 - s(1,\tilde{\xi}^s), \quad s\tilde{\xi}^s = (x_0' - y', x_0).$$

then we have $s \in (0,\tau]$. On the one hand, we have for $f(s,x_0) := s\ell(\frac{x_0}{s})$

$$s\mathcal{L}(\tilde{\xi}^s) \geq s\ell(\frac{x_0}{s}) = f(s,x_0) \geq \inf_{s'>0} f(s',x_0) = f(\tau_*(x_0),x_0) = \hat{u}(x_0) = u_+(X_0)$$

where the map τ_* is positively 1-homogeneous. From Step 4 of the proof of Lemma 3.2, we know that the map $f(\cdot,x_0) : (0,+\infty) \to \mathbb{R}$ is strictly convex. If $2\kappa|x_0| \geq 2\tau_*(x_0)$, then we have for

$$\alpha := f_s'(2\kappa|x_0|, x_0) = \ell(\xi_0) - \xi_0 \cdot D\ell(\xi_0) \geq \alpha_0 > 0 \quad \text{for} \quad \xi_0 := (2\kappa)^{-1}\frac{x_0}{|x_0|}$$

with α_0 defined in (6.3). Hence

$$\begin{aligned} f(s,x_0) &\geq f(2\kappa|x_0|, x_0) + \alpha(s - 2\kappa|x_0|) \\ &\geq f(\tau_*(x_0), x_0) + \alpha(s - 2\kappa|x_0|) \\ &= u_+(X_0) + \alpha(s - 2\kappa|x_0|) \end{aligned}$$

and then

$$s\mathcal{L}(\tilde{\xi}^s) \geq f(s, x_0) \geq u_+(X_0) + \alpha_0 \max\{0, s - 2\kappa|x_0|\}.$$

Hence

$$\begin{aligned} u(X_0) &= u(Y) + s\mathcal{L}(\tilde{\xi}^s) \\ &\geq \left\{ \inf_{([-s,0)\times \overline{B}_{sR_L}(X_0))\cap \Gamma_{\tau,2\rho}} u \right\} + u_+(X_0) + \alpha_0 \max\{0, s - 2\kappa|x_0|\} \\ &\geq u_+(X_0) + \inf_{\Gamma_{s,r+sR_L}} u + \alpha_0 \max\{0, s - 2\kappa|x_0|\} \end{aligned}$$

i.e.

$$u(X_0) \geq u_+(X_0) + \inf_{s\in(0,\tau]} \left\{ \inf_{\Gamma_{s,r+sR_L}} u + \alpha_0 \max\{0, s - 2\kappa|x_0|\} \right\}.$$

Then condition (6.5) prevents to have $Y \in \Gamma_{\tau,2\rho}$. Hence $Y \in \Gamma^{init}_{\tau,2\rho} \backslash (\{-\tau\} \times \tilde{\Gamma})$. As in the proof of Lemma 4.2, we show that we can choose $\tilde{\xi}^s$ independent on s for $s \in (0, \tau]$, and this shows (6.6). This ends the proof of the lemma.

7 Proof of Theorem 1.3: full directional derivatives

This section is fully devoted to the proof of Theorem 1.3.

Proof of Theorem 1.3
Up to redefine H, we can assume that $\lambda = 0 = p'$.
Step 1: blow-up limits and double blow-up limits
For $\varepsilon > 0$, we consider the blow-up functions

$$u^\varepsilon(t, \tilde{x}) = \varepsilon^{-1} u(\varepsilon t, \varepsilon \tilde{x}) \tag{7.1}$$

which are Lipschitz continuous, uniformly with respect to ε, with the same Lipschitz constant. By Ascoli-Arzela theorem, from any sequence $\varepsilon \to 0$, we can extract a subsequence (still denoted by ε) such that $u^\varepsilon \to u^0$ locally uniformly on compact sets of $\mathbb{R}^{1+m+d}$. Moreover by stability of viscosity solutions, the limit u^0 solves the whole space problem

$$\begin{cases} u^0_t + H(Du^0) = 0 & \text{on} \quad \mathbb{R}^{1+m} \times \Omega \quad \text{(in the viscosity sense)}, \\ u^0 = 0 & \text{on} \quad \mathbb{R}^{1+m} \times \partial\Omega \quad \text{(in the strong sense)}. \end{cases} \tag{7.2}$$

From Theorem 1.1, we know that $u^0 = u^0(x)$ with $x \in \overline{\Omega} = \mathbb{R}^d$ and from Corollary 4.5, we even know that

$$u^0(t, x', x) = \min\{\hat{u}(x), w(x)\} \quad \text{with } w \text{ concave solution of } H(Dw) = 0 \text{ on } \mathbb{R}^d \tag{7.3}$$

and convex $\hat{u}$ defined in (3.2). We can now consider the rescaling for $\mu > 0$

$$w^\mu(x) := \mu^{-1}\{w(\mu x) - w(0)\}.$$

Because w is globally Lipschitz and concave, on the one hand, we know that we have the blow-up convergence

$$w^\mu(x) \to w^0(x) := \lim_{\mu\to 0^+} \left\{ \frac{w(\mu x) - w(0)}{\mu} \right\} = \inf_{p\in D^+w(0)} p \cdot x \quad \text{as} \quad \mu \to 0^+.$$

On the other hand, for the same reason, we have the blow-down convergence

$$w^\mu(x) \to w^\infty(x) := \lim_{\mu\to +\infty} \left\{ \frac{w(\mu x) - w(0)}{\mu} \right\} \quad \text{as} \quad \mu \to +\infty.$$

Here by construction, both w^0 and w^∞ are globally Lipschitz continuous, concave, and moreover 1-homogeneous and solve $H(Dw) = 0$.

As a consequence, we get for $u^0 = u^0(x)$ that

$$(u^0)^\mu(x) = \mu^{-1} u^0(\mu x)$$

satisfies (using (7.3))

$$\begin{cases} (u^0)^\mu \to \min(\hat u, w^\infty) = w^\infty & \text{as} \quad \mu \to +\infty,\\ (u^0)^\mu \to \left\{ \begin{array}{lll} \min(\hat u, w^0) = w^0 & \text{if} \quad w(0) = 0 \\ \hat u & \text{if} \quad w(0) > 0 \end{array}\right| & \text{as} \quad \mu \to 0^+. \end{cases}$$

Notice that the limit of $(u^0)^\mu$ is then either equal to $\hat u$, or is concave and 1-homogeneous.

Step 2: setting of the problem

Our goal is to show later the uniqueness of the blow-up limit. If $\inf_{\mathbb{R}^d} H(0,\cdot) = 0 = H(0,p_0)$, then we know that the blow-up limit is $u^0(X) = p_0 \cdot x$ and then is unique. We then assume from Lemma 3.1 that (3.1) holds true. We start as follows.

Consider now two sequences $\varepsilon^i = \varepsilon^i_k \to 0$ for $i = 1, 2$, such that for rescaling (7.1), we have $u^{\varepsilon^i} \to u^i$ locally uniformly on compact sets of $\mathbb{R}^{1+m+d}$. Notice that each limit u^i has a shape as in (7.3). Then by a diagonal extraction argument, we can always find sequences $a^{\varepsilon^i} \to +\infty$ which go to infinity sufficiently slowly such that $a^{\varepsilon^i}\varepsilon^i \to 0$ and

$$u^{a^{\varepsilon^i}\varepsilon^i}(t,x',x) \to \hat u^i_\infty(x) \quad \text{with} \quad \hat u^i_\infty := \lim_{\mu\to+\infty} (u^i)^\mu$$

and we can similarly find sequences $b^{\varepsilon^i} \to 0^+$ which go to zero sufficiently slowly such that

$$u^{b^{\varepsilon^i}\varepsilon^i}(t,x',x) \to \hat u^i_0(x) \quad \text{with} \quad \hat u^i_0 := \lim_{\mu\to 0^+} (u^i)^\mu.$$

