Level-set convex Hamilton-Jacobi equations on networks

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Abstract

The paper deals with Hamilton-Jacobi equations on networks with level-set convex (in the gradient variable) Hamiltonians which can be discontinuous with respect to the space variable at vertices. First, we prove that imposing a general *vertex condition* is equivalent to imposing a specific one which only depends on Hamiltonians and an additional free paremeter, the *flux limiter*. Second, a general method for proving comparison principles for *flux-limited vertex conditions* is introduced. This method consists in constructing a *vertex test function* to be used in the doubling variable approach. With such a theory and such a method in hand, a very general existence and uniqueness results is derived for Hamilton-Jacobi equations on networks. It also opens many perspectives for the study of these equations in such a singular geometrical framework. To illustrate this fact, we derive for instance a homogenization result for networks.

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Contents

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1 Introduction

This paper is concerned with Hamilton-Jacobi (HJ) equations on networks. The contribution of this paper is two-fold: one the one hand, general vertex conditions are proved to be equivalent to one which only depends on the Hamiltonians and a free parameter (which will be referred to as the flux limiter); on the other hand, a comparison principle is proved by using such a reduction.

Let us first discuss the second contribution. It is known that the core of the theory for HJ equations lies in the proof of a strong uniqueness result, i.e. of a comparison principle. Such a uniqueness result has been staying out of reach for some time. It is related to the identified difficulty of getting uniqueness results for discontinuous Hamiltonians. The proof of the comparison principle in the Euclidian setting is based on the so-called *doubling variable technique*. It is known that, even in a one-dimension space, such a method generally fails for piecewise constant (in x) Hamiltonians at discontinuities (see the last paragraph of Subsection 1.5). Since the network setting contains the previous one, the classical doubling variable technique is known to fail at vertices [24, 1, 18].

Eventhough, we show in this paper that the doubling variable approach can still be used if a suitable *vertex test function* G at each vertex is introduced. Roughly speaking, such a test function will allow the edges of the network to exchange enough information. More precisely, the usual penalization term, $\frac{(x-y)^2}{\varepsilon}$ with $\varepsilon > 0$, is replaced with $\varepsilon G(\varepsilon^{-1}x, \varepsilon^{-1}y)$. For a general HJ equation

$$u_t + H(x, u_x) = 0$$

the vertex test function has to ("almost") satisfy (at least close to the node $x = 0$),

$$H(y, -G_y(x, y)) - H(x, G_x(x, y)) \leq 0.$$

This key inequality fills the lack of compatibility between Hamiltonians[1]. The construction of a (vertex) test function satisfying such a condition allows us to circumvent the discontinuity of $H(x, p)$ at the junction point.

[1]Compatibility conditions are assumed in [24, 1] for instance.

1.1 The junction framework

In order to explain the construction of the vertex test function and our main results, we focus in this introduction on the simplest network, which will be referred to as a *junction*, and on Hamiltonians which are constant with respect to the space variable on each edge. The case of a general network with (t,x)-dependent Hamiltonians will be presented in Section 7 below.

A *junction* is a network made of one node and a finite number of infinite edges. It is endowed with a flat metric on each edge. It can be viewed as the set of N distinct copies ($N \geq 1$) of the half-line which are glued at the origin. For $i = 1, ..., N$, each branch J_i is assumed to be isometric to $[0, +\infty)$ and

$$J = \bigcup_{i=1,...,N} J_i \qquad \text{with} \quad J_i \cap J_j = \{0\} \quad \text{for} \quad i \neq j \tag{1.1}$$

where the origin 0 is called the *junction point*. For points $x, y \in J$, $d(x, y)$ denotes the geodesic distance on J defined as

$$d(x,y) = \begin{cases} |x-y| & \text{if } x, y \text{ belong to the same branch,} \\ |x| + |y| & \text{if } x, y \text{ belong to different branches.} \end{cases}$$

For a smooth real-valued function u defined on J, $\partial_i u(x)$ denotes the (spatial) derivative of u at $x \in J_i$ and the "gradient" of u is defined as follows,

$$u_x(x) := \begin{cases} \partial_i u(x) & \text{if} \quad x \in J_i^* := J_i \setminus \{0\}, \\ (\partial_1 u(0), ..., \partial_N u(0)) & \text{if} \quad x = 0. \end{cases} \tag{1.2}$$

With such a notation in hand, we consider the following Hamilton-Jacobi equation on the junction J

$$\begin{cases} u_t + H_i(u_x) = 0 & \text{for} \quad t \in (0, +\infty) \quad \text{and} \quad x \in J_i^*, \\ u_t + F(u_x) = 0 & \text{for} \quad t \in (0, +\infty) \quad \text{and} \quad x = 0 \end{cases} \tag{1.3}$$

submitted to the initial condition

$$u(0, x) = u_0(x) \quad \text{for} \quad x \in J. \tag{1.4}$$

The second equation in (1.3) is referred to as *the junction condition*. In general, minimal assumptions are required in order to get a good notion of weak (i.e. viscosity) solutions. We shed some light on the fact that Equation (1.3) can be thought as a *system* of Hamilton-Jacobi equations associated with H_i coupled through a "dynamical" boundary condition involving F. This point of view can be useful, see Subsection 1.5. As far as junction functions are concerned, we will construct below some special ones (denoted by F_A) from the Hamiltonians H_i ($i = 1, ..., N$) and a real parameter A.

We consider the important case of Hamiltonians H_i satisfying the following structure condition: there exist numbers $p_i^0 \in \mathbb{R}$ such that for each $i = 1, ..., N$,

$$(1.5) \qquad \begin{cases} \textbf{(Continuity)} & H_i \in C(\mathbb{R}) \\ \textbf{(Level-set convexity)} & \begin{cases} H_i \text{ nonincreasing in } (-\infty, p_i^0] \\ H_i \text{ nondecreasing in } [p_i^0, +\infty) \end{cases} \\ \textbf{(Coercivity)} & \lim_{|q|\to+\infty} H_i(q) = +\infty. \end{cases}$$

1.2 First main new idea: "relevant" junction conditions

We next introduce a one-parameter family of junction conditions: given a *flux limiter* $A \in \mathbb{R} \cup \{-\infty\}$, the A-limited flux through the junction point is defined for $p = (p_1, \ldots, p_N)$ as

$$(1.6) \qquad F_A(p) = \max\left(A, \quad \max_{i=1,\ldots,N} H_i^-(p_i)\right)$$

for some given $A \in \mathbb{R} \cup \{-\infty\}$ where H_i^- is the nonincreasing part of H_i defined by

$$H_i^-(q) = \begin{cases} H_i(q) & \text{if} \quad q \le p_i^0, \\ H_i(p_i^0) & \text{if} \quad q > p_i^0. \end{cases}$$

We now consider the following important special case of (1.3),

$$(1.7) \qquad \begin{cases} u_t + H_i(u_x) = 0 & \text{for} \quad t \in (0, +\infty) \quad \text{and} \quad x \in J_i^*, \\ u_t + F_A(u_x) = 0 & \text{for} \quad t \in (0, +\infty) \quad \text{and} \quad x = 0 \end{cases}$$

We point out that the flux functions F_A associated with $A \in [-\infty, A_0]$ coincide if one chooses

$$(1.8) \qquad A_0 = \max_{i=1,\ldots,N} \min_{\mathbb{R}} H_i.$$

As annonced above, general junction conditions are proved to be equivalent to those flux-limited junction conditions. Let us be more precise: a *junction function* $F : \mathbb{R}^N \to \mathbb{R}$ should at least satisfy the following condition,

(1.9) $\qquad F : \mathbb{R}^N \to \mathbb{R}$ is continuous and non-increasing with respect to all variables.

Indeed, the monotonicity assumption on F is related to the notion of viscosity solutions that will be introduced. In particular, it is mandatory in order to construct solutions through the Perron process. It will be proven that for such a junction condition F, solving (1.3) is equivalent to solving (1.7) for some $A = A_F$ only depending on F.

The special case of convex Hamiltonians. In the special case of convex Hamiltonians H_i with different minimum values, Problem (1.7) can be viewed as the Hamilton-Jacobi-Bellman equation satisfied by the value function of an optimal control problem; see for instance [18] when $A = -\infty$. In this case, existence and uniqueness of viscosity solutions for (1.7)-(1.4) (with $A = -\infty$) have been established either with a very rigid method [18] based on an explicit Oleinik-Lax formula which does not extend easily to networks, or in cases reducing to $H_i = H_j$ for all i, j if Hamiltonians do not depend on the space variable [24, 1]. In such an optimal control framework, trajectories can stay for a while at the junction point. In this case, the running cost at the junction point equals $\min_i L_i(0) = -\max_i(\min H_i)$. In this special case, the parameter A consists in replacing the previous running cost at the junction point by $\min(-A, \min_i L_i(0))$. In Section 5, the link between our results and optimal control theory will be further investigated.

1.3 Second main new idea: the vertex test function

The goal of the present paper is to provide the reader with a general handy and flexible method to prove a comparison principle, allowing in particular to deal with Hamiltonians that are not convex with respect to the gradient variable and are possibly discontinuous with respect to the space variable at the vertices. As explained above, this method consists in combining the doubling variable technique with the construction of a vertex test function G. We took our inspiration for the construction of this function in papers like [15, 3] dealing with scalar conservation laws with discontinuous flux functions. In such papers, authors stick to the case $N = 2$. A natural family of explicit solutions of (1.7) is given by

$$u(t,x) = p_i x - \lambda t \quad \text{if} \quad x \in J_i$$

for (p, λ) in the *germ* $\mathcal{G}_A$ defined as follows,

(1.10)
$$\mathcal{G}_A = \begin{cases} \left\{(p,\lambda) \in \mathbb{R}^N \times \mathbb{R}, \quad H_i(p_i) = F_A(p) = \lambda \quad \text{for all} \quad i = 1, ..., N\right\} & \text{if } N \geq 2, \\ \{(p_1, \lambda) \in \mathbb{R} \times \mathbb{R}, \quad H_1(p_1) = \lambda \geq A\} & \text{if } N = 1. \end{cases}$$

In the special case of convex Hamiltonians satisfying $H_i'' > 0$ the vertex test function G is a regularized version[2] of the function G^0 defined as follows: for $(x, y) \in J_i \times J_j$,

$$G^0(x,y) = \sup_{(p,\lambda) \in \mathcal{G}_A} (p_i x - p_j y - \lambda)\,. \tag{1.11}$$

1.4 Main results

The main result of this paper is the following comparison principle for Hamilton-Jacobi equations on a junction.

[2]Such a function should indeed be regularized since it is not C^1 on the diagonal $\{x = y\}$ of J^2.

Theorem 1.1 (Comparison principle on a junction)**.** *Let $A \in \mathbb{R} \cup \{-\infty\}$. Assume that the Hamiltonians satisfy* (1.5) *and that the initial datum u_0 is uniformly continuous. Then for all sub-solution u and super-solution w of* (1.7)-(1.4) *satisfying for some $T > 0$ and $C_T > 0$,*

$$u(t,x) \le C_T(1+d(0,x)), \quad w(t,x) \ge -C_T(1+d(0,x)), \quad \textit{for all} \quad (t,x) \in [0,T) \times J, \tag{1.12}$$

we have

$$u \le w \quad \textit{in} \quad [0,T) \times J.$$

Our second main result sheds light on the fact that the class of junction conditions we consider are in fact quite general. Indeed, for junction function F satisfying (1.9), it is always possible to construct solutions of (1.3) by Perron method [19]. Keeping in mind that it is expected that such solutions satisfy the junction condition *in a relaxed sense* (see Definition 2.2), the next theorem states that those relaxed solutions of (1.3) are in fact solutions of (1.7) for some $A = A_F$.

Theorem 1.2 (General junction conditions reduce to F_A)**.** *Assume that the Hamiltonians satisfy* (1.5) *and that F satisfies* (1.9) *and that the initial datum u_0 is uniformly continuous. Then there exists $A_F \in \mathbb{R}$ such that any relaxed viscosity solution of* (1.3) *is in fact a viscosity solution of* (1.7) *with $A = A_F$, in the sense of Definition 2.1.*

As a consequence of the two previous results, we can prove the following one.

Theorem 1.3 (Existence and uniqueness on a junction)**.** *Assume that the Hamiltonians satisfy* (1.5)*, that F satisfies* (1.9) *and that the initial datum u_0 is uniformly continuous. Then there exists a unique relaxed viscosity solution u of* (1.7)*,* (1.4) *such that for every $T > 0$, there exists a constant $C_T > 0$ such that*

$$|u(t,x) - u_0(x)| \le C_T \quad \textit{for all} \quad (t,x) \in [0,T) \times J.$$

The network setting. We will extend easily our results to the case of networks and non-convex Hamiltonians depending on time and space and to limiting parameters A (appearing in the Hamiltonian at the junction point) depending on time and vertex, see Section 7. Noticeably, a localization procedure allows us to use the vertex test function constructed for a single junction.

In order to state the results in the network setting, we need to make precise the assumptions satisfied by the Hamiltonians associated with each edge and the limiting parameters associated with each vertex. This ends in a rather long list of assumptions. Still, when reading the proof of the comparison principle in this setting, the reader may check that the main structure properties used in the proof are gathered in the technical Lemma 7.2.

As an application of the comparison principle, we consider a model case for homogenization on a network. The network $\mathcal{N}_\varepsilon$ whose vertices are $\varepsilon\mathbb{Z}^d$ is naturally embedded in $\mathbb{R}^d$. We consider for all edges a Hamiltonian only depending on the gradient variable but which is "repeated $\varepsilon\mathbb{Z}^d$-periodically with respect to edges". We prove that when $\varepsilon \to 0$, the solution of the "oscillating" Hamilton-Jacobi equation posed in $\mathcal{N}_\varepsilon$ converges toward the unique solution of an "effective" Hamilton-Jacobi equation posed in $\mathbb{R}^d$.

A first general comment about the main results. Our proofs do not rely on optimal control interpretation (there is no representation formula of solutions for instance) but on PDE methods. We believe that the method consisting in the construction of a vertex text function is very flexible and opens many perspectives. To the best of our knowledge, it is also the first uniqueness results for a Hamilton-Jacobi equation posed on a network for Hamiltonians that are not convex with respect to the gradient variable, are possibly discontinuous at the vertices in the space variable, and with a A-limited flux condition at the junction. Even for $N = 1$ or $N = 2$ branches, our results are completely new.

1.5 Comparison with known results

Hamilton-Jacobi equations on networks. There is a growing interest in the study of Hamilton-Jacobi equations on networks. The first results were obtained in [24] for eikonal equations. Several years after this first contribution, the three papers [1, 18, 25] were published more or less simultaneously. In these three papers, the Hamiltonians are always *convex* with respect to the gradient variables and the optimal control interpretation of the equation is at the core of the proofs of comparison principles. Still, frameworks are significantly different.

First, the networks in [1] are embedded in $\mathbb{R}^2$ while in [24, 25, 18], the networks are understood as metric spaces and Hamilton-Jacobi equations are studied in such metric spaces. Recently, a general approach of eikonal equations in metric spaces has been proposed in [17].

In [1], the authors study an optimal control problem in $\mathbb{R}^2$ and impose a *state constraint*: the trajectories of the controlled system have to stay in the embedded network. From this point of view, [1] is related to [13, 14] where trajectories in $\mathbb{R}^N$ are constrained to stay in a closed set K which can have an empty interior. But as pointed out in [1], the framework from [13, 14] imply some restricting conditions on the geometry of the embedded networks.

The main contribution of [18] in compare with [1, 25] comes from the dependence of the Hamiltonians with respect to the space variable. It is continuous in [1, 25] while [18] deals with Hamiltonians that are possibly discontinuous at the junction point (but are independent of the space variable on each edge).

The reader is referred to [9] where the different notions of viscosity solutions used in [1, 18, 25] are compared; in the few cases where frameworks coincide, they are proved to be equivalent.

In [18], the comparison principle was a consequence of a super-optimality principle (in the spirit of [21] or [26, 27]) and the comparison of sub-solutions with the value function of the optimal control problem. Still, the idea of using the "fundamental solution" $\mathcal{D}$ to prove a comparison principle originates in the proof of the comparison of sub-solutions and the value function. Moreover, as explained in Subsection 3.3, the comparison principle obtained in this paper could also be proved, for $A = -\infty$ and under more restrictive assumptions on the Hamiltonians, by using this fundamental solution.

The reader is referred to [1, 18, 25] for further references about Hamilton-Jacobi equations on networks.

Networks, regional optimal control and ramified spaces. We already pointed out that the Hamilton-Jacobi equation on a network can be regarded as a system of Hamilton-Jacobi equations coupled through vertices. In this perspective, our work can be compared with studies of Hamilton-Jacobi equations posed on, say, two domains separated by a frontier where some *transmission conditions* should be imposed. This can be even more general by considering equations in ramified spaces [8]. Contributions to such problems are [6, 7] on the one hand and [23, 22] on the other hand.

We first point out that their framework is genuinely multi-dimensional while ours is monodimensional. Moreover, their approach differs from the one in papers like [1, 25, 18] and the present one since the idea is to write a Hamilton-Jacobi equation on the (lower-dimensional) frontier. Another difference is that techniques from dynamical systems play also an important role in these papers.

Still, results can be compared. Precisely, considering a framework were both results can be applied, that is to say the monodimensional one, we will prove in Section 6 that the value function U^- from [7] coincides with the solution of (1.7) for some constant A that is determined. And we prove more (in the monodimensional setting; see also extensions below): we prove that the value function U^+ from [7] coincides with the solution of (1.7) for some (distinct) constant A which is also computed.

Hamilton-Jacobi equations with discontinuous source terms. There are numerous papers about Hamilton-Jacobi equations with discontinuous Hamiltonians. The recent paper [16] considers a Hamilton-Jacobi equation where specific solutions are expected. In the one-dimensional space, it can be proved that these solutions are in fact F_A-solutions in the sense of the present paper with $A = c$ where c is a constant appearing in the HJ equation at stake in [16]. The introduction of [16] contains a rather long list of results for HJ equations with discontinuous Hamiltonians; the reader is referred to it for further details.

The contribution of the paper. In light of the review we made above, we can emphasize the main contributions of the paper: in compare with [24, 25], we deal not only with eikonal equations but with general Hamilton-Jacobi equations. In compare with [1], we are able to deal with networks with infinite number of edges, that are not embedded. In compare with [1, 18, 24, 25], we can deal with non-convex discontinuous Hamilton-Jacobi equations and we provide a flexible PDE method instead of an optimal control approach. The link with optimal control (in the spirit of [1, 6, 7]) and the link with regional control (in the spirit of [6, 7]) are thoroughly investigated. In particular, a PDE characterization of the two value functions introduced in [7] is provided, one of the two characterizations being new.

We would also like to mention that the extension of our results to a higher dimensional setting (in the spirit of [6, 7]) is now reachable for level-set convex Hamiltonians.

To finish with, a application of our results to homogenization on networks is also presented.

1.6 Organization of the article and notation

Organization of the article. The paper is organized as follows. In Section 2, we introduce the notion of viscosity solution for Hamilton-Jacobi equations on junctions, we prove that they are stable (Proposition 2.3) and we give an existence result (Theorem 2.9). In Section 3, we prove the comparison principle in the junction case (Theorem 2.9). In Section 4, we construct the vertex test function (Theorem 3.2). In Section 5, a general optimal control problem on a junction is considered and the associated value function is proved to be a solution of (1.7) for some computable constant A. In Section 6, the two value functions introduced in [7] are shown to be solutions of (1.7) for two explicit (and distinct) constants A. In Section 7, we explain how to generalize the previous results (viscosity solutions, HJ equations, existence, comparison principle) to the case of networks. In Section 8, we present a straightforward application of our results by proving a homogenization result passing from an "oscillating" Hamilton-Jacobi equation posed in a network embedded in an Euclidian space to a Hamilton-Jacobi equation in the whole space. Finally, we prove several technical results in Appendix A and we state results for stationary Hamilton-Jacobi equations in Appendix B.

Notation for a junction. A junction is denoted by J. It is made of a finite number of edges and a junction point. The N edges of a junction ($N \in \mathbb{N} \setminus \{0\}$) are isometric to $[0, +\infty)$. Given a final time $T > 0$, J_T denotes $(0, T) \times J$.

The Hamiltonians on the branches J_i of the junction are denoted by H_i; they only depend on the gradient variable. The Hamiltonian at the junction point is denoted by F_A and is defined from all H_i and a constant A which "limits" the flux of information at the junction.

Given a function $u : J \to \mathbb{R}$, its gradient at x is denoted by u_x; it is a real number if $x \neq 0$ but it is a vector of $\mathbb{R}^N$ at $x = 0$. We let $|u_x|$ denote $|\partial_i u|$ outside the junction point and $\max_{i=1,\dots,N} |\partial_i u|$ at the junction point. If now $u(t, x)$ also depends on the time $t \in (0, +\infty)$, u_t denotes the time derivative.

Notation for networks. A network is denoted by $\mathcal{N}$. It is made of vertices $n \in \mathcal{V}$ and edges $e \in \mathcal{E}$. Each edge is either isometric to $[0, +\infty)$ or to a compact interval whose length is bounded from below; hence a network is naturally endowed with a metric. The associated open (resp. closed) balls are denoted by $B(x, r)$ (resp. $\bar{B}(x, r)$) for $x \in \mathcal{N}$ and $r > 0$.

In the network case, an Hamiltonian is associated with each edge e and is denoted by H_e. It depends on time and space; moreover, the limited flux functions A can depend on time and vertices: $A_n(t)$.

Further notation. Given a metric space E, $C(E)$ denotes the space of continuous real-valued functions defined in E. A modulus of continuity is a function $\omega : [0, +\infty) \to [0, +\infty)$ which is non-increasing and $\omega(0+) = 0$.

2 Viscosity solutions on a junction

This section is devoted to viscosity solutions in the junction case. After defining them, we will discuss their stability. In order to do so, relaxed viscosity solutions of (1.3) are considered and are proved to coincide with viscosity solutions in the special case of (1.7). We will also prove that general junction conditions reduce to an F_A junction condition for some parameter A.

2.1 Definitions

In order to define viscosity solutions, we first introduce the class of test functions.

Class of test functions. For $T > 0$, set $J_T = (0,T) \times J$. We define the class of test functions on $(0,T) \times J$ by

$$C^1(J_T) = \left\{\varphi \in C(J_T),\ \text{the restriction of } \varphi \text{ to } (0,T) \times J_i \text{ is } C^1 \text{ for } i = 1, ..., N\right\}.$$

Viscosity solutions. In order to define viscosity solutions, we recall the definition of upper and lower semi-continuous envelopes u^* and u_* of a (locally bounded) function u defined on $[0,T) \times J$,

$$u^*(t,x) = \limsup_{(s,y)\to(t,x)} u(s,y) \qquad \text{and} \qquad u_*(t,x) = \liminf_{(s,y)\to(t,x)} u(s,y).$$

Definition 2.1 (Viscosity solutions)**.** Assume that the Hamiltonians satisfy (1.5) and that F satisfies (1.9) and let $u : [0,T) \times J \to \mathbb{R}$.

i) We say that u is a *sub-solution* (resp. *super-solution*) of (1.3) in $(0,T) \times J$ if for all test function $\varphi \in C^1(J_T)$ such that

$$u^* \le \varphi \quad (\text{resp.} \quad u_* \ge \varphi) \quad \text{in a neighborhood of } (t_0,x_0) \in J_T$$

with equality at (t_0,x_0) for some $t_0 > 0$, we have

$$\begin{aligned} &\varphi_t + H_i(\varphi_x) \le 0 \quad (\text{resp.} \quad \ge 0) \quad \text{at } (t_0,x_0) \qquad \text{if } x_0 \in J_i^* \\ &\varphi_t + F(\varphi_x) \le 0 \quad (\text{resp.} \quad \ge 0) \quad \text{at } (t_0,x_0) \qquad \text{if } x_0 = 0. \end{aligned} \tag{2.1}$$

ii) We say that u is a *sub-solution* (resp. *super-solution*) of (1.3), (1.4) on $[0,T) \times J$ if additionally

$$u^*(0,x) \le u_0(x) \quad (\text{resp.} \quad u_*(0,x) \ge u_0(x)) \quad \text{for all} \quad x \in J.$$

iii) We say that u is a *(viscosity) solution* if u is both a sub-solution and a super-solution.

2.2 General junction conditions and stability

An important property that we expect for viscosity solutions is their stability, either by passing to local uniform limit, or the stability of sub-solutions (resp. super-solutions) through supremum (resp. infimum). Besides, a junction condition can be seen as a boundary condition. Since it is known that upper (resp. lower) semi-limits or suprema (resp. infima) of sub-solutions are known to satisfy boundary conditions *in a viscosity sense* [20, 5]. This is the reason why, for general junction functions F, the junction condition is relaxed: at the junction point, either the junction condition or the equation is satisfied. This is the reason why the following definition is needed.

Definition 2.2 (Relaxed viscosity solutions)**.** Assume that the Hamiltonians satisfy (1.5) and that F satisfies (1.9) and let $u : [0,T) \times J \to \mathbb{R}$.

i) We say that u is a *relaxed sub-solution* (resp. *relaxed super-solution*) of (1.3) in $(0,T) \times J$ if for all test function $\varphi \in C^1(J_T)$ such that

$$u^* \le \varphi \quad (\text{resp.} \quad u_* \ge \varphi) \quad \text{in a neighborhood of } (t_0,x_0) \in J_T$$

with equality at (t_0,x_0) for some $t_0 > 0$, we have

$$\varphi_t + H_i(\varphi_x) \le 0 \quad (\text{resp.} \quad \ge 0) \quad \text{at } (t_0,x_0)$$

if $x_0 \neq 0$, and

$$\left.\begin{array}{lllll} \text{either} & \varphi_t + F(\varphi_x) \le 0 & (\text{resp.} & \ge 0) & \\ \text{or} & \varphi_t + H_i(\partial_i \varphi) \le 0 & (\text{resp.} & \ge 0) & \text{for some } i \end{array}\right| \quad \text{at } (t_0,x_0)$$

if $x_0 = 0$.

ii) We say that u is a *relaxed (viscosity) solution* if u is both a sub-solution and a super-solution.

With this definition in hand, we can now state a first stability result.

Proposition 2.3 (Stability by supremum/infimum)**.** *Assume that the Hamiltonians H_i satisfy* (1.5) *and that F satisfies* (1.9)*. Let $\mathcal{A}$ be a nonempty set and let $(u_a)_{a\in\mathcal{A}}$ be a familly of relaxed sub-solutions (resp. relaxed super-solutions) of* (1.3) *on* $(0,T) \times J$*. Let us assume that*

$$u = \sup_{a\in\mathcal{A}} u_a \quad (\textit{resp.} \quad u = \inf_{a\in\mathcal{A}} u_a)$$

is locally bounded on $(0,T) \times J$*. Then u is a relaxed sub-solution (resp. relaxed super-solution) of* (1.3) *on* $(0,T) \times J$*.*

In the following proposition, we assert that, for the special junction functions F_A, the junction condition is in fact always satisfied *in the classical sense*, that is to say in the sense of Definition 2.1.

Proposition 2.4 (F_A junction conditions are always satisfied in the classical sense)**.** *Assume that the Hamiltonians satisfy* (1.5) *and consider* $A \in \mathbb{R}$*. If* $F = F_A$*, then relaxed viscosity super-solutions (resp. relaxed viscosity sub-solutions) coincide with viscosity super-solutions (resp. viscosity sub-solutions).*

Proof of Proposition 2.4. The proof was done in [18] for the case $A = -\infty$, using the monotonicities of the H_i. We follow the same proof and omit details.

THE SUPER-SOLUTION CASE. Let u be a super-solution satisfying the junction condition in the viscosity sense and let us assume by contradiction that there exists a test function φ touching u from below at $P_0 = (t_0, 0)$ for some $t_0 \in (0, T)$, such that

$$\varphi_t + F_A(\varphi_x) < 0 \quad \text{at} \quad P_0. \tag{2.2}$$

Then we can construct a test function $\tilde{\varphi}$ satisfying $\tilde{\varphi} \leq \varphi$ in a neighborhood of P_0, with equality at P_0 such that

$$\tilde{\varphi}_t(P_0) = \varphi_t(P_0) \quad \text{and} \quad \partial_i \tilde{\varphi}(P_0) = \min(p_i^0, \partial_i \varphi(P_0)) \quad \text{for} \quad i = 1, ..., N.$$

Using the fact that $F_A(\varphi_x) = F_A(\tilde{\varphi}_x) \geq H_i^-(\partial_i \tilde{\varphi}) = H_i(\partial_i \tilde{\varphi})$ at P_0, we deduce a contradiction with (2.2) using the viscosity inequality satisfied by φ for some $i \in \{1, \ldots, N\}$.

