A priori gradient bounds for fully nonlinear parabolic equations and applications to porous medium models

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Abstract. We prove a priori gradient bounds for classical solutions of the fully nonlinear parabolic equation

$$u_t = F(D^2u, Du, u, x, t).$$

Several applications are given, including the standard porous medium equation.
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1 Introduction

Consider the general fully nonlinear parabolic problem

$$u_t = F(D^2u, Du, u, x, t), \quad (x,t) \in Q := \mathbb{T}^d \times (0, +\infty), \tag{1.1}$$

$$u(x, 0) = u_0(x), \quad x \in \mathbb{T}^d. \tag{1.2}$$

To simplify our arguments, we consider the case of the d-dimensional torus $\mathbb{T}^d := (\mathbb{R}/\mathbb{Z})^d$ for $d \geq 1$. Note that up to the price of technicalities, the case of the whole space $\mathbb{R}^d$ could be also considered. The aim of the paper is to find assumptions on F in order to obtain, for all $t \geq 0$, a gradient bound on Du of the form

$$\|Du(\cdot, t)\|_\infty \leq \|Du_0\|_\infty. \tag{1.3}$$

As an application of our general approach, we prove gradient estimate (1.3) for the weak nonnegative solution of the standard porous medium equation

$$u_t = \Delta u^m, \quad (x,t) \in Q, \tag{1.4}$$

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where $1 \leq m \leq 1 + \frac{4}{3+d}$. For such range of m, this result is new.
Similar gradient estimates are given for the problem

$$u_t = \Delta G(u), \quad (x,t) \in Q, \tag{1.5}$$

for some class of functions G, and for the doubly nonlinear problem

$$u_t = \text{div}\,\left(\psi(u, |Du|^2)Du\right), \quad (x,t) \in Q, \tag{1.6}$$

for some class of functions ψ. Our estimate will be applied to two specific examples of equation (1.6) arising in hydrology (and this was our initial motivation for this work). These two examples are the following equations

$$u_t = \text{div}\,\left(u(1-u)\frac{Du}{1+|Du|^2}\right), \quad (x,t) \in Q, \tag{1.7}$$

and

$$u_t = \text{div}\,\left(u(1-u)Du\right), \quad (x,t) \in Q. \tag{1.8}$$

Equation (1.8) derives from equation (1.7) as an approximation for small gradients. In these two equations the function u represents the height of the sharp interface between salt and fresh water in a confined aquifer, see for instance [2, 5, 11].

1.1 Main results

In this subsection, we will present our main results. To this end, we will start by an assumption on the function F appearing in equation (1.1). In order to write this assumption, we need to introduce some notation.

For two symmetric matrices $X = (x_{ij})_{1 \leq i,j \leq d}$ and $Y = (y_{ij})_{1 \leq i,j \leq d}$ in $\mathbb{R}^{d \times d}$, we denote by $X : Y$ the inner scalar product $\sum_{i,j=1,..,d} x_{ij}y_{ji} = tr(XY)$. Moreover for $p, q \in \mathbb{R}^d$, we set $(X \cdot p)_i = \sum_{j=1,..,d} x_{ij}p_j$ and $p \cdot q = \sum_{j=1,..,d} p_j q_j$. For later use, we also denote by $tr(X)$ the trace of X.

Assumption 1.1
Let $d \geq 1$ and let $\underline{u} < \overline{u}$ and $M > 0$ be three real numbers,

$$E := S^d \times \overline{B(0,M)} \times [\underline{u}, \overline{u}] \times Q \quad \subset \quad S^d \times \mathbb{R}^d \times \mathbb{R} \times Q,$$

where S^d is the space of $d \times d$-symmetric real matrices and $F = F(X, p, u, x, t)$ is a real function defined on E satisfying the following conditions:

i) ***Regularity****:* $F \in C^1(E)$.

ii) ***Degenerate parabolicity****: For all $X, Y \in S^d$, $(p, u, x, t) \in \mathbb{R}^d \times \mathbb{R} \times Q$, we have the implication*

$$\textit{if } X \leq Y, \qquad \textit{then } F(X,p,u,x,t) \leq F(Y,p,u,x,t).$$

iii) ***Differential inequality****:*

$$-D_X F : X^2 + |p|^2 D_u F + D_x F \cdot p \leq 0, \tag{1.9}$$

for all $(X, p, u, x, t) \in E$ such that $X \cdot p = 0$.