Hence up to redefine the sequences ε^i (by $a^{\varepsilon^i}\varepsilon^i \to 0$ or $b^{\varepsilon^i}\varepsilon^i \to 0$), and redefine the limit u^i, we can assume that for $i = 1, 2$

$$(7.4) \qquad u^{\varepsilon^i}(t,x',x) \to \hat u^i(x) \quad \text{as} \quad \varepsilon^i \to 0$$

with $\hat u^i$ solution of (1.1), where $\hat u^i$ is 1-homogeneous (i.e. $\hat u^i(\alpha x) = \alpha \hat u^i(x)$ for all $\alpha \geq 0$), and either $\hat u^i = \hat u$, or $\hat u^i$ is concave and then from (5.1), we see that there exists a compact set $E^i \subset \partial K$ such that

$$(7.5) \qquad \hat u^i(x) = \inf_{p\in E^i} p\cdot x = \inf_{p\in K^i} p\cdot x \quad \text{and} \quad K^i = \text{co}(E^i), \quad E^i = K^i \cap \partial K.$$

Moreover, we have the following property. For any $\eta > 0$, there exists $\varepsilon_\eta > 0$ such that for all $\varepsilon^i < \varepsilon_\eta$, we have for $X = (X', x)$ and $X' := (t, x')$

$$(7.6) \qquad |u^{\varepsilon^i}(X',x) - \hat u^i(x)| \leq \eta \quad \text{for all} \quad (X',x) \in [-1,1] \times \overline{B}_1.$$

Step 3: framework for a proof by contradiction

Step 3.1: position of the problem

We want to show that $\hat u^1 = \hat u^2$. Assume by contradiction that $\hat u^1 \not\equiv \hat u^2$, i.e.

$$(7.7) \qquad \hat u^1(e) < \hat u^2(e) \text{ for some unit vector } e \in \mathbb{S}^{d-1}.$$

In particular this forces $\hat u^1$ to be concave as in (7.5), and $\hat u^2$ is either equal to $\hat u$ or also concave as in (7.5). We distinguish two cases.

Case A: $\hat u^2$ is concave

Then, with notation in (7.5), we have $E^1 \neq E^2$. If $E^1 \subset E^2$, then (7.5) implies $\hat u^1 \geq \hat u^2$ which is not the case by assumption (7.7). Hence $E^1 \backslash E^2 \neq \emptyset$. We now choose

$$p_- \in E^1\backslash E^2, \quad \xi_- := D_x H(0,p_-) \neq 0.$$

Then from (5.3), we get (using $-\hat u(\xi_-) \leq \hat u^2(-\xi_-)$)

$$\hat u^1(-\xi_-) = -\hat u(\xi_-) < \hat u^2(-\xi_-) \quad \text{and} \quad \hat u^1(-\xi_-) = -\hat u(\xi_-) < 0 < \hat u(-\xi_-).$$

Case B: $\hat u^2 = \hat u$

Then consider any

$$p_- \in E^1, \quad \xi_- := D_x H(0,p_-) \neq 0.$$

Again from (5.3), we get

$$\hat{u}^1(-\xi_-) = -\hat{u}(\xi_-) < 0 < \hat{u}(-\xi_-) = \hat{u}^2(-\xi_-).$$

Conclusion
Hence in both cases, there exists $p_- \in E^1$ with $\xi_- = D_x H(0, p_-) \neq 0$ such that for $x_1 := -\lambda_1 \xi_-$ for $\lambda_1 > 0$, we have from ii) of Lemma 5.1

$$\Xi_-^{x_1} = \Xi_-^{x_1}(\hat{u}^1) = \{\xi_-\} \quad \text{and} \quad \hat{u}^1(x_1) < \min\left\{\hat{u}(x_1), \hat{u}^2(x_1)\right\}. \tag{7.8}$$

Step 3.2: framework for $\varepsilon^1 < \varepsilon^2$
We assume (7.8). The idea of the proof consists to use the key equality along a characteristic of velocity ξ_- for $\hat{u}^1$ approached by u^{ε_1}, and to propagate the information far away where now u^{ε_1} behaves like u^{ε_2}, i.e. like $\hat{u}^2$. This will lead to a contradiction because $\hat{u}^1(-\xi_-) \neq \hat{u}_2(-\xi_-)$.
We set $\varepsilon = \varepsilon^2$, and get from (7.6) that

$$|u^\varepsilon(X', x) - \hat{u}^2(x)| \le \eta \quad \text{for all} \quad (X', x) \in [-1, 1] \times \overline{B}_1.$$

For $\mu := \dfrac{\varepsilon^1}{\varepsilon_2} \in (0, 1)$, we have $|u^{\mu\varepsilon}(X', x) - \hat{u}^1(x)| \le \eta$ for all $(X', x) \in [-1, 1] \times \overline{B}_1$, i.e.

$$|u^\varepsilon(X', x) - \hat{u}^1(x)| \le \mu\eta \quad \text{for all} \quad (X', x) \in [-\mu, \mu] \times \overline{B}_\mu. \tag{7.9}$$

Step 4: core of the proof by contradiction
Step 4.1: first bound from above on $u^\varepsilon(X_\mu)$
Recall that u has Lipschitz constant $L > 0$, and fix $\tau_0 > 0$ such that $2\tau_0 R_L = 1$ with $R_L \ge 1$ given in Lemma 6.1. Then there exists some fixed $\lambda_1 > 0$ small enough such that

$$\tilde{x}_1 = (0, x_1) = (0, -\lambda_1 \xi_-) \in \overline{B}_1, \quad \tilde{x}_\mu := \mu \tilde{x}_1 \in \overline{B}_\mu.$$

Then for $\rho = \tau_0 R_L = 1/2$, we have for $\mu > 0$ small enough $B_\mu \subset B_\rho \subset B_{2\rho} \subset B_1$, and from (7.8), we have $\hat{u}^1(x_1) < \hat{u}(x_1)$ and then for $X_\mu := (0, \tilde{x}_\mu) = (0, 0, x_\mu)$ and $x_\mu = \mu x_1$

$$u^\varepsilon(X_\mu) \le \hat{u}^1(x_\mu) + \mu\eta = \mu\hat{u}(x_1) + \mu(\eta - \left\{\hat{u}(x_1) - \hat{u}^1(x_1)\right\}) < -\mu\eta + \hat{u}(x_\mu) = -\mu\eta + u_+(X_\mu)$$

for η small enough such that $0 < 2\eta < \hat{u}(x_1) - \hat{u}^1(x_1)$. We get

$$u^\varepsilon(X_\mu) \le -\mu\eta + u_+(X_\mu). \tag{7.10}$$

Step 4.2: effective bound from above on $u^\varepsilon(X_\mu)$
Now from (1.4) with $\lambda = p' = 0$, we deduce that for $s > 0$ and $\mu > 0$ (and using $R_L \ge 1$)

$$\inf_{\Gamma_{s,\mu+sR_L}} u = o(\mu + sR_L).$$

With notation of Lemma 6.1, we have for $X_0 := X_\mu$, $x_0 := x_\mu = \mu x_1 \in \overline{B}_\mu$, and $r := \mu$ (and $\tau := \tau_0$)

$$\begin{aligned} & \inf_{s \in (0,\tau_0]} \left\{ \inf_{\Gamma_{s,\mu+sR_L}} u + \alpha_0 \max\{0, s - 2\kappa|x_\mu|\} \right\} \\ \le \; & \inf_{s \in (0,\tau_0]} \{o(\mu + sR_L) + \alpha_0 \max\{0, s - 2\kappa|x_\mu|\}\} \\ \le \; & o(\mu + 2\kappa\mu|x_1|R_L)) \\ = \; & o(\mu). \end{aligned}$$

Hence from (7.10) for $\mu > 0$ small enough, we get

$$u^\varepsilon(X_\mu) < u_+(X_\mu) + \inf_{s \in (0,\tau_0]} \left\{ \inf_{\Gamma_{s,\mu+sR_L}} u + \alpha_0 \max\{0, s - 2\kappa|x_\mu|\} \right\}.$$

Step 4.3: properties along characteristics
Now from Lemma 6.1, there exists $\tilde{\xi}_-^\mu \in \overline{B}_{R_L}$ such that

$$u^\varepsilon(X_\mu) = u^\varepsilon(X_\mu - s(1, \tilde{\xi}_-^\mu)) + s\mathcal{L}(\tilde{\xi}_-^\mu) \quad \text{for all} \quad s \in [0, \tau_0]. \tag{7.11}$$