THE SUB-SOLUTION CASE. Let now u be a sub-solution satisfying the junction condition in the viscosity sense and let us assume by contradiction that there exists a test function φ touching u from above at $P_0 = (t_0, 0)$ for some $t_0 \in (0, T)$, such that

$$\varphi_t + F_A(\varphi_x) > 0 \quad \text{at} \quad P_0. \tag{2.3}$$

Let us define

$$I = \left\{ i \in \{1, ..., N\}, \quad H_i^-(\varphi) < F_A(\varphi_x) \quad \text{at} \quad P_0 \right\}$$

and for $i \in I$, let $q_i \geq p_i^0$ be such that

$$H_i(q_i) = F_A(\varphi_x(P_0))$$

where we have used the fact that $H_i(+\infty) = +\infty$. Then we can construct a test function $\tilde{\varphi}$ satisfying $\tilde{\varphi} \geq \varphi$ in a neighborhood of P_0, with equality at P_0, such that

$$\tilde{\varphi}_t(P_0) = \varphi_t(P_0) \quad \text{and} \quad \partial_i \tilde{\varphi}(P_0) = \begin{cases} \max(q_i, \partial_i \varphi(P_0)) & \text{if} \quad i \in I, \\ \partial_i \varphi(P_0) & \text{if} \quad i \notin I. \end{cases}$$

Using the fact that $F_A(\varphi_x) = F_A(\tilde{\varphi}_x) \leq H_i(\partial_i \tilde{\varphi})$ at P_0, we deduce a contradiction with (2.3) using the viscosity inequality for φ for some $i \in \{1, \ldots, N\}$. □

Proposition 2.5 (General junction conditions reduce to F_A)**.** *Let the Hamiltonians satisfy* (1.5) *and* F *satisfy* (1.9)*. There exists* $A_F \in \mathbb{R}$ *such that any relaxed super-solution (resp. relaxed sub-solution) of* (1.3) *is a super-solution (resp. sub-solution) of* (1.7) *with* $A = A_F$*.*

The flux limiter A_F is given by the following lemma.

Lemma 2.6 (Definitions of A_F and $\bar{p}$)**.** *Let $\bar{p}^0 = (\bar{p}_1^0, \ldots, \bar{p}_N^0)$ with $\bar{p}_i^0 \geq p_i^0$ be the minimal real number such that $H_i(\bar{p}_i^0) = A_0$ with A_0 given in (1.8).*

If $F(\bar{p}^0) \geq A_0$, then there exists a unique $A_F \in \mathbb{R}$ such that there exists $\bar{p} = (\bar{p}_1, \ldots, \bar{p}_N)$ with $\bar{p}_i \geq \bar{p}_i^0 \geq p_i^0$ such that

$$H_i(\bar{p}_i) = A = F(\bar{p}).$$

If $F(\bar{p}^0) < A_0$, we set $A_F = A_0$ and $\bar{p} = \bar{p}^0$.

In particular, we have

$$\{\forall i : p_i \geq \bar{p}_i\} \quad \Rightarrow F(p) \leq A_F, \tag{2.4}$$

$$\{\forall i : p_i \leq \bar{p}_i\} \quad \Rightarrow F(p) \geq A_F. \tag{2.5}$$

Proof of Proposition 2.5 for super-solutions. We let A denote A_F for clarity. Without loss of generality, we assume that u is lower semi-continuous. Consider a test function ϕ touching u from below at $(t, x) \in (0, +\infty) \times J$,

$$\phi \leq u \text{ in } B_R(t, x) \quad \text{ and } \quad \phi(t, x) = u(t, x)$$

for some $R > 0$. If $x \neq 0$, there is nothing to prove. We therefore assume that $x = 0$. In particular, we have

$$\phi_t(t, 0) + \max(F(\phi_x(t, 0)), \max_i H_i(\partial_i \phi(t, 0))) \geq 0. \tag{2.6}$$

We want to prove that

$$\phi_t(t, 0) + \max(A, \max_i H_i(\partial_i \phi(t, 0))) \geq 0. \tag{2.7}$$

Indeed, (2.7) implies that u is a relaxed super-solution of (1.7) and, consequently, a super-solution of (1.7).

In order to get such an inequality, we consider

$$I = \{i \in \{1, \ldots, N\} : \partial_i \phi(t, 0) \leq \bar{p}_i^0\}$$

and we distinguish two cases.

If I is empty, then we deduce from (2.4) that $F(\phi_x(t, 0)) \leq A$ and (2.7) follows from (2.6).

Assume now that I is not empty. In this case, for each $i_0 \in I$, we consider

$$p_{i_0} = \sup\{p \in \mathbb{R} : \exists r > 0, \;\; \phi(s, y) + py \leq u(s, y) \\ \text{for all } (s, y) \in (t - r, t + r) \times [0, r) \text{ with } y \in J_{i_0}\}.$$

Remark first that $p_{i_0} \geq 0$.

Lemma 2.7 (Property of p_{i_0})**.** *For each* $i_0 \in I$,

$$\phi_t(t,0) + H_{i_0}(\partial_{i_0}\phi(t,0) + p_{i_0}) \geq 0. \tag{2.8}$$

Proof. From the definition of p_{i_0}, we know that, for all $\varepsilon \in (0,R)$, there exists $\delta = \delta(\varepsilon) \in (0,\varepsilon)$ such that

$$u(s,y) \geq \phi(s,y) + (p_{i_0} - \varepsilon)y \quad \text{for all } (s,y) \in (t-\delta, t+\delta) \times [0,\delta) \text{ with } y \in J_{i_0}$$

and there exists $(t_\varepsilon, x_\varepsilon) \in B_{\delta/2}(t,0)$ such that

$$u(t_\varepsilon, x_\varepsilon) < \phi(t_\varepsilon, x_\varepsilon) + (p_{i_0} + \varepsilon)x_\varepsilon.$$

Now consider a smooth function $\Psi : \mathbb{R}^2 \to [-1,0]$ such that

$$\Psi \equiv \begin{cases} 0 & \text{in } B_{\frac{1}{2}}(0), \\ -1 & \text{outside } B_1(0) \end{cases}$$

and define

$$\Phi(s,y) = \phi(s,y) + 2\varepsilon\Psi_\delta(s,y) + \begin{cases} (p_{i_0} + \varepsilon)y & \text{if } y \in J_{i_0} \\ 0 & \text{if not} \end{cases}$$

with $\Psi_\delta(Y) = \delta\Psi(Y/\delta)$. We have

$$\Phi(s,y) \leq \phi(s,y) \leq u(s,y) \quad \text{for } (s,y) \in B_\delta(t,0) \text{ and } y \notin J_{i_0}$$

and

$$\begin{cases} \Phi(s,y) = \phi(s,y) - 2\varepsilon\delta + (p_{i_0} + \varepsilon)y \leq u(s,y) & \text{for } (s,y) \in (\partial B_\delta(t,0)) \cap (\mathbb{R} \times J_{i_0}), \\ \Phi(s,0) \leq \phi(s,0) \leq u(s,0) & \text{for } s \in (t-\delta, t+\delta) \end{cases}$$

and

$$\Phi(t_\varepsilon, x_\varepsilon) = \phi(t_\varepsilon, x_\varepsilon) + (p_{i_0} + \varepsilon)x_\varepsilon > u(t_\varepsilon, x_\varepsilon).$$

We conclude that there exists a point $(\bar{t}_\varepsilon, \bar{x}_\varepsilon) \in B_\delta(t,0) \cap \left(\mathbb{R} \times J^*_{i_0}\right)$ such that $u - \Phi$ reaches a minimum in $\overline{B_\delta(t,0)} \cap (\mathbb{R} \times J_{i_0})$. Consequently,

$$\Phi_t(\bar{t}_\varepsilon, \bar{x}_\varepsilon) + H_{i_0}(\partial_{i_0}\Phi(\bar{t}_\varepsilon, \bar{x}_\varepsilon))) \geq 0$$

which implies

$$\phi_t(\bar{t}_\varepsilon, \bar{x}_\varepsilon) + 2\varepsilon(\Psi_\delta)_t(\bar{t}_\varepsilon, \bar{x}_\varepsilon) + H_{i_0}(\partial_{i_0}\phi(\bar{t}_\varepsilon, \bar{x}_\varepsilon) + 2\varepsilon\partial_y(\Psi_\delta)(\bar{t}_\varepsilon, \bar{x}_\varepsilon) + p_{i_0} + \varepsilon) \geq 0.$$

Letting ε go to 0 yields (2.8). □

With this lemma in hand, it is now easy to get (2.7) when I is not empty. Indeed, if $\partial_{i_0}\phi(t,0)+p_{i_0}<\bar{p}_{i_0}$ for some $i_0\in I$, then

$$\begin{aligned} H_{i_0}(\partial_{i_0}\phi(t,0)+p_{i_0}) &\le \max(A, H^-_{i_0}(\partial_{i_0}\phi(t,0)+p_{i_0})) \\ &\le \max(A, H^-_{i_0}(\partial_{i_0}\phi(t,0)) \\ &\le F_A(\phi_x(t,0)). \end{aligned}$$

Combining this with (2.8) yields again (2.7).

If now for all $i_0\in I$, we have

$$\partial_{i_0}\phi(t,0)+p_{i_0}\ge \bar{p}_{i_0},$$

then the following modified test function

$$\varphi(s,y)=\phi(s,y)+\begin{cases}(\bar{p}_{i_0}-\partial_{i_0}\phi(t,0))y & \text{if } y\in J_{i_0} \text{ and } i_0\in I\\ 0 & \text{if } y\in J_i \text{ and } i\notin I\end{cases}$$

touches u from below at $(t,0)$ and for all i,

$$H_i(\partial_i\varphi(t,0))\ge A.$$

Consequently,

$$\phi_t(t,0)+\max(F(\partial_x\varphi(t,0)), \max_{i_0\in I} H_{i_0}(\bar{p}_{i_0}), \max_{i\notin I} H_i(\partial_i\phi(t,0)))\ge 0$$

which implies once again (2.7), using (2.4). The proof is now complete. $\square$

Proof of Proposition 2.5 for sub-solutions. We argue as in the super-solution case. We let A denote A_F for clarity. Without loss of generality, we assume that u est upper semi-continuous. Consider a test function ϕ touching u from above at $(t,x)\in(0,+\infty)\times J$,

$$\phi\ge u \text{ in } B_R(t,x) \quad \text{and} \quad \phi(t,x)=u(t,x)$$

for some $R>0$. If $x\ne 0$, there is nothing to prove. We therefore assume that $x=0$. In particular, we have

$$\phi_t(t,0)+\min(F(\phi_x(t,0)), \min_i H_i(\partial_i\phi(t,0)))\le 0. \tag{2.9}$$

We want to prove that

$$\phi_t(t,0)+\min(F_A(\phi_x(t,0))), \min_i H_i(\partial_i\phi(t,0)))\le 0. \tag{2.10}$$

Indeed, (2.10) implies that u is a relaxed sub-solution of (1.7) and, consequently, a sub-solution of (1.7).

Let λ denote $\phi_t(t,0)$. We can then assume that

$$\forall i \in \{1,\dots,N\}, \quad \lambda + H_i(\partial_i \phi(t,0)) > 0, \tag{2.11}$$

and we want to prove

$$\lambda + A \le 0 \quad \text{and} \quad \forall i, \lambda + H_i^-(\partial_i \phi(t,0)) \le 0. \tag{2.12}$$

In view of (2.11) and the fact that $A \ge A_0$, we can easily check that (2.12) is equivalent to

$$\lambda + A \le 0$$

and

$$\forall i, \partial_i \phi(t,0) > p_i^0. \tag{2.13}$$

We next define for each $i \in \{1,\dots,N\}$

$$p_i = \inf\{p \in \mathbb{R} : \exists r > 0, \quad \phi(s,y) + py \ge u(s,y) \text{ for } (s,y) \in (t-r,t+r)\times[0,r) \text{ with } y \in J_i\}.$$

We have $p_i \le 0$. Arguing as in Lemma 2.7, we can prove the following lemma

Lemma 2.8 (Property of p_i)**.** *For all $i \in \{1,\dots,N\}$,*

$$\lambda + H_i(\partial_i \phi(t,0) + p_i) \le 0. \tag{2.14}$$

Remark in particular that this implies that $H_i(\partial_i \phi(t,0) + p_i) < H_i(\partial_i \phi(t,0))$ which implies (2.13). For later use, we notice that this also implies

$$\lambda + A_0 \le 0. \tag{2.15}$$

We now distinguish two cases.

First, if there exists an index $i \in \{1,\dots,N\}$ such that

$$H_i(\partial_i \phi(t,0) + p_i) \ge A,$$

we conclude from (2.14) that $\lambda + A \le 0$, which is the desired inequality.

We thus can assume that

$$\forall i, H_i(\partial_i \phi(t,0) + p_i) < A,$$

which implies that

$$\partial_i \phi(t,0) + p_i < \bar{p}_i. \tag{2.16}$$

Consider the following modified test function:

$$\varphi(s,y) = \phi(s,y) + (\bar{p}_i - \partial_i \phi(t,0))y \quad \text{if } y \in J_i.$$

From (2.16), we have $\bar{p}_i - \partial_i \phi(t,0) > p_i$. By definition of p_i, we deduce that $\varphi \ge u$ on some neighborhood of $(t,0)$ with equality at $(t,0)$. Therefore we have

$$\lambda + \min(F(\varphi_x(t,0)), \min_{i\in\{1,\dots,N\}} H_i(\partial_i \varphi(t,0))) \le 0.$$

Recall that $\partial_i \varphi(t,0) = \bar{p}_i$. If $F(\bar{p}^0) \ge A_0$, we deduce from the definition of $\bar{p}$ that $\lambda + A \le 0$. Now if $F(\bar{p}^0) < A_0$, then $A = A_0$ and (2.15) provides the same conclusion. Therefore in all cases, we have $\lambda + A \le 0$. Combined with (2.13), this shows (2.12), and then (2.10) holds true. The proof is now complete. $\square$

2.3 Existence

Theorem 2.9 (Existence)**.** *Let $T > 0$ and J be the junction defined in* (1.1)*. Assume that Hamiltonians satisfy* (1.5)*, that the junction function F satisfies* (1.9) *and that the initial datum u_0 is uniformly continuous. Then there exists a generalized viscosity solution u of* (1.3)-(1.4) *in $[0,T) \times J$ and a constant $C_T > 0$ such that*

$$|u(t,x) - u_0(x)| \le C_T \quad \textit{for all} \quad (t,x) \in [0,T) \times J.$$

Proof of Theorem 2.9. The proof follows classically along the lines of Perron's method (see [19, 10]), and then we omit details. We assume without loss of generality that $A > -\infty$ (and even $A > A_0 - 1$).

Step 1: Barriers. Because of the uniform continuity of u_0, for any $\varepsilon \in (0,1]$, it can be regularized by convolution to get a modified initial data u_0^ε satisfying

$$|u_0^\varepsilon - u_0| \le \varepsilon \quad \text{and} \quad |(u_0^\varepsilon)_x| \le L_\varepsilon \tag{2.17}$$

with $L_\varepsilon \ge \max_{i=1,\dots,N} |p_i^0|$. Let

$$C_\varepsilon = \max\left(|A|, \max_{i=1,\dots,N} \max_{|p_i| \le L_\varepsilon} |H_i(p_i)|, \max_{|p_i|\le L_\varepsilon} F(p_1,\dots,p_N)\right).$$

Then the functions

$$u_\varepsilon^\pm(t,x) = u_0^\varepsilon(x) \pm C_\varepsilon t \pm \varepsilon \tag{2.18}$$

are global super and sub-solutions with respect to the initial data u_0. We then define

$$u^+(t,x) = \inf_{\varepsilon\in(0,1]} u_\varepsilon^+(t,x) \quad \text{and} \quad u^-(t,x) = \sup_{\varepsilon\in(0,1]} u_\varepsilon^-(t,x).$$

Then we have $u^- \le u^+$ with $u^-(0,x) = u_0(x) = u^+(0,x)$. Moreover, by stability of sub/super-solutions (see Proposition 2.3), we get that u^+ is a super-solution and u^- is a sub-solution of (1.3) on $(0,T) \times J$.

Step 2: Maximal sub-solution and preliminaries. Consider the set

$$S = \left\{ w : [0,T) \times J \to \mathbb{R}, \quad w \text{ is a sub-solution of (1.3) on } (0,T)\times J, \quad u^- \le w \le u^+ \right\}.$$

It contains u^-. Then the function

$$u(t,x) = \sup_{w\in S} w(t,x)$$

is a sub-solution of (1.3) on $(0,T)\times J$ and satisfies the initial condition. It remains to show that u is a super-solution of (1.3) on $(0,T) \times J$. This is classical for a Hamilton-Jacobi equation on an interval, so we only have to prove it at the junction point. We assume by contradiction that u is not a super-solution at $P_0 = (t_0, 0)$ for some $t_0 \in (0, T)$. Thanks to Proposition 2.4, this implies that there exists a test function φ satisfying $u_* \geq \varphi$ in a neighborhood of P_0 with equality at P_0, and such that

$$\left\{ \begin{array}{l} \varphi_t + F(\varphi_x) < 0, \\ \varphi_t + H_i(\partial_i \varphi) < 0, \quad \text{for} \quad i = 1, ..., N \end{array} \right| \quad \text{at} \quad P_0. \tag{2.19}$$

We also have $\varphi \leq u_* \leq u^+_*$. As usual, the fact that u^+ is a super-solution and condition (2.19) imply that we cannot have $\varphi = (u^+)_*$ at P_0. Therefore we have for some $r > 0$ small enough

$$\varphi < (u^+)_* \quad \text{on} \quad \overline{B_r(P_0)} \tag{2.20}$$

where we define the ball $B_r(P_0) = \{(t, x) \in (0, T) \times J, \quad |t - t_0|^2 + d^2(0, x) < r^2\}$. Substracting $|(t, x) - P_0|^2$ to φ and reducing $r > 0$ if necessary, we can assume that

$$\varphi < u_* \quad \text{on} \quad \overline{B_r(P_0)} \setminus \{P_0\}\,. \tag{2.21}$$

Further reducing $r > 0$, we can also assume that (2.19) still holds in $\overline{B_r(P_0)}$.

Step 3: Sub-solution property and contradiction. We claim that φ is a sub-solution of (1.3) in $B_r(P_0)$. Indeed, if ψ is a test function touching φ from above at $P_1 = (t_1, 0) \in B_r(P_0)$, then

$$\psi_t(P_1) = \varphi_t(P_1) \quad \text{and} \quad \partial_i \psi(P_1) \geq \partial_i \varphi(P_1) \quad \text{for} \quad i = 1, ..., N.$$

Using the fact that F is non-increasing with respect to all variables, we deduce that

$$\psi_t + F(\psi_x) < 0 \quad \text{at} \quad P_1$$

as desired. Defining for $\delta > 0$,

$$u_\delta = \begin{cases} \max(\delta + \varphi, u) & \text{in } B_r(P_0), \\ u & \text{outside} \end{cases}$$

and using (2.21), we can check that $u_\delta = u > \delta + \varphi$ on $\partial B_r(P_0)$ for $\delta > 0$ small enough. This implies that u_δ is a sub-solution lying above u^-. Finally (2.20) implies that $u_\delta \leq u^+$ for $\delta > 0$ small enough. Therefore $u_\delta \in S$, but is is classical to check that u_δ is not below u for $\delta > 0$, which gives a contradiction with the maximality of u. □

3 Comparison principle on a junction

This section is devoted to the proof of the comparison principle in the case of a junction (see Theorem 1.1).

It is convenient to introduce the following shorthand notation

$$H(x,p) = \begin{cases} H_i(p) & \text{for} \quad p = p_i & \text{if} \quad x \in J_i^*, \\ F_A(p) & \text{for} \quad p = (p_1, ..., p_N) & \text{if} \quad x = 0. \end{cases} \tag{3.1}$$

In particular, keeping in mind the definition of u_x (see (1.2)), Problem (1.7) on the junction can be rewritten as follows

$$u_t + H(x, u_x) = 0 \quad \text{for all} \quad (t,x) \in (0,+\infty) \times J.$$

We next make a trivial but useful observation.

Lemma 3.1. *It is enough to prove Theorem 1.1 further assuming that*

$$p_i^0 = 0 \quad \textit{for} \quad i = 1, ..., N \quad \textit{and} \quad 0 = H_1(0) \geq H_2(0) \geq ... \geq H_N(0). \tag{3.2}$$

Proof of Lemma 3.1. We can assume without loss of generality that

$$H_1(p_1^0) \geq ... \geq H_N(p_N^0).$$

Let us define

$$u(t,x) = \tilde{u}(t,x) + p_i^0 x - tH_1(p_1^0) \quad \text{for} \quad x \in J_i.$$

Then u is a solution of (1.7) if and only if $\tilde{u}$ is a solution of (1.7) with each H_i replaced with $\tilde{H}_i(p) = H_i(p + p_i^0) - H_1(p_1^0)$ and F_A replaced with $\tilde{F}_{\tilde{A}}$ constructed using the Hamiltonians $\tilde{H}_i$ and the parameter $\tilde{A} = A - H_1(p_1^0)$. □

3.1 The vertex test function

Then our key result is the following one.

Theorem 3.2 (The vertex test function – general case)**.** *Let $A \in \mathbb{R} \cup \{-\infty\}$ and $\gamma > 0$. Assume the Hamiltonians satisfy* (1.5) *and* (3.2). *Then there exists a function $G : J^2 \to \mathbb{R}$ enjoying the following properties.*

i) (Regularity)

$$G \in C(J^2) \quad \textit{and} \quad \begin{cases} G(x, \cdot) \in C^1(J) & \textit{for all} \quad x \in J, \\ G(\cdot, y) \in C^1(J) & \textit{for all} \quad y \in J. \end{cases}$$

ii) (Bound from below) $G \geq 0 = G(0,0)$.

iii) (Compatibility condition on the diagonal) *For all $x \in J$,*

$$0 \leq G(x,x) - G(0,0) \leq \gamma. \tag{3.3}$$

iv) (Compatibility condition on the gradients) *For all* $(x, y) \in J^2$,

$$H(y, -G_y(x, y)) - H(x, G_x(x, y)) \leq \gamma \tag{3.4}$$

where notation introduced in (1.2) *and* (3.1) *are used.*

v) (Superlinearity) *There exists* $g : [0, +\infty) \to \mathbb{R}$ *nondecreasing and s.t. for* $(x, y) \in J^2$

$$g(d(x, y)) \leq G(x, y) \quad \textit{and} \quad \lim_{a \to +\infty} \frac{g(a)}{a} = +\infty. \tag{3.5}$$

vi) (Gradient bounds) *For all* $K > 0$*, there exists* $C_K > 0$ *such that for all* $(x, y) \in J^2$,

$$d(x, y) \leq K \quad \Longrightarrow \quad |G_x(x, y)| + |G_y(x, y)| \leq C_K. \tag{3.6}$$

3.2 Proof of the comparison principle

We will also need the following result whose classical proof is given in Appendix for the reader's convenience.

Lemma 3.3 (A priori control)**.** *Let* $T > 0$ *and let* u *be a sub-solution and* w *be a super-solution as in Theorem 1.1. Then there exists a constant* $C = C(T) > 0$ *such that for all* $(t, x), (s, y) \in [0, T) \times J$*, we have*

$$u(t, x) \leq w(s, y) + C(1 + d(x, y)). \tag{3.7}$$

We are now ready to make the proof of comparison principle.

Proof of Theorem 1.1. The proof proceeds in several steps.

Step 1: the penalization procedure. We want to prove that

$$M = \sup_{(t,x) \in [0,T) \times J} (u(t, x) - w(t, x)) \leq 0.$$

Assume by contradiction that $M > 0$. Then for $\alpha, \eta > 0$ small enough, we have $M_{\varepsilon,\alpha} \geq M/2 > 0$ for all $\varepsilon, \nu > 0$ with

$$M_{\varepsilon,\alpha} = \sup_{(t,x),(s,y) \in [0,T) \times J} \left\{ u(t, x) - w(s, y) - \varepsilon G\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) - \frac{(t-s)^2}{2\nu} - \frac{\eta}{T-t} - \alpha \frac{d^2(0, x)}{2} \right\} \tag{3.8}$$

where the vertex test function $G \geq 0$ is given by Theorem 3.2 for a parameter γ satisfying

$$0 < \gamma < \min\left(\frac{\eta}{2T^2}, \frac{M}{4\varepsilon}\right).$$

Thanks to Lemma 3.3 and (3.5), we deduce that

$$0 < \frac{M}{2} \leq C(1 + d(x, y)) - \varepsilon g\left(\frac{d(x, y)}{\varepsilon}\right) - \frac{(t - s)^2}{2\nu} - \frac{\eta}{T - t} - \alpha\frac{d^2(0, x)}{2} \tag{3.9}$$

which implies in particular that

$$\varepsilon g\left(\frac{d(x, y)}{\varepsilon}\right) \leq C(1 + d(x, y)). \tag{3.10}$$

Because of the superlinearity of g appearing in (3.5), we know that for any $K > 0$, there exists a constant $C_K > 0$ such that for all $a \geq 0$

$$Ka - C_K \leq g(a).$$

For $K \geq 2C$, we deduce from (3.10) that

$$d(x, y) \leq \inf_{K \geq 2C} \left\{ \frac{C}{K - C} + \frac{C_K}{C}\varepsilon \right\} =: \omega(\varepsilon) \tag{3.11}$$

where ω is a concave, nondecreasing function satisfying $\omega(0) = 0$. We deduce from (3.9) and (3.11) that the supremum in (3.8) is reached at some point $(t, x, s, y) = (t_\nu, x_\nu, s_\nu, y_\nu)$.

Step 2: use of the initial condition. We first treat the case where $t_\nu = 0$ or $s_\nu = 0$. If there exists a sequence $\nu \to 0$ such that $t_\nu = 0$ or $s_\nu = 0$, then calling (x_0, y_0) any limit of subsequences of (x_ν, y_ν), we get from (3.8) and the fact that $M_{\varepsilon,\alpha} \geq M/2$ that

$$0 < \frac{M}{2} \leq u_0(x_0) - u_0(y_0) \leq \omega_0(d(x_0, y_0)) \leq \omega_0 \circ \omega(\varepsilon)$$

where ω_0 is the modulus of continuity of the initial data u_0 and ω is defined in (3.11). This is impossible for ε small enough.

Step 3: use of the equation. We now treat the case where $t_\nu > 0$ and $s_\nu > 0$. Then we can write the viscosity inequalities with $(t, x, s, y) = (t_\nu, x_\nu, s_\nu, y_\nu)$ using the shorthand notation (3.1) for the Hamiltonian,

$$\begin{aligned} \frac{\eta}{(T - t)^2} + \frac{t - s}{\nu} + H(x, G_x(\varepsilon^{-1}x, \varepsilon^{-1}y) + \alpha d(0, x)) &\leq 0, \\ \frac{t - s}{\nu} + H(y, -G_y(\varepsilon^{-1}x, \varepsilon^{-1}y)) &\geq 0. \end{aligned}$$

Substrating these two inequalities, we get

$$\frac{\eta}{T^2} \leq H(y, -G_y(\varepsilon^{-1}x, \varepsilon^{-1}y)) - H(x, G_x(\varepsilon^{-1}x, \varepsilon^{-1}y) + \alpha d(0, x)).$$

Using (3.4) with $\gamma \in \left(0, \frac{\eta}{2T^2}\right)$, we deduce for $p = G_x(\varepsilon^{-1}x, \varepsilon^{-1}y)$

$$\frac{\eta}{2T^2} \leq H(x,p) - H(x, p + \alpha d(0,x)). \tag{3.12}$$

Because of (3.6) and (3.11), we see that p is bounded for ε fixed by $|p| \leq C_{\frac{\omega(\varepsilon)}{\varepsilon}}$. Finally, for $\varepsilon > 0$ fixed and $\alpha \to 0$, we have $\alpha d(0,x) \to 0$, and using the uniform continuity of $H(x,p)$ for $x \in J$ and p bounded, we get a contradiction in (3.12). The proof is now complete. □

3.3 The vertex test function versus the fundamental solution

Recalling the definition of the germ $\mathcal{G}_A$ (see (1.10)), let us associate with any $(p,\lambda) \in \mathcal{G}_A$ the following functions for $i,j = 1, ..., N$,

$$u^{p,\lambda}(t,x,s,y) = p_i x - p_j y - \lambda(t-s) \quad \text{for} \quad (x,y) \in J_i \times J_j, \quad t,s \in \mathbb{R}.$$

The reader can check that they solve the following system,

$$\begin{cases} u_t + H(x, u_x) = 0, \\ -u_s + H(y, -u_y) = 0. \end{cases} \tag{3.13}$$

Then, for $N \geq 2$, the function $\tilde{G}^0(t,x,s,y) = (t-s)G^0\left(\frac{x}{t-s}, \frac{y}{t-s}\right)$ can be rewritten as

$$\tilde{G}^0(t,x,s,y) = \sup_{(p,\lambda)\in\mathcal{G}_A} u^{p,\lambda}(t,x,s,y) \quad \text{for} \quad (x,y) \in J \times J, \quad t-s \geq 0 \tag{3.14}$$

which satisfies

$$\tilde{G}^0(s,x,s,y) = \begin{cases} 0 & \text{if} \quad x = y, \\ +\infty & \text{otherwise.} \end{cases} \tag{3.15}$$

For $N \geq 2$ and $A > A_0$, it is possible to check (assuming (4.1)) that $\tilde{G}^0$ is a viscosity solution of (3.13) for $t - s > 0$, only outside the diagonal $\{x = y \neq 0\}$. Therefore, even if (3.14) appears as a kind of (second) Hopf formula (see for instance [4, 2]), this formula does not provide a true solution on the junction.

