SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS IN $\mathbb{R}^N$ WITH CONICAL-SHAPED LEVEL SETS

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Abstract. This article deals with the questions of the existence, of the uniqueness and of the qualitative properties of solutions of semilinear elliptic equations in $\mathbb{R}^N$. Three types of conical conditions at infinity are successively considered. This defines three frameworks: the weak framework, the strong framework and the framework of solutions with asymptots. The results are based on different kinds of sliding methods and, following the ideas of Berestycki, Nirenberg and Vega, on comparison principles in cones or in $\mathbb{R}^N$.

1 Introduction

1.1 General presentation

The starting point of this article is the paper of Bonnet and Hamel [13]. For any angle $\alpha \in]0, \pi/2]$, Bonnet and Hamel have proved the existence of solutions $(c, u) \in I\!R \times C^2(I\!R^2)$ for the following semilinear elliptic equation

$$\Delta u - c\partial_y u + f(u) = 0, \quad 0 \leq u \leq 1 \ \text{ in } I\!R^2 \tag{1.1}$$

The functions u satisfy $0 < u < 1$ and "conical" conditions at infinity:

$$\begin{cases} \forall\ \delta \in [0, \pi - \alpha[, & \liminf\limits_{|(x,y)|\to\infty,\ (x,y)\in\mathcal{C}^+(0,\delta)} u(x, y) = 1 \\ \forall\ \delta \in [0, \alpha[, & \limsup\limits_{|(x,y)|\to\infty,\ (x,y)\in\mathcal{C}^-(0,\delta)} u(x, y) = 0 \end{cases} \tag{1.2}$$

where, for any $y_0 \in I\!R$ and any $\beta \geq 0$, the lower and upper cones $\mathcal{C}^\pm(y_0, \beta)$ are defined by

$$\mathcal{C}^\pm(y_0, \beta) = \{(x, y) = (0, y_0) + \rho(\cos\varphi, \sin\varphi),\ \rho \geq 0,\ |\varphi \mp \pi/2| \leq \beta\}$$

In other words, the level sets of the solutions u have two asymptotic directions as $|x| \to +\infty$, both these directions $(\pm\sin\alpha, -\cos\alpha)$ make an angle α with the negative y-axis. Roughly speaking, the functions u are asymptotically conical-shaped far away from the origin.

The notation $\partial_y u$ means the partial derivative of the function u with respect to the variable y. The function f is lipschitz-continuous in $[0, 1]$, continuously differentiable in a left neighborhood of the point 1 and has the following profile:

$$\exists\ \theta \in (0, 1),\ f = 0 \text{ on } [0, \theta] \cup \{1\},\ f > 0 \text{ on } (\theta, 1) \text{ and } f'(1) < 0 \tag{1.3}$$

We extend f by 0 outside $[0, 1]$. Hence, f is lipschitz-continuous on $I\!R$ and from standard elliptic estimates, any bounded solution u of (1.1) is of class $C^{2,\mu}(I\!R^2)$ for any $\mu \in [0, 1[$.

Equation (1.1) arises in models of equidiffusional premixed Bunsen flames. The function u is a normalized temperature and its level sets represent the profile of a conical-shaped Bunsen flame coming out of a Bunsen burner (see

Joulin [30], Sivashinsky [36], Williams [38]). The temperature of the unburnt gases is close to 0 and that of the burnt gases is close to 1. The hot zone is above the fresh zone. The real θ is called an ignition temperature. The real c is the speed of the gases at the exit of the burner.

In the onedimensional case, equation (1.1) and conditions at infinity (1.2) reduce to the ordinary differential equation

$$\begin{cases} u_0'' - c_0 u_0' + f(u_0) = 0 \\ u_0(-\infty) = 0, \quad u_0(+\infty) = 1 \end{cases} \tag{1.4}$$

$>$From results of Aronson, Weinberger [3], Berestycki, Nicolaenko, Scheurer [10], Fife, McLeod [19], there exists a unique solution (c_0, u_0) of (1.4) such that $u_0(0) = \theta$. The functions u that are solutions of (1.4) are unique up to translation. Besides, the speed c_0 is positive and the function u_0 is increasing.

For any angle $\alpha \in]0, \pi/2]$, Bonnet and Hamel have proved in [13] that there exists a solution (c, u) of (1.1), (1.2) and that, for any solution (c, u) of (1.1), (1.2), the speed c is unique and given by the formula

$$c = \frac{c_0}{\sin \alpha}$$

This formula, which had already been used in several papers (see e.g. Sivashinsky [36]), is very natural. Indeed, consider the corresponding evolution problem. The speed c_0 is now nothing else than the projection on the directions $(\pm \cos \alpha, -\sin \alpha)$ of the vertical speed c of the curved flame moving downwards. The speed c_0 is the speed of two planar waves moving in the directions $(\pm \cos \alpha, -\sin \alpha)$ perpendicular to the half-lines making an angle α with the vertical axis. Let us mention here that in experiments, if we know the speed c of the gases at the exit of a Bunsen burner, the measurement of the angle α of the flame is used to determine the speed c_0 of planar flames (see Williams [38]).

Having recalled those results, a natural question to ask is to study the set of the solutions (c, u) of the same reaction-diffusion equation

$$\begin{matrix} \Delta u - c\partial_y u + f(u) = 0, \\ 0 \le u \le 1 \end{matrix} \quad \text{in } I\!\!R^N = \{x = (x', y) \in I\!\!R^{N-1} \times I\!\!R\} \tag{1.5}$$

in any dimension $N \geq 2$ (and especially in $I\!R^3$), with asymptotic conditions like

$$\begin{cases} \forall\ \delta \in [0, \pi - \alpha[, & \liminf\limits_{|(x,y)| \to \infty,\ (x,y) \in \mathcal{C}^+(0,\delta)} u(x,y) = 1 \\ \forall\ \delta \in [0, \alpha[, & \limsup\limits_{|(x,y)| \to \infty,\ (x,y) \in \mathcal{C}^-(0,\delta)} u(x,y) = 0 \end{cases} \tag{1.6}$$

where we set

$$\mathcal{C}^+(y_0, \beta) = \{x = (0, y_0) + \rho\nu,\ \nu \in \mathbf{S}^{N-1},\ \rho \in I\!R^+,\ \nu_N \geq \cos\beta\}$$

$$\mathcal{C}^-(y_0, \beta) = \{x = (0, y_0) + \rho\nu,\ \nu \in \mathbf{S}^{N-1},\ \rho \in I\!R^+,\ -\nu_N \geq \cos\beta\}.$$

We will also study the question of the existence of solutions of (1.5), (1.6) when the angle α is bigger than $\pi/2$.

The purpose of this article is to prove various kinds of existence, nonexistence, uniqueness, monotonicity or symmetry results for the solutions of the semilinear elliptic equation (1.5) with the unusual conical conditions at infinity like (1.6) or other ones described later. The methods used are basically the sliding method, the moving plane method and their different variances. The main difficulties really come from the unboundedness of the open set $I\!R^N$ where the reaction-diffusion equation (1.5) is set, and from the type of boundary conditions at infinity like (1.6).

There are many works dealing with the uniqueness, symmetry, monotonicity properties of solutions of semilinear elliptic equations in $I\!R^N$ with uniform conditions at infinity like $u(x) \to 0$, or other decay conditions, as $|x| \to \infty$ (e.g. Chen, Li [15], Gidas, Ni, Nirenberg [22], Gidas, Spruck [23], Li [33], Li, Ni [34], and also Amick, Fraenkel [2] for vortex rings). Other properties, like convexity (see Kawohl [31]), have also been emphasized. As far as conical conditions are concerned, results exist for semilinear elliptic equations set in cone-like domains with Dirichlet conditions at the boundary (e.g. Bandle, Essen [4] and the literature cited therein).

However, even for this single reaction-diffusion equation (1.5), problems which are set in unbounded domains with conical conditions like (1.6) – or (1.8), (1.12) below – do not seem to have been studied yet, as far as we know, but in [13]. We will set our results in different frameworks which will be defined in succession in the next subsections: the weak framework, the strong framework and the framework of solutions with asymptots. These different words are related to the asymptotic conditions at infinity that the solutions of (1.5) are required to satisfy.

1.2 The weak framework

As far as the solutions of (1.5), (1.6) are concerned, only the situation in dimension 2 and for angles $\alpha \in]0, \pi/2]$ is known: the solutions (c, u) exist and the speed c is unique. The following theorem closes the question of the existence of solutions for angles α greater than $\pi/2$, even with the weak asymptotic conditions (1.6).

Theorem 1.1 *In any dimension $N \geq 2$, there is no solution of (1.5), (1.6) with an angle $\alpha \in (\pi/2, \pi)$.*

The physical meaning of this result is that there is no flame which is pointed inside the Bunsen burner.

The question of the uniqueness of solutions of the reaction-diffusion equation (1.5) with weak asymptotic conditions (1.6) (for angles $\alpha \leq \pi/2$) is very tricky. To illustrate the difficulty, let us mention the conjecture of De Giorgi (1978, [25]) on a similar problem: if u is a solution of $\Delta u = u^3 - u$ in $I\!R^N$, fulfilling the simple limit $\lim_{x_N \to \pm\infty} u(x_1, \cdots, x_N) = \pm 1$ and such that $\partial_{x_N} u > 0$, then the level sets of u are hyperplanes. Notice that for the equation (1.5), this conjecture is not true because, in dimension 2 and for all angles $\alpha < \pi/2$, the solutions of Bonnet and Hamel satisfy the same requirements but are not planar. The De Giorgi's conjecture was proved by Modica and Mortola in [35] in the case $N = 2$ if the level sets of u are the graphs of an equilipschitzian family of functions and in any dimension N if there is a point $x \in I\!R^N$ such that $|\nabla u(x)|^2 = \frac{1}{2}(1 - u(x)^2)^2$ (see Caffarelli, Garofalo, Segala [14]). Recently Ghoussoub and Gui proved in [20] the conjecture in the case $N = 2$ without any additional requirement. For this question, see also the work of Berestycki, Caffarelli and Nirenberg in [8]. For $N \geq 3$, the conjecture is an open question. On the other hand, Ghoussoub and Gui [20] introduced the stronger uniform convergence condition, namely, $u(x_1, \cdots, x_{N-1}, x_N) \to \pm 1$ as

$x_N \to \pm\infty$ uniformly with respect to $(x_1, \cdots, x_{N-1})$, and they proved that, in dimension $N \le 3$ and under this uniformity condition, the solutions u of $\Delta u = u^3 - u$ in $\mathbb{R}^N$ only depend on x_N (this one-dimensional symmetry also works for non-linearities $f(u)$ more general than $u^3 - u$ [20]). Later, Berestycki and the authors [9], with the device of the sliding method, proved the same result in any dimension N under this uniformity condition. Farina [18] – independently and using the moving plane method – proved a similar result. Lastly, Barlow, Bass and Gui [5], using a very different method relying on probabilistic arguments, proved the same one-dimensional symmetry result as Berestycki and the authors, in any dimension N under the uniformity condition; actually, Barlow, Bass and Gui proved the result under much weaker conditions.

These remarks eventually show the difficulty of considering conditions at infinity which are somehow only simple or not globally uniform. For instance, the limits (1.6) are only uniform in cones which are strictly embedded in $\mathcal{C}^-(0,\alpha)$ or $\mathcal{C}^+(0,\pi-\alpha)$.

1.3 The strong framework

In order to define a “strong framework" related to the conical asymptotic conditions, let us study more carefully, in dimension 2 and for angles $\alpha \le \pi/2$, what kind of asymptotic conditions the solutions u^* of (1.5), (1.6) built by Bonnet and Hamel satisfy: namely a uniformity property far away from their level sets.

Theorem 1.2 *In dimension $N = 2$ and for any $\alpha \in]0,\pi/2]$, let u^* be the solution of (1.5), (1.6) in [13]. For any $\lambda \in (0,1)$, let $y = \phi_\lambda(x)$ be the level set $\{u^*(x,y) = \lambda\}$. The function ϕ_λ is of class C^1 and satisfies $\phi'_\lambda(x) \pm \cot\alpha \to 0$ as $x \to \pm\infty$. Set $\Omega^-_\lambda(y_0) = \{y < y_0 + \phi_\lambda(x)\}$ and $\Omega^+_\lambda(y_0) = \{y > y_0 + \phi_\lambda(x)\}$. Then*

$$\forall \lambda \in (0,1), \quad \begin{cases} \displaystyle\lim_{y_0 \to +\infty} \inf_{\Omega^+_\lambda(y_0)} u^* = 1 \\ \displaystyle\lim_{y_0 \to -\infty} \sup_{\Omega^-_\lambda(y_0)} u^* = 0 \end{cases} \tag{1.7}$$

The conditions (1.7) are stronger than (1.6) and allow us to define naturally the strong framework in any dimension N: we study the solutions (c,u) of

$$\begin{cases} \Delta u - c\partial_y u + f(u) = 0 \text{ in } I\!R^N \\ \displaystyle\lim_{y_0\to+\infty} \inf_{\Omega^+(y_0)} u = 1 \\ \displaystyle\lim_{y_0\to-\infty} \sup_{\Omega^-(y_0)} u = 0 \end{cases} \tag{1.8}$$

for which there exists a lipschitz-continuous function $\phi(x')$, of class C^1 far away from the origin $x'=0$, and such that

$$\lim_{|x'|\to+\infty} \left(\nabla\phi(x') + \cot\alpha \, \frac{x'}{|x'|} \right) = 0, \tag{1.9}$$

where, for any $y_0 \in I\!R$, we set $\Omega^+(y_0) = \{y > y_0 + \phi(x')\}$ and $\Omega^-(y_0) = \{y < y_0 + \phi(x')\}$.

Since the problem (1.8), (1.9) is clearly stronger than (1.5), (1.6), it has no solution if $\alpha > \pi/2$, in any dimension $N \geq 2$. On the other hand, theorem 1.2 means that it has a solution for any angle $\alpha \leq \pi/2$ in dimension $N=2$, namely the solution of Bonnet and Hamel. However, the question of the existence in any dimension $N \geq 3$ is still open.

The following theorem deals with properties of solutions (c,u) in the strong framework:

Theorem 1.3 *In any dimension $N \geq 2$, if (c,u) is a solution of (1.8), (1.9), then $\alpha \leq \pi/2$, $c = \dfrac{c_0}{\sin\alpha}$ and u is nondecreasing in each nonzero direction of $\mathcal{C}^+(0,\alpha)$. Furthermore, if $\alpha = \pi/2$, the functions u are unique and planar: up to a translation, they are equal to the planar front $u_0(y)$ in $I\!R^N$.*

Hence, as far as the uniqueness of the couples (c,u) solutions of (1.8), (1.9) is concerned, we can say more in the strong framework than in the weak framework (for which only the uniqueness for the speed in dimension 2 is known [13]). Nevertheless, the question of the uniqueness of the functions u (up to translation) remains very intricate because of the very few informations about the function ϕ.

Theorem 1.3 will be proved in section 3 thanks to the next two theorems 1.4 and 1.5, which are related to the question: "if two functions are respectively supersolution and subsolution of the same reaction-diffusion equation in a straight infinite cylinder, on which condition can the supersolution be moved over the subsolution?" A first answer to this question in dimension 1, was given by Vega (see [37]): if $\overline{u}$ and $\underline{u}$ are respectively a supersolution and a subsolution of (1.4) with $\overline{u}(+\infty) = 1$ and $\underline{u}(-\infty) = 0$, $0 < \underline{u}, \overline{u} < 1$, then there exists $t^* \in I\!R$ such that $\overline{u}(\cdot + t^*) \equiv \underline{u}(\cdot)$. The following theorem deals with the multidimensional case in straight cylinders Σ. For the sake of simplicity, we only state the case where $\Sigma = I\!R^N = \{(x', y), x' = (x_1, \cdots, x_{N-1}) \in I\!R^{N-1}, y \in I\!R\}$ (*see* [27] for additional results). We have:

Theorem 1.4 (Comparison principle in $I\!R^N$) *Let $\phi : I\!R^{N-1} \to I\!R$ be a uniformly continuous function and set*

$$\begin{cases} \Omega^+(y_0) & = \{y > y_0 + \phi(x')\} \\ \Omega^-(y_0) & = \{y < y_0 + \phi(x')\} \\ \Gamma(y_0) & = \{y = y_0 + \phi(x')\} \end{cases}$$

Consider the semilinear elliptic equation

$$I(u) := \sum_{1 \le i,j \le N} a_{ij}(x')\partial_{ij}u + \sum_{1 \le i \le N} b_i(x')\partial_i u + f(x', u) = 0 \text{ in } I\!R^N \tag{1.10}$$

for functions u such that $a \le u \le b$ and satisfying the boundary and asymptotic conditions:

$$\begin{cases} \lim\limits_{y_0 \to +\infty} \inf\limits_{\Omega^+(y_0)} u = b \\ \lim\limits_{y_0 \to -\infty} \sup\limits_{\Omega^-(y_0)} u = a \end{cases} \tag{1.11}$$

with $a < b \in I\!R$. We assume that equation (1.10) is elliptic in the sense that $c_0|\xi|^2 \le \sum_{1 \le i,j \le N} a_{ij}(x')\xi_i\xi_j \le C_0|\xi|^2$ for some $0 < c_0 \le C_0$ and for any $\xi \in I\!R^N$ and $x' \in I\!R^{N-1}$. Besides, $a_{ij} \in C^{2,\delta}$, $b_i \in C^{1,\delta}$. The function f is continuous, bounded on $I\!R^{N-1} \times [a, b]$ and such that $|f(\tilde{x}', \tilde{u}) - f(x', u)| \le C(|\tilde{x}' - x'|^\delta + |\tilde{u} - u|)$ for some constant $C > 0$. We assume that there exist $a < a' < b' < b$ such that f is nonincreasing in u for u in $[a, a']$ or $[b', b]$. For any $x' \in I\!R^{N-1}$, $f(x', \cdot)$ is extended on $I\!R$ by $f(x', u) = f(x', a)$ if $u \le a$ and $f(x', u) = f(x', b)$ if $u \ge b$.

Let $\overline{u}$ and $\underline{u}$ be two lipschitz-continuous functions, respectively super- and subersolution for (1.10), (1.11), namely:

$$\begin{cases} I(\overline{u}) \leq 0 \ \textit{in}\ I\!R^N,\ a \leq \overline{u} \leq b \ \textit{in}\ I\!R^N \ \textit{and} \ \displaystyle\lim_{y_0\to+\infty} \inf_{\Omega^+(y_0)} \overline{u} = b \\ I(\underline{u}) \geq 0 \ \textit{in}\ I\!R^N,\ a \leq \underline{u} \leq b \ \textit{in}\ I\!R^N \ \textit{and} \ \displaystyle\lim_{y_0\to-\infty} \sup_{\Omega^-(y_0)} \underline{u} = a, \end{cases}$$

the inequalities $I(\overline{u}) \leq 0 \leq I(\underline{u})$ holding in the distribution sense. For any $t \in I\!R$, set $\overline{u}^t(x', y) = \overline{u}(x', y)$ and define $\overline{u}^{-\infty}(x') = \liminf_{t\to-\infty} \overline{u}(x', t)$.