Now at the scale $\mu\varepsilon$ with $X_1 = (0, \tilde{x}_1)$, we get by change of scales $s = \mu\sigma$

$$u^\varepsilon(\mu X_1) = u^\varepsilon(\mu X_1 - \mu\sigma(1, \tilde{\xi}_-^\mu)) + \mu\sigma \mathcal{L}(\tilde{\xi}_-^\mu) \quad \text{for all} \quad \mu\sigma \in [0, \tau_0]$$

i.e.

$$u^{\mu\varepsilon}(X_1) = u^{\mu\varepsilon}(X_1 - \sigma(1, \tilde{\xi}_-^\mu)) + \sigma \mathcal{L}(\tilde{\xi}_-^\mu) \quad \text{for all} \quad \sigma \in [0, \mu^{-1}\tau_0]. \tag{7.12}$$

Step 4.4: passing to the limit
Now in the limit $\varepsilon \to 0$ with $\mu \to 0$, we have (up to extraction of subsequences)

$$u^\varepsilon \to \hat{u}^2, \quad X_\mu \to 0_{\mathbb{R}^{1+m+d}}, \quad \tilde{\xi}_-^\mu \to \tilde{\xi}_-^0 \in \overline{B}_{R_L}$$

and passing to the limit in (7.11) for $\hat{u}^i(t, x', x) = \hat{u}^i(x)$

$$\hat{u}^2(0) = \hat{u}^2(-s(1, \tilde{\xi}_-^0)) + s\mathcal{L}(\tilde{\xi}_-^0) \quad \text{for all} \quad s \in [0, \tau_0] \tag{7.13}$$

and in the limit in (7.12)

$$\hat{u}^1(X_1) = \hat{u}^1(X_1 - \sigma(1, \tilde{\xi}_-^0)) + \sigma\mathcal{L}(\tilde{\xi}_-^0) \quad \text{for all} \quad \sigma \in [0, +\infty). \tag{7.14}$$

The limit $\sigma \to +\infty$ gives

$$\hat{u}^1(-\xi_-^0) + \mathcal{L}(\tilde{\xi}_-^0) = 0 \quad \text{with} \quad \tilde{\xi}_-^0 := (\xi^{0\prime}{}_-, \xi_-^0).$$

Step 4.5: identification of the limit characteristic ξ_-^0
Now we have

$$\begin{aligned} \ell(\xi_-^0) &= \inf_{\xi' \in \mathbb{R}^m} \mathcal{L}(\xi', \xi_-^0) \\ &\le \mathcal{L}(\xi^{0\prime}{}_-, \xi_-^0) \\ &= -\hat{u}^1(-\xi_-^0) \\ &\le \hat{u}(\xi_-^0) \\ &= \inf_{\tau > 0} \tau \ell(\frac{\xi_-^0}{\tau}) \\ &\le \ell(\xi_-^0). \end{aligned}$$

Hence we have equality in all inequalities and

$$-\hat{u}^1(-\xi_-^0) = \mathcal{L}(\tilde{\xi}_-^0) = \ell(\xi_-^0). \tag{7.15}$$

Now for $x_1 = -\lambda_1 \xi_-$, relation (7.14) shows that $\xi_-^0 \in \Xi_-^{x_1} = \Xi_-^{x_1}(\hat{u}^1) = \{\xi_-\}$, i.e.

$$\xi_-^0 = \xi_- = D_x H(0, p_-) \quad \text{with} \quad p_- \in E^1 = K^1 \cap \partial K.$$

Step 4.6: contradiction
Now (7.13) also shows that

$$\hat{u}^2(-\xi_-^0) = -\mathcal{L}(\tilde{\xi}_-^0) = \hat{u}^1(-\xi_-^0)$$

where the last equality follows for instance from (7.15). For $x_1 = -\lambda_1 \xi_- = -\lambda_1 \xi_-^0$, we deduce that

$$\hat{u}^2(x_1) = \hat{u}^1(x_1)$$

which is in contradiction with (7.8).
Step 5: conclusion
We conclude that $\hat{u}^1 = \hat{u}^2$, and then 1-homogeneous blow-up limits coincide. This implies the uniqueness of the blow-up limit, which also has to be 1-homogeneous. This means that for $X = (t, x', x)$

$$\varepsilon^{-1} u(\varepsilon X) \to \hat{u}^1(x) \quad \text{locally on compact set of } \mathbb{R}^{1+m+d} \text{ as } \varepsilon \to 0$$

with $\hat{u}^1(\alpha x) = \alpha \hat{u}^1(x)$ for all $\alpha \ge 0$. This shows (1.5) and ends the proof of the theorem.

8 Proof of Theorem 1.4: strong traces of directional derivatives

This section is devoted to the proof of Theorem 1.4. We start with the following building block result.

Proposition 8.1 (Strong convergence of the blow-up gradient at the boundary)
We work under the assumptions of Theorem 1.3 with (λ, p') replaced by $(\lambda^0, {p'}^0)$. In particular, there exists $(\lambda^0, {p'}^0) \in \mathbb{R} \times \mathbb{R}^m$ and a 1-homogeneous function $u^0 : \mathbb{R}^{1+m+d} \to \mathbb{R}$ such that for $X = (t, x', x)$, we have

$$u(X) = u^0(X) + o(|X|) \quad \text{as} \quad X \to 0 \quad \text{in} \quad \mathbb{R}^{1+m+d}, \quad \text{with} \quad u^0(X) = \lambda^0 t + {p'}^0 \cdot x' + u^0(0, 0, x).$$

Recall that u is a Lipschitz continuous viscosity solution of $u_t + H(Du) = 0$ in a neighborhood of 0 in $\mathbb{R}^{1+m} \times \Omega$, with H strictly convex.
Then for $\varepsilon > 0$, the blow-up $u^\varepsilon(X) := \varepsilon^{-1} \{u(\varepsilon X) - u(0)\}$ enjoys the following strong convergence of its time-space gradient towards a 0-homogeneous function

$$(u^\varepsilon_t, Du^\varepsilon) \to (\lambda^0, Du^0) \quad \text{in} \quad L^1_{loc}(\mathbb{R}^{1+m+d}; \mathbb{R}^{1+m+d}) \quad \text{as} \quad \varepsilon \to 0. \tag{8.1}$$

Proof of Proposition 8.1
Step 1: preliminaries
Because Du^ε is uniformly bounded, we can use standard compactness of (Young) measures, and extract a subsequence (still denoted by ε) and find a family of probability measures ν_X on $\mathbb{R}^{m+d}$ for $X \in \mathbb{R}^{1+m+d}$ such that for any continuous function $F : \mathbb{R}^{m+d} \to \mathbb{R}$, we have

$$F(Du^\varepsilon) \to \bar{F} := \langle \nu_X, F \rangle = \int_{\mathbb{R}^{m+d}} F(P) d\nu_X(P) \quad \text{in} \quad L^\infty_{loc}(\mathbb{R}^{1+m+d}) \quad \text{weak} - *.$$

Notice that we also have for any measurable function $G : \mathbb{R}^{1+m+d} \times \mathbb{R}^{m+d} \to \mathbb{R}$, such that $P \mapsto G(X, P)$ is continuous on compact sets uniformly in X, and $X \mapsto G(X, P)$ is bounded uniformly for bounded P, we also have (see for instance [2])

$$G(X, Du^\varepsilon) \to \bar{G}(X) := \langle \nu_X, G(X, \cdot) \rangle = \int_{\mathbb{R}^{m+d}} G(X, P) d\nu_X(P) \quad \text{in} \quad L^\infty_{loc}(\mathbb{R}^{1+m+d}) \quad \text{weak} - *. \tag{8.2}$$

