Flux-limited solutions for quasi-convex Hamilton-Jacobi equations on networks

Cyril Imbert, Régis Monneau

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Abstract

We study Hamilton-Jacobi equations on networks in the case where Hamiltonians are quasi-convex with respect to the gradient variable and can be discontinuous with respect to the space variable at vertices. First, we prove that imposing a general *vertex condition* is equivalent to imposing a specific one which only depends on Hamiltonians and an additional free parameter, the *flux limiter*. Second, a general method for proving comparison principles is introduced. This method consists in constructing a *vertex test function* to be used in the doubling variable approach. With such a theory and such a method in hand, we present various applications, among which a very general existence and uniqueness result for quasi-convex Hamilton-Jacobi equations on networks.

AMS Classification: 35F21, 49L25, 35B51.

Keywords: Hamilton-Jacobi equations, networks, quasi-convex Hamiltonians, discontinuous Hamiltonians, viscosity solutions, flux-limited solutions, comparison principle, vertex test function, homogenization, optimal control, discontinuous running cost.

Contents

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1 Introduction

This paper is concerned with Hamilton-Jacobi (HJ) equations on networks associated with Hamiltonians that are quasi-convex and coercive in the gradient variable and possibly discontinuous at the vertices of the network in the space variable.

Space discontinuous Hamiltonians have been identified as both important/relevant and difficult to handle; in particular, a few theories/approaches (see below) were developed to study the associated HJ equations. In this paper, we show that if they are assumed to be quasi-convex and coercive in the gradient variable, then not only uniquess can be proved for very general conditions at discontinuities (referred to as *junction conditions*), but such conditions can even be classified: imposing a general junction condition reduce to impose a junction condition of optimal control type, referred to as *a flux-limited junction condition.* As far as uniqueness is concerned, a comparison principle is proved. We show that the doubling variable approach can be adapted to the discontinuous setting if we go beyond the classical test function $|x-y|^2/2$ by using a *vertex test function* instead. This vertex test function can be used to do much more, like dealing with second order terms [31] or getting error estimates for monotone schemes [33].

We point out that the present article is written in the one-dimensional setting for pedagogical reasons but our theory extends readily to higher dimensions [29].

1.1 The junction framework

We focus in this introduction and in most of the article on the simplest network, referred to as a *junction*, and on Hamiltonians which are constant with respect to the space variable on each edge. Indeed, this simple framework leads us to the main difficulties to be overcome and allows us to present the main contributions. We will see in Section 5 that the case of a general network with (t,x)-dependent Hamiltonians is only an extension of this special case.

A *junction* is a network made of one vertex and a finite number of infinite edges. It is endowed with a flat metric on each edge. It can be viewed as the set of N distinct copies ($N \geq 1$) of the half-line which are glued at the origin. For $i = 1, ..., N$, each branch J_i is assumed to be isometric to $[0, +\infty)$ and

$$J = \bigcup_{i=1,...,N} J_i \quad \text{with} \quad J_i \cap J_j = \{0\} \quad \text{for} \quad i \neq j \tag{1.1}$$

where the origin 0 is called the *junction point*. For points $x, y \in J$, $d(x, y)$ denotes the geodesic distance on J defined as

$$d(x,y) = \begin{cases} |x-y| & \text{if } x, y \text{ belong to the same branch,} \\ |x| + |y| & \text{if } x, y \text{ belong to different branches.} \end{cases}$$

For a smooth real-valued function u defined on J, $\partial_i u(x)$ denotes the (spatial) derivative of u at $x \in J_i$ and the "gradient" of u is defined as follows,

$$u_x(x) := \begin{cases} \partial_i u(x) & \text{if} \quad x \in J_i^* := J_i \setminus \{0\}, \\ (\partial_1 u(0), ..., \partial_N u(0)) & \text{if} \quad x = 0. \end{cases} \tag{1.2}$$

With such a notation in hand, we consider the following Hamilton-Jacobi equation on the junction J

$$\begin{cases} u_t + H_i(u_x) = 0 \quad \text{for} \quad t \in (0, +\infty) \quad \text{and} \quad x \in J_i^*, \\ u_t + F(u_x) = 0 \quad \text{for} \quad t \in (0, +\infty) \quad \text{and} \quad x = 0 \end{cases} \tag{1.3}$$

subject to the initial condition

$$u(0, x) = u_0(x) \quad \text{for} \quad x \in J. \tag{1.4}$$

The second equation in (1.3) is referred to as *the junction condition*. In general, minimal assumptions are required in order to get a good notion of weak (*i.e.* viscosity) solutions. We shed some light on the fact that Equation (1.3) can be thought as a *system* of Hamilton-Jacobi equations associated with H_i coupled through a "dynamical" boundary condition involving F. This point of view can be useful, see Subsection 1.5. As far as junction functions are concerned, we will construct below some special ones (denoted by F_A) from the Hamiltonians H_i ($i = 1, ..., N$) and a real parameter A.

We consider the important case of Hamiltonians H_i satisfying the following structure condition:

$$\text{For } i = 1, \ldots, N, \quad H_i \text{ continuous, quasi-convex and coercive.} \tag{1.5}$$

We recall that H_i is quasi-convex if its sub-level sets $\{p : H_i(p) \leq \lambda\}$ are convex. In particular, since H_i is also assumed to be coercive, there exist numbers $p_i^0 \in \mathbb{R}$ such that

$$\begin{cases} H_i \text{ nonincreasing in } (-\infty, p_i^0] \\ H_i \text{ nondecreasing in } [p_i^0, +\infty). \end{cases}$$

1.2 First main new idea: classification of junction conditions

In the present paper, two notions of viscosity solutions are introduced: *relaxed (viscosity) solutions* (see Definition 2.1), which can be used to deal with all junction conditions, and *flux-limited (viscosity) solutions* (see Definition 2.2) which are associated with flux-limited junction conditions. Relaxed solutions are used to prove existence and ensure stability. Flux-limited solutions satisfy the junction condition in a stronger sense and are used in order to prove uniqueness. Our first main result states that relaxed solutions for general junction conditions are in fact flux-limited solutions for some junction conditions of optimal-control type.

We now introduce the notion of flux-limited junction condition. Given a *flux limiter* $A \in \mathbb{R} \cup \{-\infty\}$, the A-limited flux through the junction point is defined for $p = (p_1, \dots, p_N)$ as

$$F_A(p) = \max\left(A, \quad \max_{i=1,\dots,N} H_i^-(p_i)\right) \tag{1.6}$$

where H_i^- is the nonincreasing part of H_i defined by

$$H_i^-(q) = \begin{cases} H_i(q) & \text{if} \quad q \le p_i^0, \\ H_i(p_i^0) & \text{if} \quad q > p_i^0. \end{cases}$$

We now consider the following important special case of (1.3),

$$\begin{cases} u_t + H_i(u_x) = 0 & \text{for} \quad t \in (0, +\infty) \quad \text{and} \quad x \in J_i^*, \\ u_t + F_A(u_x) = 0 & \text{for} \quad t \in (0, +\infty) \quad \text{and} \quad x = 0. \end{cases} \tag{1.7}$$

We point out that the flux functions F_A associated with $A \in [-\infty, A_0]$ coincide if one chooses

$$A_0 = \max_{i=1,\dots,N} \min_{\mathbb{R}} H_i. \tag{1.8}$$

As announced above, general junction conditions are proved to be equivalent to those flux-limited junction conditions. Let us be more precise: a *junction function* is an $F\colon \mathbb{R}^N \to \mathbb{R}$ satisfying

(1.9) $\qquad F : \mathbb{R}^N \to \mathbb{R}$ is continuous and non-increasing with respect to all variables.

Theorem 1.1 (General junction conditions reduce to flux-limited ones)**.** *Assume that the Hamiltonians satisfy* (1.5) *and that the junction function satisfies* (1.9) *and that the initial datum u_0 is uniformly continuous. Then there exists $A_F \in \mathbb{R}$ such that any relaxed (viscosity) solution of* (1.3) *is in fact a flux-limited (viscosity) solution of* (1.7) *with $A = A_F$.*

Remark 1.2. Assumption (1.9) is minimal, at least "natural"; indeed, monotonicity is related to the notion of viscosity solutions that will be introduced. In particular, it is needed in order to construct solutions through the Perron method [32].

Remark 1.3. We recall that relaxed and flux-limited solutions are respectively introduced in Definitions 2.1 and 2.2.

The special case of convex Hamiltonians. In the special case of convex Hamiltonians H_i with different minimum values, Problem (1.7) can be viewed as the Hamilton-Jacobi-Bellman equation satisfied by the value function of an optimal control problem; see for instance [30] when $A = -\infty$. In this case, existence and uniqueness of viscosity solutions for (1.7)-(1.4) (with $A = -\infty$) have been established either with a very rigid method [30] based on an explicit Oleinik-Lax formula which does not extend easily to networks, or in cases reducing to $H_i = H_j$ for all i, j if Hamiltonians do not depend on the space variable [40, 1]. In such an optimal control framework, trajectories can stay for a while at the junction point. In this case, the running cost at the junction point equals $-\max_i(\min H_i)$. In this special case, the parameter A consists in replacing the previous running cost at the junction point by $\min(-A, \min_i L_i(0))$. In Section 6, the link between our results and optimal control theory will be investigated.

1.3 Second main new idea: the vertex test function

The second main contribution of this paper is to provide the reader with a general yet handy and flexible method to prove a comparison principle, allowing in particular to deal with Hamiltonians that are quasi-convex and coercive with respect to the gradient variable and are possibly discontinuous with respect to the space variable at the vertices.

It is known that the core of the theory for HJ equations lies in the proof of a strong uniqueness result, *i.e.* of a comparison principle. It is also known that it is difficult to get uniqueness results for discontinuous Hamiltonians. Indeed, the standard proof of the comparison principle in the Euclidian setting is based on the so-called *doubling variable technique*; and such a method, even in the monodimensional case, generally fails for piecewise constant (in x) Hamiltonians at discontinuities (see the last paragraph of Subsection 1.5). Since the network setting contains the previous one, the classical doubling variable technique is known to fail at vertices [40, 1, 30].

Before discussing the method we develop to prove it, we state the comparison principle.

Theorem 1.4 (Comparison principle on a junction)**.** *Assume that the Hamiltonians satisfy* (1.5)*, the junction function satisfies* (1.9) *and that the initial datum u_0 is uniformly continuous. Then for all (relaxed) sub-solution u and (relaxed) super-solution v of* (1.3)-(1.4) *satisfying for some $T > 0$ and $C_T > 0$,*

$$u(t,x) \le C_T(1+d(0,x)), \quad v(t,x) \ge -C_T(1+d(0,x)), \quad \textit{for all} \quad (t,x) \in [0,T) \times J, \tag{1.10}$$

we have

$$u \le v \quad \textit{in} \quad [0,T) \times J.$$

Combining Theorems 1.1 and 1.4, we get the following one.

Theorem 1.5 (Existence and uniqueness on a junction)**.** *Assume that the Hamiltonians satisfy* (1.5)*, that F satisfies* (1.9) *and that the initial datum u_0 is uniformly continuous.*

Then there exists a unique (relaxed) viscosity solution u *of* (1.3), (1.4) *such that for every* $T > 0$*, there exists a constant* $C_T > 0$ *such that*

$$|u(t,x) - u_0(x)| \leq C_T \quad \textit{for all} \quad (t,x) \in [0,T) \times J.$$

As we previously mentioned it, we prove Theorem 1.4 by remarking that the doubling variable approach can still be used if a suitable *vertex test function* G at each vertex is introduced. Roughly speaking, such a test function will allow the edges of the network to exchange the necessary information. More precisely, the usual penalization term, $\frac{(x-y)^2}{\varepsilon}$ with $\varepsilon > 0$, is replaced with $\varepsilon G(\varepsilon^{-1}x, \varepsilon^{-1}y)$. For a general HJ equation

$$u_t + H(x, u_x) = 0,$$

the vertex test function has to (almost) satisfy,

$$H(y, -G_y(x,y)) - H(x, G_x(x,y)) \leq 0$$

(at least close to the vertex $x = 0$). This key inequality compensates for the lack of compatibility between Hamiltonians[1]. The construction of a (vertex) test function satisfying such a condition allows us to circumvent the discontinuity of $H(x,p)$ at the junction point.

As explained above, this method consists in combining the doubling variable technique with the construction of a vertex test function G. We took our inspiration for the construction of this function from papers like [26, 7] dealing with scalar conservation laws with discontinuous flux functions. In such papers, authors stick to the case $N = 2$.

A natural family of explicit solutions of (1.7) is given by

$$u(t,x) = p_i x - \lambda t \quad \text{if} \quad x \in J_i$$

for (p, λ) in the *germ* $\mathcal{G}_A$ defined as follows,
(1.11)

$$\mathcal{G}_A = \begin{cases} \{(p,\lambda) \in \mathbb{R}^N \times \mathbb{R}, \quad H_i(p_i) = F_A(p) = \lambda \quad \text{for all} \quad i = 1, ..., N\} & \text{if } N \geq 2, \\ \{(p_1, \lambda) \in \mathbb{R} \times \mathbb{R}, \quad H_1(p_1) = \lambda \geq A\} & \text{if } N = 1. \end{cases}$$

In the special case of convex Hamiltonians satisfying $H_i'' > 0$ the vertex test function G is a regularized version[2] of the function $A + G^0$, where G^0 is defined as follows: for $(x,y) \in J_i \times J_j$,

$$G^0(x,y) = \sup_{(p,\lambda) \in \mathcal{G}_A} (p_i x - p_j y - \lambda). \tag{1.12}$$

In particular, we have $A + G^0(x,x) = 0$.

[1] Compatibility conditions are assumed in [40, 1] for instance.
[2] Such a function should indeed be regularized since it is not C^1 on the diagonal $\{x = y\}$ of J^2.

1.4 The network setting

We will extend our results to the case of networks and quasi-convex Hamiltonians depending on time and space and to flux limiters A depending on time and vertex, see Section 5. Noticeably, a localization procedure allows us to use the vertex test function constructed for a single junction.

In order to state the results in the network setting, we need to make precise the assumptions satisfied by the Hamiltonians associated with each edge and the flux limiters associated with each vertex. This results in a rather long list of assumptions. Still, when reading the proof of the comparison principle in this setting, the reader may check that the main structure properties used in the proof are gathered in the technical Lemma 5.2.

As an application of the comparison principle, we consider a model case for homogenization on a network. The network $\mathcal{N}_\varepsilon$ whose vertices are $\varepsilon\mathbb{Z}^d$ is naturally embedded in $\mathbb{R}^d$. We consider for all edges a Hamiltonian only depending on the gradient variable but which is "repeated $\varepsilon\mathbb{Z}^d$-periodically with respect to edges". We prove that when $\varepsilon \to 0$, the solution of the "oscillating" Hamilton-Jacobi equation posed in $\mathcal{N}_\varepsilon$ converges toward the unique solution of an "effective" Hamilton-Jacobi equation posed in $\mathbb{R}^d$.

A first general comment about the main results. Our proofs do not rely on optimal control interpretation (there is no representation formula of solutions for instance) but on PDE methods. We believe that the construction of a vertex test function is flexible and opens many perspectives. It also sheds light on the fact that the framework of quasi-convex Hamiltonians, which is slightly more general than the one of convex ones (at least in the evolution case), deserves special attention.

1.5 Comparison with known results

Hamilton-Jacobi equations on networks. There is a growing interest in the study of Hamilton-Jacobi equations on networks. The first results were obtained in [40] for eikonal equations. Several years after this first contribution, the three papers [1, 30, 41] were published more or less simultaneously. In these three papers, the Hamiltonians are always *convex* with respect to the gradient variables and optimal control plays in important role (at least in [1, 30]). Still, frameworks are significantly different.

Recently, a general approach of eikonal equations in metric spaces has been proposed in [28, 5, 24] (see also [36]).

In [1], the authors study an optimal control problem in $\mathbb{R}^2$ and impose a *state constraint*: the trajectories of the controlled system have to stay in the embedded network. From this point of view, [1] is related to [21, 22] where trajectories in $\mathbb{R}^N$ are constrained to stay in a closed set K which can have an empty interior. But as pointed out in [1], the framework from [21, 22] implies some restricting conditions on the geometry of the embedded networks. Our approach can now handle the general case for networks.

Our approach is also used to reformulate "state constraint" solutions by Ishii and Koike [33] (see Proposition 2.15).

The reader is referred to [14] where the different notions of viscosity solutions used in [1, 30, 41] are compared; in the few cases where frameworks coincide, they are proved to be equivalent.

In [30], the comparison principle was a consequence of a super-optimality principle (in the spirit of [35] or [42, 43]) and the comparison of sub-solutions with the value function of the optimal control problem. Still, the idea of using the "fundamental solution" $\mathcal{D}$ to prove a comparison principle originates in the proof of the comparison of sub-solutions and the value function. Moreover, as explained in Subsection 3.3, the comparison principle obtained in this paper could also be proved, for $A = -\infty$ and under more restrictive assumptions on the Hamiltonians, by using this fundamental solution.

The reader is referred to [1, 30, 41] for further references about Hamilton-Jacobi equations on networks.

Networks, regional optimal control and stratified spaces. We already pointed out that the Hamilton-Jacobi equation on a network can be regarded as a system of Hamilton-Jacobi equations coupled through vertices. In this perspective, our work can be compared with studies of Hamilton-Jacobi equations posed on, say, two domains separated by a frontier where some *transmission conditions* should be imposed. Contributions to such problems are [9, 10, 38, 37, 2]. This can be even more general by considering equations in stratified spaces [12, 11].

We first point out that the framework of these works is genuinely multi-dimensional while in this paper we stick to a monodimensional setting; still, our method generalizes to a higher dimensional setting [29]. Another difference between their approach and the one presented in the present work and in papers like [1, 41, 30] is that these authors write a Hamilton-Jacobi equation on the frontier (which is lower-dimensional). Another difference is that techniques from dynamical systems play also an important role. We mention that the techniques from [2] can be applied to treat the cases considered in our work.

Still, results can be compared. Precisely, considering a framework were both results can be applied, that is to say the monodimensional one, we will prove in Section 7 that the value function U^- from [10] coincides with the solution of (1.7) for some constant A that is determined. And we prove more (in the monodimensional setting; see also extensions below): we prove that the value function U^+ from [10] coincides with the solution of (1.7) for some (distinct) constant A which is also computed.

Hamilton-Jacobi equations with discontinuous source terms. There are numerous papers about Hamilton-Jacobi equations with discontinuous Hamiltonians. The first contribution is due to Dupuis [19]; see also [18, 25, 16, 17]. The recent paper [27] considers a Hamilton-Jacobi equation where specific solutions are expected. In the one-dimensional space, it can be proved that these solutions are in fact flux-limited solutions in the sense of the present paper with $A = c$ where c is a constant appearing in the HJ equation at stake in [27]. The introduction of [27] contains a rather long list of results for HJ equations with discontinuous Hamiltonians; the reader is referred to it for further details.

Contributions of the paper. In light of the review we made above, we can emphasize the main contributions of the paper: compared to [40, 41], we deal not only with eikonal equations but with general Hamilton-Jacobi equations. In contrast to [1], we are able to deal with networks with infinite number of edges, that are not embedded. In constrast to [1, 30, 40, 41], we can deal with quasi-convex (but not necessarily convex) discontinuous Hamilton-Jacobi equations with general junctions conditions. For such equations, flux-limited solutions are introduced and a flexible PDE framework is developed instead of an optimal control approach. Eventhough, the link with optimal control (in the spirit of [1, 9, 10]) and with regional control (in the spirit of [9, 10]) are thoroughly investigated. In particular, a PDE characterization of the two value functions introduced in [10] is provided, one of the two characterizations being new.

Several applications are also developed: the extension to the network setting and some homogenization results.

Perspectives. More homogenization results were recently obtained in [23]. An example of applications of this result is the case where a periodic Hamiltonian $H(x, p)$ is perturbed by a compactly supported function of the space variable $f(x)$, say. Such a situation is considered in lectures by Lions at Collège de France [34]. Rescaling the solution, the expected effective Hamilton-Jacobi equation is supplemented with a junction condition which keeps memory of the compact perturbation.

We would also like to mention that our results extend to a higher dimensional setting (in the spirit of [9, 10]) for quasi-convex Hamiltonians [29].

1.6 Organization of the article and notation

Organization of the article. The paper is organized as follows. In Section 2, we introduce the notion of viscosity solution for Hamilton-Jacobi equations on junctions, we prove that they are stable (Proposition 2.4) and we give an existence result (Theorem 2.14). In Section 3, we prove the comparison principle in the junction case (Theorem 2.14). In Section 4, we construct the vertex test function (Theorem 3.2). In Section 6, a general optimal control problem on a junction is considered and the associated value function is proved to be a solution of (1.7) for some computable constant A. In Section 7, the two value functions introduced in [10] are shown to be solutions of (1.7) for two explicit (and distinct) constants A. In Section 5, we explain how to generalize the previous results (viscosity solutions, HJ equations, existence, comparison principle) to the case of networks. In Section 8, we present a straightforward application of our results by proving a homogenization result passing from an "oscillating" Hamilton-Jacobi equation posed in a network embedded in an Euclidian space to a Hamilton-Jacobi equation in the whole space. Finally, we prove several technical results in Appendix A and we state results for stationary Hamilton-Jacobi equations in Appendix B.

Notation for a junction. A junction is denoted by J. It is made of a finite number of edges and a junction point. The N edges of a junction, $J_1, \dots, J_N$ ($N \in \mathbb{N} \setminus \{0\}$) are isometric to $[0, +\infty)$. The open edge is denoted by $J_i^* = J_i \setminus \{0\}$. Given a final time $T > 0$, J_T denotes $(0, T) \times J$.

The Hamiltonians on the branches J_i of the junction are denoted by H_i; they only depend on the gradient variable. The Hamiltonian at the junction point is denoted by F_A and is defined from all H_i and a constant A which "limits" the flux of information at the junction.

Given a function $u : J \to \mathbb{R}$, its gradient at x is denoted by u_x; it is a real number if $x \neq 0$ but it is a vector of $\mathbb{R}^N$ at $x = 0$. We let $|u_x|$ denote $|\partial_i u|$ outside the junction point and $\max_{i=1,\dots,N} |\partial_i u|$ at the junction point. If now $u(t, x)$ also depends on the time $t \in (0, +\infty)$, u_t denotes the time derivative.

Notation for networks. A network is denoted by $\mathcal{N}$. It is made of vertices $n \in \mathcal{V}$ and edges $e \in \mathcal{E}$. Each edge is either isometric to $[0, +\infty)$ or to a compact interval whose length is bounded from below; hence a network is naturally endowed with a metric. The associated open (resp. closed) balls are denoted by $B(x, r)$ (resp. $\bar{B}(x, r)$) for $x \in \mathcal{N}$ and $r > 0$.

In the network case, an Hamiltonian is associated with each edge e and is denoted by H_e. It depends on time and space; moreover, the limited flux functions A can depend on time t and the vertex n: $A_n(t)$.

Further notation. Given a metric space E, $C(E)$ denotes the space of continuous real-valued functions defined in E. A modulus of continuity is a function $\omega : [0, +\infty) \to [0, +\infty)$ which is non-increasing and $\omega(0+) = 0$.

2 Relaxed and flux-limited solutions

This section starts with the introduction of two notions of viscosity solutions in the junction case and of their studies. Relaxed (viscosity) solutions are first introduced; they are defined for general junction conditions. They naturally satisfy good stability properties (see for instance Proposition 2.4). Flux-limited solutions are associated with flux-limited junction conditions. They satisfy the junction condition in a stronger sense (see Proposition 2.5). The main contribution of this section is the proof of Theorem 1.1. It relies on the observation that the set of test functions for flux-limited solutions can be reduced drastically: it is enough to consider test functions with fixed space slopes (Theorem 2.7).

2.1 Definitions

In order to introduce the two notions of viscosity solution which will be used in the remaining of the paper, we first introduce the class of test functions. For $T > 0$, set $J_T = (0, T) \times J$.

We define the class of test functions on $(0,T)\times J$ by

$$C^1(J_T)=\left\{\varphi\in C(J_T),\ \text{the restriction of } \varphi \text{ to } (0,T)\times J_i \text{ is } C^1 \text{ for } i=1,...,N\right\}.$$

We (classically) say that a test function ϕ touches a function u from below (respectively from above) at (t,x) if $u-\phi$ reaches a minimum (respectively maximum) at (t,x) in a neighborhood of it.

We recall the definition of upper and lower semi-continuous envelopes u^* and u_* of a (locally bounded) function u defined on $[0,T)\times J$,

$$u^*(t,x)=\limsup_{(s,y)\to(t,x)} u(s,y) \qquad \text{and} \qquad u_*(t,x)=\liminf_{(s,y)\to(t,x)} u(s,y).$$

Definition 2.1 (Relaxed solutions)**.** Assume that the Hamiltonians satisfy (1.5) and that F satisfies (1.9) and let $u:[0,T)\times J\to\mathbb{R}$.

i) We say that u is a *relaxed sub-solution* (resp. *relaxed super-solution*) of (1.3) in $(0,T)\times J$ if for all test function $\varphi\in C^1(J_T)$ touching u^* from above (resp. from below= at $(t_0,x_0)\in J_T$, we have

$$\varphi_t+H_i(\varphi_x)\le 0 \quad (\text{resp.} \quad \ge 0) \quad \text{at } (t_0,x_0)$$

if $x_0\neq 0$, and

$$\left.\begin{array}{lll} \text{either} & \varphi_t+F(\varphi_x)\le 0 & (\text{resp.} \quad \ge 0) \\ \text{or} & \varphi_t+H_i(\partial_i\varphi)\le 0 & (\text{resp.} \quad \ge 0) \quad \text{for some } i \end{array}\right| \quad \text{at } (t_0,x_0)$$

if $x_0=0$.

ii) We say that u is a *relaxed sub-solution* (resp. *relaxed super-solution*) of (1.3), (1.4) on $[0,T)\times J$ if additionally

$$u^*(0,x)\le u_0(x) \quad (\text{resp.} \quad u_*(0,x)\ge u_0(x)) \quad \text{for all} \quad x\in J.$$

iii) We say that u is a *relaxed solution* if u is both a relaxed sub-solution and a relaxed super-solution.

We give a second definition of viscosity solutions in the case of flux-limited junction functions F_A: the junction condition is satisfied "in a classical sense" for test functions touching sub- and super-solutions at the junction point.

Definition 2.2 (Flux-limited solutions)**.** Assume that the Hamiltonians satisfy (1.5) and let $u:[0,T)\times J\to\mathbb{R}$.

i) We say that u is a *flux-limited sub-solution* (resp. *flux-limited super-solution*) of (1.7) in $(0,T)\times J$ if for all test function $\varphi\in C^1(J_T)$ touching u^* from above (resp. from below) at $(t_0,x_0)\in J_T$, we have

$$\begin{array}{llll} \varphi_t+H_i(\varphi_x)\le 0 & (\text{resp.} \quad \ge 0) & \text{at } (t_0,x_0) & \text{if } x_0\in J_i^* \\ \varphi_t+F(\varphi_x)\le 0 & (\text{resp.} \quad \ge 0) & \text{at } (t_0,x_0) & \text{if } x_0=0. \end{array} \tag{2.1}$$

ii) We say that u is a *flux-limited sub-solution* (resp. *flux-limited super-solution*) of (1.7), (1.4) on $[0,T)\times J$ if additionally

$$u^*(0,x)\le u_0(x)\quad \text{(resp.}\quad u_*(0,x)\ge u_0(x))\quad \text{for all}\quad x\in J.$$

iii) We say that u is a *flux-limited solution* if u is both a flux-limited sub-solution and a flux-limited super-solution.

2.2 The "weak continuity" condition for sub-solutions

If F not only satisfies (1.9) but is also *semi-coercive*, that is to say if

$$F(p)\to +\infty \quad \text{as} \quad \max_i\, (\max(0,-p_i))\to +\infty \tag{2.2}$$

then any F-relaxed sub-solution satisfies a "weak continuity" condition at the junction point. Precisely, the following lemma holds true.