Theorem 1.2 ***(A priori gradient bound for Problem (1.1))***
Suppose that F satisfies Assumption 1.1 and let us assume the existence of a function $u \in C^3(\overline{Q})$ solution of the parabolic problem (1.1), (1.2) such that for all $t \geq 0$, we have

$$\underline{u} \leq u(\cdot,t) \leq \overline{u} \qquad \text{and} \qquad \|Du(\cdot,t)\|_{L^\infty(\mathbb{T}^d)} \leq M; \tag{1.10}$$

then for all $t \geq 0$

$$\|Du(\cdot,t)\|_{L^\infty(\mathbb{T}^d)} \leq \|Du_0\|_{L^\infty(\mathbb{T}^d)}. \tag{1.11}$$

Moreover, if F satisfies the following condition

$$F(0,0,C,x,t) = 0, \text{ for any constant } C \in [\underline{u},\overline{u}]; \tag{1.12}$$

then for all $t \geq 0$

$$\min_{\mathbb{T}^d} u_0 \leq u(\cdot,t) \leq \max_{\mathbb{T}^d} u_0. \tag{1.13}$$

Remark 1.3 ***(Generalization)***
Up to adapt the proofs with certain technicalities, it would also be possible to get a similar result for the same equation on $\Omega \times (0,T)$ with $\Omega = \mathbb{R}^d$ or $\mathbb{T}^d$, and $T > 0$.

Corollary 1.4 ***(A priori gradient bounds for Problem (1.5))***
Let $d \geq 1$ and $G \in C^3([\underline{u},\overline{u}])$, satisfying $G' \geq 0$ on $[\underline{u},\overline{u}]$ and

$$\frac{d-1}{4}(G'')^2 \leq -G^{(3)}G' \quad \text{ on } [\underline{u},\overline{u}]. \tag{1.14}$$

Let us assume the existence of $u \in C^3(\overline{Q})$ solution of (1.5), (1.2) such that u and Du satisfy condition (1.10) for some $M > 0$; then for all $t \geq 0$, u and Du satisfy a priori bounds (1.13) and (1.11).

The next corollary gives a new result for the standard porous medium equation (1.4).

Corollary 1.5 ***(Application to the standard porous medium equation)***
Let $d \geq 1$, $1 \leq m \leq 1 + \dfrac{4}{3+d}$ and $u_0 \in W^{1,\infty}(\mathbb{T}^d)$ with u_0 nonnegative. Then there exists a unique nonnegative weak solution $u \in L^\infty(0,+\infty; W^{1,\infty}(\mathbb{T}^d)) \cap C([0,\infty); L^1(\mathbb{T}^d))$ of Problem (1.4), (1.2). Moreover, u satisfies bounds (1.13) and (1.11).

1.2 Brief review of the literature

When F does not depend on u, certain gradient estimates of the form (1.3) do exist in the literature for solutions of Problem (1.1), (1.2). For instance (see [4]), such estimates hold true for solutions of the equation

$$u_t = \operatorname{div}(\psi_0(|Du|^2)Du),$$

in any dimension $d \geq 1$ and under some assumptions on ψ_0. Notice also that Assumption 1.1 iii) is always satisfied if F does not depend on u neither on x.
Let us notice that in [13], a uniform gradient estimate is given for bounded solutions of the elliptic equation

$$\lambda u + F_0(D^2u, Du, x) = 0, \quad x \in \mathbb{R}^d,$$

where $\lambda > 0$ and under some hypothesis on F_0. Such elliptic equation can be seen as a time discretization of evolution equation of type (1.1) when F does not depend on u.
In [15], a uniform gradient bound is given for the general quasilinear equation

$$u_t = a(x,t,u,Du) + \sum_{i,j=1}^{d} a_{ij}(x,t,u,Du)\partial_{x_i x_j} u, \quad (x,t) \in \mathcal{O} \times (0,T),$$

where $\mathcal{O}$ is an open bounded domain of $\mathbb{R}^d$ and T is small enough. This gradient bound depends both on the gradient of the initial data and on a bound on the gradient of the boundary of $\mathcal{O}$.