Because u is Lipschitz continuous, we have in particular almost everywhere

$$u^\varepsilon_t + H(Du^\varepsilon) = 0, \quad u^0_t + H(Du^0) = 0.$$

Step 2: limit of a nonnegative integral
We set

$$0 \le \Psi(X, P) := H(P) - H(P^0) - (P - P^0) \cdot DH(P^0) \quad \text{with} \quad P^0 = Du^0(X) \tag{8.3}$$

where the nonnegativity of Ψ (a.e. in X) follows from the convexity of H. Now for any test function $0 \le \varphi \in C^\infty_c(\mathbb{R}^{1+m+d})$, we consider the following integral

$$0 \le I^\varepsilon := \int_{\mathbb{R}^{1+m+d}} \varphi(X) \; \Psi(X, Du^\varepsilon(X)) \; dX.$$

On the other hand, setting $B^\varepsilon := -(Du^\varepsilon - Du^0) \cdot DH(Du^0)$, we get

$$\begin{aligned} I^\varepsilon &= \int_{\mathbb{R}^{1+m+d}} \varphi \left\{ B^\varepsilon + H(Du^\varepsilon) - H(Du^0) \right\} \; dX \\ &= \int_{\mathbb{R}^{1+m+d}} \left\{ \varphi B^\varepsilon + \varphi_t (u^\varepsilon - u^0) \right\} \; dX \end{aligned} \tag{8.4}$$

where we have used the PDE for the last line. From the strong uniform convergence of u^ε towards u^0, we also get $\int \varphi_t (u^\varepsilon - u^0) \; dX \to 0$. On the other hand, we split $M := DH(Du^0) \in L^\infty$ in two parts $M_\delta := \rho_\delta \star M$ and $\bar{M}_\delta := M - M_\delta$, where ρ_δ is a standard mollifier. We get $M_\delta \in C^1$ with bounded gradient, and $\bar{M}_\delta \to 0$ in $L^1_{loc}(\mathbb{R}^{1+m+d})$. Hence we write

$$B^\varepsilon = -(Du^\varepsilon - Du^0) \cdot M = -(Du^\varepsilon - Du^0) \cdot M_\delta - (Du^\varepsilon - Du^0) \cdot \bar{M}_\delta =: B^\varepsilon_\delta + \bar{B}^\varepsilon_\delta$$

and get

$$\int_{\mathbb{R}^{1+m+d}} \varphi B^\varepsilon_\delta \; dX = \int_{\mathbb{R}^{1+m+d}} (u^\varepsilon - u^0) \; \mathrm{div}(\varphi M_\delta) \; dX \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

We also have

$$|\int_{\mathbb{R}^{1+m+d}} \varphi \bar{B}^\varepsilon_\delta \; dX| \le 2\mathrm{Lip}(Du) \int_{\mathbb{R}^{1+m+d}} \varphi |\bar{M}_\delta| \; dX \to 0 \quad \text{as} \quad \delta \to 0.$$

This shows that we also get $\int \varphi B^\varepsilon \; dX \to 0$. Therefore we get

$$I^\varepsilon \to 0 = I^0 := \int_{\mathbb{R}^{1+m+d}} \varphi(X)\overline{\Psi}(X) \; dX$$

with $0 \le \overline{\Psi}(X) = \int_{\mathbb{R}^{m+d}} \Psi(X,P) \; d\nu_X(P)$ for a.e. $X \in \mathbb{R}^{1+m+d}$, where the nonnegativity of Ψ follows again from the convexity of H.

Step 3: consequence

Step 1 implies $\varphi\overline{\Psi} = 0$ a.e. for all test function $\varphi \ge 0$. Therefore we get $\overline{\Psi} = 0$ a.e. on $\mathbb{R}^{1+m+d}$. Now the strict convexity of H implies that $\mathrm{supp}(\nu_X) \subset \left\{P^0(X)\right\}$ and then

$$\nu_X(P) = \delta_0(P - P^0(X)) \quad \text{for a.e.} \quad X \in \mathbb{R}^{1+m+d}.$$

We then deduce that

$$Du^\varepsilon \to P^0 = Du^0 \quad \text{in} \quad L^1_{loc}(\mathbb{R}^{1+m+d};\mathbb{R}^d)$$

not only for the subsequence, but also for the full sequence $\varepsilon \to 0$, because any limit Young measure is a unique Dirac mass. Finally, writing again $u^\varepsilon_t - u^0_t = H(Du^\varepsilon) - H(Du^0)$ and using the fact that H is locally Lipschitz, we get the convergence $u^\varepsilon_t \to u^0_t = \lambda^0$ in L^1_{loc}. This ends the proof of the lemma.

Proof of Theorem 1.4

We precise the notation with the ball $B'_1 \subset \mathbb{R}^{1+m}$ that we distinguish from the ball $B_1 \subset \mathbb{R}^d$.

Step 1: preliminaries

For $X' = (t,x') \in B'_1$ and $(X',0) \in \Gamma$, we consider the tangential gradient $(u_t, D_{x'}u)(X',0)$ of the Lipschitz continuous function $X' \mapsto u(X',0)$. From Rademacher's theorem, we know that the tangential gradient exists a.e.. Now from Theorem 1.3 (up to a rescaling for each $X' = (t,x')$ by a factor $\mu > 0$ depending on X' such that $(X',0) + (-\mu,\mu) \times B_\mu \subset B'_1 \times B_1$), we deduce that

$$u(X+Y) - u(X) = Bu(X)(Y) + o(|Y|) \quad \text{for a.e.} \quad X = (t,x',0) \in \Gamma$$

where $Bu(X) : \mathbb{R}^{1+m+d} \to \mathbb{R}$ is 1-homogeneous and Lipschitz such that for $Y = (s,y',y)$

$$Bu(X)(Y) = \lambda s + P' \cdot y' + Bu(X)(0,y) \quad \text{with} \quad \lambda := u_t(X',0), \quad P' := D_{x'}u(X',0).$$

This shows (1.6). We now set the gradients for a.e. $X' \in B'_1$ and all $x \in B_1$

$$P^0(X',x) := D(Bu(X',0))(0,x), \quad P(X',x) := Du(X',x).$$

Step 2: rescaling and extraction of some Young measure

For $\varepsilon > 0$, we consider the anisotropic rescaling

$$P^\varepsilon(X',x) := P(X',\varepsilon x).$$

Because Du is bounded, we can use standard compactness of (Young) measures, and extract a subsequence (still denoted by ε) and find a family of probability measures ν_X on $\mathbb{R}^{m+d}$ for $X = (X',x) \in Q_0 = B'_1 \times B_1$ such that for any continuous function $F : \mathbb{R}^{m+d} \to \mathbb{R}$, we have

$$F(P^\varepsilon) \to \bar{F} := \langle \nu_X, F \rangle = \int_{\mathbb{R}^d} F(P) d\nu_X(P) \quad \text{in} \quad L^\infty_{loc}(Q_0) \quad \text{weak} - *.$$

Step 3: characterization of the Young measure

Our goal is to show that the limit Young measure ν_X is a Dirac mass of center $P^0(X)$.

Let us consider a test function $0 \leq \varphi \in C_c^\infty(B_1(0))$, and let us consider the following integral which is well defined for $\varepsilon > 0$ small enough (because φ has compact support in the unit ball) for $Y = (Y', y) \in B_1' \times B_1$

$$J^\varepsilon := |B_1'|^{-1} \int_{B_1'} \varphi(X') \left\{ \int_{B_1' \times B_1} |P((X',0) + \varepsilon Y) - P^0(X', y)| \, dY \right\} \, dX'.$$

From Proposition 8.1, we have for the special case $X = (X', 0)$ for almost every $X' \in B_1'$

$$u_X^\varepsilon(Y) := \varepsilon^{-1} \left\{ u(X + \varepsilon Y) - u(X) \right\} \to u_X^0(Y)$$

and for fixed $X = (X', 0)$

$$P(X + \varepsilon Y) = Du_X^\varepsilon(Y) \to Du_X^0(Y) = (D_{X'} u_X^0(Y', 0), D_x u_X^0(0, y)) = P^0(X', y) \quad \text{in} \quad L^1_{loc}(\mathbb{R}_Y^{1+m+d}).$$

