Isaacs equations with Lyapunov functional

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June 10, 2026

Abstract

In this short Note, we study a class of evolutive Hamilton-Jacobi-Isaacs equations in dimension 2 with Hamiltonians in separate variables. These equations arise in differential games. We show that there is an unexpected Lyapunov functional given by the integral of some convex function of the gradient of the solution, related to the Hamiltonian.

Keywords: Isaacs equation, Lyapunov functional, Hamilton-Jacobi equations, L^p a priori estimates.

1 Introduction

In this Note, let us consider viscosity solutions $u(t,x,y)$ with $t \in (0,T)$ and $(x,y) \in \mathbb{R}^2$, for $T > 0$ of the following Hamilton-Jacobi equation of Isaacs' type

$$u_t + H_1(u_x) - H_2(u_y) = 0 \quad \text{on} \quad (0,T) \times \Omega \tag{1.1}$$

with initial data

$$u(0,\cdot) = u_0 \quad \text{on} \quad \{0\} \times \Omega, \tag{1.2}$$

where Ω is either $\mathbb{R}^2$ or a two-dimensional torus.

We will be interested in viscosity solutions, for which we refer the reader to [4] for first order equations, and also [7] for second order equations. Up to the knowledge of the author, no Lyapunov functional was known for Hamilton-Jacobi equations in dimension bigger than one. Still we have the following simple but unexpected result (at least for the author).

Theorem 1.1 (A Lyapunov functional)
For $j = 1,2$, assume that

$$H_j : \mathbb{R} \to \mathbb{R} \text{ is } \textbf{convex} \text{ such that } 0 = \inf_{\mathbb{R}} H_j \text{ and } \liminf_{|q| \to +\infty} \frac{H_j(q)}{|q|} = +\infty. \tag{1.3}$$

Assume that $\Omega = \mathbb{R}^2$ and that the initial data u_0 is continuous with compact support. Then the unique viscosity solution u of (1.1)-(1.2) is such that the following quantity

$$E(u(t,\cdot)) := \int_\Omega (H_1(u_x) + H_2(u_y))(t,\cdot)\ dxdy \tag{1.4}$$

is finite nonnegative and nonincreasing in time for $t \in [0,T)$, provided that $E(u_0)$ is finite.

In particular, Theorem 1.1 provides a sort of Sobolev (or Orlicz) estimate on the solution which is preserved by the PDE. Unfortunately, we have no generalization of Theorem 1.1 in higher dimensions.

Remark 1.2 (The meaning of $E(u_0)$)
Recall that we assume here the growth of H_1, H_2 to be superlinear at infinity. Then notice that finite $E(u_0)$ requires that the distributional derivatives satisfy $(u_0)_x, (u_0)_y \in L^1_{loc}(\mathbb{R}^2)$, in order to make sense to the integral with values in $[0,+\infty]$. Moreover, when the integral is finite, we say that $E(u_0)$ is finite. The same is required for $E(u(t,\cdot))$.

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Remark 1.3 *For the limit case $H_1(z) = H_2(z) = |z|$, the result of Theorem 1.1 is still true, but $E(u(t,\cdot))$ has to be interpreted for elements $u_x(t,\cdot), u_y(t,\cdot), \in \mathcal{D}'(\Omega)$ which are measures, i.e. with $E(u(t,\cdot))$ replaced by $|u_x(t,\cdot)|(\Omega) + |u_y(t,\cdot)|(\Omega)$.*

Remark 1.4 (Isaacs equation)
Equation (1.1) is a Hamilton-Jacobi equation of Isaacs' type. Indeed, if we denote by H_j^* the Legendre-Fenchel transform of H_j, then we can define the Hamiltonian H and introduce some Lagrangian $\mathcal{L}$

$$H(p,q) := H_1(p) - H_2(q) \quad \text{and} \quad \mathcal{L}(p,q,p',q') := \{p'p - H_1^*(p')\} - \{q'q - H_2^*(q')\}$$

such that the Hamiltonian satisfies

$$H(p,q) = \sup_{p'\in\mathbb{R}} \inf_{q'\in\mathbb{R}} \mathcal{L}(p,q,p',q') = \inf_{q'\in\mathbb{R}} \sup_{p'\in\mathbb{R}} \mathcal{L}(p,q,p',q'). \tag{1.5}$$

In differential games theory, Condition (1.5) exchanging inf/sup *(or* min/max*), is called the "minimax" condition of Isaacs. We refer the reader for instance the original book [9], classical references like [8] for representation formulas in Dynamic Programming, and the book [3], and also more recent developments like [1] on the ergodic problem and in particular the survey [6] on differential games.*

Up to the knowledge of the author, the long time convergence of the solutions of (1.1) seems to remain an open problem in general (see for instance [4]).