Then the set $I = \{t \in I\!R,\ \forall s \geq t,\ \overline{u}^s \geq \underline{u} \ in \ I\!R^N\}$ is not empty. Let $t^* := \inf I$. We have $\overline{u}^{t^*} \geq \underline{u}$ in $I\!R^N$ and, if $t^* > -\infty$, then $\inf_{\Gamma(y_0)} (\overline{u}^{t^*} - \underline{u}) = 0$ for any $y_0 \in I\!R$.

Notice that theorem 1.4 nolonger works if $a = -\infty$ and $b = +\infty$. For instance, consider the equation $u'' = 0$ in $I\!R$ with $u(-\infty) = -\infty$, $u(+\infty) = +\infty$ and take $\overline{u}(x) = x$ and $\underline{u}(x) = 2x$.

A consequence of theorem 1.4 is the monotonicity result:

Theorem 1.5 *Under the assumptions of theorem 1.4, if u is a solution of (1.10), (1.11), then u is increasing in y.*

1.4 The framework of solutions with asymptots

In this subsection, we consider the solutions (c, u) of (1.8), (1.9) for which there exists a function $\phi(x')$ such that $\sup_{x'\in I\!R^{N-1}} |\phi(x') + \cot\alpha\ |x'|| < \infty$. This is equivalent to look for the solutions (c, u) of

$$\begin{cases} \Delta u - c\partial_y u + f(u) = 0,\ 0 \leq u \leq 1 \ \text{ in } I\!R^N \\ \displaystyle\lim_{y\to+\infty} \inf_{\mathcal{C}^+(y,\pi-\alpha)} u = 1 \\ \displaystyle\lim_{y\to-\infty} \sup_{\mathcal{C}^-(y,\alpha)} u = 0 \end{cases} \tag{1.12}$$

This problem is then a particular case of the weak and strong frameworks described in the previous sections. Hence, theorems 1.1 and 1.3 work. Furthermore, we have:

Theorem 1.6 *If $N \geq 3$ and if $\alpha \neq \pi/2$, then (1.12) has no solution.*

Theorem 1.7 *In dimension $N = 2$ and for any $\alpha \le \pi/2$, the solutions (c, u) of (1.12) are unique, up to a translation in (x, y) for u. We have $c = c_0/\sin\alpha$. Besides,*

(i) there exists a real x_0 such that u is symmetric with respect to the vertical line $\{x = x_0\}$,

(ii) for any $\lambda \in (0, 1)$, the level set $\{u(x, y) = \lambda\}$ has two asymptots parallel to both half-lines $\{y = -\cot\alpha|x|,\ x \ge 0\}$ and $\{y = -\cot\alpha|x|,\ x \le 0\}$; besides, (1.8) works for any $\phi = \phi_\lambda$,

(iii) there exist two reals $t_\pm$ such that for any sequence $x_n \to \pm\infty$, the functions $u_n(x, y) = u(x + x_n, y - \cot\alpha\ |x_n|)$ go to the planar fronts $u_0(\pm\cos\alpha\ x + \sin\alpha\ y + t_\pm)$ as $x_n \to \pm\infty$, uniformly on compact subsets of $I\!R^2$,

(iv) up to some translation, any solution u of (1.12) is equal to the solution given in [13].

Part (iii) gives meaning to the expression "solutions with asymptots". Eventually, the only possible solutions of (1.12) occur if $N = 2$ and if $\alpha \le \pi/2$ and turn out to be those of Bonnet and Hamel. The price to pay as counterpart of this uniqueness result is that we do not know a priori if (1.12) has a solution in dimension 2 and for angles $\alpha \le \pi/2$, whereas the functions in [13] are solutions in the weak and strong frameworks.

The following theorems give two sufficient conditions for problem (1.12) to have solutions in dimension 2.

Theorem 1.8 (Existence result for some angles α and for some functions f) *Let f be a function satisfying (1.3) and such that $c_0^2 > 4/9 \sup\limits_{[0,1]} f'$. Assume that the restriction of f is C^1 on $[\theta, 1]$. There exists $\alpha_0 \in (0, \pi/2)$ such that for any angle $\alpha \in (0, \alpha_0)$, problem (1.12) in dimension $N = 2$ has a solution u. Besides, for any $\varepsilon > 0$, there are some functions f such that $\alpha_0 \ge \pi/2 - \varepsilon$.*

The last assertion in theorem 1.8 implies that for any angle $\alpha \in (0, \pi/2)$, there are functions f satisfying (1.3) and such that problem (1.12) has a solution. This existence result is in strong contrast with the nonexistence result in dimension $N \ge 3$ (theorem 1.6).

Theorem 1.9 *If $N = 2$ and $\alpha \in]0, \pi/2]$, if (c, u) is a solution of (1.5), (1.6)* (necessarily with $c = \frac{c_0}{\sin\alpha}$) *such that*

$$\exists\ \underline{y} \in I\!R,\ \exists\ \underline{\xi} \in (0,1),\ u \geq \underline{\xi}\ on\ \partial\mathcal{C}^-(\underline{y}, \alpha), \tag{1.13}$$

$$\exists\ \overline{y} \in I\!R,\ \exists\ \overline{\xi} \in (0,1),\ u \leq \overline{\xi}\ in\ \mathcal{C}^-(\overline{y}, \alpha), \tag{1.14}$$

then (c, u) is also a solution of (1.12).

As a consequence, in dimension $N = 2$, if two solutions u_1 and u_2 satisfy (1.5) with the weak conditions (1.6) and the nondegeneracy assumptions (1.13), (1.14), with maybe different values for $\underline{y}$, $\underline{\xi}$, $\overline{y}$ and $\overline{\xi}$, then they are equal up to translation. The functions built in [13] satisfy (1.14). Finally, a necessary and sufficient condition for the exitence of solutions of (1.12) in dimension 2 with angles $\alpha \leq \pi/2$ is that the functions of Bonnet and Hamel solving (1.5), (1.6) satisfy (1.13).

Theorem 1.9 is proved in section 4.1.3 by the construction of sub- and super-solutions defined on sets rotating around a fixed point. Lastly, notice that the new conditions (1.13), (1.14) are similar, for semilinear elliptic problems studied by Berestycki, Caffarelli and Nirenberg [6], [7] or Esteban and Lions [17], to the Dirichlet condition $u = 0$ on the boundary of the domain. For the case of a half-space, it was especially proved in [6] that the solutions of a certain class of semilinear elliptic equations are unique and planar.

Summary. We present in the following table a summary of the existence, nonexistence, uniqueness and monotonicity results concerning the solutions (c, u) of the semilinear elliptic equation (1.5) in the weak framework (1.6), in the strong framework (1.8), (1.9) and in the framework of solutions with asymptots (1.12). In this table, the words existence and nonexistence mean the existence and nonexistence of a couple (c, u); uniqueness for the functions u is understood to be uniqueness up to translation. The numbers in parentheses refer to the sections in which the results are proved. Lastly, theorems 1.4 and 1.5 are proved in section 5. The results of this article, as well as further ones, have been announced in [27] and [28].

Conjectures. The nonexistence result in theorem 1.6 actually sheds some light on the difficulty of the question of the existence of solutions of equation (1.5) in dimension $N \geq 3$ and with angles $\alpha \leq \pi/2$. We conjecture that the existence holds in the strong framework (a fortiori, this would imply the existence in the weak framework). Moreover, we conjecture that the question of the uniqueness for the functions u works in the strong framework and that, henceforth, equation (1.5) is well-posed in the strong framework in any dimension $N \geq 2$ and for angles $\alpha \leq \pi/2$.

<table>
<tr><th colspan="2">Solutions (c, u) of:</th><th>in $I\!R^2$</th><th>in $I\!R^N$, $N \geq 3$</th></tr>
<tr><td>Weak framework: (1.5), (1.6)</td><td>$\frac{\pi}{2} < \alpha < \pi$</td><td colspan="2">Nonexistence (2)</td></tr>
<tr><td rowspan="3">Strong framework: (1.8), (1.9)</td><td rowspan="2">$0 < \alpha < \frac{\pi}{2}$</td><td>Existence (3)</td><td></td></tr>
<tr><td colspan="2">Uniqueness of c: $c = c_0 / \sin \alpha$,
monotonicity for u (3)</td></tr>
<tr><td>$\alpha = \frac{\pi}{2}$</td><td colspan="2">Unique solution (c_0, u_0) (3)</td></tr>
<tr><td rowspan="2">Solutions with asymptots: (1.12)</td><td>$0 < \alpha < \frac{\pi}{2}$</td><td>Uniqueness (4.1.1)

Existence
for some f (4.1.2)
or under
nondegeneracy
assumption (4.1.3)</td><td>Nonexistence
(4.2)</td></tr>
<tr><td>$\alpha = \frac{\pi}{2}$</td><td colspan="2">Unique solution (c_0, u_0) (4.1.1)</td></tr>
</table>

2 Weak framework: nonexistence with angles $\alpha > \pi/2$

2.1 Dimension $N = 2$

This section is devoted to proving that there is no solution (c, u) of (1.5), (1.6) in dimension $N = 2$ when the angle α is bigger than $\pi/2$ (theorem 1.1 for $N = 2$). We divide the proof into several steps.

Lemma 2.1 *In any dimension $N \geq 2$, if $\pi/2 < \alpha < \pi$ and if (c, u) is a solution of (1.5), (1.6), then $c < c_0$.*

Proof. Let (c, u) be a solution of (1.5), (1.6) with $\alpha \in (\pi/2, \pi)$. By using a suitable supersolution, we will prove that $c < c_0$.

Let us suppose that $c \geq c_0$. The function $\overline{u}(x, y) = u_0(y)$ satisfies

$$\Delta\overline{u} - c\partial_y\overline{u} + f(\overline{u}) = (c_0 - c)u_0'(y) \leq 0 \text{ in } I\!R^N$$

since u_0 is increasing. Besides, $\lim\limits_{y_0 \to +\infty} \inf\limits_{\{y > y_0\}} \overline{u} = 1$. On the other hand, since $\alpha > \pi/2$ and u satisfies (1.6), it comes that $\lim\limits_{y_0 \to -\infty} \sup\limits_{\{y < y_0\}} u = 0$. Hence, u and $\overline{u}$ are respectively sub- and supersolutions for (1.5) and (1.11) in the sense of theorem 1.4 applied in $I\!R^2$ with $\phi(x) \equiv 0$.

By the comparison principle in theorem 1.4, there exists a real t such that $\overline{u}^t = u_0^t \geq u$ in $I\!R^2$. Besides, the infimum t^* of such t's is necessarily finite. Henceforth, we get $\inf\limits_{x'} (u_0(y_0 + t^*) - u(x', y_0)) = 0$ for any $y_0 \in I\!R$. Take $y_0 = 0$. The infimum cannot be attained at a finite point, otherwise the functions $u_0(y + t^*)$ and $u(x', y)$ would be identical by the strong maximum principle. There exists then a sequence $x_n' \to \pm\infty$ such that $u_0(t^*) - u(x_n', 0) \to 0$ as $n \to \infty$, that is to say $u(x_n', 0) \to u_0(t^*) > 0$. This is in contradiction with (1.6) and completes the proof of lemma 2.1.

In dimension $N = 2$, let (c, u) be a solution of (1.5), (1.6) for an angle $\alpha \geq \pi/2$. Let us choose a real $\theta' \in (\theta, 1)$ and call $\theta'' = (\theta' + 1)/2$, $\theta'' \in (\theta', 1)$. Let us define $\alpha' = \pi - \alpha$ and fix temporarily an angle β such that $0 < \beta < \alpha'$.

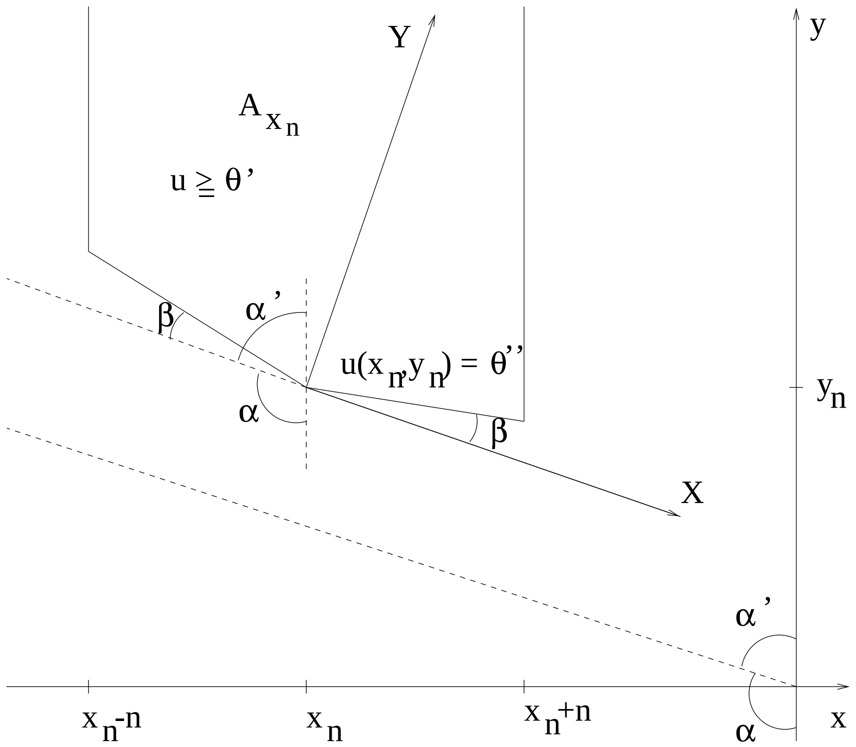

Figure 1: The set $\mathcal{A}_{x_n}$

By (1.6), we have $u(x,y) \to 1$ as $y \to +\infty$ and $u(x,y) \to 0$ as $y \to -\infty$ for any $x \in \mathbb{R}$. Since u is continuous, we can therefore define the functions $\phi_-(x) = \min\{y, u(x,y) = \theta''\}$ and $\phi_+(x) = \max\{y, u(x,y) = \theta''\}$.

Let us fix temporarily a integer $n \in \mathbb{N}$. For any $x_0 \in \mathbb{R}$, let us define the set

$$\mathcal{A}_{x_0} = \{x_0 - n \leq x \leq x_0,\ y \geq \phi_+(x_0) - \cot(\alpha' - \beta)(x - x_0)\}$$
$$\cup\ \{x_0 \leq x \leq x_0 + n,\ y \geq \phi_+(x_0) - \cot(\alpha' + \beta)(x - x_0)\}$$

Lemma 2.2 *There exists a real $x_n \leq -n/2$ such that for all (x,y) in $\mathcal{A}_{x_n}$, $u(x,y) \geq \theta'$*

Proof. Assume not. By (1.6) applied in $\mathcal{C}^+(0,\delta)$ (for some $0 < \delta < \alpha'$), there exists a real y_0' such that

$$\forall -n/2 \leq x \leq n/2,\ \forall y \geq y_0' - \cot(\alpha' + \beta)x,\ \ u(x,y) > \theta'' \tag{2.1}$$

Besides, once again by (1.6) applied this time in $\mathcal{C}^-(0, \alpha - \beta/2)$, we have $u(x, y_0' - \cot(\alpha' + \beta)x) \to 0$ as $x \to -\infty$. Hence, there exists a real $x_0 < -n/2$ such that $\phi_+(x_0) \geq y_0' - \cot(\alpha' + \beta)x_0$. Set $(x_0, y_0) = (x_0, \phi_+(x_0))$.

Since we have supposed that lemma 2.2 does not work, there exists a point (x_1, y_1') in $\mathcal{A}_{x_0}$ such that $u(x_1, y_1') < \theta'$. By definition of $\mathcal{A}_{x_0}$ and of $\phi_+(x_1)$, it comes that the point $(x_1, y_1) = (x_1, \phi_+(x_1))$ is in $\mathcal{A}_{x_0}$. In particular, this point is in the set

$$E = \{x \leq x_0,\ y \geq y_0 - \cot(\alpha' - \beta)(x - x_0)\} \cup \{x_0 \leq x,\ y \geq y_0 - \cot(\alpha' + \beta)(x - x_0)\}$$

$$= \{y \geq \max\ [\ y_0 - \cot(\alpha' - \beta)(x - x_0), y_0 - \cot(\alpha' + \beta)(x - x_0)\ ],\ x \in \mathbb{R}\}$$

and $\mathcal{A}_{x_1} \subset E$. Since $|x_1 - x_0| \leq n$, (2.1) yields that $x_1 < -n/2$.

On the other hand, since $u(x_0, y) \geq \theta'' > \theta'$ for any $y \geq y_0$ and since u is globally Lipschitz-continuous (by standard elliptic estimates), it comes that there exists a constant $\eta > 0$ (not depending on x_0), such that $|x_0 - x_1| \geq \eta$.

By induction, there exists a sequence of points $(x_k, y_k) = (x_k, \phi_+(x_k))$ such that $(x_k, y_k) \in \mathcal{A}_{x_{k-1}}$, $(x_k, y_k) \in E$, $x_k < -n/2$ and $|x_k - x_{k-1}| \geq \eta$ for any $k \in \mathbb{N}^*$. Since $|x_k - x_{k-1}| \geq \eta > 0$ for any k, it comes that there is an infinite number of k's such that $x_k < x_{k-1}$. For such k's, we actually have $x_k \leq x_{k-1} - \eta$ and

$$\begin{aligned} y_k &\geq y_{k-1} - \cot(\alpha' - \beta)(x_k - x_{k-1}) \\ &\geq y_{k-1} + (\cot(\alpha' + \beta) - \cot(\alpha' - \beta))(x_k - x_{k-1}) - \cot(\alpha' + \beta)(x_k - x_{k-1}) \end{aligned}$$

Set $\eta' = (\cot(\alpha' - \beta) - \cot(\alpha' + \beta))\eta$, η' is positive because $0 < \alpha' - \beta < \alpha' + \beta < \pi$. If $x_k < x_{k-1}$, we get $y_k \geq y_{k-1} + \eta' - \cot(\alpha' + \beta)(x_k - x_{k-1})$. On the other hand, if $x_k > x_{k-1}$, we have $y_k \geq y_{k-1} - \cot(\alpha' + \beta)(x_k - x_{k-1})$ because $(x_k, y_k) \in \mathcal{A}_{x_{k-1}}$. Call $N(k)$ the number of l's in $\{1, \cdots, k\}$ such that $x_l < x_{l-1}$. By an immediate induction, we deduce that

$$\begin{aligned} y_k &\geq y_0 + \eta' N(k) - \cot(\alpha' + \beta)(x_k - x_0) \\ &\geq y_0 + \eta' N(k) - \cot(\alpha' + \beta)(-n/2 - x_0) \end{aligned}$$

because $x_k \leq -n/2$ for any k. Hence, since we noticed that $N(k) \to +\infty$ as $k \to +\infty$, it comes that $y_k \to +\infty$ as $k \to +\infty$. Besides, we recall that

$(x_k, y_k) \in E$, whence $y_k \geq y_0 - \cot(\alpha' - \beta)(x_k - x_0)$. These facts imply that the points (x_k, y_k) are in $\mathcal{C}^+(0, \alpha' - \beta/2)$ for k large enough and satisfy $u(x_k, y_k) = \theta'' < 1$. This is in contradiction with (1.6) and completes the proof of lemma 2.2.