Lemma 2.3 ("Weak continuity" condition at the junction point)**.** *Assume that the Hamiltonians satisfy* (1.5) *and that F satisfies* (1.9) *and* (2.2)*. Then any relaxed sub-solution u of* (1.3) *satisfies for all $t\in(0,T)$ and all $i\in\{1,\dots,N\}$,*

$$u(t,0)=\limsup_{(s,y)\to(t,0),y\in J_i^*} u(s,y).$$

Proof. Since u is upper semi-continuous, we know that for all $t\in(0,T)$ and i,

$$u(t,0)\ge \limsup_{(s,y)\to(t,0),y\in J_i^*} u(s,y).$$

Assume that there exists t^* and i_0 such that

$$u(t^*,0)> \limsup_{(s,y)\to(t^*,0),y\in J_{i_0}^*} u(s,y).$$

Since u is upper semi-continuous, we know that we can find t_0 arbitrarily close to t^* such that $u(t_0,0)$ is arbitrarily close to $u(t^*,0)$ and such that there exists a C^1 function $\Psi(t)$ (strictly) touching $u(t,0)$ from above at t_0. In particular, we can ensure

$$u(t_0,0)> \limsup_{(s,y)\to(t_0,0),y\in J_{i_0}^*} u(s,y) \tag{2.3}$$

and

$$\begin{cases} u(t,0) & <\Psi(t) \quad \text{for } t\in[t_0-r_0,t_0+r_0]\setminus\{t_0\}\\ u(t_0,0) & =\Psi(t_0).\end{cases}$$

In particular, since $(\Psi-u)(t_0\pm r_0,0)>0$, there exist $\delta_1>0$ and $r_1>0$ small enough such that

$$u(t_0\pm r_0,x)+\delta_1\le \Psi(t_0\pm r_0)\quad \text{for } x\in B(0,r_1)\subset J. \tag{2.4}$$

We now consider the test function $\phi(t,x) = \Psi(t) + p_i x$ for $x \in J_i$. We claim that for $i \neq i_0$ and for $p_i = p_i(r_1)$ large enough, $u - \phi$ reaches its maximum M_i on $Q_0 = [t_0 - r_0, t_0 + r_0] \times [0, r_1] \subset (0,T) \times J_i$ at $(t_0, 0)$. We first remark that $M_i \geq u(t_0,0) - \Psi(t_0) = 0$. Moreover, for $(t_0 \pm r_0, x)$ and $x \in [0, r_1]$, (2.4) implies that

$$u(t_0 \pm r_0, x) - \Psi(t_0 \pm r_0) - p_i x \leq -\delta_1 < M_i.$$

For $(t,x) \in Q_0$ and $x = r_1$, we have for p_i large enough

$$u(t,x) - \Psi(t) - p_i x \leq \|u^+\|_{L^\infty(Q_0)} + \|\Psi\|_{L^\infty([t_0 - r_0, t_0 + r_0])} - p_i r_1 < M_i.$$

Hence the supremum is reached either for $x = 0$ or x in the interior of Q_0. In the latter case, this yields the viscosity inequality

$$\Psi'(t) + H_i(p_i) \leq 0$$

which cannot hold true for large p_i. We conclude that

$$\begin{cases} u(t,x) & < \Psi(t) + p_i x \quad \text{in } Q_0 \setminus \{(t_0, 0)\} \\ u(t_0, 0) & = \Psi(t_0). \end{cases}$$

We now get

$$\begin{cases} u(t,x) < \Psi(t) + p_i x & \text{in } [t_0 - r_0, t_0 + r_0] \times [0, r_1] \setminus \{(t_0,0)\} \text{ with } p_i > 0 \text{ if } i \neq i_0 \\ u(t,x) < \Psi(t) + p_{i_0} x & \text{in } [t_0 - r_0, t_0 + r_0] \times [0, r_1] \setminus \{(t_0,0)\} \text{ with } p_{i_0} < 0 \text{ if } i = i_0 \\ u(t_0, 0) = \Psi(t_0). \end{cases}$$

where we have used (2.3) for any negative p_{i_0} and any small enough $r_1 = r_1(p_{i_0})$. This implies that

$$\Psi'(t_0) + F(p_1, \dots, p_{i_0}, \dots, p_N) \leq 0$$

which cannot hold true for p_{i_0} very negative because of (2.2). The proof is now complete. □

2.3 General junction conditions and stability

The first stability result is concerned with the supremum of relaxed sub-solutions. Such a result is used in the Perron process to construct relaxed solutions. Its proof is standard so we skip it.

Proposition 2.4 (Stability by supremum/infimum)**.** *Assume that the Hamiltonians H_i satisfy* (1.5) *and that F satisfies* (1.9)*. Let $\mathcal{A}$ be a nonempty set and let $(u_a)_{a \in \mathcal{A}}$ be a familly of relaxed sub-solutions (resp. relaxed super-solutions) of* (1.3) *on $(0,T) \times J$. Let us assume that*

$$u = \sup_{a \in \mathcal{A}} u_a \quad (\textit{resp.} \quad u = \inf_{a \in \mathcal{A}} u_a)$$

is locally bounded on $(0,T) \times J$. Then u is a relaxed sub-solution (resp. relaxed super-solution) of (1.3) *on $(0,T) \times J$.*

In the following proposition, we assert that, for the special junction functions F_A, the junction condition is in fact always satisfied *in the classical (viscosity) sense*, that is to say in the sense of Definition 2.2 (and not Definition 2.1).

Proposition 2.5 (flux-limited junction conditions are satisfied in the classical sense)**.** *Assume that the Hamiltonians satisfy* (1.5) *and consider* $A \in \mathbb{R}$*. If* $F = F_A$*, then relaxed super-solutions (resp. relaxed sub-solutions) coincide with flux-limited super-solutions (resp. flux-limited sub-solutions).*

Proof of Proposition 2.5. The proof was done in [30] for the case $A = -\infty$, using the monotonicities of the H_i. We follow the same proof and omit details.

The super-solution case. Let u be a relaxed super-solution satisfying the junction condition in the viscosity sense and let us assume by contradiction that there exists a test function φ touching u from below at $P_0 = (t_0, 0)$ for some $t_0 \in (0, T)$, such that

$$\varphi_t + F_A(\varphi_x) < 0 \quad \text{at} \quad P_0. \tag{2.5}$$

Then we can construct a test function $\tilde{\varphi}$ satisfying $\tilde{\varphi} \le \varphi$ in a neighborhood of P_0, with equality at P_0 such that

$$\tilde{\varphi}_t(P_0) = \varphi_t(P_0) \quad \text{and} \quad \partial_i \tilde{\varphi}(P_0) = \min(p_i^0, \partial_i \varphi(P_0)) \quad \text{for} \quad i = 1, ..., N.$$

Using the fact that $F_A(\varphi_x) = F_A(\tilde{\varphi}_x) \ge H_i^-(\partial_i \tilde{\varphi}) = H_i(\partial_i \tilde{\varphi})$ at P_0, we deduce a contradiction with (2.5) using the viscosity inequality satisfied by φ for some $i \in \{1, \ldots, N\}$.

The sub-solution case. Let now u be a sub-solution satisfying the junction condition in the viscosity sense and let us assume by contradiction that there exists a test function φ touching u from above at $P_0 = (t_0, 0)$ for some $t_0 \in (0, T)$, such that

$$\varphi_t + F_A(\varphi_x) > 0 \quad \text{at} \quad P_0. \tag{2.6}$$

Let us define

$$I = \left\{ i \in \{1, ..., N\}, \quad H_i^-(\partial_i \varphi) < F_A(\varphi_x) \quad \text{at} \quad P_0 \right\}$$

and for $i \in I$, let $q_i \ge p_i^0$ be such that

$$H_i(q_i) = F_A(\varphi_x(P_0))$$

where we have used the fact that $H_i(+\infty) = +\infty$. Then we can construct a test function $\tilde{\varphi}$ satisfying $\tilde{\varphi} \ge \varphi$ in a neighborhood of P_0, with equality at P_0, such that

$$\tilde{\varphi}_t(P_0) = \varphi_t(P_0) \quad \text{and} \quad \partial_i \tilde{\varphi}(P_0) = \begin{cases} \max(q_i, \partial_i \varphi(P_0)) & \text{if} \quad i \in I, \\ \partial_i \varphi(P_0) & \text{if} \quad i \notin I. \end{cases}$$

Using the fact that $F_A(\varphi_x) = F_A(\tilde{\varphi}_x) \le H_i(\partial_i \tilde{\varphi})$ at P_0, we deduce a contradiction with (2.6) using the viscosity inequality for φ for some $i \in \{1, \ldots, N\}$. □

The last stability result is concerned with sub-solutions of the Hamilton-Jacobi equation away from the junction point and which satisfy the "weak continuity" condition. The following proposition asserts that such a "weak continuity" is stable under upper semi-limit.

Proposition 2.6 (Stability of the "weak continuity" condition)**.** *Consider a family of Hamiltonians H^ε satisfying* (1.5)*. We also assume that the coercivity of the Hamiltonians is uniform in ε. Let u^ε be a family of subsolutions of*

$$u_t + H_i^\varepsilon(u_x) = 0 \quad \text{in } (0,T) \times J_i^*$$

for all $i = 1, \dots, N$ such that, for all i,

$$u^\varepsilon(t,0) = \limsup_{(s,y)\to(t,0), y\in J_i^*} u^\varepsilon(s,y). \tag{2.7}$$

If the upper semi-limit $\bar{u} = \limsup^* u^\varepsilon$ is everywhere finite, then it satisfies for all i

$$\bar{u}(t,0) = \limsup_{(s,y)\to(t,0), y\in J_i^*} \bar{u}(s,y).$$

Proof. We argue by contradiction by assuming that there exists i_0 and $t^* \in (0,T)$ such that

$$\bar{u}(t^*,0) > \limsup_{(s,y)\to(t_0,0), y\in J_{i_0}^*} \bar{u}(s,y).$$

Our goal is first to use a perturbation argument to get a test function $\Psi(t)$ touching strictly $\bar{u}$ from above at a time t_0 where the previous inequality still hold true. Using the upper semi-continuity of $\bar{u}$, we can keep $\bar{u}$ away from $\Psi(t)$ in a neighborhood of the point corresponding to the boundary of the time interval where $\bar{u}$ and Ψ are strictly separed. From the definition of $\bar{u}$, we also get a sequence of points $(t_\varepsilon, x_\varepsilon)$ realizing the value $\bar{u}(t_0,0)$. Considering now $\Psi(t) + px$ for p positive and very large, we use the sequence $(t_\varepsilon, x_\varepsilon)$ in order to get a contact point of u^ε with this test-function away from $x = 0$. This will lead to the desired contradiction since p is arbitrarily large.

We now make precise how to use the previous strategy. Since $\bar{u}$ is upper semi-continuous, we know that we can find t_0 arbitrarily close to t^* such that $\bar{u}(t_0,0)$ is arbitrarily close to $\bar{u}(t^*,0)$ and such that there exists a C^1 function $\psi(t)$ (strictly) touching $\bar{u}(t,0)$ from above at t_0. In particular, we can ensure

$$\bar{u}(t_0,0) > \limsup_{(s,y)\to(t_0,0), y\in J_{i_0}^*} \bar{u}(s,y) \tag{2.8}$$

and

$$\begin{cases} \bar{u}(t,0) & < \Psi(t) \quad \text{for } t \in [t_0 - r_0, t_0 + r_0] \setminus \{t_0\} \\ \bar{u}(t_0,0) & = \Psi(t_0). \end{cases}$$

In particular, since $(\Psi - \bar{u})(t_0 \pm r_0, 0) > 0$, there exist $\delta_1 > 0$ and $r_1 > 0$ such that

$$\bar{u}(t_0 \pm r_0, x) + 2\delta_1 \leq \Psi(t_0 \pm r_0) \quad \text{for } x \in B(x_0, r_1) \subset J.$$

Since $\bar{u}$ is the upper relaxed-limit of u^ε, this implies in particular that for ε small enough,

$$u^\varepsilon(t_0 \pm r_0, x) + \delta_1 \leq \Psi(t_0 \pm r_0) \quad \text{for } x \in B(x_0, r_1) \subset J. \tag{2.9}$$

We claim that

$$\Psi(t_0, 0) = \bar{u}(t_0, 0) > \limsup_{\varepsilon \to 0, s \to t_0} u^\varepsilon(s, 0).$$

Indeed, if the previous inequality is replaced with an equality, this would contradict (2.7). In particular, reducing r_0 and δ_0 if necessary, we can further assume that for $\varepsilon \in]0, \varepsilon_0[$,

$$\forall t \in [t_0 - r_0, t_0 + r_0] \setminus \{t_0\}, \quad u^\varepsilon(t, 0) + \delta_0 \leq \Psi(t_0). \tag{2.10}$$

Let $(t_\varepsilon, x_\varepsilon) \to (t_0, 0)$ be such that

$$\bar{u}(t_0, 0) = \lim_{\varepsilon \to 0} u^\varepsilon(t_\varepsilon, x_\varepsilon).$$

By (2.10), we know that $x_\varepsilon \neq 0$ for ε small enough. We also know that there exists j_0 such that $x_\varepsilon \in J_{j_0}^*$ for ε small enough (along a subsequence) with $j_0 \neq i_0$. Indeed, if $x_\varepsilon \in J_{i_0}^*$ (at least along a subsequence), then

$$\bar{u}(t_0, 0) = \lim u^\varepsilon(t_\varepsilon, x_\varepsilon) \leq \limsup \bar{u}(t_\varepsilon, x_\varepsilon) \leq \limsup_{(s,y) \to (t_0, 0), y \in J_{i_0}^*} \bar{u}(s, y)$$

which is in contradiction with (2.8).

We now consider $\Psi(t) + px$ with $p > 0$ and we consider the point $(s^\varepsilon, y^\varepsilon)$ where the maximum of $u^\varepsilon - \Psi(t) - px$ is reached in $Q_0 = [t_0 - r_0, t_0 + r_0] \times [0, r_1] \subset (0, T) \times J_{j_0}$. Remark that for $x = 0$ and $(t, x) \in Q_0$, (2.10) implies that

$$u^\varepsilon(t, 0) - \Psi(t) \leq -\delta_0 < 0.$$

Analogously, for $t = t_0 \pm r_0$ and $(t, x) \in Q_0$, (2.9) implies that

$$u^\varepsilon(t_0 \pm r_0, x) - \Psi(t_0 \pm r_0) - px \leq -\delta_1 < 0.$$

Finally, for $x = r_1$ and $(t, x) \in Q_0$, we have for ε small and some $\delta_2 > 0$,

$$u^\varepsilon(t, 0) - \Psi(t) - pr_1 \leq \bar{u}(t, 0) + \delta_2 + \|\Psi\|_\infty - pr_1.$$

Since $\bar{u}$ is locally bounded from above (because it is upper semi-continuous), we conclude that we can choose p large (depending on $\delta_2 + \|\Psi\|_\infty$ and a local bound of $\bar{u}$ from above) such that for $x = r_1$ and $(t, x) \in Q_0$, we have for ε small and some $\delta_2 > 0$,

$$u^\varepsilon(t, 0) - \Psi(t) - pr_1 \leq -\delta_1.$$

Finally, the maximum M^ε of $u^\varepsilon - \Psi(t) - px$ in Q_0 satisfies

$$M^\varepsilon \le u^\varepsilon(t_\varepsilon, x_\varepsilon) - \Psi(t_\varepsilon) - px_\varepsilon \to \bar{u}(t_0, 0) - \Psi(t_0) = 0.$$

We conclude that $(s^\varepsilon, y^\varepsilon)$ belongs to the interior of Q_0 which entails

$$\Psi'(s_\varepsilon) + H^\varepsilon_{j_0}(p) \le 0$$

which cannot hold true for p very large because of the uniform coercivity of $H^\varepsilon_{j_0}$. The proof is now complete. □

2.4 Reducing the set of test functions

We show in this subsection, that to check the flux-limited junction condition, it is sufficient to consider very specific test functions. This important property is useful both from a theoretical point of view and from the point of view of applications.

We consider functions satisfying a Hamilton-Jacobi equation in $J \setminus \{0\}$, that is to say, solutions of

$$u_t + H_i(u_x) = 0 \quad \text{for} \quad (t,x) \in (0,T) \times J_i^* \tag{2.11}$$

for $i = 1, \dots, N$. The non-increasing part H_i^- of the Hamiltonian H_i is used in the definition of flux-limited junction conditions. In the next theorem, the non-decreasing part H_i^+ is needed. It is defined by

$$H_i^+(q) = \begin{cases} H_i(q) & \text{if} \quad q \ge p_i^0, \\ H_i(p_i^0) & \text{if} \quad q < p_i^0 \end{cases}$$

where we recall that p_i^0 is a point realizing the minimum of H_i.

Theorem 2.7 (Reduced set of test functions)**.** *Assume that the Hamiltonians satisfy* (1.5) *and consider $A \in [A_0, +\infty[$ with A_0 given in* (1.8)*. Given arbitrary solutions $p_i^A \in \mathbb{R}$, $i = 1, \dots, N$, of*

$$H_i(p_i^A) = H_i^+(p_i^A) = A, \tag{2.12}$$

let us fix any time independent test function $\phi_0(x)$ satisfying

$$\partial_i \phi_0(0) = p_i^A.$$

Given a function $u : (0,T) \times J \to \mathbb{R}$, the following properties hold true.

i) *If for all $i = 1, \dots, N$, u is an upper semi-continuous sub-solution of* (2.11) *and satisfies*

$$u(t,0) = \limsup_{(s,y)\to(t,0),\, y\in J_i^*} u(s,y), \tag{2.13}$$

then u is a A_0-flux limited sub-solution.

ii) *Given $A > A_0$ and $t_0 \in (0,T)$, if for all $i = 1,\dots,N$, u is an upper semi-continuous sub-solution of* (2.11) *and satisfies* (2.13) *and for any test function φ touching u from above at $(t_0, 0)$ with*

$$\varphi(t,x) = \psi(t) + \phi_0(x) \tag{2.14}$$

for some $\psi \in C^1(0;+\infty)$, we have

$$\varphi_t + F_A(\varphi_x) \le 0 \quad \textit{at} \quad (t_0,0),$$

then u is a A-flux-limited sub-solution at $(t_0, 0)$.

iii) *Given $t_0 \in (0,T)$, if u is lower semi-continuous super-solution of* (2.11) *and if for any test function φ touching u from below at $(t_0, 0)$ satisfying* (2.14)*, we have*

$$\varphi_t + F_A(\varphi_x) \ge 0 \quad \textit{at} \quad (t_0,0), \tag{2.15}$$

then u is a A-flux-limited super-solution at $(t_0, 0)$.

Remark 2.8. Theorem 2.7 exhibits (necessary and) sufficient conditions for sub- and super-solutions of (2.11) to be flux-limited solutions. After proving Theorem 2.7, we realized that this result shares some similarities with the way of checking the entropy condition at the junction for conservation law equations associated to bell-shaped fluxes. Indeed it is known that it is sufficient to check the entropy condition only with one particular stationary solution of the Riemann solver (see [13, 7, 6]).

Counter-example 1. The set of test functions can be reduced to a single one for flux-limited sub-solution only if the "weak continuity" condition (2.13) is imposed. Indeed, if this condition is not satisfied, then the conclusion is false. Consider for instance Hamiltonians reaching their minimum at $p_i^0 = 0$ and such that $A_0 = 0$ and consider $A \ge A_0 = 0$ such that $AT < 1$ and consider

$$u(t,x) = \begin{cases} 1 - At & \text{for } (t,x) \in (-T,T) \times \{0\} \\ 0 & \text{elsewhere.} \end{cases}$$

We remark that u does not satisfy (2.13) but it trivially satisfies (2.11). Now consider $p_i^\varepsilon \le 0$ such that $H_i(p_i^\varepsilon) = \varepsilon^{-1}$; the test function defined as

$$\phi(t,x) = 1 - At + p_i^\varepsilon x \quad \text{for} \quad x \in J_i$$

touches u from above at any $(t, 0)$ and if u were a A-flux-limited solution, we would get

$$-A + A \vee \varepsilon^{-1} \le 0$$

which is false for ε small enough. If now u is touched from above by a test function $\psi(t) + \phi_0(x)$ at $(t, 0)$, then $\psi'(t) = -A$ so that

$$\psi'(t) + A \le 0.$$

In order to prove this result, the two following technical lemmas are needed.

Lemma 2.9 (Super-solution property for the critical slope on each branch)**.** *Let $u : (0,T)\times J_i \to \mathbb{R}$ be a lower super-solution of* (2.11) *for some $i = 1,\dots,N$. Let ϕ be a test function touching u from below at some point $(t_0, 0)$ with $t_0 \in (0,T)$. Consider the following* critical slope

$$\bar{p}_i = \sup\{\bar{p} \in \mathbb{R} : \exists r > 0,\ \phi(t,x)+\bar{p}x \le u(t,x) \text{ for } (t,x) \in (t_0-r, t_0+r)\times[0,r) \text{ with } x \in J_i\}.$$

If $\bar{p}_i < +\infty$, then we have

$$\phi_t + H_i(\partial_i\phi + \bar{p}_i) \ge 0 \quad \text{at} \quad (t_0, 0) \quad \text{with} \quad \bar{p}_i \ge 0. \tag{2.16}$$

Proof. From the definition of $\bar{p}_i$, we know that, for all $\varepsilon > 0$ small enough, there exists $\delta = \delta(\varepsilon) \in (0,\varepsilon)$ such that

$$u(s,y) \ge \phi(s,y) + (\bar{p}_i - \varepsilon)y \quad \text{for all } (s,y) \in (t-\delta, t+\delta)\times[0,\delta) \text{ with } y \in J_i$$

and there exists $(t_\varepsilon, x_\varepsilon) \in B_{\delta/2}(t,0)$ such that

$$u(t_\varepsilon, x_\varepsilon) < \phi(t_\varepsilon, x_\varepsilon) + (\bar{p}_i + \varepsilon)x_\varepsilon.$$

Now consider a smooth function $\Psi : \mathbb{R}^2 \to [-1, 0]$ such that

$$\Psi \equiv \begin{cases} 0 & \text{in } B_{\frac{1}{2}}(0), \\ -1 & \text{outside } B_1(0) \end{cases}$$

and define

$$\Phi(s,y) = \phi(s,y) + 2\varepsilon\Psi_\delta(s,y) + \begin{cases} (\bar{p}_i + \varepsilon)y & \text{if } y \in J_i \\ 0 & \text{if not} \end{cases}$$

with $\Psi_\delta(s,y) = \delta\Psi(s/\delta, y/\delta)$. We have

$$\Phi(s,y) \le \phi(s,y) \le u(s,y) \quad \text{for } (s,y) \in B_\delta(t,0) \text{ and } y \notin J_i$$

and

$$\begin{cases} \Phi(s,y) = \phi(s,y) - 2\varepsilon\delta + (\bar{p}_i+\varepsilon)y \le u(s,y) & \text{for } (s,y) \in (\partial B_\delta(t,0)) \cap (\mathbb{R}\times J_i)\,, \\ \Phi(s,0) \le \phi(s,0) \le u(s,0) & \text{for } s \in (t-\delta, t+\delta) \end{cases}$$

and

$$\Phi(t_\varepsilon, x_\varepsilon) = \phi(t_\varepsilon, x_\varepsilon) + (\bar{p}_i + \varepsilon)x_\varepsilon > u(t_\varepsilon, x_\varepsilon).$$

We conclude that there exists a point $(\bar{t}_\varepsilon, \bar{x}_\varepsilon) \in B_\delta(t,0) \cap (\mathbb{R}\times J_i^*)$ such that $u - \Phi$ reaches a minimum in $\overline{B_\delta(t,0)} \cap (\mathbb{R}\times J_i)$. Consequently,

$$\Phi_t(\bar{t}_\varepsilon, \bar{x}_\varepsilon) + H_i(\partial_i\Phi(\bar{t}_\varepsilon, \bar{x}_\varepsilon)) \ge 0$$

which implies

$$\phi_t(\bar{t}_\varepsilon, \bar{x}_\varepsilon) + 2\varepsilon(\Psi_\delta)_t(\bar{t}_\varepsilon, \bar{x}_\varepsilon) + H_i(\partial_i\phi(\bar{t}_\varepsilon, \bar{x}_\varepsilon) + 2\varepsilon\partial_y(\Psi_\delta)(\bar{t}_\varepsilon, \bar{x}_\varepsilon) + \bar{p}_i + \varepsilon) \ge 0.$$

Letting ε go to 0 yields (2.16). This ends the proof of the lemma. □

Lemma 2.10 (Sub-solution property for the critical slope on each branch)**.** *Let $u : (0,T) \times J_i \to \mathbb{R}$ be a sub-solution of* (2.11) *for some $i = 1, \dots, N$. Let ϕ be a test function touching u from above at some point $(t_0, 0)$ with $t_0 \in (0,T)$. Consider the following* critical slope,

$$\bar{p}_i = \inf\{\bar{p} \in \mathbb{R} : \exists r > 0,\ \phi(t,x)+\bar{p}x \geq u(t,x) \text{ for } (t,x) \in (t_0-r, t_0+r)\times[0,r) \text{ with } x \in J_i\}.$$

If u satisfies (2.13) *then $-\infty < \bar{p}_i \leq 0$ and*

$$\phi_t + H_i(\partial_i \phi + \bar{p}_i) \leq 0 \quad \text{at} \quad (t_0, 0). \tag{2.17}$$

Proof. We only prove that $\bar{p}_i > -\infty$ since this is the only main difference with the proof of the previous lemma.

Assume that $\overline{p}_i = -\infty$. This implies that there exists $p_n \to -\infty$ and $r_n > 0$ such that $\phi + p_n x \geq u$ in $B_n = (t_0 - r_n, t_0 + r_n) \times [0, r_n) \subset \mathbb{R} \times J_i$. Remark first that, replacing ϕ with $\phi + (t-t_0)^2 + x^2$ if necessary, we can assume that

$$u(t,x) < \phi(t,x) + p_n x \text{ if } (t,x) \neq (t_0, 0). \tag{2.18}$$

In particular, there exits $\delta_n > 0$ such that $\phi + p_n x \geq u + \delta_n$ on on $\partial B_n \setminus \{(t_0,0)\}$, where we recall that by definition of ∂B_n (inside J_T) does not contain $(t_0 - r_0, t_0 + r_0) \times \{0\}$. Since u satisfies (2.13), there exists $(t_\varepsilon, x_\varepsilon) \to (t_0, 0)$ such that $x_\varepsilon \in J_i^*$ and $u(t_0, 0) = \lim_{\varepsilon \to 0} u(t_\varepsilon, x_\varepsilon)$.

We now introduce the following perturbed test function

$$\Psi(t,x) = \phi(t,x) + p_n x + \frac{\eta}{x}$$

where $\eta = \eta(\varepsilon)$ is a small parameter to be chosen later. Let $(s_\varepsilon, y_\varepsilon)$ realizing the infimum of $\Psi - u$ in B_n. In particular,

$$(\phi + p_n(\cdot) - u)(s_\varepsilon, y_\varepsilon) \leq \Psi(s_\varepsilon, y_\varepsilon) - u(s_\varepsilon, y_\varepsilon) \leq \Psi(t_\varepsilon, x_\varepsilon) - u(t_\varepsilon, x_\varepsilon) \to 0 \quad \text{as} \quad \varepsilon \to 0 \tag{2.19}$$

as soon as $\eta(\varepsilon) = o(x_\varepsilon)$. In particular, in view of (2.18), this implies that $(s_\varepsilon, y_\varepsilon) \to (t_0, 0)$ as $\varepsilon \to 0$. Since u is a subsolution of (2.11), we know that

$$\phi_t(s_\varepsilon, y_\varepsilon) + H_i\left(\phi_x(s_\varepsilon, y_\varepsilon) + p_n - \frac{\eta}{y_\varepsilon^2}\right) \leq 0.$$

Hence we can pass to the limit as $\varepsilon \to 0$ in the viscosity inequality and get

$$\phi_t(t_0, 0) + H_i(\phi_x(t_0, 0) + p_n^0) \leq 0$$

where $p_n^0 = \liminf_{\varepsilon \to 0} p_n - \frac{\eta}{x_\varepsilon^2} \in [-\infty, 0]$. The previous inequality implies in particular that $p_n^0 > -\infty$ and p_n^0 is bounded from below by a constant C which only depends on H_i and ϕ_t, ϕ_x at $(t_0, 0)$. But this also implies that $p_n \geq C$ which is the desired contradiction. The proof of the finiteness of $\overline{p}_i$ is now complete. □

We are now ready to make the proof of Theorem 2.7.

Proof of Theorem 2.7. We first prove the results concerning sub-solutions and then turn to super-solutions.

Sub-solutions. Let u be a sub-solution of (2.11). Let ϕ be a test function touching u from above at $(t_0, 0)$. Let $\phi_t(t_0, 0) = -\lambda$. We want to show

$$F_A(\phi_x) \le \lambda \quad \text{at} \quad (t_0, 0). \tag{2.20}$$

Notice that by Lemma 2.10, for all $i = 1, \ldots, N$, there exists $\bar{p}_i \le 0$ such that

$$H_i(\partial_i \phi + \bar{p}_i) \le \lambda \quad \text{at} \quad (t_0, 0). \tag{2.21}$$

In particular, we deduce that

$$A_0 \le \lambda. \tag{2.22}$$

Inequality (2.21) also implies that at $(t_0, 0)$

$$\begin{aligned} F_A(\phi_x) &= \max(A, \max_{i=1,\ldots,N} H_i^-(\partial_i \phi)) \\ &\le \max(A, \max_{i=1,\ldots,N} H_i^-(\partial_i \phi + \bar{p}_i)) \\ &\le \max(A, \max_{i=1,\ldots,N} H_i(\partial_i \phi + \bar{p}_i)) \\ &\le \max(A, \lambda). \end{aligned}$$

In particular for $A = A_0$, this implies the desired inequality (2.20). Assume now that (2.20) does not hold true. Then we have

$$A_0 \le \lambda < A.$$

Then (2.21) implies that

$$\partial_i \phi(t_0, 0) + \bar{p}_i < p_i^A = \partial_i \phi_0(0).$$

Let us consider the modified test function

$$\varphi(t, x) = \phi(t, 0) + \phi_0(x) \quad \text{for} \quad x \in J$$

which is still a test function touching u from above at $(t_0, 0)$ (in a small neighborhood). This test function φ satisfies in particular (2.14). Because $A > A_0$, we then conclude that

$$\varphi_t + F_A(\varphi_x) \le 0 \quad \text{at} \quad (t_0, 0)$$

i.e.

$$-\lambda + A \le 0$$

which gives a contradiction. Therefore (2.20) holds true.