Remark 1.5 *For instance for $H_1(z) = H_2(z) = |z|^2$, $\lambda_1 = 1 = \lambda_2$, we see that $u_\infty(x,y) := \alpha\ \phi(x+y)$ with $\phi(z) := \inf_{k\in\mathbb{Z}} (z+k)^2$ and parameter $\alpha \geq 0$ is a family of stationary viscosity solutions of (1.1). This example shows explicitly the non uniqueness of the solutions (even up to addition of constants).*

As an illustration of our Lyapunov method, let us consider solutions $u = u(x,y)$ to the following stationary equation for $p \in [1,+\infty)$ on the 2-dimensional torus Ω

$$u + H(u_x, u_y) = f \quad \text{on} \quad \Omega := \mathbb{R}^2/\Lambda \quad \text{with} \quad \Lambda := \mathbb{Z}\lambda_1 e_x + \mathbb{Z}\lambda_2 e_y, \quad \lambda_j > 0, \quad j = 1,2 \tag{1.6}$$

where (e_x, e_y) is the standard orthonormal basis of $\mathbb{R}^2$, where $f = f(x,y)$ is a given function for $(x,y) \in \Omega$. With classical methods, it is easy to show estimates like $|u_x|_{L^\infty(\Omega)} \leq |f_x|_{L^\infty(\Omega)}$ and $|u_y|_{L^\infty(\Omega)} \leq |f_y|_{L^\infty(\Omega)}$. Here with our method, we get furthermore the following result.

Proposition 1.6 (A priori estimate on the torus for stationary solutions)
Let $\lambda_1, \lambda_2 > 0$, and let Ω be the 2-dimensional torus defined in (1.6). Assume that H satisfies (1.3).
i) (First estimate)
Assume u and f to be continuous real functions on Ω and that u is a viscosity solution of (1.6). Then we have

$$E(u) \leq E(f) \tag{1.7}$$

for E defined in (1.4), and where the left hand side is bounded, as soon as the right hand side is (and the derivatives of u, f are taken in the distributional sense).
ii) (A refined estimate)
Assume furthermore that

$$\text{for any } R > 0, \text{ there exists } \delta_R > 0 \text{ such that } H_1'', H_2'' \geq 2\delta_R \quad \text{on} \quad (-R,R). \tag{1.8}$$

and that $|f_x|_{L^\infty(\Omega)}, |f_y|_{L^\infty(\Omega)} \leq R$. Then we have the following refinement of (1.7) for $Du := (u_x, u_y)$

$$E(u) + \delta_R \int_\Omega |Du - Df|^2\ dxdy \leq E(f). \tag{1.9}$$

2 The proofs

Proof of Theorem 1.1
The core of the proof is contained in Step 2.1. The remaining parts of the proof can be seen as technical steps to justify Step 2.1.
The existence and uniqueness of a solution u follows from the general theory of viscosity solutions (see [4] and more generally [7]). Without loss of generality (up to replace $u(t,\cdot)$ by $u(t,\cdot)-\lambda t$ for some suitable constant λ), we can assume moreover that $H_j(0)=0=\inf H_j$ for $j=1,2$. We now assume the following restrictive condition

$$H \text{ and } u_0 \text{ are smooth,} \tag{2.1}$$

that we will remove later on.
Step 1: finite velocity of propagation and reduction to a periodic problem
By assumption on u_0, there exists some radii $R_0, R_1>0$ such that

$$\text{supp}(u_0)\subset\subset \bar{B}_{R_0}(0) \quad \text{and} \quad Du_0(x)\in B_{R_1}(0) \quad \text{for a.e. } x\in\mathbb{R}^2.$$

Then define $L:=\sup_{P\in B_{R_1}(0)}|DH(P)|$. The classical comparison principle (with finite velocity of propagation) shows here that

$$\text{supp}(u(t,\cdot))\subset\subset \bar{B}_{R_0+Lt}(0). \tag{2.2}$$

Indeed, it comes from the existence of a cone of dependence

$$\mathcal{C}=\mathcal{C}_{t_0,x_0}:=\left\{(s,x)\in(-\infty,t_0]\times\mathbb{R}^2, \quad |x-x_0|\le L(t_0-s)\right\} \quad \text{with} \quad \mathcal{C}^t:=\mathcal{C}\cap\{s=t\}$$