In [14], a bound on $D(G^\alpha(u))$ for some $\alpha > \frac{1}{2}$ is given for solutions of (1.5), (1.2). The bound is given by a large constant depending on the initial data.
Applying this result to the standard porous medium equation (1.4), (1.2), with the choice $G(u) = u^m$ for m in a certain range, this gives for all $t \geq 0$

$$\|Du(\cdot,t)\|_\infty \leq M_0(\|Du_0\|_\infty), \tag{1.15}$$

where $M_0(\|Du_0\|_\infty)$ is a large constant depending on the initial data.
Notice that a particular application of the general results in [10] also gives a similar bound for different range of values of m.
For the same equation, in any dimension $d \geq 1$, it is possible to deduce from [16] certain interior gradient estimates assuming some local integrability of the gradient of a power of the solution.
For the porous medium equation, the only known gradient bound of the form (1.3) is given in [17, Proposition 15.4, p. 359] where the author proves the general result in dimension $d = 1$ and for $t \geq 0$

$$\|Du^{m-1}(\cdot,t)\|_\infty \leq \|Du_0^{m-1}\|_\infty.$$

In particular, for $m = 2$, this gives an estimate of the form (1.3). Notice that our corollary (1.5) gives gradient estimate of the form (1.3) at least in dimension $d = 1$ for $1 \leq m \leq 2$.

We underline the fact that in this paper, we focus on gradient estimates of the form (1.3) and not of the less precise form (1.15).

1.3 Organization of the paper

In section 2, we prove the main results. Section 3 is dedicated to equation (1.6) and hydrological models (1.7) and (1.8).

2 Proof of the main results

Before proving Theorem 1.2, we start by some notation and recall the maximum principle for solutions u of Problem (1.1).

Notation
i) For any $i,j,k = 1,..,d$, we consider the following derivatives

$$u_i = D_{x_i} u = \frac{\partial u}{\partial x_i} \qquad u_{it} = D_t u_i = \frac{\partial^2 u}{\partial t \partial x_i}, \qquad u_{ij} = D^2_{x_i x_j} u = \frac{\partial^2 u}{\partial x_i \partial x_j}.$$

ii) For $T > 0$, we denote by $C^{2,1}(\mathbb{T}^d \times [0,T])$ the space of continuous functions such that the derivatives u_t, u_i and u_{ij} exist and are continuous on $\mathbb{T}^d \times [0,T]$ for $i,j = 1,\dots,d$.

Proposition 2.1 ***(Maximum principle for solutions of** (1.1)**)***
Let $T > 0$ and F be a function satisfying conditions i) and ii) of Assumption 1.1. Assume the existence of u and $v \in C^{2,1}(\mathbb{T}^d \times (0,T]) \cap C(\mathbb{T}^d \times [0,T])$ two solutions of equation (1.1) satisfying condition (1.10). If $u(\cdot,0) \leq v(\cdot,0)$, then $u(\cdot,t) \leq v(\cdot,t)$ for all $t \in [0,T]$.

This result seems standard, even if we have no references for this specific result. For sake of convenience, a short proof is given in the appendix.