Super-solutions. Let u be a super-solution of (2.11). Let ϕ be a test function touching u from below at $(t_0, 0)$. Let $\phi_t(t_0, 0) = -\lambda$. We want to show

$$F_A(\phi_x) \geq \lambda \quad \text{at} \quad (t_0, 0). \tag{2.23}$$

Notice that by Lemma 2.9, there exists $\bar{p}_i \geq 0$ for $i = 1, \ldots, N$ such that

$$H_i(\partial_i \phi + \bar{p}_i) \geq \lambda \quad \text{at} \quad (t_0, 0). \tag{2.24}$$

Note that (2.23) holds true if $\lambda \leq A$ or if there exists one index i such that $H_i^-(\partial_i \phi + \bar{p}_i) = H_i(\partial_i \phi + \bar{p}_i)$. Assume by contradiction that (2.23) does not hold true. Then we have in particular

$$A_0 \leq A < \lambda \leq H_i^+(\partial_i \phi + \bar{p}_i) \quad \text{at} \quad (t_0, 0), \quad \text{for} \quad i = 1, \ldots, N. \tag{2.25}$$

From the fact that $H_i^-(\partial_i \phi + \bar{p}_i) < H_i(\partial_i \phi + \bar{p}_i)$ for all index i, we deduce in particular that

$$\partial_i \phi(t_0, 0) + \bar{p}_i > p_i^A = \partial_i \phi_0(0).$$

We then introduce the modified test function

$$\varphi(t, x) = \phi(t_0, 0) + \phi_0(x) \quad \text{for} \quad x \in J$$

which is a test function touching u from below at $(t_0, 0)$ (this is a test function below u in a small neighborhood of $(t_0, 0)$). This test function φ satisfies in particular (2.14). We then conclude that

$$\varphi_t + F_A(\varphi_x) \geq 0 \quad \text{at} \quad (t_0, 0)$$

i.e.

$$-\lambda + A \geq 0$$

which gives a contradiction. Therefore (2.23) holds true. This ends the proof of the theorem. □

2.5 An additional characterization of flux-limited sub-solutions

As an application of Theorem 2.7, we give an equivalent characterization of sub-solutions in terms of the properties of its trace at the junction point $x = 0$.

Theorem 2.11 (Equivalent characterization of flux-limited sub-solutions)**.** *Assume that the Hamiltonians H_i satisfy* (1.5)*. Let $u : (0, T) \times J \to \mathbb{R}$ be an upper semi-continuous sub-solution of* (2.11)*. If u is a A-flux-limited sub-solution then for any function $\psi \in C^1(0, T)$ such that ψ touches $u(\cdot, 0)$ from above at $t_0 \in (0, T)$, we have*

$$\psi_t + A \leq 0 \quad \text{at} \quad t_0. \tag{2.26}$$

Conversely, if (2.26) *holds true for any ψ as above and if u satisfies for all i,*

$$u(t, 0) = \limsup_{(s,y) \to (t,0), y \in J_i^*} u(s, y),$$

then u is a A-flux-limited sub-solution.

Proof of Theorem 2.11. We successively prove that the condition is necessary and sufficient.

Necessary condition. Let $\psi \in C^1(0,T)$ touching $u(\cdot,0)$ from above at $(t_0,0)$ with $t_0 \in (0,T)$. As usual, we can assume without loss of generality that the contact point is strict. Let $\varepsilon > 0$ small enough in order to satisfy

$$\frac{1}{\varepsilon} > p_i^A \tag{2.27}$$

where p_i^A is chosen as in (2.12). Let

$$\phi(t,x) = \psi(t) + \frac{x}{\varepsilon} \quad \text{for} \quad x \in J_i \quad \text{for} \quad i = 1,\dots,N.$$

For $r > 0, \delta > 0$, let

$$\Omega := (t_0 - r, t_0 + r) \times B_\delta(0)$$

where $B_\delta(0)$ is the ball in J centered at 0 and of radius δ. From the upper semi-continuity of u, we can choose r, δ small enough, and then ε small enough, so that

$$\sup_{\Omega}(u-\phi) > \sup_{\partial\Omega}(u-\phi).$$

Therefore there exists a point $P_\varepsilon = (t_\varepsilon, x_\varepsilon) \in \Omega$ such that we have

$$\sup_{\Omega}(u-\phi) = (u-\phi)(P_\varepsilon).$$

If $x_\varepsilon \in J_i^*$, then we have

$$\phi_t + H_i(\partial_i \phi) \le 0 \quad \text{at} \quad P_\varepsilon$$

i.e.

$$\psi'(t_\varepsilon) + H_i(\varepsilon^{-1}) \le 0.$$

This is impossible for ε small enough, because of the coercivity of H_i. Therefore we have $x_\varepsilon = 0$, and get

$$\phi_t + F_A(\phi_x) \le 0 \quad \text{at} \quad P_\varepsilon.$$

Because of (2.27), we deduce that $F_A(\phi_x) = A$ and then

$$\psi'(t_\varepsilon) + A \le 0 \quad \text{with} \quad t_\varepsilon \in (t_0 - r, t_0 + r).$$

In the limit $r \to 0$, we get the desired inequality (2.26).

Sufficient condition. Let $\phi(t,x)$ be a test function touching u from above at $(t_0,0)$ for some $t_0 \in (0,T)$. From Theorem 2.7, we know that we can assume that ϕ satisfies (2.14). Then $\phi(t,0)$ touches $u(t,0)$ from above at t_0. Therefore we have by assumption

$$\phi_t(t_0,0) + A \le 0.$$

Because of (2.14), we get the desired inequality

$$\phi_t + F_A(\phi_x) \le 0 \quad \text{at} \quad (t_0,0).$$

This ends the proof of the theorem. □

2.6 General junction conditions reduce to flux-limited ones

Proposition 2.12 (General junction conditions reduce to flux-limited ones)**.** *Let the Hamiltonians satisfy* (1.5) *and* F *satisfy* (1.9)*. There exists* $A_F \in \mathbb{R}$ *such that*

- *any relaxed super-solution of* (1.3) *is an* A_F*-flux-limited super-solution and any relaxed sub-solution of* (1.3) *such that for all* $i = 1, \dots, N$*,*

$$u(t,0) = \limsup_{(s,y)\to(t,0), y\in J_i^*} u(s,y)$$

  is a A_F*-flux-limited sub-solution;*

- *any* A_F*-flux-limited sub-solution is a relaxed sub-solution of* (1.3)*.*

Counter-example 2. If the "weak continuity" condition does not hold, then the conclusion of the proposition is false. Indeed, consider $N = 1$ and $H_1(p) = |p|$ and $F \equiv 0$. In this case $A_0 = 0$ and $A_F = 0$. Then the function

$$u(t,x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x > 0 \end{cases}$$

is a relaxed solution of (1.3) but it does not satisfy the "weak continuity" condition. Moreover, it is not a 0-flux-limited sub-solution: indeed, $\phi(t,x) = 1 + p_i x$ for $x \in J_i$ touches u from above and $\phi_t + F_A(\phi_x) = F_A(p)$ which is not necessarily non-positive since p can be chosen arbitrarily.

The flux limiter A_F is given by the following lemma.

Lemma 2.13 (Definitions of A_F and $\bar{p}$)**.** *Let* $\bar{p}^0 = (\bar{p}_1^0, \dots, \bar{p}_N^0)$ *with* $\bar{p}_i^0 \geq p_i^0$ *be the minimal real number such that* $H_i(\bar{p}_i^0) = A_0$ *with* A_0 *given in (1.8).*

If $F(\bar{p}^0) \geq A_0$*, then there exists a unique* $A_F \in \mathbb{R}$ *such that there exists* $\bar{p} = (\bar{p}_1, \dots, \bar{p}_N)$ *with* $\bar{p}_i \geq \bar{p}_i^0 \geq p_i^0$ *such that*

$$H_i(\bar{p}_i) = H_i^+(\bar{p}_i) = A_F = F(\bar{p}).$$

If $F(\bar{p}^0) < A_0$*, we set* $A_F = A_0$ *and* $\bar{p} = \bar{p}^0$*.*

In particular, we have

$$\{\forall i : p_i \geq \bar{p}_i\} \quad \Rightarrow F(p) \leq A_F, \tag{2.28}$$

$$\{\forall i : p_i \leq \bar{p}_i\} \quad \Rightarrow F(p) \geq A_F. \tag{2.29}$$

Proof of Proposition 2.12. Let A denote A_F. We first prove that relaxed super-solutions are flux-limited solutions. We only do the proof for super-solutions since it is very similar for sub-solutions.

Without loss of generality, we assume that u is lower semi-continuous. Consider a test function ϕ touching u from below at $(t, x) \in (0, +\infty) \times J$,

$$\phi \leq u \text{ in } B_R(t, x) \quad \text{ and } \quad \phi(t, x) = u(t, x)$$

for some $R > 0$. If $x \neq 0$, there is nothing to prove. We therefore assume that $x = 0$. In particular, we have

$$\phi_t(t, 0) + \max(F(\phi_x(t, 0)), \max_i H_i(\partial_i \phi(t, 0))) \geq 0. \tag{2.30}$$

By Theorem 2.7, we can assume that the test function satisfies

$$\partial_i \phi(t, 0) = \bar{p}_i \tag{2.31}$$

where $\bar{p}_i$ is given in Lemma 2.13. We now want to prove that

$$\phi_t(t, 0) + A \geq 0.$$

This follows immediately from (2.30), (2.31) and the definition of $\bar{p}_i$ in Lemma 2.13.

We now prove that flux-limited sub-solutions are relaxed sub-solutions. Once again, we only do the proof for sub-solutions since it is very similar for super-solutions. Consider a test function ϕ touching u from above at $(t, 0)$. Then

$$A_F \vee \max_i H_i^-(p_i) \leq \lambda$$

with $p_i = \partial_i \phi(t, 0)$ and $\lambda = -\phi_t(t, 0)$. We distinguish three cases.

Assume first that for all i, $p_i \geq \pi_i^+(A_F)$. Then $F(p) \leq F(\pi^+(A_F)) = A_F \leq \lambda$.

If there exists i_0 such that $p_{i_0} < \pi_{i_0}^+(A_F)$ and $H_{i_0}(p_{i_0}) \leq A_F$, we have $H_{i_0}(p_{i_0}) \leq \lambda$.

If there exists i_0 such that $p_{i_0} < \pi_{i_0}^+(A_F)$ and $H_{i_0}(p_{i_0}) > A_F$, then we have $H_{i_0}(p_{i_0}) = H_{i_0}^-(p_{i_0}) \leq \lambda$. The proof is now complete. □

2.7 Existence of solutions

Theorem 2.14 (Existence)**.** *Let $T > 0$ and J be the junction defined in* (1.1)*. Assume that Hamiltonians satisfy* (1.5)*, that the junction function F satisfies* (1.9) *and that the initial datum u_0 is uniformly continuous. Then there exists a relaxed viscosity solution u of* (1.3)-(1.4) *in $[0, T) \times J$ and a constant $C_T > 0$ such that*

$$|u(t, x) - u_0(x)| \leq C_T \quad \textit{for all} \quad (t, x) \in [0, T) \times J.$$

Proof of Theorem 2.14. The proof follows classically along the lines of Perron's method (see [32, 15]), and then we omit details.

Step 1: Barriers. Because of the uniform continuity of u_0, for any $\varepsilon \in (0,1]$, it can be regularized by convolution to get a modified initial data u_0^ε satisfying

$$|u_0^\varepsilon - u_0| \le \varepsilon \quad \text{and} \quad |(u_0^\varepsilon)_x| \le L_\varepsilon \tag{2.32}$$

with $L_\varepsilon \ge \max_{i=1,\dots,N} |p_i^0|$. Indeed, if we consider $u_i : \mathbb{R} \to \mathbb{R}$ such that $u_i(x) = (u_0)|_{J^i}(x)$ for $x \ge 0$ and $u_i(x) = u_i(0)$ for $x < 0$, we can get u_i^ε such that $|u_i^\varepsilon - u_0| \le \varepsilon/2$ on J_i and $|(u_i^\varepsilon)_x| \le L_\varepsilon$. In particular, $|u_i(0) - u_0(0)| \le \varepsilon/2$. We can now define $u_0^\varepsilon(x) = u_i(x) - u_i(0) + u_0(0)$ and get u_0^ε such that (2.32) holds true. Let

$$C_\varepsilon = \max\left(\max_{i=1,\dots,N} \max_{|p_i| \le L_\varepsilon} |H_i(p_i)|, \max_{|p_i|\le L_\varepsilon} F(p_1,\dots,p_N) \right).$$

Then the functions

$$u_\varepsilon^\pm(t,x) = u_0^\varepsilon(x) \pm C_\varepsilon t \pm \varepsilon \tag{2.33}$$

are global super and sub-solutions with respect to the initial data u_0. We then define

$$u^+(t,x) = \inf_{\varepsilon\in(0,1]} u_\varepsilon^+(t,x) \quad \text{and} \quad u^-(t,x) = \sup_{\varepsilon\in(0,1]} u_\varepsilon^-(t,x).$$

Then we have $u^- \le u^+$ with $u^-(0,x) = u_0(x) = u^+(0,x)$. Moreover, by stability of sub/super-solutions (see Proposition 2.4), we get that u^+ is a super-solution and u^- is a sub-solution of (1.3) on $(0,T) \times J$.

Step 2: Maximal sub-solution and preliminaries. Consider the set

$$S = \left\{ w : [0,T) \times J \to \mathbb{R}, \quad w \text{ is a sub-solution of (1.3) on } (0,T)\times J, \quad u^- \le w \le u^+ \right\}.$$

It contains u^-. Then the function

$$u(t,x) = \sup_{w\in S} w(t,x)$$

is a sub-solution of (1.3) on $(0,T)\times J$ and satisfies the initial condition. It remains to show that u is a super-solution of (1.3) on $(0,T) \times J$. This is classical for a Hamilton-Jacobi equation on an interval, so we only have to prove it at the junction point. We assume by contradiction that u is not a super-solution at $P_0 = (t_0, 0)$ for some $t_0 \in (0,T)$. This implies that there exists a test function φ satisfying $u_* \ge \varphi$ in a neighborhood of P_0 with equality at P_0, and such that

$$\left\{ \begin{array}{l} \varphi_t + F(\varphi_x) < 0, \\ \varphi_t + H_i(\partial_i \varphi) < 0, \quad \text{for} \quad i = 1,\dots,N \end{array} \right| \quad \text{at} \quad P_0. \tag{2.34}$$

We also have $\varphi \le u_* \le u_*^+$. As usual, the fact that u^+ is a super-solution and condition (2.34) imply that we cannot have $\varphi = (u^+)_*$ at P_0. Therefore we have for some $r > 0$ small enough

$$\varphi < (u^+)_* \quad \text{on} \quad \overline{B_r(P_0)} \tag{2.35}$$

where we define the ball $B_r(P_0) = \{(t,x) \in (0,T) \times J, \quad |t-t_0|^2 + d^2(0,x) < r^2\}$. Substracting $|(t,x) - P_0|^2$ to φ and reducing $r > 0$ if necessary, we can assume that

$$\varphi < u_* \quad \text{on} \quad \overline{B_r(P_0)} \setminus \{P_0\}\,. \tag{2.36}$$

Further reducing $r > 0$, we can also assume that (2.34) still holds in $\overline{B_r(P_0)}$.

Step 3: Sub-solution property and contradiction. We claim that φ is a sub-solution of (1.3) in $B_r(P_0)$. Indeed, if ψ is a test function touching φ from above at $P_1 = (t_1, 0) \in B_r(P_0)$, then

$$\psi_t(P_1) = \varphi_t(P_1) \quad \text{and} \quad \partial_i\psi(P_1) \geq \partial_i\varphi(P_1) \quad \text{for} \quad i = 1, ..., N.$$

Using the fact that F is non-increasing with respect to all variables, we deduce that

$$\psi_t + F(\psi_x) < 0 \quad \text{at} \quad P_1$$

as desired. Defining for $\delta > 0$,

$$u_\delta = \begin{cases} \max(\delta + \varphi, u) & \text{in } B_r(P_0), \\ u & \text{outside} \end{cases}$$

and using (2.36), we can check that $u_\delta = u > \delta + \varphi$ on $\partial B_r(P_0)$ for $\delta > 0$ small enough. This implies that u_δ is a sub-solution lying above u^-. Finally (2.35) implies that $u_\delta \leq u^+$ for $\delta > 0$ small enough. Therefore $u_\delta \in S$, but is is classical to check that u_δ is not below u for $\delta > 0$, which gives a contradiction with the maximality of u. □

2.8 Further properties of flux-limited solutions

In this section, we focus on properties of solutions of the following equation

$$u_t + H(u_x) = 0 \tag{2.37}$$

for a single Hamiltonian satisfying (1.5). We start with the following result, which is strongly related to the reformulation of state constraints from [33], and its use in [3].

Proposition 2.15 (Reformulation of state constraints)**.** *Assume that H satisfies* (1.5). *Let $u : (0,T) \times [a,b] \to \mathbb{R}$. If u satisfies*

$$\begin{cases} u_t + H(u_x) = 0 & \textit{for} \quad (t,x) \in (0,T) \times (a,b), \\ u_t + H^-(u_x) = 0 & \textit{for} \quad (t,x) \in (0,T) \times \{a\}\,, \\ u_t + H^+(u_x) = 0 & \textit{for} \quad (t,x) \in (0,T) \times \{b\} \end{cases} \tag{2.38}$$

in the viscosity sense if and only if

$$\begin{cases} u_t + H(u_x) \geq 0 & \textit{for} \quad (t,x) \in (0,T) \times \overline{\Omega}, \\ u_t + H(u_x) \leq 0 & \textit{for} \quad (t,x) \in (0,T) \times \Omega \end{cases} \tag{2.39}$$

in the viscosity sense and

$$u(t,c) = \limsup_{(s,y)\to(t,c),\, y\in]a,b[} u(s,y) \quad \textit{for} \quad c = a, b. \tag{2.40}$$

Proof of Proposition 2.15. Remark first that only boundary conditions should be studied.

We first prove that (2.39) implies (2.38). From Theorem 2.7-i), we deduce that the viscosity sub-solution inequality is satisfied on the boundary for (2.38) with the choice $A = A_0 = \min H$.

Let us now consider a test function φ touching u_* from below at the boundary (t_0, x_0). We want to show that u_* is a viscosity super-solution for (2.38) at (t_0, x_0). By Theorem 2.7, it is sufficient to check the inequality assuming that

$$\varphi(t,x) = \psi(t) + \phi(x)$$

with

$$\begin{cases} H(\phi_x) = H^+(\phi_x) = A_0 & \text{at} \quad x_0 \quad \text{if} \quad x_0 = a, \\ H(\phi_x) = H^-(\phi_x) = A_0 & \text{at} \quad x_0 \quad \text{if} \quad x_0 = b. \end{cases}$$

(The second equality involves H^- instead of H^+ because, locally around b, the domain looks like $]b-\varepsilon, b]$ and not $[b, b+\varepsilon[$.) Remark that we have in all cases $H(\phi_x) = H^+(\phi_x) = H^-(\phi_x)$ at x_0. We then deduce from the fact that u_* is a viscosity super-solution of (2.39), that u_* is also a viscosity super-solution of (2.38) at (t_0, x_0).

We now prove that (2.38) implies (2.39). The second line of (2.39) is easy to get. As far as the first line is concerned, it follows from the fact that $H \geq H^\pm$. This ends the proof of the proposition. □

Proposition 2.16 (Classical viscosity solutions are also solutions "at one point")**.** *Assume that H satisfies* (1.5) *and consider a classical Hamilton-Jacobi equation posed in the whole line,*

$$u_t + H(u_x) = 0 \quad \textit{for all} \quad (t,x) \in (0,T) \times \mathbb{R} \tag{2.41}$$

i) (Sub-Solutions) *Let $u : (0,T) \times \mathbb{R} \to \mathbb{R}$ be a sub-solution of (2.41). Then u satisfies*

$$u_t(t,0) + \max(H^+(u_x(t,0^-)), H^-(u_x(t,0^+))) \leq 0. \tag{2.42}$$

ii) (Super-Solutions) *Let $u : (0,T) \times \mathbb{R} \to \mathbb{R}$ be a super-solution of (2.41). Then u satisfies*

$$u_t(t,0) + \max(H^+(u_x(t,0^-)), H^-(u_x(t,0^+))) \geq 0. \tag{2.43}$$

Remark 2.17. We remark that the reverse implication holds true since, when testing with C^1 function, $u_x(t,0^-) = u_x(t,0^+)$ and $H = \max(H^+, H^-)$.

Proof. **Sub-solutions.** In order to apply Theorem 2.7-i), we first remark that the following lemma, whose proof is postponed, implies that u satisfies the "weak continuity" condition (2.13) with the choice $H_2 = H_3 = H$ and $H_1(p) = H(-p)$.

Lemma 2.18 ("weak continuity" condition with C^1 test functions)**.** *Given two Hamiltonians H_1, H_2 satisfying* (1.5) *and H_3 continuous and coercive, let $u : (0,T) \times \mathbb{R} \to \mathbb{R}$ be upper semi-continuous such that all C^1 function ϕ touching u from above at (t,x) satisfies*

$$\begin{cases} \phi_t(t,x) + H_1(\phi_x(t,x)) \leq 0 & \text{if } x < 0, \\ \phi_t(t,x) + H_2(\phi_x(t,x)) \leq 0 & \text{if } x > 0, \\ \phi_t(t,x) + H_3(\phi_x(t,x)) \leq 0 & \text{if } x = 0. \end{cases}$$

Then for all $t \in (0,T)$,

$$u(t,0) = \limsup_{(s,y)\to(t,0), y>0} u(s,y) = \limsup_{(s,y)\to(t,0), y<0} u(s,y).$$

Thanks to Theorem 2.7-i, we deduce that u is a A_0-flux-limited sub-solution with $A_0 = \min H$, which implies (2.42).

Super-solutions. We do not have to use Lemma 2.18, but instead we have to check (2.15) with $A = A_0$ and a good choice of a test function ϕ_0 on $J = J_1 \cup J_2$. Indeed, we simply choose

$$\phi(x) = \begin{cases} \phi_0(y) & \text{for} \quad y = x \in J_1 & \text{if} \quad x \geq 0, \\ \phi_0(y) & \text{for} \quad y = -x \in J_2 & \text{if} \quad x \leq 0, \end{cases}$$

such that ϕ is C^1 on $\mathbb{R}$ and $H(\phi_0'(0)) = \min H = A_0$. This implies (2.43) and ends the proof of the proposition. □

We now prove Lemma 2.18.

Proof of Lemma 2.18. Assume first that there exists t^* such that

$$u(t^*,0) > \limsup_{(s,y)\to(t^*,0), y>0} u(s,y) \quad \text{and} \quad u(t^*,0) > \limsup_{(s,y)\to(t^*,0), y<0} u(s,y).$$

Since $u(t,0)$ is upper semi-continuous, there exists t_0 arbitrarily close to t^* with $u(t_0,0)$ arbitrarily close to $u(t^*,0)$ such that there exists a C^1 function $\Psi(t)$ (strictly) touching $u(t,0)$ from above at $(t_0,0)$. In particular, we can get δ_0 and r_0 such that

$$u(t_0,0) \geq u(s,y) + \delta_0 \text{ for } (s,y) \in B_{r_0}(t,0), y \neq 0.$$

In this first case, the test function $\Psi(t) + px$ (with p arbitrary) touches u from above at $(t_0,0)$. This implies

$$\Psi'(t_0) + H_3(p) \leq 0$$

which contradicts the coercivity of H_3.

Assume now that

$$u(t^*,0) = \limsup_{(s,y)\to(t^*,0), y\geq 0} u(s,y) \quad \text{and} \quad u(t^*,0) > \limsup_{(s,y)\to(t^*,0), y<0} u(s,y).$$

In this case, we can argue as in the proof of Lemma 2.3, the intervals $(-\infty, 0]$ and $[0, +\infty)$ playing the role of J_i for $i \neq i_0$ and J_{i_0} respectively; in particular, we construct a test function $\Psi(t) + px$ with p very negative and get a contradiction with the coercivity of H_2.

The remaining case is similar to the previous one. The proof is now complete. □

Proposition 2.19 (Restriction of sub-solutions are sub-solutions)**.** *Assume that H satisfies* (1.5)*. Let $u : (0,T) \times \mathbb{R} \to \mathbb{R}$ be upper semi-continuous satisfying*

$$u_t + H(u_x) \leq 0 \quad \textit{for all} \quad (t,x) \in (0,T) \times \mathbb{R} \tag{2.44}$$

Then the restriction w of u to $(0,T) \times [0, +\infty)$ satisfies

$$\begin{cases} w_t + H(w_x) \leq 0 & \textit{for all} \quad (t,x) \in (0,T) \times (0,+\infty)\,, \\ w_t + H^-(w_x) \leq 0 & \textit{for all} \quad (t,x) \in (0,T) \times \{0\}\,. \end{cases}$$

Proof of Proposition 2.19. We simply have to study w at the boundary. From Proposition 2.16, we know that u satisfies in the viscosity sense

$$u_t + \max(H^+(u_x(t,0^-)), H^-(u_x(t,0^+))) \leq 0.$$

By Theorem 2.11 with two branches, we deduce that $v(t) = u(t,0)$ satisfies

$$v_t + \min H \leq 0.$$

Again by Theorem 2.11 (now with one branch) and because $v(t) = w(t,0)$, we deduce that w satisfies

$$w_t + H^-(w_x) \leq 0 \quad \text{for all} \quad (t,0) \in (0,T) \times \{0\}$$

which ends the proof. □

Remark 2.20. Notice that the restriction of a super-solution of (2.37) may not be a super-solution on the boundary, as shown by the following example: for $H(p) = |p| - 1$, the solution $u(t,x) = x$ solves $u_t + H(u_x) = 0$ in $\mathbb{R}$ but does not solve $u_t + H^-(u_x) \geq 0$ at $x = 0$.

3 Comparison principle on a junction

This section is devoted to the proof of the comparison principle in the case of a junction (see Theorem 1.4). In view of Propositions 2.12 and 2.5, it is enough to consider sub- and super-solutions (in the sense of Definition 2.2) of (1.7) for some $A = A_F$.

It is convenient to introduce the following shorthand notation

$$H(x,p) = \begin{cases} H_i(p) & \text{for} \quad p = p_i & \text{if} \quad x \in J_i^*, \\ F_A(p) & \text{for} \quad p = (p_1, ..., p_N) & \text{if} \quad x = 0. \end{cases} \tag{3.1}$$

In particular, keeping in mind the definition of u_x (see (1.2)), Problem (1.7) on the junction can be rewritten as follows

$$u_t + H(x, u_x) = 0 \quad \text{for all} \quad (t,x) \in (0,+\infty) \times J.$$

We next make a trivial but useful observation.

Lemma 3.1. *It is enough to prove Theorem 1.4 further assuming that*

$$(3.2) \qquad p_i^0 = 0 \quad \textit{for} \quad i = 1, ..., N \quad \textit{and} \quad 0 = H_1(0) \geq H_2(0) \geq ... \geq H_N(0).$$

Proof. We can assume without loss of generality that

$$H_1(p_1^0) \geq ... \geq H_N(p_N^0).$$

Let us define

$$u(t,x) = \tilde{u}(t,x) + p_i^0 x - tH_1(p_1^0) \quad \text{for} \quad x \in J_i.$$

Then u is a solution of (1.7) if and only if $\tilde{u}$ is a solution of (1.7) with each H_i replaced with $\tilde{H}_i(p) = H_i(p + p_i^0) - H_1(p_1^0)$ and F_A replaced with $\tilde{F}_{\tilde{A}}$ constructed using the Hamiltonians $\tilde{H}_i$ and the parameter $\tilde{A} = A - H_1(p_1^0)$. □

3.1 The vertex test function

Then our key result is the following one.