Lemma 2.3 *In dimension $N = 2$, if $\pi/2 \leq \alpha < \pi$ and if (c, u) is a solution of (1.5), (1.6), then $c \geq c_0/\sin\alpha$.*

Lemmas 2.1 and 2.3 clearly imply theorem 1.1 in dimension $N = 2$: there is no solution (c, u) of (1.5), (1.6) if $\alpha > \pi/2$.

Proof of lemma 2.3. Let us assume that $c < c_0/\sin\alpha = c_0/\sin\alpha'$. Let $\beta_n > 0$ be a sequence such that $\beta_n \to 0$ as $n \to +\infty$. For any $n \in I\!N$, by lemma 2.2, there exists a real x_n such that $u(x, y) \geq \theta'$ for any $(x, y) \in \mathcal{A}_{x_n}$, where $\mathcal{A}_{x_n}$ is defined with the angle β_n. Set $y_n = \phi_+(x_n)$. We have $u(x_n, y_n) = \theta''$ for any n. Define the functions $u_n(x, y) = u(x + x_n, y + y_n)$ in $I\!R^2$. By standard elliptic estimates and Sobolev injections, up to extraction of some subsequence, there exists a function u_∞ solving (1.5) such that $u_n \to u_\infty$ in $W^{2,p}_{loc}(I\!R^2)$ for any $1 < p < \infty$. We have $u_\infty(0, 0) = \theta''$.

Besides, by definition of (x_n, y_n), it is the case that $u_n(x, y) \geq \theta'$ for any (x, y) such that $-n \leq x \leq 0$ and $y \geq -\cot(\alpha' - \beta_n)\ x$, or $0 \leq x \leq n$ and $y \geq -\cot(\alpha' + \beta_n)\ x$. Passing to the limit $n \to +\infty$, we get

$$u_\infty(x, y) \geq \theta',\ \forall\ (x, y) \in \{y \geq -\cot\alpha'\ x,\ x \in I\!R\}$$

Let us change the coordinates and call $X = \sin\alpha'\ x - \cos\alpha'\ y$, $Y = \cos\alpha'\ x + \sin\alpha'\ y$: the positive X-axis is in the direction of $(\sin\alpha', -\cos\alpha')$ and the positive Y-axis is in the direction of $(\cos\alpha', \sin\alpha')$. Set $\tilde{u}(X, Y) = u_\infty(x, y)$. In the (X, Y) coordinates, the function $\tilde{u}$ satisfies

$$I(\tilde{u}) := \Delta\tilde{u} + c\cos\alpha'\ \partial_X\tilde{u} - c\sin\alpha'\ \partial_Y\tilde{u} + f(\tilde{u}) = 0 \text{ in } I\!R^2 \tag{2.2}$$

and $\overline{u}(0, 0) = \theta''$, $\overline{u}(X, Y) \geq \theta'$ if $Y \geq 0$. Let us finally define $\overline{u}(Y) = \inf_{X \in I\!R} \tilde{u}(X, Y)$. As an infimum of lipschitz-continuous solutions of (2.2), this function $\overline{u}(Y)$ is lipschitz-continuous and satisfies

$$\overline{u}'' - c\sin\alpha'\ \overline{u}' + f(\overline{u}) \leq 0 \text{ in } I\!R$$

Besides, it is the case that $\overline{u}(Y) \geq \theta'$ if $Y \geq 0$ and $\theta' \leq \overline{u}(0) \leq \theta''$.

For a real $\varepsilon > 0$ small enough, that will be chosen later, let $(c_\varepsilon, u_\varepsilon)$ be the unique solution of $u_\varepsilon'' - c_\varepsilon u_\varepsilon' + f(u_\varepsilon) = 0$ in $I\!R$, $u_\varepsilon(0) = \theta'$, and $u_\varepsilon(-\infty) = -\varepsilon$, $u_\varepsilon(+\infty) = 1$ (the function f has been extended by 0 outside the interval $[0,1]$). From [11], we know that $c_\varepsilon < c_0$ and that $c_\varepsilon \to c_0$ as $\varepsilon \to 0$. Since we have assumed that $c_0 > c\sin\alpha'$, we choose $\varepsilon > 0$ such that $c_\varepsilon > c\sin\alpha'$.

Let us now define the function $\underline{u}(Y) = u_\varepsilon(Y)$. Since u_ε is increasing, this function $\underline{u} = u_\varepsilon$ satisfies

$$\underline{u}'' - c\sin\alpha'\ \underline{u}' + f(\underline{u}) = (c_\varepsilon - c\sin\alpha')u_\varepsilon' > 0 \text{ in } I\!R$$

Let us now use the following lemma:

Lemma 2.4 *Let f be a lipschitz-continuous function such that $f > 0$ in $(\theta, 1)$ and $f(1) = 0$. Let v be a lipschitz-continuous function defined in $I\!R^+$ such that*

$$\begin{cases} v'' + \lambda v' + f(v) \leq 0 \text{ in } I\!R^+ \\ 1 \geq v \geq \theta' > \theta \text{ in } I\!R^+ \end{cases}$$

for a real λ. Then v is nondecreasing in $I\!R^+$ and $v(+\infty) = 1$.

In particular, taking $v = \overline{u}$, we get $\overline{u}(+\infty) = 1$. On the other hand, $\underline{u}(-\infty) = -\varepsilon$. Since $-\varepsilon \leq \underline{u}, \overline{u} \leq 1$ in $I\!R$ and since $\overline{u}$, $\underline{u}$ are respectively lipschitz-continuous super- and subsolution for the equation $z'' - c\sin\alpha'\ z' + f(z) = 0$, where the function f satisfies the assumptions of theorem 1.4 (f is nonincreasing in a neighbourhood of $-\varepsilon$ and 1), the comparison principle in theorem 1.4, applied in $I\!R$, yields that there exists a real t such that $\overline{u}^t(Y) = \overline{u}(Y + t) \geq \underline{u}(Y)$ in $I\!R$ (notice here that this result could have been obtained directly by the results of Vega in [37]). The infimum t^* of such t's is necessarily finite, otherwise $1 > \theta'' \geq \overline{u}(0) \geq \lim_{t \to -\infty} \underline{u}(-t) = 1$. Theorem 1.4 then yields furthermore that $\overline{u}^{t^*} \equiv \underline{u}$ in $I\!R$. This is impossible because $\underline{u}(-\infty) = -\varepsilon < 0$ and $\overline{u} \geq 0$ in $I\!R$. This completes the proof of lemma 2.3.

Proof of lemma 2.4 Let $x_0 \geq 0$. Assume that there exists a real $x_1 > x_0$ such that $v(x_1) < v(x_0)$. Then $v \leq v(x_1)$ on $[x_1, +\infty[$, because, if it were not so, we obtain a contradiction with the maximum principle (by considering the minimum of v on $[x_0, x_2]$ where $x_2 > x_1$ and $v(x_2) > v(x_1)$). More generally, we get that v is nonincreasing in $[x_1, +\infty[$. Set $\gamma = \lim_{+\infty} v$. We get $\theta' \leq \gamma \leq 1$. Consider the new functions $v_n(x) = v(x+n)$. Then $v_n \to v_\infty \equiv \gamma$ and v_∞ verifies $v''_\infty + \lambda v'_\infty + f(v_\infty) \leq 0$, that is to say $f(\gamma) \leq 0$. This implies that $\gamma = 1$, but this is impossible because $\gamma \leq v(x_1) < v(x_0) \leq 1$. Eventually, we conclude that $\forall x_0 \geq 0,\ \forall x_1 \geq x_0,\ v(x_1) \geq v(x_0)$. This means that v is nondecreasing in $I\!R^+$. As above, we also get that $f(v(+\infty)) \leq 0$ and $v(+\infty) = 1$.

2.2 Dimension $N \geq 3$

Our aim in this section is to prove theorem 1.1 in dimensions $N \geq 3$. Let us fix an angle $\pi/2 < \alpha < \pi$ and suppose that there is a solution (c, u) of (1.5), (1.6). From lemma 2.1, we know that $c < c_0$. The following lemma states that $c \geq c_0/\sin\alpha$: this implies the desired result.

Lemma 2.5 *In any dimension $N \geq 3$, if $\pi/2 \leq \alpha < \pi$ and if (c, u) is a solution of (1.5), (1.6), then $c \geq c_0/\sin\alpha$.*

Proof. We will proceed as in section 2.1 and make a strong use of the comparison principles stated in theorem 1.4. Let (c, u) be a solution of (1.5), (1.6) for an angle $\alpha \geq \pi/2$. Let us suppose that $c < c_0/\sin\alpha$.

Let $SO(N-1)$ be the group of rotations in $I\!R^{N-1}$. For any $\rho \in SO(N-1)$, the function $u_\rho(x', y) = u(\rho(x'), y)$ is also a solution of $\Delta u_\rho - c\partial_y u_\rho + f(u_\rho) = 0$ in $I\!R^N$. Besides, by standard elliptic estimates, the function u is globally lipschitz-continuous in $I\!R^N$. Hence, the function

$$v(x', y) = \inf_{\rho \in SO_{N-1}} u_\rho(x', y)$$

is globally lipschitz-continuous and satisfies $\Delta v - \partial_y v + f(v) \leq 0$ in $I\!R^N$ in the distribution sense. By definition of v, there exists then a globally lipschitz-continuous function $\tilde{v}$ defined in $I\!R^+ \times I\!R$ such that $v(x', y) = \tilde{v}(r, y)$ where $r = \sqrt{x_1^2 + \cdots + x_{N-1}^2}$.

Define $w(x,y) = \tilde{v}(|x|,y)$ for $(x,y) \in I\!R^2$. The function w is globally lipschitz-continuous in $I\!R^2$ and is a solution of

$$\Delta w + \frac{N-2}{x}\partial_x w - c\partial_y w + f(w) \leq 0 \text{ in } I\!R^* \times I\!R$$

in the distribution sense. Besides, since the function u fulfills the asymptotic conditions (1.6), it is easy to see that the function w satisfies the analogous conditions (1.6) in $I\!R^2$.

Henceforth, with the same notations as in section 2.1, lemma 2.2 does work, that is to say that for any sequence $\beta_n \to 0$, $\beta_n > 0$, there exists a point (x_n, y_n) such that $x_n \leq -n/2$, $y_n = \phi_+(x_n)$, $w(x_n,y_n) = \theta''$ and $w \geq \theta'$ in $\mathcal{A}_{x_n}$ where

$$\mathcal{A}_{x_n} = \{x_n - n \leq x \leq x_n,\ y \geq y_n - \cot(\alpha' - \beta_n)(x - x_n)\}$$

$$\cup\ \{x_n \leq x \leq x_n + n,\ y \geq y_n - \cot(\alpha' + \beta_n)(x - x_n)\}$$

Since the function w is globally lipschitz-continuous in $I\!R^2$, it comes from Arzela-Ascoli's theorem that the functions $u_n(x,y) = w(x + x_n, y + y_n)$ converge locally to a lipschitz-continuous function u_∞, up to extraction of some subsequence. We have $u_\infty(0,0) = \theta''$ and $u_\infty(x,y) \geq \theta'$ if $y \geq -\cot\alpha'\ x$.

Since $x_n \to -\infty$ and w is globally lipschitz-continuous in $I\!R^2$, the terms $\frac{N-2}{x+x_n}\partial_x w(x+x_n, y+y_n)$ converge locally to 0. Hence, in the distribution sense, the function u_∞ satisfies

$$\Delta u_\infty - c\partial_y u_\infty + f(u_\infty) \leq 0 \text{ in } I\!R^2$$

With the same notations as in section 2.1, the remainder of the proof of lemma 2.3 works. We get a contradiction by comparing, in a new system of cartesian coordinates (X,Y), the function $\overline{u}(Y) = \inf_{X \in I\!R} \tilde{u}(X,Y)$ – where $\tilde{u}(X,Y) = u_\infty(x,y)$ – with a suitable subsolution $\underline{u}(Y)$, and eventually by using the comparison principle of theorem 1.4.

Remark 2.6 *In any dimension $N \geq 2$, it is proved in [27] that if (c,u) is a solution of (1.5), (1.6) with an angle $\alpha \leq \pi/2$, then $c \geq c_0/\sin\alpha$.*

3 Strong framework

In this section we will prove theorems 1.2 and 1.3 dealing with the existence of solutions (c, u) of (1.8), (1.9) in dimension $N = 2$ for angles $\alpha \leq \pi/2$, with the uniqueness of the speed c and with the monotonicity properties for the functions u in any dimension $N \geq 2$.

3.1 Dimension $N = 2$, angles $\alpha \leq \pi/2$: existence result

Our aim in this section is to prove that, in dimension $N = 2$ and for any angle $\alpha \leq \pi/2$, the function u^* built by Bonnet and Hamel in [13], solving (1.5), (1.6) with the unique speed $c = c_0/\sin\alpha$ is actually also a solution of (1.8), (1.9), where the graph $\{y = \phi(x)\}$ can be any of the level sets $\{y = \phi_\lambda(x)\}$ $(0 < \lambda < 1)$. Let us fix $\alpha \in]0, \pi/2]$. We recall that the function u^* is a solution of

$$\begin{cases} \Delta u^* - c\partial_y u^* + f(u^*) = 0 \text{ in } I\!R^2 \\ \forall\, \delta \in [0, \pi - \alpha[, \quad \liminf\limits_{|(x,y)|\to\infty,\ (x,y)\in\mathcal{C}^+(0,\delta)} u^*(x,y) = 1 \\ \forall\, \delta \in [0, \alpha[, \quad \limsup\limits_{|(x,y)|\to\infty,\ (x,y)\in\mathcal{C}^-(0,\delta)} u^*(x,y) = 0 \end{cases}$$

where $c = c_0/\sin\alpha$. Moreover we know that $\forall \tau \in \mathcal{C}^+(0,\alpha)^o,\ \partial_\tau u^* > 0$ in $I\!R^2$. Fix a real $\lambda \in (0,1)$, the level set $\{u^*(x,y) = \lambda\}$ is the graph $\{y = \phi_\lambda(x)\}$ of a lipschitz-continuous function ϕ_λ with lipschitz-norm $\leq \cot\alpha$. By the implicit function theorem, the function ϕ_λ is of class C^1. Besides, it is proved in [13] that for any sequence $x_n \to \pm\infty$, the functions $u_k(x,y) = u^*(x + x_k, y + \phi_\lambda(x_k))$ go to the planar function $u_0(\pm\cos\alpha\ x + \sin\alpha\ y + u_0^{-1}(\lambda))$ in $W^{2,p}_{loc}(I\!R^2)$ for any $1 < p < \infty$ (and in particular in C^1_{loc}). This implies that $\phi'_\lambda(x) \pm \cot\alpha \to 0$ as $x \to \pm\infty$.

Suppose that $\lim\limits_{y_0\to+\infty} \inf\limits_{\Omega^+_\lambda(y_0)} u^* = \overline{\xi} < 1$. Therefore, there exists a sequence $(x_k, y_k) \in I\!R^2$ such that $y_k - \phi_\lambda(x_k) \to +\infty$ and $u^*(x_k, y_k) \to \overline{\xi} < 1$. The reals x_k cannot be bounded because, for any real $A \geq 0$, $\lim\limits_{y\to+\infty} \inf\limits_{[-A,A]\times[y,+\infty[} u^* = 1$ by (1.6). Hence, up to extraction of some subsequence, and by symmetry of u^* in x, we may assume that $x_k \to -\infty$. Set $u^*_k(x,y) = u^*(x_k + x, y + \phi_\lambda(x_k))$.

From a result in [13], it is the case that:

$$u_k^*(x,y) \to u_\infty^*(x,y) = u_0(-\cos\alpha\ x + \sin\alpha\ y + u_0^{-1}(\lambda))$$

uniformly on compact subsets of $\mathbb{R}^2$. Let y_0 be such that $u_0(\sin\alpha\ y_0 + u_0^{-1}(\lambda)) = \overline{\xi} + \delta < 1$ for some $\delta > 0$. Then for k large enough, we have both $u_k^*(0,y_0) \geq \overline{\xi} + \frac{2\delta}{3}$, $u_k^*(0, y_k - \phi_\lambda(x_k)) \leq \overline{\xi} + \frac{\delta}{3}$ and $y_k - \phi_\lambda(x_k) > y_0$. This is impossible because u^* is increasing in y.

This proves that $\lim\limits_{y_0 \to +\infty} \inf\limits_{\Omega_\lambda^+(y_0)} u^* = 1$. Similarly we get $\lim\limits_{y_0 \to -\infty} \sup\limits_{\Omega_\lambda^+(y_0)} u^* = 0$.

3.2 Dimension $N \geq 2$: monotonicity properties for the functions u

Fix an angle $\alpha \leq \pi/2$ and consider a solution (c,u) of (1.8) for which there exists a function ϕ satisfying (1.9): $\nabla\phi(x') + \cot\alpha \dfrac{x'}{|x'|} \to 0$ as $|x'| \to +\infty$. Let us fix a direction τ in $\mathcal{C}^+(0,\alpha)^o$. Choose a set of vectors $(\tau^1, \cdots, \tau^{N-1})$ such that $(\tau^1, \cdots, \tau^{N-1}, \tau)$ is a frame and define the new cartesian coordinates $X_i = \tau^i \cdot (x', y)$ $(1 \leq i \leq N-1)$, $Y = \tau \cdot (x', y)$. Let us note $X' = (X_1, \cdots, X_{N-1})$. The function $\tilde{u}(X', Y) = u(x', y)$ satisfies $\Delta\tilde{u} - c\tilde{\tau} \cdot \nabla\tilde{u} + f(\tilde{u}) = 0$ in $\mathbb{R}^N = \{(X', Y)\}$ where $\tilde{\tau}$ is the constant vector $\tilde{\tau} = (\tilde{\tau}_1, \cdots, \tilde{\tau}_{N-1}, \tilde{\tau}_N)$. Besides, since u is a solution of (1.8) and the function $\phi(x')$ satisfies (1.9), it is easy to see that there exists a lipschitz-continuous function $\tilde{\phi}(X')$ such that $\tilde{u}$ satisfies (1.8), (1.9) with the function $\tilde{\phi}$ (the set $\{Y = \tilde{\phi}(X')\}$ is not necessarily equal to the set $\{y = \phi(x)\}$ but we can choose a real R large enough such that $\{Y = \tilde{\phi}(X'),\ |X'| \geq R\}$ is a subset of $\{y = \phi(x)\}$). Theorem 1.5 applied in $\mathbb{R}^N = \mathbb{R}^{N-1} \times \mathbb{R}$ implies that the function $\tilde{u}$ is increasing in Y. This means that the function u is increasing in any direction τ of the interior of $\mathcal{C}^+(0,\alpha)$. By continuity, u is also nondecreasing in any nonzero direction of $\mathcal{C}^+(0,\alpha)$.