which implies for $t\ge 0$ the inclusion $\mathbb{R}^2\backslash\text{supp}(u(t,\cdot))\supset \bigcup_{\mathcal{C}^0\subset\ \mathbb{R}^2\backslash\bar{B}_{R_0}(0)}\mathcal{C}^t$.
From (2.2), we can consider the "periodic" cell

$$\Omega_R:=[-R,R)^2 \quad \text{with} \quad R:=R_0+LT$$

and periodize the initial data into $u_0^R(x):=u_0(x+2kR)$ for the unique $k\in\mathbb{Z}^2$ such that $x+2kR\in\Omega_R$. Then it is easy to check that the solution u^R to (1.1) with initial data u_0^R satisfies

$$u^R(t,x)=u(t,x) \quad \text{for all} \quad (t,x)\in[0,T)\times\Omega_R.$$

Step 2: the vanishing viscosity approach
Step 2.1: the ε-viscosity
We then consider a vanishing viscosity approximation u^ε of the solution u, solving

$$u_t^\varepsilon+H(u_x^\varepsilon,u_y^\varepsilon)=\varepsilon\Delta u^\varepsilon \quad \text{on} \quad (0,T)\times\Omega \tag{2.3}$$

where here $\Omega=\mathbb{R}^2$, and with initial data u_0^R. We set

$$v:=u_x^\varepsilon, \quad w:=u_y^\varepsilon.$$

Taking the x and y derivatives, we get the following two equations

$$\left\{\begin{array}{l} v_t+H_v'v_x+H_w'w_x=\varepsilon\Delta v\\ w_t+H_v'v_y+H_w'w_y=\varepsilon\Delta w.\end{array}\right. \tag{2.4}$$

We see this system as an hyperbolic system, for which we look for an entropy. To this end, we set (with some abuse of notation)

$$\beta(v,w):=H_1(v)+H_2(w)$$

and we multiply the first line of (2.4) by $\beta_v'=H_1'(v)$ and the second line by $\beta_w'=H_2'(w)$. We get

$$\begin{aligned} &\beta_t+\beta_v'H_v'v_x+\beta_w'H_w'w_y+\{\beta_w'H_v'+\beta_v'H_w'\}\,u_{xy}^\varepsilon\\ =\ &\varepsilon\left\{\text{div}(H_1'(v)\nabla v)+\text{div}(H_2'(w)\nabla w)\right\}-\varepsilon\left\{H_1''(v)|\nabla v|^2+H_2''(w)|\nabla w|^2\right\}.\end{aligned}$$

If there exists a couple of functions $(A, B) = (A, B)(v, w)$ such that

$$(2.5)\qquad \begin{cases} A'_v & := & \beta'_v H'_v, \\ B'_w & := & \beta'_w H'_w, \\ A'_w + B'_v & := & \beta'_w H'_v + \beta'_v H'_w \end{cases}$$

then we get

$$(2.6)\qquad \beta_t + A_x + B_y = \varepsilon\Delta\beta - \varepsilon\left\{H''_1(v)|\nabla v|^2 + H''_2(w)|\nabla w|^2\right\} \le \varepsilon\Delta\beta.$$

Notice that (2.5) is satisfied for

$$A = A(v), \quad B = B(w) \quad \text{with} \quad \begin{cases} A'_v & = & (H'_1(v))^2, \\ B'_w & = & -(H'_2(w))^2, \\ A'_w + B'_v & = & 0 = \beta'_w H'_v + \beta'_v H'_w. \end{cases}$$

The integration of (2.6) on the periodic cell Ω_R gives $\frac{d}{dt}\left(\int_{\Omega_R} \beta(v, w)(t, \cdot)\, dxdy\right) \le 0$. This implies

$$\int_{\Omega_R} \beta(u^\varepsilon_x, u^\varepsilon_y)(t, \cdot)\, dxdy \le \int_{\Omega_R} \beta((u_0)_x, (u_0)_y)\, dxdy \quad \text{for all} \quad t \in [0, T).$$

Step 2.2: the limit $\varepsilon \to 0$
Because u^R is continuous, it is classical for viscosity solutions that for any $\delta \in (0, T)$, we have

$$u^\varepsilon \to u^R \quad \text{uniformly on} \quad [0, T - \delta] \times \Omega_R \quad \text{as} \quad \varepsilon \to 0.$$

This implies for $t \in [0, T)$ that $(u^\varepsilon_x, u^\varepsilon_y)(t, \cdot) \to (u^R_x, u^R_y)(t, \cdot)$ in $\mathcal{D}'(\mathbb{R}^2)$ as $\varepsilon \to 0$. We also have for $t \in [0, T)$ that $\sup_{(x,y)\in\mathbb{R}^2} |(u^\varepsilon_x, u^\varepsilon_y)(t, x, y)| \le R_1$. Hence we conclude that

$$(u^\varepsilon_x, u^\varepsilon_y)(t, \cdot) \rightharpoonup (u^R_x, u^R_y)(t, \cdot) \quad \text{in} \quad (\text{weak-}L^2(\Omega_R))^2 \quad \text{as} \quad \varepsilon \to 0.$$