Theorem 3.2 (The vertex test function – general case)**.** *Let $A \in \mathbb{R} \cup \{-\infty\}$ and $\gamma > 0$. Assume the Hamiltonians satisfy* (1.5) *and* (3.2)*. Then there exists a function $G : J^2 \to \mathbb{R}$ enjoying the following properties.*

i) (Regularity)

$$G \in C(J^2) \quad \textit{and} \quad \begin{cases} G(x,\cdot) \in C^1(J) & \textit{for all} \quad x \in J, \\ G(\cdot,y) \in C^1(J) & \textit{for all} \quad y \in J. \end{cases}$$

ii) (Bound from below) $G \geq 0 = G(0,0)$.

iii) (Compatibility condition on the diagonal) *For all $x \in J$,*

$$(3.3) \qquad 0 \leq G(x,x) - G(0,0) \leq \gamma.$$

iv) (Compatibility condition on the gradients) *For all $(x,y) \in J^2$,*

$$(3.4) \qquad H(y, -G_y(x,y)) - H(x, G_x(x,y)) \leq \gamma$$

where notation introduced in (1.2) *and* (3.1) *are used.*

v) (Superlinearity) *There exists $g : [0,+\infty) \to \mathbb{R}$ nondecreasing and s.t. for $(x,y) \in J^2$*

$$(3.5) \qquad g(d(x,y)) \leq G(x,y) \quad \textit{and} \quad \lim_{a \to +\infty} \frac{g(a)}{a} = +\infty.$$

vi) (Gradient bounds) *For all $K > 0$, there exists $C_K > 0$ such that for all $(x,y) \in J^2$,*

$$(3.6) \qquad d(x,y) \leq K \quad \Longrightarrow \quad |G_x(x,y)| + |G_y(x,y)| \leq C_K.$$

Remark 3.3. The vertex test function G is obtained as a regularized version of a function G^0 which is C^1 except on the diagonal $x = y$. It is in fact possible to check directly that G^0 does not satisfy the viscosity inequalities on the diagonal in the sense of Proposition 2.16 (when it is not C^1 on the diagonal).

3.2 Proof of the comparison principle

We will also need the following result whose classical proof is given in Appendix for the reader's convenience.

Lemma 3.4 (A priori control). *Let $T > 0$ and let u be a sub-solution and v be a super-solution as in Theorem 1.4. Then there exists a constant $C = C(T) > 0$ such that for all $(t,x),(s,y) \in [0,T) \times J$, we have*

$$u(t,x) \leq v(s,y) + C(1 + d(x,y)). \tag{3.7}$$

We are now ready to make the proof of comparison principle.

Proof of Theorem 1.4. As explained at the beginning of the current section, in view of Propositions 2.12 and 2.5, it is enough to consider sub- and super-solutions (in the sense of Definition 2.2) of (1.7) for some $A = A_F$.

The remaining of the proof proceeds in several steps.

Step 1: the penalization procedure. We want to prove that

$$M = \sup_{(t,x)\in[0,T)\times J} (u(t,x) - v(t,x)) \leq 0.$$

Assume by contradiction that $M > 0$. Then for $\alpha, \eta > 0$ small enough, we have $M_{\varepsilon,\alpha} \geq 3M/4 > 0$ for all $\varepsilon, \nu > 0$ with

$$M_{\varepsilon,\alpha} = \sup_{(t,x),(s,y)\in[0,T)\times J} \left\{ u(t,x) - v(s,y) - \varepsilon G\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) - \frac{(t-s)^2}{2\nu} - \frac{\eta}{T-t} - \alpha \frac{d^2(0,x)}{2} \right\} \tag{3.8}$$

where the vertex test function $G \geq 0$ is given by Theorem 3.2 for a parameter γ satisfying

$$0 < \gamma < \min\left(\frac{\eta}{2T^2}, \frac{M}{8\varepsilon}\right).$$

Since $M_{\varepsilon,\alpha} \geq 3M/4$, the supremum can be taken over points (x,y) such that the corresponding value is larger than $M/2$. Thanks to Lemma 3.4 and (3.5), these points satisfy

$$0 < \frac{M}{2} \leq C(1 + d(x,y)) - \varepsilon g\left(\frac{d(x,y)}{\varepsilon}\right) - \frac{(t-s)^2}{2\nu} - \frac{\eta}{T-t} - \alpha \frac{d^2(0,x)}{2} \tag{3.9}$$

which implies in particular that

$$\varepsilon g\left(\frac{d(x,y)}{\varepsilon}\right) \leq C(1 + d(x,y)). \tag{3.10}$$

Because of the superlinearity of g appearing in (3.5), we know that for any $K > 0$, there exists a constant $C_K > 0$ such that for all $a \geq 0$

$$Ka - C_K \leq g(a).$$

For $K \geq 2C$, we deduce from (3.10) that

$$d(x,y) \leq \inf_{K \geq 2C} \left\{ \frac{C}{K-C} + \frac{C_K}{C}\varepsilon \right\} =: \omega(\varepsilon) \tag{3.11}$$

where ω is a concave, nondecreasing function satisfying $\omega(0) = 0$. We deduce from (3.9) and (3.11) that the supremum in (3.8) is reached at some point $(t,x,s,y) = (t_\nu, x_\nu, s_\nu, y_\nu)$.

Step 2: use of the initial condition. We first treat the case where $t_\nu = 0$ or $s_\nu = 0$. If there exists a sequence $\nu \to 0$ such that $t_\nu = 0$ or $s_\nu = 0$, then calling (x_0, y_0) any limit of subsequences of (x_ν, y_ν), we get from (3.8) and the fact that $M_{\varepsilon,\alpha} \geq M/2$ that

$$0 < \frac{M}{2} \leq u_0(x_0) - u_0(y_0) \leq \omega_0(d(x_0,y_0)) \leq \omega_0 \circ \omega(\varepsilon)$$

where ω_0 is the modulus of continuity of the initial data u_0 and ω is defined in (3.11). This is impossible for ε small enough.

Step 3: use of the equation. We now treat the case where $t_\nu > 0$ and $s_\nu > 0$. Then we can write the viscosity inequalities with $(t,x,s,y) = (t_\nu, x_\nu, s_\nu, y_\nu)$ using the shorthand notation (3.1) for the Hamiltonian,

$$\begin{aligned} \frac{\eta}{(T-t)^2} + \frac{t-s}{\nu} + H(x, G_x(\varepsilon^{-1}x, \varepsilon^{-1}y) + \alpha d(0,x)) &\leq 0, \\ \frac{t-s}{\nu} + H(y, -G_y(\varepsilon^{-1}x, \varepsilon^{-1}y)) &\geq 0. \end{aligned}$$

Substracting these two inequalities, we get

$$\frac{\eta}{T^2} \leq H(y, -G_y(\varepsilon^{-1}x, \varepsilon^{-1}y)) - H(x, G_x(\varepsilon^{-1}x, \varepsilon^{-1}y) + \alpha d(0,x)).$$

Using (3.4) with $\gamma \in \left(0, \frac{\eta}{2T^2}\right)$, we deduce for $p = G_x(\varepsilon^{-1}x, \varepsilon^{-1}y)$

$$\frac{\eta}{2T^2} \leq H(x,p) - H(x, p + \alpha d(0,x)). \tag{3.12}$$

Because of (3.6) and (3.11), we see that p is bounded for ε fixed by $|p| \leq C_{\frac{\omega(\varepsilon)}{\varepsilon}}$. Finally, for $\varepsilon > 0$ fixed and $\alpha \to 0$, we have $\alpha d(0,x) \to 0$, and using the uniform continuity of $H(x,p)$ for $x \in J$ and p bounded, we get a contradiction in (3.12). The proof is now complete. □

3.3 The vertex test function versus the fundamental solution

Recalling the definition of the germ $\mathcal{G}_A$ (see (1.11)), let us associate with any $(p,\lambda) \in \mathcal{G}_A$ the following functions for $i,j = 1,...,N$,

$$u^{p,\lambda}(t,x,s,y) = p_i x - p_j y - \lambda(t-s) \quad \text{for} \quad (x,y) \in J_i \times J_j, \quad t,s \in \mathbb{R}.$$

The reader can check that they solve the following system,

$$\begin{cases} u_t + H(x, u_x) = 0, \\ -u_s + H(y, -u_y) = 0. \end{cases} \tag{3.13}$$

Then, for $N \geq 2$, the function $\tilde{G}^0(t, x, s, y) = (t-s)G^0\left(\frac{x}{t-s}, \frac{y}{t-s}\right)$ can be rewritten as

$$\tilde{G}^0(t, x, s, y) = \sup_{(p,\lambda)\in\mathcal{G}_A} u^{p,\lambda}(t, x, s, y) \quad \text{for} \quad (x, y) \in J \times J, \quad t - s \geq 0 \tag{3.14}$$

which satisfies

$$\tilde{G}^0(s, x, s, y) = \begin{cases} 0 & \text{if} \quad x = y, \\ +\infty & \text{otherwise.} \end{cases} \tag{3.15}$$

For $N \geq 2$ and $A > A_0$, it is possible to check (at least in the smooth convex case – see (4.1) below) that $\tilde{G}^0$ is a viscosity solution of (3.13) for $t-s > 0$, only outside the diagonal $\{x = y \neq 0\}$. Therefore, even if (3.14) appears as a kind of (second) Hopf formula (see for instance [8, 4]), this formula does not provide a true solution on the junction.

On the other hand, under more restrictive assumptions on the Hamiltonians and for $A = A_0$ and $N \geq 2$ (see [30]), there is a natural viscosity solution of (3.13) with the same initial conditions (3.15), which is $\mathcal{D}(t, x, s, y) = (t - s)\mathcal{D}_0\left(\frac{x}{t-s}, \frac{y}{t-s}\right)$ where $\mathcal{D}_0$ is a cost function defined in [30] following an optimal control interpretation. The function $\mathcal{D}_0$ is not C^1 in general (but it is semi-concave) and it is much more difficult to study it and to use it in comparison with G^0. Nevertheless, under suitable restrictive assumptions on the Hamiltonians, it would be also possible to replace in our proof of the comparison principle the term $\varepsilon G(\varepsilon^{-1}x, \varepsilon^{-1}y)$ in (3.8) by $\varepsilon\mathcal{D}_0(\varepsilon^{-1}x, \varepsilon^{-1}y)$.

4 Construction of the vertex test function

This section is devoted to the proof of Theorem 3.2. Our construction of the vertex test function G is follows the same pattern as the particular subcase of normalized convex Hamiltonians H_i.

4.1 The case of smooth convex Hamiltonians

Assume that the Hamiltonians H_i satisfy the following assumptions for $i = 1, ..., N$,

$$\begin{cases} H_i \in C^2(\mathbb{R}) \quad \text{with} \quad H_i'' > 0 \quad \text{on} \quad \mathbb{R}, \\ H_i' < 0 \quad \text{on} \quad (-\infty, 0) \quad \text{and} \quad H_i' > 0 \quad \text{on} \quad (0, +\infty), \\ \displaystyle\lim_{|p|\to+\infty} \frac{H_i(p)}{|p|} = +\infty. \end{cases} \tag{4.1}$$

It is useful to associate with each H_i satisfying (4.1) its partial inverse functions $\pi_i^\pm$:

$$\text{for } \lambda \geq H_i(0), \quad H_i(\pi_i^\pm(\lambda)) = \lambda \quad \text{such that} \quad \pm\,\pi_i^\pm(\lambda) \geq 0. \tag{4.2}$$

Assumption (4.1) implies that $\pi_i^\pm \in C^2(\min H_i, +\infty) \cap C([\min H_i, +\infty))$ thanks to the inverse function theorem.

We recall that G^0 is defined, for $i, j = 1, ..., N$, by

$$G^0(x,y) = \sup_{(p,\lambda)\in\mathcal{G}_A} (p_i x - p_j y - \lambda) \quad \text{if} \quad (x,y) \in J_i \times J_j$$

where $\mathcal{G}_A$ is defined in (1.11). Replacing A with $\max(A, A_0)$ if necessary, we can always assume that $A \geq A_0$ with A_0 given by (1.8).

Proposition 4.1 (The vertex test function – the smooth convex case)**.** *Let $A \geq A_0$ with A_0 given by* (1.8) *and assume that the Hamiltonians satisfy* (4.1)*. Then G^0 satisfies*

i) (Regularity)

$$G^0 \in C(J^2) \quad \textit{and} \quad \begin{cases} G^0 \in C^1(\{(x,y) \in J \times J, \quad x \neq y\}), \\ G^0(0,\cdot) \in C^1(J) \quad \textit{and} \quad G^0(\cdot,0) \in C^1(J); \end{cases}$$

ii) (Bound from below) $G^0 \geq G^0(0,0) = -A$;

iii) (Compatibility conditions) (3.3) *holds with $\gamma = 0$ for all $x \in J$ and* (3.4) *holds with $\gamma = 0$ for (x,y) such that either $x \neq y$ or $x = y = 0$;*

iv) (Superlinearity) (3.5) *holds for some $g = g^0$;*

v) (Gradient bounds) (3.6) *holds only for $(x,y) \in J^2$ such that $x \neq y$ or $(x,y) = (0,0)$;*

vi) (Saturation close to the diagonal) *For $i \in \{1, ..., N\}$ and for $(x,y) \in J_i \times J_i$, we have $G^0(x,y) = \ell_i(x-y)$ with $\ell_i \in C(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\})$ and*

$$\ell_i(a) = \begin{cases} a\pi_i^+(A) - A & \textit{if} \quad 0 \leq a \leq z_i^+ \\ a\pi_i^-(A) - A & \textit{if} \quad z_i^- \leq a \leq 0 \end{cases}$$

where $(z_i^-, z_i^+) := (H_i'(\pi_i^-(A)), H_i'(\pi_i^+(A)))$ and the functions $\pi_i^\pm$ are defined in (4.2)*. Moreover $G^0 \in C^1(J_i \times J_i)$ if and only if $\pi_i^+(A) = 0 = \pi_i^-(A)$.*

Remark 4.2. The compatibility condition (3.4) for $x \neq y$, is in fact an equality with $\gamma = 0$ when $N \geq 2$.

The proof of this proposition is postponed until Subsection 4.4. With such a result in hand, we can now prove Theorem 3.2 in the case of smooth convex Hamiltonians.

Lemma 4.3 (The case of smooth convex Hamiltonians)**.** *Assume that the Hamiltonians satisfy* (4.1)*. Then the conclusion of Theorem 3.2 holds true.*

Proof. We note that the function $G^0 + A$ satisfies all the properties required for G, except on the diagonal $\{(x,y) \in J \times J, x = y \neq 0\}$ where G^0 may not be C^1. To this end, we first introduce the set I of indices such that $G^0 \notin C^1(J_i \times J_i)$. We know from Proposition 4.1 vi) that

$$I = \left\{ i \in \{1, ..., N\}, \quad \pi_i^+(A) > \pi_i^-(A) \right\}.$$

For $i \in I$, we are going to contruct a regularization $\tilde{G}^{0,i}$ of G^0 in a neighbourhood of the diagonal $\{(x,y) \in J_i \times J_i,\ x = y \neq 0\}$.

Step 1: Construction of $\tilde{G}^{0,i}$ for $i \in I$. Let us define

$$L_i(a) = \begin{cases} a\pi_i^+(A) & \text{if} \quad a \geq 0, \\ a\pi_i^-(A) & \text{if} \quad a \leq 0. \end{cases}$$

We first consider a convex C^1 function $\tilde{L}_i : \mathbb{R} \to \mathbb{R}$ coinciding with L_i outside (z_i^-, z_i^+), that we choose such that

$$0 \leq \tilde{L}_i - L_i \leq 1. \tag{4.3}$$

Then for $\varepsilon \in (0,1]$, we define

$$\ell_i^\varepsilon(a) := \begin{cases} \varepsilon \tilde{L}_i\left(\frac{a}{\varepsilon}\right) - A & \text{if} \quad a \in [\varepsilon z_i^-, \varepsilon z_i^+], \\ \ell_i(a) & \text{otherwise.} \end{cases}$$

which is a $C^1(\mathbb{R})$ (and convex) function. We now consider a cut-off function ζ satisfying for some constant $B > 0$

$$\begin{cases} \zeta \in C^\infty(\mathbb{R}), \\ \zeta' \geq 0, \\ \zeta > 0 \quad \text{in} \quad (0, +\infty), \\ \zeta = 0 \quad \text{in} \quad (-\infty, 0], \\ \zeta = 1 \quad \text{in} \quad [B, +\infty), \\ \pm z_i^\pm \zeta' < 1 \quad \text{in} \quad (0, +\infty) \end{cases} \tag{4.4}$$

and for $\varepsilon \in (0,1]$, we define for $(x,y) \in J_i \times J_i$:

$$\tilde{G}^{0,i}(x,y) = \ell_i^{\varepsilon\zeta(x+y)}(x-y).$$

Step 2: First properties of $\tilde{G}^{0,i}$. By construction, we have $\tilde{G}^{0,i} \in C^1((J_i \times J_i) \setminus \{(0,0)\})$. Moreover we have

$$\tilde{G}^{0,i} = G^0 \quad \text{on} \quad (J_i \times J_i) \setminus \delta_i^\varepsilon$$

where

$$\delta_i^\varepsilon = \left\{(x,y) \in J_i \times J_i, \quad \varepsilon z_i^- \zeta(x+y) < x - y < \varepsilon z_i^+ \zeta(x+y)\right\}$$

is a neighborhood of the diagonal

$$\{(x,y) \in J_i \times J_i, \quad x = y \neq 0\}.$$

Because of (4.3), we also have

$$0 \leq G^0 - \tilde{G}^{0,i} \leq \varepsilon. \tag{4.5}$$

As a consequence of (4.4), we have in particular

$$(J_i \times J_i) \setminus \delta_i^\varepsilon \quad \supset \quad (J_i \times \{0\}) \cup (\{0\} \times J_i)$$

and moreover $\tilde{G}^{0,i}$ coincides with G^0 on a neighborhood of $(J_i^* \times \{0\}) \cup (\{0\} \times J_i^*)$, which implies that

$$\tilde{G}^{0,i} = G^0, \quad \tilde{G}_x^{0,i} = G_x^0 \quad \text{and} \quad \tilde{G}_y^{0,i} = G_y^0 \quad \text{on} \quad (J_i \times \{0\}) \cup (\{0\} \times J_i). \tag{4.6}$$

Step 3: Computation of the gradients of $\tilde{G}^{0,i}$. For $(x, y) \in \delta_i^\varepsilon$, we have

$$\begin{cases} \tilde{G}_x^{0,i}(x,y) & = (\ell_i^{\varepsilon\zeta(x+y)})'(x-y) + \varepsilon\zeta'(x+y)\ \xi_i\left(\dfrac{x-y}{\varepsilon\zeta(x+y)}\right) \\ -\tilde{G}_y^{0,i}(x,y) & = (\ell_i^{\varepsilon\zeta(x+y)})'(x-y) - \varepsilon\zeta'(x+y)\ \xi_i\left(\dfrac{x-y}{\varepsilon\zeta(x+y)}\right) \end{cases}$$

with

$$\xi_i(b) = \tilde{L}_i(b) - b\tilde{L}_i'(b)$$

while if $(x, y) \in (J_i \times J_i) \setminus \delta_i^\varepsilon$ we have

$$\tilde{G}_x^{0,i}(x,y) = -\tilde{G}_y^{0,i}(x,y).$$

Given $\gamma > 0$, and using the local uniform continuity of H_i, we see that we have for ε small enough

$$H_i(\tilde{G}_x^{0,i})) \le H_i(-\tilde{G}_y^{0,i}) + \gamma \quad \text{in} \quad J_i^* \times J_i^*$$

and using (4.6), we get

$$H(x, \tilde{G}_x^{0,i}(x,y)) - H(y, -\tilde{G}_y^{0,i}(x,y)) \le \gamma \quad \text{for all} \quad (x,y) \in J_i \times J_i. \tag{4.7}$$

Step 4: Definition of G. We set for $(x, y) \in J_i \times J_j$:

$$G(x,y) = \begin{cases} G^0(x,y) + A & \text{if} \quad i \neq j \quad \text{or} \quad i = j \notin I, \\ \tilde{G}^{0,i}(x,y) + A & \text{if} \quad i = j \in I. \end{cases}$$

From the properties of G^0, we recover all the expected properties of G with $g(a) = g^0(a)+A$. In particular from Proposition 4.1-(iii), (4.7) and (4.5), we respectively get the compatibility condition for the Hamiltonians (3.4) and the compatibility condition on the diagonal (3.3) for ε small enough. As far as (3.5) is concerned, we remark that $G(x, y)$ coincide with $G^0(x, y) + A$ when $d(x, y)$ is large. As far as (3.6) is concerned, G_x and G_y coincide with G_x^0 and G_y^0 if $x \in J_i$ and $y \in J_j$ with $i \neq j$; hence we can apply Proposition 4.1-(v). In the case where x and y belongs to the same branch, $G(x, y)$ is a smooth function of $x - y$ when $x + y \ge 1$ (since $\zeta(r) = 1$ for $r \ge 1$). In particular, G_x and G_y are bounded as soon as $|x - y|$ is so. Finally, when $x + y \le 1$, (x, y) is in a compact set and G_x and G_y are also bounded. □

4.2 The general case

Let us consider a slightly stronger assumption than (1.5), namely

$$\begin{cases} H_i \in C^2(\mathbb{R}) \quad \text{with} \quad H_i''(p_i^0) > 0, \\ H_i' < 0 \quad \text{on} \quad (-\infty, p_i^0) \quad \text{and} \quad H_i' > 0 \quad \text{on} \quad (p_i^0, +\infty), \\ \lim\limits_{|q|\to+\infty} H_i(q) = +\infty. \end{cases} \tag{4.8}$$

We will also use the following technical result which allows us to reduce certain non-convex Hamiltonians to convex Hamiltonians.

Lemma 4.4 (From non-convex to convex Hamiltonians)**.** *Given Hamiltonians H_i satisfying* (4.8) *and* (3.2)*, there exists a function $\beta : \mathbb{R} \to \mathbb{R}$ such that the functions $\beta \circ H_i$ satisfy* (4.1) *for $i = 1, ..., N$. Moreover, we can choose β such that*

$$\beta \quad \textit{is convex}, \quad \beta \in C^2(\mathbb{R}), \quad \beta(0) = 0 \quad \textit{and} \quad \beta' \geq \delta > 0. \tag{4.9}$$

Proof. Recalling (4.2), it is easy to check that $(\beta \circ H_i)'' > 0$ if and only if we have

$$(\ln \beta')'(\lambda) > -\frac{H_i''}{(H_i')^2} \circ \pi_i^{\pm}(\lambda) \quad \text{for} \quad \lambda \geq H_i(0). \tag{4.10}$$

Because $H_i''(0) > 0$, we see that the right hand side is negative for λ close enough to $H_i(0)$.

Then it is easy to choose a function β satisfying (4.10) and (4.9). Indeed, since we impose $\beta(0) = 0$, we only need to find a non-decreasing C^1 function β' bounded from below by some $\delta > 0$. Let β' be written in the form e^B. We impose $(e^B)(0) = \delta$ and (4.9) is satisfied if B' is bounded from below in $[H_i(0), +\infty)$ by a given function which is negative at $H_i(0)$. The subtle point is that β should not depend on i. It is enough to take the supremum of these lower bounds, add a small constant which preserves the "room" at $H_i(0)$ and consider a smooth function above this supremum.

Finally, compositing β with another convex increasing function which is superlinear at $+\infty$ if necessary, we can ensure that $\beta \circ H_i$ is superlinear. □

Lemma 4.5 (The case of smooth Hamiltonians)**.** *Theorem 3.2 holds true if the Hamiltonians satisfy* (4.8).

Proof. We assume that the Hamiltonians H_i satisfy (4.8). Thanks to Lemma 3.1, we can further assume that they satisfy (3.2). Let β be the function given by Lemma 4.4. If u solves (1.7) on $(0, T) \times J$, then u is also a viscosity solution of

$$\begin{cases} \bar{\beta}(u_t) + \hat{H}_i(u_x) = 0 & \text{for} \quad t \in (0, T) \quad \text{and} \quad x \in J_i^*, \\ \bar{\beta}(u_t) + \hat{F}_{\hat{A}}(u_x) = 0 & \text{for} \quad t \in (0, T) \quad \text{and} \quad x = 0 \end{cases} \tag{4.11}$$

with $\hat{F}_{\hat{A}}$ constructed as F_A where H_i and A are replaced with $\hat{H}_i$ and $\hat{A}$ defined as follows

$$\hat{H}_i = \beta \circ H_i, \quad \hat{A} = \beta(A)$$

and $\bar{\beta}(\lambda) = -\beta(-\lambda)$. We can then apply Theorem 3.2 in the case of smooth convex Hamiltonians (namely Lemma 4.3) to construct a vertex test function $\hat{G}$ associated to problem (4.11) for every $\hat{\gamma} > 0$. This means that we have with $\hat{H}(x, p) = \beta(H(x, p))$,

$$\hat{H}(y, -G_y) \leq \hat{H}(x, G_x) + \hat{\gamma}.$$

This implies

$$H(y, -G_y) \leq \beta^{-1}(\beta(H(x, G_x)) + \hat{\gamma}) \leq H(x, G_x) + \hat{\gamma}|(\beta^{-1})'|_{L^\infty(\mathbb{R})}.$$

Because of the lower bound on β' given by Lemma 4.4, we get $|(\beta^{-1})'|_{L^\infty(\mathbb{R})} \leq 1/\delta$ which yields the compatibility condition (3.4) with $\gamma = \hat{\gamma}/\delta$ arbitrarily small. □

We are now in position to prove Theorem 3.2 in the general case.

Proof of Theorem 3.2. Let us now assume that the Hamiltonians only satisfy (1.5). In this case, we simply approximate the Hamiltonians H_i by other Hamiltonians $\tilde{H}_i$ satisfying (4.8) such that

$$|H_i - \tilde{H}_i| \leq \gamma.$$

We then apply Theorem 3.2 to the Hamiltonians $\tilde{H}_i$ and construct an associated vertex test function $\tilde{G}$ also for the parameter γ. We deduce that

$$H(y, -\tilde{G}_y) \leq H(x, \tilde{G}_x) + 3\gamma$$

with $\gamma > 0$ arbitrarily small, which shows again the compatibility condition on the Hamiltonians (3.4) for the Hamiltonians H_i's. The proof is now complete in the general case. □

Remark 4.6 (A variant in the proof of construction of G^0). When the Hamiltonians are not convex, it is also possible to use the function β from Lemma 4.4 in a different way by defining directly the function G^0 as follows

$$\tilde{G}^0(x, y) = \sup_{(p,\lambda)\in\mathcal{G}_A} (p_i x - p_j y - \beta(\lambda)).$$

4.3 A special function

In order to prove Proposition 4.1, we first need to study a special function $\mathfrak{G}$. Precisely, we define the following convex function for $z = (z_1, ..., z_N) \in \mathbb{R}^N$,

$$\mathfrak{G}(z) = \sup_{(p,\lambda)\in\mathcal{G}_A} (p \cdot z - \lambda).$$

We remark that if $\pm z_i \geq 0$ then the supremum will select $\pm p_i \geq 0$ if the two vectors $(p_1, \ldots, \pm p_i, \ldots p_N)$ belong to the germ $\mathcal{G}_A$. Moreover, in view of the definition of the germ, see (1.11), we know that $(p, \lambda) \in \mathcal{G}_A$ if and only if $p_i = \pi_i^{\sigma_i}(\lambda)$ for some $\sigma_i \in \{-, +\}$, $\lambda \geq A$ and $(\sigma_1, \ldots, \sigma_N) \neq (+, \ldots, +)$ for $\lambda > A$. These facts explain why we will assume that $\sigma \neq (+, \ldots, +)$ in the two next lemmas.