In particular, if (c,u) is a solution of (1.8), (1.9) with the angle $\alpha = \pi/2$, we get that u is both nondecreasing and nonincreasing in any nonzero direction τ in $\mathbb{R}^N$ such that $\tau_N = 0$. This implies that the function u only depends on y and satisfies (1.4) with the speed c. Henceforth, we deduce that $u = u(y) = u_0(y)$ (up to translation) and that $c = c_0$.

3.3 Dimension $N \geq 2$: uniqueness of the speed

Let α be an angle in $]0, \pi/2]$ and (c, u) be a solution of (1.8) and (1.9) for some function $\phi(x')$. We want to prove that the speed c is unique and given by the same formula as in dimension $N = 2$: $c = c_0/\sin\alpha$.

Let us consider the sequence $(x'_n, y_n) = (-n, 0, \cdots, 0, \phi(x'_n))$ and define the functions $u_n(x', y) = u(x' + x'_n, y + y_n)$ in $I\!R^N$. By standard elliptic estimates and Sobolev injections, up to extraction of some subsequence, the sequence (u_n) converges in $W^{2,p}_{loc}(I\!R^N)$ (for any $1 < p < \infty$) to a function u_∞ solving (1.5). We now claim that

$$\begin{array}{lll} \lim\limits_{y_0 \to +\infty} & \inf\limits_{\substack{y \geq y_0 + \cot\alpha\ x_1 \\ (x_2,\cdots,x_{N-1}) \in I\!R^{N-2}}} & u_\infty(x', y) = 1 \\ \lim\limits_{y_0 \to -\infty} & \sup\limits_{\substack{y \leq y_0 + \cot\alpha\ x_1 \\ (x_2,\cdots,x_{N-1}) \in I\!R^{N-2}}} & u_\infty(x', y) = 0 \end{array} \tag{3.1}$$

Let us prove the formula for $y_0 \to +\infty$ (the proof of the other one is similar). Let $\varepsilon > 0$. Since u satisfies the asymptotic conditions in (1.8), there exists a real y_0 such that $u(x', y) \geq 1 - \varepsilon$ if $y \geq y_0 + \phi(x')$. Fix any point (x', y) such that $y \geq y_0 + 1 + \cot\alpha\ x_1$, $(x_2, \cdots, x_{N-1}) \in I\!R^{N-2}$. From the finite increment theorem, we have that $\phi(x' + x'_n) - \phi(x'_n) = \nabla\phi(x'_n + t_n x') \cdot x'$ with some $t_n \in [0, 1]$. Hence, by (1.9) and since $x'_n = (-n, 0, \cdots, 0)$, it comes that $\phi(x' + x'_n) - \phi(x'_n) \to \cot\alpha\ x_1$ as $n \to +\infty$. This implies that $y + y_n = y + \phi(x'_n) \geq y_0 + \phi(x' + x'_n)$ and that $u_n(x', y) \geq 1 - \varepsilon$ for n large enough. The limit $n \to +\infty$ gives the desired result.

In the new coordinates $X_1 = \sin\alpha\ x_1 + \cos\alpha\ y$, $X_2 = x_2, \cdots, X_{N-1} = x_{N-1}$, $Y = -\cos\alpha\ x_1 + \sin\alpha\ y$, the function $\tilde{u}(X', Y) = u(x', y)$ satisfies the equation

$$\Delta\tilde{u} - c\cos\alpha\ \partial_{X_1}\tilde{u} - c\sin\alpha\ \partial_Y\tilde{u} + f(\tilde{u}) = 0 \text{ in } I\!R^N$$

and $\liminf\limits_{Y \to +\infty} \tilde{u} = 1$, $\limsup\limits_{Y \to -\infty} \tilde{u} = 0$ by (3.1). With the same arguments as in the previous subsection, by using theorem 1.5, we get that the function $\tilde{u}$ is then increasing in any direction τ such that $\tau_N > 0$. By continuity, $\tilde{u}$ is constant in any direction τ such that $\tau_N = 0$, that is to say that $\tilde{u} = \tilde{u}(Y)$ and that $\tilde{u}(Y)$ is a solution of $\tilde{u}'' - c\sin\alpha\ \tilde{u}' + f(\tilde{u}) = 0$ in $I\!R$ and $\tilde{u}(-\infty) = 0$, $\tilde{u}(+\infty) = 1$. By the uniqueness of the speed for the onedimensional equation (1.4), this eventually implies that $c = c_0/\sin\alpha$.

4 Solutions with asymptots.

4.1 Dimension $N = 2$

4.1.1 Uniqueness of solutions with asymptots

In this section, we prove theorem 1.7 dealing with the uniqueness and the qualitative properties of the solutions (c, u) of (1.12) in dimension $N = 2$ for angles $\alpha \leq \pi/2$ (for angles $\alpha > \pi/2$, there is no solution by theorem 1.1). Notice that the "planar" case $\alpha = \pi/2$ has already been treated in section 3 concerning the strong framework. Let now (c, u) be a solution of (1.12) for an angle $\alpha < \pi/2$. Hence, (c, u) is a solution in the weak and strong frameworks. In particular, $c = c_0/\sin\alpha$ ([13]) and u is nondecreasing in any nonzero direction of $\mathcal{C}^+(0, \alpha)$ (theorem 1.3).

Asymptotic planar behaviour in the directions $y = -\cot\alpha\ |x|$, $x \to \pm\infty$. Consider a sequence $x_n \to -\infty$ and define

$$u_n(x, y) = u(x + x_n, y - \cot\alpha\ |x_n|)$$

>From standard elliptic estimates, up to extraction of some subsequence, the sequence (u_n) converges in $W^{2,p}_{loc}$ norms to some function u^-. From (1.12), this function u^- is a solution of

$$\begin{cases} \Delta u^- - c\partial_y u^- + f(u^-) = 0 \text{ in } \mathbb{R}^2 \\ \lim\limits_{y_0 \to +\infty} \inf\limits_{\{y \geq y_0 + \cot\alpha\ x\}} u^-(x, y) = 1 \\ \lim\limits_{y_0 \to -\infty} \sup\limits_{\{y \leq y_0 + \cot\alpha\ x\}} u^-(x, y) = 0 \end{cases}$$

In the new coordinates

$$\begin{cases} X = \sin\alpha\ x + \cos\alpha\ y \\ Y = -\cos\alpha\ x + \sin\alpha\ y, \end{cases}$$

(the positive X-axis is in the direction of $(\sin\alpha, \cos\alpha)$ and the positive Y-axis is in the direction of $(-\cos\alpha, \sin\alpha)$) the function $\tilde{u}^-(X, Y) = u^-(x, y)$ satisfies

$$\begin{cases} \Delta\tilde{u}^- - c_0\partial_Y\tilde{u}^- - c_0\cot\alpha\ \partial_X\tilde{u}^- + f(\tilde{u}^-) = 0 \text{ in } \mathbb{R}^2 \\ \lim\limits_{Y \to +\infty} \inf\limits_{X} \tilde{u}^-(X, Y) = 1 \\ \lim\limits_{Y \to -\infty} \sup\limits_{X} \tilde{u}^-(X, Y) = 0 \end{cases} \tag{4.1}$$

In a similar way, we can define a function u^+ for any sequence $x_n \to +\infty$ and write a similar problem involving the new coordinates $X = -\sin\alpha\ x + \cos\alpha\ y$, $Y = \cos\alpha\ x + \sin\alpha\ y$. The asymptotic behaviour of u in the directions $\{y = -\cot\alpha\ |x|,\ x \to \pm\infty\}$ is given by the following proposition, which corresponds to part (iv) of theorem 1.7:

Proposition 4.1 *There exist two reals $t^\pm$ such that $u^\pm(x,y) = u_0(\pm\cos\alpha\ x + \sin\alpha\ y + t^\pm) = u_0(Y + t^\pm)$. Besides, the reals $t^\pm$ do not depend on the sequences $x_n \to \pm\infty$ and are equal to $t^\pm = u_0^{-1}(\lim_{x\to\pm\infty} u(x, -\cot\alpha\ |x|))$*

Proof. The fact that the functions $\tilde{u}^\pm(X,Y)$ only depend on Y can be done with the same device as in section 3.2: for instance, by using theorem 1.5, we deduce that $\tilde{u}^-$ is increasing in any direction τ such that $\tau_Y > 0$. Then, it only depends on Y and satisfies (1.4), whence $\tilde{u}^-(X,Y) = u_0(Y + t^-)$. On the other hand, the function u is nondecreasing in both directions $(\pm\sin\alpha, \cos\alpha)$. We can therefore define the limits $\lim_{x\to\pm\infty} u(x, -\cot\alpha\ |x|) = \theta^\pm \in [0,1]$. Now, for any sequences $(x_n) \to \pm\infty$, the functions $u(x + x_n, y - \cot\alpha\ |x_n|)$ locally converge to the planar fronts $u_0(\pm\cos\alpha\ x + \sin\alpha\ y + u_0^{-1}(\theta^\pm))$. Thus, the reals $t^\pm$ do not depend on the sequences (x_n) and are then equal to $u_0^{-1}(\theta^\pm)$. Notice that this especially proves that $\theta^\pm$ cannot be 0 or 1.

Proof of theorem 1.7. Let α be any fixed angle in $(0, \pi/2)$ and u any solution of (1.12) in dimension $N = 2$. For any $x_0 \in I\!R$, the shifted function

$$u_{x_0}(x,y) = u(x + x_0, y)$$

is also a solution of (1.12). From proposition 4.1, we know that the functions $(u_{x_0})_n = u(x + x_0 + x_n, y - \cot\alpha\ |x_n|)$ go to the planar fronts $u_0(\pm\cos\alpha\ x + \sin\alpha\ y + (t_{x_0})^\pm)$ for some $(t_{x_0})^\pm \in I\!R$ and for any sequences $x_n \to \pm\infty$. It is easy to check that $(t_{x_0})^\pm = t^\pm \pm \cos\alpha\ x_0$ where $t^\pm$ are defined in proposition 4.1 for the function u. Taking $x_0 = (t^- - t^+)/(2\cos\alpha)$, we get

$$(t_{x_0})^- = (t_{x_0})^+$$

We will use the following definition: for any solution u of (1.12), we say that u is "*asymptotically symmetric in x*" if the reals $t^\pm$ defined in proposition 4.1 are equal.

Now consider another solution u' for (1.12). According to the preceding arguments, there exists a real x_0' such that the function $u'_{x_0'}(x,y) = u'(x+x_0',y)$ is asymptotically symmetric in x.

Both functions u_{x_0} and $u'_{x_0'}$ are solutions of problem (1.12), which is of the same kind as (1.10), (1.11) with $\phi(x) = -\cot\alpha\ |x|$. Henceforth, theorem 1.4 can be applied with $(\overline{u}, \underline{u}) = (u_{x_0}, u'_{x_0'})$ and with $(\overline{u}, \underline{u}) = (u'_{x_0'}, u_{x_0})$. There exist two reals $\underline{t} \le \overline{t}$ such that $u^{\underline{t}}_{x_0} \le u'_{x_0'} \le u^{\overline{t}}_{x_0}$. Define $\underline{t}^* = \sup\{t,\ u^t_{x_0} \le u'_{x_0'} \text{ in } I\!\!R^2\}$ and $\overline{t}^* = \inf\{t,\ u^t_{x_0} \ge u'_{x_0'} \text{ in } I\!\!R^2\}$. These reals are finite since $u_{x_0}(0,t) \to 1$ or 0 as $t \to \pm\infty$ and $0 < u'_{x_0'}(0,0) < 1$. We have

$$u^{\underline{t}^*}_{x_0} \le u'_{x_0'} \le u^{\overline{t}^*}_{x_0} \text{ in } I\!\!R^2$$

The strong maximum principle yields that either $u^{\underline{t}^*}_{x_0} < u'_{x_0'}$ or $u^{\underline{t}^*}_{x_0} \equiv u'_{x_0'}$, and either $u'_{x_0'} < u^{\overline{t}^*}_{x_0}$ or $u'_{x_0'} \equiv u^{\overline{t}^*}_{x_0}$. Let us suppose that $u^{\underline{t}^*}_{x_0} < u'_{x_0'} < u^{\overline{t}^*}_{x_0}$ in $I\!\!R^2$. From the last assertion in theorem 1.4 and since $u'_{x_0'}$ and u_{x_0} are asymptotically symmetric in x, it follows that:

$$\begin{cases} \lim\limits_{x_n \to \pm\infty} u'_{x_0'}(x+x_n, y-\cot\alpha\ |x_n|) - u^{\underline{t}^*}_{x_0}(x+x_n, y-\cot\alpha\ |x_n|) = 0 \\ \lim\limits_{x_n \to \pm\infty} u^{\overline{t}^*}_{x_0}(x+x_n, y-\cot\alpha\ |x_n|) - u'_{x_0'}(x+x_n, y-\cot\alpha\ |x_n|) = 0 \end{cases} \tag{4.2}$$

>From our choice of x_0, the functions $u_{x_0}(x+x_n, y-\cot\alpha\ |x_n|)$ go to the planar front $u_0(-\cos\alpha\ x + \sin\alpha\ y + t)$ as $x_n \to -\infty$, where $t = t^-_{x_0} = t^+_{x_0}$. From (4.2), we deduce that $\sin\alpha\ \underline{t}^* + t = \sin\alpha\ \overline{t}^* + t$. This means that $\underline{t}^* = \overline{t}^*$ and that $u'_{x_0'} \equiv u^{\underline{t}^*}_{x_0}$ in $I\!\!R^2$. Finally, we always have

$$u'(x+x_0', y) \equiv u(x+x_0, y+t^*) \text{ in } I\!\!R^2$$

where t^* is equal to $\underline{t}^*$ or $\overline{t}^*$ (actually, we then have $\underline{t}^* = \overline{t}^*$). This proves that the solutions of (1.12) are unique up to translation.

Let us now prove that any solution u of (1.12) is symmetric with respect to some line $\{x = x_0\}$, that is part (i) of theorem 1.7. Indeed, for any solution u of (1.12), the function $u'(x,y) = u(-x,y)$ is itself a solution of (1.12). Hence, there exists a couple $(t_1,t_2) \in I\!\!R^2$ such that $u(-x,y) = u(x+t_1, y+t_2)$ in $I\!\!R^2$. At the point $(-t_1/2, 0)$, it comes that $u(t_1/2, 0) = u(t_1/2, t_2)$. Since u is increasing in y, we deduce that $t_2 = 0$ and finally that u is symmetric with respect to the line $\{x = t_1/2\}$.

Let us now study the level sets of u and prove part (ii) of theorem 1.7. For any $\lambda \in (0,1)$, the level set $\{u(x,y)=\lambda\}$ is a curve $y=\phi_\lambda(x)$ since $u(x,y)$ is strictly increasing in y and goes to 0 and 1 as $y \to \pm\infty$, for any $x \in I\!R$. From the behaviour of u in the asymptotic directions $y=-\cot\alpha\ |x|$ (proposition 4.1) and since u_0 is strictly increasing, there exists a unique real t such that $u(x, t-\cot\alpha\ |x|) \to \lambda$ as $x \to -\infty$. Besides, the level set $\{u(x,y)=\lambda\}$ is below the line $y=t+\cot\alpha\ x$ since u is increasing in the direction $(\sin\alpha, \cos\alpha)$. From the asymptotic behaviour of u, it also comes that $\lim_{x\to-\infty} u(x, t-\varepsilon-\cot\alpha\ |x|) < \lambda$ for any $\varepsilon > 0$. We deduce that the level set $\{u(x,y)=\lambda\}$ is above the line $y=t-\varepsilon+\cot\alpha\ x$ for $-x$ large enough. With the same arguments as $x \to +\infty$, this proves that the level set $\{u(x,y)=\lambda\}$ is asymptotic to the half-line $y=-\cot\alpha\ |x|+t$ as $x\to-\infty$ (resp. $y=-\cot\alpha\ |x|+t'$ as $x\to+\infty$. As in the proof of theorem 1.2 in section 3, the function u is a solution of (1.8), (1.9) for any function $\phi=\phi_\lambda$.

Let us now prove that u is equal, up to some translation, to the function u^* built in [13] and solving (1.5), (1.6). First of all, we know that there exists a real t_1 such that the function $u_{t_1}=u(x+t_1,y)$ is symmetric in x. Since u^* is nonincreasing in both directions $(\pm\sin\alpha, -\cos\alpha)$, the level set $\{u^*=u^*(0,0)\}$ is not below $\partial\mathcal{C}^-(0,\alpha)$. Hence, theorem 1.2 yields that $\lim_{y\to-\infty} \sup_{\mathcal{C}^-(y,\alpha)} u^* = 0$. Apply theorem 1.4 with $\overline{u}=u_{t_1}$ and $\underline{u}=u^*$: there exists a real t such that $u^t_{t_1} \geq u^*$ in $I\!R^2$. The infimum t^* of such t's is finite because $u(t_1,t)\to 0$ as $t\to-\infty$ and $u^*(0,0)>0$. Henceforth, theorem 1.4 yields that

$$\forall y_0 \in I\!R, \quad \inf_{\partial C^-(y_0,\alpha)} (u^{t^*}_{t_1} - u^*) = 0 \tag{4.3}$$

>From the strong maximum principle, we have either $u^{t^*}_{t_1} \equiv u^*$ or $u^{t^*}_{t_1} > u^*$ in $I\!R^2$. In any case, and since both u_{t_1} and u^* are symmetric in x, (4.3) implies that

$$\lim_{x_n\to\pm\infty} (u^{t^*}_{t_1}(x_n, y+\cot\alpha\ |x|-\cot\alpha\ |x_n|) - u^*(x_n, y+\cot\alpha\ |x|-\cot\alpha\ |x_n|)) = 0$$

for any $(x,y)\in I\!R^2$. On the one hand, u^* is nondecreasing in both directions $(\pm\sin\alpha, \cos\alpha)$, hence $u^*(x,y) \geq \lim_{x_n\to\pm\infty} u^*(x_n, y+\cot\alpha\ |x|-\cot\alpha\ |x_n|)$. On the other hand, since u_{t_1} is a solution of (1.12), proposition 4.1 implies that

$\lim_{x_n \to \pm\infty} u_{t_1}^{t^*}(x_n, y + \cot\alpha\ |x| - \cot\alpha\ |x_n|) = u_0(\sin\alpha\ y + \cos\alpha\ |x| + t_0)$ for some real t_0. Eventually, $u^*(x,y) \geq u_0(\sin\alpha\ y + \cos\alpha\ |x| + t_0)$ in $I\!R^2$. Thus, the function u^* verifies both asymptotic conditions in (1.12). Since the solutions of (1.12) are unique modulo translation, we conclude that u and u^* are equal up to a translation.