On the other hand, under more restrictive assumptions on the Hamiltonians and for $A = A_0$ and $N \geq 2$ (see [18]), there is a natural viscosity solution of (3.13) with the same initial conditions (3.15), which is $\mathcal{D}(t,x,s,y) = (t-s)\mathcal{D}_0\left(\frac{x}{t-s}, \frac{y}{t-s}\right)$ where $\mathcal{D}_0$ is a cost function defined in [18] following an optimal control interpretation. The function $\mathcal{D}_0$ is not C^1 in general (but it is semi-concave) and it is much more difficult to study it and to use it in comparison with G^0. Nevertheless, under suitable restrictive assumptions on the Hamiltonians, it would be also possible to replace in our proof of the comparison principle the term $\varepsilon G(\varepsilon^{-1}x, \varepsilon^{-1}y)$ in (3.8) by $\varepsilon\mathcal{D}_0(\varepsilon^{-1}x, \varepsilon^{-1}y)$.

4 Construction of the vertex test function

This section is devoted to the proof of Theorem 3.2. Our construction of the vertex test function G is modelled on the particular subcase of normalized convex Hamiltonians H_i.

4.1 The case of smooth convex Hamiltonians

Assume that the Hamiltonians H_i satisfy the following assumptions for $i = 1, ..., N$,

$$\left\{\begin{array}{l} H_i \in C^2(\mathbb{R}) \quad \text{with} \quad H_i'' > 0 \quad \text{on} \quad \mathbb{R}, \\ H_i' < 0 \quad \text{on} \quad (-\infty, 0) \quad \text{and} \quad H_i' > 0 \quad \text{on} \quad (0, +\infty), \\ \displaystyle\lim_{|p|\to+\infty} \frac{H_i(p)}{|p|} = +\infty. \end{array}\right. \tag{4.1}$$

It is useful to associate with each H_i satisfying (4.1) its partial inverse functions $\pi_i^{\pm}$:

$$\text{for } \lambda \geq H_i(0), \quad H_i(\pi_i^{\pm}(\lambda)) = \lambda \quad \text{such that} \quad \pm\, \pi_i^{\pm}(\lambda) \geq 0. \tag{4.2}$$

Assumption (4.1) implies that $\pi_i^{\pm} \in C^2(\min H_i, +\infty) \cap C([\min H_i, +\infty))$ thanks to the inverse function theorem.

We recall that G^0 is defined, for $i, j = 1, ..., N$, by

$$G^0(x, y) = \sup_{(p,\lambda)\in\mathcal{G}_A} (p_i x - p_j y - \lambda) \quad \text{if} \quad (x, y) \in J_i \times J_j$$

where $\mathcal{G}_A$ is defined in (1.10). Replacing A with $\max(A, A_0)$ if necessary, we can always assume that $A \geq A_0$ with A_0 given by (1.8).

Proposition 4.1 (The vertex test function – the smooth convex case). *Let $A \geq A_0$ with A_0 given by* (1.8) *and assume that the Hamiltonians satisfy* (4.1). *Then G^0 satisfies*

i) (Regularity)

$$G^0 \in C(J^2) \quad \textit{and} \quad \left\{\begin{array}{l} G^0 \in C^1(\{(x, y) \in J \times J, \quad x \neq y\}), \\ G^0(0, \cdot) \in C^1(J) \quad \textit{and} \quad G^0(\cdot, 0) \in C^1(J); \end{array}\right.$$

ii) (Bound from below) $G^0 \geq G^0(0, 0) = -A$;

iii) (Compatibility conditions) (3.3) *and* (3.4) *hold with $\gamma = 0$;*

iv) (Superlinearity) (3.5) *holds for some $g = g^0$;*

v) (Gradient bounds) (3.6) *holds only for $(x, y) \in J^2$ such that $x \neq y$ or $(x, y) = (0, 0)$;*

vi) (Saturation close to the diagonal) *For* $i \in \{1, ..., N\}$ *and for* $(x, y) \in J_i \times J_i$, *we have* $G^0(x, y) = \ell_i(x - y)$ *with* $\ell_i \in C(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\})$ *and*

$$\ell_i(a) = \begin{cases} +a\pi_i^+(A) - A & \text{if} \quad 0 \le a \le z_i^+ \\ -a\pi_i^-(A) - A & \text{if} \quad z_i^- \le a \le 0 \end{cases}$$

where $(z_i^-, z_i^+) := (H_i'(\pi_i^-(A)), H_i'(\pi^+(A)))$ *and the functions* $\pi_i^\pm$ *are defined in* (4.2). *Moreover* $G^0 \in C^1(J_i \times J_i)$ *if and only if* $\pi_i^+(A) = 0 = \pi_i^-(A)$.

Remark 4.2. The compatibility condition (3.4) is in fact an equality with $\gamma = 0$ when $N \ge 2$.

The proof of this proposition is postponed until Subsection 4.4. With such a result in hand, we can now prove Theorem 3.2 in the case of smooth convex Hamiltonians.

Lemma 4.3 (The case of smooth convex Hamiltonians)**.** *Assume that the Hamiltonians satisfy* (4.1). *Then the conclusion of Theorem 3.2 holds true.*

Proof. We note that the function G^0 satisfies all the properties required for G, except on the diagonal $\{(x, y) \in J \times J, x = y \neq 0\}$ where G^0 may not be C^1. To this end, we first introduce the set I of indices such that $G^0 \notin C^1(J_i \times J_i)$. We know from Proposition 4.1 vi) that

$$I = \left\{ i \in \{1, ..., N\}, \quad \pi_i^+(A) > \pi_i^-(A) \right\}.$$

For $i \in I$, we are going to contruct a regularization $\tilde{G}^{0,i}$ of G^0 in a neighbourhood of the diagonal $\{(x, y) \in J_i \times J_i,\ x = y \neq 0\}$.

Step 1: Construction of $\tilde{G}^{0,i}$ for $i \in I$. Let us define

$$L_i(a) = \begin{cases} a\pi_i^+(A) & \text{if} \quad a \ge 0, \\ -a\pi_i^-(A) & \text{if} \quad a \le 0. \end{cases}$$

We first consider a convex C^1 function $\tilde{L}_i : \mathbb{R} \to \mathbb{R}$ coinciding with L_i outside (z_i^-, z_i^+), that we choose such that

$$0 \le \tilde{L}_i - L_i \le 1. \tag{4.3}$$

Then for $\varepsilon \in (0, 1]$, we define

$$\ell_i^\varepsilon(a) := \begin{cases} \varepsilon \tilde{L}_i\left(\frac{a}{\varepsilon}\right) - A & \text{if} \quad a \in [\varepsilon z_i^-, \varepsilon z_i^+], \\ \ell_i(a) & \text{otherwise}. \end{cases}$$

which is a $C^1(\mathbb{R})$ (and convex) function. We now consider a cut-off function ζ satisfying for some constant $B > 0$

$$\begin{cases} \zeta \in C^\infty(\mathbb{R}), \\ \zeta' \ge 0, \\ \zeta = 0 \quad \text{in} \quad (-\infty, 0], \\ \zeta = 1 \quad \text{in} \quad [B, +\infty), \\ \pm z_i^\pm \zeta' < 1 \quad \text{in} \quad (0, +\infty) \end{cases} \tag{4.4}$$

and for $\varepsilon \in (0,1]$, we define for $(x,y) \in J_i \times J_i$:

$$\tilde{G}^{0,i}(x,y) = \ell_i^{\varepsilon\zeta(x+y)}(x-y).$$

Step 2: First properties of $\tilde{G}^{0,i}$. By construction, we have $\tilde{G}^{0,i} \in C^1((J_i \times J_i) \setminus \{0\})$. Moreover we have

$$\tilde{G}^{0,i} = G^0 \quad \text{on} \quad (J_i \times J_i) \setminus \delta_i^\varepsilon$$

where

$$\delta_i^\varepsilon = \left\{(x,y) \in J_i \times J_i, \quad \varepsilon z_i^- \zeta(x+y) < x - y < \varepsilon z_i^+ \zeta(x+y)\right\}$$

is a neighborhood of the diagonal

$$\left\{(x,y) \in J_i \times J_i, \quad x = y \neq 0\right\}.$$

Because of (4.3), we also have

$$G^0 \le \tilde{G}^{0,i} \le \varepsilon. \tag{4.5}$$

As a consequence of (4.4), we have in particular

$$(J_i \times J_i) \setminus \delta_i^\varepsilon \quad \supset \quad (J_i \times \{0\}) \cup (\{0\} \times J_i)$$

and moreover $\tilde{G}^{0,i}$ coincides with G^0 on a neighborhood of $(J_i^* \times \{0\}) \cup (\{0\} \times J_i^*)$, which implies that

$$\tilde{G}^{0,i} = G^0, \quad \tilde{G}^{0,i}_x = G^0_x \quad \text{and} \quad \tilde{G}^{0,i}_y = G^0_y \quad \text{on} \quad (J_i \times \{0\}) \cup (\{0\} \times J_i). \tag{4.6}$$

Step 3: Computation of the gradients of $\tilde{G}^{0,i}$. For $(x,y) \in \delta_i^\varepsilon$, we have

$$\begin{cases} \tilde{G}^{0,i}_x(x,y) &= (\ell_i^{\varepsilon\zeta(x+y)})'(x-y) + \varepsilon\zeta'(x+y)\ \xi_i\left(\dfrac{x-y}{\varepsilon\zeta(x+y)}\right) \\ -\tilde{G}^{0,i}_y(x,y) &= (\ell_i^{\varepsilon\zeta(x+y)})'(x-y) - \varepsilon\zeta'(x+y)\ \xi_i\left(\dfrac{x-y}{\varepsilon\zeta(x+y)}\right) \end{cases}$$

with

$$\xi_i(b) = \tilde{L}_i(b) - b\tilde{L}_i'(b)$$

while if $(x,y) \in (J_i \times J_i) \setminus \delta_i^\varepsilon$ we have

$$\tilde{G}^{0,i}_x(x,y) = -\tilde{G}^{0,i}_y(x,y).$$

Given $\gamma > 0$, and using the local uniform continuity of H_i, we see that we have for ε small enough

$$H_i(\tilde{G}^{0,i}_x)) \le H_i(-\tilde{G}^{0,i}_y) + \gamma \quad \text{in} \quad J_i^* \times J_i^*$$

and using (4.6), we get

$$H(x, \tilde{G}^{0,i}_x(x,y)) - H(y, -\tilde{G}^{0,i}_y(x,y)) \le \gamma \quad \text{for all} \quad (x,y) \in J_i \times J_i. \tag{4.7}$$

Step 4: Definition of G. We set for $(x,y)\in J_i\times J_j$:

$$G(x,y)=\begin{cases} G^0(x,y)-G^0(0,0) & \text{if} \quad i\neq j \quad \text{or} \quad i=j\notin I,\\ \tilde{G}^{0,i}(x,y)-G^0(0,0) & \text{if} \quad i=j\in I.\end{cases}$$

From the properties of G^0, we recover all the expected properties of G with $g(a)=g^0(a)-G^0(0,0)$. In particular from (4.7) and (4.5), we respectively get the compatibility condition for the Hamiltonians (3.4) and the compatibility condition on the diagonal (3.3) for ε small enough. $\square$

4.2 The general case

Let us consider a slightly stronger assumption than (1.5), namely

$$\text{(4.8)} \qquad \begin{cases} H_i\in C^2(\mathbb{R}) \quad \text{with} \quad H_i''(p_i^0)>0,\\ H_i'<0 \quad \text{on} \quad (-\infty,p_i^0) \quad \text{and} \quad H_i'>0 \quad \text{on} \quad (p_i^0,+\infty),\\ \lim\limits_{|q|\to+\infty} H_i(q)=+\infty.\end{cases}$$

We will also use the following technical result which allows us to reduce certain non-convex Hamiltonians to convex Hamiltonians.

Lemma 4.4 (From non-convex to convex Hamiltonians)**.** *Given Hamiltonians H_i satisfying* (4.8) *and* (3.2)*, there exists a function $\beta:\mathbb{R}\to\mathbb{R}$ such that the functions $\beta\circ H_i$ satisfy* (4.1) *for $i=1,...,N$. Moreover, we can choose β such that*

$$\text{(4.9)} \qquad \beta \quad \textit{is convex}, \quad \beta\in C^2(\mathbb{R}), \quad \beta(0)=0 \quad \textit{and} \quad \beta'\geq\delta>0.$$

Proof. Recalling (4.2), it is easy to check that $(\beta\circ H_i)''>0$ if and only if we have

$$\text{(4.10)} \qquad (\ln\beta')'(\lambda)>-\frac{H_i''}{(H_i')^2}\circ\pi_i^{\pm}(\lambda) \quad \text{for} \quad \lambda\geq H_i(0).$$

Because $H_i''(0)>0$, we see that the right hand side is negative for λ close enough to $H_i(0)$. Then it is easy to choose a function β satisfying (4.10) and (4.9). Finally, compositing β with another convex increasing function which is superlinear at $+\infty$ if necessary, we can ensure that $\beta\circ H_i$ superlinear. $\square$

Lemma 4.5 (The case of smooth Hamiltonians)**.** *Theorem 3.2 holds true if the Hamiltonians satisfy* (4.8).

Proof. We assume that the Hamiltonians H_i satisfy (4.8). Thanks to Lemma 3.1, we can further assume that they satisfy (3.2). Let β be the function given by Lemma 4.4. If u solves (1.7) on $(0,T)\times J$, then u is also a viscosity solution of

$$\text{(4.11)} \qquad \begin{cases} \bar\beta(u_t)+\hat H_i(u_x)=0 & \text{for} \quad t\in(0,T) \quad \text{and} \quad x\in J_i^*,\\ \bar\beta(u_t)+\hat F_{\hat A}(u_x)=0 & \text{for} \quad t\in(0,T) \quad \text{and} \quad x=0\end{cases}$$

with $\hat{F}_{\hat{A}}$ constructed as F_A where H_i and A are replaced with $\hat{H}_i$ and $\hat{A}$ defined as follows

$$\hat{H}_i = \beta \circ H_i, \quad \hat{A} = \beta(A)$$

and $\bar{\beta}(\lambda) = -\beta(-\lambda)$. We can then apply Theorem 3.2 in the case of smooth convex Hamiltonians to construct a vertex test function $\hat{G}$ associated to problem (4.11) for every $\hat{\gamma} > 0$. This means that we have with $\hat{H}(x,p) = \beta(H(x,p))$,

$$\hat{H}(y, -G_y) \le \hat{H}(x, G_x) + \hat{\gamma}.$$

This implies

$$H(y, -G_y) \le \beta^{-1}(\beta(H(x,G_x)) + \hat{\gamma}) \le H(x, G_x) + \hat{\gamma}|(\beta^{-1})'|_{L^\infty(\mathbb{R})}.$$

Because of the lower bound on β' given by Lemma 4.4, we get $|(\beta^{-1})'|_{L^\infty(\mathbb{R})} \le 1/\delta$ which yields the compatibility condition (3.4) with $\gamma = \hat{\gamma}/\delta$ arbitrarily small. $\square$

We are now in position to prove Theorem 3.2 in the general case.

Proof of Theorem 3.2. Let us now assume that the Hamiltonians only satisfy (1.5). In this case, we simply approximate the Hamiltonians H_i by other Hamiltonians $\tilde{H}_i$ satisfying (4.8) such that

$$|H_i - \tilde{H}_i| \le \gamma.$$

We then apply Theorem 3.2 to the Hamiltonians $\tilde{H}_i$ and construct an associated vertex test function $\tilde{G}$ also for the parameter γ. We deduce that

$$H(y, -\tilde{G}_y) \le H(x, \tilde{G}_x) + 3\gamma$$

with $\gamma > 0$ arbitrarily small, which shows again the compatibility condition on the Hamiltonians (3.4) for the Hamiltonians H_i's. The proof is now complete in the general case. $\square$

Remark 4.6 (A variant in the proof of construction of G^0). When the Hamiltonians are not convex, it is also possible to use the function β from Lemma 4.4 in a different way by defining directly the function G^0 as follows

$$\tilde{G}^0(x,y) = \sup_{(p,\lambda)\in\mathcal{G}_A} \left(p_i x - p_j y - \beta(\lambda)\right).$$

4.3 A special function

In order to prove Proposition 4.1, we first need to study a special function $\mathfrak{G}$. Precisely, we define the following convex function for $z = (z_1, ..., z_N) \in \mathbb{R}^N$,

$$\mathfrak{G}(z) = \sup_{(p,\lambda)\in\mathcal{G}_A} (p \cdot z - \lambda).$$

We then consider the following subsets of $\mathbb{R}^N$,

$$Q_\sigma = \{z = (z_1, \dots, z_N) \in \mathbb{R}^N : \quad \sigma_i z_i \geq 0, \quad i = 1, \dots, N\}$$

$$\Delta_\sigma = \{z = (z_1, \dots, z_N) \in Q_\sigma : \sum_{i=1}^{N} \frac{\sigma_i z_i}{\bar{z}_i^\sigma(A)} \leq 1\}$$

where $\bar{z}_i^\sigma(A) = \sigma_i H_i'(\pi_i^{\sigma_i}(A)) \geq 0$ and the functions $\pi_i^\pm$ are defined in (4.2). We also make precise that we use the following convenient convention,

$$\frac{\bar{z}_i}{\bar{z}_i^\sigma(A)} = \begin{cases} 0 & \text{if} \quad \bar{z}_i = 0, \\ +\infty & \text{if} \quad \bar{z}_i > 0 \quad \text{and} \quad \bar{z}_i^\sigma(A) = 0. \end{cases} \tag{4.12}$$

Lemma 4.7 (The function $\mathfrak{G}$ in Q_σ)**.** *Under the assumptions of Proposition 4.1, we have, for any $\sigma \in \{\pm\}^N$ with $\sigma \neq (+, \dots, +)$ if $N \geq 2$:*

i) *$\mathfrak{G}$ is C^1 on Q_σ (up to the boundary).*

ii) *For all $z \in Q_\sigma$, there exists a unique $\lambda = \mathfrak{L}(z) \geq A$ such that*

$$\begin{aligned} \mathfrak{G}(z) &= p \cdot z - \lambda \\ \nabla \mathfrak{G}(z) &= p = (p_1, \dots, p_N) \\ p_i &= \pi_i^{\sigma_i}(\lambda) \end{aligned}$$

with $(p, \lambda) \in \mathcal{G}_A$.

iii) *For all $z \in Q_\sigma$, $\mathfrak{L}(z) = A$ if and only if $z \in \Delta_\sigma$. In particular, $\mathfrak{G}$ is linear in Δ_σ.*

Before giving global properties of $\mathfrak{G}$, we introduce the set

$$\bar{\Omega} = \begin{cases} \mathbb{R} & \text{if } N = 1, \\ \mathbb{R}^N \setminus (0, +\infty)^N & \text{if } N \geq 2. \end{cases} \tag{4.13}$$

Lemma 4.8 (Global properties of $\mathfrak{G}$ and $\mathfrak{L}$)**.** *Under the assumptions of Proposition 4.1, the function $\mathfrak{G}$ is convex and finite in $\mathbb{R}^N$, reaches its minimum $-A$ at 0 and the function $\mathfrak{L}$ is continuous in $\bar{\Omega}$.*

Proof of Lemmas 4.7 and 4.8. Let $\sigma \in \{\pm\}^N$ and $z \in Q_\sigma$. We set

$$\pi^\sigma(\lambda) = (\pi_1^{\sigma_1}(\lambda), ..., \pi_N^{\sigma_N}(\lambda)).$$

Using the fact that $\pi^\sigma(A) \in \mathcal{G}_A$, we get $\mathfrak{G}(z) \geq \mathfrak{G}(0) = -A$.

Step 1: Explicit expression of $\mathfrak{G}$. For $\sigma \neq (+, \dots, +)$ if $N \geq 2$, we have

$$(p, \lambda) \in \mathcal{G}_A \cap (Q_\sigma \times \mathbb{R}) \quad \Longleftrightarrow \quad \lambda \geq A \quad \text{and} \quad p = \pi^\sigma(\lambda). \tag{4.14}$$

This implies in particular that

$$\mathfrak{G}(z) = \sup_{\lambda \geq A} \, (z \cdot \pi^\sigma(\lambda) - \lambda). \tag{4.15}$$

Step 2: Optimization. Because of the superlinearity of the Hamiltonians H_i (see (4.1)), we have for $z \neq 0$,

$$\lim_{\lambda \to +\infty} f^\sigma(\lambda) = -\infty \quad \text{for} \quad f^\sigma(\lambda) := z \cdot \pi^\sigma(\lambda) - \lambda.$$

Therefore the supremum in (4.15) is reached for some $\lambda \in [A, +\infty)$, i.e.

$$\mathfrak{G}(z) = z \cdot \pi^\sigma(\lambda) - \lambda.$$

Then we have $\lambda = A$ or $\lambda > A$ and $(f^\sigma)'(\lambda) = 0$. Note that for $\lambda > A_0$, we can rewrite $(f^\sigma)'(\lambda) = 0$ as

$$\sum_{i=1,\ldots,N} \frac{\bar{z}_i}{\bar{z}_i^\sigma} = 1 \quad \text{with} \quad \begin{cases} \bar{z}_i = \sigma_i z_i \geq 0, \\ \\ \bar{z}_i^\sigma = \bar{z}_i^\sigma(\lambda) := \sigma_i H_i'(\pi_i^{\sigma_i}(\lambda)) > 0. \end{cases}$$

Moreover, we have

$$(\bar{z}_i^\sigma)'(\lambda) = \frac{H_i''(\pi_i^{\sigma_i}(\lambda))}{\sigma_i H_i'(\pi_i^{\sigma_i}(\lambda))} > 0$$

where the strict inequality follows from the strict convexity of Hamiltonians, see (4.1). Moreover, by definition of $\bar{z}_i^\sigma$, we have

$$\lim_{\lambda \to +\infty} \bar{z}_i^\sigma(\lambda) = +\infty$$

because H_i is convex and superlinear.

Step 3: Foliation and definition of $\mathfrak{L}$. Let us consider the sets

$$P^\sigma(\lambda) = \begin{cases} \left\{ \bar{z} \in [0, +\infty)^N, \quad \displaystyle\sum_{i=1,\ldots,N} \frac{\bar{z}_i}{\bar{z}_i^\sigma(\lambda)} = 1 \right\} & \text{if} \quad \lambda > A, \\ \\ \left\{ \bar{z} \in [0, +\infty)^N, \displaystyle\sum_{i=1,\ldots,N} \frac{\bar{z}_i}{\bar{z}_i^\sigma(A)} \leq 1 \right\} & \text{if} \quad \lambda = A \end{cases} \tag{4.16}$$

(keeping in mind convention (4.12)). Because for $\lambda > A$, the intersection points of the piece of hyperplane $P(\lambda)$ with each axis $\mathbb{R}e_i$ are $\bar{z}_i^0(\lambda)e_i$, we deduce that we can write the partition

$$[0, +\infty)^N = \bigcup_{\lambda \geq A} P^\sigma(\lambda)$$

where $P^\sigma(\lambda)$ gives a foliation by hyperplanes for $\lambda > A$. Then we can define for $z \in Q_\sigma$,

$$\mathfrak{L}^\sigma(z) = \{\lambda \quad \text{such that} \quad \bar{z} \in P^\sigma(\lambda) \quad \text{for} \quad \bar{z}_i = \sigma_i z_i \quad \text{for} \quad i = 1, \ldots, N\}.$$

From our definition, we get that the function $\mathfrak{L}^\sigma$ is continuous on Q_σ and satisfies $\mathfrak{L}^\sigma(0) = A$. For $z \in Q_\sigma$ such that $z_{i_0} = 0$, we see from the definition of P^σ given in (4.16) that the value of $\mathfrak{L}^\sigma(z)$ does not depend on the value of σ_{i_0}. Therefore we can glue up all the $\mathfrak{L}^\sigma$ in a single continuous function $\mathfrak{L}$ defined for $z \in \bar{\Omega}$ by

$$\mathfrak{L}(z) = \mathfrak{L}^\sigma(z) \quad \text{if} \quad z \in Q_\sigma.$$

which satisfies $\mathfrak{L}(0) = A$.

Step 4: Regularity of $\mathfrak{G}$ and computation of the gradients. For $z \in Q_\sigma \subset \bar{O}mega$, we have

$$\mathfrak{G}(z) = \sup_{\lambda \geq A} (z \cdot \pi^\sigma(\lambda) - \lambda)$$

where the supremum is reached only for $\lambda = \mathfrak{L}(z)$. Moreover $\mathfrak{G}$ is convex in $\mathbb{R}^N$. We just showed that the subdifferential of $\mathfrak{G}$ on the interior of Q_σ is the singleton $\{\pi^\sigma(\lambda)\}$ with $\lambda = \mathfrak{L}(z)$. This implies that $\mathfrak{G}$ is derivable on the interior of Q_σ and

$$\nabla \mathfrak{G}(z) = \pi^\sigma(\lambda) \quad \text{with} \quad \lambda = \mathfrak{L}(z).$$

The fact that the maps π^σ and $\mathfrak{L}$ are continuous implies that $\mathfrak{G}_{|Q_\sigma}$ is C^1. □

4.4 Proof of Proposition 4.1

We now turn to the proof of Proposition 4.1.

Proof of Proposition 4.1. By definition of G^0, we have

$$G^0(x,y) = \mathfrak{G}(Z(x,y)) \quad \text{with} \quad Z(x,y) := xe_i - ye_j \in \bar{\Omega} \quad \text{if} \quad (x,y) \in J_i \times J_j$$

where $(e_1, ..., e_N)$ is the canonical basis of $\mathbb{R}^N$ and $\bar{\Omega}$ is defined in (4.13).

Step 1: Regularity. Then Lemmas 4.7 and 4.8 imply immediately that $G^0 \in C(J^2)$ and $G^0 \in C^1(R)$ for each region R given by

$$R = \begin{cases} J_i \times J_j & \text{if} \quad i \neq j, \\ T_i^\pm = \{(x,y) \in J_i \times J_i, \quad \pm(x-y) \geq 0\} & \text{if} \quad i = j. \end{cases} \tag{4.17}$$

This regularity of $\mathfrak{G}$ implies in particular the regularity of G^0 given in i).

Step 2: Computation of the gradients. We also deduce from Lemma 4.8 that

$$\Lambda(x,y) := \mathfrak{L}(Z(x,y))$$

defines a continuous map $\Lambda : J^2 \to [A,+\infty)$ which satisfies

$$\Lambda(x,x) = A \tag{4.18}$$

because of Lemma 4.7-iii) and $Z(x,x) = 0$. Moreover, for each R given by (4.17) and for all $(x,y) \in R \subset J_i \times J_j$ we have

$$G^0(x,y) = p_i x - p_j y - \lambda$$

and

$$(G^0_{|R})_x(x,y) = p_i \quad \text{and} \quad (G^0_{|R})_y(x,y) = -p_j$$

with $\lambda = \Lambda(x,y)$ and $(p,\lambda) \in \mathcal{G}_A$ and

$$(p_i, p_j) = \begin{cases} (\pi_i^+(\lambda), \pi_j^-(\lambda)) & \text{if} \quad R = J_i \times J_j \quad & \text{with} \quad i \neq j, \\ (\pi_i^\pm(\lambda), \pi_i^\pm(\lambda)) & \text{if} \quad R = T_i^\pm & \text{with} \quad i = j. \end{cases} \tag{4.19}$$

Step 3: Checking the compatibility condition on the gradients. Let us consider $(x,y) \in J^2$ with $x = y = 0$ or $x \neq y$. We have

$$(\partial_i G^0(\cdot,y))(x) \in \left\{\pi_i^\pm(\lambda)\right\} \quad \text{and} \quad -(\partial_j G^0(x,\cdot))(y) \in \left\{\pi_j^\pm(\lambda)\right\} \quad \text{with} \quad \lambda = \Lambda(x,y) \geq A.$$

We claim that

$$H(x, G^0_x(x,y)) = \lambda. \tag{4.20}$$

It is clear except in the special case where

$$x = 0 \quad \text{and} \quad (\partial_i G^0(\cdot,y))(0) = \pi_i^+(\lambda) \quad \text{for all} \quad i = 1,...,N \tag{4.21}$$

If $0 \neq y \in J_j$, then $(x,y) = (0,y) \in T_j^-$ and $(\partial_j G^0(\cdot,y))(0) = \pi_j^-(\lambda)$. Therefore (4.21) only happens if $y = 0$ and then

$$H(0, G^0_x(0,0)) = A$$

which still implies (4.20), because $\lambda = \Lambda(0,0) = A$.