Proof of Theorem 1.2.
Bounds (1.13) on u are a direct application of Proposition 2.1 and condition (1.12), comparing the solutions with both the minimum and the maximum of u_0 which are two other constant solutions of the same equation.
The uniform bound on Du will be done in the following four steps:

Step 1: Differential equation for $|Du|^2$
Differentiating equation (1.1) yields

$$u_{it} = D_X F : D^2 u_i + D_p F \cdot D u_i + u_i D_u F + D_{x_i} F. \tag{2.1}$$

Multiplying by u_i, we get

$$\left(\frac{u_i^2}{2}\right)_t = D_X F : u_i D^2 u_i + D_p F \cdot u_i D u_i + u_i^2 D_u F + u_i D_{x_i} F. \tag{2.2}$$

Since

$$D(|Du|^2/2) = D^2 u \cdot Du \tag{2.3}$$

and

$$D^2(|Du|^2/2) = \sum_i u_i D^2 u_i + (D^2 u)^2, \tag{2.4}$$

setting $w := |Du|^2/2$, by summation on i in (2.2), we obtain

$$w_t = D_X F \cdot (D^2 w - (D^2 u)^2) + D_p F \cdot Dw + 2w D_u F + D_x F \cdot Du, \tag{2.5}$$

where we used equality (2.4) for the first term and (2.3) for the second one.

Step 2: Differential equation at the maximum of $|Du|^2$
Now, for all $t \geq 0$, set $M(t) := \max_{\mathbb{T}^d} w(\cdot,t)$. Let $t_0 > 0$ and (x_0,t_0) be a point such that $w(x_0,t_0) = M(t_0)$. At (x_0,t_0), we have

$$Dw(x_0,t_0) = 0, \tag{2.6}$$

and

$$D^2 w(x_0,t_0) \leq 0. \tag{2.7}$$

From Assumption 1.1 ii), we deduce that $D_X F \geq 0$ and then $D_X F : (D^2 w(x_0,t_0)) \leq 0$ by (2.7). Similarly, we get $D_p F \cdot Dw(x_0,t_0) = 0$ by (2.6). This implies that

$$w_t \leq -D_X F : (D^2 u)^2 + 2w D_u F + D_x F \cdot Du. \tag{2.8}$$

Step 3: Inequality $M' \leq 0$
In this step, we prove that, in the viscosity sense, for all $t > 0$, we have

$$M'(t) \leq 0. \tag{2.9}$$

Let V be a neighborhood of $t_0 \in (0, +\infty)$ and $\phi \in C^1(0, +\infty)$ verifying

$$\begin{cases} M(t) \leq \phi(t) & \text{on V,} \\ M(t_0) = \phi(t_0); \end{cases}$$

then $w(x_0, t) \leq \phi(t)$ and $w(x_0, t_0) = \phi(t_0)$, which implies that $\phi'(t_0) = w_t(x_0, t_0)$. Note that (2.6) and (2.3) imply that $X \cdot p = 0$ with $p = Du(x_0, t_0)$ and $X = D^2u(x_0, t_0)$. Therefore, we deduce from (2.8) and Assumption 1.1 iii) that

$$\phi'(t_0) = w_t(x_0, t_0) \leq 0.$$

Thus, inequality (2.9) is satisfied in the viscosity sense.

Step 4: Conclusion
We deduce that $M(t) \leq M(0)$ for all $t \geq 0$, i.e. for all $(x, t) \in Q$, we have

$$w(x, t) = \frac{|Du(x, t)|^2}{2} \leq M(t) \leq M(0) = \max_{\mathbb{T}^d} \frac{|Du(\cdot, 0)|^2}{2};$$

which ends the proof of Theorem 1.2. □

Remark 2.2 ***(Reformulation of (1.9) for quasilinear problems)***
i) If F is in quasilinear divergence form, independent from x, (for an equation $u_t = div(a(Du, u, t))$) i.e. for F as follows

$$F(X, p, u, x, t) = D_p a(p, u, t) : X + D_u a(p, u, t) \cdot p$$

for some C^2-vector field a, then (1.9) is equivalent to

$$-D_p a : X^2 + |p|^2 D^2_{up} a : X + |p|^2 D^2_{uu} a \cdot p \leq 0. \tag{2.10}$$

ii) If F is quasilinear but not in divergence form, independent from x, i.e.

$$F(X, p, u, x, t) = A(p, u) : X + B(p, u)$$

for some C^1-matrix A and C^1-function B, then (1.9) is equivalent to

$$-A : X^2 + |p|^2 D_u A : X + |p|^2 D_u B \leq 0. \tag{2.11}$$

Proof of Corollary 1.4.
Corollary 1.4 follows from the application of Theorem 1.2, once we check Assumption 1.1 and condition (1.12). Condition (1.12) is straightforward. We now check Assumption 1.1.