4.1.2 Existence of solutions with asymptots for angles $\alpha < \pi/2$ and for some functions f

In this section, our purpose is to build, in dimension $N = 2$, a solution u of (1.12) which is over a given subsolution v fulfilling $\lim_{y\to+\infty} \inf_{\mathcal{C}^+(y,\pi-\alpha)} v = 1$. To do that, we consider a sequence of solutions u_n over v in bounded domains Ω_n covering the plane $I\!R^2$. For some functions f and some angles $\alpha \leq \pi/2$, this process leads to a solution of (1.12). We recall that $c = c_0/\sin\alpha$ is the unique possible speed for (1.12).

Define $\phi(x) = -\dfrac{1}{c_0 \sin\alpha}\ \ln(\cosh(c_0\cos\alpha\ x))$ and

$$v(x,y) = u_0(\sin\alpha\ (y - \phi(x)))$$

where $u_0(z)$ is the solution of the onedimensional equation (1.4). An easy calculation shows that

$$\begin{aligned} I(v)\ &:= \Delta v - c\partial_y v + f(v) \\ &= [c_0 \sin^2\alpha\ (1+\phi'^2) - \sin\alpha\ \phi'' - c_0] u_0'(\sin\alpha\ (y-\phi(x))) \\ &\qquad +[1 - \sin^2\alpha\ (1+\phi'^2)]\ f(v) \\ &= \cos^2\alpha \cosh^{-2}(c_0\cos\alpha\ x)\ f(v)\ =: f_1(x,y) \\ &\geq 0 \end{aligned}$$

Consider two sequences of positive reals a_n, b_n. Set $\Omega_{a_n,b_n} = \{(x,y),\ (x, y - \phi(x)) \in (-a_n, a_n) \times (-b_n, b_n)\}$ and let Ω_n be a smooth domain embedded in Ω_{a_n,b_n} in such a way that $\Omega_{a_n,b_n} \backslash \Omega_n \subset \bigcup_{i=1,\cdots,4} B(C_i, r)$ where C_i, $i = 1,\cdots,4$ are the four corners of Ω_{a_n,b_n} and $B(C_i,r)$ are the balls with centers C_i and radius $r > 0$ fixed.

The function $v \leq 1$ satisfies $I(v) \geq 0 = I(1)$ and the function f is lipschitz-continuous. Hence, by a classical iterative method, there exists a unique function u_n in Ω_n solving:

$$\begin{cases} I(u_n) = \Delta u_n - c\partial_y u_n + f(u_n) = 0 & \text{in } \Omega_n \\ v \le u_n \le 1 & \text{in } \Omega_n \\ u_n = v & \text{on } \partial\Omega_n \end{cases} \tag{4.4}$$

Furthermore, since v is not a solution of (1.5) in Ω_n ($f(v) \not\equiv 0$), the strong maximum principle yields that $u_n > v$ in Ω_n.

For some given reals μ_1 and μ_2 which will be chosen later, set

$$w_n(x,y) = (u_n(x,y) - v(x,y))\ e^{\mu_1 y + \mu_2 \phi(x)} > 0 \text{ in } \Omega_n$$

By a straightforward calculation, this function $w_n \ge 0$ is a solution of

$$\begin{aligned} \tilde{I}(w_n) \ &= \Delta w_n - (2\mu_1 + c)\partial_y w_n - 2\mu_2\phi'(x)\partial_x w_n \\ &\quad +(K_n(x,y) + \lambda(x))w_n \\ &= -f_2(x,y) \end{aligned} \tag{4.5}$$

where
$$\begin{cases} K_n(x,y) &= \dfrac{f(v + e^{-\mu_1 y - \mu_2\phi(x)} w_n) - f(v)}{e^{-\mu_1 y - \mu_2 \phi(x)} w_n} \\ \lambda(x) &= \mu_1^2 + c\mu_1 + \mu_2^2 \phi'(x)^2 - \mu_2\phi''(x) \\ f_2(x,y) &= f_1(x,y) e^{\mu_1 y + \mu_2\phi(x)} \end{cases}$$

Besides, from standard elliptic estimates, we have $\|w_n\|_{W^{2,2}(\Omega_n)} \le C_n$ for some constant C_n.

Let us multiply equation (4.5) by w_n and integrate in Ω_n. We get:

$$\int_{\Omega_n} \Delta w_n\ w_n - (2\mu_1 + c)\partial_y w_n\ w_n - 2\mu_2\phi'(x)\partial_x w_n\ w_n + (K_n(x,y) + \lambda(x))w_n^2$$
$$= \ -\int_{\Omega_n} f_2 w_n$$

By integrating by parts over this smooth domain Ω_n, and using the fact that $w_n = 0$ on $\partial\Omega_n$, we eventually get

$$\int_{\Omega_n} \|\nabla w_n\|^2 + (\Lambda(x) - K_n(x,y))w_n^2 \ = \ \int_{\Omega_n} f_2 w_n \tag{4.6}$$

where $\Lambda(x) = -\lambda(x) - \mu_2\phi''(x) = -\mu_1^2 - c\mu_1 - \mu_2^2\phi'(x)^2$.

Proposition 4.2 *If there exist reals μ_1, μ_2 and Λ_0 such that $\mu_1 < 0$, $\mu_1+\mu_2 \leq 0$, $\Lambda_0 > 0$,*

$$\forall n, \quad \Lambda(x) - K_n(x,y) \geq \Lambda_0 > 0 \text{ in } \Omega_n$$

and if $f_2 \in L^2(I\!\!R^2)$, then there exists a solution of (1.12).

Proof. Assume that all the requirements of proposition 4.2 are satisfied. From (4.6), we have $\|w_n\|_{W^{1,2}(\Omega_n)} \leq C\|f_2\|_{L^2(I\!\!R^2)}$ where $C = \min(\Lambda_0, 1)^{-1/2}$. Taking the limit a_n, $b_n \to +\infty$ and r_n constant, up to extraction of some subsequence, the functions w_n's go locally to some function $w \geq 0$ in L^2. By (4.5) and the definition of K_n, this function w satisfies, in the distribution sense:

$$\Delta w - (2\mu_1 + c)\partial_y w - 2\mu_2\phi'(x)\partial_x w + \lambda(x)w$$

$$+(f(v + e^{\mu_1 y+\mu_2\phi(x)}w) - f(v))e^{\mu_1 y+\mu_2\phi(x)} = -f_2(x,y)$$

Besides, we have $\|w\|_{W^{1,2}(I\!\!R^2)} \leq C\|f_2\|_{L^2(I\!\!R^2)}$.

Set $u = v + e^{-\mu_1 y-\mu_2\phi(x)}w$. The functions u_n's defined in the previous section go to u in $L^2_{loc}(I\!\!R^2)$. The function u is such that $0 < v \leq u \leq 1$ in $I\!\!R^2$ and is a solution of (1.5) in $I\!\!R^2$. From standard elliptic estimates, the function u is a classical solution of (1.5) and has a bounded norm in $C^1(I\!\!R^2)$. We also have:

$$\|(u-v)e^{\mu_1 y+\mu_2\phi(x)}\|_{W^{1,2}(I\!\!R^2)} \leq C\|f_2\|_{L^2(I\!\!R^2)} \tag{4.7}$$

Since $u \geq v$, it comes that $\lim\limits_{y\to+\infty} \inf\limits_{\mathcal{C}^+(y,\pi-\alpha)} u = 1$. To prove that u is actually a solution of (1.12), the only thing that remains to be proved is that $\lim\limits_{y\to-\infty} \inf\limits_{\mathcal{C}^-(y,\alpha)} u = 0$. If this is not the case and since the function u is globally lipschitz-continuous, there exist two reals ε, $r > 0$ and a sequence of points (x_n, y_n) such that $y_n + \cot\alpha\ |x_n| \to -\infty$ and $u(x,y) \geq \varepsilon > 0$ in the ball $B_n = B((x_n,y_n), r)$ of center (x_n, y_n) and fixed radius $r > 0$. By definition of ϕ, it also comes that $y_n - \phi(x_n) \to -\infty$. For any point $(x,y) \in B_n$, we have $e^{\mu_1 y+\mu_2\phi(x)} = e^{\mu_1(y-\phi(x))}e^{(\mu_1+\mu_2)\phi(x)} \geq e^{\mu_1(y-\phi(x))}$ since $\mu_1 + \mu_2 \leq 0$ and $\phi \leq 0$. Lastly, since $v \to 0$ uniformly in B_n and $\mu_1 < 0$, it comes that $(u-v)e^{\mu_1 y+\mu_2\phi(x)}$ uniformly goes to $+\infty$ in B_n. This is contradiction with (4.7).

We conclude that the function u satisfies the conical conditions in (1.12). It is then a solution of (1.12).

In the following lemmas, we prove that the requirements in proposition 4.2 are satisfied for some ranges of functions f and of angles α.

Lemma 4.3 *Remember that* $\mu_0 = \frac{\sqrt{c_0^2+4|f'(1)|}-c_0}{2}$ *and that* $1-u_0(y) \sim C_0 e^{-\mu_0 y}$ *as* $y \to +\infty$ *for some positive constant* C_0 *(see Fife, McLeod [19]). We have:*

$$\left(f_2 \in L^2(I\!R^2)\right) \Longleftrightarrow \begin{cases} \mu_1 < \mu_0 \sin\alpha \\ \mu_1 + \mu_2 + 2c_0 \sin\alpha > 0 \end{cases}$$

Proof. Let us remember that $f_2(x,y) = \cos^2\alpha \cosh^{-2}(c_0 \cos\alpha\, x) f(u_0(\sin\alpha(y-\phi(x))))e^{\mu_1 y+\mu_2\phi(x)}$ and $\cosh^{-2}(c_0\cos\alpha\ x) = e^{2c_0\sin\alpha\phi(x)}$. Setting $\tilde{y} = y - \phi(x)$, we get by using a straightforward calculation:

$$\int_{I\!R^2} f_2^2 = \cos^4\alpha \int_{I\!R} f^2(u_0(\sin\alpha\tilde{y}))e^{2\mu_1\tilde{y}}d\tilde{y} \int_{I\!R} e^{2(2c_0\sin\alpha+\mu_1+\mu_2)\phi(x)}dx$$

For the first integral in $\tilde{y}$, since $f \equiv 0$ on $[0,\theta]$ and $u_0(y) \le \theta$ for $y \le 0$, we only have to study the behaviour as $\tilde{y} \to +\infty$. As $\tilde{y} \to +\infty$, we have $f^2(u_0(\sin\alpha\tilde{y}))e^{2\mu_1\tilde{y}} \sim C_0^2 f'(1)^2 e^{2(\mu_1-\mu_0\sin\alpha)\tilde{y}}$. Hence, the integral in $\tilde{y}$ converges if and only if $\mu_1 < \mu_0\sin\alpha$.

On the other hand, $\phi(x) = -\cot\alpha\ |x| + \ln 2/c_0\sin\alpha + o(1)$ as $x \to \pm\infty$. It follows that the integral $\int_{I\!R} e^{2(2c_0\sin\alpha+\mu_1+\mu_2)\phi(x)}dx$ converges if and only if $\mu_1 + \mu_2 + 2c_0\sin\alpha > 0$.

Lemma 4.4 *There is a continuous decreasing function* $k :]0,\pi/2] \to [1/4, 9/4[$ *such that for any angle* α *in* $]0,\pi/2]$*, there exists a pair* (μ_1,μ_2) *such that* $\mu_1 < 0$*,* $\mu_1+\mu_2+2c_0\sin\alpha \ge 0$ *and* $\Lambda(x) \ge c_0^2 k(\alpha)$ *for any* $x \in I\!R$*.*

Proof. From the definition of $\phi(x)$, we have $h(\mu_1,\mu_2) := \inf_x \Lambda(x) = -\mu_1^2 - c\mu_1 - \mu_2^2\cot^2\alpha$. In order that Λ satisfy the requirement of proposition 4.2, we will maximize this function h in the set $\mathcal{H} = \{\mu_1 \le 0,\ \mu_1+\mu_2+2c_0\sin\alpha \ge 0\}$. It is easy to see that this maximum is equal to

$$\max_{\mathcal{H}} h = \max\Big(\max_{\mu_1 \le -2c_0\sin\alpha}(-\mu_1^2 - c\mu_1 - (\mu_1+2c_0\sin\alpha)^2\cot^2\alpha),$$

$$\max_{-2c_0\sin\alpha \le \mu_1 \le 0}(-\mu_1^2 - c\mu_1)\Big)$$

After some easy calculations, we find that if $\pi/6 \le \alpha \le \pi/2$, then $\max_{\mathcal{H}} h = c_0^2/(4\sin^2\alpha) > 0$ and is reached for $(\mu_1^*, \mu_2^*) = (-c_0/(2\sin\alpha), 0)$, and if $0 < \alpha \le \pi/6$, then $\max_{\mathcal{H}} h = c_0^2(3-4\sin^2\alpha)^2/4 > 0$ and this maximum is reached for $(\mu_1^*, \mu_2^*) = (-c_0 \sin\alpha\,(5-4\sin^2\alpha)/2,\ c_0\sin\alpha\,(1-4\sin^2\alpha)/2)$. The function $g(\alpha) = -\mu_1^* - \mu_2^*$ is that defined in theorem 1.8. For any angle α, we can see that $\mu_2^* \ge 0$.

Lastly, it is easy to check that the function $k(\alpha) := \max_{\mathcal{H}} h\ /c_0^2$ is decreasing and ranges within $[1/4, 9/4[$ as α is in $]0, \pi/2]$.

Proof of theorem 1.8. Let us first recall that the speed c_0 of the planar wave solving (1.4) only depends on f and let us note it $c_0(f)$. We first claim that

$$\frac{1}{4} < \frac{\sup\ f'}{c_0^2(f)} \tag{4.8}$$

Indeed, let $\tilde{f}$ be the restriction of f in the interval $[\theta, 1]$. From the classical results of Kolomogorov, Petrovskii, Piskunov [32], or later Aronson, Weinberger [3] and Fife, McLeod [19], there exists a positive real c^* such that for any $c \ge c^*$, there is a unique solution u of $u'' - cu' + \tilde{f}(u) = 0$, $u(-\infty) = \theta$, $u(+\infty) = 1$, and there is no solution for $c < c^*$. Besides, it is the case that

$$2\sqrt{\tilde{f}'(\theta)} \le c^* \le 2 \sup_{u\in[\theta,1]} \sqrt{\frac{\tilde{f}(u)}{u-\theta}} \tag{4.9}$$

(see also Hadeler and Rothe [26]). Since $\tilde{f}(\theta) = f(\theta) = 0$, for any $u \in (\theta, 1)$, there exists $s \in (\theta, u)$ such that $\dfrac{\tilde{f}(u)}{u-\theta} = f'(s) \le \sup\limits_{[0,1]} f'$. Consequently, we get $c^* \le 2\sqrt{\sup\limits_{[0,1]} f'}$. On the other hand, from Berestycki, Nirenberg [11], it is the case that $c_0(f) < c^*$ (this could also be done with the comparison principle in thereom 1.4). This finally gives the inequality (4.8).

Let us now assume that the function f satisfies

$$\frac{\sup\ f'}{c_0^2(f)} < \frac{9}{4}$$

and define $\alpha_0 = \sup\{\alpha \in]0, \pi/2],\ k(\alpha) > \sup\limits_{[0,1]} f'\ /c_0^2(f)\}$; this angle α_0 exists and is positive since $\sup k(\alpha) = 9/4$; besides, $\alpha_0 < \pi/2$ by (4.8) and since $k(\pi/2) = 1/4$.

Let now α be in $(0,\alpha_0)$. By the standard elliptic estimates and Sobolev injections, it is the case that the functions u_n defined by (4.4) converge in the spaces $W^{2,p}_{loc}(\mathbb{R}^2)$, up to extraction of some subsequence, to a function u such that $u \geq v$ and u solves (1.5).

>From lemma 4.4, we get $c_0^2 k(\alpha) > \sup f'$. By continuity of the function $h(\mu_1,\mu_2)$, there exists a pair (μ_1,μ_2) such that $\mu_1 < 0 (< \mu_0 \sin\alpha)$, $\mu_1+\mu_2+2c_0\sin\alpha > 0$, $\mu_2 > 0$ and $\Lambda(x) > \sup f' + \varepsilon$ for some $\varepsilon > 0$ and for any $x \in \mathbb{R}$. From lemma 4.3, the function f_2 is in $L^2(\mathbb{R}^2)$. Furthermore, for any n, we have

$$K_n(x,y) = \frac{f(v+w_n)-f(v)}{w_n} \leq \sup_{[0,1]} f' \text{ in } \Omega_n$$

since $w_n > 0$ in Ω_n. Hence, $\Lambda(x) - K_n(x,y) \geq \varepsilon > 0$ in Ω_n. All the assumptions of proposition 4.2 are satisfied, therefore the function u is a solution of (1.12).

Let us now prove that α_0 can be taken as close to $\pi/2$ as possible if the function f is well-chosen. Having (4.8) in mind, we will actually prove that for any $\eta > 0$, there exists a function f_δ such that

$$\frac{1}{4} < \frac{\sup f'_\delta}{c_0^2(f_\delta)} < \frac{1}{4} + \eta \tag{4.10}$$

Owing to the definition of α_0 and since $k(\pi/2) = 1/4$, this will give the last assertion in theorem 1.8.