Hence on the one hand, from the Lebesgue dominated convergence theorem, we get $J^\varepsilon \to 0$. On the other hand, consider the change of variable $Z' = X' + \varepsilon Y'$. We get

$$J^\varepsilon = |B_1'|^{-1} \int_{B_1' \times B_1} \left\{ \int_{B_1'} \varphi(Z' - \varepsilon Y') \; |P^\varepsilon(Z', y) - P^0(Z' - \varepsilon Y', y)| \, dZ' \right\} \, dY.$$

We now introduce

$$\hat{J}^\varepsilon := |B_1'|^{-1} \int_{B_1' \times B_1} \left\{ \int_{B_1'} \varphi(Z') \; |P^\varepsilon(Z', y) - P^0(Z', y)| \, dZ' \right\} \, dY$$

which satisfies

$$\hat{J}^\varepsilon - J^\varepsilon \to 0$$

from the continuity of translations in L^1 for the term P^0 and from the uniform continuity for the factor φ. Hence $\hat{J}^\varepsilon \to 0$ with (for $z = y$ and $Z = (Z', z)$)

$$\hat{J}^\varepsilon = \int_{B_1' \times B_1} \varphi(Z') \; |P^\varepsilon(Z) - P^0(Z)| \, dZ.$$

By density of continuous functions in $L^1(B_1' \times B_1)$, it is easy to justify by aproximations (of P^0) that we have (for a subsequence still denoted by ε) the following limit (as in (8.2))

$$\hat{J}^\varepsilon \to 0 = \hat{J}^0 := \int_{B_1' \times B_1} \varphi(Z') \left\{ \int_{\mathbb{R}^d} |P - P^0(Z)| \, d\nu_Z(P) \right\} \, dZ.$$

Because $\varphi \geq 0$, this implies $\text{supp}(\nu_Z) \subset \left\{ P^0(Z) \right\}$, and then

$$\nu_Z(P) = \delta_0(P - P^0(Z)) \quad \text{for a.e.} \quad Z \in B_1' \times B_1.$$

Step 4: conclusion
From the uniqueness and the expression of the Young measure ν_Z, we deduce that we have

$$P^\varepsilon \to P^0 \quad \text{in} \quad L^1(B_1' \times B_1)$$

not only for the extracted subsequence, but also for the whole sequence ε (even for a continuous parameter $\varepsilon \to 0$). Finally, the convergence of $u_t(X', \varepsilon x)$ follows from the PDE, the uniform bounds on the gradient, the L^1 convergence of the gradient P^ε, and the continuity of H. This shows convergence (1.7) of the time-space gradient. This ends the proof of the theorem.

9 Proof of Proposition 1.7: a counter-example

Proof of Proposition 1.7
Step 1: the rotation of the kink
For $x = (x_1, x_2) \in \mathbb{R}^2$, let us consider the kink function $U(x) := -|x_1|$, which is a viscosity solution of $h(|DU|) = 1$ because $h(1) = 1$. We now introduce polar coordinates $x = (x_1, x_2) = \Phi(r, \theta) := (r\cos\theta, r\sin\theta)$. Given a C^1 function $\theta_0 : (0, 1] \to \mathbb{R}$, we define

$$u(x) = (U \circ \Phi)(r, \theta - \theta_0(r)) = -r|\cos(\alpha)|, \quad \text{for} \quad \alpha := \theta - \theta_0(r).$$

For $e_r = \frac{x}{|x|}$, $e_\theta := e_r^\perp$, we have

$$-Du(x) = \{|\cos\alpha| + r\theta_0'(r) \cdot \text{sign}(\cos\alpha)\sin\alpha\}\, e_r - r\,\{\text{sign}(\cos\alpha)\sin\alpha\}\,\frac{e_\theta}{r} \quad \text{a.e.}$$

and

$$|Du|^2 = 1 + (r\theta_0')^2 + r\theta_0'\sin(2\alpha) > 0.$$

Assume that

$$r\theta_0'(r) \to 0 \quad \text{as} \quad r \to 0^+.$$

Then we have $h(\sqrt{1 + (r\theta_0')^2 + r\theta_0'\sin(2\alpha)}) \to h(1) = 1$ as $r \to 0^+$. Defining

$$a(x) := \frac{1}{h(\sqrt{1 + (r\theta_0')^2 + r\theta_0'\sin(2\alpha)})} \quad \text{for} \quad \alpha := \theta - \theta_0(r) \quad \text{and} \quad (r, \theta) := \Phi^{-1}(x).$$

we see that $a \in C(B_1)$ and u is Lipschitz continuous. Moreover u is also C^1 except on the curve

$$\Gamma_0 := \{\Phi(r, \theta), \quad (r, \theta) \in (0, 1] \times \mathbb{R}, \quad \theta = \theta(r) \mod \pi, \quad r > 0\} \cup \{0_{\mathbb{R}^2}\}$$

We have $D^+u(x) \neq \emptyset$ for all $x \in \Gamma_0$, while $D^-u(x) = \emptyset$. Because $H(x, P)$ is convex in P, we can easily show that u is a Lipschitz continuous viscosity solution of

$$a(x)h(|Du|) = 1 \quad \text{on} \quad B_1 \subset \mathbb{R}^2.$$

On the other hand, if we choose θ_0 such that

$$\theta_0(r) \to +\infty \quad \text{as} \quad r \to 0$$

then we get a counter-example. We can for instance choose $\theta_0(r) := \sqrt{-\ln\left(\frac{r}{2}\right)}$.
Step 2: checking strict convexity of g_0
We set

$$g_0(p) := h(|p|)$$

which is C^1 because $h'(0) = 0$. For $p \neq 0$, we have $Dg_0(p) = h'(|p|)\dfrac{p}{|p|}$, while $Dg_0(0) = 0$. We also have for $p \neq 0$ with $h \in C^2$

$$D^2 g_0(p) = h''(|p|)\hat{p} \otimes \hat{p} + \frac{h'(|p|)}{|p|} \cdot \{Id - \hat{p} \otimes p\} \quad \text{with} \quad \hat{p} := \frac{p}{|p|}$$

which is continuous as $p \to 0$ with limit $D^2 g_0(0) = h''(0) \cdot Id$. Hence $g_0 \in C^2$ with $D^2 g_0 > 0$.
Step 3: proof of (1.10)
Assume by contradiction that (1.10) is false. Then for any $\delta > 0$, there exists sequences $\mathbb{R}^2 \setminus \{0\} \ni b \to 0$, and $(x_b)_b$ with $x_b \in \overline{B}_{\frac{1}{2}}$, such that

$$u(x_b + b) + u(x_b - b) - 2u(x_b) > \delta|b|.$$

Defining the blow-up u_b with moving center x_b, and the normal vector n_b

$$u_b(x) := |b|^{-1}\{u(x_b + x) - u(x_b)\}, \quad n_b := \frac{b}{|b|}$$

up to extraction of a subsequence, we get $u_b \to u_0$ and $n_b \to n_0$ and

$$H(x_0, Du_0) = 0 \quad \text{on} \quad \mathbb{R}^2, \quad \text{and} \quad u_0(n_0) + u_0(-n_0) - 2u_0(0) \geq \delta > 0 \tag{9.1}$$

Then Liouville-type Theorem 1.1 (precisely (4.11)) implies that u_0 is concave which is in contradiction with (9.1). Finally the fact that ω_0 is not Dini integrable follows from Theorem 2.1.10 on page 35 in Cannarsa, Sinestrari [10], and from the fact that semiconcavity implies the existence of directional derivatives (see Theorem 3.2.1 on page 55 in [10]). This ends the proof of the proposition.

10 Proof of Theorem 1.8: generalization to variable coefficients

We start with the following result.