For $\sigma = (\sigma_1, \ldots, \sigma_N) \in \{\pm\}^N$, we consider the following subsets of $\mathbb{R}^N$,

$$Q_\sigma = \{z = (z_1, \ldots, z_N) \in \mathbb{R}^N : \quad \sigma_i z_i \geq 0, \quad i = 1, \ldots, N\}$$
$$\Delta_\sigma = \{z = (z_1, \ldots, z_N) \in Q_\sigma : \sum_{i=1}^N \frac{\sigma_i z_i}{\bar{z}_i^\sigma(A)} \leq 1\}$$

where $\bar{z}_i^\sigma(A) = \sigma_i H_i'(\pi_i^{\sigma_i}(A)) \geq 0$ and the functions $\pi_i^\pm$ are defined in (4.2). We also make precise that we use the following convenient convention,

$$\frac{\bar{z}_i}{\bar{z}_i^\sigma(A)} = \begin{cases} 0 & \text{if} \quad \bar{z}_i = 0, \\ +\infty & \text{if} \quad \bar{z}_i > 0 \quad \text{and} \quad \bar{z}_i^\sigma(A) = 0. \end{cases} \tag{4.12}$$

Lemma 4.7 (The function $\mathfrak{G}$ in Q_σ)**.** *Under the assumptions of Proposition 4.1, we have, for any $\sigma \in \{\pm\}^N$ with $\sigma \neq (+,\dots,+)$ if $N \geq 2$:*

i) *$\mathfrak{G}$ is C^1 on Q_σ (up to the boundary).*

ii) *For all $z \in Q_\sigma$, there exists a unique $\lambda = \mathfrak{L}(z) \geq A$ such that*

$$\mathfrak{G}(z) = p \cdot z - \lambda$$
$$\nabla\mathfrak{G}(z) = p = (p_1, \dots, p_N)$$
$$p_i = \pi_i^{\sigma_i}(\lambda)$$

with $(p,\lambda) \in \mathcal{G}_A$. In particular, p_i is unique.

iii) *For all $z \in Q_\sigma$, $\mathfrak{L}(z) = A$ if and only if $z \in \Delta_\sigma$. In particular, $\mathfrak{G}$ is linear in Δ_σ: for $z \in \Delta_\sigma$, $\mathfrak{G}(z) = \sum_i \pi_i^{\sigma_i}(A)z - A$.*

Before giving global properties of $\mathfrak{G}$, we introduce the set

$$\bar{\Omega} = \begin{cases} \mathbb{R} & \text{if } N = 1, \\ \mathbb{R}^N \setminus (0,+\infty)^N & \text{if } N \geq 2. \end{cases} \tag{4.13}$$

Lemma 4.8 (Global properties of $\mathfrak{G}$ and $\mathfrak{L}$)**.** *Under the assumptions of Proposition 4.1, the function $\mathfrak{G}$ is convex and finite in $\mathbb{R}^N$, reaches its minimum $-A$ at 0 and the function $\mathfrak{L}$ is continuous in $\bar{\Omega}$.*

Proof of Lemmas 4.7 and 4.8. Let $\sigma \in \{\pm\}^N$ and $z \in Q_\sigma$. We set

$$\pi^\sigma(\lambda) = (\pi_1^{\sigma_1}(\lambda), ..., \pi_N^{\sigma_N}(\lambda)).$$

Using the fact that $(\pi^\sigma(A), A) \in \mathcal{G}_A$, we get $\mathfrak{G}(z) \geq \mathfrak{G}(0) = -A$.

Step 1: Explicit expression of $\mathfrak{G}$. For $\sigma \neq (+,\dots,+)$ if $N \geq 2$, we have

$$(p,\lambda) \in \mathcal{G}_A \cap (Q_\sigma \times \mathbb{R}) \iff \lambda \geq A \quad \text{and} \quad p = \pi^\sigma(\lambda). \tag{4.14}$$

This implies in particular that

$$\mathfrak{G}(z) = \sup_{\lambda \geq A} (z \cdot \pi^\sigma(\lambda) - \lambda). \tag{4.15}$$

Step 2: Optimization. Because of the superlinearity of the Hamiltonians H_i (see (4.1)), we have for $z \neq 0$,

$$\lim_{\lambda\to+\infty} f^\sigma(\lambda) = -\infty \quad \text{for} \quad f^\sigma(\lambda) := z \cdot \pi^\sigma(\lambda) - \lambda.$$

Therefore the supremum in (4.15) is reached for some $\lambda \in [A, +\infty)$, *i.e.*

$$\mathfrak{G}(z) = z \cdot \pi^\sigma(\lambda) - \lambda.$$

Then we have $\lambda = A$ or $\lambda > A$ and $(f^\sigma)'(\lambda) = 0$. Note that for $\lambda > A_0$, we can rewrite $(f^\sigma)'(\lambda) = 0$ as

$$\sum_{i=1,\ldots,N} \frac{\bar{z}_i}{\bar{z}_i^\sigma} = 1 \quad \text{with} \quad \begin{cases} \bar{z}_i = \sigma_i z_i \geq 0, \\ \\ \bar{z}_i^\sigma = \bar{z}_i^\sigma(\lambda) := \sigma_i H_i'(\pi_i^{\sigma_i}(\lambda)) > 0. \end{cases}$$

Moreover, we have

$$(\bar{z}_i^\sigma)'(\lambda) = \frac{H_i''(\pi_i^{\sigma_i}(\lambda))}{\sigma_i H_i'(\pi_i^{\sigma_i}(\lambda))} > 0$$

where the strict inequality follows from the strict convexity of Hamiltonians, see (4.1). Moreover, by definition of $\bar{z}_i^\sigma$, we have

$$\lim_{\lambda \to +\infty} \bar{z}_i^\sigma(\lambda) = +\infty$$

because H_i is convex and superlinear.

Step 3: Foliation and definition of $\mathfrak{L}$. Let us consider the sets

$$P^\sigma(\lambda) = \begin{cases} \left\{ \bar{z} \in [0, +\infty)^N, \quad \sum_{i=1,\ldots,N} \frac{\bar{z}_i}{\bar{z}_i^\sigma(\lambda)} = 1 \right\} & \text{if} \quad \lambda > A, \\ \\ \left\{ \bar{z} \in [0, +\infty)^N, \sum_{i=1,\ldots,N} \frac{\bar{z}_i}{\bar{z}_i^\sigma(A)} \leq 1 \right\} & \text{if} \quad \lambda = A \end{cases} \tag{4.16}$$

(keeping in mind convention (4.12)). Because for $\lambda > A$, the intersection points of the piece of hyperplane $P^\sigma(\lambda)$ with each axis $\mathbb{R}e_i$ are $\bar{z}_i^\sigma(\lambda)e_i$, we deduce that we can write the partition (see Figure 1)

$$[0, +\infty)^N = \bigcup_{\lambda \geq A} P^\sigma(\lambda)$$

where $P^\sigma(\lambda)$ gives a foliation by hyperplanes for $\lambda > A$. Then we can define for $z \in Q_\sigma$,

$$\mathfrak{L}^\sigma(z) = \{\lambda \quad \text{such that} \quad \bar{z} \in P^\sigma(\lambda) \quad \text{for} \quad \bar{z}_i = \sigma_i z_i \quad \text{for} \quad i = 1, \ldots, N\}.$$

From our definition, we get that the function $\mathfrak{L}^\sigma$ is continuous on Q_σ and satisfies $\mathfrak{L}^\sigma(0) = A$. For $z \in Q_\sigma$ such that $z_{i_0} = 0$, we see from the definition of P^σ given in (4.16) that the value of $\mathfrak{L}^\sigma(z)$ does not depend on the value of σ_{i_0}. Therefore we can glue up all the $\mathfrak{L}^\sigma$ in a single continuous function $\mathfrak{L}$ defined for $z \in \bar{\Omega}$ by

$$\mathfrak{L}(z) = \mathfrak{L}^\sigma(z) \quad \text{if} \quad z \in Q_\sigma.$$

which satisfies $\mathfrak{L}(0) = A$.

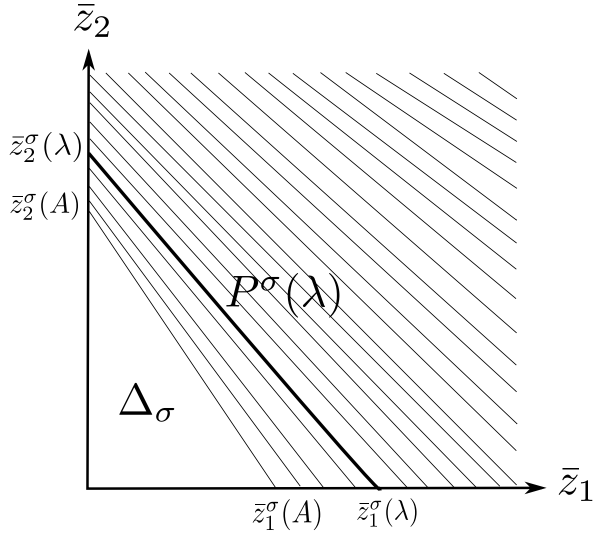

Figure 1: The foliation of $[0,+\infty)^2$ $(N=2)$ with sets $P^\sigma(\lambda)$ for $\lambda \geq A$.

Step 4: Regularity of $\mathfrak{G}$ and computation of the gradients. For $z \in Q_\sigma \subset \bar{\Omega}$, we have

$$\mathfrak{G}(z) = \sup_{\lambda \geq A} (z \cdot \pi^\sigma(\lambda) - \lambda)$$

where the supremum is reached only for $\lambda = \mathfrak{L}(z)$. Moreover $\mathfrak{G}$ is convex in $\mathbb{R}^N$. We just showed that the subdifferential of $\mathfrak{G}$ on the interior of Q_σ is the singleton $\{\pi^\sigma(\lambda)\}$ with $\lambda = \mathfrak{L}(z)$. This implies that $\mathfrak{G}$ is differentiable in the interior of Q_σ and

$$\nabla\mathfrak{G}(z) = \pi^\sigma(\lambda) \quad \text{with} \quad \lambda = \mathfrak{L}(z).$$

The fact that the maps π^σ and $\mathfrak{L}$ are continuous implies that $\mathfrak{G}_{|Q_\sigma}$ is C^1. □

4.4 Proof of Proposition 4.1

We now turn to the proof of Proposition 4.1.

Proof of Proposition 4.1. By definition of G^0, we have

$$G^0(x,y) = \mathfrak{G}(Z(x,y)) \quad \text{with} \quad Z(x,y) := xe_i - ye_j \in \bar{\Omega} \quad \text{if} \quad (x,y) \in J_i \times J_j$$

where $(e_1, ..., e_N)$ is the canonical basis of $\mathbb{R}^N$ and $\bar{\Omega}$ is defined in (4.13).

Step 1: Regularity. Then Lemmas 4.7 and 4.8 imply immediately that $G^0 \in C(J^2)$ and $G^0 \in C^1(R)$ for each region R given by

$$R = \begin{cases} J_i \times J_j & \text{if} \quad i \neq j, \\ T_i^\pm = \{(x,y) \in J_i \times J_i, \quad \pm(x-y) \geq 0\} & \text{if} \quad i = j. \end{cases} \tag{4.17}$$

This regularity of $\mathfrak{G}$ implies in particular the regularity of G^0 given in i).

Step 2: Computation of the gradients. We also deduce from Lemma 4.8 that

$$\Lambda(x,y) := \mathfrak{L}(Z(x,y))$$

defines a continuous map $\Lambda : J^2 \to [A,+\infty)$ which satisfies

$$\Lambda(x,x) = A \tag{4.18}$$

because of Lemma 4.7-iii) and $Z(x,x) = 0$. Moreover, for each R given by (4.17) and for all $(x,y) \in R \subset J_i \times J_j$ we have

$$G^0(x,y) = p_i x - p_j y - \lambda$$

and

$$(G^0_{|R})_x(x,y) = p_i \quad \text{and} \quad (G^0_{|R})_y(x,y) = -p_j$$

with $\lambda = \Lambda(x,y)$ and $(p,\lambda) \in \mathcal{G}_A$ and

$$(p_i,p_j) = \begin{cases} (\pi_i^+(\lambda), \pi_j^-(\lambda)) & \text{if} \quad R = J_i \times J_j \quad & \text{with} \quad i \neq j, \\ (\pi_i^\pm(\lambda), \pi_i^\pm(\lambda)) & \text{if} \quad R = T_i^\pm & \text{with} \quad i = j. \end{cases} \tag{4.19}$$

Step 3: Checking the compatibility condition on the gradients. Let us consider $(x,y) \in J^2$ with $x = y = 0$ or $x \neq y$. We have

$$(\partial_i G^0(\cdot,y))(x) \in \left\{\pi_i^\pm(\lambda)\right\} \quad \text{and} \quad -(\partial_j G^0(x,\cdot))(y) \in \left\{\pi_j^\pm(\lambda)\right\} \quad \text{with} \quad \lambda = \Lambda(x,y) \geq A.$$

We claim that

$$H(x, G^0_x(x,y)) = \lambda. \tag{4.20}$$

If $x \neq 0$, then $H(x, G^0_x(x,y)) = H_i(\pi_i^\pm(\lambda)) = \lambda$. If $x = 0$ and there exists i such that $\sigma_i = -$, then $H_i^-(\partial_i^x G^0(0,y)) = H_i^-(\pi_i^-(\lambda)) = \lambda$ and $H_j^-(\partial_j^x G^0(0,y)) = H_j^-(\pi_j^{\sigma_j}(\lambda)) \leq \lambda$. Hence we also have in this case that (4.20) holds true. We are left with treating the case where

$$x = 0 \quad \text{and} \quad (\partial_i G^0(\cdot,y))(0) = \pi_i^+(\lambda) \quad \text{for all} \quad i = 1, ..., N \tag{4.21}$$

If $0 \neq y \in J_j$, then $(x,y) = (0,y) \in T_j^-$ and $(\partial_j G^0(\cdot,y))(0) = \pi_j^-(\lambda)$. Therefore (4.21) only happens if $y = 0$ and then

$$H(0, G^0_x(0,0)) = A$$

which still implies (4.20), because $\lambda = \Lambda(0,0) = A$.

In view of (4.20), (3.4) with equality and $\gamma = 0$ is equivalent to

$$H(y, -G^0_y(x,y)) = \lambda. \tag{4.22}$$

Arguing like we did to get (4.20), we can treat all cases except the following one

$$y = 0 \quad \text{and} \quad -(\partial_j G^0(x,\cdot))(0) = \pi_j^+(\lambda) \quad \text{for all} \quad j = 1, ..., N. \tag{4.23}$$

If $x \in J_i$ and $N \geq 2$, then we can find $j \neq i$ such that $-(\partial_j G^0(x,\cdot))(0) = \pi_j^-(\lambda)$. Therefore (4.23) only happens if $N = 1$ and then

$$H(0, -G^0_y(x,0)) = A \leq \lambda.$$

Step 4: Superlinearity. In view of the definition of G^0, we deduce from (4.19) that for all $\lambda \geq A$,

$$G^0(x,y) \geq \begin{cases} x\pi_i^+(\lambda) - y\pi_j^-(\lambda) - \lambda & \text{if} \quad i \neq j, \\ (x-y)\pi_i^\pm(\lambda) - \lambda & \text{if} \quad i = j \quad \text{and} \quad \pm(x-y) \geq 0 \end{cases}$$

Setting

$$\pi^0(\lambda) := \min_{\pm,\ i=1,\dots,N} \pm\pi_i^\pm(\lambda) \geq 0,$$

we get

$$G^0(x,y) \geq d(x,y)\pi^0(\lambda) - \lambda.$$

From the definition (4.2) of $\pi_i^\pm$ and the assumption (4.1) on the Hamiltonians, we deduce that

$$\pi^0(\lambda) \to +\infty \quad \text{as} \quad \lambda \to +\infty$$

which implies that for any $K \geq 0$, there exists a constant $C_K \geq 0$ such that

$$G^0(x,y) \geq Kd(x,y) - C_K.$$

Therefore we get (3.5) with

$$g^0(a) = \sup_{K \geq 0}(Ka - C_K).$$

Step 5: Gradient bounds. Note that

$$\sum_{i=1,\dots,N} |Z_i(x,y)| = d(x,y).$$

Because each component of the gradients of G^0 are equal to one of the $\left\{\pi_k^\pm(\lambda)\right\}_{\pm,k=1,\dots,N}$ with $\lambda = \mathfrak{L}(Z(x,y))$, we deduce (3.6) from the continuity of $\mathfrak{L}$ and of the maps $\pi_k^\pm$.

Step 6: Saturation close to the diagonal. Point vi) in Proposition 4.1 follows from Lemma 4.7-iii), from the definition of $\mathfrak{G}$ and from the regularity of G^0. In particular, for $(x,y) \in T_i^\pm$, $Z = (x-y)e_i$ belongs to $P^\sigma(\lambda) \cup \Delta_\sigma$ with $\sigma_i = \pm$. Hence, Lemma 4.7-iii) implies that $G^0(x,y) = \pi_i^\pm(A)(x-y) - A$ for $\pm(x-y) \in [0, \pm z_i^\pm]$ with $z_i^\pm = H_i'(\pi_i^\pm(A))$. We recall that $\bar{z}_i^\sigma = \pm z_i^\pm \geq 0$ appears in the definition of $P^\sigma(\lambda)$ and Δ_σ. □

4.5 A second vertex test function

In this subsection, we propose a construction of a second vertex test function $G^\sharp$ (see Theorem 4.12 below), that can be seen as a kind of approximation of the original vertex test function G. This test function is somehow less natural than our previous test function, but it has the advantage that it is easier to check its properties. Moreover, it can be useful in applications.

We introduce the following

Definition 4.9 (Piecewise C^1 Regularity)**.** We say that a function u belongs to $C^{1,\sharp}(J)$, if $u \in C(J)$, and if for any branch J_i for $i = 1, \dots, N$, there exists a sequence of points $(a^i_k)_{k\in\mathbb{N}}$ on the branch J_i satisfying

$$0 = a^i_0 < a^i_1 < \cdots < a^i_k < a^i_{k+1} \to +\infty \quad \text{as} \quad k \to +\infty$$

such that

$$u_{|[a^i_k, a^i_{k+1}]} \in C^1\left([a^i_k, a^i_{k+1}]\right) \quad \text{for all} \quad k \in \mathbb{N}, \quad i = 1, \dots, N.$$

The smooth convex case

Following what we did in order to construct the first vertex test function, we first construct $G^\sharp$ in the smooth convex case and we then derive the general case by approximation. In the smooth convex case, we first consider

$$G^{0,\sharp}(x,y) = \sup_{k\in\mathbb{N}} \left(\sup_{(p,\lambda_k)\in\mathcal{G}_A} (p_i x - p_j y - \lambda_k) \right) \quad \text{if} \quad (x,y) \in J_i \times J_j \tag{4.24}$$

for an increasing sequence $(\lambda_k)_{k\in\mathbb{N}}$ satisfying for some constant $\gamma_0 > 0$

$$\begin{cases} \lambda_0 = A \quad \text{and} \quad \lambda_k \to +\infty \quad \text{as} \quad k \to +\infty \\ \lambda_{k+1} - \lambda_k \le \gamma_0 \quad \text{for all} \quad k \ge 0. \end{cases} \tag{4.25}$$

Lemma 4.10 (Piecewise linearity)**.** *The function $G^{0,\sharp}$ is piecewise linear. More precisely,*

- *For $(x,y) \in J_i \times J_i$,*

  $$G^{0,\sharp}(x,y) = \ell_i(x-y)$$

  with $\ell_i \in C(\mathbb{R})$ and

  $$\ell_i(a) = \begin{cases} a\pi_i^+(\lambda_k) - \lambda_k & \text{if} \quad z_i^{k,+} \le a \le z_i^{k+1,+} \\ a\pi_i^-(\lambda_k) - \lambda_k & \text{if} \quad z_i^{k+1,-} \le a \le z_i^{k,-} \end{cases} \quad \Bigg| \quad \text{for all} \quad k \ge 0$$

  and

  $$z_i^{0,\pm} = 0 \quad \text{and} \quad z_i^{k+1,\pm} = \frac{\lambda_{k+1} - \lambda_k}{\pi_i^\pm(\lambda_{k+1}) - \pi_i^\pm(\lambda_k)} \quad \text{for all} \quad k \ge 0 \tag{4.26}$$

  (recall that $\pi_i^\pm$ is defined in (4.2)). We have in particular for all $k \ge 1$

  $$z_i^{k+1,-} < z_i^{k,-} < z_i^{0,-} = 0 = z_i^{0,+} < z_i^{k,+} < z_i^{k+1,+}. \tag{4.27}$$

- *For $(x,y) \in J_i \times J_j$ with $i \ne j$,*

  $$G^{0,\sharp}(x,y) = x\pi_i^+(\lambda_k) - y\pi_i^-(\lambda_k) - \lambda_k$$

  for $(x,y) \in \Delta^k_{ij}$ with

  $$\Delta^k_{ij} = \left\{ (x,y) \in J_i \times J_j, \quad \frac{x}{z_i^{k,+}} - \frac{y}{z_j^{k,-}} \ge 1, \quad \frac{x}{z_i^{k+1,+}} - \frac{y}{z_j^{k+1,-}} \le 1 \right\}. \tag{4.28}$$

Proof. Remark that $\lambda_k = H_i(\pi_i^{\pm}(\lambda_k))$. Therefore the definition of $z_i^{k,\pm}$ and the convexity of H_i imply inequalities (4.27). It is then easy to check the explicit expressions of $G^{0,\sharp}$. $\square$

We recall that if $u \in C^{1,\sharp}(J)$ and u is not C^1 at a point $x \in J_i^*$, then Proposition 2.16 can be used in order to understand H as follows

$$H(x, u_x) = \max\left(H_i^+(\partial_i u(x^-)), H_i^-(\partial_i u(x^+))\right). \tag{4.29}$$

This interpretation will be used to check inequality (3.4) at points where $G^{0,\sharp}(x,y)$ is not C^1 with $(x,y) \in J_i \times J_j$ with $i \neq j$.

Proposition 4.11 (The second vertex test function – the smooth convex case)**.** *Let $A \geq A_0$ with A_0 given by* (1.8) *and assume that the Hamiltonians satisfy* (4.1)*. Let $(\lambda_k)_{k\in\mathbb{N}}$ be any increasing sequence satisfying (4.25) for some given $\gamma_0 > 0$. Then the function $G^{0,\sharp} : J^2 \to \mathbb{R}$ defined in (4.24) satisfies properties ii) and iv) listed in Proposition 4.1, together with the following properties*

i') (Regularity)

$$G^{0,\sharp} \in C(J^2) \quad \textit{and} \quad \begin{cases} G^{0,\sharp}(x,\cdot) \in C^{1,\sharp}(J) & \textit{for all} \quad x \in J, \\ G^{0,\sharp}(\cdot,y) \in C^{1,\sharp}(J) & \textit{for all} \quad y \in J. \end{cases}$$

iii') (Compatibility conditions) *On the one hand,* (3.3) *holds with $\gamma = 0$ for all $x \in J$. On the other hand,* (3.4) *holds with $\gamma = \gamma_0$, for all $(x,y) \in J^2$, except possibly for all points on the diagonals $x = y \in J_i^*$ for $i \in \{1, \dots, N\}$.*

Moreover, at points $(x,y) \in J_i \times J_j$ with $i \neq j$, where the functions $G^{0,\sharp}(x,\cdot)$ or $G^{0,\sharp}(\cdot,y)$ are not C^1, inequality (3.4) *has to be understood using convention (4.29);*

v') (Gradient bounds) *Estimate* (3.6) *holds for all $(x,y) \in J^2$ if we understand it as a bound for both left and right derivatives, at points where the functions $G^{0,\sharp}(x,\cdot)$ and $G^{0,\sharp}(\cdot,y)$ are not C^1.*

Proof. The regularity i') follows immediately for the previous lemma. Moreover points ii) and iv) listed in Proposition 4.1 follow easily, and similarly for the gradient bounds v'). Also (3.3) holds clearly for $\gamma = 0$.

The only important point is to check inequality (3.4) in iii') with $\gamma = \gamma_0$.

Step 1: checking on $J_i^* \times J_i^*$ Inequality (3.4) is clearly true for $(x,y) \in J_i^* \times J_i^*$, if $x - y \neq z_i^{k,\pm}$. Let us check it if $x - y = z_i^{k+1,\pm} \neq 0$. We distinguish two cases.

Case 1: $(x,y) \in J_i^* \times J_i^*$ with $x - y = z_i^{k+1,+} > 0$. The only novelty here is that the function $G^{0,\sharp}$ is not C^1 at those points, and we have to use interpretation (4.29) to compute it. We get

$$\begin{aligned} H(x, G_x^{0,\sharp}(x,y)) &= \max(H_i^+(G_x^{0,\sharp}(x^-,y)), H_i^-(G_x^{0,\sharp}(x^+,y))) \\ &= \max(H_i^+(\pi_i^+(\lambda_k)), H_i^-(\pi_i^+(\lambda_{k+1}))) \\ &= \lambda_k \end{aligned} \tag{4.30}$$

and

$$H(y,-G_y^{0,\sharp}(x,y)) \begin{aligned}[t] &= \max(H_i^+(-G_y^{0,\sharp}(x,y^-)), H_i^-(-G_y^{0,\sharp}(x,y^+))) \\ &= \max(H_i^+(\pi_i^+(\lambda_{k+1})), H_i^-(\pi_i^+(\lambda_k))) \\ &= \lambda_{k+1}. \end{aligned} \tag{4.31}$$

This implies inequality (3.4) for $\gamma = \gamma_0 \geq \lambda_{k+1} - \lambda_k$.

Case 2: $(x,y) \in J_i^* \times J_i^*$ with $x - y = z_i^{k+1,-} < 0$. We compute

$$H(x,G_x^{0,\sharp}(x,y)) \begin{aligned}[t] &= \max(H_i^+(G_x^{0,\sharp}(x^-,y)), H_i^-(G_x^{0,\sharp}(x^+,y))) \\ &= \max(H_i^+(\pi_i^-(\lambda_{k+1})), H_i^-(\pi_i^-(\lambda_k))) \\ &= \lambda_k \end{aligned} \tag{4.32}$$

and

$$H(y,-G_y^{0,\sharp}(x,y)) \begin{aligned}[t] &= \max(H_i^+(-G_y^{0,\sharp}(x,y^-)), H_i^-(-G_y^{0,\sharp}(x,y^+))) \\ &= \max(H_i^+(\pi_i^-(\lambda_k)), H_i^-(\pi_i^-(\lambda_{k+1}))) \\ &= \lambda_{k+1} \end{aligned} \tag{4.33}$$

which gives the result.

Step 2: checking on Δ_{ij}^k for $i \neq j$. This inequality is also obviously true if $(x,y) \in \operatorname{Int} \Delta_{ij}^k$ for $i \neq j$. We then distinguish six cases.

Case 1: $x = y = 0$. This case is similar to the study of G^0 and we get immediately

$$H(0,-G_y^{0,\sharp}(0,0)) = -A = H(0,G_x^{0,\sharp}(0,0)).$$

Case 2: $(x,y) \in \Delta_{ij}^k$ with $y = 0$ and $z_i^{k,+} < x < z_i^{k+1,+}$.

$$H(0,-G_y^{0,\sharp}(x,0)) = \lambda_k = H(x,G_x^{0,\sharp}(x,0)).$$

Case 3: $(x,y) \in \Delta_{ij}^k$ with $x = 0$ and $-z_j^{k,-} < y < -z_j^{k+1,-}$.

$$H(y,-G_y^{0,\sharp}(0,y)) = \lambda_k = H(0,G_x^{0,\sharp}(0,y)).$$

Case 4: $(x,y) \in (\partial\Delta_{ij}^k) \setminus ((J_i \times \{0\}) \cup (\{0\} \times J_j))$. Let us consider the subcase where $\dfrac{x}{z_i^{k+1,+}} - \dfrac{y}{z_j^{k+1,-}} = 1$ (the other case with $k+1$ replaced by k being of course similar). We compute again:

$$H(x,G_x^{0,\sharp}(x,y)) \begin{aligned}[t] &= \max(H_i^+(G_x^{0,\sharp}(x^-,y)), H_i^-(G_x^{0,\sharp}(x^+,y))) \\ &= \max(H_i^+(\pi_i^+(\lambda_k)), H_i^-(\pi_i^+(\lambda_{k+1}))) \\ &= \lambda_k \end{aligned}$$

and

$$H(y,-G_y^{0,\sharp}(x,y)) \begin{aligned}[t] &= \max(H_j^+(-G_y^{0,\sharp}(x,y^-)), H_j^-(-G_y^{0,\sharp}(x,y^+))) \\ &= \max(H_j^+(\pi_j^-(\lambda_k)), H_j^-(\pi_j^-(\lambda_{k+1}))) \\ &= \lambda_{k+1}. \end{aligned}$$

This implies again inequality (3.4) for $\gamma = \gamma_0 \geq \lambda_{k+1} - \lambda_k$.

Case 5: $(x,y) \in \Delta_{ij}^k$ with $y = 0$ and $x = z_i^{k+1,+}$. Again, we check easily that $H(0, -G_y^{0,\sharp}(x,0)) = \lambda_{k+1}$, and $H(x, G_x^{0,\sharp}(x,0)) = \lambda_k$, as in Case 4.