To do that, let us consider the functions f_δ defined by $f_\delta = 0$ in $[0,\delta]$ and $f_\delta(u) = (u-\delta)(1-u)$ in $[\delta,1]$. This function satisfies (1.3) with $\theta = \delta$ and is C^1 in $[\delta,1]$. Let $c_{0,\delta}$ be the unique admissible speed for (1.4) with the function f_δ and let c^*_δ be the minimal speed for the solutions of $u'' - cu' + f_\delta(u) = 0$, $u(-\infty) = \delta$, $u(+\infty) = 1$. From the results above, we know that $c_{0,\delta} < c^*_\delta$. Besides, by (4.9) and since the restriction $\tilde{f}_\delta$ of f_δ on $[\delta,1]$ is concave, we have $c^*_\delta = 2\sqrt{\tilde{f}_\delta{}'(\delta)} = 2\sqrt{1-\delta} \leq 2$. Let δ_n be a sequence converging to 0 and let u_{δ_n} be the unique function solving (1.4) with f_{δ_n} (and the speed c_{0,δ_n}) such that $u_{\delta_n}(0) = 1/2$. Since the speeds c_{0,δ_n} are in $[0,2]$, up to extraction of some subsequence, there exists a real $c \in [0,2]$ and a nondecreasing function u in $\mathbb{R}$ solving $u'' - cu' + u(1-u) = 0$, $u(0) = 1/2$. It then comes that $u(-\infty) = 0$, $u(+\infty) = 1$ and that c is greater than or equal to the minimal speed corresponding to the nonlinearity $u(1-u)$, namely 2 by (4.9). Finally, we conclude that $c_{0,\delta} \to 2$ as $\delta \to 0$ (the limit does not depend on the sequence $\delta_n \to 0$). On the other hand, we have $\sup_{[0,1]} f'_\delta = 1 - \delta$. Hence, (4.10) is true when $\delta > 0$ is small enough. This completes the proof of theorem 1.8.

4.1.3 Existence of solutions with asymptots under a nondegeneracy assumption

The aim of this section is to prove theorem 1.9, that is to say that if u is a solution of (1.5), (1.6), (1.13), (1.14) in dimension 2 and with an angle $\alpha \le \pi/2$, then u is also a solution of (1.12). Notice that the speed is necessarily equal to $c = c_0/\sin\alpha$. The proof is divided into several lemmas.

Lemma 4.5 *Let u be a supersolution of (1.5) such that $u > 0$ in $I\!\!R^2$ and define*

$$v(y) = \inf_{\partial\mathcal{C}^-(y/sin\alpha,\alpha)} u$$

The function v is nondecreasing, lipschitz-continuous and verifies $v'' - c_0 v' + f(v) \le 0$ in $I\!\!R$.

Proof. Let us suppose for the time being that the function v is proved to be nondecreasing. The function

$$w(x,y) = \inf\left(\inf_{\lambda\in I\!\!R} u(x+\lambda, y+\lambda\cot\alpha),\ \inf_{\lambda\in I\!\!R} u(x+\lambda, y-\lambda\cot\alpha)\right)$$

is then equal to the function $v(\sin\alpha\ y + \cos\alpha\ |x|)$. Since the functions inside the infimum are uniformly lipschitz-continuous and supersolutions of (1.5) in $I\!\!R^2$, the function w is lipschitz-continuous and satisfies $\Delta w - c\partial_y w + f(w) \le 0$ in $I\!\!R^2$ in the distribution sense. This yields that the function v is also lipschitz-continuous and satisfies $v'' - c_0 v' + f(v) \le 0$ in $I\!\!R$.

To prove that v is nondecreasing, it is enough to prove, because of the invariance of the problem by translation, that if $u \ge \underline{\xi}$ on $\partial\mathcal{C}^-(0,\alpha) = \partial\mathcal{C}^+(0,\pi-\alpha)$, then $u \ge \underline{\xi}$ in $\mathcal{C}^+(0,\pi-\alpha)$, for any fixed but arbitrary $\underline{\xi} \in (0,1)$.

Fix any $N \in I\!\!N^*$, $a \ge \frac{2N}{c} > 0$ and $0 < \varepsilon < \underline{\xi}$. Set $\varepsilon_1 = \sqrt{\frac{Nc}{2a}e^{-N}}$ and $\varepsilon_2 = \frac{N}{a}$ and consider the function

$$z(x,y) = (\underline{\xi} - \varepsilon)\ \cos(\varepsilon_1 x)\ 1_{\{\varepsilon_1 x \in (-\frac{\pi}{2},\frac{\pi}{2})\}}\ (1 - e^{\varepsilon_2 y})^+$$

In the set $\{z > 0\}$, we have:

$$\begin{aligned} I(z) &:= \Delta z - c\partial_y z + f(z) \\ &\geq -\varepsilon_1^2 z + (\underline{\xi} - \varepsilon)\cos(\varepsilon_1 x)(-\varepsilon_2^2 + c\varepsilon_2)e^{\varepsilon_2 y} \\ &\geq (\underline{\xi} - \varepsilon)\cos(\varepsilon_1 x)\ K(y) \end{aligned}$$

where

$$K(y) = [\varepsilon_2(c - \varepsilon_2) + \varepsilon_1^2]e^{\varepsilon_2 y} - \varepsilon_1^2 \geq \frac{Nc}{2a}(e^{\frac{N}{a}y} - e^{-N})$$

owing to the choice of ε_1 and ε_2. Hence, $I(z) \geq 0$ in $\{-a \leq y \leq 0\} \cap \{z > 0\}$. For any $t \in [0, a]$, the function $z^t(x, y) = z(x, y + \frac{\pi}{2\varepsilon_1}\cot\alpha - t)$ satisfies $I(z^t) \geq 0$ in $\{z^t > 0\} \cap \mathcal{C}^+(0, \pi - \alpha)$. On the other hand, we get by construction:

$$\forall\ t \in [0, a], \quad z^t \leq \underline{\xi} - \varepsilon < \underline{\xi} \leq u \text{ on } \partial\mathcal{C}^+(0, \pi - \alpha) \tag{4.11}$$

We will now apply a sliding method. We first observe that $z^0 = 0 \leq u$ in $\mathcal{C}^+(0, \pi - \alpha)$. Let us define $t^* = \sup\ \{t \in [0, a], z^t \leq u \text{ in } \mathcal{C}^+(0, \pi - \alpha)\}$. We immediately get $z^{t^*} \leq u$ in $\mathcal{C}^+(0, \pi - \alpha)$. If $t^* < a$, then there are two sequences $a > t_n \searrow t^*$ and $P_n \in \mathcal{C}^+(0, \pi - \alpha)$ such that $z^{t_n}(P_n) > u(P_n)$. By definition of z^t and of z, the points P_n are bounded. Up to extraction of a subsequence, we can assume that $P_n \to P_\infty$. It then follows that $z(P_\infty) = u(P_\infty) > 0$, whence $t^* > 0$. By (4.11), the point P_∞ cannot be on $\partial\mathcal{C}^+(0, \pi - \alpha)$. Hence, from the strong maximum principle and since u is a supersolution of (1.5), it comes that $z^{t^*} \equiv u$ in the connected component Ω of $\{z^{t^*} > 0\} \cap \mathcal{C}^+(0, \pi - \alpha) = (-\frac{\pi}{2\varepsilon_1}, \frac{\pi}{2\varepsilon_1}) \times (-\infty, t^* - \frac{\pi}{2\varepsilon_1}\cot\alpha) \cap \mathcal{C}^+(0, \pi - \alpha)$ containing P_∞. This is impossible because on $\partial\Omega \subset \{z^{t^*} = 0\} \cup \partial\mathcal{C}^+(0, \pi - \alpha)$.

This finally yields that $t^* = a$. This means that

$$z^a = (\underline{\xi} - \varepsilon)\ \cos\left(\sqrt{\frac{Nc}{2ae^N}}x\right)\ 1_{\{|x| < \frac{\pi}{2}\sqrt{\frac{2ae^N}{Nc}}\}}\ \left(1 - e^{\frac{N}{a}(y + \frac{\pi}{2}\sqrt{\frac{2ae^N}{Nc}}\cot\alpha\ - a)}\right)^+$$
$$\leq u \text{ in } \mathcal{C}^+(0, \pi - \alpha)$$

By successively taking the limits $a \to +\infty$, $N \to +\infty$ and $\varepsilon \to 0$, we find $u \geq \underline{\xi}$ in $\mathcal{C}^+(0, \pi - \alpha)$. This completes the proof of the lemma.

Consider now any $\xi \in (0,1)$. For any reals η_1, η_2 with $|\eta_1| < \min(\theta, \xi)$ and $|\eta_2| < \min(1-\theta, 1-\xi)$, define the functions

$$f^{\eta_2}(u) = \begin{cases} f(u) & \text{if } u \in]-\infty, 1-2|\eta_2|] \\ f(1-2|\eta_2|)\dfrac{(1+\eta_2-u)}{\eta_2+2|\eta_2|} & \text{if } u \in [1-2|\eta_2|, 1+\eta_2] \\ 0 & \text{if } u \in [1+\eta_2, +\infty[\end{cases}$$

These functions are lipschitz-continuous and satisfy (1.3) on $[\eta_1, 1+\eta_2]$. Hence, there exists a unique pair $(c_0^{\eta_1,\eta_2}, u_0^{\eta_1,\eta_2}) \in I\!R \times C^2(I\!R)$ solving

$$\begin{cases} (u_0^{\eta_1,\eta_2})'' - c_0^{\eta_1,\eta_2}(u_0^{\eta_1,\eta_2})' + f^{\eta_2}(u_0^{\eta_1,\eta_2}) = 0 \text{ on } I\!R \\ u_0^{\eta_1,\eta_2}(-\infty) = \eta_1, \ u_0^{\eta_1,\eta_2}(+\infty) = 1+\eta_2 \\ u_0^{\eta_1,\eta_2}(0) = \xi \end{cases} \tag{4.12}$$

Besides, $c_0^{\eta_1,\eta_2} > 0$. These solutions also depend on y_0 and ξ but we do not mention these items in the notations. Since f is C^1 in a left neighborhood of 1 and $f'(1) < 0$, it is easy to check that the functions f^{η_2} are nondecreasing in η_2 for $|\eta_2|$ small enough.

Lemma 4.6 *There exist two intervals $(-r_1, r_1)$, $(-r_2, r_2)$ with $r_1 < \min(\theta, \xi)$, $r_2 < \min(1-\theta, 1-\xi)$ and which do not depend on y_0, such that:*
(i) the function $(\eta_1, \eta_2) \longmapsto c_0^{\eta_1,\eta_2}$ is continuous and increasing in η_1 and η_2,
(ii) for any compact set K in $I\!R$, the function $(\eta_1, \eta_2) \longmapsto u_0^{\eta_1,\eta_2}$ is continuous in $W^{2,p}(K)$ for any $1 < p < +\infty$,
(iii) there exists a continuous and decreasing function g, which is a bijection from $(-r_1, r_1)$ into $(-r_2, r_2)$ and such that $g(0) = 0$ and $c_0^{\eta_1, g(\eta_1)} = c_0$.

This lemma states natural properties fulfilled by the speeds $c_0^{\eta_1,\eta_2}$ and the functions $u_0^{\eta_1,\eta_2}$. The proof is straightforward and done in [27], we do not give it here. In a few words, it is based on results of Berestycki and Nirenberg [11] and on the onedimensional version of theorem 1.4.

Let now u be a solution of (1.5) fulfilling (1.6), (1.13) and (1.14). In the new system of cartesian coordinates $X = \sin\alpha\ x + \cos\alpha\ y$, $Y = -\cos\alpha\ x + \sin\alpha\ y$, the function $\tilde{u}(X, Y) = u(x, y)$ satisfies:

$$\Delta\tilde{u} - c_0\partial_Y\tilde{u} - c_0\cot\alpha\partial_X\tilde{u} + f(\tilde{u}) = 0 \text{ in } I\!R^2$$

and the conical conditions

$$\begin{cases} \forall \delta \in (0, \pi - \alpha), & \displaystyle\liminf_{\|\nu\| \to \infty,\ \nu \in \mathcal{C}(-(\pi-2\alpha)+\delta, \pi-\delta)} \tilde{u} = 1 \\ \forall \delta \in (0, \alpha), & \displaystyle\limsup_{\|\nu\| \to \infty,\ \nu \in \mathcal{C}(-\pi+\delta, -(\pi-2\alpha)-\delta)} \tilde{u} = 0 \end{cases} \tag{4.13}$$

where we define $\mathcal{C}(\beta, \beta') = \{(X, Y) = \rho(\cos\phi, \sin\phi),\ \rho \geq 0,\ \phi \in [\beta, \beta']\}$.

Lemma 4.7 *With the notations of lemma 4.6, for any $\eta_1 \in (0, r_1)$ and any $\eta_2 \in (g(\eta_1), 0)$, there exists a real $\underline{t}$ such that*

$$\tilde{u}(X, Y) \geq u_0^{\eta_1,\eta_2}(Y + \underline{t}) \text{ in the quadrant } \{Y \geq \underline{y}\sin\alpha,\ X \leq \underline{y}\cos\alpha\}$$

where $u_0^{\eta_1,\eta_2}$ is a solution of (4.12) with $\xi = \underline{\xi}/2$.

Lemma 4.8 *With the notations of lemma 4.6, for any $\eta_1 \in (0, r_1)$ and any $\eta_2 \in (-r_2, g(\eta_1))$, there exists a real $\overline{t}$ such that*

$$\tilde{u}(X, Y) \leq u_0^{\eta_1,\eta_2}(Y + \overline{t}) \text{ in the cone } \{Y \leq \overline{y}\sin\alpha,\ x \leq 0\}$$

where $u_0^{\eta_1,\eta_2}$ is a solution of (4.12) with $\xi = (1 + \overline{\xi})/2$.

We will only prove lemma 4.7, by comparing $\tilde{u}$ with suitable subsolutions in cones rotating around a fixed point. The proof of lemma 4.8 can be done with the same kind of arguments, by using this time super-solutions.

Proof of lemma 4.7. By hypothesis, we have $\tilde{u} \geq \underline{\xi} > 0$ on the half-line $\{Y = \underline{y}\sin\alpha,\ X \leq \underline{y}\cos\alpha\}$. Set $\underline{Y} = \underline{y}\sin\alpha$ and $\underline{X} = \underline{y}\cos\alpha$. By a translation of the origin in the direction y, we can always assume that $\underline{y} = \underline{Y} = \underline{X} = 0$. From the monotonicity lemma 4.5, we have

$$\tilde{u} \geq \underline{\xi} \text{ in the half-plane } \{Y \geq 0\}$$

(and also in $\{X \geq Y\cot 2\alpha\}$). We divide the proof of lemma 4.7 into three steps and use a sliding method by rotation around a fixed point.

Step 1: construction of a subsolution in the quadrant $\mathcal{Q} = \{Y \geq 0,\ X \leq 0\}$. Set $\xi = \underline{\xi}/2$. Let $r_1 > 0$ be given in lemma 4.6. For any fixed $0 < \eta_1 < r_1$ and

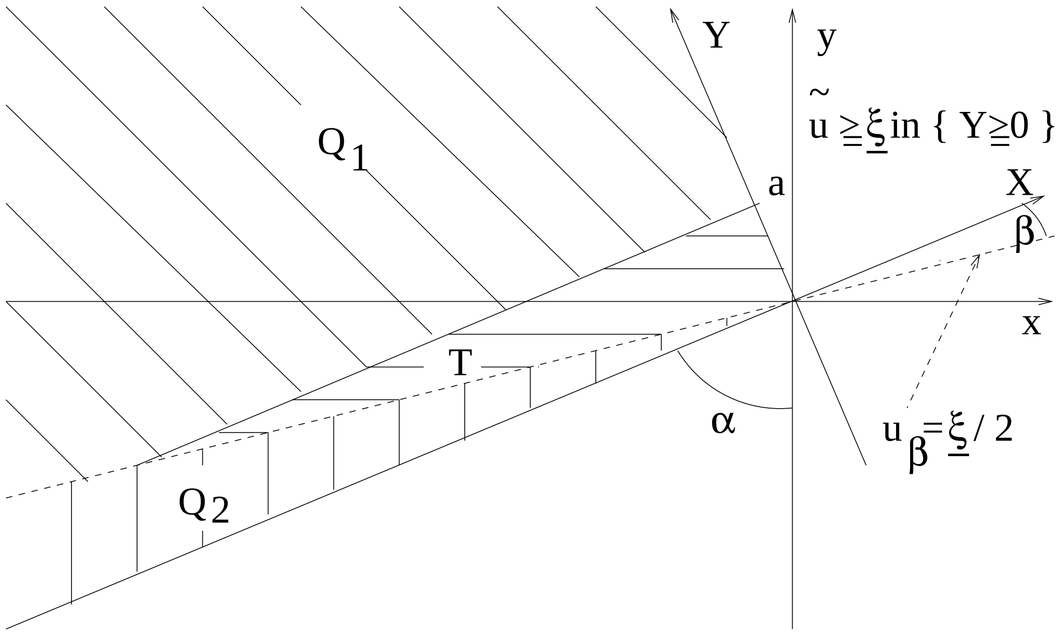

Figure 2: The quadrant $\mathcal{Q}$

$g(\eta_1) < \eta_2 < 0$ (the function g was defined in lemma 4.6), note $u_1 = u_0^{\eta_1,\eta_2}$. This function u_1 satisfies (4.12).

First of all, since $\tilde{u}(0,Y) \geq \underline{\xi}$ if $Y \geq 0$, $\tilde{u}(0,Y) \to 1$ as $Y \to +\infty$, and $u_1(Y) \to \eta_1 < \underline{\xi}$ (resp. $u_1(Y) \to 1 + \eta_1 < 1$) as $Y \to -\infty$ (resp. $Y \to +\infty$), there exists a real $t_0 > 0$ large enough such that $u_1(Y - t_0) < \tilde{u}(0,Y)$ if $Y \geq 0$.

For any $\beta > 0$ small enough, let us define a family of planar functions

$$u_\beta(X,Y) = u_1(\sin\beta\ X + \cos\beta\ Y - t_0)$$

whose level sets are lines parallel to $\{Y = -\tan\beta\ X\}$ (see the joint figure). We have:

$$\begin{aligned} I(u_\beta) &:= \Delta u_\beta - c_0 \partial_Y u_\beta - c_0 \cot\alpha\ \partial_X u_\beta + f(u_\beta) \\ &= (c_0^{\eta_1,\eta_2} - c_0\cos\beta - c_0\cot\alpha\sin\beta)u_1' \ + \ f(u_1) - f^{\eta_2}(u_1) \end{aligned}$$

We know that $f \geq f^{\eta_2}$ and $u_1' \geq 0$. Since $\eta_2 \in (g(\eta_1), 0)$, lemma 4.6 yields $c_0^{\eta_1,\eta_2} > c_0$. Hence, for $0 < \beta < \beta_0$ small enough, we get $I(u_\beta) \geq 0$ in $I\!R^2$.