Lemma 10.1 (Modulus of strict convexity)
Assume that $\mathcal{L} : \mathbb{R}^{m+d} \to \mathbb{R}$ is strictly convex and C^1. Moreover assume that for some $R > 0$, there exists a continuous modulus $\tilde{\omega}_R : (0, +\infty) \to (0, +\infty)$ with $\tilde{\omega}_R(0^+) = 0$, such that

$$(P - Q, D\mathcal{L}(P) - D\mathcal{L}(Q)) \geq |P - Q|\tilde{\omega}_R(|P - Q|) \quad \text{for all} \quad P, Q \in \overline{B}_R. \tag{10.1}$$

Then we have

$$\lambda\mathcal{L}(P) + (1-\lambda)\mathcal{L}(Q) - \mathcal{L}(\lambda P + (1-\lambda)Q) \geq \lambda(1-\lambda)\tilde{\Omega}_R(|P-Q|), \quad \text{for all} \quad \lambda \in [0,1], \quad P, Q \in \overline{B}_R \tag{10.2}$$

with

$$\tilde{\Omega}_R(r) := \int_0^r \tilde{\omega}_R(s)\ ds.$$

Proof of Lemma 10.1
For $\lambda \in [0,1]$, for $P, Q \in \overline{B}_R$, and $P_\lambda := \lambda P + (1-\lambda)Q$, and $\bar{P} := P - Q$, we have

$$\begin{aligned}
& \lambda\mathcal{L}(P) + (1-\lambda)\mathcal{L}(Q) - \mathcal{L}(P_\lambda) \\
= \ & \lambda(1-\lambda)(P-Q) \cdot \int_0^1 ds\ D\mathcal{L}(P_\lambda + s(P - P_\lambda)) + (1-\lambda)\lambda(Q-P) \cdot \int_0^1 ds\ D\mathcal{L}(P_\lambda + s(Q - P_\lambda)) \\
= \ & \lambda(1-\lambda) \int_0^1 \frac{ds}{s}\ (s\bar{P}) \cdot \left\{ D\mathcal{L}(P_\lambda + s(1-\lambda)\bar{P}) - D\mathcal{L}(P_\lambda - s\lambda\bar{P}) \right\}.
\end{aligned}$$

Setting

$$p_s := P_\lambda + s(1-\lambda)\bar{P}, \quad q_s := P_\lambda - s\lambda\bar{P}, \quad \text{with} \quad p_s - q_s = s\bar{P}$$

we get $p_s, q_s \in \overline{B}_R$ and

$$\begin{aligned}
& \lambda\mathcal{L}(P) + (1-\lambda)\mathcal{L}(Q) - \mathcal{L}(P_\lambda) \\
= \ & \lambda(1-\lambda) \int_0^1 \frac{ds}{s}\ (p_s - q_s) \cdot \{D\mathcal{L}(p_s) - D\mathcal{L}(q_s)\} \\
\geq \ & \lambda(1-\lambda) \int_0^1 ds|\bar{P}|\tilde{\omega}_R(s|\bar{P}|) \\
= \ & \lambda(1-\lambda)\tilde{\Omega}_R(|\bar{P}|)
\end{aligned}$$

where we have applied (10.1) in the third line. This shows (10.2), which ends the proof of the lemma.

Proof of Theorem 1.8
Part A: Proof of generalization of Theorem 1.3
Up to redefine H, we can assume that $\lambda = 0 = p'$.
Steps 1 to 3: as in the proof of Theorem 1.3
These steps are identical to the ones of the proof of Theorem 1.3, except that after the first blow-up, equation (7.2) has to be replaced by the following

$$\left\{ \begin{array}{rll} u_t^0 + H_0(Du^0) = 0 & \text{on} \quad \mathbb{R}^{1+m} \times \Omega & \text{(in the viscosity sense)} \\ u^0 = 0 & \text{on} \quad \mathbb{R}^{1+m} \times \partial\Omega & \text{(in the strong sense).} \end{array} \right. \tag{10.3}$$

where in (7.2) the Hamiltonian $H(P)$ has been replaced by

$$H_0(P) = H(0_{\mathbb{R}^{1+m+d}}, P) \quad \text{for} \quad P \in \mathbb{R}^{m+d}.$$

Again u^0 is unique and linear if $\inf_{\mathbb{R}^d} H_0(0_{\mathbb{R}^m}, \cdot) = 0$. We then assume that H_0 satisfies (3.1), and set

$$K := \left\{ p \in \mathbb{R}^d, \quad H_0(0_{\mathbb{R}^m}, p) \le 0 \right\}.$$

We recall the obtained framework. For any $\eta > 0$, there exists $\varepsilon_\eta > 0$ such that for all $i = 1, 2$ and $\varepsilon^i < \varepsilon_\eta$, we have for $X = (X', x)$ and $X' := (t, x')$

$$|u^{\varepsilon^i}(X', x) - \hat{u}^i(x)| \le \eta \quad \text{for all} \quad X \in [-1, 1] \times \overline{B}_1.$$

For $\varepsilon = \varepsilon^2 > \varepsilon^1$ and $\mu = \dfrac{\varepsilon^1}{\varepsilon^2} \in (0, 1)$, we have in particular

$$\begin{cases} |u^\varepsilon(X', x) - \hat{u}^2(x)| \le \eta & \text{for all} \quad X \in [-1, 1] \times \overline{B}_1, \\ |u^\varepsilon(X', x) - \hat{u}^1(x)| \le \mu\eta & \text{for all} \quad X \in [-\mu, \mu] \times \overline{B}_\mu. \end{cases}$$

Moreover, we have $\hat{u}^1(x) = \inf_{p \in E^1} p \cdot x$ with the compact set $E^1 \subset \partial K$. There exists $p_- \in E^1$ with $\xi_- = D_x H(0, p_-)$ such that for $x_1 := -\lambda_1 \xi_-$ for $\lambda_1 > 0$ and with notation of Lemma 5.1, we have

$$\Xi_-^{x_1} = \Xi_-^{x_1}(\hat{u}^1) = \{\xi_-\} \quad \text{and} \quad \hat{u}^1(x_1) < \min\left\{ \hat{u}(x_1), \hat{u}^2(x_1) \right\}.$$

Step 4: core of the proof by contradiction
Step 4.1: first bound from above on $u^\varepsilon(X_\mu)$
We proceed as in Step 4.1 of the proof of Theorem 1.3. Recall that u has Lipschitz constant $L > 0$, and fix $\tau_0 > 0$ such that $2\tau_0 R_L = 1$ with $R_L \ge 1$ given in (1.12). Then for $\rho = \tau_0 R_L = 1/2$, and for $\mu > 0$ small enough, we get

$$u^\varepsilon(X_\mu) \le -\mu\eta + u_+(X_\mu) \tag{10.4}$$

where $X_\mu := (0, \tilde{x}_\mu) = (0, 0, x_\mu)$ and $x_\mu = \mu x_1$ with $\tilde{x}_1 = (0, x_1) = (0, -\lambda_1 \xi_-) \in \overline{B}_1$ and $\tilde{x}_\mu := \mu \tilde{x}_1 \in \overline{B}_\mu$. Starting from now, the proof differs from the one of Theorem 1.3.
Step 4.2: strict convexity quantified
From Lemma 10.1 (applied for frozen X), we have with $P_\lambda = \lambda P + (1 - \lambda) Q$

$$\lambda \mathcal{L}(X, P) + (1-\lambda)\mathcal{L}(X, Q) - \mathcal{L}(X, P_\lambda) \ge \lambda(1-\lambda)\tilde{\Omega}_R(|P - Q|), \quad \text{for all } \lambda \in [0, 1],\ P, Q \in \overline{B}_R,\ X \in Q_0. \tag{10.5}$$

Step 4.3: minimization and dyadic estimate
Then we still have the following representation formula (identified with the solution, because the comparison principle still arises for Lipschitz continuous solutions, even for low regularity in X, here continuity). Now for $\rho = \dfrac{1}{2}$ and all $\tau \in (0, \tau_0)$, and $X = (0, \tilde{x}) \notin \Gamma$ with $\tilde{x} \in B_\rho$, we have

$$u(X) = \inf_{s \in [-\tau, 0),\ \gamma(s) \in \Gamma^D_{\tau, 2\rho},\ \gamma(0) = X,\ \gamma((s, 0)) \subset B_\rho \setminus \tilde{\Gamma}} \left\{ u(\gamma(s)) + \int_s^0 d\sigma\ \mathcal{L}(\sigma, \gamma(\sigma), \dot{\gamma}(\sigma)) \right\}$$

where $\tilde{\Gamma} := \mathbb{R}^m \times \{0_{\mathbb{R}^d}\}$. Notice in particular that the L-Lipschitz continuity of u imposes (for R_L given in (1.12))

$$|\dot{\gamma}(\sigma)| \le R_L \tag{10.6}$$

and to have short hand notation, we set

$$\omega := \omega_R, \quad \tilde{\Omega} := \tilde{\Omega}_R, \quad R := 1 + R_L.$$

From the convexity of $\mathcal{L}$ in the variable $\dot{\gamma}(\sigma)$ and its continuity in $X = (\sigma, \gamma(\sigma))$, the existence of a minimizer γ_* with "optimal exit time" s_* is known.