In view of (4.20), (3.4) with equality and $\gamma = 0$ is equivalent to

$$H(y, -G^0_y(x,y)) = \lambda. \tag{4.22}$$

This is clear except possibly in the special case where

$$y = 0 \quad \text{and} \quad -(\partial_j G^0(x,\cdot))(0) = \pi_j^+(\lambda) \quad \text{for all} \quad j = 1,...,N. \tag{4.23}$$

If $x \in J_i$ and $N \geq 2$, then we can find $j \neq i$ such that $-(\partial_j G^0(x,\cdot))(0) = \pi_j^-(\lambda)$. Therefore (4.23) only happens if $N = 1$ and then

$$H(0, -G^0_y(x,0)) = A \leq \lambda.$$

Step 4: Superlinearity. In view of the definition of G^0, we deduce from (4.19) that

$$G^0(x,y) \geq \begin{cases} x\pi_i^+(\lambda) - y\pi_j^-(\lambda) - \lambda & \text{if} \quad i \neq j, \\ (x-y)\pi_i^\pm(\lambda) - \lambda & \text{if} \quad i = j \quad \text{and} \quad \pm(x-y) \geq 0 \end{cases}$$

Setting

$$\pi^0(\lambda) := \min_{\pm,\ i=1,\dots,N} \pm\pi_i^\pm(\lambda) \geq 0,$$

we get

$$G^0(x,y) \geq d(x,y)\pi^0(\lambda) - \lambda.$$

From the definition (4.2) of $\pi_i^\pm$ and the assumption (4.1) on the Hamiltonians, we deduce that

$$\pi^0(\lambda) \to +\infty \quad \text{as} \quad \lambda \to +\infty$$

which implies that for any $K \geq 0$, there exists a constant $C_K \geq 0$ such that

$$G^0(x,y) \geq Kd(x,y) - C_K.$$

Therefore we get (3.5) with

$$g^0(a) = \sup_{K\geq 0}(Ka - C_K).$$

Step 5: Gradient bounds. Note that

$$\sum_{i=1,\dots,N} |Z_i(x,y)| = d(x,y).$$

Because each component of the gradients of G^0 are equal to one of the $\left\{\pi_k^\pm(\lambda)\right\}_{\pm,k=1,\dots,N}$ with $\lambda = \mathfrak{L}(Z(x,y))$, we deduce (3.6) from the continuity of $\mathfrak{L}$ and of the maps $\pi_k^\pm$.

Step 6: Saturation close to the diagonal. Point vi) in Proposition 4.1 follows from Lemma 4.7-iii), from the definition of $\mathfrak{G}$ and from the regularity of G^0. □

5 First application: link with optimal control theory

This section is devoted to the study of the value function of an optimal control problem associated with trajectories running over the junction.

5.1 An optimal control problem

As before, we consider a junction $J = \bigcup_{i=1,\dots,N} J_i$. We consider compact metric spaces $\mathbb{A}_i$ and functions $b_i, \ell_i : \mathbb{A}_i \to \mathbb{R}$ for $i = 0, \dots, N$. The sets $\mathbb{A}_i$ are the sets of controls on each branch J_i^* for $i = 1, \dots, N$, while the set $\mathbb{A}_0$ is the set of controls at the junction point $x = 0$. The functions b_i represent the dynamics and the ℓ_i are the running cost functions.

To keep the presentation simple, we do not consider more general cases where the dynamics and the running cost functions could depend on (t,x). We then define the general set of controls:

$$\mathbb{A} = \mathbb{A}_0 \times \cdots \times \mathbb{A}_N$$

and define for $\alpha = (\alpha_0, \ldots, \alpha_N) \in \mathbb{A}$ and $x \in J$,

$$b(x,\alpha) = \begin{cases} b_i(\alpha_i) & \text{if} \quad x \in J_i^*, \\ b_0(\alpha_0) & \text{if} \quad x = 0. \end{cases}$$

Similarly, we define

$$\ell(x,\alpha) = \begin{cases} \ell_i(\alpha_i) & \text{if} \quad x \in J_i^*, \\ \ell_0(\alpha_0) & \text{if} \quad x = 0. \end{cases}$$

We then define the set of admissible dynamics

$$\mathcal{T}_{t,x} = \left\{ \begin{array}{l} (X(\cdot),\alpha(\cdot)) \in \mathrm{Lip}(0,t;J) \times L^\infty(0,t;\mathbb{A}), \\ \begin{cases} X(t) = x, \\ \dot{X}(s) = b(X(s),\alpha(s)) \quad \text{for a.e.} \quad s \in (0,t) \end{cases} \end{array} \right\}. \tag{5.1}$$

For $i = 0, \ldots, N$, we assume the following

$$\begin{cases} b_i \text{ and } \ell_i \text{ are continuous and bounded} \\ \text{for } i \neq 0, \quad \{(b_i(\alpha_i), \ell_i(\alpha_i)) : \alpha_i \in \mathbb{A}_i\} \text{ is closed and convex} \\ \text{for } i \neq 0, \quad B_i = \{b_i(\alpha_i) : \alpha_i \in \mathbb{A}_i\} \text{ contains } [-\delta, \delta]. \end{cases} \tag{5.2}$$

Then we consider the value function of the optimal control problem,

$$u(t,x) = \inf_{(X(\cdot),\alpha(\cdot)) \in \mathcal{T}_{t,x}} E(X,\alpha) \tag{5.3}$$

with

$$E(X,\alpha) = u_0(X(0)) + \int_0^t \ell(X(s),\alpha(s))\, ds$$

where the initial data u_0 is assumed to be globally Lispschitz continuous on J. We define for $i = 1, \ldots, N$ and $p_i \in \mathbb{R}$,

$$H_i(p_i) = \sup_{\alpha_i \in \mathbb{A}_i} \left(b_i(\alpha_i) p_i - \ell_i(\alpha_i) \right).$$

It is easy to check that the Hamiltonians H_i satisfy Assumption (1.5). For $i = 0$, we define

$$H_0 = \begin{cases} \sup_{\alpha_0 \in \mathbb{A}_{00}} \left(-\ell_0(\alpha_0) \right) & \text{if} \quad \mathbb{A}_{00} \neq \emptyset, \\ -\infty & \text{if} \quad \mathbb{A}_{00} = \emptyset \end{cases}$$

with

$$\mathbb{A}_{00} = \left\{ \alpha_0 \in \mathbb{A}_0, \quad b_0(\alpha_0) = 0 \right\}.$$

5.2 A reduced representation formula

Proposition 5.1 (A reduced representation formula)**.** *Let L_i denote the Legendre-Fenchel transform of H_i. Then*

$$u(t,x) = \inf_{X(\cdot)\in S_{t,x}(0,t)} \left\{ u_0(X(0)) + \int_0^t L(X(s),\dot{X}(s))\, ds \right\} \tag{5.4}$$

with

$$S_{t,x}(0,t) = \{X(\cdot) \in Lip(t_0,t;J), \quad \textit{with} \quad X(t)=x\}$$

and

$$L(x,p) = \begin{cases} L_i(p) & \textit{if} \quad x \in J_i^*, \\ \min\left(-H_0, \min_{i=1,\dots,N} (L_i(p))\right) & \textit{if} \quad x=0. \end{cases}$$

Moreover,

- *the infimum in* (5.4) *is a minimum and there exist optimal trajectories that are straight lines in each open branch J_i^*;*
- *the function u is continuous.*

Proof. The proof proceeds in several steps.

Step 1: reduced controls. Consider $i \neq 0$. By assumption, the set $\{(b_i(\alpha_i), \ell_i(\alpha_i)) : \alpha_i \in \mathbb{A}_i\}$ is convex. In particular, for each $\alpha_i \in \mathbb{A}_i$, the set

$$I(b_i(\alpha_i)) = \{(\ell_i(\beta_i)) : \beta_i \in \mathbb{A}_i, b_i(\beta_i) = b_i(\alpha_i)\}$$

is a compact interval. Hence, for all $\alpha_i \in \mathbb{A}_i$, there exists $\bar{\alpha}_i \in \mathbb{A}_i$ such that

$$b_i(\bar{\alpha}_i) = b_i(\alpha_i) \quad \text{and} \quad \ell_i(\bar{\alpha}_i) = \min_{l\in I(\alpha_i)} l.$$

In other words,

$$\forall \beta_i \in \mathbb{A}_i, \quad b_i(\beta_i) = b_i(\alpha_i) \Rightarrow \ell_i(\beta_i) \geq \ell_i(\bar{\alpha}_i).$$

In particular,

$$\ell_i(\bar{\alpha}_i) \leq \ell_i(\alpha_i). \tag{5.5}$$

This implies that

$$\begin{aligned} H_i(p_i) &= \sup_{\alpha_i\in\mathbb{A}_i} (b_i(\alpha_i)p_i - \ell_i(\alpha_i)) \\ &= \sup_{c_i\in B_i} (c_i p_i - \ell_i(\bar{\alpha}_i)). \end{aligned}$$

This means that

$$L_i(c_i) = \begin{cases} \ell_i(\bar{\alpha}_i) & \text{if } c_i \in B_i \\ +\infty & \text{if not.} \end{cases} \tag{5.6}$$

Similarly, for $i = 0$, if $\mathbb{A}_{00} \neq \emptyset$, then there exists $\bar{\alpha}_0 \in \mathbb{A}_{00}$ such that

$$0 = b_0(\bar{\alpha}_0) \quad \text{and} \quad \ell_0(\bar{\alpha}_0) = -H_0 \leq \ell_0(\alpha_0) \quad \text{for all} \quad \alpha_0 \in \mathbb{A}_{00}.$$

If now $\mathbb{A}_{00} = \emptyset$, we simply choose $\bar{\alpha}_0 = \alpha_0$.

Step 2: equivalent dynamics. Now let us consider a trajectory $(X(\cdot), \alpha(\cdot)) \in \mathcal{T}_{t,x}$, and let us define

$$\mathbb{T}_0 = \{s \in (0,t), \quad X(s) = 0\}.$$

In particular by Stampacchia theorem, we have

$$0 = b_0(\alpha_0(s)) \quad \text{for a.e.} \quad s \in \mathbb{T}_0,$$

i.e.

$$\alpha_0(s) \in \mathbb{A}_{00} \quad \text{for a.e.} \quad s \in \mathbb{T}_0. \tag{5.7}$$

Case $\mathbb{A}_{00} \neq \emptyset$. Then we define the new control $\bar{\alpha}(s) = (\bar{\alpha}_0(s), \dots, \bar{\alpha}_N(s)) \in \mathbb{A}$ as above for which we have

$$\dot{X}(s) = b(X(s), \bar{\alpha}(s)) \quad \text{for a.e.} \quad s \in (0,t).$$

This shows that $(X(\cdot), \bar{\alpha}(\cdot)) \in \mathcal{T}_{t,x}$. Moreover, (5.5) and (5.6) imply that

$$l(X(s), \alpha(s)) \geq l(X(s), \bar{\alpha}(s)) = \tilde{L}(X(s), \dot{X}(s))$$

with

$$\tilde{L}(x,p) = \begin{cases} L_i(p) & \text{if} \quad x \in J_i^*, \\ -H_0 & \text{if} \quad x = 0. \end{cases}$$

This implies that

$$u(t,x) = \inf_{X(\cdot) \in S_{t,x}(0,t)} \left\{ u_0(X(0)) + \int_0^t \tilde{L}(X(s), \dot{X}(s))\, ds \right\}. \tag{5.8}$$

We remark that $\tilde{L}$ may not be lower semi-continuous at $x = 0$. But L is lower semi-continuous and lies below $\tilde{L}$, so we get

$$u(t,x) \geq V(t,x) \tag{5.9}$$

with

$$V(t,x) = \inf_{X(\cdot) \in S_{t,x}(0,t)} \left\{ u_0(X(0)) + \int_0^t L(X(s), \dot{X}(s))\, ds \right\}.$$

Case $\mathbb{A}_{00} = \emptyset$. Then from (5.7), we see that the Lebesgue measure of $\mathbb{T}_0$ is zero. Hence, (5.8) and (5.9) still hold true.

Step 3: Minimizing sequences. Given $(t,x) \in (0,+\infty) \times J$, let us consider a minimizing sequence $(X^\varepsilon(\cdot), \alpha^\varepsilon(\cdot)) \in \mathcal{T}_{t,x}$. By replacing α^ε with $\bar{\alpha}^\varepsilon$, we can assume that

$$u(t,x) + \varepsilon \geq E(X^\varepsilon, \alpha^\varepsilon) = u_0(X^\varepsilon(0)) + \int_0^t \tilde{L}(X^\varepsilon(s), \dot{X}^\varepsilon(s))\ ds \geq u(t,x).$$

If there exists an interval (t_a, t_b) such that

$$X^\varepsilon(s) \in J_i^* \quad \text{for all} \quad s \in (t_a, t_b),$$

then using the convexity of L_i, we deduce that

$$\int_{t_a}^{t_b} L_i(\dot{X}^\varepsilon(s))\ ds \geq \int_{t_a}^{t_b} L_i(\dot{\tilde{X}}^\varepsilon(s))\ ds$$

with

$$\tilde{X}^\varepsilon(s) = \left(\frac{t_b - s}{t_b - t_a}\right) X^\varepsilon(t_a) + \left(\frac{s - t_a}{t_b - t_a}\right) X^\varepsilon(t_b)$$

and

$$\dot{\tilde{X}}^\varepsilon(s) \in B_i$$

because B_i is a compact interval. Considering maximal intervals of type (t_a, t_b), we see that we can replace X^ε by a new curve $\tilde{X}^\varepsilon$ which is particularly simple, because it is a straight line in each open branch J_i^* and such that

$$E(X^\varepsilon, \alpha^\varepsilon) \geq u_0(\tilde{X}^\varepsilon(0)) + \int_0^t L(\tilde{X}^\varepsilon(s), \dot{\tilde{X}}^\varepsilon(s))\ ds.$$

By assumption, X^ε is uniformly Lipschitz continuous, and then $\tilde{X}^\varepsilon$ is also uniformly Lipschitz continuous, and by Ascoli-Arzelà theorem, there exists a subsequence such that $\tilde{X}^\varepsilon$ converges uniformly on $[0,t]$ to X^0. The limit curve X^0 is also simple, i.e. it is a straight line on each open branch J_i^* with $\dot{X}^0 \in B_i$ when $X^0 \in J_i^*$. Passing to the limit, we easily deduce (from the lower semi-continuity of L) that

$$u(t,x) \geq u_0(X^0(0)) + \int_0^t L(X^0(s), \dot{X}^0(s))\ ds.$$

Step 3: Reverse inequality. We now want to prove that we have the reverse inequality

$$u(t,x) \leq u_0(X^0(0)) + \int_0^t L(X^0(s), \dot{X}^0(s))\ ds \tag{5.10}$$

We distinguish two cases.

Case $L(0,0) = -H_0 < \min_{i=1,\dots,N} L_i(0)$. In this case, we simply use the fact that $L(X^0(s), \dot{X}^0(s)) = \tilde{L}(X^0(s), \dot{X}^0(s)) = l(X^0(s), \alpha^0(s))$ and $(X^0(\cdot), \alpha^0(\cdot)) \in \mathcal{T}_{t,x}$. This implies (5.10).

Case $L(0,0) = L_{i_0}(0) < -H_0$ for some $i_0 \in \{1, \dots, N\}$. The only interesting case is when there is a nonempty maximal interval (t_a, t_b) such that $X^0(s) = 0$ for $s \in (t_a, t_b)$ (otherwise we can apply the reasoning of the previous case). By continuity, we have $X^0(t_a) = 0 = X^0(t_b)$.Then for every $\varepsilon > 0$ small enough, we can find three controls $\alpha^a_{i_0}, \alpha^b_{i_0}, \alpha^c_{i_0} \in \mathbb{A}_{i_0}$ such that

$$b_{i_0}(\alpha^a_{i_0}) > 0, \quad b_{i_0}(\alpha^b_{i_0}) < 0$$

and

$$X^\varepsilon(s) = \begin{cases} X^0(t_a) + (s - t_a) b_{i_0}(\alpha^a_{i_0}) & \text{for} \quad s \in (t_a, t^\varepsilon_a], \quad \text{with} \quad t^\varepsilon_a = t_a + \varepsilon \\ X^0(t_b) + (s - t_b) b_{i_0}(\alpha^b_{i_0}) & \text{for} \quad s \in [t^\varepsilon_b, t_b), \quad \text{with} \quad t^\varepsilon_b = t_b - \varepsilon \\ \left(\frac{t^\varepsilon_b - s}{t^\varepsilon_b - t^\varepsilon_a}\right) X^\varepsilon(t^\varepsilon_a) + \left(\frac{s - t^\varepsilon_a}{t^\varepsilon_b - t^\varepsilon_a}\right) X^\varepsilon(t^\varepsilon_b), & \text{with} \quad b_{i_0}(\alpha^c_{i_0}) = \dot{X}^\varepsilon(s) \to 0 \end{cases}$$

This implies that there exists a modified controlled tractory $(X^\varepsilon(\cdot), \alpha^\varepsilon(\cdot)) \in \mathcal{T}_{t,x}$ such that

$$u(t,x) \le E(X^\varepsilon, \alpha^\varepsilon) \to u_0(X^0(0)) + \int_0^t L(X^0(s), \dot{X}^0(s))\, ds$$

which implies (5.10).

Step 4: Conclusion. It remains to justify that u is continuous. This is a consequence of the Lipschitz continuity of u_0 and the fact that L is bounded (where it is finite). The proof is now complete. □

5.3 Derivation of the Hamilton-Jacobi-Bellman equation

Theorem 5.2 (The value function is an F_A-solution)**.** *The value function u defined by* (5.3) *is the unique solution of* (1.7) *with $A = H_0$ supplemented with the initial condition* (1.4)*.*

Proof. Deriving the Hamilton-Jacobi-Bellman equation outside the junction point is quite standard. This is the reason why we will focus on the junction condition. As in the standard case, it relies on the dynamic programming principle.

Step 1: Dynamic programming principle. For $t_0 < t$, we set,

$$S_{t,x}(t_0, t) = \{X(\cdot) \in \mathrm{Lip}(t_0, t; J), \quad \text{with} \quad X(t) = x\}.$$

For $t > 0$ and $h \in (0, t)$, we can easily show, splitting the interval $(0,t)$ into $(0, t-h)$ and $(t-h, t)$ that we have the following dynamic programming principle,

$$u(t,x) = \inf_{X(\cdot) \in S_{t,x}(t-h,t)} \left\{ u(t-h, X(t-h)) + \int_{t-h}^t L(X(s), \dot{X}(s))\, ds \right\}. \tag{5.11}$$

In particular, if $X \in S_{t,x}(t-h,t)$ is optimal,

$$u(t,x) = u(t-h, X(t-h)) + \int_{t-h}^{t} L(X(s), \dot{X}(s))\, ds,$$

then, for $s \in (t-h,t)$, splitting the interval $(t-h,t)$ into $(t-h,s)$ and (s,t), we get

$$u(t,x) = u(s, X(s)) + \int_{s}^{t} L(X(s), \dot{X}(s))\, ds \quad \text{for all} \quad s \in (t-h,t). \tag{5.12}$$

Step 2: u is a super-solution at the junction point. Consider any test function φ such that

$$\varphi \le u \text{ in } (0,+\infty) \times J \quad \text{and} \quad \varphi = u \text{ at } (t,0).$$

Let us consider an optimal trajectory X which is a straight line in each open branch J_i^*. Then there exists a small interval $(t-\varepsilon,t)$ such that for all $s \in (t-\varepsilon,t)$,

$$\begin{cases} \text{either} & \dot{X}(s) = q,\ X(s) \in J_i^* \quad \text{for some } (q,i) \in (-\infty,0) \times \{1,\dots,N\}, \\ \text{or} & X(s) = 0. \end{cases} \tag{5.13}$$

In the first case, we have for $s \in (t-\varepsilon,t)$,

$$L(X(s), \dot{X}(s)) = L_i(q).$$

Applying the super-optimality equality (5.12), we deduce that

$$\varphi(t, X(t)) = u(t,0) = u(s, X(s)) + \int_s^t L(X(\tau), \dot{X}(\tau))\, d\tau \ge \varphi(s, X(s)) + \int_s^t L_i(q)\, d\tau.$$

This implies that

$$\frac{\varphi(t,X(t)) - \varphi(s,X(s))}{t-s} - L_i(q) \ge 0.$$

Passing to the limit $s \to t^-$, we deduce that at $(t,0)$,

$$\varphi_t + q\partial_i \varphi - L_i(q) \ge 0 \quad \text{and} \quad q \le 0.$$

This implies that

$$\varphi_t(t,0) + F_A(\varphi_x(t,0)) \ge 0$$

with $A = H_0$.

In the second case, we have

$$L(X(s), \dot{X}(s)) = -H_0.$$

Arguing as above, we deduce

$$\varphi_t + H_0 \ge 0$$

which implies again

$$\varphi_t(t,0) + F_A(\varphi_x(t,0)) \ge 0$$

with $A = H_0$.

Step 3: u is a sub-solution at the junction point. Consider any test function φ such that

$$\varphi \geq u \text{ in } (0,+\infty)\times J \quad \text{and} \quad \varphi = u \text{ at } (t,0).$$

We us fix $i = 1,\dots,N$ and consider the special trajectory for $s \in (t-\varepsilon, t)$,

$$X^q(s) = X(t) + (s-t)q \in J_i$$

for some arbitrary $q < 0$. We get in particular from the dynamic programming principle that

$$\begin{aligned}\varphi(t,X(t)) = u(t,0) &\leq u(s,X^q(s)) + \int_s^t L(X^q(\tau),q)\,d\tau \\ &\leq \varphi(s,X^q(s)) + \int_s^t L_i(q)\,d\tau.\end{aligned}$$

This implies

$$\frac{\varphi(t,X(t)) - \varphi(s,X(s))}{t-s} - L_i(q) \leq 0.$$

Passing to the limit $s \to t^-$, we deduce that at $(t,0)$,

$$\varphi_t + q\partial_i\varphi - L_i(q) \leq 0.$$

Now choose $q = 0$ i.e. $X(s) = X(t)$ for $s \in (t-\varepsilon,t)$. Arguing as above we get at $(t,0)$,

$$\varphi_t + H_0 \leq 0.$$

Since $i \in \{1,\dots,N\}$ and $q < 0$ are arbitrary, we get at $(t,0)$,

$$\varphi_t + F_A(\varphi_x) \leq 0$$

with $A = H_0$. This is the viscosity sub-solution inequality at the junction point.The proof is now complete. □

6 Second application: link with regional control

In this section, we shed light on the consequence of our results on the interpretation of the results from [7] when both frameworks coincide. Roughly speaking, the one-dimensional framework from [7] reduces to our framework with two branches. In this case, the value function U^- defined by (2.7) in [7] (see also (6.4) in the present paper) and characterized in Theorem 4.4 in [7] corresponds to the unique solution of (1.7) for some $A = H_T$ which is exhibited below. Similarly, the function U^+ defined by (2.8) in [7] (see also (6.5) in the present paper) corresponds to the unique solution of (1.7) for some $A = H_T^{reg}$. This is shown in the first subsection. In the other subsections, we compute H_T and H_T^{reg} and provide a general relation between (reformulated) F_A-solutions and Ishii solutions.

6.1 The framework

The one dimensional framework of [7] corresponds to

$$\Omega_1 = (-\infty, 0), \quad \mathcal{H} = \{0\}, \quad \Omega_2 = (0, +\infty).$$

In this case, $(\mathbf{H}_\Omega)$ in [7] is satisfied. We refer to this framework as *the common framework*.

Hamiltonians. As far as the Hamiltonian is concerned, the (t, x)-dependence is not relevant for what we discuss now; for this reason we consider the simplified case of convex Hamiltonians given for $i = 1, 2$ by

$$H_i(p) = \sup_{\alpha_i \in A_i} (-b_i(\alpha_i)p - \ell_i(\alpha_i))$$

for some compact metric space A_i and $b_i, \ell_i : A_i \to \mathbb{R}$. In this simplified framework, $(\mathbf{H}_C)$ reduces to the following assumptions for $i = 1, 2$:

$$\left\{ \begin{array}{l} b_i \text{ and } \ell_i \text{ are continuous and bounded} \\ \{(b_i(\alpha_i), \ell_i(\alpha_i)) : \alpha_i \in A_i\} \text{ is closed and convex} \\ B_i = \{b_i(\alpha_i) : \alpha_i \in A_i\} \text{ contains } [-\delta, \delta]. \end{array} \right. \tag{6.1}$$

In particular, we see that B_i is a compact interval. Introducing the Legendre-Fenchel transform L_i of H_i, it is possible to see that this problem can be reformulated by assuming that for $i = 1, 2$

$$H_i(p) = \sup_{q \in B_i} (qp - L_i(q))$$

where $L_i : B_i \to \mathbb{R}$ is convex where we recall that B_i is a compact interval containing $[-\delta, \delta]$. Indeed the graph of L_i on B_i is the lower boundary of the closed convex set $\{(b_i(\alpha_i), \ell_i(\alpha_i)) : \alpha_i \in A_i\}$ in the plane $\mathbb{R}^2$. In particular, we see that H_i is convex, Lipschitz continuous and $H_i(p) \to +\infty$ as $|p| \to +\infty$. This last fact comes from the fact that $\pm\delta \in B_i$. Moreover H_i reaches its minimum at any convex subgradient p_i^0 of L_i at 0 and satisfies

$$\left\{ \begin{array}{lll} H_i & \text{is non-increasing on} & (-\infty, p_i^0], \\ H_i & \text{is non-decreasing on} & [p_i^0, +\infty). \end{array} \right.$$

Hence, H_i satisfies (1.5).

Translation of F_A-solutions in the real line setting. The notion of solutions $\tilde{u}(t, x)$ from Section 2 on two branches $J_1 \cup J_2$ with two Hamiltonians

$$\tilde{H}_1(q) = H_1(-q) \quad \text{and} \quad \tilde{H}_2(q) = H_2(q)$$

is translated in the common framework to functions u defined for $(t, x) \in [0, +\infty) \times \mathbb{R}$ by

$$u(t, x) = \left\{ \begin{array}{ll} \tilde{u}(t, x) & \text{for} \quad 0 \le x \in J_2, \\ \tilde{u}(t, -x) & \text{for} \quad 0 \le -x \in J_1. \end{array} \right.$$

Then $\tilde{u}$ solves (1.7) with Hamiltonians $\tilde{H}_i$, if and only if u solves

$$(6.2)\qquad \begin{cases} u_t + H_1(u_x) = 0 & \text{for} \quad (t,x) \in (0,+\infty)\times(-\infty,0),\\ u_t + H_2(u_x) = 0 & \text{for} \quad (t,x) \in (0,+\infty)\times(0,+\infty),\\ u_t + \check{F}_A(u_x(t,0^-),u_x(t,0^+)) = 0 & \text{for} \quad (t,x) \in (0,+\infty)\times\{0\} \end{cases}$$

with

$$\check{F}_A(q_1,q_2) = \max(A, H_1^+(q_1), H_2^-(q_2))$$

where

$$H_i^-(q) = \begin{cases} H_i(q) & \text{if} \quad q < p_i^0,\\ H_i(p_i^0) & \text{if} \quad q \le p_i^0 \end{cases} \quad \text{and} \quad H_i^+(q) = \begin{cases} H_i(p_i^0) & \text{if} \quad q \le p_i^0,\\ H_i(q) & \text{if} \quad q > p_i^0. \end{cases}$$

Viscosity inequalities are now naturally written by considering, for u, test functions $\phi : [0,+\infty)\times\mathbb{R} \to \mathbb{R}$ that are continuous, and C^1 in $[0,+\infty)\times(-\infty,0]$ and in $[0,+\infty)\times[0,+\infty)$.