Let us compare u_β and $\tilde{u}$ on $\partial\mathcal{Q}$. Since u_1 is increasing, for any $X \leq 0$, $u_\beta(X,0) = u_1(\sin\beta\ X - t_0) \leq u_1(0) = \underline{\xi}/2 < \tilde{u}(X,0)$. Besides, for any $Y \geq 0$, $u_\beta(0,Y) = u_1(\cos\beta\ (Y - t_0) \leq u_1(Y - t_0) < \tilde{u}(0,Y)$ by our choice of t_0.

Hence, $I(u_\beta) \geq 0 = I(\tilde{u})$ in $\mathcal{Q}$ and $u_\beta \leq \tilde{u}$ on $\partial\mathcal{Q}$.

Step 2: sliding method. We now want to prove, by a sliding method, that for any $\beta \in (0, \beta_0)$ we have $u_\beta \leq \tilde{u}$ in $\mathcal{Q}$. We first slide u_β to the right (in X-direction), and define, for any $t \leq 0$,

$$u_{\beta,t}(X,Y) = u_\beta(X+t,Y) = u_1(\sin\beta\ (X+t) + \cos\beta\ Y - t_0)$$

Since u_1 is increasing, we have $u_{\beta,t} \leq u_\beta$ in $I\!R^2$ for any $t \leq 0$.

The function $\tilde{u}$ satisfies (4.13) and $1+\eta_2 < 1$. Hence, there exists a real $a > 0$ such that $\tilde{u} > 1+\eta_2$ in the set $\mathcal{Q}_1 = \{(X,Y) \in \mathcal{C}(\pi/2, \pi-\beta),\ Y \geq a\}$. Now $\mathcal{Q}$ can be divided into three regions: $\mathcal{Q}_1$, $\mathcal{Q}_2 = \mathcal{C}(\pi-\beta,\pi)$ and the triangle

$$T = \{(X,Y) \in\ \mathcal{C}(\pi/2, \pi-\beta),\ 0 < Y < a\}$$

In the closed set $\mathcal{Q}_1$, we have $\tilde{u}(X,Y) > 1+\eta_2 \geq u_{\beta,t}$ since $u_1 \leq 1+\eta_2$. In $\mathcal{Q}_2$, we have $u_{\beta,t}(X,Y) \leq u_\beta(X,Y) \leq u_1(\sin\beta\ X + \cos\beta\ Y) \leq u_1(0) = \underline{\xi}/2$ since $\sin\beta\ X + \cos\beta\ Y \leq 0$ in $\mathcal{Q}_2$. But $\tilde{u} \geq \underline{\xi}$ in $\mathcal{Q}$, whence $\tilde{u}(X,Y) > u_{\beta,t}$ in the closed set $\mathcal{Q}_2$ for any $t \leq 0$. Lastly, as $t \to -\infty$, $u_{\beta,t}$ goes to $\eta_1 < \xi$ uniformly in the bounded set T. Then, there exists a real $-t_1$ large enough such that $\tilde{u} \geq u_{\beta,t_1}$ in $\overline{T}$.

Let us now slide u_{β,t_1} to the left (in $-X$-direction), and define $t^* = \sup\{t \leq 0,\ \tilde{u} \geq u_{\beta,t}$ in $\mathcal{Q}\}$. Let us assume $t^* < 0$. We have $\tilde{u} \geq u_{\beta,t^*}$ in $\mathcal{Q}$. By the strong maximum principle, either $\tilde{u} > u_{\beta,t^*}$ or $\tilde{u} \equiv u_{\beta,t^*}$ in the interior of T. But $\tilde{u} > u_{\beta,t^*}$ on the three edges of the triangle T: this is true for the 2 edges lying on $\mathcal{Q}_1$ and $\mathcal{Q}_2$ from the arguments above, and it is also true on the edge $\{(0,Y),\ 0 \leq Y \leq a\}$ by definition of t_0 in step 1. Hence, this yields $\tilde{u} > u_{\beta,t^*}$ in $\overline{T}$, and even, by continuity, $\tilde{u} > u_{\beta,t^*+\varepsilon}$ for some $\varepsilon \in (0, -t^*)$ small enough. We eventually get $\tilde{u} > u_{\beta,t^*+\varepsilon}$ in $\mathcal{Q}$ which is in contradiction with the definition of t^*. We therefore conclude that $t^* = 0$ and $\tilde{u}(X,Y) \geq u_\beta$ in $\mathcal{Q}$.

Step 3: conclusion. From step 2, we get

$$\tilde{u}(X,Y) \geq u_1(\sin\beta\ X + \cos\beta\ Y - t_0) \text{ in } \mathcal{Q}$$

Remember that t_0 was choosen once and for all at the beginning of step 1, and does not depend on β. The passage to the limit $\beta \to 0$ gives the result of lemma 4.7 with $\underline{t} = -t_0$.

Proof of theorem 1.9. Let us first notice that similar results as those of lemmas 4.7 and 4.8 can be stated in the system of cartesian coordinates $\tilde{X} = -\sin\alpha\ x + \cos\alpha\ y$, $\tilde{Y} = \cos\alpha\ x + \sin\alpha\ y$.

Since $\liminf_{\|\nu\| \to +\infty,\ \nu \in \mathcal{C}^+(0,\pi-\alpha)} u(\nu) = 1$ and from the result in lemma 4.7 – and the same one in variables $(\tilde{X}, \tilde{Y})$ –, we then deduce $\lim_{y \to +\infty} \inf_{\mathcal{C}^+(y,\pi-\alpha)} u \geq 1 + g(\eta_1)$. Since this is true for any $\eta_1 \in (0, r_1)$ and $g(\eta_1) \to 0$ as $\eta_1 \to 0$, we get the conical condition in (1.12) on upper cones. Similarly, from the result in lemma 4.8 and the same one in variables $(\tilde{X}, \tilde{Y})$, we get $\lim_{y \to -\infty} \sup_{\mathcal{C}^-(y,\alpha)} u \leq \eta_1$ for any $\eta_1 \in (0, r_1)$. Finally, u satisfies both conical conditions in (1.12).

4.2 Dimension $N \geq 3$: nonexistence of solutions with asymptots if $\alpha \neq \pi/2$

This section is devoted to the proof of theorem 1.6: in dimension $N \geq 3$ and for any angle $\alpha \neq \pi/2$, there is no solution (c, u) of (1.12). From theorem 1.1, there is no solution if $\alpha > \pi/2$. From theorem 1.3, if $\alpha = \pi/2$, the couple (c_0, u_0) is the unique solution.

Let $N \geq 3$ and $\alpha < \pi/2$ be fixed and suppose that there is a solution (c, u) of (1.12). Since (c, u) is then also a solution of (1.8), (1.9) (strong framework with $\phi(x') = -\cot\alpha\ |x'|$), theorem 1.3 implies that the speed c is equal to $c_0/\sin\alpha$.

In section 4.1.2, we proved the existence of solutions with asymptots in dimension 2 by considering a subsolution $v(x, y)$ of the type $v(x, y) = u_0(\sin\alpha\ (y - \phi(x)))$ where ϕ was even and given by $\phi(x) = -\frac{1}{c_0 \sin\alpha} \ln\cosh(c_0 \cos\alpha\ x)$. In dimension $N \geq 3$, consider

$$v(x', y) = u_0(\sin\alpha\ (y - \phi(r))), \quad r = |x'| \tag{4.14}$$

where ϕ is a given function of class C^2 in $\mathbb{R}^+$ such that $\phi(0) = \phi'(0) = 0$. A straightforward calculation shows that

$$\Delta v - c\partial_y v + f(v)$$
$$= \sin\alpha \left[c_0 \sin\alpha(1+\phi'^2(r)) - \phi''(r) - \frac{N-2}{r}\phi'(r) - c\right] u_0'(\sin\alpha\ (y-\phi(r)))$$
$$+ \left[1 - \sin^2\alpha\ (1+\phi'^2(r))\right] f(v) \quad \text{in } I\!R^N$$

We now require that the function v be a subsolution of (1.12). Since $c = c_0/\sin\alpha$, it suffices that $|\phi'| \leq \cot\alpha$, $\phi'(+\infty) = -\cot\alpha$ and

$$\begin{cases} \phi'' + \dfrac{N-2}{r}\phi' - c_0 \sin\alpha\ \phi'^2 + c_0 \cos\alpha \cot\alpha = 0 \text{ in } I\!R^+ \\ \phi(0) = \phi'(0) = 0 \end{cases} \tag{4.15}$$

Let us notice that, in the case $N = 2$, the function

$$\phi(r) = -\frac{1}{c_0 \sin\alpha}\ \ln\cosh(c_0 \cos\alpha\ r)$$

is the unique solution of (4.15) such that $-\cot\alpha < \phi' \leq 0$ in $I\!R^+$. Besides, it is asymptotic to the line $y = \dfrac{\ln 2}{c_0 \sin\alpha} - \cot\alpha\ r$ as $r \to +\infty$ and $\int_0^{+\infty} |\cot\alpha + \phi'(r)| dr < +\infty$.

The situation is very different for the dimensions $N \geq 3$. Indeed, we have:

Lemma 4.9 *For $N \geq 3$ and $0 < \alpha < \pi/2$, there exists a solution ϕ of (4.15) in $I\!R^+$ such that $-\cot\alpha < \phi'(r) < 0$ for any $r > 0$, and $\phi''(r) < 0$ for any $r \geq 0$. Moreover, ϕ is analytic in r^2 and $\displaystyle\int_0^{+\infty} |\cot\alpha + \phi'(r)|\ dr = +\infty$. In particular, the function ϕ has no asymptot as $r \to +\infty$.*

Postponing the proof of lemma 4.9, the function v defined by (4.14) verifies

$$\lim_{y\to-\infty} \sup_{\mathcal{C}^-(y,\alpha)} v = 0$$

because $\phi(r) \geq -\cot\alpha\ r$ for any $r \geq 0$. From the comparison principle (theorem 1.4), the set $I = \{t,\ \forall s \geq t,\ u^t \geq v\}$ is not empty. Besides, since $\lim\limits_{t\to-\infty} u(0,t) = 0$, we have $t^* = \inf I > -\infty$. Then theorem 1.4 also yields that

$$\forall y_0 \in I\!R, \quad \inf_{\partial\mathcal{C}^-(y_0,\alpha)} (u^{t^*} - v) = 0$$

Since $\phi(r)+\cot\alpha\, r \to +\infty$ as $r \to +\infty$, we have $\lim_{\lambda\to+\infty} \sup_{|(x',y)|\geq\lambda,\ (x',y)\in\partial\mathcal{C}^-(y_0,\alpha)} v = 0$ for any $y_0 \in I\!R$. On the other hand, $\lim_{\lambda\to+\infty} \inf_{|(x',y)|\geq\lambda,\ (x',y)\in\partial\mathcal{C}^-(y_0,\alpha)} u^{t^*} \geq 1/2$ for y_0 large enough by the uniform asymptotic conditions in (1.12). Let us fix y_0 (large enough). There is then a point $P_0 \in \partial\mathcal{C}^-(y_0,\alpha)$ such that $u^{t^*}(P_0) = v(P_0)$. The strong maximum principle yields that $u^{t^*} \equiv v$ in $I\!R^N$. This is impossible on $\partial\mathcal{C}^-(y_0,\alpha)$. Hence, there is no solution (c,u) to (1.12) if $N \geq 3$ and $\alpha < \pi/2$.

Proof of lemma 4.9. In $I\!R^{N-1}$, for any $R > 0$, let B_R be the open ball centered at the origin and with radius R. Let w_R be the unique solution of the Dirichlet problem

$$\begin{cases} \Delta w_R - c_0^2\cos^2\alpha\; w_R = 0, & x' \text{ in } B_R \\ \qquad w_R = 1 \text{ on } \partial B_R \end{cases}$$

Since the constants 0 and 1 are respectively strict sub- and supersolutions of this problem, we have $0 < w_R < 1$ in B_R. By the device of moving planes, as Gidas, Ni and Nirenberg did in [21], we can prove that the function w_R is radial, $w_R = w_R(r)$, $r = |x'|$, and that $w_R'(r) > 0$ for any $r > 0$.

Let us now define the function $z_R(x') = \dfrac{w_R(r)}{w_R(0)}$ in B_R. This function $z_R = z_R(r)$ satisfies $z_R \geq z_R(0) = 1$ in B_R, $z_R'(0) = 0$ and $z_R'(r) > 0$ for any $r > 0$. From Harnack inequality, standard elliptic estimates and Sobolev injections, there exists a radial function $z = z(r)$ defined in $I\!R^{N-1}$ such that $z_R \to z$ locally in $I\!R^{N-1}$. The function z satisfies $z \geq z(0) = 1$, $z'(0) = 0$, $z'(r) \geq 0$ in $I\!R^+$ and

$$z'' + \frac{N-2}{r}z' - c_0^2\cos^2\alpha\; z = 0 \text{ in } I\!R^+$$

Define the function $\phi(r) = -\dfrac{1}{c_0\sin\alpha}\ln z$. Since z satisfies $\Delta z - c_0^2\cos^2\alpha\; z = 0$ in $I\!R^{N-1}$, it is analytic in x' and radial. Hence, the function ϕ is analytic in r^2. Besides, $\phi(0) = \phi'(0) = 0$, $\phi(r) \leq 0$, $\phi'(r) \leq 0$ in $I\!R^+$ and ϕ satisfies

$$\phi'' + \frac{N-2}{r}\phi' - c_0\sin\alpha\;\phi'^2 + c_0\cos\alpha\cot\alpha = 0 \text{ in } I\!R^+,$$

that is to say equation (4.15).

Let us now prove the other assertions stated in lemma 4.9. Let us suppose that there exists a real r_0 such that $\phi'' \geq 0$ in $[r_0, +\infty[$. By (4.15), the function ϕ'' cannot be identically 0 in $[r_0, +\infty[$ (otherwise, ϕ' should be a nonzero constant, this is impossible because of the term $\dfrac{N-2}{r}\phi'$). Hence, the function ϕ' has a limit $\phi'(+\infty)$ such that $\phi'(r_0) < \phi'(+\infty) \leq 0$. By (4.15), the function ϕ'' has a limit $\phi''(+\infty)$, which turns out to be 0 since $\phi'(+\infty)$ exists. Finally, equation (4.15) at $+\infty$ gives $\phi'(+\infty) = -\cot\alpha$. Since $\phi'(r_0) < \phi'(+\infty)$ and $\phi'(0) = 0$, there exists then a real $r_1 > 0$ such that $\phi'(r_1) < -\cot\alpha$ and $\phi''(r_1) = 0$. This is in contradiction with equation (4.15) at the point r_1. Hence, at this stage, we conclude that for any $r_0 > 0$, there exists a real $r_1 \geq r_0$ such that $\phi''(r_1) < 0$.

Let us now assume that there exists a real $r_0 \geq 0$ such that $\phi''(r_0) \geq 0$. First of all, from equation (4.15) at the point 0 and since $\phi'(0) = 0$, we have $\phi''(0) = -\dfrac{c_0 \cos^2\alpha}{(N-1)\sin\alpha} < 0$. In particular, we have $r_0 > 0$. >From the previous paragraph, there exists $r_1 > r_0$ such that $\phi''(r_1) < 0$. Hence, there exists a real $r_2 > 0$ such that $\phi''(r_2) \geq 0$ and $\phi'''(r_2) = 0$. On the other hand, we have $\phi''' + \frac{N-2}{r}\phi'' - \frac{N-2}{r^2}\phi' - 2c_0 \sin\alpha\ \phi'\phi'' = 0$ in $(0, +\infty)$. At the point r_2, we have $\phi''(r_2) \geq 0$ and $\phi'(r_2) \leq 0$ (by definition of z and ϕ), and both $\phi''(r_2)$ and $\phi'(r_2)$ cannot be 0 by (4.15). Hence, $\phi'''(r_2) = 0$ is impossible. This proves that $\phi'' < 0$ in $I\!R^+$, whence $-\cot\alpha < \phi' < 0$ in $I\!R^+$.

Let us now prove that the integral $\displaystyle\int_0^{+\infty} (\phi' + \cot\alpha)dr$ is infinite. We recall that $\phi' + \cot\alpha > 0$ is $I\!R^+$ and suppose that $\displaystyle\int_0^{+\infty} (\phi' + \cot\alpha)dr < +\infty$. By (4.15), we get

$$
\begin{aligned}
0 &= \int_1^{+\infty} \left[\phi'' + \frac{N-2}{r}\phi' - c_0 \sin\alpha\ \phi'^2 + c_0 \cos\alpha \cot\alpha\right]\ dr \\
&= -\int_1^{+\infty} \Big[\ \phi'' + \frac{N-2}{r}(\phi' + \cot\alpha) - \frac{N-2}{r}\cot\alpha \\
&\qquad\qquad -c_0 \sin\alpha\ (\phi' + \cot\alpha)^2 + 2c_0 \cos\alpha\ (\phi' + \cot\alpha)\Big]\ dr
\end{aligned}
$$

In the right hand side, all the integrals converge but $\displaystyle\int_1^{+\infty} \frac{N-2}{r}\cot\alpha\ dr$. This gives a contradiction and finally proves that the integral $\displaystyle\int_0^{+\infty} (\phi' + \cot\alpha)dr$ is infinite.

5 Appendix: comparison principles in $\mathbb{R}^N$

This section is devoted to the proof theorems 1.4 and 1.5. The basic tool is the sliding method. With the notations and assumptions of theorem 1.4, let us define the elliptic operator $L(x') = a_{ij}(x')\partial_{ij} + b_i(x')\partial_i$. We are given a function $\phi : \mathbb{R}^{N-1} \longrightarrow \mathbb{R}$, uniformly continuous in $\mathbb{R}^{N-1}$ and we recall that

$$\begin{cases} \Omega^+(y_0) = \{y > y_0 + \phi(x'), x' \in \mathbb{R}^{N-1}\} \\ \Omega^-(y_0) = \{y < y_0 + \phi(x'), x' \in \mathbb{R}^{N-1}\} \\ \Gamma(y_0) = \{y = y_0 + \phi(x'), x' \in \mathbb{R}^{N-1}\} \end{cases}$$

For any $t \in \mathbb{R}$, we use the notation $w^t(x', y) = w(x', y + t)$ and $w^{+\infty}(x') := \limsup_{y \to +\infty} w(x', y)$, $w^{-\infty}(x') := \liminf_{y \to -\infty} w(x', y)$ for $x' \in \mathbb{R}^{N-1}$.

Consider first two lipschitz-continuous functions, $\underline{u}$ and $\overline{u}$, respectively sub- and supersolutions of (1.10), but only in a subset $\Omega \subset \mathbb{R}^N$: $L(x')\underline{u} + f(x', \underline{u}) \geq 0$ in Ω, and $L(x')\overline{u} + f(x', \overline{u}) \leq 0$ in Ω. We assume that $a \leq \underline{u} \leq b$, $a \leq \overline{u} \leq b$ in Ω.

