Recent results on dislocations dynamics and homogenization

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Dislocations are defects present in crystals that are one the main explanation of the plastic behaviour of metals. We are interested in the collective behaviour of such defects and present here an homogenization result in its simplest form.

Consider dislocations represented as points of coordinates y_i in one space dimension, satisfying the following system of ODEs

$$\dot{y}_i = F(y_i) - \sum_{j\in\{1,\ldots,N_\varepsilon\}\setminus\{i\}} \frac{1}{y_i - y_j} \quad \text{for } i = 1, \ldots, N_\varepsilon \tag{0.1}$$

where F is a 1-periodic given force, and the term $1/(y_i - y_j)$ take into account the long range interactions between dislocations i and j.

We introduce the following rescaled "cumulative distribution function"

$$u^\varepsilon(t,x) = \epsilon \left(\sum_{i=1}^{N_\varepsilon} H(x - \varepsilon y_i(t/\varepsilon)) \right)$$

where $H(x)$ is the Heavyside function which is equal to 1 for positive x and zero for negative x.

The effective equation satisfied by the limit solution u^0 is

$$\begin{cases} \partial_t u^0 = \overline{H}^0 \left(-(-\Delta)^{\frac{1}{2}}[u^0(t,\cdot)], \nabla u^0 \right) & \text{in} \quad (0,+\infty)\times \mathbf{R}, \\ u^0(0,x) = u_0(x) & \text{on} \quad \mathbf{R} \end{cases} \tag{0.2}$$

for some suitable initial condition u_0, and where the half Laplacian is defined for $v = v(x)$ by

$$Lv(x) := -(-\Delta)^{\frac{1}{2}} v(x) = c \int_{\mathbf{R}} \frac{dz}{z^2} \left\{ v(x+z) - v(x) - zv'(x) 1_{\{|z|\le 1\}} \right\}$$

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for some positive constant c. This operator is also defined by Fourier transform by

$$\widehat{Lv}(\xi) = -|\xi|\widehat{v}(\xi)$$

Then we have the following homogenization result for our particle system.

Theorem 0.1 (Homogenization of the particle system)
Assume that F is 1-periodic and Lipschitz continuous. Then there exists a continuous function $\overline{H}^0$, non-decreasing in its first argument such that the following holds. Let us consider a given nondecreasing function $u_0 \in W^{2,\infty}(\mathbf{R})$ and let us denote by $\lfloor z \rfloor$ the floor integer part of a real number z.

Assume that $y_1(0) < ... < y_{N_\epsilon}(0)$ are given by the discontinuities of the function

$$u^\varepsilon(0,x) = u_0^\varepsilon(x) := \varepsilon\lfloor \frac{u_0(x)}{\varepsilon} \rfloor.$$

Then u^ε converges towards the unique solution u^0 of (0.2).

Here the function $\overline{H}^0$ appearing in the homogenized equation, can be interpreted as the effective "visco-plastic law" satisfied by the macroscopic material. Even if this model is very simple, this result contains the features that we expect for a more realistic model.

References

[1] N. Forcadel, C. Imbert, R. Monneau, *Homogenization of the dislocation dynamics and of some particle systems with two-body interactions*, preprint hal-00140545.