Step 1: Checking Assumption 1.1 i) and ii)
Equation (1.5) is a particular case of (1.1) for

$$F(X, p, u, x, t) := G'(u)\, tr(X) + G''(u)|p|^2.$$

As $G \in C^3$ and $G' \geq 0$, we get that F satisfies Assumption 1.1 i) and ii).

Step 2: Checking Assumption 1.1 iii)
Using (2.10) with $a(u,p,t) = G'(u)p$ inequality (1.9) can be written as

$$-G'(u)\,tr(X^2) + |p^2|G''(u)\,tr(X) + |p|^4 G'''(u) \leq 0. \tag{2.12}$$

We have to check (2.12) for all $(X,p,u,x,t) \in E$ such that $X \cdot p = 0$.

Step 2.1: A preliminary result
We prove that

$$G''' \leq 0 \quad \text{ on } [\underline{u}, \overline{u}]. \tag{2.13}$$

If $G'(u) > 0$, by hypothesis (1.14), $G^{(3)}(u) \leq 0$. Now, if $G'(u) = 0$, we consider two cases:
Case 1. Suppose that there exists a sequence $(u_k)_k$ in $[\underline{u}, \overline{u}]$ converging to u such that $G'(u_k) > 0$, then $G^{(3)}(u_k) \leq 0$, and by continuity of $G^{(3)}$, we get $G^{(3)}(u) \leq 0$.
Case 2. Suppose that $G'(u) = 0$ in a neighborhood of u, then, using the regularity C^3 of G, $G''(u) = G^{(3)}(u) = 0$ in this neighborhood.
In both cases $G^{(3)}(u) \leq 0$.

Step 2.2: The core of the analysis
For $p = 0$, inequality (2.12) is satisfied since $G'(u) \geq 0$. We now consider $p \neq 0$:
Case 1. : $d = 1$
The identity $X \cdot p = 0$ implies that $X = 0$, hence condition (2.12) is satisfied provided that $G^{(3)}(u) \leq 0$, which is implied by (2.13).
Case 2. : $d \geq 2$
Subcase 2.1. : $G'(u) = 0$
By hypothesis (1.14), we have $G''(u) = 0$, and using (2.13), we have $G^{(3)}(u) \leq 0$. This implies (2.12).
Subcase 2.2. : $G'(u) > 0$
We set

$$a = G'(u),\ -b = |p|^2 G''(u) \text{ and } -c = |p|^4 G^{(3)}(u). \tag{2.14}$$

For any $(X,p) \in S^d \times \overline{B(0,M)}$ such that $X \cdot p = 0$, up to change the orthonormal basis of $\mathbb{R}^d$, X can be written as a diagonal matrix of eigenvalues $0, \lambda_1, .., \lambda_{d-1}$ in the direct decomposition $p \oplus p^\perp$. Writing

$$\Lambda = (0, \lambda_1, ..., \lambda_{d-1}) \text{ and } e = (0, 1, ..., 1)\,,$$

then (2.12) can be rewritten as

$$c + be \cdot \Lambda + a\Lambda^2 \geq 0. \tag{2.15}$$

Since

$$|be \cdot \Lambda| = \left| \frac{be}{\sqrt{2a}} \cdot \sqrt{2a}\Lambda \right| \leq \frac{1}{2}\left(2a\Lambda^2 + \frac{|be|^2}{2a}\right) \leq a\Lambda^2 + \frac{|be|^2}{4a},$$

we obtain that (2.15) is true provided that $(d-1)b^2 \leq 4ac$, which is given by the hypothesis (1.14) on G. $\square$