Lemma 5.1 *Let $\Omega = \Omega^-(y_1)$ for some $y_1 \in \mathbb{R}$ and assume that*

$$\begin{cases} \underline{u} \leq a' \text{ in } \Omega^-(y_1) \text{ and } \lim_{y \to -\infty} \sup_{\Omega^-(y)} \underline{u} = a, \\ \underline{u} \leq \overline{u} \text{ on } \Gamma(y_1) \end{cases} \tag{5.1}$$

Let $I_1 = \{t \in \mathbb{R}^-,\ \forall s \in [t, 0],\ \overline{u}^s \geq \underline{u} \text{ on } \Gamma(y_1)\}$. We have $0 \in I_1$ and $\forall t \in I_1$, $\overline{u}^t \geq \underline{u}$ in $\Omega^-(y_1)$. Let $t^* = \inf I_1$. It is the case that $\overline{u}^{t^*} \geq \underline{u}$ in $\Omega^-(y_1)$. Furthermore, if $t^* \neq -\infty$, then $\inf_{\Gamma(y_1)}(\overline{u}^{t^*} - \underline{u}) = 0$.

Lemma 5.2 *Let $\Omega = \Omega^+(y_2)$ for some $y_2 \in \mathbb{R}$ and assume that*

$$\begin{cases} \overline{u} \geq b' \text{ in } \Omega^+(y_2) \text{ and } \lim_{y \to +\infty} \inf_{\Omega^+(y)} \overline{u} = b, \\ \underline{u} \leq \overline{u} \text{ on } \Gamma(y_2) \end{cases} \tag{5.2}$$

Let $I_2 = \{t \in \mathbb{R}^+,\ \forall s \in [0, t],\ \overline{u} \geq \underline{u}^s \text{ on } \Gamma(y_2)\}$. We have $0 \in I_2$ and $\forall t \in I_2$, $\overline{u} \geq \underline{u}^t$ in $\Omega^+(y_2)$. Let $t^* = \sup I_2$. It is the case that $\overline{u} \geq \underline{u}^{t^*}$ in $\Omega^+(y_2)$. Furthermore, if $t^* \neq +\infty$, then $\inf_{\Gamma(y_2)}(\overline{u} - \underline{u}^{t^*}) = 0$.

Proof. We only prove lemma 5.2 (lemma 5.1 is equivalent to it by changing y in $-y$). We will use an argument developped by Vega [37].

Let I_2 be defined as in lemma 5.2. By (5.2), we have $0 \in I_2$. Consider now any $t \in I_2$. We want to prove that $\overline{u} \geq \underline{u}^t$ in $\Omega = \Omega^+(y_2)$. During the proof of this lemma, we use the notations

$$\overline{u}_\varepsilon = \overline{u} + \varepsilon$$

for any $\varepsilon > 0$. Let us set

$$\varepsilon^* = \inf\{\varepsilon > 0,\ \overline{u}_\varepsilon \geq \underline{u}^t \text{ in } \Omega^+(y_2)\}$$

We have $\varepsilon^* < +\infty$ because $\overline{u}_{\varepsilon=b-a} \geq \underline{u}^t$. Furthermore, $\overline{u}_{\varepsilon^*} \geq \underline{u}^t$ in $\Omega^+(y_2)$.

Let us suppose that $\varepsilon^* > 0$. We can then find a sequence $\varepsilon_k \nearrow \varepsilon^*$ and points $(x'_k, y_k) \in \Omega^+(y_2)$ such that

$$\overline{u}_{\varepsilon_k}(x'_k, y_k) = \overline{u}(x'_k, y_k) + \varepsilon_k < \underline{u}^t(x'_k, y_k) \tag{5.3}$$

Hence $\overline{u}(x'_k, y_k) < b - \varepsilon_k$. Since $\varepsilon_k \to \varepsilon^* > 0$ and by (5.2), there exists a real $y'_2 > y_2$ such that, for k large enough, $(x'_k, y_k) \in \Omega^+(y_2)\backslash\Omega^+(y'_2)$, that is to say

$$y_2 + \phi(x'_k) < y_k \leq y'_2 + \phi(x'_k) \tag{5.4}$$

Let $\Omega_0 = \Omega^+(y_2)\backslash\overline{\Omega^+(y'_2 + 1)}$. We now move the origin to (x'_k, y_k) and consider the limit problem as $k \to +\infty$. To do this, we define the sets $\Omega_k = \Omega_0 - (x'_k, y_k)$. We also define $\overline{u}_k(x', y) = \overline{u}(x'_k + x', y_k + y)$, $\underline{u}_k(x', y) = \underline{u}(x'_k + x', y_k + y)$, $L_k(x') = L(x'_k + x')$, $f_k(x', \cdot) = f(x'_k + x', \cdot)$ and $\tau_k(x') = \tau(x'_k + x')$.

Remember that we have

$$L(x')\overline{u} + f(x', \overline{u}) \leq 0 \text{ in } \Omega_0$$

On the other hand, $\overline{u}_{\varepsilon_k} = \overline{u} + \varepsilon_k \geq \overline{u} \geq b'$ in $\Omega^+(y_2)$. Since f is nonincreasing in u over $[b', +\infty[$, we get $f(x', \overline{u}) \geq f(x', \overline{u}_{\varepsilon_k})$ in $\Omega^+(y_2)$ and consequently:

$$L_k(x')(\overline{u}_k + \varepsilon_k) + f_k(x', \overline{u}_k + \varepsilon_k) \leq 0 \text{ in } \Omega_k$$

where $\Omega_k = \Omega_0 - (x'_k, y_k) = \{(x', y),\ y_2 + \phi(x'_k + x') - y_k < y < y'_2 + 1 + \phi(x'_k + x') - y_k\}$.

For any compact set K in $\mathbb{R}^{N-1}$ containing 0, by (5.4) and the uniform continuity of ϕ in $\mathbb{R}^{N-1}$, the functions $x' \mapsto \phi(x'^k + x') - y^k$ are bounded and uniformly continuous in the compact set K. By Ascoli's theorem we can then assume that, up to extraction of some subsequence, $\phi(x'^k + x') - y^k \to \phi_\infty(x')$ uniformly in the compact subsets of $\mathbb{R}^{N-1}$; the function ϕ_∞ is uniformly continuous in $\mathbb{R}^{N-1}$.

Similarly, we can assume that $\overline{u}_k$ and $\underline{u}_k$ converge to two lipschitz-continuous functions $\overline{u}_\infty$ and $\underline{u}_\infty$ locally in the set

$$\Omega_\infty = \{y_2 + \phi_\infty(x') < y < y_2' + 1 + \phi_\infty(x')\}$$

Both these functions $\overline{u}_\infty$ and $\underline{u}_\infty$ can be lipschitz-continuously extended in $\overline{\Omega_\infty}$. We can also assume that $L_k(x')(\overline{u}_k + \varepsilon_k) \to L_\infty(x')(\overline{u}_\infty + \varepsilon^*)$, $L_k(x')(\underline{u}_k^t) \to L_\infty(x')(\underline{u}_\infty^t)$ in the distribution sense in Ω_∞. The limit operator L_∞ can be written as $L_\infty(x') = a_{ij,\infty}(x')\partial_{ij} + b_{i,\infty}(x')\partial_i$ where $a_{ij,\infty}$ and $b_{i,\infty}$ have the same regularity as a_{ij} and b_i in $\mathbb{R}^{N-1}$. In the same way, the functions $f_k(x', \overline{u}_k + \varepsilon_k)$ and $f_k(x', \underline{u}^t)$ converge locally in Ω_∞ to two uniformly continuous functions $\overline{f}_\infty(x', y)$ and $\underline{f}_\infty(x', y)$. These functions can be extended in $\overline{\Omega_\infty}$ and satisfy

$$|\overline{f}_\infty - \underline{f}_\infty| \leq C|\overline{u}_\infty + \varepsilon^* - \underline{u}_\infty^t| \text{ in } \overline{\Omega_\infty} \tag{5.5}$$

where the constant $C > 0$ is defined in theorem 1.4.

By summarizing all the previous facts, we get that

$$\begin{cases} L_\infty(x')(\overline{u}_\infty + \varepsilon^*) + \overline{f}_\infty(x', y) \leq 0 \text{ in } \Omega_\infty \\ L_\infty(x')\underline{u}_\infty^t + \underline{f}(x', y) \geq 0 \text{ in } \Omega_\infty \\ \overline{u}_\infty + \varepsilon^* \geq \underline{u}_\infty^t \text{ in } \overline{\Omega}_\infty \end{cases}$$

Moreover we have $0 \in \overline{\Omega_\infty}$. By (5.3), we have that $\overline{u}_k(0,0) + \varepsilon_k \leq \underline{u}_k^t(0,0)$, and passing to the limit $k \to +\infty$, we get that $\overline{u}_\infty(0,0) + \varepsilon^* \leq \underline{u}_\infty^t(0,0)$. On the other hand, since $\overline{u} + \varepsilon^* \geq \underline{u}^t$ in $\mathbb{R}^N$, it immediately comes that $\overline{u}_\infty + \varepsilon^* \geq \underline{u}_\infty^t$ in $\mathbb{R}^N$. Finally, $\overline{u}_\infty(0,0) + \varepsilon^* = \underline{u}_\infty^t(0,0)$. Besides, we observe that $\overline{u}_\infty + \varepsilon^* \geq \underline{u}_\infty^t + \varepsilon^* > \underline{u}_\infty^t$ on $\Gamma_\infty(y_2) = \{y = y_2 + \phi_\infty(x')\}$ and that $0 \leq y_2' + \phi_\infty(0)$ by passage to the limit in (5.4). Hence, the point $(0,0)$ cannot be on the bottom or on the top boundary of Ω_∞, that is to say that $(0,0) \in \Omega_\infty$.

The function $z = \overline{u}_\infty + \varepsilon^* - \underline{u}^t_\infty$ verifies

$$\begin{cases} L_\infty(x')z + d_\infty(x', y)z \le 0 \text{ in } \Omega_\infty \\ z \ge 0 \text{ in } \overline{\Omega_\infty}, \ z(0,0) = 0, \ (0,0) \in \Omega_\infty \end{cases}$$

where $d_\infty(x', y) = \frac{\overline{f}_\infty(x',y) - \underline{f}_\infty(x',y)}{z(x',y)}$ is a bounded function by (5.5). The strong maximum principle implies $z \equiv 0$ in $\overline{\Omega_\infty}$. This is in contradiction with the fact that $z \ge \varepsilon^* > 0$ on $\Gamma_\infty(y_2)$.

This finally shows that $\varepsilon^* = 0$, that is to say:

$$\forall t \in I_2, \quad \overline{u} \ge \underline{u}^t \text{ in } \Omega^+(y_2)$$

Define now $t^* = \sup I_2$. We have $\overline{u} \ge \underline{u}^{t^*}$ in $\Omega^+(y_2)$ and especially on $\Gamma(y_2)$. Let us consider the case $t^* < +\infty$ and assume that $\overline{u} - \underline{u}^{t^*} \ge m > 0$ on $\Gamma(y_2)$ for some $m > 0$. Since $\underline{u}$ is lipschitz-continuous, there exists $\eta_0 > 0$ such that $\overline{u} - \underline{u}^{t^*+\eta} \ge m/2 > 0$ on $\Gamma(y_2)$ for any $\eta \in [0, \eta_0]$. This would be in contradiction with the definition of t^*. Thus, $\inf\limits_{\Gamma(y_2)} (\overline{u} - \underline{u}^{t^*}) = 0$. This completes the proof of lemma 5.2.

Proof of theorem 1.4. Under the assumptions of theorem 1.4, there exist $y_1 \le y_2 \in \mathbb{R}$ such that

$$\overline{u} \ge b' \text{ in } \Omega^+(y_2)$$

$$\underline{u} \le a' \text{ in } \Omega^-(y_1)$$

Let $t_0 = y_2 - y_1$. For any $t \ge t_0$, we have $\overline{u}^t \ge b' > a' \ge \underline{u}$ on $\Gamma(y_1)$. From lemmas 5.1 and 5.2 we get that

$$\forall t \ge t_0, \ \overline{u}^t \ge \underline{u} \text{ in } \mathbb{R}^N$$

Define $t^* = \inf\{t \in \mathbb{R}, \ \overline{u}^t \ge \underline{u} \text{ in } \mathbb{R}^N\}$. It comes that $\overline{u}^{t^*} \ge \underline{u}$ in $\mathbb{R}^N$. Let us consider the case $t^* \ne -\infty$ and suppose that

$$\exists y_0 \in \mathbb{R}, \ \inf_{\Gamma(y_0)} (\overline{u}^{t^*} - \underline{u}) \ge \delta > 0 \tag{5.6}$$

On the other hand, there exist $y_1^* < y_0 < y_2^*$ such that

$$\begin{cases} \overline{u}^{t^*} \ge (b + b')/2 \text{ in } \Omega^+(y_2^*) \\ \underline{u} \le a' \text{ in } \Omega^-(y_1^*) \end{cases}$$

If $m := \inf_{\Omega^-(y_2^*)\backslash\Omega^-(y_1^*)}(\overline{u}^{t^*} - \underline{u}) = 0$, then there exists a sequence (x'_k, y_k) in $\overline{\Omega^-(y_2^*)\backslash\Omega^-(y_1^*)}$ such that $\overline{u}^{t^*}(x'^k, y^k) - \underline{u}(x'^k, y^k) \to 0$. With the same notations and arguments as in the proof of lemma 5.2, up to extraction of some subsequence, the functions $\phi(x_k + x') - y_k$ converge to some uniformly continuous function $\phi_\infty(x')$ in the compact subsets of $I\!R^{N-1}$. Similarly, the functions $z_k(x', y) = \overline{u}^{t^*}(x'_k+x', y_k+y) - \underline{u}(x'_k+x', y_k+y)$ converge to a function $z_\infty \geq 0$, locally in $I\!R^N$. On the one hand, we have $z_\infty(0,0) = 0$ and conclude that $z \equiv 0$ in $I\!R^N$. On the other hand, by (5.6), we have $z_k(0, \phi(x'_k) + y_0 - y_k) \geq \delta > 0$; by passage to the limit, it comes that $z_\infty(0, \phi_\infty(0) + y_0) \geq \delta$. This is impossible.

Consequently $m > 0$, that is to say:

$$\inf_{\Omega^-(y_2^*)\backslash\Omega^-(y_1^*)} (\overline{u}^{t^*} - \underline{u}) > 0$$

Since both $\overline{u}$ and $\underline{u}$ are lipschitz-continuous, this property is still true with $t^* - \eta$ instead of t^* for any $\eta \in [0, \eta_0]$, $\eta_0 > 0$ small enough. From our choice of y_2^*, we can also choose η_0 in such a way that $\overline{u}^{t^*-\eta} \geq b'$ in $\Omega^+(y_2^*)$ for any $\eta \in [0, \eta_0]$. From lemmas 5.1 and 5.2, we deduce that $\overline{u}^{t^*-\eta} - \underline{u} \geq 0$ in $I\!R^N$ for any $\eta \in [0, \eta_0]$. This contradicts the definition of t^*.

Finally, this proves that if $t^* > -\infty$, then:

$$\forall y \in I\!R, \quad \inf_{\Gamma(y)} (\overline{u}^{t^*} - \underline{u}) = 0$$

Proof of theorem 1.5. This theorem asserts that every solution u of (1.10), (1.11) is increasing in y. We are very grateful to H. Berestycki for a simple version of its proof.

Let u be a solution of (1.10), (1.11). By standard elliptic estimates, this function is class C^1 in $I\!R^N$. Take $\underline{u} = \overline{u} = u$. By theorem 1.4, the set $I = \{t \in I\!R^+,\ \forall s \geq t,\ u^s \geq u \text{ in } I\!R^N\}$ is not empty. Set $t^* = \inf I$ and suppose that $t^* > 0$.

Fix a real $y_0 \in I\!R$. By theorem 1.4, $\inf_{\Gamma(y_0)}(u^{t^*} - u) = 0$. There exists then a sequence $(x'_k, y_k) \in \Gamma(y_0)$ such that $u^{t^*}(x'_k, y_k) - u(x'_k, y_k) < 0$. As in the proof of theorem 1.4, the functions $\phi(x'_k + x') - y_k$ and $u_k = u(x'_k + x', y_k + y)$ go respectively to $\phi_\infty(x')$ and $u_\infty(x', y)$ locally in $I\!R^{N-1}$ and $I\!R^N$. We have

$u_\infty^{t^*} \geq u_\infty$ in $I\!R^N$ and $u_\infty^{t^*}(0,0) = u_\infty(0,0)$. We conclude similarly that $u_\infty^{t^*} \equiv u_\infty$ in $I\!R^N$.

For any $y_1 \in I\!R$, set $\Omega_\infty^+(y_1) = \{y > y_1 + \phi_\infty(x')\}$, $\Omega_\infty^-(y_1) = \{y < y_1 + \phi_\infty(x')\}$ and $\Gamma_\infty(y_1) = \{y = y_1 + \phi_\infty(x')\}$. By definition, we have $(0,0) \in \Gamma(y_0)$ and, from the uniform limits (1.11), it also comes that

$$\lim_{y \to +\infty} \inf_{\Omega_\infty^+(y)} u_\infty = b$$

$$\lim_{y \to -\infty} \sup_{\Omega_\infty^-(y)} u_\infty = a$$

This is in contradiction with the t^*-periodicity of the function u_∞ in the direction $(0, \cdots, 0, 1)$. This yields that $t^* = 0$. For any $t > 0$, we have $u^t \geq u$ in $I\!R^N$. With the strong maximum principle, we conclude as above that $u^t > u$ in $I\!R^N$. In other words, u is increasing in y.

In [27], with similar arguments, the following Liouville theorem is proved:

Theorem 5.3 *([27]) Under the assumptions of theorem 1.4, but without any reference to a function ϕ, if u is a lipschitz-continuous function solving (1.10) and such that $a \leq u \leq b$, and if $f(x', u)$ is nondecreasing in $u \in [a,b]$ for any $x' \in I\!R^{N-1}$, then $u \equiv u(x')$.*

Remark 5.4 *Theorems 1.4, 1.5, 5.3 also work in infinite straight cylinders $\Sigma = \omega \times I\!R$, where ω is a smooth bounded or unbounded subset of $I\!R^{N-1}$, with Neumann or Dirichlet type conditions on $\partial\Sigma$ (see [27] for more details).*

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CAPTIONS FOR THE FIGURES

FIG 1: The set $\mathcal{A}_{x_n}$

FIG 2: The quadrant $\mathcal{Q}$