Case 6: $(x,y) \in \Delta_{ij}^k$ with $x = 0$ and $y = -z_j^{k+1,-}$. We have $H(y, -G_y^{0,\sharp}(0,y)) = \lambda_{k+1}$ as in Case 4, and $H(0, G_x^{0,\sharp}(0,y)) = \lambda_k$. □

The general case

Then we have the following

Theorem 4.12 (The second vertex test function)**.** *Let $A \in \mathbb{R} \cup \{-\infty\}$ and $\gamma > 0$. Assume that the Hamiltonians satisfy* (1.5) *and* (3.2)*. Then there exists a function $G^\sharp : J^2 \to \mathbb{R}$ enjoying properties ii) to vi) listed in Theorem 3.2, and property i') given in Proposition 4.11.*
In particular, at points (different from the origin) where functions $G^\sharp(x,\cdot)$ and $G^\sharp(\cdot,y)$ are not C^1, we get bounds (3.6) on both left and right derivatives. Moreover, at those points, inequality (3.4) has to be interpreted in the sense of Proposition 2.16. Moreover, there exists some $\varepsilon > 0$ such that we have

$$G^\sharp = G^{0,\sharp} \quad \text{on} \quad J^2 \backslash \delta_\varepsilon \quad \text{with} \quad \delta_\varepsilon = \left\{ (x,y) \in \bigcup_{i=1,\dots,N} J_i^* \times J_i^*, \quad |x-y| \leq \varepsilon \right\} \tag{4.34}$$

where $G^{0,\sharp}$ is given in Proposition 4.11, with $\gamma = \gamma_0$.

Proof of Theorem 4.12. In the smooth convex case, we define $G^\sharp$ as in (4.34). On $J_i^* \times J_i^*$, we simply define $G^\sharp$ as a regularization of $G^{0,\sharp}$ along each line $x = y \in J_i^*$, following the procedure described in the proof of Lemma 4.3 for $\varepsilon \leq \gamma = \gamma_0$. The general case follows by approximation. □

Remark 4.13. With the help of Proposition 2.16, it is straighforward to check that the proof of the comparison principle works as well with this second vertex test function $G^\sharp$ given by Theorem 4.12.

5 Extension to networks

5.1 Definition of a network

A general abstract network $\mathcal{N}$ is characterized by the set $\mathcal{E}$ of its *edges* and the set $\mathcal{V}$ of its vertices (or nodes). It is endowed with a distance.

Edges. $\mathcal{E}$ is a finite or countable set of edges. Each edge $e \in \mathcal{E}$ is assumed to be either isometric to the half line $[0, +\infty)$ with $\partial e = \{e^0\}$ (where the endpoint e^0 can be identified to $\{0\}$), or to a compact interval $[0, l_e]$ with

$$\inf_{e \in \mathcal{E}} l_e > 0 \tag{5.1}$$

and $\partial e = \{e^0, e^1\}$. Condition (5.1) implies in particular that the network is complete. The endpoints $\{e^0\}, \{e^1\}$ can respectively be identified to $\{0\}$ and $\{l_e\}$. The *interior* e^* of an edge e refers to $e \setminus (\partial e)$.

Vertices. It is convenient to see vertices of the network as a partition of the sets of all edge endpoints,

$$\bigcup_{e \in \mathcal{E}} \partial e = \bigcup_{n \in \mathcal{V}} n;$$

we assume that each set n only contains a finite number of endpoints.

Here each $n \in \mathcal{V}$ can be identified as a vertex (or node) of the network as follows. For every $x, y \in \bigcup_{e \in \mathcal{E}} e$, we define the equivalence relation:

$$x \sim y \quad \Longleftrightarrow (x = y \quad \text{or} \quad x, y \in n \in \mathcal{V})$$

and we define the network as the quotient

$$\mathcal{N} = \left(\bigcup_{e \in \mathcal{E}} e \right) / \sim \quad = \quad \left(\bigcup_{e \in \mathcal{E}} e^* \right) \cup \mathcal{V}. \tag{5.2}$$

We also define for $n \in \mathcal{V}$

$$\mathcal{E}_n = \{e \in \mathcal{E}, \quad n \in \partial e\}$$

and its partition $\mathcal{E}_n = \mathcal{E}_n^- \cup \mathcal{E}_n^+$ with

$$\mathcal{E}_n^- = \left\{ e \in \mathcal{E}_n, n = e^0 \right\}, \quad \mathcal{E}_n^+ = \left\{ e \in \mathcal{E}_n, n = e^1 \right\}.$$

Distance. We also define the distance function $d(x, y) = d(y, x)$ as the minimal length of a continuous path connecting x and y on the network, using the metric of each edge (either isometric to $[0, +\infty)$ of to a compact interval). Note that, because of our assumptions, if $d(x, y) < +\infty$, then there is only a finite number of minimal paths.

Remark 5.1. For any $\varepsilon > 0$, there is a bound (depending on ε) on the number of minimal paths connecting x to y for all $y \in B(\bar{y}, \varepsilon) = \{y \in \mathcal{N}, \quad d(\bar{y}, y) < \varepsilon\}$.

5.2 Hamilton-Jacobi equations on a network

Given a Hamiltonian H_e on each edge $e \in \mathcal{E}$, we consider the following HJ equation on the network $\mathcal{N}$,

$$(5.3)\qquad \begin{cases} u_t + H_e(t, x, u_x) = 0 & \text{for} \quad t \in (0, +\infty) \quad \text{and} \quad x \in e^*, \\ u_t + F_A(t, x, u_x) = 0 & \text{for} \quad t \in (0, +\infty) \quad \text{and} \quad x = n \in \mathcal{V} \end{cases}$$

supplemented with an initial condition

$$(5.4)\qquad u(0, x) = u_0(x) \quad \text{for} \quad x \in \mathcal{N}.$$

The limited flux functions F_A associated with the Hamiltonians H_e are defined below. We first make precise the meaning of u_x in (5.3).

Gradients of real functions. For a real function u defined on the network $\mathcal{N}$, we denote by $\partial_e u(x)$ the (spatial) derivative of u at $x \in e$ and define the "gradient" of u by

$$u_x(x) := \begin{cases} \partial_e u(x) & \text{if} \quad x \in e^* = e \setminus (\partial e), \\ ((\partial_e u(x))_{e \in \mathcal{E}_n^-}, (\partial_e u(x))_{e \in \mathcal{E}_n^+}) & \text{if} \quad x = n \in \mathcal{V} \end{cases}.$$

The norm $|u_x|$ simply denotes $|\partial_e u|$ for $x \in e^*$ or $\max\{|\partial_e u| : e \in \mathcal{E}_n\}$ at the vertex $x = n$.

Limited flux functions. We also define for $(t, x) \in \mathbb{R} \times \partial e$,

$$H_e^-(t, x, q) = \begin{cases} H_e(t, x, q) & \text{if} \quad q \le p_e^0(t, x), \\ H_e(t, x, p_e^0(t, x)) & \text{if} \quad q > p_e^0(t, x) \end{cases}$$

and

$$H_e^+(t, x, q) = \begin{cases} H_e(t, x, p_e^0(t, x)) & \text{if} \quad q \le p_e^0(t, x), \\ H_e(t, x, q) & \text{if} \quad q > p_e^0(t, x). \end{cases}$$

Given limiting functions $(A_n)_{n \in \mathcal{V}}$, we define for $p = (p_e)_{e \in \mathcal{E}_n}$,

$$F_A(t, n, p) = \max\left(A_n(t), \quad \max_{e \in \mathcal{E}_n^-} H_e^-(t, n, p_e), \quad \max_{e \in \mathcal{E}_n^+} H_e^+(t, n, p_e) \right).$$

In particular, for each $n \in \mathcal{V}$, the functions $F_A(t, n, \cdot)$ are the same for all $A_n(t) \in [-\infty, A_n^0(t)]$ with

$$(5.5)\qquad A_n^0(t) := \max\left(\max_{e \in \mathcal{E}_n^-} H_e^-(t, n, p_e^0(t, n)), \quad \max_{e \in \mathcal{E}_n^+} H_e^+(t, n, p_e^0(t, n)) \right).$$

A shorthand notation. As in the junction case, we introduce

$$(5.6)\qquad H_{\mathcal{N}}(t, x, p) = \begin{cases} H_e(t, x, p) & \text{for} \quad p \in \mathbb{R}, & t \in \mathbb{R}, & \text{if} \quad x \in e^*, \\ F_A(t, x, p) & \text{for} \quad p = (p_e)_{e \in \mathcal{E}_n} \in \mathbb{R}^{\text{Card } \mathcal{E}_n}, & t \in \mathbb{R}, & \text{if} \quad x = n \in \mathcal{V} \end{cases}$$

in order to rewrite (5.3) as

$$(5.7)\qquad u_t + H_{\mathcal{N}}(t, x, u_x) = 0 \quad \text{for all} \quad (t, x) \in (0, +\infty) \times \mathcal{N}.$$

5.3 Assumptions on the Hamiltonians

For each $e \in \mathcal{E}$, we consider a Hamiltonian $H_e : [0,+\infty) \times e \times \mathbb{R} \to \mathbb{R}$ satisfying

- **(H0)** (Continuity) $H_e \in C([0,+\infty) \times e \times \mathbb{R})$.
- **(H1)** (Uniform coercivity) For all $T > 0$,

$$\lim_{|q|\to+\infty} H_e(t,x,q) = +\infty$$

  uniformly with respect to $t \in [0,T]$ and $x \in e \in \mathcal{E}$.

- **(H2)** (Uniform bound on the Hamiltonians for bounded gradients) For all $T, L > 0$, there exists $C_{T,L} > 0$ such that

$$\sup_{t\in[0,T],\ p\in[-L,L], x\in\mathcal{N}\setminus\mathcal{V}} |H_{\mathcal{N}}(t,x,p)| \leq C_{T,L}.$$

- **(H3)** (Uniform modulus of continuity for bounded gradients) For all $T, L > 0$, there exists a modulus of continuity $\omega_{T,L}$ such that for all $|p|, |q| \leq L$, $t \in [0,T]$ and $x \in e \in \mathcal{E}$,

$$|H_e(t,x,p) - H_e(t,x,q)| \leq \omega_{T,L}(|p-q|).$$

- **(H4)** (Quasi-convexity) For all $n \in \mathcal{V}$, there exists a (possibly discontinuous) function $t \mapsto p_e^0(t,n)$ such that

$$\begin{cases} H_e(t,n,\cdot) & \text{is nonincreasing on} \quad (-\infty, p_e^0(t,n)], \\ H_e(t,n,\cdot) & \text{is nondecreasing on} \quad [p_e^0(t,n), +\infty). \end{cases}$$

- **(H5)** (Uniform modulus of continuity in time) For all $T > 0$, there exists a modulus of continuity $\bar{\omega}_T$ such that for all $t, s \in [0,T]$, $p \in \mathbb{R}$, $x \in e \in \mathcal{E}$,

$$H_e(t,x,p) - H_e(s,x,p) \leq \bar{\omega}_T\left(|t-s|(1 + \max(H_e(s,x,p), 0))\right).$$

- **(H6)** (Uniform continuity of A^0) For all $T > 0$, there exists a modulus of continuity $\bar{\omega}_T$ such that for all $t, s \in [0,T]$ and $n \in \mathcal{V}$,

$$|A_n^0(t) - A_n^0(s)| \leq \bar{\omega}_T(|t-s|).$$

As far as flux limiters are concerned, the following assumptions will be used.

- **(A0)** (Continuity of A) For all $T > 0$ and $n \in \mathcal{V}$, $A_n \in C([0,T])$.
- **(A1)** (Uniform bound on A) For all $T > 0$, there exists a constant $C_T > 0$ such that for all $t \in [0,T]$ and $n \in \mathcal{V}$

$$|A_n(t)| \leq C_T.$$

- **(A2)** (Uniform continuity of A) For all $T > 0$, there exists a modulus of continuity $\bar{\omega}_T$ such that for all $t, s \in [0, T]$ and $n \in \mathcal{V}$,

$$|A_n(t) - A_n(s)| \leq \bar{\omega}_T(|t - s|).$$

The proof of the following technical lemma is postponed until appendix.

Lemma 5.2 (Estimate on the difference of Hamiltonians)**.** *Assume that the Hamiltonians satisfy* (H0)-(H4) *and* (A0)-(A1)*. Then for all $T > 0$, there exists a constant $C_T > 0$ such that*

$$|p_e^0(t,x)| \leq C_T \quad \text{for all} \quad t \in [0,T], \quad x \in \partial e, \quad e \in \mathcal{E}, \tag{5.8}$$

$$|A_n^0(t)| \leq C_T \quad \text{for all} \quad t \in [0,T], \quad n \in \mathcal{V}. \tag{5.9}$$

If we assume moreover (H5)-(H6) *and* (A2)*, then there exists a modulus of continuity $\tilde{\omega}_T$ such that for all $t, s \in [0, T]$, and x, p*

$$H_{\mathcal{N}}(t,x,p) - H_{\mathcal{N}}(s,x,p) \leq \tilde{\omega}_T(|t-s|(1 + \max(0, H_{\mathcal{N}}(s,x,p)))). \tag{5.10}$$

Remark 5.3. From the proof, the reader can check that Assumptions (H5)-(H6) and (A2) in the statement of Theorem 5.8 can in fact be replaced with (5.10).

Remark 5.4 (Example of Hamiltonians with uniform modulus of time continuity)*.* Condition on the uniform modulus of continuity in time in (H5) is for instance satisfied by Hamiltonians of the type for $q > 0$ and $\delta > 0$ such that for all $x \in e \in \mathcal{E}$ we have

$$H_e(t,x,p) = c_e(t,x)|p|^q \quad \text{with} \quad 0 < \delta \leq c_e(t,x) \leq 1/\delta$$

with c_e Lipschitz continuous in time and continuous in space.

5.4 Viscosity solutions on a network

Class of test functions. For $T > 0$, set $\mathcal{N}_T = (0,T) \times \mathcal{N}$. We define the class of test functions on $(0,T) \times \mathcal{N}$ by

$$C^1(\mathcal{N}_T) = \left\{\varphi \in C(\mathcal{N}_T), \text{ the restriction of } \varphi \text{ to } (0,T) \times e \text{ is } C^1, \text{ for all } e \in \mathcal{E}\right\}.$$

Definition 5.5 (Viscosity solutions)**.** Assume the Hamiltonians satisfy *(H0)-(H4)* and *(A0)-(A1)* and let $u : [0,T) \times \mathcal{N} \to \mathbb{R}$.

i) We say that u is a *sub-solution* (resp. *super-solution*) of (1.7) in $(0,T) \times \mathcal{N}$ if for all test function $\varphi \in C^1(\mathcal{N}_T)$ such that

$$u^* \leq \varphi \quad \text{(resp.} \quad u_* \geq \varphi) \quad \text{in a neighborhood of } (t_0, x_0) \in \mathcal{N}_T$$

with equality at (t_0, x_0), we have

$$\varphi_t + H_{\mathcal{N}}(t,x,\varphi_x) \leq 0 \quad \text{(resp.} \quad \geq 0) \quad \text{at } (t_0, x_0).$$

ii) We say that u is a *sub-solution* (resp. *super-solution*) of (1.7), (1.4) in $[0,T)\times\mathcal{N}$ if additionally

$$u^*(0,x) \le u_0(x) \quad \text{(resp.} \quad u_*(0,x) \ge u_0(x)) \quad \text{for all} \quad x \in \mathcal{N}.$$

iii) We say that u is a *(viscosity) solution* if u is both a sub-solution and a super-solution.

Remark 5.6 (Touching sub-solutions with semi-concave functions). When proving the comparison principle in the network setting, sub-solutions (resp. super-solutions) will be touched from above (resp. from below) by functions that will not be C^1, but only semi-concave (resp. semi-convex). We recall that a function is semi-concave if it is the sum of a concave function and a smooth (C^2 say) function. But it is a classical observation that, at a point where a semi-concave function is not C^1, we can replace the semi-concave function by a C^1 test function touching it from above.

As in the case of a junction (see Proposition 2.4), viscosity solutions are stable through supremum/infimum. We also have the following existence result.

Theorem 5.7 (Existence on a network)**.** *Assume* (H0)-(H4) *and* (A0)-(A1) *on the Hamiltonians and assume that the initial data u_0 is uniformly continuous on $\mathcal{N}$. Let $T>0$. Then there exists a viscosity solution u of* (5.7),(5.4) *on $[0,T)\times\mathcal{N}$ and a constant $C_T>0$ such that*

$$|u(t,x)-u_0(x)| \le C_T \quad \textit{for all} \quad (t,x)\in[0,T)\times\mathcal{N}.$$

Proof. The proof follows along the lines of the ones of Theorem 1.4. The main difference lies in the construction of barriers. We proceed similarly and get a regularized initial data u_0^ε satisfying

$$|u_0^\varepsilon - u_0| \le \varepsilon \quad \text{and} \quad |(u_0^\varepsilon)_x| \le L_\varepsilon.$$

Then the functions

$$u_\varepsilon^\pm(t,x) = u_0^\varepsilon(x) \pm C_\varepsilon t \pm \varepsilon \tag{5.11}$$

are global super and sub-solutions with respect to the initial data u_0 if C_ε is chosen as follows,

$$C_\varepsilon = \max\left(\sup_{t\in[0,T]}\sup_{n\in\mathcal{V}}|\max(A_n(t),A_n^0(t))|, \sup_{t\in[0,T]}\sup_{e\in\mathcal{E}}\sup_{x\in e,\ |p_e|\le L_\varepsilon}|H_e(t,x,p_e)|\right); \tag{5.12}$$

indeed, we use (5.9) in Lemma 5.2 to bound the first terms in (5.12). □

5.5 Comparison principle on a network

Theorem 5.8 (Comparison principle on a network)**.** *Assume the Hamiltonians satisfy* (H0)-(H6) *and* (A0)-(A2) *and assume that the initial data u_0 is uniformly continuous* *on $\mathcal{N}$. Let $T > 0$. Then for all sub-solution u and super-solution w of* (5.7), (5.4) *in* $[0,T) \times \mathcal{N}$, *satisfying for some $C_T > 0$ and some $x_0 \in \mathcal{N}$*

(5.13)

$$u(t,x) \le C_T(1 + d(x_0,x)), \quad w(t,x) \ge -C_T(1 + d(x_0,x)), \quad \textit{for all} \quad (t,x) \in [0,T) \times \mathcal{N},$$

we have

$$u \le w \quad on \quad [0,T) \times \mathcal{N}.$$

As a straighforward corollary of Theorems 5.8 and 5.7, we get

Corollary 5.9 (Existence and uniqueness)**.** *Under the assumptions of Theorem 5.8, there exits a unique viscosity solution u of* (5.7), (5.4) *in* $[0,T) \times \mathcal{N}$ *such that there exists a constant $C > 0$ with*

$$|u(t,x) - u_0(x)| \le C \quad \textit{for all} \quad (t,x) \in [0,T) \times \mathcal{N}.$$

In order to prove Theorem 5.8, we first need two technical lemmas that are proved in appendix.

Lemma 5.10 (A priori control – the network case)**.** *Let $T > 0$ and let u be a sub-solution and w be a super-solution as in Theorem 5.8. Then there exists a constant $C = C(T) > 0$ such that for all* $(t,x),(s,y) \in [0,T) \times \mathcal{N}$, *we have*

$$u(t,x) \le w(s,y) + C(1 + d(x,y)). \tag{5.14}$$

Lemma 5.11 (Uniform control by the initial data)**.** *Under the assumptions of Theorem 5.8, for any $T > 0$ and $C_T > 0$, there exists a modulus of continuity $f : [0,T) \to [0,+\infty]$ satisfying $f(0^+) = 0$ such that for all sub-solution u (resp. super-solution w) of* (5.7), (5.4) *on* $[0,T) \times \mathcal{N}$, *satisfying* (5.13) *for some $x_0 \in \mathcal{N}$, we have for all* $(t,x) \in [0,T) \times \mathcal{N}$,

$$u(t,x) \le u_0(x) + f(t) \quad (\textit{resp.} \quad w(t,x) \ge u_0(x) - f(t))\,. \tag{5.15}$$

We can now turn to the proof of Theorem 5.8. The proof is similar the comparison principle on a junction (Theorem 1.4). Still, a space localization procedure has to be performed in order to "reduce" to the junction case. From a technical point of view, a noticeable difference is that we will fix the time penalization (for some parameter ν small enough), and then will first take the limit $\varepsilon \to 0$ (ε being the parameter for the space penalization), and then take the limit $\alpha \to 0$ (α being the penalizaton parameter to keep the optimization points at a finite distance).

Proof of Theorem 5.8. Let $\eta > 0$ and $\theta > 0$ and consider

$$M(\theta) = \sup \left\{ u(t,x) - w(s,x) - \frac{\eta}{T-t}, \quad x \in \mathcal{N}, \quad t,s \in [0,T), \quad |t-s| \le \theta \right\}.$$

We want to prove that

$$M = \lim_{\theta \to 0} M(\theta) \le 0.$$

Assume by contradiction that $M > 0$. From Lemma 5.10 we know that M is finite.

Step 1: The localization procedure. Let ψ denote $\frac{d^2(x_0,\cdot)}{2}$.

Lemma 5.12 (Localization)**.** *The supremum*

$$M_\alpha = \sup_{\substack{t,s\in[0,T],t<T\\ x\in\mathcal{N}}} \left\{ u(t,x) - w(s,x) - \alpha\psi(x) - \frac{\eta}{T-t} - \frac{(t-s)^2}{2\nu} \right\}$$

is reached for some point $(t_\alpha, s_\alpha, x_\alpha)$*. Moreover, for* α *and* ν *small enough, we have the following localization estimates*

$$M_\alpha \geq 3M/4 > 0 \tag{5.16}$$

$$d(x_0, x_\alpha) \leq \frac{C}{\sqrt{\alpha}} \tag{5.17}$$

$$0 < \tau_\nu \leq t_\alpha, s_\alpha \leq T - \frac{\eta}{2C} \tag{5.18}$$

$$\lim_{\nu\to 0}\left(\limsup_{\alpha\to 0} \frac{(t_\alpha - s_\alpha)^2}{2\nu}\right) = 0 \tag{5.19}$$

where C *is a constant which does not depend on* α*,* ε*,* ν *and* η*.*

Proof of Lemma 5.12. Choosing α small enough, we have (5.16) for all $\nu > 0$. Because the network is complete for its metric, the supremum in the definition of M_α is reached at some point $(t_\alpha, s_\alpha, x_\alpha)$. From Lemma 5.10, we deduce that

$$0 < \frac{3M}{4} \leq M_\alpha \leq C - \alpha\psi(x_\alpha) - \frac{\eta}{T-t_\alpha} - \frac{(t_\alpha - s_\alpha)^2}{2\nu}$$

and then

$$\alpha\psi(x_\alpha) + \frac{\eta}{T-t_\alpha} + \frac{(t_\alpha - s_\alpha)^2}{2\nu} \leq C. \tag{5.20}$$

This implies (5.17) changing C if necessary.

On the one hand, we get from (5.20) the second inequality in (5.18) by choosing ν such that $\sqrt{2\nu C} \leq \eta/2C$. On the other hand, we get from Lemma 5.11

$$0 < M_\alpha \leq f(t_\alpha) + f(s_\alpha) - \frac{\eta}{T}.$$

In particular,

$$\frac{\eta}{T} \leq 2f(\tau + \sqrt{2\nu C})$$

where $\tau = \min(t_\alpha, s_\alpha)$. If both τ and ν are too small, we get a contradiction. Hence the first inequality in (5.18) holds for some constant τ_ν depending on ν but not on α, ε and η.

We now turn to the proof of (5.19). We know that for any $\delta > 0$, there exists $\theta(\delta) > 0$ (with $\theta(\delta) \to 0$ as $\delta \to 0$) and $(t^\delta, s^\delta, x^\delta) \in [0,T) \times [0,T) \times \mathcal{N}$ such that

$$u(t^\delta, x^\delta) - w(s^\delta, x^\delta) - \frac{\eta}{T-t^\delta} \geq M - \delta \quad \text{and} \quad |t^\delta - s^\delta| \leq \theta(\delta).$$

Then from (5.20) we deduce that

$$M(\sqrt{2\nu C}) - \frac{(t_\alpha - s_\alpha)^2}{2\nu} \geq M_\alpha \geq M - \delta - \alpha\psi(x^\delta) - \frac{|\theta(\delta)|^2}{2\nu}$$

and then

$$\limsup_{\alpha\to 0} \frac{(t_\alpha - s_\alpha)^2}{2\nu} \leq M(\sqrt{2\nu C}) - M + \delta + \frac{|\theta(\delta)|^2}{2\nu}.$$

Taking the limit $\delta \to 0$, we get

$$\limsup_{\alpha\to 0} \frac{(t_\alpha - s_\alpha)^2}{2\nu} \leq M(\sqrt{2\nu C}) - M$$

which yields the desired result. □

Step 2: Reduction when x_α is a vertex. We adapt here Lemma 3.1.

Lemma 5.13 (Reduction)**.** *Assume that $x_\alpha = n \in \mathcal{V}$. Without loss of generality, we can assume that $\mathcal{E}_n^+ = \emptyset$ and $p_e^0(t_\alpha, x_\alpha) = 0$ for each $e \in \mathcal{E}_n$ with $n = x_\alpha$.*

Proof of Lemma 5.13. The orientation of the edges $e \in \mathcal{E}_n$ can be changed in order to reduce to the case $\mathcal{E}_n^+ = \emptyset$. In particular, for $p = (p_e)_{e\in\mathcal{E}_n}$,

$$F_A(t, n, p) = \max\left(A_n(t), \quad \max_{e\in\mathcal{E}_n^-} H_e^-(t, n, p_e)\right).$$

We can then argue as in Lemma 3.1. This means that we redefine the Hamiltonians (and the flux limiter A_n) only locally for $e \in \mathcal{E}_n$. Using (5.8), we can check that the new Hamiltonians (locally for $e \in \mathcal{E}_n$) and A_n still satisfy *(H0)-(H6)* and *(A0)-(A2)* (with the same modulus of continuity, and with some different controlled constants $C_{T,L}$). We also have (5.13) with some controlled different constants. □

Step 3: The penalization procedure. We now consider for $\varepsilon > 0$ and $\gamma \in (0, 1)$

$$M_{\alpha,\varepsilon} = \sup_{\substack{(t,x),(s,y)\in[0,T]\times\overline{B(x_\alpha,r)} \\ t<T}} \left\{ u(t,x) - w(s,y) - \alpha\psi(x) - \frac{\eta}{T-t} \right.$$
$$\left. - \frac{(t-s)^2}{2\nu} - G_\varepsilon^{\alpha,\gamma}(x,y) - \varphi^\alpha(t,s,x) \right\}$$

where the function φ^α

$$\varphi^\alpha(t,s,x) = \frac{1}{2}\left(|t-t_\alpha|^2 + |s-s_\alpha|^2 + d^2(x,x_\alpha)\right)$$

will help us to localize the problem around $(t_\alpha, s_\alpha, x_\alpha)$, and $B(x_\alpha, r)$ is the open ball of radius $r = r(\alpha) > 0$ centered at x_α; besides, we choose $r \in (0, 1)$ small enough such that

$B(x_\alpha, r) \subset e$ if $x_\alpha \in e \setminus \mathcal{V}$. Lemma A.2 ensures that ψ and φ^α are semi-concave and therefore can be used as test functions, see Remark 5.6.

We choose

$$G_\varepsilon^{\alpha,\gamma}(x,y) = \varepsilon G^{\alpha,\gamma}(\varepsilon^{-1}x, \varepsilon^{-1}y)$$

with

$$G^{\alpha,\gamma}(x,y) = \begin{cases} \dfrac{(x-y)^2}{2} & \text{if} \quad x_\alpha \in \mathcal{N} \setminus \mathcal{V}, \\ G^{x_\alpha,\gamma}(x,y) & \text{if} \quad x_\alpha \in \mathcal{V}, \end{cases}$$

where $G^{x_\alpha,\gamma} \geq 0$ is the vertex test function of parameter $\gamma > 0$ given by Theorem 3.2, built on the junction problem associated to the vertex x_α at time t_α, *i.e.* associated to junction problem for the Hamiltonian $H_\mathcal{V}^{t_\alpha,x_\alpha}$ given by

$$H_\mathcal{V}^{t_\alpha,n}(x,p) := \begin{cases} H_e(t_\alpha, n, p) & \text{if} \quad x \in e \setminus \{n\} \quad \text{with} \quad e \in \mathcal{E}_n, \\ F_A(t_\alpha, n, p) & \text{if} \quad x = n. \end{cases} \tag{5.21}$$

The supremum in the definition of $M_{\alpha,\varepsilon}$ is reached at some point $(t,x),(s,y) \in [0,T] \times \overline{B(x_\alpha, r)}$ with $t < T$. These maximizers satisfy the following penalization estimates.

Lemma 5.14 (Penalization)**.** *For $\varepsilon \in (0,1)$ and $\gamma \in (0, M/4)$, we have*

$$M_{\alpha,\varepsilon} \geq M_\alpha - \varepsilon\gamma \geq M/2 > 0 \tag{5.22}$$

$$d(x,y) \leq \omega(\varepsilon) \tag{5.23}$$

$$0 < \tau_\nu \leq s, t \leq T - \sigma_\eta$$

for some modulus of continuity ω (depending on α and γ) and τ_ν and σ_η not depending on (ε, γ). Moreover,

$$(t,s,x,y) \to (t_\alpha, s_\alpha, x_\alpha, x_\alpha) \quad \text{as } (\varepsilon,\gamma) \to (0,0).$$

In particular, we have $x, y \in B(x_\alpha, r)$ for $\varepsilon, \gamma > 0$ small enough.