Ishii solutions on the real line. In [7], Ishii solutions are considered. A function u is said to be a Ishii solution if it solves

$$(6.3)\qquad \begin{cases} u_t + H_1(u_x) = 0 & \text{for} \quad (t,x) \in (0,+\infty)\times(-\infty,0),\\ u_t + H_2(u_x) = 0 & \text{for} \quad (t,x) \in (0,+\infty)\times(0,+\infty),\\ u_t + \min(H_1(u_x),H_2(u_x)) \le 0 & \text{for} \quad (t,x) \in (0,+\infty)\times\{0\},\\ u_t + \max(H_1(u_x),H_2(u_x)) \ge 0 & \text{for} \quad (t,x) \in (0,+\infty)\times\{0\}. \end{cases}$$

For such solutions, test functions are C^1 in space and time, and not piecewise C^1; moreover, if the test function ϕ touches u^* from above at $(t,0)$ (resp. u_* from below at $(t,0)$) then

$$\partial_t\phi(t,0) + \min(H_1(\partial_x\phi(t,0)), H_2(\partial_x\phi(t,0))) \le 0$$

$$(\text{resp. } \partial_t\phi(t,0) + \max(H_1(\partial_x\phi(t,0)), H_2(\partial_x\phi(t,0))) \ge 0).$$

Tangential Hamiltonians. Using notation similar to the one of [7], we define

$$\begin{aligned} \mathcal{A} &= B_1 \times B_2 \times [0,1],\\ \mathcal{A}_0 &= \{(\alpha_1,\alpha_2,\mu) \in \mathcal{A} : \alpha_1\alpha_2 \le 0 \text{ and } 0 = \mu\alpha_1 + (1-\mu)\alpha_2\},\\ \mathcal{A}_0^{\mathrm{reg}} &= \{(\alpha_1,\alpha_2,\mu) \in \mathcal{A} : \alpha_1 \le 0, \alpha_2 \ge 0 \text{ and } 0 = \mu\alpha_1 + (1-\mu)\alpha_2\}. \end{aligned}$$

In the common framework, the tangential Hamiltonians H_T and H_T^{reg} given in [7] reduce to constants, and we can see that we can write them as follows

$$\begin{aligned} H_T &= \sup_{(\alpha_1,\alpha_2,\mu)\in\mathcal{A}_0} (-\mu L_1(\alpha_1) - (1-\mu)L_2(\alpha_2)),\\ H_T^{\mathrm{reg}} &= \sup_{(\alpha_1,\alpha_2,\mu)\in\mathcal{A}_0^{\mathrm{reg}}} (-\mu L_1(\alpha_1) - (1-\mu)L_2(\alpha_2)). \end{aligned}$$

The value functions U^- and U^+. We consider the following initial condition

$$u(0,x) = g(x) \quad \text{for} \quad x \in \mathbb{R}$$

with g bounded and globally Lipschitz continuous.

For $a = (\alpha_1, \alpha_2, \mu) \in \mathcal{A}$, we define

$$\begin{cases} b_{\mathcal{H}}(a) = \mu b_1(\alpha_1) + (1-\mu) b_2(\alpha_2), \\ \ell_{\mathcal{H}}(a) = \mu b_1(\alpha_1) + (1-\mu) b_2(\alpha_2) \end{cases}$$

and for $x \in \mathbb{R}$, we set

$$b(x,a) = \begin{cases} b_1(\alpha_1) & \text{if} \quad x \in (-\infty, 0) = \Omega_1, \\ b_2(\alpha_2) & \text{if} \quad x \in (0, +\infty) = \Omega_2, \\ b_{\mathcal{H}}(a) & \text{if} \quad x \in \mathcal{H} = \{0\} \end{cases}$$

and

$$\ell(x,a) = \begin{cases} \ell_1(\alpha_1) & \text{if} \quad x \in (-\infty, 0) = \Omega_1, \\ \ell_2(\alpha_2) & \text{if} \quad x \in (0, +\infty) = \Omega_2, \\ \ell_{\mathcal{H}}(a) & \text{if} \quad x \in \mathcal{H} = \{0\}\,. \end{cases}$$

We consider admissible controlled trajectories starting from the point from $(0,x)$ and ending at time $t > 0$ defined by

$$\mathcal{T}_{t,x} = \left\{ \begin{array}{l} (X(\cdot), a(\cdot)) \in \mathrm{Lip}(0,t;\mathbb{R}) \times L^\infty(0,t;\mathcal{A}) \quad \text{such that} \\ \begin{cases} X(0) = x, \\ \dot{X}(s) = b(X(s), a(s)) \quad \text{for a.e.} \quad s \in (0,t) \end{cases} \end{array} \right\}$$

and define the set of regular controlled trajectories as

$$\mathcal{T}^{reg}_{t,x} = \left\{ \begin{array}{l} (X(\cdot), a(\cdot)) \in \mathcal{T}_{x,t} \quad \text{such that} \\ a(s) \in \mathcal{A}_0^{reg} \quad \text{for a.e.} \quad s \in (0,t) \quad \text{such that} \quad X(s) = 0 \end{array} \right\}.$$

Notice that the definition of $\mathcal{T}_{t,x}$ differs from the one given in (5.1), where now X takes the value x at time 0 instead at time t. Then we define

$$U^-(x,t) = \inf_{(X(\cdot),a(\cdot)) \in \mathcal{T}_{t,x}} \left\{ g(X(t)) + \int_0^t \ell(X(s), a(s))\, ds \right\} \tag{6.4}$$

and

$$U^+(x,t) = \inf_{(X(\cdot),a(\cdot)) \in \mathcal{T}^{reg}_{t,x}} \left\{ g(X(t)) + \int_0^t \ell(X(s), a(s))\, ds \right\}. \tag{6.5}$$

Then we have the following characterization of U^- and U^+:

Theorem 6.1 (Characterization of U^- and U^+)). *Under the previous assumptions, U^- is the unique $\check{F}_A$-solution with initial data g for $A = H_T$. Similarly, U^+ is the unique $\check{F}_A$-solution with initial data g for $A = H_T^{reg}$.*

Proof. Theorem 6.1 is a straightforward application of Theorem 5.2. □

Corollary 6.2 (Conditions for uniqueness of Ishii solution). *Recall that $H_T \geq H_T^{reg}$. Under the previous assumptions, if $H_T = H_T^{reg}$, then there is uniqueness of the Ishii solution with initial data g. If $H_T > H_T^{reg}$, then there exists an initial data g such that there are two different Ishii solutions with the same initial data g.*

Proof. Given an initial data g which is assumed to be bounded and Lipshitz continuous, and under the previous assumptions, it is known in [7] that the minimal Ishii solution is U^- and that the maximal Ishii solution is U^+. When $H_T = H_T^{reg}$, then Theorem 6.1 implies that both U^- and U^+ are $\check{F}_A$-solutions with the same initial data g and with the same $A = H_T = H_T^{reg}$. The uniqueness of the $\check{F}_A$-solutions implies that $U^- = U^+$ and then this is the unique Ishii solution.

On the contrary, if $H_T > H_T^{reg}$, then

$$U^-(t,x) = -At + p_1 x 1_{\{x<0\}} + p_2 x 1_{\{x\geq 0\}}$$

is a $\check{F}_A$-solution with $A = H_T$ with initial data $g(x) = U^-(0,x)$ (which is may be not bounded here) if

$$H_T = A = H_1(p_1) = H_2(p_2), \quad p_2 \geq p_2^0, \quad p_1 \leq p_1^0.$$

On the other hand, U^- is not a $\check{F}_{H_T^{reg}}$-solution because $\check{F}_{H_T^{reg}}(p_1,p_2) = H_T^{reg} < H_T$. Using Theorem 5.2, we can check that U^- and U^+ are given respectively by the representation formula (6.4) and (6.5). This shows the result if we allow unbounded initial data g.

To adapt it to bounded initial data, we can simply truncate the initial data between two constants $-R$ and R and consider

$$g_R(x) = \min(R, \max(-R, g(x))).$$

Let us call respectively U_R^- and U_R^+ the solutions associated to the initial data g_R. In particular for R large enough, there exists $r > 0$ such that

$$g_R(x) = g(x) \quad \text{for} \quad |x| \leq r.$$

Then we see from the representation formula (6.4) and the fact that the trajectories propagate with finite velocity (because of the bound $|b_i| \leq c$) that

$$U_R^-(t,x) = U^-(t,x) \quad \text{for} \quad |x| \leq r - ct.$$

We then conclude as above that U_R^- is not a $\check{F}_{H_T^{reg}}$-solution for $x = 0$ and $t \in (0, r/c)$ if $H_T^{reg} < H_T$, and therefore $U_R^+ \neq U_R^-$. □

6.2 Computation of tangential Hamiltonians

We consider

$$A^* = \max_{q \in \mathrm{ch}\left[p_1^0, p_2^0\right]} (\min(H_1(q), H_2(q))).$$

with the chord

$$\mathrm{ch}\left[p_1^0, p_2^0\right] = [\min(p_1^0, p_2^0), \max(p_1^0, p_2^0)].$$

We also recall that $A_0 = \max_{i=1,2}\left(\min_{q\in\mathbb{R}} H_i(q)\right) = \max_{i=1,2} H_i(p_i^0)$.

Proposition 6.3. (Characterization of H_T)

$$H_T = \max(A^*, A_0).$$

Proof. Reduction. Let A_c denote $\max(A^*, A_0)$. Remark that there exists $p_c \in \mathbb{R}$ such that $A_c = H_{i_c}(p_c)$ for some $i_c \in \{1, 2\}$. We then consider

$$\tilde{H}_i(p) = H_i(p_c + p) - A_c.$$

In this case, using obvious notation, $\tilde{A}_c = 0$ and $\tilde{p}_c = 0$. We are going to prove that

$$\tilde{H}_T = 0.$$

Remark that

$$\begin{aligned}
\tilde{L}_i(p) &= \sup_q (pq - \tilde{H}_i(q)) \\
&= \sup_q (pq - H_i(p_c + q)) + A_c \\
&= \sup_q (pq - H_i(q)) - p_c p + A_c \\
&= L_i(p) - p_c p + A_c.
\end{aligned}$$

Then

$$\begin{aligned}
\tilde{H}_T &= \sup_{(\alpha_1,\alpha_2,\mu)\in A_0} (-\mu \tilde{L}_1(\alpha_1) - (1-\mu)\tilde{L}_2(\alpha_2)) \\
&= \sup_{(\alpha_1,\alpha_2,\mu)\in A_0} (-\mu L_1(\alpha_1) - (1-\mu)L_2(\alpha_2)) - A_c \\
&= H_T - A_c.
\end{aligned}$$

Hence, it is enough to prove

$$\tilde{H}_T = 0.$$

From now on, we assume that $A_c = 0$ and $p_c = 0$. We distinguish two cases.

First case. Assume first that $0 = A_c = A^* \geq A_0$. Then $0 = A^* = H_1(p^*) = H_2(p^*) = H_{i_c}(p_c)$ with $p^* \in \mathrm{ch}\,[p_1^0, p_2^0]$. Choosing initially $p_c = p^*$, we can assume that

$A^* = H_1(0) = H_2(0) = 0$. In particular, $L_1 \geq 0$ and $L_2 \geq 0$. Hence $H_T \leq 0$. To get the reverse inequality, we observe that there exists $\alpha_i^* \in \partial H_i(0)$, $i = 1, 2$, with

$$\alpha_1^* \alpha_2^* \leq 0.$$

Indeed, if this is not true, this implies that for all $\alpha_i \in \partial H_i(0)$,

$$\alpha_1 \alpha_2 > 0$$

which is impossible because H_1 and H_2 cross at p^*.

Pick now $\mu \in [0, 1]$ such that $\mu \alpha_1^* + (1 - \mu)\alpha_2^* = 0$. Then $(\alpha_1^*, \alpha_2^*, \mu) \in \mathcal{A}_0$ and consequently,

$$H_T \geq -\mu L_1(\alpha_1^*) - (1 - \mu)L_2(\alpha_2^*) = \mu H_1(0) + (1 - \mu)H_2(0) = 0.$$

Hence $H_T = 0$ in the first case, as desired.

Second case. We now assume that $0 = A_c = A_0 > A^*$. In this case, there exists $a \in \{1, 2\}$ such that

$$\min H_a = H_a(0) = 0,$$

with the initial choice $p_c = p_a^0$. This implies in particular

$$L_a \geq L_a(0) = 0.$$

Moreover, for $b \neq a$,

$$\min L_b = -H_b(0) \geq 0,$$

we we have used the fact that $A^* < A_0$. Hence, $L_a \geq 0$ and $L_b \geq 0$ and consequently, $H_T \leq 0$. Moreover with $\alpha_i^* \in \partial H_i(0)$, we have, $(0, \alpha_2^*, 1) \in \mathcal{A}_0$ when $a = 1$ and $(\alpha_1^*, 0, 0) \in \mathcal{A}_0$ when $a = 2$. Hence, in both cases,

$$H_T \geq -L_a(0) = 0.$$

Hence $H_T = 0$ in the second case too. The proof is now complete. □

Proposition 6.4 (Characterization of H_T^{reg})**.**

$$H_T^{reg} = \begin{cases} H_T & \text{if} \quad p_2^0 < p_1^0, \\ A_0 & \text{if} \quad p_2^0 \geq p_1^0, \end{cases} \tag{6.6}$$

where we recall that $p_i^0 \in \operatorname{argmin} H_i$ *for* $i = 1, 2$.

Proof. The proof is similar to the proof of Proposition 6.3. We make precise how to adapt it.

Reduction. Let A_c denote the right hand side of (6.6). Then the reduction to the case $A_c = 0$ and $p_c = 0$ is completely analogous. We now have to prove that $H_T^{reg} = 0$.

First case. Assume first that $0 = A_c = A^* \geq A_0$. Note that this case only makes sense either when $p_2^0 < p_1^0$ or when $p_2^0 \geq p_1^0$ and $0 = A_c = A^* = A_0$. Similarly, we get

$H_T^{reg} \leq 0$. To get the reverse inequality, we observe that there exists $\alpha_i^* \in \partial H_i(0)$, $i = 1, 2$, with

$$\alpha_1^* \alpha_2^* \leq 0.$$

We deduce that we can choose $\alpha_2^* \geq 0$ and $\alpha_1^* \leq 0$, both in the case $p_2^0 < p_1^0$ and the case $p_2^0 \geq p_1^0$ and $0 = A_c = A^* = A_0$. This implies that we can find $(\alpha_1^*, \alpha_2^*, \mu) \in \mathcal{A}_0^{reg}$ and similarly, we conclude that $H_T^{reg} \geq 0$. Hence $H_T = 0$ in the first case, as desired.

Second case. We now assume that $0 = A_c = A_0$. We set again for some $a \in \{1, 2\}$:

$$\min H_a = H_a(0) = 0.$$

From our definition of a, we have again

$$L_a \geq L_a(0) = 0 \quad \text{and} \quad p_a^0 = 0.$$

We first prove that $H_T^{reg} \leq 0$. In order to do so, we now distinguish three subcases.

Assume first $p_2^0 < p_1^0$. Then we can assume that $A_0 > A^*$ (otherwise we have $A_0 = A^*$ and we can apply the first case). Then we deduce, as in the proof of Proposition 6.3, that $H_T^{reg} \leq 0$.

Assume now that $p_2^0 \geq p_1^0$ and $a = 1$. We deduce that $0 = p_1^0 \leq p_2^0$. But because H_2 is minimal at p_2^0, we have $0 \in \partial H_2(p_2^0)$, and we deduce that $0 \leq p_2^0 \in \partial L_2(0)$. This implies that $L_2 \geq L_2(0) = -H_2(p_2^0) \geq 0$ on $\mathbb{R}^+$. By definition of H_T^{reg}, this implies that $H_T^{reg} \leq 0$.

Assume finally that $p_2^0 \geq p_1^0$ and $a = 2$. This subcase is symmetric with respect to the previous one. We deduce that $0 = p_2^0 \geq p_1^0$. But because H_1 is minimal at p_1^0, we deduce that $0 \geq p_1^0 \in \partial L_1(0)$. This implies that $L_1 \geq L_1(0) = -H_1(p_1^0) \geq 0$ on $\mathbb{R}^-$. Again, by definition of H_T^{reg}, this implies that $H_T^{reg} \leq 0$.

We now prove that $H_T^{reg} \geq 0$. To do so pick some $(0, \alpha_2, 1) \in \mathcal{A}_0^{reg}$ when $a = 1$ and some $(\alpha_1, 0, 0) \in \mathcal{A}_0^{reg}$ when $a = 2$. Hence, in both cases, we get

$$H_T^{reg} \geq -L_a(0) = 0.$$

Hence $H_T = 0$ in the second case too. The proof is now complete. □

6.3 Junction-type solutions and Ishii solutions

We say that u is a $\check{F}_A$-sub-solution (resp. $\check{F}_A$-super-solution) if it is a sub-solution (resp. super-solution) of (6.2). We also say that u is a Ishii sub-solution (resp. super-solution) if it is a sub-solution (resp. super-solution) of

$$\begin{cases} u_t + H_1(u_x) = 0 & \text{for} \quad (t,x) \in (0,+\infty) \times (-\infty, 0) \\ u_t + H_2(u_x) = 0 & \text{for} \quad (t,x) \in (0,+\infty) \times (0, +\infty) \\ u_t + \min(H_1(u_x), H_2(u_x)) = 0 & \text{for} \quad (t,x) \in (0,+\infty) \times \{0\} \end{cases}$$

$$\left(\text{resp.} \begin{cases} u_t + H_1(u_x) = 0 & \text{for} \quad (t,x) \in (0,+\infty) \times (-\infty, 0) \\ u_t + H_2(u_x) = 0 & \text{for} \quad (t,x) \in (0,+\infty) \times (0, +\infty) \\ u_t + \max(H_1(u_x), H_2(u_x)) = 0 & \text{for} \quad (t,x) \in (0,+\infty) \times \{0\} \end{cases} \right).$$

Then we have the following result.

Proposition 6.5 (Relation between $\check{F}_A$ and Ishii sub/super-solutions)**.** *Every $\check{F}_A$-sub-solution (resp. $\check{F}_A$-super-solution) is a Ishii sub-solution (resp. Ishii super-solution) if $A \geq H_T^{reg}$ (resp. $A \leq H_T$).*
Moreover for every $A \in [A_0, H_T^{reg})$, there exists a $\check{F}_A$-sub-solution which is not a Ishii sub-solution. For every $A > H_T$, there exists a $\check{F}_A$-super-solution which is not a Ishii super-solution.

Remark 6.6. If we use the expressions of Propositions 6.3 and 6.4 as definitions of H_T and H_T^{reg}, then the result of Proposition 6.5 still holds true for general Hamiltonians satisfying (1.5), and not only for particular convex Hamiltonians coming from optimal control theory. In particular, we deduce that every $\check{F}_A$-solution is a Ishii solution for each $A \in [H_T^{reg}, H_T]$.

Proof. We treat successively sub-solutions and super-solutions.

Sub-Solutions. Let u be a $\check{F}_A$-sub-solution with $A \geq H_T^{reg}$. Consider a C^1 function $\phi : \mathbb{R} \to \mathbb{R}$ touching u from above at $(t, 0)$ for some $t > 0$. Then

$$\lambda + \check{F}_A(q, q) \leq 0$$

where $\lambda = \partial_t \phi(t, 0)$ and $q = \partial_x \phi(t, 0)$. In particular, $\lambda + A \leq 0$. We want to prove that

$$\lambda + \min(H_1(q), H_2(q)) \leq 0.$$

If $q \leq p_2^0$, then

$$\min(H_1(q), H_2(q)) \leq H_2^-(q) \leq \check{F}_A(q, q) \leq -\lambda.$$

Similarly, if $q \geq p_1^0$, then

$$\min(H_1(q), H_2(q)) \leq H_1^+(q) \leq \check{F}_A(q, q) \leq -\lambda.$$

If $p_2^0 < p_1^0$, and $q \in [p_2^0, p_1^0]$, then by definition of A^*, we have

$$\min(H_1(q), H_2(q)) \leq A^* \leq H_T = H_T^{reg} \leq A \leq -\lambda.$$

This shows that u is a Ishii sub-solution.

If $A^* \leq A_0$ or $p_2^0 \geq p_1^0$, there is nothing additional to prove. Assume now that $p_2^0 < p_1^0$ with $H_T^{reg} = A^* > A_0$, and we claim that for any $A \in [A_0, H_T^{reg}) = [A_0, A^*)$, there exists a $\check{F}_A$-sub-solution which is not an Ishii sub-solution. Indeed, let us consider $p^* \in [p_2^0, p_1^0]$ such that

$$A^* = H_1(p^*) = H_2(p^*).$$

Then there exists $p_2^0 \leq p_2 < p^* < p_1 \leq p_1^0$ such that

$$A = H_1(p_1) = H_2(p_2) = \check{F}_A(p_1, p_2) \tag{6.7}$$

Let us now consider

$$u(t, x) = -At + p_1 x 1_{\{x<0\}} + p_2 x 1_{\{x \geq 0\}}$$

In particular u is $\check{F}_A$-sub-solution because of (6.7). Now the test function $\phi(t,x) = -At + p^* x$ touches u at $(t,0)$ from above and does not satisfy the inequality

$$\partial_t \phi(t,0) + \min(H_1(\partial_x \phi(t,0)), H_2(\partial_x \phi(t,0))) \le 0.$$

This shows that u is not a Ishii sub-solution.

SUPER-SOLUTIONS. Let u be a $\check{F}_A$-super-solution with $A \le H_T$. Consider a C^1 function $\phi : \mathbb{R} \to \mathbb{R}$ touching u from below at $(t,0)$ for some $t > 0$. Then

$$\lambda + F_A(q,q) \ge 0$$

where $\lambda = \partial_t \phi(t,0)$ and $q = \partial_x \phi(t,0)$. Without loss of generality, we can assume that $A \ge A_0$. We want to prove that

$$\lambda + \max(H_1(q), H_2(q)) \ge 0.$$

If $F_A(q,q) = A$, then we deduce from Lemma 6.7 below that

$$0 \le \lambda + A \le \lambda + H_T \le \lambda + \max(H_1(q), H_2(q)).$$

If now $F_A(q,q) = H_1^+(q)$, then

$$0 \le \lambda + F_A(q,q) \le \lambda + H_1(q) \le \lambda + \max(H_1(q), H_2(q)).$$

If finally $F_A(q,q) = H_2^-(q)$, then

$$0 \le \lambda + F_A(q,q) \le \lambda + H_2(q) \le \lambda + \max(H_1(q), H_2(q)).$$

This shows that u is a Ishii super-solution.

Assume next that $A > H_T$. If $A^* \ge A_0$, let $p^* \in \mathrm{ch}\,[p_1^0, p_2^0]$ such that

$$A^* = H_1(p^*) = H_2(p^*).$$

Let us choose an index $a \in \{1,2\}$ such that

$$\max_{i=1,2} H_i(p_i^0) = H_a(p_a^0).$$

Then we set

$$\bar{p} = \begin{cases} p^* & \text{if} \quad A^* \ge A_0, \\ p_1 & \text{if} \quad A^* < A_0 \quad \text{and} \quad a = 1, \\ p_2 & \text{if} \quad A^* < A_0 \quad \text{and} \quad a = 2. \end{cases}$$

In particular we have

$$\max(H_1(\bar{p}), H_2(\bar{p})) = H_T. \tag{6.8}$$

Then for $A > H_T$, there exist $p_2 \ge \max(p_1^0, p_2^0) \ge \bar{p} \ge \min(p_1^0, p_2^0) \ge p_1$ such that

$$H_2(p_2) = A = H_1(p_1).$$

Let us now define

$$u(t,x) = -At + p_1 x 1_{\{x<0\}} + p_2 x 1_{\{x\ge 0\}}.$$

Then u is a $\check{F}_A$-super-solution because $\check{F}_A(p_1,p_2) = A$. Now the test function $\phi(t,x) = -At + \bar{p}x$ touches u at $(t,0)$ from below and does not satisfy the inequality

$$\partial_t \phi(t,0) + \max(H_1(\partial_x \phi(t,0)), H_2(\partial_x \phi(t,0))) \ge 0$$

because of (6.8). This shows that u is not a Ishii super-solution. This achieves the proof. □

In the previous proof, we used the following elementary lemma.

Lemma 6.7 (Bound from above for H_T)**.** *For all $q \in \mathbb{R}$, $H_T \le \max(H_1(q), H_2(q))$.*

Proof. We recall that $H_T = \max(A^*, A_0)$. Assume first that $\max(A^*, A_0) = A_0$, then $A_0 = \min H_a$ for some $a \in \{1,2\}$. In particular, for all $q \in \mathbb{R}$, we have $H_T = A_0 \le H_a(q) \le \max(H_1(q), H_2(q))$.
If now $\max(A^*, A_0) = A^* > A_0$, then there exists $p^* \in [p_i^0, p_j^0]$ for some $i, j \in \{1,2\}$ $(i \ne j)$, such that

$$A^* = H_i(p^*) = H_j(p^*).$$

Moreover, H_j is non-increasing in $(-\infty, p^*]$ hence

$$H_j(q) \ge A^* \text{ for } q \le p^*;$$

similarly, H_i is non-decreasing in $[p^*, +\infty)$ hence

$$H_i(q) \ge A^* \text{ for } q \ge p^*.$$

This implies the expected inequality. □

This proposition allows to recover the following result:

Corollary 6.8 (Characterization of U^-)**.** *The function U^- is the unique $\check{F}_A$-solution u with initial data g and $A = H_T$.*

Proof. Proposition 6.5 implies in particular that if u is the (unique) $\check{F}_A$-solution with initial data g and $A = H_T$, then u is also a Ishii solution. But $\check{F}_A \ge A$ implies that u satisfies

$$\partial_t u + H_T \le 0 \text{ in } (0,+\infty) \times \mathbb{R}$$

in the viscosity sense (with C^1 space-time test functions). In order to apply the results from [7], we need more, namely,

$$\partial_t v + H_T \le 0 \text{ in } (0,+\infty) \tag{6.9}$$

where $v(t) = u(t, 0)$. Remark that v is continuous since so is u. To get such an inequality, consider a test function $\phi \in C^1_t$ touching strictly from above the function v at some time $t_0 > 0$. We can consider the space-time test function

$$\phi_\varepsilon(t, x) = \phi(t) + \frac{|x|}{\varepsilon}$$

and deduce from the coercivity of the Hamiltonians H_i that, locally close to $(t_0, 0)$, a point $(t_\varepsilon, x_\varepsilon)$ of maximum of $u - \phi_\varepsilon$ is reached for $x_\varepsilon = 0$ for $\varepsilon > 0$ small enough. We then get the expected result as ε tends to 0.

Now we can use the results from [7] to get that U^- is the unique Ishii solution such that (6.9) holds true. Then $U^- = u$. □

7 First extension: networks

7.1 Definition of a network

A general abstract network $\mathcal{N}$ is characterized by the set ($\mathcal{E}$ of its *edges* and the set $\mathcal{V}$) of its "nodes". It is endowed with a distance.

Edges. $\mathcal{E}$ is a finite or countable set of edges. Each edge $e \in \mathcal{E}$ is assumed to be either isometric to the half line $[0, +\infty)$ with $\partial e = \{e^0\}$ (where the endpoint e^0 can be identified to $\{0\}$), or to a compact interval $[0, l_e]$ with

$$\inf_{e \in \mathcal{E}} l_e > 0 \tag{7.1}$$

and $\partial e = \{e^0, e^1\}$. Condition (7.1) implies in particular that the network is complete. The endpoints $\{e^0\}, \{e^1\}$ can respectively be identified to $\{0\}$ and $\{l_e\}$. The *interior* e^* of an edge e refers to $e \setminus (\partial e)$.

Vertices (or nodes). It is convenient to see vertices of the network as a partition of the sets of all edge endpoints,

$$\bigcup_{e \in \mathcal{E}} \partial e = \bigcup_{n \in \mathcal{V}} n;$$

we assume that each set n only contains a finite number of endpoints.

Here each $n \in \mathcal{V}$ can be identified as a vertex (or node) of the network as follows. For every $x, y \in \bigcup_{e \in \mathcal{E}} e$, we define the equivalence relation:

$$x \sim y \quad \Longleftrightarrow (x = y \quad \text{or} \quad x, y \in n \in \mathcal{V})$$

and we define the network as the quotient

$$\mathcal{N} = \left(\bigcup_{e \in \mathcal{E}} e \right) / \sim \quad = \quad \left(\bigcup_{e \in \mathcal{E}} e^* \right) \cup \mathcal{V}. \tag{7.2}$$

We also define for $n \in \mathcal{V}$

$$\mathcal{E}_n = \{e \in \mathcal{E}, \quad n \in \partial e\}$$

and its partition $\mathcal{E}_n = \mathcal{E}_n^- \cup \mathcal{E}_n^+$ with

$$\mathcal{E}_n^- = \left\{e \in \mathcal{E}_n, n = e^0\right\}, \quad \mathcal{E}_n^+ = \left\{e \in \mathcal{E}_n, n = e^1\right\}.$$

Distance. We also define the distance function $d(x, y) = d(y, x)$ as the minimal length of a continuous path connecting x and y on the network, using the metric of each edge (either isometric to $[0, +\infty)$ of to a compact interval). Note that, because of our assumptions, if $d(x, y) < +\infty$, then there is only a finite number of minimal paths.