Notice also that the functional

$$\begin{array}{rlcl} \Phi: & (L^2(\Omega_R))^2 & \to & \mathbb{R} \cup \{+\infty\} \\ & (q, r) & \mapsto & \int_{\Omega_R} \beta(q, r)\, dxdy \end{array}$$

is convex. We now state the following result which will be proven below:

Claim A: Φ is lower semi-continuous for the strong topology of $(L^2(\Omega_R))^2$.

Then it is known (see [5]) that Φ is also lower semi-continuous for the weak topology of $(L^2(\Omega_R))^2$. This implies that $\Phi((u^R_x, u^R_y)(t, \cdot)) \le \liminf_{\varepsilon\to 0} \Phi((u^\varepsilon_x, u^\varepsilon_y)(t, \cdot)) \le \Phi((u_0)_x, (u_0)_y)$ and then

$$(2.7)\qquad E(u(t, \cdot)) = \Phi((u_x, u_y)(t, \cdot)) = \Phi((u^R_x, u^R_y)(t, \cdot)) \le \Phi((u_0)_x, (u_0)_y) = E(u_0).$$

The general case (i.e. without assuming (2.1)) follows by smooth approximation of the convex functions H_1, H_2 and of the initial data u_0. For the approximation of the initial data, notice that if $0 \le \rho \in C^\infty_c(\mathbb{R}^2)$ is of integral 1, and for $\eta > 0$, we can define the regularized initial data $u^\eta_0 := \rho_\eta \star u_0$ obtained by convolution of u_0 by the mollifier $\rho_\eta = \eta^{-2}\rho(\eta\cdot)$. For $\eta > 0$ small enough we still have $\text{supp}(u^\eta_0) \subset\subset \bar{B}_{R_0}(0)$. From the convexity of β, we then deduce for $Du_0 := ((u_0)_x, (u_0)_y)$ that

$$E(u^\eta_0) = \int_{\Omega_R} \beta(\rho_\eta \star Du_0) = \int_{\mathbb{R}^2} \beta(\rho_\eta \star Du_0) \le \int_{\mathbb{R}^2} \rho_\eta \star (\beta(Du_0)) = \left(\int_{\mathbb{R}^2} \rho_\eta\right) \cdot \left(\int_{\mathbb{R}^2} \beta(Du_0)\right) = E(u_0)$$

which implies from Claim A that $E(u^\eta_0) \to E(u_0)$ as $\eta \to 0$.
Step 3: proof of Claim A
The result for Φ follows from showing that the convex functional for $j = 1, 2$

$$\begin{array}{rlcl} \Phi_j: & L^2(\Omega_R) & \to & \mathbb{R} \cup \{+\infty\} \\ & q & \mapsto & \int_{\Omega_R} H_j(q)\, dxdy \end{array}$$

is lower semicontinuous for the strong topolopy of $L^2(\Omega_R)$. To this end, let us consider a sequence $q^n \to q$ in $L^2(\Omega_R)$ and assume by contradiction that

$$\Phi_j(q) > \lim_{n\to+\infty} \Phi_j(q^n). \tag{2.8}$$

For $K \geq 0$, define the truncation

$$T_K(a) := \begin{cases} K & \text{if} \quad a > K, \\ a & \text{if} \quad a \in [-K, K], \\ -K & \text{if} \quad a < -K. \end{cases}$$

Then

$$\Phi_j(T_K(q)) \to \Phi_j(q) \quad \text{as} \quad K \to +\infty.$$

Up to extract a subsequence (still denoted by $(q^n)_n$), we also have $q^n \to q$ a.e. on Ω_R, and then

$$\Phi_j(T_K(q^n)) \to \Phi_j(T_K(q)) \quad \text{as} \quad n \to +\infty.$$

Case 1: $\Phi_j(q) < +\infty$
Given any $\eta > 0$, let us now fix K large enough such that $\Phi_j(T_K(q)) \geq -\eta/2+\Phi_j(q)$ and let us also fix n large enough such that $\Phi_j(T_K(q^n)) \geq -\eta/2 + \Phi_j(T_K(q))$. This shows that $\Phi_j(q^n) \geq \Phi_j(T_K(q^n)) \geq -\eta + \Phi_j(q)$ and then

$$\Phi_j(q) \leq \liminf_{n\to+\infty} \Phi_j(q^n) \tag{2.9}$$

which gives a contradiction with (2.8), and then proves (2.9).
Case 2: $\Phi_j(q) = +\infty$
This case is a simple adaptation of Case 1, still concluding to (2.9). We then deduce Claim A.