Proof of Lemma 5.14. For all $\varepsilon, \nu > 0$, the compatibility on the diagonal (3.3) of the vertex test function $G^{x_\alpha,\gamma}$ yields the first inequality in (5.22). Then for $\varepsilon \in (0,1]$, with a choice of γ such that $0 < \gamma < M/4$, we have the second one.

Bound on $d(x,y)$**.** Remark that

$$\varepsilon g\left(\frac{d(x,y)}{\varepsilon}\right) \leq G_\varepsilon^{x_\alpha,\gamma}(x,y)$$

where

$$g(a) = \begin{cases} \dfrac{a^2}{2} & \text{if} \quad x_\alpha \in \mathcal{N} \setminus \mathcal{V}, \\ g^{x_\alpha,\gamma}(a) & \text{if} \quad x_\alpha \in \mathcal{V}, \end{cases}$$

and where $g^{x_\alpha,\gamma}$ is the superlinear function associated to $G^{x_\alpha,\gamma}$ and given by Theorem 3.2. Thanks to Lemma 5.10, we deduce that the maximiser $(t,x),(s,y)$ satisfies

$$
(5.24)\qquad \begin{aligned} 0 < M/2 &\le C(1+d(x,y)) - G_\varepsilon^{\alpha,\gamma}(x,y) - \frac{(t-s)^2}{2\nu} - \frac{\eta}{T-t} - \alpha\psi(x) \\ &\le C(1+d(x,y)) - \varepsilon g\left(\frac{d(x,y)}{\varepsilon}\right) - \frac{(t-s)^2}{2\nu} - \frac{\eta}{T-t} - \alpha\psi(x) \end{aligned}
$$

which implies in particular that

$$\varepsilon g\left(\frac{d(x,y)}{\varepsilon}\right) \le C(1+d(x,y)).$$

This gives (5.23) as in Step 1 of the proof of Theorem 1.4.

First time estimate. From (5.24) with $G_\varepsilon^{\alpha,\gamma} \ge 0$ and (5.23), we deduce in particular that for $\varepsilon \in (0,1]$

$$0 < M/2 \le C' - \frac{(t-s)^2}{2\nu} - \frac{\eta}{T-t}.$$

This implies in particular that

$$
(5.25)\qquad T-t \ge \frac{\eta}{C'}, \quad T-s \ge \frac{\eta}{C'} - \sqrt{2\nu C'} \ge \frac{\eta}{2C'} =: \sigma_\eta > 0
$$

for $\nu > 0$ small enough, and up to redefine σ_η for the new constant $C' \ge C$.

Second time estimate. From Lemma 5.11, we have with

$$
\begin{aligned} 0 < M/2 &\le f(t) + f(s) + u_0(x) - u_0(y) - \tfrac{\eta}{T} - \tfrac{(t-s)^2}{2\nu} \\ &\le f(t) + f(s) + \omega_0 \circ \omega(\varepsilon) - \frac{\eta}{T} - \frac{(t-s)^2}{2\nu} \end{aligned}
$$

where ω_0 is the modulus of continuity of u_0. Let us choose $\varepsilon > 0$ small enough such that

$$
(5.26)\qquad \omega_0 \circ \omega(\varepsilon) \le \frac{M}{2}.
$$

As in the proof of Lemma 5.12, for $\tau = \min(t,s)$, we get

$$\frac{\eta}{T} \le 2f(\tau + \sqrt{2\nu C'}).$$

For ν small enough (with η fixed), we then get a contradiction if τ converges to 0 as ν does.

Convergence of maximizers. Because of (5.22) and using the fact that $G_\varepsilon^{\alpha,\gamma} \geq 0$, we get for $\varepsilon \in (0,1]$

$$M_\alpha - \gamma \leq M_{\alpha,\varepsilon} \leq u(t,x) - w(s,y) - \alpha\psi(x) - \frac{\eta}{T-t} - \frac{(t-s)^2}{2\nu} - \varphi^\alpha(t,s,x).$$

Extracting a subsequence if needed, we can assume

$$(t,x,s,y) \to (\bar{t},\bar{x},\bar{s},\bar{x}) \quad \text{as} \quad (\varepsilon,\gamma) \to (0,0)$$

for some $\bar{t},\bar{s} \in [\tau_\nu, T-\sigma_\eta]$, $\bar{x} \in \overline{B(x_\alpha,r)}$. We get

$$M_\alpha \leq u(\bar{t},\bar{x}) - w(\bar{s},\bar{x}) - \alpha\psi(\bar{x}) - \frac{\eta}{T-\bar{t}} - \frac{(\bar{t}-\bar{s})^2}{2\nu} - \varphi^\alpha(\bar{t},\bar{s},\bar{x}) \leq M_\alpha - \varphi^\alpha(\bar{t},\bar{s},\bar{x})$$

which implies that $(\bar{t},\bar{s},\bar{x}) = (t_\alpha, s_\alpha, x_\alpha)$. □

Step 4: Viscosity inequalities. Then we can write the viscosity inequalities at (t,x) and (s,y) using the shorthand notation (5.6),

$$\frac{\eta}{(T-t)^2} + \frac{t-s}{\nu} + (t-t_\alpha) + H_{\mathcal{N}}(t,x,p_x^{\alpha,\gamma,\varepsilon} + \alpha\psi_x(x) + \varphi_x^\alpha(t,s,x)) \leq 0 \tag{5.27}$$

$$\frac{t-s}{\nu} - (s-s_\alpha) + H_{\mathcal{N}}(s,y,p_y^{\alpha,\gamma,\varepsilon}) \geq 0$$

where

$$\begin{cases} p_x^{\alpha,\gamma,\varepsilon} = G_x^{\alpha,\gamma}(\varepsilon^{-1}x,\varepsilon^{-1}y), \\ p_y^{\alpha,\gamma,\varepsilon} = -G_y^{\alpha,\gamma}(\varepsilon^{-1}x,\varepsilon^{-1}y). \end{cases}$$

We choose ε,γ small enough such that (Lemma 5.14) we have

$$|t-t_\alpha|, \quad |s-s_\alpha| \quad \leq \frac{\eta}{4T^2}.$$

Substracting the two viscosity inequalities, we get

$$\frac{\eta}{2T^2} \leq H_{\mathcal{N}}(s,y,p_y^{\alpha,\gamma,\varepsilon}) - H_{\mathcal{N}}(t,x,p_x^{\alpha,\gamma,\varepsilon} + \alpha\psi_x(x) + \varphi_x^\alpha(t,s,x)). \tag{5.28}$$

Step 5: Gradient estimates. We deduce from (5.27) that

$$\tilde{p}_x^{\alpha,\gamma,\varepsilon} = p_x^{\alpha,\gamma,\varepsilon} + \alpha\psi_x(x) + \varphi_x^\alpha(t,s,x)$$

satisfies

$$H_{\mathcal{N}}(t,x,\tilde{p}_x^{\alpha,\gamma,\varepsilon}) \leq \frac{s-t}{\nu} + t_\alpha - t \leq \frac{T}{\nu} + T. \tag{5.29}$$

Hence (H1) implies that there exists a constant C'_ν (independent of α, ε, γ, but depending on η, ν) such that

$$\begin{cases} |\tilde{p}_x^{\alpha,\gamma,\varepsilon}| \le C'_\nu & \text{if} \quad x \neq x_\alpha \quad \text{or} \quad x_\alpha \notin \mathcal{V}, \\ \tilde{p}_x^{\alpha,\gamma,\varepsilon} \ge -C'_\nu & \text{if} \quad x = x_\alpha \quad \text{and} \quad x_\alpha \in \mathcal{V}. \end{cases}$$

From (5.17), we deduce that

$$|\alpha\psi_x(x) + \varphi_x^\alpha(t,s,x)| \le C\sqrt{\alpha} + d(x,x_\alpha) \le C \tag{5.30}$$

for $\alpha \le 1$ (using (5.17)). Therefore, we have for some constant C_ν (independent of α, ε, γ):

$$\begin{cases} |p_x^{\alpha,\gamma,\varepsilon}| \le C_\nu & \text{if} \quad x \neq x_\alpha \quad \text{or} \quad x_\alpha \notin \mathcal{V}, \\ p_x^{\alpha,\gamma,\varepsilon} \ge -C_\nu & \text{if} \quad x = x_\alpha \quad \text{and} \quad x_\alpha \in \mathcal{V}. \end{cases}$$

From the compatibility condition of the Hamiltonians satisfied by $G^{\alpha,\gamma}$ if $x_\alpha \in \mathcal{V}$, or the definition of $G^{\alpha,\gamma}$ if $x_\alpha \notin \mathcal{V}$, we have in both cases,

$$H^{t_\alpha,x_\alpha}(y, p_y^{\alpha,\gamma,\varepsilon}) \le H^{t_\alpha,x_\alpha}(x, p_x^{\alpha,\gamma,\varepsilon}) + \gamma \tag{5.31}$$

where

$$H^{t_\alpha,x_\alpha}(x,p) = \begin{cases} H_\mathcal{V}^{t_\alpha,n}(x,p) & \text{if} \quad x_\alpha = n \in \mathcal{V}, \\ H_e(t_\alpha, x_\alpha, p) & \text{if} \quad x_\alpha \notin \mathcal{V}, x_\alpha \in e^*. \end{cases}$$

We deduce that $p_y^{\alpha,\gamma,\varepsilon}$ satisfies (modifying C_ν if necessary)

$$\begin{cases} |p_y^{\alpha,\gamma,\varepsilon}| \le C_\nu & \text{if} \quad y \neq x_\alpha \quad \text{or } x_\alpha \notin \mathcal{V}, \\ p_y^{\alpha,\gamma,\varepsilon} \ge -C_\nu & \text{if} \quad y = x_\alpha \quad \text{and} \quad x_\alpha \in \mathcal{V}. \end{cases}$$

For $z = x, y \in \mathcal{V}$, $p_z^{\alpha,\gamma,\varepsilon}$ is a vector and its components are only bounded from below, see above. But when writing viscosity inequalities, they appear as variables of the non-increasing part of Hamiltonians. Hence, if they are too large, they can be replaced with the point minimizing the Hamiltonian, without changing the viscosity inequalities. This is the reason why we truncate each component of this vector by a well chosen constant K. Precisely, we define for $z = x, y$,

$$\bar{p}_z^{\alpha,\gamma,\varepsilon} = \begin{cases} (\min(K, (p_z^{\alpha,\gamma,\varepsilon})_{\tilde{z}}))_{\tilde{z}\in x_\alpha} & \text{if} \quad z = x_\alpha \quad \text{and} \quad x_\alpha \in \mathcal{V} \\ p_z^{\alpha,\gamma,\varepsilon} & \text{if not.} \end{cases}$$

with, in the case where $x_\alpha \in \mathcal{V}$, the constant K given by

$$K = \max_{e \in \mathcal{E}_{x_\alpha}} (p_e^0(s,x_\alpha), p_e^0(t_\alpha,x_\alpha), p_e^0(t,x_\alpha) + C)) \le C_T + C$$

(C comes from (5.30) and C_T from (5.8)). We then have

$$|\bar{p}_z^{\alpha,\gamma,\varepsilon}| \le C_\nu + C_T + C =: C_{\nu,T}$$

and

$$\frac{\eta}{2T^2} \leq H_{\mathcal{N}}(s,y,\bar{p}_y^{\alpha,\gamma,\varepsilon}) - H_{\mathcal{N}}(t,x,\bar{p}_x^{\alpha,\gamma,\varepsilon} + \alpha\psi_x(x) + \varphi_x^\alpha(t,s,x)), \tag{5.32}$$

$$H_{\mathcal{N}}(t,x,\bar{p}_x^{\alpha,\gamma,\varepsilon} + \alpha\psi_x(x) + \varphi_x^\alpha(t,s,x)) \leq \frac{s-t}{\nu} + t_\alpha - t \leq \frac{T}{\nu} + T, \tag{5.33}$$

$$H^{t_\alpha,x_\alpha}(y,\bar{p}_y^{\alpha,\gamma,\varepsilon}) \leq H^{t_\alpha,x_\alpha}(x,\bar{p}_x^{\alpha,\gamma,\varepsilon}) + \gamma. \tag{5.34}$$

Step 6: The limit $(\varepsilon,\gamma) \to (0,0)$ and conclusion as $\alpha \to 0$. Up to a subsequence, we get in the limit $(\varepsilon,\gamma) \to (0,0)$ for $z = x, y$:

$$\bar{p}_z^{\alpha,\gamma,\varepsilon} \to \bar{p}_z^\alpha \quad \text{with} \quad |\bar{p}_z^\alpha| \leq C_{\nu,T}.$$

Moreover, passing to the limit in (5.32) and (5.33), we get respectively

$$\frac{\eta}{2T^2} \leq H_{\mathcal{N}}(s_\alpha,x_\alpha,\bar{p}_y^\alpha) - H_{\mathcal{N}}(t_\alpha,x_\alpha,\bar{p}_x^\alpha + \alpha\psi_x(x_\alpha))$$

and

$$H_{\mathcal{N}}(t_\alpha,x_\alpha,\bar{p}_x^\alpha + \alpha\psi_x(x_\alpha)) \leq \frac{s_\alpha - t_\alpha}{\nu} \leq \frac{T}{\nu}.$$

On the other hand, passing to the limit in (5.34) gives

$$H^{t_\alpha,x_\alpha}(x_\alpha,\bar{p}_y^\alpha) \leq H^{t_\alpha,x_\alpha}(x_\alpha,\bar{p}_x^\alpha).$$

Because

$$H_{\mathcal{N}}(t_\alpha,x_\alpha,p) = H^{t_\alpha,x_\alpha}(x_\alpha,p)$$

we get for any p,

$$\frac{\eta}{2T^2} \leq I_1 + I_2$$

with

$$\begin{aligned} I_1 &= H_{\mathcal{N}}(s_\alpha,x_\alpha,\bar{p}_x^\alpha) - H_{\mathcal{N}}(s_\alpha,x_\alpha,\bar{p}_x^\alpha + \alpha\psi_x(x_\alpha)),\\ I_2 &= H_{\mathcal{N}}(s_\alpha,x_\alpha,\bar{p}_x^\alpha + \alpha\psi_x(x_\alpha)) - H_{\mathcal{N}}(t_\alpha,x_\alpha,\bar{p}_x^\alpha + \alpha\psi_x(x_\alpha)). \end{aligned}$$

Thanks to (H3) and (5.17), we have $|\alpha\psi_x(x_\alpha)| \leq C_{\nu,T}$ and we thus get

$$I_1 \leq \omega_{T,2C_{\nu,T}}(\alpha\psi_x(x_\alpha)) \leq \omega_{T,2C_\nu}(C\sqrt{\alpha}). \tag{5.35}$$

Now thanks to Lemma 5.2, we also have

$$\begin{aligned} I_2 &\leq \tilde{\omega}_T(|t_\alpha - s_\alpha|(1 + \max(H_{\mathcal{N}}(t_\alpha,x_\alpha,\bar{p}_x^\alpha + \alpha\psi_x(x_\alpha)),0)))\\ &\leq \tilde{\omega}_T(|t_\alpha - s_\alpha|(1 + \max(\frac{s_\alpha - t_\alpha}{\nu},0))). \end{aligned}$$

Then taking first the limit $\alpha \to 0$ and then taking the limit $\nu \to 0$, we use (5.19) to get the desired contradiction. This achieves the proof of Theorem 5.8. □

6 First application: link with optimal control theory

This section is devoted to the study of the value function of an optimal control problem associated with trajectories running over the junction.

6.1 Assumptions on dynamics and running costs

As before, we consider a junction $J = \bigcup_{i=1,\dots,N} J_i$. We consider compact metric spaces $\mathbb{A}_i$ for $i = 0,\dots,N$ and functions $b_i, \ell_i : [0,T] \times J_i \times \mathbb{A}_i \to \mathbb{R}$ for $i = 1,\dots,N$ and $b_0, \ell_0 : [0,T] \times \mathbb{A}_0 \to \mathbb{R}$. The sets $\mathbb{A}_i$ are the sets of controls on each branch J_i^* for $i = 1,\dots,N$, while the set $\mathbb{A}_0$ is the set of controls at the junction point $x = 0$. The functions b_i represent the dynamics and the ℓ_i's are the running cost functions.

For $i = 1,\dots,N$, we follow [10] by assuming the following

$$
(6.1)\qquad \begin{cases} b_i \text{ and } \ell_i \text{ are continuous and bounded} \\ b_i \text{ is uniformly continuous w.r.t. } (t,x) \text{ uniformly w.r.t. } \alpha_i \\ \ell_i \text{ is uniformly continuous w.r.t. } (t,x) \text{ uniformly w.r.t. } \alpha_i \\ \mathcal{B}_i(t,x) := \{(b_i(t,x,\alpha_i), \ell_i(t,x,\alpha_i)) : \alpha_i \in \mathbb{A}_i\} \text{ is closed and convex} \\ B_i(t,x) = \{b_i(t,x,\alpha_i) : \alpha_i \in \mathbb{A}_i\} \text{ contains } [-\delta,\delta] \end{cases}
$$

for some δ independent of (t,x).

It is easy to check the following lemmas.

Lemma 6.1 (Hamiltonians)**.** *Assume (6.1). Then given $i \in \{1,\dots,N\}$, the Hamiltonian H_i defined by*

$$
H_i(t,x,p_i) = \sup_{\alpha_i \in \mathbb{A}_i} \left(b_i(t,x,\alpha_i)p_i - \ell_i(t,x,\alpha_i) \right)
$$

satisfies Assumption (1.5).

Lemma 6.2 (Non-increasing Hamiltonians)**.** *Assume (6.1). Given $i \in \{1,\dots,N\}$, then the non-increasing part of $H_i(t,0,p_i)$ with respect to p_i, is given by*

$$
\begin{aligned}
H_i^-(t,p_i) &= \sup_{\alpha_i \in \mathbb{A}_i^-} \left(b_i(t,0,\alpha_i)p_i - \ell_i(t,0,\alpha_i) \right) \\
&= \sup_{\alpha_i \in \mathbb{A}_i^<} \left(b_i(t,0,\alpha_i)p_i - \ell_i(t,0,\alpha_i) \right)
\end{aligned}
$$

where $\mathbb{A}_i^- = \{\alpha_i \in \mathbb{A}_i : b_i(t,0,\alpha_i) \le 0\}$ and $\mathbb{A}_i^< = \{\alpha_i \in \mathbb{A}_i : b_i(t,0,\alpha_i) < 0\}$.

As far as the dynamics and running costs at the junction point are concerned, we also assume that

$$
(6.2)\qquad b_0 \text{ and } l_0 \text{ are continuous bounded}, \quad \mathbb{A}_0 \subset \mathbb{R}^{d_0}
$$

for some $d_0 \geq 1$, and define

$$B_0(t) = \{b_0(t, \alpha_0) : \alpha_0 \in \mathbb{A}_0\}.$$

We also define

$$A_0(t) = \max_{i=1,\dots,N} \min_{p\in\mathbb{R}} H_i(t, 0, p). \tag{6.3}$$

We set

$$H_0(t) = \begin{cases} \sup_{\alpha_0\in\mathbb{A}_0(t)} (-\ell_0(t, \alpha_0)) & \text{if} \quad \mathbb{A}_0(t) \neq \emptyset, \\ -\infty & \text{if} \quad \mathbb{A}_0(t) = \emptyset \end{cases} \tag{6.4}$$

with

$$\mathbb{A}_0(t) = \{\alpha_0 \in \mathbb{A}_0, \quad b_0(t, \alpha_0) = 0\}, \tag{6.5}$$

and we assume that

$$\bar{H}_0 : t \mapsto \max(H_0(t), A_0(t)) \text{ is continuous in } [0, T]. \tag{6.6}$$

6.2 The value function

We then define the general set of controls,

$$\mathbb{A} = \mathbb{A}_0 \times \cdots \times \mathbb{A}_N$$

and define for $\alpha = (\alpha_0, \dots, \alpha_N) \in \mathbb{A}$ and $(t, x) \in [0, T] \times J$,

$$b(t, x, \alpha) = \begin{cases} b_i(t, x, \alpha_i) & \text{if} \quad x \in J_i^*, \\ b_0(t, \alpha_0) & \text{if} \quad x = 0. \end{cases}$$

Similarly, we define

$$\ell(t, x, \alpha) = \begin{cases} \ell_i(t, x, \alpha_i) & \text{if} \quad x \in J_i^*, \\ \ell_0(t, \alpha_0) & \text{if} \quad x = 0. \end{cases}$$

For $0 \leq s < t \leq T$ and $y, x \in J$, we define the set of admissible dynamics

$$\mathcal{T}_{s,y}^{t,x} = \left\{ \begin{array}{l} (X(\cdot), \alpha(\cdot)) \in \mathrm{Lip}([s, t]; J) \times L^\infty([s, t]; \mathbb{A}), \\ \begin{cases} X(s) = y, \quad X(t) = x, \\ \dot{X}(\tau) = b(\tau, X(\tau), \alpha(\tau)) \quad \text{for a.e.} \quad \tau \in (s, t) \end{cases} \end{array} \right\}. \tag{6.7}$$

Then we consider the value function of the optimal control problem,

$$u(t, x) = \inf_{z\in J} \inf_{(X(\cdot),\alpha(\cdot))\in\mathcal{T}_{0,z}^{t,x}} E_0^t(X, \alpha) \tag{6.8}$$

with

$$E_0^t(X,\alpha) = u_0(X(0)) + \int_0^t \ell(\tau, X(\tau), \alpha(\tau))\, d\tau$$

where the initial datum u_0 is assumed to be globally Lipschitz continuous.

Note that if $\mathcal{T}_{0,z}^{t,x} = \emptyset$, then we have $\inf_{\mathcal{T}_{0,z}^{t,x}} (\dots) = +\infty$. More generally and for later use, we set

$$E_s^t(X,\alpha) = u(s, X(s)) + \int_s^t \ell(\tau, X(\tau), \alpha(\tau))\, d\tau. \tag{6.9}$$

6.3 Dynamic programming principle

The following result is expected and quite standard.

Proposition 6.3 (Dynamic programming principle)**.** *For all $x \in J$, $t \in (0,T]$ and $s \in [0,t)$, the value function u defined in (6.8) satisfies*

$$u(t,x) = \inf_{y\in J} \inf_{(X(\cdot),\alpha(\cdot))\in \mathcal{T}_{s,y}^{t,x}} E_s^t(X,\alpha)$$

where E_s^t and $\mathcal{T}_{s,y}^{t,x}$ are defined respectively in (6.9) ansd (6.7).

Proof. Let $V(t,x)$ denote the right hand side of the desired equality. Consider $(X(\cdot),\alpha(\cdot)) \in \mathcal{T}_{0,z}^{s,y}$ and $(\tilde X(\cdot), \tilde\alpha(\cdot)) \in \mathcal{T}_{s,y}^{t,x}$. Then

$$(\bar X(\tau), \bar\alpha(\tau)) = \begin{cases} (X(\tau),\alpha(\tau)) & \text{if } \tau \in [0,s] \\ (\tilde X(\tau), \tilde\alpha(\tau)) & \text{if } \tau \in (s,t] \end{cases}$$

lies in $\mathcal{T}_{0,z}^{t,x}$. In particular,

$$\begin{aligned} u(t,x) &\le u_0(z) + \int_0^t \ell(\tau, \bar X(\tau), \bar\alpha(\tau))\, d\tau \\ &\le u_0(z) + \int_0^s \ell(\tau, X(\tau), \alpha(\tau))\, d\tau + \int_s^t \ell(\tau, \tilde X(\tau), \tilde\alpha(\tau))\, d\tau. \end{aligned}$$

Taking the infimum, first with respect to $(X(\cdot),\alpha(\cdot))$ and z, and then with respect to $(\tilde X(\cdot), \tilde\alpha(\cdot))$ yields $u(t,x) \le V(t,x)$.

To get the reversed inequality, consider, for all $\varepsilon > 0$, an admissible dynamics $(X^\varepsilon(\cdot), \alpha^\varepsilon(\cdot)) \in \mathcal{T}_{0,z}^{t,x}$ such that

$$\begin{aligned} u(t,x) &\ge u_0(X^\varepsilon(0)) + \int_0^t \ell(\tau, X^\varepsilon(\tau), \alpha^\varepsilon(\tau))\, d\tau - \varepsilon \\ &\ge u_0(X^\varepsilon(0)) + \int_0^s \ell(\tau, X^\varepsilon(\tau), \alpha^\varepsilon(\tau))\, d\tau + \int_s^t \ell(\tau, X^\varepsilon(\tau), \alpha^\varepsilon(\tau))\, d\tau - \varepsilon \\ &\ge u(s, X^\varepsilon(s)) + \int_s^t \ell(\tau, X^\varepsilon(\tau), \alpha^\varepsilon(\tau))\, d\tau - \varepsilon \\ &\ge V(t,x) - \varepsilon. \end{aligned}$$

Since ε is arbitrary, we conclude. $\square$

6.4 Derivation of the Hamilton-Jacobi-Bellman equation

We will show that the value function u solves the following problem

$$\begin{cases} u_t + H_i(t,x,u_x) = 0 & \text{for all} \quad (t,x) \in (0,T) \times J_i^*, \\ u_t + F_{\bar{H}_0(t)}(t,u_x) = 0 & \text{for all} \quad (t,x) \in (0,T) \times \{0\} \end{cases} \tag{6.10}$$

with

$$F_{\bar{H}_0(t)}(t,u_x(t,0^+)) := \max\left(\bar{H}_0(t), \max_{i=1,\dots,N} H_i^-(t,\partial_i u(t,0^+))\right)$$

and with initial condition

$$u(0,x) = u_0(x) \quad \text{for all} \quad x \in J. \tag{6.11}$$

We also consider the following condition for $i = 1,\dots,N$

$$b_i \text{ is Lipschitz continuous w.r.t. } t \text{ uniformly w.r.t. } (x,\alpha_i). \tag{6.12}$$

Theorem 6.4 (The value function is a flux-limited solution)**.** *Assume (6.1), (6.2) and (6.6). Let us also consider H_i, H_i^- and $\bar{H}_0$ respectively defined in Lemmas 6.1 and 6.2 and in (6.6). Assume also that the initial datum u_0 is globally Lipschitz on J.*

i) (Existence)*. The value function u defined by* (6.8) *is a solution of* (6.10)*,* (6.11)*.*

ii) (Uniqueness)*. If we assume moreover (6.12), then u is the unique solution of* (6.10)*,* (6.11)*.*

In order to prove this theorem, two technical results are needed. Their proofs is postponed until the end of the proof of Theorem 6.4.

Lemma 6.5 (A measurable selection result)**.** *Assume that b_0 and ℓ_0 satisfy (6.2). For some $[a,b] \subset (0,T)$, let us also assume that*

$$\emptyset \neq \mathbb{A}_0(\tau) := \{\alpha_0 \in \mathbb{A}_0, \quad b_0(\tau,\alpha_0) = 0\} \quad \textit{for all} \quad \tau \in [a,b]$$

and that

$$\tau \mapsto H_0(\tau) := \sup_{\alpha_0 \in \mathbb{A}_0(\tau)} (-\ell_0(\tau,\alpha_0)) \quad \textit{is continuous on} \quad [a,b].$$

Then there exists a mesurable selection $\bar{\alpha}_0 \in L^\infty([a,b];\mathbb{A}_0)$ such that

$$\bar{\alpha}_0(\tau) \in \mathbb{A}_0(\tau) \quad \textit{and} \quad H_0(\tau) = -\ell_0(\tau,\bar{\alpha}_0(\tau)) \quad \textit{for a.e.} \quad \tau \in [a,b].$$

Proposition 6.6 (Checking assumptions for the comparison principle)**.** *Assume (6.1), (6.2), (6.6) and (6.12). Let us also consider H_i, H_i^- and $\bar{H}_0$ respectively defined in Lemmas 6.1 and 6.2 and in (6.6). Using notation from Section 5 on networks, let us consider the network $\mathcal{N} = J$, with edges $\mathcal{E} = \{J_1, \dots, J_N\} = \mathcal{E}_n^-$ where the unique vertex n is identified to the junction point 0. We set $H_e(t,x,p) := H_i(t,x,p)$ and $H_e^-(t,p) = H_i^-(t,p)$ for $e = J_i$ for each $i = 1, \quad ,N$. We also set $A_n(t) := \bar{H}_0(t)$. Then assumptions (H0)-(H6) and (A0)-(A2) are satisfied.*

Proof of Theorem 6.4. We will show that u^* is a super-solution and u_* is a sub-solution on $(0,T) \times J$. Deriving the Hamilton-Jacobi-Bellman equation outside the junction point is known and standard. This is the reason why we will focus on the junction condition. As in the standard case, it relies on the dynamic programming principle.