Remark 7.1. For any $\varepsilon > 0$, there is a bound (depending on ε) on the number of minimal paths connecting x to y for all $y \in B(\bar{y}, \varepsilon) = \{y \in \mathcal{N}, \quad d(\bar{y}, y) < \varepsilon\}$.

7.2 Hamilton-Jacobi equations on a network

Given a Hamiltonian H_e on each edge $e \in \mathcal{E}$, we consider the following HJ equation on the network $\mathcal{N}$,

$$\left\{\begin{array}{llll} u_t + H_e(t, x, u_x) = 0 & \text{for} \quad t \in (0, +\infty) & \text{and} & x \in e^*, \\ u_t + F_A(t, x, u_x) = 0 & \text{for} \quad t \in (0, +\infty) & \text{and} & x = n \in \mathcal{V} \end{array}\right. \tag{7.3}$$

submitted to an initial condition

$$u(0, x) = u_0(x) \quad \text{for} \quad x \in \mathcal{N}. \tag{7.4}$$

The limited flux functions F_A associated with the Hamiltonians H_e are defined below. We first make precise the meaning of u_x in (7.3).

Gradients of real functions. For a real function u defined on the network $\mathcal{N}$, we denote by $\partial_e u(x)$ the (spatial) derivative of u at $x \in e$ and define the "gradient" of u by

$$u_x(x) := \begin{cases} \partial_e u(x) & \text{if} \quad x \in e^* = e \setminus (\partial e), \\ ((\partial_e u(x))_{e \in \mathcal{E}_n^-}, (\partial_e u(x))_{e \in \mathcal{E}_n^+}) & \text{if} \quad x = n \in \mathcal{V} \end{cases}.$$

The norm $|u_x|$ simply denotes $|\partial_e u|$ for $x \in e^*$ or $\max\{|\partial_e u| : e \in \mathcal{E}_n$ at the vertex $x = n$.

Limited flux functions. We also define for $(t, x) \in \mathbb{R} \times \partial e$,

$$H_e^-(t, x, q) = \left\{\begin{array}{ll} H_e(t, x, q) & \text{if} \quad q \leq p_e^0(t, x), \\ H_e(t, x, p_e^0(t, x)) & \text{if} \quad q > p_e^0(t, x) \end{array}\right.$$

and

$$H_e^+(t, x, q) = \left\{\begin{array}{ll} H_e(t, x, p_e^0(t, x)) & \text{if} \quad q \leq p_e^0(t, x), \\ H_e(t, x, q) & \text{if} \quad q > p_e^0(t, x). \end{array}\right.$$

Given limiting functions $(A_n)_{n\in\mathcal{V}}$, we define for $p=(p_e)_{e\in\mathcal{E}_n}$,

$$F_A(t,n,p)=\max\left(A_n(t),\quad \max_{e\in\mathcal{E}_n^-} H_e^-(t,n,p_e),\quad \max_{e\in\mathcal{E}_n^+} H_e^+(t,n,-p_e)\right).$$

In particular, for each $n\in\mathcal{V}$, the functions $F_A(t,n,\cdot)$ are the same for all $A_n(t)\in[-\infty,A_n^0(t)]$ with

$$A_n^0(t):=\max\left(\max_{e\in\mathcal{E}_n^-} H_e^-(t,n,p_e^0(t,n)),\quad \max_{e\in\mathcal{E}_n^+} H_e^+(t,n,p_e^0(t,n))\right). \tag{7.5}$$

A shorthand notation. As in the junction case, we introduce

$$H_{\mathcal{N}}(t,x,p)=\begin{cases} H_e(t,x,p) & \text{for}\quad p\in\mathbb{R}, \qquad t\in\mathbb{R}, \quad \text{if}\quad x\in e^*,\\ F_A(t,x,p) & \text{for}\quad p=(p_e)_{e\in\mathcal{E}_n}\in\mathbb{R}^{\text{Card }\mathcal{E}_n}, \qquad t\in\mathbb{R}, \quad \text{if}\quad x=n\in\mathcal{V}\end{cases} \tag{7.6}$$

in order to rewrite (7.3) as

$$u_t+H_{\mathcal{N}}(t,x,u_x)=0\quad\text{for all}\quad (t,x)\in(0,+\infty)\times\mathcal{N}. \tag{7.7}$$

7.3 Assumptions on the Hamiltonians

For each $e\in\mathcal{E}$, we consider a Hamiltonian $H_e:[0,+\infty)\times e\times\mathbb{R}\to\mathbb{R}$ satisfying

- **(H0)** (Continuity) $H_e\in C([0,+\infty)\times e\times\mathbb{R})$.
- **(H1)** (Uniform coercivity) For all $T>0$,

  $$\lim_{|q|\to+\infty} H_e(t,x,q)=+\infty$$

  uniformly with respect to $t\in[0,T]$ and $x\in e\subset\mathcal{N}$ and $e\subset\mathcal{N}$.
- **(H2)** (Uniform bound on the Hamiltonians for bounded gradients) For all $T,L>0$, there exists $C_{T,L}>0$ such that

  $$\sup_{t\in[0,T],\ p\in[-L,L],x\in\mathcal{N}\setminus\mathcal{V}} |H_{\mathcal{N}}(t,x,p)|\le C_{T,L}.$$
- **(H3)** (Uniform modulus of continuity for bounded gradients) For all $T,L>0$, there exists a modulus of continuity $\omega_{T,L}$ such that for all $|p|,|q|\le L$, $t\in[0,T]$ and $x\in e\in\mathcal{E}$,

  $$|H_e(t,x,p)-H_e(t,x,q)|\le\omega_{T,L}(|p-q|).$$
- **(H4)** (Level-set convexity) For all $n\in\mathcal{V}$, there exists a continuous function $t\mapsto p_e^0(t,n)$ such that

  $$\begin{cases} H_e(t,n,\cdot) & \text{is nonincreasing on}\quad (-\infty,p_e^0(t,n)],\\ H_e(t,n,\cdot) & \text{is nondecreasing on}\quad [p_e^0(t,n),+\infty).\end{cases}$$

- **(H5)** (Uniform modulus of continuity in time) For all $T > 0$, there exists a modulus of continuity $\bar{\omega}_T$ such that for all $t, s \in [0, T]$, $p \in \mathbb{R}$, $x \in e \in \mathcal{E}$,

$$H_e(t, x, p) - H_e(s, x, p) \leq \bar{\omega}_T \left(|t - s|(1 + \max(H_e(s, x, p), 0))\right).$$

- **(H6)** (Uniform continuity of A^0) For all $T > 0$, there exists a modulus of continuity $\bar{\omega}_T$ such that for all $t, s \in [0, T]$ and $n \in \mathcal{V}$,

$$|A_n^0(t) - A_n^0(s)| \leq \bar{\omega}_T(|t - s|).$$

As far as flux limiters are concerned, the following assumptions will be used.

- **(A0)** (Continuity of A) For all $T > 0$ and $n \in \mathcal{V}$, $A_n \in C([0, T])$.
- **(A1)** (Uniform bound on A) For all $T > 0$, there exists a constant $C_T > 0$ such that for all $t \in [0, T]$ and $n \in \mathcal{V}$

$$|A_n(t)| \leq C_T.$$

- **(A2)** (Uniform continuity of A) For all $T > 0$, there exists a modulus of continuity $\bar{\omega}_T$ such that for all $t, s \in [0, T]$ and $n \in \mathcal{V}$,

$$|A_n(t) - A_n(s)| \leq \bar{\omega}_T(|t - s|).$$

The proof of the following technical lemma is postponed until appendix.

Lemma 7.2 (Estimate on the difference of Hamiltonians). *Assume that the Hamiltonians satisfy* (H0)-(H4) *and* (A0)-(A1). *Then for all $T > 0$, there exists a constant $C_T > 0$ such that*

$$|p_e^0(t, x)| \leq C_T \quad \text{for all} \quad t \in [0, T], \quad x \in \partial e, \quad e \in \mathcal{E}, \tag{7.8}$$

$$|A_n^0(t)| \leq C_T \quad \text{for all} \quad t \in [0, T], \quad n \in \mathcal{V}. \tag{7.9}$$

If we assume moreover (H5)-(H6) *and* (A2), *then there exists a modulus of continuity $\tilde{\omega}_T$ such that for all $t, s \in [0, T]$, and x, p*

$$H_{\mathcal{N}}(t, x, p) - H_{\mathcal{N}}(s, x, p) \leq \tilde{\omega}_T(|t - s|(1 + \max(0, H_{\mathcal{N}}(s, x, p)))). \tag{7.10}$$

Remark 7.3. The reader can check that Assumptions (H5)-(H6) and (A2) in the statement of Theorem 7.8 can in fact be replaced with (7.10).

Remark 7.4 (Example of Hamiltonians with uniform modulus of time continuity). Condition on the uniform modulus of continuity in time in (H5)-(H6) is for instance satisfied by Hamiltonians of the type for $q > 0$ and $\delta > 0$ such that for all $x \in e \in \mathcal{E}$ we have

$$H_e(t, x, p) = c_e(t, x)|p|^q \quad \text{with} \quad 0 < \delta \leq c_e(t, x) \leq 1/\delta$$

with c_e uniformly continuous in time and continuous in space.

7.4 Viscosity solutions on a network

Class of test functions. For $T > 0$, set $\mathcal{N}_T = (0,T) \times \mathcal{N}$. We define the class of test functions on $(0,T) \times \mathcal{N}$ by

$$C^1(\mathcal{N}_T) = \left\{\varphi \in C(\mathcal{N}_T), \text{ the restriction of } \varphi \text{ to } (0,T) \times e \text{ is } C^1, \text{ for all } e \in \mathcal{E}\right\}.$$

Definition 7.5 (Viscosity solutions)**.** Assume the Hamiltonians satisfy *(H0)-(H4)* and *(A0)-(A1)* and let $u : [0,T) \times \mathcal{N} \to \mathbb{R}$.

i) We say that u is a *sub-solution* (resp. *super-solution*) of (1.7) in $(0,T) \times \mathcal{N}$ if for all test function $\varphi \in C^1(\mathcal{N}_T)$ such that

$$u^* \leq \varphi \quad (\text{resp.} \quad u_* \geq \varphi) \quad \text{in a neighborhood of } (t_0, x_0) \in \mathcal{N}_T$$

with equality at (t_0, x_0), we have

$$\varphi_t + H_{\mathcal{N}}(t, x, \varphi_x) \leq 0 \quad (\text{resp.} \quad \geq 0) \quad \text{at } (t_0, x_0).$$

ii) We say that u is a *sub-solution* (resp. *super-solution*) of (1.7), (1.4) in $[0,T) \times \mathcal{N}$ if additionally

$$u^*(0,x) \leq u_0(x) \quad (\text{resp.} \quad u_*(0,x) \geq u_0(x)) \quad \text{for all} \quad x \in \mathcal{N}.$$

iii) We say that u is a *(viscosity) solution* if u is both a sub-solution and a super-solution.

Remark 7.6 (Touching sub-solutions with semi-concave functions). When proving the comparison principle in the network setting, sub-solutions (resp. super-solutions) will be touched from above (resp. from below) by functions that will not be C^1, but only semi-concave (resp. semi-convex). We recall that a function is semi-concave if it is the sum of a concave function and a smooth (C^2 say) function. But it is a classical observation that, at a point where a semi-concave function is not C^1, we can replace the semi-concave function by a C^1 test function touching it from above.

As in the case of a junction (see Proposition 2.3), viscosity solutions are stable through supremum/infimum. We also have the following existence result.

Theorem 7.7 (Existence on a network)**.** *Assume* (H0)-(H4) *and* (A0)-(A1) *on the Hamiltonians and assume that the initial data u_0 is uniformly continuous on $\mathcal{N}$. Let $T > 0$. Then there exists a viscosity solution u of* (7.7),(7.4) *on* $[0,T) \times \mathcal{N}$ *and a constant $C_T > 0$ such that*

$$|u(t,x) - u_0(x)| \leq C_T \quad \textit{for all} \quad (t,x) \in [0,T) \times \mathcal{N}.$$

Proof. The proof follows along the lines of the ones of Theorem 1.1. The main difference lies in the construction of barriers. We proceed similarly and get a regularized initial data u_0^ε satisfying

$$|u_0^\varepsilon - u_0| \leq \varepsilon \quad \text{and} \quad |(u_0^\varepsilon)_x| \leq L_\varepsilon.$$

Then the functions

$$u_\varepsilon^\pm(t,x) = u_0^\varepsilon(x) \pm C_\varepsilon t \pm \varepsilon \tag{7.11}$$

are global super and sub-solutions with respect to the initial data u_0 if C_ε is chosen as follows,

$$C_\varepsilon = \max\left(\sup_{t\in[0,T]}\sup_{n\in\mathcal{V}}|\max(A_n(t), A_n^0(t))|, \sup_{t\in[0,T]}\sup_{e\in\mathcal{E}}\sup_{x\in e,\ |p_e|\le L_\varepsilon}|H_e(t,x,p_e)|\right); \tag{7.12}$$

indeed, we use (7.9) in Lemma 7.2 to bound the first terms in (7.12). □

7.5 Comparison principle on a network

Theorem 7.8 (Comparison principle on a network)**.** *Assume the Hamiltonians satisfy* (H0)-(H6) *and* (A0)-(A2) *and assume that the initial data u_0 is uniformly continuous on $\mathcal{N}$. Let $T > 0$. Then for all sub-solution u and super-solution w of* (7.7)*,* (7.4) *in $[0,T)\times\mathcal{N}$, satisfying for some $C_T > 0$ and some $x_0 \in \mathcal{N}$*

$$u(t,x) \le C_T(1 + d(x_0,x)), \quad w(t,x) \ge -C_T(1+d(x_0,x)), \quad \textit{for all} \quad (t,x)\in[0,T)\times\mathcal{N}, \tag{7.13}$$

we have

$$u \le w \quad \textit{on} \quad [0,T)\times\mathcal{N}.$$

As a straighforward corollary of Theorems 7.8 and 7.7, we get

Corollary 7.9 (Existence and uniqueness)**.** *Under the assumptions of Theorem 7.8, there exits a unique viscosity solution u of* (7.7)*,* (7.4) *in $[0,T)\times\mathcal{N}$ such that there exists a constant $C > 0$ with*

$$|u(t,x) - u_0(x)| \le C \quad \textit{for all} \quad (t,x)\in[0,T)\times\mathcal{N}.$$

In order to prove Theorem 7.8, we first need two technical lemmas that are proved in appendix.

Lemma 7.10 (A priori control – the network case)**.** *Let $T > 0$ and let u be a sub-solution and w be a super-solution as in Theorem 7.8. Then there exists a constant $C = C(T) > 0$ such that for all $(t,x),(s,y)\in[0,T)\times\mathcal{N}$, we have*

$$u(t,x) \le w(s,y) + C(1 + d(x,y)). \tag{7.14}$$

Lemma 7.11 (Uniform control by the initial data)**.** *Under the assumptions of Theorem 7.8, for any $T > 0$ and $C_T > 0$, there exists a modulus of continuity $f : [0,T) \to [0,+\infty]$ satisfying $f(0^+) = 0$ such that for all sub-solution u (resp. super-solution w) of* (7.7)*,* (7.4) *on $[0,T)\times\mathcal{N}$, satisfying* (7.13) *for some $x_0\in\mathcal{N}$, we have for all $(t,x)\in[0,T)\times\mathcal{N}$,*

$$u(t,x) \le u_0(x) + f(t) \quad (\textit{resp.} \quad w(t,x) \ge u_0(x) - f(t)). \tag{7.15}$$

We can now turn to the proof of Theorem 7.8. The proof is similar the comparison principle on a junction (Theorem 1.1). Still, a space localization procedure has to be performed in order to "reduce" to the junction case. From a technical point of view, a noticeable difference is that we will fix the time penalization (for some parameter ν small enough), and then will first take the limit $\varepsilon \to 0$ (ε being the parameter for the space penalization), and then take the limit $\alpha \to 0$ (α being the penalizaton parameter to keep the optimization points at a finite distance).

Proof of Theorem 7.8. Let $\eta > 0$ and $\theta > 0$ and consider

$$M(\theta) = \sup \left\{ u(t,x) - w(s,x) - \frac{\eta}{T-t}, \quad x \in \mathcal{N}, \quad t,s \in [0,T), \quad |t-s| \leq \theta \right\}.$$

We want to prove that

$$M = \lim_{\theta \to 0} M(\theta) \leq 0.$$

Assume by contradiction that $M > 0$. From Lemma 7.10 we know that M is finite.

Step 1: The localization procedure. Let ψ denote $\frac{d^2(x_0,\cdot)}{2}$.

Lemma 7.12 (Localization)**.** *The supremum*

$$M_\alpha = \sup_{\substack{t,s\in[0,T],t<T \\ x \in \mathcal{N}}} \left\{ u(t,x) - w(s,x) - \alpha\psi(x) - \frac{\eta}{T-t} - \frac{(t-s)^2}{2\nu} \right\}$$

is reached for some point $(t_\alpha, s_\alpha, x_\alpha)$. Moreover, for α and ν small enough, we have the following localization estimates

$$M_\alpha \geq 3M/4 > 0 \tag{7.16}$$

$$d(x_0, x_\alpha) \leq \frac{C}{\sqrt{\alpha}} \tag{7.17}$$

$$0 < \tau_\nu \leq t_\alpha, s_\alpha \leq T - \frac{\eta}{2C} \tag{7.18}$$

$$\lim_{\nu \to 0} \left(\limsup_{\alpha \to 0} \frac{(t_\alpha - s_\alpha)^2}{2\nu} \right) = 0 \tag{7.19}$$

where C is a constant which does not depend on α, ε, ν and η.

Proof of Lemma 7.12. Choosing α small enough, we have (7.16) for all $\nu > 0$. Because the network is complete for its metric, the supremum in the definition of M_α is reached at some point $(t_\alpha, s_\alpha, x_\alpha)$. From Lemma 7.10, we deduce that

$$0 < \frac{3M}{4} \leq M_\alpha \leq C - \alpha\psi(x_\alpha) - \frac{\eta}{T - t_\alpha} - \frac{(t_\alpha - s_\alpha)^2}{2\nu}$$

and then

$$\alpha\psi(x_\alpha) + \frac{\eta}{T - t_\alpha} + \frac{(t_\alpha - s_\alpha)^2}{2\nu} \leq C. \tag{7.20}$$

This implies (7.17) changing C if necessary.

On the one hand, we get from (7.20) the second inequality in (7.18) by choosing ν such that $\sqrt{2\nu C} \leq \eta/2C$. On the other hand, we get from Lemma 7.11

$$0 < M_\alpha \leq f(t_\alpha) + f(s_\alpha) - \frac{\eta}{T}.$$

In particular,

$$\frac{\eta}{T} \leq 2f(\tau + \sqrt{2\nu C})$$

where $\tau = \min(t_\alpha, s_\alpha)$. If both τ and ν are too small, we get a contradiction. Hence the first inequality in (7.18) holds for some constant τ_ν depending on ν but not on α, ε and η.

We now turn to the proof of (7.19). We know that for any $\delta > 0$, there exists $\theta(\delta) > 0$ (with $\theta(\delta) \to 0$ as $\delta \to 0$) and $(t^\delta, s^\delta, x^\delta) \in [0,T) \times [0,T) \times \mathcal{N}$ such that

$$u(t^\delta, x^\delta) - w(s^\delta, x^\delta) - \frac{\eta}{T - t^\delta} \geq M - \delta \quad \text{and} \quad |t^\delta - s^\delta| \leq \theta(\delta).$$

Then from (7.20) we deduce that

$$M(\sqrt{2\nu C}) - \frac{(t_\alpha - s_\alpha)^2}{2\nu} \geq M_\alpha \geq M - \delta - \alpha\psi(x^\delta) - \frac{|\theta(\delta)|^2}{2\nu}$$

and then

$$\limsup_{\alpha\to 0} \frac{(t_\alpha - s_\alpha)^2}{2\nu} \leq M(\sqrt{2\nu C}) - M + \delta + \frac{|\theta(\delta)|^2}{2\nu}.$$

Taking the limit $\delta \to 0$, we get

$$\limsup_{\alpha\to 0} \frac{(t_\alpha - s_\alpha)^2}{2\nu} \leq M(\sqrt{2\nu C}) - M$$

which yields the desired result. □

Step 2: Reduction when x_α is a vertex. We adapt here Lemma 3.1.

Lemma 7.13 (Reduction)**.** *Assume that $x_\alpha = n \in \mathcal{V}$. Without loss of generality, we can assume that $\mathcal{E}_n^+ = \emptyset$ and $p_e^0(t_\alpha, x_\alpha) = 0$ for each $e \in \mathcal{E}_n$ with $n = x_\alpha$.*

Proof of Lemma 7.13. The orientation of the edges $e \in \mathcal{E}_n$ can be changed in order to reduce to the case $\mathcal{E}_n^+ = \emptyset$ In particular, for $p = (p_e)_{e\in\mathcal{E}_n}$,

$$F_A(t, n, p) = \max\left(A_n(t), \quad \max_{e\in\mathcal{E}_n^-} H_e^-(t, n, p_e)\right).$$

We can then argue as in Lemma 3.1. This means that we redefine the Hamiltonians (and the flux limiter A_n) only locally for $e \in \mathcal{E}_n$. Therefore we can assume that Using (7.8), we can check that the new Hamiltonians (locally for $e \in \mathcal{E}_n$) and A_n still satisfy *(H0)-(H6)* and *(A0)-(A2)* (with the same modulus of continuity, and with some different controlled constants $C_{T,L}$). We also have (7.13) with some controlled different constants. $\square$

Step 3: The penalization procedure. We now consider for $\varepsilon > 0$ and $\gamma \in (0,1)$

$$M_{\alpha,\varepsilon} = \sup_{\substack{(t,x),(s,y)\in[0,T]\times\overline{B(x_\alpha,r)}\\ t<T}} \Big\{ u(t,x) - w(s,y) - \alpha\psi(x) - \frac{\eta}{T-t} - \frac{(t-s)^2}{2\nu} - G_\varepsilon^{\alpha,\gamma}(x,y) - \varphi^\alpha(t,s,x) \Big\}$$

where the function φ^α

$$\varphi^\alpha(t,s,x) = \frac{1}{2}\left(|t-t_\alpha|^2 + |s-s_\alpha|^2 + d^2(x,x_\alpha)\right)$$

will help us to localize the problem around $(t_\alpha, s_\alpha, x_\alpha)$, and $B(x_\alpha, r)$ is the open ball of radius $r = r(\alpha) > 0$ centered at x_α; besides, we choose $r \in (0,1)$ small enough such that $B(x_\alpha, r) \subset e$ if $x_\alpha \in e \setminus \mathcal{V}$. Lemma A.2 ensures that ψ and φ^α are semi-concave and therefore can be used as test functions, see Remark 7.6.

We choose

$$G_\varepsilon^{\alpha,\gamma}(x,y) = \varepsilon G^{\alpha,\gamma}(\varepsilon^{-1}x, \varepsilon^{-1}y)$$

with

$$G^{\alpha,\gamma}(x,y) = \begin{cases} \dfrac{(x-y)^2}{2} & \text{if } \; x_\alpha \in \mathcal{N}\setminus\mathcal{V}, \\ G^{x_\alpha,\gamma}(x,y) & \text{if } \; x_\alpha \in \mathcal{V}, \end{cases}$$

where $G^{x_\alpha,\gamma} \geq 0$ is the vertex test function of parameter $\gamma > 0$ given by Theorem 3.2, built on the junction problem associated to the vertex x_α at time t_α, *i.e.* associated to junction problem for the Hamiltonian $H_{\mathcal{V}}^{t_\alpha,x_\alpha}$ given by

$$H_{\mathcal{V}}^{t_\alpha,n}(x,p) := \begin{cases} H_e(t_\alpha,n,p) & \text{if } \; x \in e\setminus\{n\} \quad \text{with} \quad e \in \mathcal{E}_n, \\ F_A(t_\alpha,n,p) & \text{if } \; x = n. \end{cases} \tag{7.21}$$

The supremum in the definition of $M_{\alpha,\varepsilon}$ is reached at some point $(t,x),(s,y) \in [0,T] \times \overline{B(x_\alpha,r)}$ with $t < T$. These maximizers satisfy the following penalization estimates.

Lemma 7.14 (Penalization). *For $\varepsilon \in (0,1)$ and $\gamma \in (0, M/4)$, we have*

$$M_{\alpha,\varepsilon} \geq M_\alpha - \varepsilon\gamma \geq M/2 > 0 \tag{7.22}$$

$$d(x,y) \leq \omega(\varepsilon) \tag{7.23}$$

$$0 < \tau_\nu \leq s, t \leq T - \sigma_\eta$$

for some modulus of continuity ω (depending on α and γ) and τ_ν and σ_η not depending on (ε,γ). Moreover,

$$(t,s,x,y) \to (t_\alpha, s_\alpha, x_\alpha, x_\alpha) \quad \text{as } (\varepsilon,\gamma)\to(0,0).$$

In particular, we have $x,y\in B(x_\alpha,r)$ for $\varepsilon,\gamma>0$ small enough.

Proof of Lemma 7.14. For all $\varepsilon,\nu>0$, the compatibility on the diagonal (3.3) of the vertex test function $G^{x_\alpha,\gamma}$ yields the first inequality in (7.22). Then for $\varepsilon\in(0,1]$, with a choice of γ such that $0<\gamma<M/4$, we have the second one.

Bound on $d(x,y)$. Remark that

$$\varepsilon g\left(\frac{d(x,y)}{\varepsilon}\right) \le G_\varepsilon^{x_\alpha,\gamma}(x,y)$$

where

$$g(a)=\begin{cases}\dfrac{a^2}{2} & \text{if} \quad x_\alpha\in\mathcal{N}\setminus\mathcal{V},\\ g^{x_\alpha,\gamma}(a) & \text{if} \quad x_\alpha\in\mathcal{V},\end{cases}$$

and where $g^{x_\alpha,\gamma}$ is the superlinear function associated to $G^{x_\alpha,\gamma}$ and given by Theorem 3.2. Thanks to Lemma 7.10, we deduce that

$$\begin{aligned}0<M/2 &\le C(1+d(x,y)) - G_\varepsilon^{\alpha,\gamma}(x,y) - \frac{(t-s)^2}{2\nu} - \frac{\eta}{T-t} - \alpha\psi(x)\\ &\le C(1+d(x,y)) - \varepsilon g\left(\frac{d(x,y)}{\varepsilon}\right) - \frac{(t-s)^2}{2\nu} - \frac{\eta}{T-t} - \alpha\psi(x)\end{aligned} \tag{7.24}$$

which implies in particular that

$$\varepsilon g\left(\frac{d(x,y)}{\varepsilon}\right) \le C(1+d(x,y)).$$

This gives (7.23) as in Step 1 of the proof of Theorem 1.1.

First time estimate. From (7.24) with $G_\varepsilon^{\alpha,\gamma}\ge 0$ and (7.23), we deduce in particular that for $\varepsilon\in(0,1]$

$$0<M/2\le C' - \frac{(t-s)^2}{2\nu} - \frac{\eta}{T-t}.$$

This implies in particular that

$$T-t\ge\frac{\eta}{C'},\quad T-s\ge\frac{\eta}{C'}-\sqrt{2\nu C'}\ge\frac{\eta}{2C'}=:\sigma_\eta>0 \tag{7.25}$$

for $\nu>0$ small enough, and up to redefine σ_η for the new constant $C'\ge C$.

Second time estimate. From Lemma 7.11, we have with

$$\begin{aligned} 0 < M/2 &\le f(t) + f(s) + u_0(x) - u_0(y) - \tfrac{\eta}{T} - \tfrac{(t-s)^2}{2\nu} \\ &\le f(t) + f(s) + \omega_0 \circ \omega(\varepsilon) - \frac{\eta}{T} - \frac{(t-s)^2}{2\nu} \end{aligned}$$

where ω_0 is the modulus of continuity of u_0. Let us choose $\varepsilon > 0$ small enough such that

$$\omega_0 \circ \omega(\varepsilon) \le \frac{M}{2}. \tag{7.26}$$

As in the proof of Lemma 7.12, for $\tau = \min(t, s)$, we get

$$\frac{\eta}{T} \le 2f(\tau + \sqrt{2\nu C'}).$$

For ν small enough (with η fixed), we then get a contradiction if τ does not converge to 0 as ν does.