Proof of Corollary 1.5.
For the existence and uniqueness of the weak solution u, we refer to [17, Theorem 9.25, p. 218]. This solution is constructed as the limit in $L^\infty(0, +\infty; L^1(\mathbb{T}^d))$

of a sequence u_ϵ in $L^\infty(0,+\infty;W^{1,\infty}(\mathbb{T}^d))\cap C([0,\infty);L^1(\mathbb{T}^d))$ of smooth positive solutions satisfying the same equation with initial data $u_{0\epsilon}=\epsilon+\rho_\epsilon * u_0$ for some mollifier ρ_ϵ with $u_0\geq 0$, which implies that

$$\epsilon+\min_{\mathbb{T}^d}u_0\leq u_{0\epsilon}\leq\epsilon+\max_{\mathbb{T}^d}u_0 \quad \text{ and } \quad \|Du_{0\epsilon}\|_{L^\infty(\mathbb{T}^d)}\leq\|Du_0\|_{L^\infty(\mathbb{T}^d)}.$$

For $1\leq m\leq 1+\frac{4}{3+d}$, the function $G(u):=u^m$ satisfies conditions of Corollary 1.4 with $\underline{u}=\epsilon+\min\limits_{\mathbb{T}^d}u_0$ and $\bar{u}=\epsilon+\max\limits_{\mathbb{T}^d}u_0$. Because the solutions are known to be smooth for any $T>0$, there exists $M=M(T)$ such that for all $t\in[0,T]$, we have $\|Du(\cdot,t)\|_{\mathbb{T}^d}\leq M$. Hence, applying a version of Corollary 1.4 for finite time interval $(0,T)$, we obtain for $t\in[0,T]$

$$\epsilon+\min_{\mathbb{T}^d}u_0\leq u_\epsilon(\cdot,t)\leq\epsilon+\max_{\mathbb{T}^d}u_0$$

and

$$\|Du_\epsilon(\cdot,t)\|_{L^\infty(\mathbb{T}^d)}\leq\|Du_{0\epsilon}\|_{L^\infty(\mathbb{T}^d)}\leq\|Du_0\|_{L^\infty(\mathbb{T}^d)}.$$

Because $T>0$ is arbitrary, we recover the bound for all time $t\geq 0$.
Now $u_\epsilon\to u$ as ϵ goes to zero and we recover the expected bounds (1.13) and (1.11) for u. □

Remark 2.3 *Notice that bounds (1.13) and (1.11) can also be deduced from Corollary 3.3 below, for smooth solutions of the doubly nonlinear diffusion equation*

$$u_t=\Delta_p(u^m),\quad (x,t)\in Q. \tag{2.16}$$

This works for $p\geq 1$, satisfying $0\leq(m-1)(p-1)\leq\frac{4}{3+d}$ and where the operator $\Delta_p v$ is defined by $\Delta_p v=div(|\nabla v|^{p-2}\nabla v)$.

3 Applications to models in hydrology

In this section, we apply Theorem 1.2 to hydrological models (1.7) and (1.8). To this end, we will first prove a priori bounds on gradient of solutions of Problem (1.6). We begin by assumptions on the function ψ

Assumption 3.1
Let $d\geq 1$, $\underline{u}\leq\overline{u}$, $L>0$ and $\psi:=\psi(u,s)$ be a real function satisfying the following conditions

i) ***Regularity:*** $\psi\in C^2([\underline{u},\overline{u}]\times[0,L])$.

ii) ***Degenerate parabolicity:*** *ψ satisfies conditions*

$$\psi\geq 0 \qquad \text{on } [\underline{u},\overline{u}]\times[0,L], \tag{3.1}$$
$$\psi+2sD_s\psi\geq 0 \qquad \text{on } [\underline{u},\overline{u}]\times[0,L]. \tag{3.2}$$

iii) ***Differential inequality:***

$$\frac{(d-1)}{4}(D_u\psi)^2\leq-\psi D^2_{uu}\psi \qquad \text{on } [\underline{u},\overline{u}]\times[0,L]. \tag{3.3}$$