Step 4: conclusion
We can now repeat Step 2 with $(0, u_0)$ replaced by $(s, u(s, \cdot))$ with $s \in [0, T)$ and then deduce from (2.7)

$$E(u(t, \cdot)) \leq E(u(s, \cdot)) \quad \text{for all} \quad t \in [s, T).$$

This ends the proof of the theorem.

Proof of Proposition 1.6
Step 1: proof of i)
Again, the general theory insures that the viscosity solution exists and is unique. We then start to consider the vanishing viscosity approximation of the PDE, as in Step 2.1 of the proof of Theorem 1.1 (with similar notation), and with the solution u^ε of

$$u^\varepsilon + H(u^\varepsilon_x, u^\varepsilon_y) = f + \varepsilon\Delta u^\varepsilon \quad \text{on} \quad \Omega$$

and assume f to be smooth (up to remove the smoothness of f after getting the a priori estimates, as usual). Similarly to the derivation of (2.4), we get for $v := u^\varepsilon_x$ and $w := u^\varepsilon_y$

$$\begin{cases} v + H'_v v_x + H'_w w_x = f_x + \varepsilon\Delta v \\ w + H'_v v_y + H'_w w_y = f_y + \varepsilon\Delta w. \end{cases} \tag{2.10}$$

Setting again $\beta(v, w) := H_1(v) + H_2(w)$ for $H(v, w) = H_1(v) - H_2(w)$, we proceed like in in Step 2.1 of the proof of Theorem 1.1, and multiply the first line of (2.10) by β'_v and the second line by β'_w, to deduce now

$$v\beta'_v + w\beta'_w + A_x + B_y = \beta'_v f_x + \beta'_w f_y + \varepsilon\Delta\beta - \varepsilon\left\{H''_1(v)|\nabla v|^2 + H''_2(w)|\nabla w|^2\right\} \tag{2.11}$$

Using Taylor expansion with integral reminder, we get

$$\begin{aligned} H_1(f_x) - H_1(v) &= \int_0^1 ds\ H'_1(v + s(f_x - v)) \cdot (f_x - v) \\ &= \int_0^1 ds\ H'_1(v) \cdot (f_x - v) + \int_0^1 ds \int_0^1 dt\ H''_1(v + st(f_x - v)) \cdot s(f_x - v)^2 \\ &= \int_0^1 ds\ H'_1(v) \cdot (f_x - v) + \int_0^1 ds \int_0^s d\tau\ H''_1(v + \tau(f_x - v)) \cdot (f_x - v)^2 \end{aligned}$$

Hence integrating (2.11) over Ω, we get

$$\begin{array}{ll} 0 & \geq \int_\Omega \ \beta'_v(v-f_x)+\beta'_w(w-f_y)\ dxdy \\ & = \int_\Omega \ \{\beta(v,w)-\beta(f_x,f_y)+D_0\}\ dxdy \end{array}$$

with

$$0 \leq D_0 := D_0(v,w;f_x,f_y) := \int_0^1 ds \int_0^s d\tau\ \left\{H_1''(v+\tau(f_x-v))\cdot(f_x-v)^2 + H_2''(w+\tau(f_y-w))\cdot(f_y-w)^2\right\}$$

where the nonnegativity of D_0 follows from the convexity of H_1, H_2. We get

$$E(u)+\int_\Omega D_0\ dxdy \leq E(f). \tag{2.12}$$

This implies the result of point i) as in the proof of Theorem 1.1.

Step 2: proof of ii)

If $|f_x|_{L^\infty(\Omega)}, |f_y|_{L^\infty(\Omega)} \leq R$, we can easily deduce from standard maximum principle techniques, that

$$|u_x|_{L^\infty(\Omega)}, |u_y|_{L^\infty(\Omega)} \leq R \tag{2.13}$$

From assumption (1.8), we see that $D_0 \geq \delta_R \left\{(u_x-f_x)^2+(u_y-f_y)^2\right\}$ which implies (1.9). This ends the proof of the proposition.

Aknowledgements

The author thanks J. Dolbeault, T. Lelièvre and G. Stoltz for providing him good working conditions. The author also thanks P. Cardaliaguet for motivating discussions on the topic. This research was partially funded by l'Agence Nationale de la Recherche (ANR), project ANR-22-CE40-0010 COSS.

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