Case A: $\gamma_*(s_*) \in \Gamma_{\tau,2\rho}$
Recall that $X = (0, \tilde{x})$ with $\tilde{x} = (x', x)$ and $x \neq 0$. Then we have

$$|x| \leq |\tilde{x} - \gamma_*(s_*)| = \left| \int_{s_*}^0 \dot{\gamma}_*(\sigma)\ d\sigma \right| \leq R_L |s_*| \quad \text{with} \quad s_* \in [-\tau, 0).$$

This shows that for $\tau > 0$ small enough, Case A is ruled out.
Case B: $\gamma_*(s_*) \notin \Gamma_{\tau,2\rho}$
Because $\gamma_*(s_*) \notin \Gamma_{\tau,2\rho}$, then $s_* = \tau$ and

$$u(X) - u(-\tau, \gamma_*(-\tau)) = \int_{-\tau}^0 d\sigma\ \mathcal{L}(\sigma, \gamma_*(\sigma), \dot{\gamma}_*(\sigma)) \geq \int_{-\tau}^0 d\sigma\ \mathcal{L}(0, \tilde{x}, \dot{\gamma}_*(\sigma)) - \tau\omega(\tau R)$$

where we have used (10.6) in the last inequality.

Setting the approximate characteristic velocity $\tilde{\xi}_{\tilde{x}}^\tau := \dfrac{\tilde{x} - \gamma_*(-\tau)}{\tau}$, we get by convexity of $\tilde{\xi} \mapsto \mathcal{L}(X, \tilde{\xi})$ that

$$u(X) - u(-\tau, \gamma_*(-\tau)) \geq \tau \mathcal{L}(X, \tilde{\xi}_{\tilde{x}}^\tau) - \tau\omega(\tau R).$$

Similarly for $2\tau \in (0, \tau_0)$, we have (by monotonicity of ω_R)

$$u(-\tau, \gamma_*(-\tau)) - u(-2\tau, \gamma_*(-2\tau)) \geq \tau \mathcal{L}(X, \tilde{\xi}_{\gamma_*(-\tau)}^\tau) - \tau\omega(2\tau R), \quad \tilde{\xi}_{\gamma_*(-\tau)}^\tau := \frac{\gamma_*(-\tau) - \gamma_*(-2\tau)}{\tau}$$

and then

$$u(X) - u(-2\tau, \gamma_*(-2\tau)) \geq \tau \mathcal{L}(X, \tilde{\xi}_{\tilde{x}}^\tau) + \tau \mathcal{L}(X, \tilde{\xi}_{\gamma_*(-\tau)}^\tau) - 2\tau\omega(2\tau R).$$

Now notice also that

$$\begin{aligned} u(X) - u(-2\tau, \gamma_*(-2\tau)) &= \textstyle\int_{-2\tau}^0 d\sigma\ \mathcal{L}(\sigma, \gamma_*(\sigma), \dot{\gamma}_*(\sigma)) \\ &= \inf_{\gamma(0)=\tilde{x},\ \gamma(-2\tau)=\gamma_*(-2\tau),\ |\dot{\gamma}(\sigma)| \leq R} \int_{-2\tau}^0 d\sigma\ \mathcal{L}(\sigma, \gamma(\sigma), \dot{\gamma}(\sigma)) \\ &\leq 2\tau\omega(2\tau R) + \inf_{\gamma(0)=\tilde{x},\ \gamma(-2\tau)=\gamma_*(-2\tau),\ |\dot{\gamma}(\sigma)| \leq R} \left\{ \int_{-2\tau}^0 d\sigma\ \mathcal{L}(X, \dot{\gamma}(\sigma)) \right\} \\ &\leq 2\tau\omega(2\tau R) + \int_{-2\tau}^0 d\sigma\ \mathcal{L}(X, \tilde{\xi}_{\tilde{x}}^{2\tau}) \end{aligned}$$

with

$$\tilde{\xi}_{\tilde{x}}^{2\tau} := \frac{\tilde{x} - \gamma_*(-2\tau)}{2\tau} = \frac{1}{2} \left\{ \tilde{\xi}_{\tilde{x}}^\tau + \tilde{\xi}_{\gamma_*(-\tau)}^\tau \right\}.$$

Hence we get $\tau \mathcal{L}(X, \tilde{\xi}_{\tilde{x}}^\tau) + \tau \mathcal{L}(X, \tilde{\xi}_{\gamma_*(-\tau)}^\tau) - 2\tau\omega(2\tau R) \leq 2\tau\omega(2\tau R) + 2\tau \mathcal{L}(X, \tilde{\xi}_{\tilde{x}}^{2\tau})$. Therefore

$$2\omega(2\tau R) \geq \frac{1}{2} \left\{ \mathcal{L}(X, \tilde{\xi}_{\tilde{x}}^\tau) + \mathcal{L}(X, \tilde{\xi}_{\gamma_*(-\tau)}^\tau) \right\} - \mathcal{L}(X, \tilde{\xi}_{\tilde{x}}^{2\tau}) \geq \frac{1}{4} \tilde{\Omega}(|\tilde{\xi}_{\tilde{x}}^\tau - \tilde{\xi}_{\gamma_*(-\tau)}^\tau|)$$

where the last inequality follows from (10.5) for $\lambda = \frac{1}{2}$. Notice that $\tilde{\xi}_{\tilde{x}}^\tau - \tilde{\xi}_{\tilde{x}}^{2\tau} = \frac{1}{2} \left\{ \tilde{\xi}_{\tilde{x}}^\tau - \tilde{\xi}_{\gamma_*(-\tau)}^\tau \right\}$. Hence

$$2\omega(2\tau R) \geq \frac{1}{4} \tilde{\Omega}(2|\tilde{\xi}_{\tilde{x}}^\tau - \tilde{\xi}_{\tilde{x}}^{2\tau}|).$$

Because $\tilde{\Omega}$ is convex, C^1 with $\tilde{\Omega}'(0) = 0 = \tilde{\Omega}(0)$, we have the chord inequality $\tilde{\Omega}(\theta r) \leq \theta \tilde{\Omega}(r)$ for all $\theta \in [0, 1]$. Hence

$$\omega(2\tau R) \geq \frac{1}{8} \tilde{\Omega}(2|\tilde{\xi}_{\tilde{x}}^\tau - \tilde{\xi}_{\tilde{x}}^{2\tau}|) \geq \tilde{\Omega}(\frac{1}{4} |\tilde{\xi}_{\tilde{x}}^\tau - \tilde{\xi}_{\tilde{x}}^{2\tau}|)$$

and

$$|\tilde{\xi}_{\tilde{x}}^\tau - \tilde{\xi}_{\tilde{x}}^{2^{-1}\tau}| \leq 4(\tilde{\Omega}^{-1} \circ \omega)(\tau R) \quad \text{for all} \quad \tau \in (0, \tau_0).$$

Step 4.4: conclusion by dyadic sums
If $f : [0, 1) \to [0, +\infty)$ is non-decreasing, then we have

$$\int_0^1 \frac{ds}{s}\ f(s) = \sum_{j \in \mathbb{N}} \int_{2^{-(j+1)}}^{2^{-j}} \frac{ds}{s}\ f(s) \geq \sum_{j \in \mathbb{N}} \frac{1}{2} f(2^{-(j+1)}).$$

Therefore we get

$$\sum_{j\in\mathbb{N}} |\tilde{\xi}_{\tilde{x}}^{2^{-j}\tau} - \tilde{\xi}_{\tilde{x}}^{2^{-(j+1)}\tau}| \le 4\sum_{j\in\mathbb{N}} (\tilde{\Omega}^{-1}\circ\omega)(2^{-j}\tau R) \le 8\int_0^1 \frac{ds}{s}(\tilde{\Omega}^{-1}\circ\omega)(2\tau sR) = 8\int_0^{2\tau R} \frac{ds}{s}(\tilde{\Omega}^{-1}\circ\omega)(s).$$

When Dini condition (1.13) holds true, this implies that the characteristic velocity converges

$$\tilde{\xi}_{\tilde{x}}^{2^{-j}\tau} \to \tilde{\xi}_{\tilde{x}}^{0} \quad \text{as} \quad j \to +\infty.$$

This is then sufficient to imply the non rotation of the blow-up limit, and then the convergence of the blow-up to a unique limit, along the same lines as the remaining part of the proof of Theorem 1.3, using equality along the characteristic curve γ_*:

$$u(X) = u(-\tau, \gamma_*(-\tau)) + \int_{-\tau}^0 d\sigma\ \mathcal{L}(\sigma, \gamma_*(\sigma), \dot{\gamma}_*(\sigma)) \quad \text{for all} \quad \tau \in (0, \tau_0).$$

We skip the details.