Remark 3.2 *Hypothesis (3.3) on ψ is similar to hypothesis (1.14) on G.*

Corollary 3.3 ***(A priori gradient bounds for solutions of Problem (1.6))*** *Suppose that ψ satisfies Assumption 3.1. Let us assume the existence of a function $u \in C^3(\overline{Q})$ solution of Problem (1.6), (1.2) satisfying condition (1.10) for $M = \sqrt{L}$. Then for all $t \geq 0$ we have a priori bounds (1.13) and (1.11).*

Proof of Corollary 3.3.
We need only to check Assumption 1.1 and condition (1.12). Condition (1.12) is straightforward.

Step 1: Checking Assumption 1.1 i) and ii)
Equation (1.5) is a particular case of (1.1) for

$$F(X, p, u, x, t) := (\psi(u, |p|^2)Id + 2D_s\psi(u, |p|^2)p \otimes p) : X + D_u\psi(u, |p|^2)|p|^2.$$

As $\psi \in C^2([\underline{u}, \overline{u}] \times [0, L])$ and satisfies (3.1) and (3.2), we see that F satisfies Assumption 1.1 i) and ii).

Step 2: Checking Assumption 1.1 iii)
Inequality (1.9) of Assumption 1.1 iii) can be written as

$$-(\psi Id + 2D_s\psi\, p \otimes p) : X^2 + |p|^2(D_u\psi Id + 2D^2_{us}\psi(u, |p|^2)p \otimes p) : X + |p|^4 D^2_{uu}\psi \leq 0. \tag{3.4}$$

for all $(X, p, u, x, t) \in E$ such that $X \cdot p = 0$. This is equivalent to

$$-\psi\, tr(X^2) + |p|^2 D_u\psi\, tr(X) + |p|^4 D^2_{uu}\psi \leq 0, \tag{3.5}$$

since $X \cdot p = 0$. In order to check Assumption 1.1 iii) we have to check (3.5) for all $(X, p, u, x, t) \in E$ such that $X \cdot p = 0$.
Setting now

$$a = \psi, \quad -b = |p|^2 D_u\psi \text{ and } -c = |p|^2 D^2_{uu}\psi,$$

inequality (3.5) reads as

$$c + b\, tr(X) + a\, tr(X^2) \geq 0, \tag{3.6}$$

that we have to check for all $(X, p, u, x, t) \in E$ such that $X \cdot p = 0$. In Step 2.1 and Step 2.2 of the proof Corollary 1.4, we can replace in (2.14) the function $G'(u)$ by the function $\psi^s(u) := \psi(u, s)$, for each given $s = |p|^2$. Then the reasoning there applies here without any change and shows that (3.6) holds true under Assumption 3.5 iii). □

Corollary 3.4 ***(Application to model (1.7))***
Let

$$d \geq 1, \quad \delta = (2 + 2d)^{-\frac{1}{2}}, \quad \underline{u} = \frac{1}{2} - \delta, \quad \overline{u} = \frac{1}{2} + \delta \quad \text{and} \quad M = 1. \tag{3.7}$$

Let us assume the existence of $u \in C^3(\overline{Q})$, solution of Problem (1.7), (1.2), such that u and Du satisfy condition (1.10). Then for all $t \geq 0$, u and Du satisfy the a priori bounds (1.13) and (1.11).

Remark 3.5 ***(Weak radial solutions of Problem (1.7))***
Assume (3.7) and let $B(0,R)$ be a ball of radius $R>0$. Then we have, by [9] for $d>1$ and by [5] for $d=1$, existence and uniqueness of weak radial solutions of Problem (1.7), (1.2) with Neumann boundary conditions. Moreover the solutions satisfy uniform bounds similar to (1.13) and (1.11), but on the ball $B(0,R)$.
Therefore Corollary 3.4 appears to be a kind of extension of this result to the non radial case, and for smooth solutions.
Finally let us mention an existence and uniqueness result in [9] for radial nondecreasing solutions with $u(x,t)\in[\frac{1}{2},1]$.