Convergence of maximizers. Because of (7.22) and using the fact that $G_\varepsilon^{\alpha,\gamma} \ge 0$, we get for $\varepsilon \in (0, 1]$

$$M_\alpha - \gamma \le M_{\alpha,\varepsilon} \le u(t, x) - w(s, y) - \alpha\psi(x) - \frac{\eta}{T-t} - \frac{(t-s)^2}{2\nu} - \varphi^\alpha(t, s, x).$$

Extracting a subsequence if needed, we can assume

$$(t, x, s, y) \to (\bar t, \bar x, \bar s, \bar x) \quad \text{as} \quad (\varepsilon, \gamma) \to (0, 0)$$

for some $\bar t, \bar s \in [\tau_\nu, T - \sigma_\eta]$, $\bar x \in \overline{B(x_\alpha, r)}$. We get

$$M_\alpha \le u(\bar t, \bar x) - w(\bar s, \bar x) - \alpha\psi(\bar x) - \frac{\eta}{T - \bar t} - \frac{(\bar t - \bar s)^2}{2\nu} - \varphi^\alpha(\bar t, \bar s, \bar x) \le M_\alpha - \varphi^\alpha(\bar t, \bar s, \bar x)$$

which implies that $(\bar t, \bar s, \bar x) = (t_\alpha, s_\alpha, x_\alpha)$. □

Step 4: Viscosity inequalities. Then we can write the viscosity inequalities at (t, x) and (s, y) using the shorthand notation (7.6),

$$\begin{aligned} \frac{\eta}{(T-t)^2} + \frac{t-s}{\nu} + (t - t_\alpha) + H_{\mathcal N}(t, x, p_x^{\alpha,\gamma,\varepsilon} + \alpha\psi_x(x) + \varphi_x^\alpha(t, s, x)) &\le 0 \\ \frac{t-s}{\nu} - (s - s_\alpha) + H_{\mathcal N}(s, y, p_y^{\alpha,\gamma,\varepsilon}) &\ge 0 \end{aligned} \tag{7.27}$$

where

$$\begin{cases} p_x^{\alpha,\gamma,\varepsilon} = G_x^{\alpha,\gamma}(\varepsilon^{-1}x, \varepsilon^{-1}y), \\ p_y^{\alpha,\gamma,\varepsilon} = -G_y^{\alpha,\gamma}(\varepsilon^{-1}x, \varepsilon^{-1}y). \end{cases}$$

We choose ε, γ small enough such that (Lemma 7.14) we have

$$|t - t_\alpha|, \quad |s - s_\alpha| \quad \le \frac{\eta}{4T^2}.$$

Substracting the two viscosity inequalities, we get

$$\frac{\eta}{2T^2} \le H_{\mathcal N}(s, y, p_y^{\alpha,\gamma,\varepsilon}) - H_{\mathcal N}(t, x, p_x^{\alpha,\gamma,\varepsilon} + \alpha\psi_x(x) + \varphi_x^\alpha(t, s, x)). \tag{7.28}$$

Step 5: Gradient estimates. We deduce from (7.27) that

$$\tilde{p}_x^{\alpha,\gamma,\varepsilon} = p_x^{\alpha,\gamma,\varepsilon} + \alpha\psi_x(x) + \varphi_x^\alpha(t,s,x)$$

satisfies

$$H_{\mathcal{N}}(t,x,\tilde{p}_x^{\alpha,\gamma,\varepsilon}) \le \frac{s-t}{\nu} + t_\alpha - t \le \frac{T}{\nu} + T. \tag{7.29}$$

Hence (H1) implies that there exists a constant C'_ν (independent of α, ε, γ, but depending on η,ν) such that

$$\begin{cases} |\tilde{p}_x^{\alpha,\gamma,\varepsilon}| \le C'_\nu & \text{if} \quad x \ne x_\alpha \quad \text{or} \quad x_\alpha \notin \mathcal{V}, \\ \tilde{p}_x^{\alpha,\gamma,\varepsilon} \ge -C'_\nu & \text{if} \quad x = x_\alpha \quad \text{and} \quad x_\alpha \in \mathcal{V}. \end{cases}$$

From (7.17), we deduce that

$$|\alpha\psi_x(x) + \varphi_x^\alpha(t,s,x)| \le C\sqrt{\alpha} + d(x,x_\alpha) \le C \tag{7.30}$$

for $\alpha \le 1$ (using (7.17)). Therefore, we have for some constant C_ν (independent of α, ε, γ):

$$\begin{cases} |p_x^{\alpha,\gamma,\varepsilon}| \le C_\nu & \text{if} \quad x \ne x_\alpha \quad \text{or} \quad x_\alpha \notin \mathcal{V}, \\ p_x^{\alpha,\gamma,\varepsilon} \ge -C_\nu & \text{if} \quad x = x_\alpha \quad \text{and} \quad x_\alpha \in \mathcal{V}. \end{cases}$$

From the compatibility condition of the Hamiltonians satisfied by $G^{\alpha,\gamma}$ if $x_\alpha \in \mathcal{V}$, or the definition of $G^{\alpha,\gamma}$ if $x_\alpha \notin \mathcal{V}$, we have in both cases,

$$H^{t_\alpha,x_\alpha}(y,p_y^{\alpha,\gamma,\varepsilon}) \le H^{t_\alpha,x_\alpha}(x,p_x^{\alpha,\gamma,\varepsilon}) + \gamma \tag{7.31}$$

where

$$H^{t_\alpha,x_\alpha}(x,p) = \begin{cases} H_{\mathcal{V}}^{t_\alpha,n}(x,p) & \text{if} \quad x_\alpha = n \in \mathcal{V}, \\ H_e(t_\alpha,x_\alpha,p) & \text{if} \quad x_\alpha \notin \mathcal{V}, x_\alpha \in e^*. \end{cases}$$

We deduce that $p_y^{\alpha,\gamma,\varepsilon}$ satisfies (modifying C_ν if necessary)

$$\begin{cases} |p_y^{\alpha,\gamma,\varepsilon}| \le C_\nu & \text{if} \quad y \ne x_\alpha \quad \text{or } x_\alpha \notin \mathcal{V}, \\ p_y^{\alpha,\gamma,\varepsilon} \ge -C_\nu & \text{if} \quad y = x_\alpha \quad \text{and} \quad x_\alpha \in \mathcal{V}. \end{cases}$$

Defining for $z = x, y$,

$$\bar{p}_z^{\alpha,\gamma,\varepsilon} = \begin{cases} \left(\min\left(K, (p_z^{\alpha,\gamma,\varepsilon})_{\tilde{z}}\right)\right)_{\tilde{z}\in x_\alpha} & \text{if} \quad z = x_\alpha \quad \text{and} \quad x_\alpha \in \mathcal{V} \\ p_z^{\alpha,\gamma,\varepsilon} & \text{if not.} \end{cases}$$

with, in the case where $x_\alpha \in \mathcal{V}$, the constant K given by

$$K = \max_{e\in\mathcal{E}_{x_\alpha}}(p_e^0(s,x_\alpha), p_e^0(t_\alpha,x_\alpha), p_e^0(t,x_\alpha) + C)) \le C_T + C$$

(C comes from (7.30) and C_T from (7.8)), we have

$$|\bar{p}_z^{\alpha,\gamma,\varepsilon}| \le C_\nu + C_T + C =: C_{\nu,T}$$

and

$$\frac{\eta}{2T^2} \le H_{\mathcal{N}}(s,y,\bar{p}_y^{\alpha,\gamma,\varepsilon}) - H_{\mathcal{N}}(t,x,\bar{p}_x^{\alpha,\gamma,\varepsilon} + \alpha\psi_x(x) + \varphi_x^\alpha(t,s,x)), \tag{7.32}$$

$$H_{\mathcal{N}}(t,x,\bar{p}_x^{\alpha,\gamma,\varepsilon} + \alpha\psi_x(x) + \varphi_x^\alpha(t,s,x)) \le \frac{s-t}{\nu} + t_\alpha - t \le \frac{T}{\nu} + T, \tag{7.33}$$

$$H^{t_\alpha,x_\alpha}(y,\bar{p}_y^{\alpha,\gamma,\varepsilon}) \le H^{t_\alpha,x_\alpha}(x,\bar{p}_x^{\alpha,\gamma,\varepsilon}) + \gamma. \tag{7.34}$$

Step 6: The limit $(\varepsilon,\gamma) \to (0,0)$ and conclusion as $\alpha \to 0$. Up to a subsequence, we get in the limit $(\varepsilon,\gamma) \to (0,0)$ for $z = x, y$:

$$\bar{p}_z^{\alpha,\gamma,\varepsilon} \to \bar{p}_z^\alpha \quad \text{with} \quad |\bar{p}_z^\alpha| \le C_{\nu,T}.$$

Moreover, passing to the limit in (7.32) and (7.33), we get respectively

$$\frac{\eta}{2T^2} \le H_{\mathcal{N}}(s_\alpha,x_\alpha,\bar{p}_y^\alpha) - H_{\mathcal{N}}(t_\alpha,x_\alpha,\bar{p}_x^\alpha + \alpha\psi_x(x_\alpha))$$

and

$$H_{\mathcal{N}}(t_\alpha,x_\alpha,\bar{p}_x^\alpha + \alpha\psi_x(x_\alpha)) \le \frac{s_\alpha - t_\alpha}{\nu} \le \frac{T}{\nu}.$$

On the other hand, passing to the limit in (7.34) gives

$$H^{t_\alpha,x_\alpha}(x_\alpha,\bar{p}_y^\alpha) \le H^{t_\alpha,x_\alpha}(x_\alpha,\bar{p}_x^\alpha).$$

Because

$$H_{\mathcal{N}}(t_\alpha,x_\alpha,p) = H^{t_\alpha,x_\alpha}(x_\alpha,p)$$

we get for any p,

$$\frac{\eta}{2T^2} \le I_1 + I_2$$

with

$$\begin{aligned}
I_1 &= H_{\mathcal{N}}(s_\alpha,x_\alpha,\bar{p}_x^\alpha) - H_{\mathcal{N}}(s_\alpha,x_\alpha,\bar{p}_x^\alpha + \alpha\psi_x(x_\alpha)),\\
I_2 &= H_{\mathcal{N}}(s_\alpha,x_\alpha,\bar{p}_x^\alpha + \alpha\psi_x(x_\alpha)) - H_{\mathcal{N}}(t_\alpha,x_\alpha,\bar{p}_x^\alpha + \alpha\psi_x(x_\alpha)).
\end{aligned}$$

Thanks to (H3) and (7.17), we have $|\alpha\psi_x(x_\alpha)| \le C_{\nu,T}$ and we thus get

$$I_1 \le \omega_{T,2C_{\nu,T}}(\alpha\psi_x(x_\alpha)) \le \omega_{T,2C_\nu}(C\sqrt{\alpha}). \tag{7.35}$$

Now thanks to Lemma 7.2, we also have

$$\begin{aligned}
I_2 &\le \tilde{\omega}_T(|t_\alpha - s_\alpha|(1 + \max(H_{\mathcal{N}}(t_\alpha,x_\alpha,\bar{p}_x^\alpha + \alpha\psi_x(x_\alpha)),0)))\\
&\le \tilde{\omega}_T(|t_\alpha - s_\alpha|(1 + \max(\frac{s_\alpha - t_\alpha}{\nu},0))).
\end{aligned}$$

Then taking first the limit $\alpha \to 0$ and then taking the limit $\nu \to 0$, we use (7.19) to get the desired contradiction. This achieves the proof of Theorem 7.8. $\square$

8 Third application: a homogenization result for a network

In this section, we present an application of the comparison principle of viscosity sub- and super-solutions on networks.

8.1 A homogenization problem

We consider the simplest periodic network generated by $\varepsilon\mathbb{Z}^d$. Hence, the network is naturally embedded in $\mathbb{R}^d$. Let us be more precise now. At scale $\varepsilon = 1$, the edges are the following subsets of $\mathbb{R}^d$: for $k, l \in \mathbb{Z}^d$, $|k-l| = 1$,

$$e_{k,l} = \{\theta k + (1-\theta)l : \theta \in [0,1]\}.$$

If $(e_1, \ldots, e_d)$ denotes the canonical basis of $\mathbb{R}^d$, then for $l = k + e_i$, $e_{k,l}$ is oriented in the direction of e_i. The network $\mathcal{N}_\varepsilon$ at scale $\varepsilon > 0$ is the one corresponding to

$$\begin{cases} \mathcal{E}_\varepsilon = \{\varepsilon e_{k,l}, k, l \in \mathbb{Z}^d, |k-l| = 1\} \\ \mathcal{V}_\varepsilon = \varepsilon\mathbb{Z}^d \end{cases}$$

endowed with the metric induced by the Euclidian norm. We next consider the following "oscillating" Hamilton-Jacobi equation on this network

$$\begin{cases} u_t^\varepsilon + H_{\frac{e}{\varepsilon}}(u_x^\varepsilon) = 0, & t > 0,\ x \in e^*, e \in \mathcal{E}_\varepsilon, \\ u_t^\varepsilon + F_A(\frac{x}{\varepsilon}, u_x^\varepsilon) = 0, & t > 0,\ x \in \mathcal{V}_\varepsilon \end{cases} \tag{8.1}$$

(for some $A \in \mathbb{R}$) submitted to the initial condition

$$u^\varepsilon(0, x) = u_0(x), \qquad x \in \mathcal{N}_\varepsilon. \tag{8.2}$$

For $m \in \mathbb{Z}^d$, it is convenient to define

$$\varepsilon e_{k,l} + \varepsilon m = \varepsilon e_{k+m,l+m}.$$

Assumptions on H for the homogenization problem For each $e \in \mathcal{N}_1$, we associate a Hamiltonian H_e and we assume

- **(H'0)** (Continuity) For all $e \in \mathcal{E}_1$, $H_e \in C(\mathbb{R})$.
- **(H'1)** (Coercivity) $e \in \mathcal{E}_1$,
$$\liminf_{|q| \to +\infty} H_e(q) = +\infty.$$
- **(H'2)** (Level-set convexity) For all $e \in \mathcal{E}_1$, there exists a $p_e^0 \in \mathbb{R}$ such that
$$\begin{cases} H_e & \text{is nonincreasing on} \quad (-\infty, p_e^0], \\ H_e & \text{is nondecreasing on} \quad [p_e^0, +\infty). \end{cases}$$
- **(H'3)** (Periodicity) For all $m \in \mathbb{Z}^d$, $H_{e+m}(p) = H_e(p)$.

A homogenization result The goal of this section is to prove the following convergence result for the oscillating Hamilton-Jacobi equation.

Theorem 8.1. *Assume* (H'0)-(H'3). *Let u_0 be Lipschitz continuous and u^ε be the solution of* (8.1)-(8.2). *There exists a continuous function $\bar{H} : \mathbb{R}^d \to \mathbb{R}$ such that u^ε converges locally uniformly towards the unique solution u^0 of*

$$u^0_t + \bar{H}(\nabla_x u^0) = 0, \qquad t > 0, x \in \mathbb{R}^d \tag{8.3}$$

$$\bar{u}^0(0,x) = u_0(x), \qquad x \in \mathbb{R}^d. \tag{8.4}$$

Remark 8.2. The meaning of the convergence u^ε towards u^0 is

$$\lim_{\substack{(s,y)\to(t,x)\\ y\in\mathcal{N}_\varepsilon}} u^\varepsilon(s,y) = u^0(t,x).$$

8.2 The cell problem

Keeping in mind the definitions of networks and derivatives of functions defined on networks, solving the cell problem consists in finding specific global solutions of (8.1) for $\varepsilon = 1$, *i.e.*

$$\begin{cases} w_t + H_e(w_y) = 0, & t \in \mathbb{R},\ y \in e^*, e \in \mathcal{E}_1, \\ w_t + F_A(y, w_y) = 0, & t \in \mathbb{R},\ y \in \mathcal{V}_1. \end{cases} \tag{8.5}$$

Precisely, for some $P \in \mathbb{R}^d$, we look for solutions $w(t,y) = \lambda t + P \cdot y + v(y)$ with a $\mathbb{Z}^d$-periodic function v; in other words, we look for (λ, v) such that

$$\begin{cases} \lambda + H_e((P\cdot y + v)_y) = 0, & y \in e^*, e \in \mathcal{E}_1, \\ \lambda + F_A(y, (P\cdot y + v)_y) = 0, & y \in \mathcal{V}_1. \end{cases} \tag{8.6}$$

Theorem 8.3. *For all $P \in \mathbb{R}^d$ there exists $\lambda \in \mathbb{R}$ and a $\mathbb{Z}^d$-periodic solution v of* (8.6). *Moreover, the function $\bar{H}$ which maps P to $-\lambda$ is continuous.*

Proof. We consider the following $\mathbb{Z}^d$-periodic stationary problem

$$\begin{cases} \alpha v^\alpha + H_e((P\cdot y + v^\alpha)_y) = 0, & y \in e^*, e \in \mathcal{E}_1, \\ \alpha v^\alpha + F_A(y, (P\cdot y + v^\alpha)_y) = 0, & y \in \mathcal{V}_1. \end{cases} \tag{8.7}$$

We consider

$$C = \max_{e\in\mathcal{E}_1} |H_e((P\cdot y)_y)|.$$

Then the existence result and the comparison principle for the stationary equation (see Appendix B) imply that there exists a (unique) $\mathbb{Z}^d$-periodic solution v^α of (8.7) such that

$$|\alpha v^\alpha| \le C.$$

Since H_e is coercive, this implies that there exists a constant $\tilde{C}$ such that for all $\alpha > 0$, v_α is Lipschitz-continuous and

$$|v_y^\alpha| \leq \tilde{C};$$

in other words, the family $(v^\alpha)_{\alpha>0}$ is equi-Lipschitz continuous. We then consider

$$\tilde{v}_\alpha = v_\alpha - v_\alpha(0).$$

By Arzelà-Ascoli theorem, there exists $\alpha_n \to 0$ such that $\tilde{v}^n := \tilde{v}_{\alpha_n}$ converges uniformly towards v. Moreover, we can also assume that

$$\alpha_n v_{\alpha_n}(0) \to \lambda.$$

Passing to the limit into the equation yields that (λ, v) solves the cell problem (8.6).

The continuity of λ is completely classical too. Consider $P_n \to P_\infty$ as $n \to \infty$ and consider (λ_n, v_n) solving (8.6). We proved above that

$$|\lambda_n| \leq C.$$

Hence, arguing as above, we can extract a subsequence from (λ_n, v_n) converging towards $(\lambda_\infty, v_\infty)$. Passing to the limit into the equation implies that $(\lambda_\infty, v_\infty)$ solves the cell problem (8.6). The uniqueness of λ yields the continuity of $\bar{H}$. The proof is now complete. $\square$

8.3 Proof of convergence

Before proving the convergence, we state without proof the following elementary lemma.

Lemma 8.4 (Barriers)**.** *There exists $C > 0$ such that for all $\varepsilon > 0$,*

$$|u^\varepsilon(t, x) - u_0(x)| \leq Ct.$$

We can now turn to the proof of convergence.

Proof of Theorem 8.1. We classically consider the relaxed semi-limits

$$\begin{cases} \overline{u}(t, x) = \limsup_{\substack{\varepsilon\to 0,(s,y)\to(t,x) \\ y\in\mathcal{N}_\varepsilon}} u^\varepsilon(s, y), \\ \underline{u}(t, x) = \liminf_{\substack{\varepsilon\to 0,(s,y)\to(t,x) \\ y\in\mathcal{N}_\varepsilon}} u^\varepsilon(s, y). \end{cases}$$

In order to prove convergence of u^ε towards u^0, it is enough to prove that $\overline{u}$ is a sub-solution of (8.3) and $\underline{u}$ is a super-solution of (8.3). We only prove that $\overline{u}$ is a sub-solution since the proof for $\underline{u}$ is very similar.

We consider a test function φ touching (strictly) $\overline{u}$ from above at (t_0, x_0): there exists $r_0 > 0$ such that for all $(t, x) \in B_{r_0}(t_0, x_0)$, $(t, x) \neq (t_0, x_0)$,

$$\varphi(t, x) > \overline{u}(t, x)$$

and $\varphi(t_0, x_0) = \overline{u}(t_0, x_0)$. We argue by contradiction by assuming that there exists $\theta > 0$ such that

$$\partial_t \varphi(t_0, x_0) - \lambda = \partial_t \varphi(t_0, x_0) + \bar{H}(\nabla_x \varphi(t_0, x_0)) = \theta > 0. \tag{8.8}$$

We then consider the following "perturbed test" function $\varphi^\varepsilon \colon \mathbb{R}^+ \times \mathcal{N}_\varepsilon \to \mathbb{R}$ [12],

$$\varphi^\varepsilon(t, x) = \varphi(t, x) + \varepsilon v(\varepsilon^{-1} x)$$

where (λ, v) solves the cell problem (8.6) for $P = \nabla_x \varphi(t_0, x_0)$.

Lemma 8.5. *For $r \leq r_0$ small enough, the function φ^ε is a super-solution of* (8.1) *in $B((t_0, x_0), r) \subset \mathcal{N}_\varepsilon$ and $\varphi^\varepsilon \geq u^\varepsilon + \eta_r$ in $\partial B(t_0, x_0), r)$ for some $\eta_r > 0$.*

Proof. Consider a test function ψ touching φ^ε from below at $(t, x) \in]0, +\infty[\times \mathcal{N}_\varepsilon$. Then the function

$$\psi_\varepsilon(s, y) = \varepsilon^{-1}(\psi(s, \varepsilon y) - \varphi(s, \varepsilon y))$$

touches v from below at $y = \frac{x}{\varepsilon} \in e$. In particular,

$$\partial_t \psi(t, x) = \partial_t \varphi(t, x), \tag{8.9}$$

$$\lambda + H_{\mathcal{N}_1}(y, \varphi_x(t_0, x_0) + \psi_x(t, x) - \varphi_x(t, x)) \geq 0. \tag{8.10}$$

Combine now (8.8), (8.9) and (8.10) and get

$$\partial_t \psi(t, x) + H_{\mathcal{N}_1}(y, \psi_x(t, x)) \geq \theta + E$$

where

$$E = (\varphi_t(t, x) - \varphi_t(t_0, x_0)) + (H_{\mathcal{N}_1}(y, \psi_x(t, x)) - H_{\mathcal{N}_1}(y, \psi_x(t, x) + \varphi_x(t_0, x_0) - \varphi_x(t, x))).$$

The fact that φ is C^1 implies that we can choose $r > 0$ small enough so that for all $(t, x) \in B((t_0, x_0), r)$,

$$E \geq -\theta.$$

Moreover, since φ is strictly above $\overline{u}$, we conclude that $\varphi^\varepsilon \geq u^\varepsilon + \eta_r$ on $\partial B((t_0, x_0), r)$ for some $\eta_r > 0$. This achieves the proof of the lemma. $\square$

From the lemma, we deduce thanks to the (localized) comparison principle that

$$\varphi^\varepsilon(t, x) \geq u^\varepsilon(t, x) + \eta_r.$$

In particular, this implies

$$u(t_0, x_0) = \varphi(t_0, x_0) \geq u(t_0, x_0) + \eta_r > u(t_0, x_0)$$

which is the desired contradiction. $\square$

8.4 Qualitative properties of the effective Hamiltonian

Proposition 8.6. *If for all $e \in \mathcal{E}_1$, $p \mapsto H_e(p)$ is convex, then so is $\bar{H} : \mathbb{R}^d \to \mathbb{R}$.*

This proposition is a simple consequence of (8.11) which is a classical characterization of λ if (λ, v) solves the cell problem.

Proposition 8.7. *Let (λ, v) be a solution of the cell problem (8.6). Then*

$$\begin{aligned} &\lambda = \inf\{\overline{\lambda} : \exists \text{ a } \mathbb{Z}^d\text{-periodic super-solution } \overline{v} \text{ of } (8.6) \text{ with } \lambda = \overline{\lambda}\}, \\ &\lambda = \sup\{\underline{\lambda} : \exists \text{ a } \mathbb{Z}^d\text{-periodic sub-solution } \underline{v} \text{ of } (8.6) \text{ with } \lambda = \underline{\lambda}\}. \end{aligned} \tag{8.11}$$

9 Remarks for schemes on the real line (two branches)

9.1 Recalling the framework

We consider two C^1 Hamiltonians H_1, H_2 satisfying the level-set convexity condition (1.5), and consider a solution $u(t,x)$ of

$$\left\{ \begin{array}{lll} u_t + H_1(u_x) = 0 & \text{for} & (t,x) \in (0,+\infty) \times (-\infty, 0), \\ u_t + H_2(u_x) = 0 & \text{for} & (t,x) \in (0,+\infty) \times (0,+\infty), \\ u_t + \check{F}(u_x(t,0^-), u_x(t,0^+)) = 0 & \text{for} & (t,x) \in (0,+\infty) \times \{0\} \end{array} \right. \tag{9.1}$$

with

(9.2)

$$\left\{ \begin{array}{l} \check{F} : \mathbb{R}^2 \to \mathbb{R} \quad \text{is Lipschitz continuous on bounded sets,} \\ \\ \check{F}(p_1,p_2) \quad \text{is nonincreasing in } p_2 \text{ and nondecreasing in } p_1, \\ \\ \text{there exists a constant } K \in \mathbb{R}, \text{ and reals } p_1^K, p_2^K \text{ such that} \quad \check{F}(p_1,p_2) \le K \Longrightarrow \left\{ \begin{array}{l} p_2 \ge p_2^K, \\ p_1 \le p_1^K \end{array} \right. \end{array} \right.$$

We also assume that

$$u(0,x) = u_0(x) \quad \text{for} \quad x \in \mathbb{R} \tag{9.3}$$

with u_0 globally Lipschitz continuous such that there exists two constants $L_a \le L_b$ with

$$L_a h \le u_0(x+h) - u_0(x) \le L_b h, \quad \text{for all} \quad h \ge 0, \quad x \in \mathbb{R} \tag{9.4}$$

Considering

$$\tilde{u}(t,x) = \left\{ \begin{array}{lll} u(t,-x) & \text{if} & x \in J_1, \\ u(t,x) & \text{if} & x \in J_2, \end{array} \right.$$

we will say that u is a relaxed viscosity solution of (9.1), (9.3), if $\tilde{u}$ is a relaxed viscosity solution of the associated PDE that it satisfies. We recall that each Hamiltonian H_α is minimal in p_α^0 and recall that we set

$$A_0 := \max_{\alpha=1,2} H_\alpha(p_\alpha^0)$$

Following Proposition 2.5, if $\check{F}(p_1^0, p_2^0) < A_0$, then we set

$$A = A_0$$

and if $\check{F}(p_1^0, p_2^0) \geq A_0$, we define $A \geq A_0$ as the unique real such that there exists $p_2 \geq p_2^0$ and $p_1 \leq p_1^0$ such that

$$A = H_1(p_1) = H_2(p_2) = \check{F}(p_1, p_2).$$

Then rephrasing Proposition 2.5, we know that u is a viscosity solution with the same initial data (9.3) and solves

$$\text{(9.5)} \qquad \begin{cases} u_t + H_1(u_x) = 0 & \text{for} \quad (t,x) \in (0,+\infty) \times (-\infty, 0), \\ u_t + H_2(u_x) = 0 & \text{for} \quad (t,x) \in (0,+\infty) \times (0, +\infty), \\ u_t + \check{F}_A(u_x(t,0^-), u_x(t,0^+)) = 0 & \text{for} \quad (t,x) \in (0,+\infty) \times \{0\} \end{cases}$$

where we recall that

$$\check{F}_A(q_1, q_2) = \max(A, H_1^+(q_1), H_2^-(q_2))$$

where we recall that

$$H_\alpha^-(q) = \begin{cases} H_\alpha(q) & \text{if} \quad q < p_\alpha^0, \\ H_\alpha(p_\alpha^0) & \text{if} \quad q \leq p_\alpha^0 \end{cases} \quad \text{and} \quad H_\alpha^+(q) = \begin{cases} H_\alpha(p_\alpha^0) & \text{if} \quad q \leq p_\alpha^0, \\ H_\alpha(q) & \text{if} \quad q > p_\alpha^0. \end{cases}$$

9.2 The scheme

We now consider an approximation U_i^n of $u(n\Delta t, i\Delta x)$ for some time step and space step $\Delta t, \Delta x > 0$, which has the following initial data

$$\text{(9.6)} \qquad U_i^0 = u_0(i\Delta x) \quad \text{for} \quad i \in \mathbb{Z}$$

We set

$$p_{i,+}^n = \frac{U_{i+1}^n - U_i^n}{\Delta x}, \quad p_{i,-}^n = \frac{U_i^n - U_{i-1}^n}{\Delta x}$$

and assume that $(U_i^n)_{n,i}$ solves the following scheme for $n \geq 0$:

$$\text{(9.7)} \qquad \frac{U_i^{n+1} - U_i^n}{\Delta t} = \begin{cases} -\max\left(H_2^-(p_{i,+}^n),\ H_2^+(p_{i,-}^n)\right), & \text{for} \quad i \geq 1, \\ -\max\left(H_1^-(p_{i,+}^n),\ H_1^+(p_{i,-}^n)\right), & \text{for} \quad i \leq -1, \\ -\check{F}(p_{i,-}^n, p_{i,+}^n) & \text{for} \quad i = 0, \end{cases}$$

We now define

$$m_0^0 = \min\left(\inf_{\alpha=1,2;\ p\in[L_a,L_b]} -H_\alpha(p), \qquad \min_{(p_1,p_2)\in[L_a,L_b]^2} -\check{F}(p_1,p_2)\right)$$

and for $\alpha = 1, 2$

$$\begin{cases} p_\alpha^{0,+} = (H_\alpha^+)^{-1}(-m_0^0), \\ p_\alpha^{0,-} = (H_\alpha^-)^{-1}(-m_0^0) \end{cases}$$

We set (with the notation of (9.2))

$$\begin{cases} \overline{p}_1^0 = \max(p_1^{-m_0^0}, p_1^{0,+}) \\ \overline{p}_2^0 = p_2^{0,+} \\ \underline{p}_1^0 = p_1^{0,-} \\ \underline{p}_1^0 = \min(p_2^{-m_0^0}, p_2^{0,-}) \end{cases}$$

Because $\check{F}$ is Lipschitz continuous on bounded sets, and because of the monotonicity of $\check{F}$, we deduce that there exists $L^+, L^- \geq 0$ such that
(9.8)
$$\check{F}(q_1,q_2)-\check{F}(p_1,p_2) \geq L^+\min(0,p_2-q_2)-L^-\max(0,p_1-q_1), \quad \text{for all} \quad (p_1,p_2),(q_1,q_2) \in \left[\underline{p}_1^0, \overline{p}_1^0\right]\times\left[\underline{p}_2^0, \overline{p}_2^0\right]$$

We consider the following CFL condition

$$\frac{\Delta x}{\Delta t} \geq \max(L^+ + L^-,\ \max_{\alpha=1,2}\ \max_{p_\alpha\in\left[\underline{p}_\alpha^0, \overline{p}_\alpha^0\right]} |H'_\alpha(p_\alpha)|) \tag{9.9}$$

Theorem 9.1 (Convergence of the scheme)**.** *Under the previous assumptions, let*

$$\varepsilon = (\Delta t, \Delta x)$$

and let us define

$$u^\varepsilon(n\Delta t, i\Delta x) = U_i^n \tag{9.10}$$

Then u^ε does converge locally uniformly to the unique solution u of (9.5) with initial data (9.3).