Part B: Proof of generalization of Theorem 1.4

Step 1: in the adaptation of the proof of Proposition 8.1

Once we know the convergence of the blow-up $u^\varepsilon \to u^0$ locally uniformly, the proof is very similar to the one of Theorem 1.4. We just have to replace everywhere $H(P)$ by $H(0,P)$ (and then in particular in the definition (8.3) of $\Psi(X,P)$, and in the expression of B^ε). The definition of the integral I^ε is unchanged. Because of the relations

$$u^\varepsilon + H(\varepsilon X, Du^\varepsilon) = 0 \quad \text{and} \quad u_t^0 + H(0, Du^0) = 0$$

we just have in the second line of (8.4), to introduce the error term $A^\varepsilon := -\{H(\varepsilon X, Du^\varepsilon) - H(0, Du^\varepsilon)\}$ which converges uniformly to zero, and then does not affect the reasoning.

Step 2: in the adaptation of the proof of Theorem 1.4

The proof is unchanged (notice that to show the convergence of $u_t(X', \varepsilon x)$ we have to use the PDE with the X-dependence, but the same argument applies).

This ends the proof of the theorem.

11 Appendix

In this appendix, we grasp together some results on exposed points of closed convex sets, that are useful in the main part of the paper.

We recall the following definitions.

Definition 11.1 (Extreme points and exposed points)
Let $K \subset \mathbb{R}^d$ be a convex compact set.

i) (Extreme point)
We say that $p_0 \in K$ is an extreme point K and denote it by $p_0 \in ext\ K$, if there are no two different points $p_1, p_2 \in K$ such that $p_0 = \lambda p_1 + (1-\lambda)p_2$ for some $\lambda \in (0,1)$.

ii) (Exposed point)
We say that $p_0 \in K$ is an exposed point of K and denote it by $p_0 \in exp\ K$, if there exists a closed half space $\Pi^\pm_{p_0,n} := \left\{p \in \mathbb{R}^d, \quad (p - p_0, \pm n) \ge 0\right\}$ for some non-zero vector n (we can in particular choose a unit vector $n \in \mathbb{S}^{d-1}$) such that

$$K \subset \Pi^-_{p_0,n}, \quad \Pi^+_{p_0,n} \cap K = \{p_0\}.$$

We get immediately exp $K \subset$ ext K, and have the following complementary result.

Lemma 11.2 (Exposed points of convex compact sets)
Let $K \subset \mathbb{R}^d$ be a convex compact set.

i)(Extreme-exposed relation)
We have

$$exp\ K \subset ext\ K \subset \overline{exp\ K}. \tag{11.1}$$

ii) (Property of exposed points)
We have

$$K = \overline{co}(exp\ K). \tag{11.2}$$

iii) (Support functions)
For every $x \in \mathbb{R}^d$, we have

$$\max_{p\in K} p\cdot x = \max_{p\in \overline{exp\ K}} p\cdot x = \sup_{p\in exp\ K} p\cdot x. \tag{11.3}$$

Furthermore the solution-set of the first problem is the convex hull of the solution-set of the second problem

$$\underset{p\in K}{Argmax}\quad p\cdot x = co\left(\underset{p\in \overline{exp\ K}}{Argmax}\quad p\cdot x\right). \tag{11.4}$$

Proof of Lemma 11.2
The results are more or less classical. For completness of the argument we give some details.
Step 1: proof of i)
The first inclusion in (11.1) is straightforward, and the second inclusion is Straszewicz's theorem [19] (see also theorem 1.4.7 on page 18 in Schneider [19], or theorem 18.6 on page 167 in Rockafeller [18]).
Step 2: proof of ii)
The classical Minkowski theorem (see Theorem 2.3.4 on page 42 in [12]) claims that if K is a compact convex set, then $K = \text{co}(\text{ext } K)$. Then (11.1) implies $K = \text{co}(\text{ext } K) \subset \overline{\text{co}}(\text{exp } K) \subset K$, which shows (11.2).
Step 3: proof of iii)
Let $\varphi(p) := p\cdot x$. From Proposition 2.4.6 on page 46 in [12], we know that point iii) is true for the set $\overline{\text{exp } K}$ replaced by ext K, i.e.

$$\max_K \varphi = \max_{\text{ext } K} \varphi \quad \text{and} \quad \underset{K}{\text{Argmax}}\ \varphi = \text{co}\left(\underset{\text{ext } K}{\text{Argmax}}\ \varphi\right). \tag{11.5}$$

Then (11.1) implies that

$$\sup_{\text{exp } K} \varphi \le \sup_{\text{ext } K} \varphi = \max_{\text{ext } K} \varphi \le \sup_{\overline{\text{exp } K}} \varphi = \sup_{\text{exp } K} \varphi$$

where the last equality follows from the continuity of φ. This shows (11.3). Now $\text{co}\left(\underset{\text{ext } K}{\text{Argmax}}\ \varphi\right) = \text{co}\left(\underset{\overline{\text{exp } K}}{\text{Argmax}}\ \varphi\right)$ and (11.5) show (11.4). This ends the proof of the lemma.

Lemma 11.3 (Exposed point and pointwise C^1 support function)
Let $K \subset \mathbb{R}^d$ be a convex compact set and its support function $v(x) := \sup_{p\in K} p\cdot x$. Assume that p_0 is an exposed point of K with admissible unit normal n, i.e for $\Pi^{\pm}_{p_0,n} := \{p \in \mathbb{R}^d,\ (p - p_0, \pm n) \ge 0\}$, we have

$$K \subset \Pi^-_{p_0,n}, \quad K \cap \Pi^+_{p_0,n} = \{p_0\}. \tag{11.6}$$

Then the subdifferential of the convex function v satisfies

$$\partial v(n) = \{p_0\} \tag{11.7}$$

i.e. v is C^1 at n.

Proof of Lemma 11.3
Notice that the Legendre-Fenchel transform of v is $v^* = (+\infty)1_{\mathbb{R}^d\setminus K}$. Moreover for p_0 as in (11.6), we have $n \in \partial v^*(p_0)$ and then

$$v^*(p_0) + v(n) = p_0\cdot n, \quad v^*(p_0) = 0, \quad p_0 \in \partial v(n). \tag{11.8}$$

Then (11.6) implies that for all $p \in \mathbb{R}^d \setminus \{p_0\}$, we have $\sup_{x\in\mathbb{R}^d} \{p \cdot x - v(x)\} = v^*(p) > v^*(p_0) + n \cdot (p - p_0)$. Hence there exists $x_p \in \mathbb{R}^d$ such that

$$p \cdot x_p - v(x_p) > v^*(p_0) + n \cdot (p - p_0) = n \cdot p - v(n)$$

where we have used (11.8) in the equality. This means $v(x_p) < v(n) + p \cdot (x_p - n)$ and then $p \notin \partial v(n)$. Hence (11.8) implies $\partial v(n) = \{p_0\}$, which ends the proof of the lemma.

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