Proof of Corollary 3.4.
We simply apply Corollary 3.3 with $\psi(u,|p|^2)=h(u)f(|p|^2)$, $h(u)=u(1-u)$ and $f(s)=1/(1+s)$. To this end, we have to check Assumption 3.1.
Assumption 3.1 i) and ii) are satisfied for $0\le \underline{u}\le \bar{u}\le 1$ and $L=1=M^2$. Indeed

$$\psi(s)+2sD_s\psi(s)=h(u)(f(s)+2sf'(s))=h(u)\frac{(1-s)}{(1+s)^2}\ge 0,$$

for all $(u,s)\in[\underline{u},\overline{u}]\times[0,L]$.
Inequality (3.2) of Assumption 3.1 iii) can be written as

$$\frac{(d-1)}{4}\left(\frac{(h'(u))^2}{-h(u)\,h''(u)}\right)\le 1. \tag{3.8}$$

Setting $u=\frac{1}{2}+v$, (3.8) is equivalent to

$$\frac{(d-1)}{2}\le\frac{\frac{1}{4}-v^2}{v^2}, \tag{3.9}$$

which means that $|v|\le(2+2d)^{-\frac{1}{2}}=\delta$, i.e. $u\in[\underline{u},\overline{u}]$. This shows that Assumption 3.1 iii) holds true. $\square$

We also have

Corollary 3.6 ***(Application to model (1.8))***
Let $d\ge 1$, $\delta=(2+2d)^{-\frac{1}{2}}$, $\underline{u}=\frac{1}{2}-\delta$, $\overline{u}=\frac{1}{2}+\delta$, and $M>0$ be a real number. Let us assume the existence of $u\in C^3(\overline{Q})$, solution of Problem (1.8), (1.2), such that u and Du satisfy condition (1.10). Then for all $t\ge 0$, u and Du satisfy a priori bounds (1.13) and (1.11).

The proof of Corollary 3.6 is similar to the one of Corollary 3.4, taking the same function $h(u)=u(1-u)$ but with $f(s)=1$. Corollary 3.6 can also be obtained from Corollary 1.4.

4 Appendix

Proof of Proposition 2.1.
Let $z:=u-v$, then we have

$$\begin{aligned} z_t = \;& F(D^2u,Du,u,x,t)-F(D^2v,Du,u,x,t)\\ &+F(D^2v,Du,u,x,t)-F(D^2v,Dv,u,x,t)\\ &+F(D^2v,Dv,u,x,t)-F(D^2v,Dv,v,x,t). \end{aligned}$$

By mean of Taylor expansion, z satisfies the following problem

$$\begin{cases} z_t = A(x,t) : D^2 z + B(x,t) \cdot Dz + C(x,t) z, & (x,t) \in \mathbb{T}^d \times (0,T), \\ z(x,0) = u(x,0) - v(x,0), & x \in \mathbb{T}^d, \end{cases}$$

where

$$\begin{cases} A(x,t) = \int_0^1 D_X F(\theta D^2 u + (1-\theta) D^2 v, Du, u, x, t) d\theta, \\ B(x,t) = \int_0^1 D_p F(D^2 v, \theta Du + (1-\theta) Dv, u, x, t) d\theta, \\ C(x,t) = \int_0^1 D_u F(D^2 v, Dv, \theta u + (1-\theta) v, x, t) d\theta. \end{cases}$$

Note that Assumption 1.1 ii) implies that $D_X F \geq 0$ and then $A \geq 0$. Now, since $z(\cdot, 0) \leq 0$, the standard maximum principle implies that $z(\cdot, t) \leq 0$ for all $t \in [0,T]$ (see, for instance, [6, Theorem 9, p. 369] applied to $w := e^{-\lambda t} z$ for a large $\lambda > 0$). $\square$

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