Proof.
Step 1: $\inf_{j\in\mathbb{Z}} W_j^n$ is nondecreasing in n
The only change with respect to the proof in [11] is to take into account the general junction condition $\check{F}$ for $i = 0$. Setting

$$W_i^n = \frac{U_i^{n+1} - U_i^n}{\Delta t},$$

we get for $i = 0$

$$\begin{aligned}\frac{W_i^{n+1} - W_i^n}{\Delta t} &= \frac{1}{\Delta t}\left(\check{F}(p_{i,-}^n, p_{i,+}^n) - \check{F}(p_{i,-}^{n+1}, p_{i,+}^{n+1})\right)\\ &\geq \frac{1}{\Delta t}\left(L^+ \min(0, p_{i,+}^{n+1} - p_{i,+}^n) - L^- \max(0, p_{i,-}^{n+1} - p_{i,-}^n)\right)\\ &= \frac{1}{\Delta x}\left(L^+ \min(0, W_{i+1}^n - W_i^n) - L^- \max(0, W_i^n - W_{i-1}^n)\right)\end{aligned}$$

i.e. for $i = 0$

$$W_i^{n+1} \geq W_i^n + \frac{\Delta t}{\Delta x}\left(L^+ \min(0, W_{i+1}^n - W_i^n) - L^- \max(0, W_i^n - W_{i-1}^n)\right) \geq \inf_{j\in\mathbb{Z}} W_j^n$$

if

$$\frac{\Delta t}{\Delta x}(L^+ + L^-) \leq 1$$

Step 2: Bound on the gradients
If we know that for $n \geq 0$:

$$W_i^n \geq m_0^0 \tag{9.11}$$

For $i \neq 0$, we deduce that

$$p_\alpha^{0,-} \leq p_{i,+}^n \quad \text{with} \quad \begin{cases} \alpha = 2 & \text{if} \quad i \geq 1,\\ \alpha = 1 & \text{if} \quad i \leq -1\end{cases}$$

and

$$p_{i,-}^n \leq p_\alpha^{0,+} \quad \text{with} \quad \begin{cases} \alpha = 2 & \text{if} \quad i \geq 1,\\ \alpha = 1 & \text{if} \quad i \leq -1\end{cases}$$

This gives in particular:

$$p_1^{0,-} \leq p_{-1,+}^n, \quad p_{0,+}^n \leq p_2^{0,+}$$

To get the missing bounds on $p_{-1,+}^n$ and $p_{0,+}^n$, we simply notice that for $i = 0$, we deduce from (9.11) that

$$-\check{F}(p_{-1,+}^n, p_{0,+}^n) \geq m_0^0$$

From the third line of (9.2), we deduce that

$$p_{-1,+}^n \leq p_1^{-m_0^0}, \quad p_{0,+}^n \geq p_2^{-m_0^0}$$

This implies in particular that

$$\begin{cases} p_{i,+}^n \in \left[\underline{p}_2^0, \overline{p}_2^0\right] & \text{for} \quad i \geq 0,\\ \\ p_{i,+}^n \in \left[\underline{p}_1^0, \overline{p}_1^0\right] & \text{for} \quad i \leq -1\end{cases} \tag{9.12}$$

Step 3: convergence. Proceeding as in [11], we can show that we have both the bound from below on the discrete time derivative (9.11) and the gradient bounds (9.12). From the scheme, we deduce that the discrete time derivative is bounded, uniformly with respect to $\varepsilon = (\Delta t, \Delta x) \to (0,0)$. We also easily deduce that any Lipschitz continuous limit u of u^ε (defined in (9.10)) as $\varepsilon \to (0,0)$ is a relaxed viscosity solution of (9.1) with initial data (9.3). Because we also know that u is a $\check{F}_A$-solution (i.e. solution of (9.5)), we deduce from the uniqueness that u is unique. We also deduce that this u would have been obtained more easily with a similar scheme with $\check{F}$ replaced by $\check{F}_A$. □

9.3 Application to schemes for conservation laws

We can consider the discrete space derivative of the scheme (9.7). Let us call

$$\rho_i^n = p_{i,+}^n$$

Then $(\rho_i^n)_{n,i}$ solves the following scheme for $n \geq 0$

$$\frac{\rho_i^{n+1} - \rho_i^n}{\Delta t} = \begin{cases} g_2(\rho_{i-1}^n, \rho_i^n) - g_2(\rho_i^n, \rho_{i+1}^n), & \text{for} \quad i \geq 1, \\ g_1(\rho_{i-1}^n, \rho_i^n) - g_1(\rho_i^n, \rho_{i+1}^n), & \text{for} \quad i \leq -2, \\ g_1(\rho_{i-1}^n, \rho_i^n) - \check{F}(\rho_i^n, \rho_{i+1}^n), & \text{for} \quad i = -1, \\ \check{F}(\rho_{i-1}^n, \rho_i^n) - g_2(\rho_i^n, \rho_{i+1}^n), & \text{for} \quad i = 0 \end{cases} \tag{9.13}$$

with for $\alpha = 1,2$

$$g_\alpha(a,b) = \max\left(H_\alpha^-(b),\ H_\alpha^+(a)\right)$$

with initial data

$$\rho_i^0 = \frac{u_0((i+1)\Delta x) - u_0(i\Delta x)}{\Delta x} \tag{9.14}$$

As a by-product of Theorem 9.1, we deduce that the numerical solution ρ_i^n converges to the function u_x in particular in the sense of distributions (where u is the limit solution associated to the Hamilton-Jacobi scheme (9.7), given in Theorem 9.1), for a general junction condition $\check{F}$ satisfying (9.2) (which is not standard).

A Appendix: proofs of some technical results

A.1 Technical results on a junction

In order to prove Lemma 3.3, we need the following one.

Lemma A.1 (A priori control at the same time)**.** *Let $T > 0$ and let u be a sub-solution and w be a super-solution as in Theorem 1.1. Then there exists a constant $C_T > 0$ such that for all $t \in [0,T), x, y \in J$, we have*

$$u(t,x) \leq w(t,y) + C_T(1 + d(x,y)). \tag{A.1}$$

We first derive Lemma 3.3 from Lemma A.1.

Proof of Lemma 3.3. Let us fix some $\varepsilon > 0$ and let us consider the sub-solution u_ε^- and super-solutions u_ε^+ defined in (2.18). Using (2.17), we see that we have for all $(t,x), (s,y) \in [0,T) \times J$

$$u_\varepsilon^+(t,x) - u_\varepsilon^-(s,y) \leq 2C_\varepsilon T + 2\varepsilon + L_\varepsilon d(x,y) \tag{A.2}$$

We first apply Lemma A.1 to control $u(t,x) - u_\varepsilon^+(t,x)$, and then apply Lemma A.1 to control $u_\varepsilon^-(s,y) - w(s,y)$. Finally we get the control on $u(t,x) - w(s,y)$, using (A.2). □

We now turn to the proof of Lemma A.1.

Proof of Lemma A.1. Let us define

$$\varphi(x,y) = \sqrt{1 + d^2(x,y)}.$$

Then $\varphi \in C^1(J^2)$ and satisfies

$$|\varphi_x(x,y)|, \; |\varphi_y(x,y)| \leq 1. \tag{A.3}$$

For constants $C_1, C_2 > 0$ to be chosen, let us consider

$$M = \sup_{t \in [0,T), \; x,y \in J} \left(u(t,x) - w(t,y) - C_2 t - C_1 \varphi(x,y) \right).$$

The result follows if we show that M is non-positive for C_1 and C_2 large enough. Assume by contradiction that $M > 0$ for any C_1 and C_2. Then for $\eta, \alpha > 0$ small enough, we have $M_{\alpha,\eta} \geq M/2 > 0$ with

$$M_{\eta,\alpha} = \sup_{t \in [0,T), \; x,y \in J} \left(u(t,x) - w(t,y) - C_2 t - C_1 \varphi(x,y) - \frac{\eta}{T-t} - \alpha \frac{d^2(x_0,x)}{2} \right). \tag{A.4}$$

From (1.12), we have

$$0 < \frac{M}{2} \leq C_T(2 + d(0,x) + d(0,y)) - C_2 t - C_1 \varphi(x,y) - \frac{\eta}{T-t} - \alpha \frac{d^2(x_0,x)}{2}$$

which shows that the supremum in (A.4) is reached at a point (t,x,y), assuming $C_1 > C_T$. Moreover, we have (for $0 < \alpha \leq 1$)

$$\alpha d(0,x) \leq C = C(C_T). \tag{A.5}$$

From the uniform continuity of the initial data u_0, there exists a constant $C_0 > 0$ such that

$$u_0(x) - u_0(y) \leq C_0 \varphi(x, y)$$

and therefore $t > 0$, assuming $C_1 > C_0$. Then the classical time penalization (or doubling variable technique) implies the existence of $a, b \in \mathbb{R}$ (that play the role of u_t and v_t) such that we have the following viscosity inequalities

$$\begin{cases} a + H\left(x, C_1\varphi_x(x,y) + \alpha d(x_0, x)\right) \leq 0, \\ b + H(y, -C_1\varphi_y(x,y)) \geq 0 \end{cases}$$

(using the shorthand notation (3.1) and writing $\alpha d(x_0, x)$ for $\alpha\left(d^2(x_0,x)/2\right)_x$ for the purposes of notation) with $a - b = C_2 + \eta(T-t)^{-2}$. Substracting these inequalities yields

$$C_2 + \frac{\eta}{(T-t)^2} \leq H(y, -C_1\varphi_y(x,y)) - H\left(x, C_1\varphi_x(x,y) + \alpha d(0,x)\right). \tag{A.6}$$

Using bounds (A.3) and (A.5), this yields a contradiction in (A.6) for C_2 large enough. □

A.2 Technical results on a network

Proof of Lemma 7.2

Proof of Lemma 7.2. (H1) and (H2) imply the uniform boundedness of the $p_e^0(t,x)$, i.e. (7.8). We also notice that because of (7.8), there exists a constant $C_0 > 0$ such that for all $t \in [0,T]$, $e \in \mathcal{E}$ and $n \in \partial e$,

$$|H_e(t, n, p_e^0(t,n))| \leq C_0 \tag{A.7}$$

from which (7.9) is easily derived.

We now turn to the proof of (7.10). In view of the definition of F_A and (A2), (H5), we see that it is enough to prove that for all for $n \in \mathcal{V}$, $t, s \in [0,T]$, $p = (p_e)_{e\in\mathcal{E}_n} \in \mathbb{R}^{\text{Card } \mathcal{E}_n}$, $x \in \mathcal{V}$,

$$A_n^0(t,p) - A_n^0(s,p) \leq \tilde{\omega}_T\Big(|t-s|(1 + \max(0, A_n^0(s,p)))\Big). \tag{A.8}$$

where

$$A_n^0(t,p) = \max_{e\in\mathcal{E}_n^-} H_e^-(t,n,p_e) \geq A_n^0(t)$$

or

$$A_n^0(t,p) = \max_{e\in\mathcal{E}_n^+} H_e^+(t,n,p_e) \geq A_n^0(t).$$

We only treat the first case, since the second case reduces to the first one by a simple change of orientation of the network.

We have

$$A_n^0(a,p) = H_{e_a}^-(a, x, p_{e_a}) \quad \text{for} \quad a = t, s.$$

Let us assume that we have (otherwise there is nothing to prove)

$$0 \le I(t,s) := A_n^0(t,p) - A_n^0(s,p).$$

We also have

$$H_{e_s}^-(t,n,p_{e_s}) \le A_n^0(t,p) = H_{e_t}^-(t,n,p_{e_t})$$

and

$$H_{e_t}^-(s,n,p_{e_t}) \le A_n^0(s,p) = H_{e_s}^-(s,n,p_{e_s}).$$

We now distinguish three cases.

Case 1: $H_{e_t}^-(s,n,p_{e_t}) < H_{e_t}(s,n,p_{e_t})$. We first note that

$$0 \le I(t,s) \le A_n^0(t,p) - A_n^0(s). \tag{A.9}$$

Let us define

$$\tau = \begin{cases} \inf\left\{\sigma \in [t,s], \quad H_{e_t}^-(\sigma,n,p_{e_t}) < H_{e_t}(\sigma,n,p_{e_t})\right\} & \text{if} \quad t < s, \\ \sup\left\{\sigma \in [s,t], \quad H_{e_t}^-(\sigma,n,p_{e_t}) < H_{e_t}(\sigma,n,p_{e_t})\right\} & \text{if} \quad t \ge s. \end{cases}$$

Let us consider a optimizing sequence $\sigma_k \to \tau$ such that

$$H_{e_t}^-(\sigma_k,n,p_{e_t}) < H_{e_t}(\sigma_k,n,p_{e_t}).$$

Then we have

$$H_{e_t}^-(\sigma_k,n,p_{e_t}) = H_{e_t}(\sigma_k,n,p_{e_t}^0(\sigma_k,n)) \le A_n^0(\sigma_k) \le A_n^0(\sigma_k,p).$$

Then passing to the limit $k \to +\infty$, we get

$$H_{e_t}^-(\tau,n,p_{e_t}) = H_{e_t}(\tau,n,p_{e_t}^0(\tau,n)) \le A_n^0(\tau) \le A_n^0(\tau,p). \tag{A.10}$$

If $\tau = t$, then (A.10) implies that $A_n^0(t,p) = A_n^0(t)$ (keeping in mind the definition of p_{e_t}).
Subcase 1.1: $\tau \ne t$. This shows that

$$H_{e_t}(\tau,n,p_{e_t}) \le A_n^0(\tau) \quad \text{and} \quad H_{e_t}(t,n,p_{e_t}) \ge A_n^0(t).$$

We now choose some $\bar\tau$ in between t and τ such that

$$H_{e_t}(\bar\tau,n,p_{e_t}) = A_n^0(\bar\tau)$$

and estimate, using (A.9) and (A.7) and (H5)-(H6),

$$\begin{aligned} 0 \le I(t,s) &\le \{A_n^0(t,p) - H_{e_t}(\bar\tau,n,p_{e_t})\} + \{A_n^0(\bar\tau) - A_n^0(s)\} \\ &\le \{H_{e_t}(t,n,p_{e_t}) - H_{e_t}(\bar\tau,n,p_{e_t})\} + \{A_n^0(\bar\tau) - A_n^0(s)\} \\ &\le \bar\omega_T(|t-\bar\tau|(1+\max(A_n^0(\bar\tau),0))) + \bar\omega_T(|\bar\tau - s|) \\ &\le \bar\omega_T(|t-s|(1+C_0)) + \bar\omega_T(|t-s|). \end{aligned}$$

Subcase 1.2: $\tau = t$. Then $A_n^0(t,p) = A_n^0(t)$. Using (A.9), this gives directly

$$0 \le I(t,s) \le A_n^0(t) - A_n^0(s) \le \bar\omega_T(|t-s|).$$

Case 2: $H^-_{e_t}(s,n,p_{e_t}) = H_{e_t}(s,n,p_{e_t})$ **and** $H^-_{e_t}(t,n,p_{e_t}) = H_{e_t}(t,n,p_{e_t})$**.** We have

$$\begin{aligned} 0 \le I(t,s) &= H^-_{e_t}(t,n,p_{e_t}) - A^0_n(s,p) \\ &\le H^-_{e_t}(t,n,p_{e_t}) - H^-_{e_t}(s,n,p_{e_t}) \\ &= H_{e_t}(t,n,p_{e_t}) - H_{e_t}(s,n,p_{e_t}) \\ &\le \bar\omega_T(|t-s|(1+\max(H_{e_t}(s,n,p_{e_t}),0))) \\ &\le \bar\omega_T(|t-s|(1+\max(H^-_{e_t}(s,n,p_{e_t}),0))) \\ &\le \bar\omega_T(|t-s|(1+\max(A^n_0(s,p),0))). \end{aligned}$$

Case 3: $H^-_{e_t}(s,n,p_{e_t}) = H_{e_t}(s,n,p_{e_t})$ **and** $H^-_{e_t}(t,n,p_{e_t}) < H_{e_t}(t,n,p_{e_t})$**.** Then

$$p^0_{e_t}(t,n) < p_{e_t} \le p^0_{e_t}(s,n).$$

Because of (A.7) and the uniform bound on the Hamiltonians for bounded gradients, (H2), we deduce that

$$H_{e_t}(s,n,p_{e_t}) \le C_1$$

for some constant $C_1 > 0$ only depending on our assumptions. Therefore, we have

$$\begin{aligned} 0 \le I(t,s) &= H^-_{e_t}(t,n,p_{e_t}) - A^0_n(s,p) \\ &\le H^-_{e_t}(t,n,p_{e_t}) - H^-_{e_t}(s,n,p_{e_t}) \\ &< H_{e_t}(t,n,p_{e_t}) - H_{e_t}(s,n,p_{e_t}) \\ &\le \bar\omega_T(|t-s|(1+C_1)). \end{aligned}$$

The proof is now complete. □

Semi-concavity of the distance

In order to prove Lemmas 7.10 and 7.11, we need the following one.

Lemma A.2 (Semi-concavity of φ and d^2)**.** *Let $\mathcal{N}$ be a network defined in* (7.2) *with edges $\mathcal{E}$ and vertices $\mathcal{V}$. Let*

$$\varphi(x,y) = \sqrt{1 + d^2(x,y)}$$

where d is the distance function on the network $\mathcal{N}$. Then $\varphi(x,\cdot)$ and $\varphi(\cdot,y)$ are 1*-Lipschitz for all $x,y \in \mathcal{N}$. Moreover φ and d^2 are semi-concave on $e_a \times e_b$ for all $e_a, e_b \in \mathcal{E}$.*

Proof of Lemma A.2. The Lipschitz properties of φ are trivial. Since $r \mapsto r^2$ and $r \mapsto \sqrt{1+r^2}$ are smooth increasing functions in $\mathbb{R}^+$, the result follows from the fact that the distance function d itself is semi-concave; it is even the minimum of a finite number of smooth functions.

If $e_a = e_b$, then $d^2(x,y) = (x-y)^2$ which implies that $\varphi \in C^1(e_a \times e_a)$. Then we only consider the cases where $e_a \ne e_b$.

Case 1: e_a **and** e_b **isometric to** $[0,+\infty)$**.** Then for $(x,y) \in e_a \times e_b$, we have

$$d(x,y) = x + y + d(e^0_a, e^0_b)$$

which implies that $\varphi \in C^1(e_a \times e_b)$.

Case 2: e_a isometric to $[0,+\infty)$ and e_b isometric to $[0,l_b]$. Reversing the orientation of e_b if necessary, we can assume that

$$d_0 := d(e_a^0, e_b^0) \leq d(e_a^0, e_b^1) =: d_1$$

and then for $(x,y) \in e_a \times e_b$, we have

$$d(x,y) = x + \min(d_0 + y, d_1 + (l_b - y)) = \min(d_0 + x + y, d_1 + x + (l_b - y)).$$

Then φ is the minimum of two C^1 functions, it is semi-concave.

Case 3: e_a and e_b isometric to $[0,l_a]$ and $[0,l_b]$. Changing the orientations of both e_a and e_b if necessary, we can assume that

$$d(e_a^0, e_b^0) = \min_{i,j=0,1} d_{ij} \quad \text{with} \quad d_{ij} = d(e_a^i, e_b^j).$$

Therefore

$$d(x,y) = \min(d_{00} + x + y, d_{01} + x + (l_b - y), d_{10} + (l_a - x) + y, d_{11} + (l_a - x) + (l_b - y))$$

and again φ is the minimum of four C^1 functions, it is therefore semi-concave. □

Proof of Lemma 7.10

Proof of Lemma 7.10. We first prove (7.14) for $t = s$ by adapting in a straightforward way the proof of Lemma A.1. The only difference is that for any $e_a, e_b \in \mathcal{E}$, the function

$$\varphi(x,y) = \sqrt{1 + d^2(x,y)}$$

may not be $C^1(e_a \times e_b)$. But Lemma A.2 and Remark 7.6 ensure that this is harmless. The remaining of the proof of Lemma A.1 is unchanged. In particular the uniform bound on the Hamiltonians for bounded gradients is used, see (H2).

Now (7.14) is obtained for $t \neq s$ by following the proof of Lemma 3.3 and using the barriers given in the proof of Theorem 7.7. □

Proof of Lemma 7.11

Proof of Lemma 7.11. We do the proof for sub-solutions (the proof for super-solutions being similar). We consider the following barrier (similar to the ones in the proof of Theorem 7.7)

$$u_\varepsilon^+(t,x) = u_0^\varepsilon(x) + K_\varepsilon t + \varepsilon$$

with

$$|u_0^\varepsilon - u_0| \leq \varepsilon \quad \text{and} \quad |(u_0^\varepsilon)_x| \leq L_\varepsilon$$

and $K_\varepsilon \geq C_\varepsilon$ with C_ε given in (7.12). It is enough to prove that for all $(t,x) \in [0,T) \times \mathcal{N}$,

$$u(t,x) \leq u_\varepsilon^+(t,x)$$

for a suitable choice of $K_\varepsilon \geq C_\varepsilon$ in order to conclude. Indeed, this implies

$$u(t,x) \leq u_0(x) + f(t)$$

with

$$f(t) = \min_{\varepsilon > 0}(K_\varepsilon t + \varepsilon)$$

which is non-negative, non-decreasing, concave and $f(0) = 0$.

We consider for $0 < \tau \leq T$,

$$M = \sup_{(t,x) \in [0,\tau) \times \mathcal{N}} (u - u_\varepsilon^+)(t,x)$$

and assume by contradiction that $M > 0$. We know by Lemma 7.10 that M is finite. Then for any $\alpha, \eta > 0$ small enough, we have $M_\alpha \geq M/2 > 0$ with

$$M_\alpha = \sup_{(t,x) \in [0,\tau) \times \mathcal{N}} \left\{ u(t,x) - u_\varepsilon^+(t,x) - \frac{\eta}{\tau - t} - \alpha \psi(x) \right\}.$$

(we recall that $\psi = d^2(x_0, \cdot)/2$). By the sublinearity of u and u_ε^+, we know that this supremum is reached at some point (t,x). Moreover $t > 0$ since $u(0,x) \leq u_0(x) \leq u_\varepsilon^+(0,x)$.

This implies in particular that

$$\begin{aligned}
0 < M/2 \leq M_\alpha &= u(t,x) - u_\varepsilon^+(t,x) - \frac{\eta}{\tau - t} - \alpha \frac{d^2(x_0,x)}{2} \\
&\leq C_T(1 + d(x_0,x)) - u_0^\varepsilon(x_0) + L_\varepsilon d(x,x_0) - \alpha \frac{d^2(x_0,x)}{2} \\
&\leq C_T(1 + d(x_0,x)) + |u_0(x_0)| + \varepsilon + L_\varepsilon d(x,x_0) - \alpha \frac{d^2(x_0,x)}{2} \\
&\leq R_\varepsilon(1 + d(x_0,x)) - \alpha \frac{d^2(x_0,x)}{2}
\end{aligned}$$

with

$$R_\varepsilon = C_T + \max(L_\varepsilon, |u_0(x_0)| + \varepsilon).$$

Then $z = \alpha d(x_0,x)$ satisfies

$$\frac{z^2}{2} \leq R_\varepsilon \alpha + R_\varepsilon z \leq R_\varepsilon \alpha + R_\varepsilon^2 + \frac{z^2}{4}$$

which implies that for $\alpha \leq 1$,

$$\alpha d(x_0,x) \leq 2\sqrt{R_\varepsilon + R_\varepsilon^2}. \tag{A.11}$$

Writing the sub-solution viscosity inequality, we get

$$K_\varepsilon + H_\mathcal{N}(t,x,(u_0^\varepsilon)_x(x) + \alpha \psi_x(x)) \leq 0$$

We get a contradiction for the choice

$$K_\varepsilon = 1+ \max\left(\sup_{t\in[0,T]}\sup_{n\in\mathcal{V}}|\max(A_n(t),A_n^0(t))|,\ \sup_{t\in[0,T]}\sup_{e\in\mathcal{E}}\sup_{x\in e}\sup_{|p_e|\le L_\varepsilon+2\sqrt{R_\varepsilon+R_\varepsilon^2}}|H_e(t,x,p_e)|\right).$$

□

B Appendix: stationary results for networks

This short section is devoted to the statement of an existence and uniqueness result for the following stationary HJ equation posed on a network $\mathcal{N}$ satisfying (7.1),

$$u + H_{\mathcal{N}}(x,u_x) = 0 \quad \text{for all} \quad x\in\mathcal{N}. \tag{B.1}$$

For each $e\in\mathcal{E}$, we consider a Hamiltonian $H_e : e\times\mathbb{R}\to\mathbb{R}$ satisfying

- **(H0-s)** (Continuity) $H_e \in C(e\times\mathbb{R})$.
- **(H1-s)** (Uniform coercivity)

$$\liminf_{|q|\to+\infty} H_e(x,q) = +\infty$$

  uniformly with respect to $x\in e$, $e\in\mathcal{E}$.
- **(H2-s)** (Uniform bound on the Hamiltonians for bounded gradients) For all $L>0$, there exists $C_L>0$ such that

$$\sup_{p\in[-L,L],x\in\mathcal{N}\setminus\mathcal{V}}|H_{\mathcal{N}}(x,p)|\le C_L.$$

- **(H3-s)** (Uniform modulus of continuity for bounded gradients) For all $L>0$, there exists a modulus of continuity ω_L such that for all $|p|,|q|\le L$ and $x\in e\in\mathcal{E}$,

$$|H_e(x,p)-H_e(x,q)|\le\omega_L(|p-q|).$$

- **(H4-s)** (Level-set convexity) For all $n\in\mathcal{V}$, there exists a $p_e^0(n)$ such that

$$\begin{cases} H_e(n,\cdot) & \text{is nonincreasing on} \quad (-\infty,p_e^0(n)],\\ H_e(n,\cdot) & \text{is nondecreasing on} \quad [p_e^0(n),+\infty).\end{cases}$$

As far as flux limiters are concerned, the following assumptions will be used.

- **(A1-s)** (Uniform bound on A) There exists a constant $C > 0$ such that for all $n \in \mathcal{V}$,

$$|A_n| \leq C.$$

The following result is a straightforward adaptation of Corollary 7.9. Proofs are even simpler since the time dependance was an issue when proving the comparison principle in the general case.

Theorem B.1 (Existence and uniqueness – stationary case)**.** *Assume* (H0-s)-(H4-s) *and* (A1-s)*. Then there exists a unique sublinear viscosity solution u of* (B.1) *in $\mathcal{N}$.*

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