Step 1: the super-solution property. Consider any test function φ such that

$$\varphi \le u_* \text{ in } (0,+\infty) \times J \quad \text{and} \quad \varphi = u_* \text{ at } (\bar{t},0) \quad \text{with} \quad \bar{t} \in (0,T).$$

Our goal is to show that

$$\varphi_t(\bar{t},0) + F_{\bar{H}_0(\bar{t})}(\bar{t},\varphi_x(\bar{t},0^+)) \ge 0 \tag{6.13}$$

The proof of this inequality proceeds in several substeps.

Step 1.1: the basic optimal control inequality. Let $(t_n,x_n) \in (0,T) \times J$ be such that

$$(t_n,x_n) \to (\bar{t},0) \quad \text{and} \quad u(t_n,x_n) \to u_*(\bar{t},0) \quad \text{as} \quad n \to +\infty.$$

Let $s \in (0,\bar{t})$. Then the dynamic programming principle yields

$$u(t_n,x_n) = \inf_{y \in J} \inf_{(X(\cdot),\alpha(\cdot)) \in \mathcal{T}_{s,y}^{t_n,x_n}} \left\{ u(s,X(s)) + \int_s^{t_n} \ell(\tau,X(\tau),\alpha(\tau))\, d\tau \right\}$$

This implies that

$$\varphi(t_n,x_n) + o_n(1) \ge \inf_{y \in J} \inf_{(X(\cdot),\alpha(\cdot)) \in \mathcal{T}_{s,y}^{t_n,x_n}} \left\{ \varphi(s,X(s)) + \int_s^{t_n} \ell(\tau,X(\tau),\alpha(\tau))\, d\tau \right\}$$

where $o_n(1) \to 0$ as $n \to +\infty$. Therefore, we have

$$S_n := \sup_{y \in J} \sup_{(X(\cdot),\alpha(\cdot)) \in \mathcal{T}_{s,y}^{t_n,x_n}} K_s^{t_n}(X,\alpha) \ge -o_n(1) \tag{6.14}$$

where

$$K_s^{t_n}(X,\alpha) := \varphi(t_n,X(t_n)) - \varphi(s,X(s)) - \int_s^{t_n} \ell(\tau,X(\tau),\alpha(\tau))\, d\tau \tag{6.15}$$

with

$$\varphi(t_n, X(t_n)) - \varphi(s, X(s)) = \int_s^{t_n} d\tau \ \{\varphi_t(\tau, X(\tau)) + \varphi_x(\tau, X(\tau))b(\tau, X(\tau), \alpha(\tau))\}.$$

Here, we take the convention that the product $\varphi_x b$ equals 0 if $X(\tau) = 0$. This makes sense for almost every τ, because by Stampacchia's truncation theorem, we have

$$0 = \dot{X}(\tau) = b(\tau, X(\tau), \alpha(\tau)) = b_0(\tau, \alpha_0(\tau)) \quad \text{a.e. on} \quad \{\tau \in (s, t_n),\ X(\tau) = 0\} \tag{6.16}$$

which implies in particular

$$\alpha_0(\tau) \in \mathbb{A}_0(\tau) \quad \text{a.e. on} \quad \{\tau \in (s, t_n),\ X(\tau) = 0\} \tag{6.17}$$

where $\mathbb{A}_0$ is defined in (6.5). This shows that we can write

$$K_s^{t_n}(X, \alpha) = \int_s^{t_n} d\tau \ \kappa(\tau, X(\tau), \alpha(\tau))$$

with for $(\tau, x) \in (0, T) \times J$ and $\beta = (\beta_0, \ldots, \beta_N) \in \mathbb{A}$:

$$\kappa(\tau, x, \beta) = \varphi_t(\tau, x) + \varphi_x(\tau, x)b(\tau, x, \beta) - \ell(\tau, x, \beta)$$

with the convention that

$$\left\{ \begin{array}{l} \varphi_x(\tau, x)b(\tau, x, \beta) = 0 \\ \beta_0 \in \mathbb{A}_0(\tau) \end{array} \right| \quad \text{if} \quad x = 0.$$

STEP 1.2: FREEZING THE COEFFICIENTS. We now freeze the coefficients at the point $(\bar{t}, 0) \in (0, T) \times J$, defining for any $(\tau, x) \in (0, T) \times J$ and $\beta \in \mathbb{A}$:

$$\bar{\kappa}(\tau, x, \beta) := \left\{ \begin{array}{ll} \varphi_t(\bar{t}, 0) + \partial_i \varphi(\bar{t}, 0) b_i(\bar{t}, 0, \beta_i) - \ell_i(\bar{t}, 0, \beta_i) & \text{if} \quad x \in J_i^*, \\ \varphi_t(\bar{t}, 0) - \ell_0(\tau, \beta_0) & \text{if} \quad x = 0, \end{array} \right. \tag{6.18}$$

with the convention that $\beta_0 \in \mathbb{A}_0(\tau)$ if $x = 0$. From structural assumptions (6.1) and (6.2), there exists a (monotone continuous) modulus of continuity ω (depending only on φ and the quantities b_i, ℓ_i for $i = 0, \ldots, N$) such that

$$|\bar{\kappa}(\tau, x, \beta) - \kappa(\tau, x, \beta)| \leq \omega(|\bar{t} - \tau| + d(x, 0)) \quad \text{for all} \quad (\tau, x, \beta) \in (0, T) \times J \times \mathbb{A}.$$

Since trajectories are uniformly Lipschitz, there exists a constant $C_0 > 0$ such that for all $\tau \in (s, t_n)$,

$$d(X(\tau), 0) \leq d(x_n, 0) + C_0|t_n - \tau| = o_n(1) + C_0|\bar{t} - \tau|.$$

Defining

$$\bar{K}_s^{t_n}(X, \alpha) = \int_s^{t_n} d\tau \ \bar{\kappa}(\tau, X(\tau), \alpha(\tau)) \tag{6.19}$$

we get that

$$(6.20) \qquad |\bar{K}_s^{t_n}(X,\alpha) - K_s^{t_n}(X,\alpha)| \le |t_n - s|\omega(o_n(1) + C_1|\bar{t} - s|) \quad \text{with} \quad C_1 = 1 + C_0.$$

STEP 1.3: APPLICATION TO A QUASI-OPTIMIZER. Let us consider a quasi-optimizer $(X^n, \alpha^n) \in \mathcal{T}_{s,y_n}^{t_n,x_n}$ for some $y_n \in J$ such that

$$K_s^{t_n}(X^n, \alpha^n) \ge S_n - o_n(1).$$

By (6.14) and estimate (6.20), this implies

$$(6.21) \qquad \bar{K}_s^{t_n}(X^n, \alpha^n) \ge -o_n(1) - |t_n - s|\omega(o_n(1) + C_1|\bar{t} - s|).$$

In order to evaluate $\bar{K}_s^{t_n}(X^n, \alpha^n)$, we naturally define the following sets. Let

$$\mathbb{T}_0^n = \{\tau \in (s, t_n), \quad X^n(\tau) = 0\}$$

which is a (relative) closed set of (s, t_n), and let us set for $i = 1, \dots, N$:

$$\mathbb{T}_i^n = \{\tau \in (s, t_n), \quad X^n(\tau) \in J_i^*\}$$

which are open sets. We have

$$\bar{K}_s^{t_n}(X^n, \alpha^n) = \sum_{i=0,\dots,N} \bar{K}_i^n \quad \text{with} \quad \bar{K}_i^n := \int_{\mathbb{T}_i^n} d\tau \ \bar{\kappa}(\tau, X^n(\tau), \alpha^n(\tau)).$$

We next study each term $\bar{K}_i^n$ of the previous sum.

STEP 1.3.1: CONVERGENCE FOR $i = 1, \dots, N$. We now use an argument that we found in [10]. For $i = 1, \dots, N$, by convexity of the set $\mathcal{B}_i(\bar{t}, 0)$ defined in (6.1), we deduce that there exists some $\bar{\alpha}_i^n \in \mathbb{A}_i$ such that

$$(6.22) \qquad \frac{1}{|\mathbb{T}_i^n|} \int_{\mathbb{T}_i^n} d\tau \ (b_i(\bar{t}, 0, \alpha^n(\tau)), \ell_i(\bar{t}, 0, \alpha^n(\tau))) = (b_i(\bar{t}, 0, \bar{\alpha}_i^n), \ell_i(\bar{t}, 0, \bar{\alpha}_i^n))$$

and then

$$\bar{K}_i^n = |\mathbb{T}_i^n| \left\{ \varphi_t(\bar{t}, 0) + \partial_i \varphi(\bar{t}, 0) b_i(\bar{t}, 0, \bar{\alpha}_i^n) - \ell_i(\bar{t}, 0, \bar{\alpha}_i^n) \right\}.$$

Moreover, decomposing the set $\mathbb{T}_i^n$ in a (at most countable) union of intervals (a_k, b_k) (with possibly $a_k = s$ or $b_k = t_n$ for some particular value of k), we see that we have with $x_n = X(t_n)$

$$(6.23) \qquad \int_{\mathbb{T}_i^n} d\tau \ b_i(\bar{t}, 0, \alpha^n(\tau)) = \int_{\mathbb{T}_i^n} d\tau \ \dot{X}^n(\tau) = \begin{cases} 0 - X^n(s) & \text{if} \quad X^n(t_n) \notin J_i^*, \qquad X^n(s) \in J_i^*, \\ X(t_n) - X^n(s) & \text{if} \quad X^n(t_n) \in J_i^*, \qquad X^n(s) \in J_i^*, \\ X(t_n) - 0 & \text{if} \quad X^n(t_n) \in J_i^*, \qquad X^n(s) \notin J_i^*. \end{cases}$$

Up to a subsequence, we have $\bar{\alpha}_i^n \to \bar{\alpha}_i$, $|\mathbb{T}_i^n| \to T_i$ for some $T_i \geq 0$. It is convenient to write T_i as $|\mathbb{T}_i|$. Remark in particular that we have

$$\sum_{i=0}^{N} |\mathbb{T}_i| = \bar{t} - s.$$

Next, we get that the sequence of trajectories $X^n(\cdot)$ converges uniformly to some $X(\cdot)$ such that

$$|\mathbb{T}_i| b_i(\bar{t}, 0, \bar{\alpha}_i) = \begin{cases} 0 - X(s) & \text{if} \quad X(s) \in J_i^*, \\ 0 & \text{if} \quad X(s) \notin J_i^* \end{cases}$$

and therefore

$$b_i(\bar{t}, 0, \bar{\alpha}_i) \leq 0 \quad \text{if} \quad |\mathbb{T}_i| \neq 0.$$

This implies

$$\bar{K}_i^n \to \bar{K}_i$$

with

$$\begin{array}{rcl} \bar{K}_i & := & |\mathbb{T}_i| \left\{ \varphi_t(\bar{t}, 0) + \partial_i \varphi(\bar{t}, 0) b_i(\bar{t}, 0, \bar{\alpha}_i) - \ell_i(\bar{t}, 0, \bar{\alpha}_i) \right\} \\ & \leq & |\mathbb{T}_i| \left\{ \varphi_t(\bar{t}, 0) + H_i^-(\bar{t}, \partial_i \varphi(\bar{t}, 0)) \right\} \\ & \leq & |\mathbb{T}_i| \left\{ \varphi_t(\bar{t}, 0) + F_{\bar{H}_0(t)}(t, \varphi_x(t, 0^+)) \right\}. \end{array} \tag{6.24}$$

STEP 1.3.2: CONVERGENCE FOR $i = 0$. We have

$$\bar{K}_0^n = \int_{\mathbb{T}_0^n} d\tau \; \bar{\kappa}(\tau, X^n(\tau), \alpha^n(\tau)) = \int_{\mathbb{T}_0^n} d\tau \; \left\{ \varphi_t(\bar{t}, 0) - \ell_0(\tau, \alpha_0^n(\tau)) \right\}.$$

Because of (6.17), we know that $\alpha_0^n(\tau) \in \mathbb{A}_0(\tau)$ for almost every $\tau \in \mathbb{T}_0^n$ which implies

$$\bar{K}_0^n \leq \int_{\mathbb{T}_0^n} d\tau \; \left\{ \varphi_t(\bar{t}, 0) + H_0(\tau) \right\} \leq \int_{\mathbb{T}_0^n} d\tau \; \left\{ \varphi_t(\bar{t}, 0) + \bar{H}_0(\tau)) \right\}$$

where H_0 and $\bar{H}_0$ are defined in (6.4) and (6.6) respectively. Since the function $\bar{H}_0$ is assumed to be continuous, see (6.6), there exists some (monotone continuous) modulus of continuity, that we still denote by ω, such that

$$\bar{K}_0^n \leq |\mathbb{T}_0^n| \left\{ \varphi_t(\bar{t}, 0) + \bar{H}_0(t_n) + \omega(|t_n - s|) \right\}$$

Up to a subsequence, we have $|\mathbb{T}_0^n| \to |\mathbb{T}_0|$ and then

$$\begin{array}{rcl} \limsup_{n \to +\infty} \bar{K}_0^n & \leq & |\mathbb{T}_0| \left\{ \varphi_t(\bar{t}, 0) + \bar{H}_0(\bar{t}) + \omega(|\bar{t} - s|) \right\} \\ & \leq & |\mathbb{T}_0| \left\{ \varphi_t(\bar{t}, 0) + F_{\bar{H}_0(t)}(t, \varphi_x(t, 0^+)) + \omega(|\bar{t} - s|) \right\}. \end{array} \tag{6.25}$$

STEP 1.4: CONCLUSION. From (6.21) on the one hand, and from (6.24), (6.25) on the other hand, we deduce that

$$\begin{aligned} -|\bar{t} - s| \omega(C_1 |\bar{t} - s|) &\leq \limsup_{n \to +\infty} \sum_{i=0,\ldots,N} \bar{K}_i^n \\ &\leq \left(\sum_{i=0,\ldots,N} |\mathbb{T}_i| \right) \left\{ \varphi_t(\bar{t}, 0) + F_{\bar{H}_0(t)}(t, \varphi_x(t, 0^+) \right\} + |\mathbb{T}_0| \omega(|\bar{t} - s|). \end{aligned}$$

Using the fact that $\sum_{i=0,\dots,N} |\mathbb{T}_i| = |\bar{t}-s|$ and $C_1 \geq 1$, and dividing by $|\bar{t}-s|$, we deduce that

$$-2\omega(C_1|\bar{t}-s|) \leq \varphi_t(\bar{t},0) + F_{\bar{H}_0(t)}(t,\varphi_x(t,0^+)).$$

Passing to the limit $s \to \bar{t}$, we deduce (6.13).

Step 2: the sub-solution property. Consider any test function φ such that

$$\varphi \geq u^* \text{ in } (0,+\infty)\times J \quad \text{and} \quad \varphi = u^* \text{ at } (\bar{t},0) \in (0,T)\times J, \quad \text{with} \quad \bar{t} \in (0,T).$$

Our goal is to show that

$$\varphi_t(\bar{t},0) + F_{\bar{H}_0(\bar{t})}(\bar{t},\varphi_x(\bar{t},0^+)) \leq 0. \tag{6.26}$$

Step 2.1: the basic optimal control inequality. Let $(t_n,x_n) \in (0,T)\times J$ such that

$$(t_n,x_n) \to (\bar{t},0) \quad \text{and} \quad u(t_n,x_n) \to u^*(\bar{t},0) \quad \text{as} \quad n \to +\infty.$$

From the dynamic programming principle, we get that for all $(s,y) \in (0,t_n)\times J$ and all $(X(\cdot),\alpha(\cdot)) \in \mathcal{T}_{s,y}^{t_n,x_n}$,

$$u(t_n,x_n) \leq E_s^{t_n}(X,\alpha) = u(s,X(s)) + \int_s^{t_n} \ell(\tau,X(\tau),\alpha(\tau))\,d\tau.$$

This implies

$$\varphi(t_n,x_n) - o_n(1) \leq \varphi(s,X(s)) + \int_s^{t_n} \ell(\tau,X(\tau),\alpha(\tau))\,d\tau$$

i.e.

$$K_s^{t_n}(X,\alpha) \leq o_n(1)$$

with $K_s^{t_n}(X,\alpha)$ defined in (6.15).

Step 2.2: freezing the coefficients. Using (6.20), this implies

$$\int_s^{t_n} d\tau\ \bar{\kappa}(\tau,X(\tau),\alpha(\tau)) = \bar{K}_s^{t_n}(X,\alpha) \leq o_n(1) + |t_n-s|\omega(o_n(1)+C_1|\bar{t}-s|) \tag{6.27}$$

with $\bar{\kappa}$ defined in (6.18).

Step 2.3: inequalities for $i_0 = 1,\dots,N$. For each $i=1,\dots,N$, let us choose some $\bar{\alpha}_i, \underline{\alpha}_i \in \mathbb{A}_i$ such that

$$b_i(\bar{t},0,\bar{\alpha}_i) < 0 \quad \text{and} \quad b_i(\bar{t},0,\underline{\alpha}_i) > 0. \tag{6.28}$$

We now fix some index $i_0 \in \{1,\dots,N\}$.

Assume first that $x_n \in J_j^*$ with $j \neq i_0$. Then we look for a solution with terminal condition $X^n(t_n) = x_n$, which solves backward the following ODE

$$\dot{X}^n(\tau) = b_j(\tau,X^n(\tau),\underline{\alpha}_j) \quad \text{for} \quad \tau < t_n$$

up to the first time τ_n^j where X^n reaches the junction point, where τ_n^j is precisely defined by

$$\tau_n^j \in (0,t_n) \quad \text{such that} \quad X^n(\tau_n^j)=0 \quad \text{and} \quad X^n(\tau)\in J_j^* \quad \text{for all} \quad \tau\in(\tau_n^j,t_n]. \tag{6.29}$$

Note that such a trajectory $X^n(\cdot)$ always exists, even if it may not be unique, because b_j is not Lipschitz in the space variable x. By assumption (6.28) and the continuity of b_j, we know that we will have $\tau_n^j \to \bar t$ as $n\to+\infty$. Then we consider some $\alpha^n(\cdot)\in L^\infty([s,t_n];\mathbb{A})$ such that

$$\left\{\begin{array}{ll} \alpha_{i_0}^n(\tau)=\bar\alpha_{i_0} & \text{if} \quad \tau\in[s,\tau_n^j],\\ \alpha_j^n(\tau)=\underline{\alpha}_j & \text{if} \quad \tau\in(\tau_n^j,t_n]. \end{array}\right.$$

Assume now that $x_n\in J_{i_0}$. In this case, we require

$$\alpha_{i_0}^n(\tau)=\bar\alpha_{i_0} \quad \text{for all} \quad \tau\in[s,t_n].$$

In both cases, we call $X^n(\cdot)$ a trajectory such that $(X^n,\alpha^n)\in\mathcal{T}_{s,X^n(s)}^{t_n,x_n}$.

Up to a subsequence, we get that X^n converges uniformly towards some X, and α^n converges to $\alpha=\bar\alpha_{i_0}$, such that (using (6.27)),

$$|\bar t-s|\left\{\varphi_t(\bar t,0)+\partial_{i_0}\varphi(\bar t,0)b_{i_0}(\bar t,0,\bar\alpha_{i_0})-\ell_{i_0}(\bar t,0,\bar\alpha_{i_0})\right\}=\bar K_s^{\bar t}(X,\alpha)\le|\bar t-s|\omega(C_1|\bar t-s|).$$

Dividing by $|\bar t-s|$ and passing to the limit $s\to\bar t$, and taking the supremum on $\bar\alpha_{i_0}\in\mathbb{A}_{i_0}$ such that $b_{i_0}(\bar t,0,\bar\alpha_{i_0})<0$, we get

$$\varphi_t(\bar t,0)+H_{i_0}^-(\bar t,\partial_{i_0}\varphi(\bar t,0))\le 0. \tag{6.30}$$

Step 2.4: inequality for $i_0=0$. We now assume that (6.26) does not hold true. Then (6.30) implies that

$$\varphi_t(\bar t,0)+H_0(\bar t)>0 \tag{6.31}$$

and

$$H_0(\bar t)=\bar H_0(\bar t)>\max_{i=1,\dots,N}H_i^-(\bar t,\partial_i\varphi(\bar t,0^+))\ge A_0(\bar t).$$

By continuity of $\bar H_0=\max(H_0,A_0)$ with A_0 continuous defined in (6.3), we deduce that there exists some $s_0<\bar t$ such that H_0 is continuous on $[s_0,\bar t]$. In particular, we have $\mathbb{A}_0(\tau)\ne\emptyset$ for all $\tau\in[s_0,\bar t]$. By Lemma 6.5, there exists a measurable selection $\bar\alpha_0\in L^\infty([s_0,\bar t];\mathbb{A}_0)$ such that

$$\bar\alpha_0(\tau)\in\mathbb{A}_0(\tau) \quad \text{and} \quad H_0(\tau)=-\ell_0(\tau,\bar\alpha_0(\tau)) \quad \text{for a.e.} \quad \tau\in[s_0,\bar t].$$

If $x_n\in J_j^*$, we now use the defintion of τ_n^j given in (6.29) and consider some $\alpha^n(\cdot)\in L^\infty([s_0,t_n];\mathbb{A})$ such that

$$\left\{\begin{array}{ll} \alpha_j^n(\tau)=\underline{\alpha}_j & \text{if} \quad \tau\in(\tau_n^j,t_n],\\ \alpha_0^n(\tau)=\bar\alpha_0(\tau) & \text{if} \quad \tau\in[s_0,\tau_n^j]. \end{array}\right.$$

If $x_n = 0$, then we simply choose some $\alpha^n(\cdot) \in L^\infty([s_0, t_n]; \mathbb{A})$ such that

$$\alpha_0^n(\tau) = \bar{\alpha}_0(\tau) \quad \text{if} \quad \tau \in [s_0, t_n].$$

Let $s \in [s_0, \bar{t})$. In any cases, we call again $X^n(\cdot)$ a trajectory such that $(X^n, \alpha^n) \in \mathcal{T}^{t_n, x_n}_{s, X^n(s)}$. Similarly to Step 2.3, up to a subsequence, we get that X^n converges uniformly towards $X = 0$, and α^n converges to $\alpha = \bar{\alpha}_{i_0}$, such that (using (6.27)):

$$\begin{aligned}
|\bar{t} - s|\omega(C_1|\bar{t} - s|) &\geq \bar{K}_s^{\bar{t}}(X, \alpha) \\
&= \int_s^{\bar{t}} d\tau \ \{\varphi_t(\bar{t}, 0) - \ell_0(\tau, \bar{\alpha}_0(\tau))\} \\
&= \int_s^{\bar{t}} d\tau \ \{\varphi_t(\bar{t}, 0) + H_0(\tau))\} \\
&\geq |\bar{t} - s| \{\varphi_t(\bar{t}, 0) + H_0(\bar{t}) - \omega(|\bar{t} - s|))\}
\end{aligned}$$

where ω still denotes some modulus of continuity of H_0 on $[s_0, \bar{t}]$. Dividing by $|\bar{t} - s|$ and passing to the limit $s \to \bar{t}$, we get

$$\varphi_t(\bar{t}, 0) + H_0(\bar{t}) \leq 0$$

which contradicts (6.31). This finally shows that (6.26) holds true.

Step 3: checking the initial condition and a priori bounds. From the fact that u_0 is continuous and the fact that b_i, ℓ_i are bounded for $i = 0, \ldots, N$, we deduce easily from the representation formula (6.8) that the value function u satisfies

$$u^*(0, x) = u_0(x) = u_*(0, x) \quad \text{for all} \quad x \in J.$$

Again from the representation formula (6.8), the fact that b_i, ℓ_i are bounded for $i = 0, \ldots, N$, and the fact that u_0 is globally Lipschitz continuous, we also easily see that there exists a constant $C > 0$ such that $|u(t, x) - u_0(x)| \leq Ct$. In particular

$$|u(t, x)| \leq C_T(1 + d(x, 0)) \quad \text{for all} \quad (t, x) \in [0, T] \times J. \tag{6.32}$$

Step 4: conclusion. The previous steps show that u solves (6.10) with initial condition (6.11). We also have the sublinear property (6.32). Then, we apply Proposition 6.6 (which is postponed) which claims that our PDE satisfies the assumptions of Corollary 5.9. This implies the indentification of the function u to the unique solution of (6.10), (6.11). This ends the proof of the theorem. □

We now turn to proofs of Lemma 6.5 and Proposition 6.6.

Proof of Lemma 6.5. We consider the map $f : [a, b] \times \mathbb{A}_0 \to \mathbb{R}^2$ defined by

$$f(\tau, \alpha_0) = (b_0(\tau, \alpha_0), H_0(\tau) + \ell_0(\tau, \alpha_0))$$

Recall that by (6.2), we have $\mathbb{A}_0 \subset \mathbb{R}^{d_0}$, with $\mathbb{A}_0$ compact. Then we define the multifunction $\Gamma : [a,b] \rightrightarrows \mathbb{R}^{d_0}$ defined by

$$\Gamma(\tau) = \{\alpha_0 \in \mathbb{A}_0, \quad f(\tau,\alpha_0) = (0,0)\}$$

Because f is continuous, $\Gamma(\tau)$ is closed. Moreover our assumptions guarantee that $\Gamma(\tau)$ is nonempty. We recall (see [39], page 314, beginning of section 2) that Γ is said to be $\mathcal{L}$-measurable (Lebesgue measurable) if and only if its graph

$$G(\Gamma) = \left\{(\tau,\alpha_0) \in [a,b] \times \mathbb{R}^{d_0}, \quad \alpha_0 \in \Gamma(\tau)\right\}$$

is $\mathcal{L} \otimes \mathcal{B}$-mesurable, *i.e.* belongs to the σ-algebra generated by the product of Lebesgue sets in $[a,b]$ and Borel sets in $\mathbb{R}^{d_0}$. Here $G(\Gamma) = f^{-1}((0,0))$ is a closed set of $[a,b] \times \mathbb{R}^{d_0}$, so this set is obviously $\mathcal{L} \otimes \mathcal{B}$-measurable. We now apply the mesurable selection result cited as the corollary on page 315 in [39]. This result states that for any $\mathcal{L}$-measurable multifunction $\Gamma : [a,b] \rightrightarrows \mathbb{R}^{d_0}$, which is closed-valued with $\Gamma(\tau)$ nonempty for almost every $\tau \in [a,b]$, there exists a $\mathcal{L}$-measurable function $\bar{\alpha}_0 : [a,b] \to \mathbb{R}^{d_0}$ such that

$$\bar{\alpha}_0(\tau) \in \Gamma(\tau) \quad \text{for almost every} \quad \tau \in [a,b]$$

This implies the result stated in the lemma and ends its proof. □

Proof of Proposition 6.6. We check successively all assumptions.
STEP 1: CHECKING (H0) AND (H3). We set

$$P = (t,x,p) \quad \text{and} \quad \Phi_i(\alpha_i, P) = pb_i(t,x,\alpha_i) - \ell_i(t,x,\alpha_i).$$

We recall that

$$H_i(P) = \sup_{\alpha_i \in \mathbb{A}_i} \Phi_i(\alpha_i, P) = \Phi_i(\bar{\alpha}_i(P), P).$$

Let $P' = (t',x',p')$. We assume that

$$|p|, |q| \leq L.$$

Using the fact that b_i, ℓ_i are uniformly continuous with respect to (t,x), uniformly with respect to $\alpha_i \in \mathbb{A}_i$, we deduce that there exists a modulus of continuity $\omega_{T,L}$ such that

$$H_i(P') \geq \Phi_i(\bar{\alpha}_i(P), P') \geq \Phi_i(\bar{\alpha}_i(P), P) - \omega_{T,L}(|P-P'|) = H_i(P) - \omega_{T,L}(|P-P'|).$$

Exchanging P and P', we get the reverse inequality, which yields

$$|H_i(P') - H_i(P)| \leq \omega_{T,L}(|P-P'|) \tag{6.33}$$

In particular, this gives the continuity of H_i.
STEP 2: CHECKING (H1). By assumption (6.1), there exists some $\delta > 0$ and controls $\alpha_i^{\pm} = \alpha_i^{\pm}(t,x)$ such that

$$\pm b_i(t,x,\alpha_i^{\pm}) \geq \delta > 0.$$

Using the fact that ℓ_i is bounded, this implies that

$$H_i(t,x,p) \geq \delta|p| - C \tag{6.34}$$

for some constant $C > 0$.
STEP 3: CHECKING (H2). Again, using the boundedness of b_i and ℓ_i, we get the uniform coercivity estimate

$$|H_i(t,x,p)| \leq C(|p|+1). \tag{6.35}$$

STEP 4: CHECKING (H4). The quasi-convexity of $H_i(t,x,\cdot)$ follows from its convexity.
STEP 5: CHECKING (H5). We write with $p' = p$, $x' = x$, $\bar{\alpha}_i := \bar{\alpha}_i(P')$

$$\begin{aligned} H_i(P') - H_i(P) &= \Phi(\bar{\alpha}_i(P'), P') - H_i(P) \\ &\leq \Phi(\bar{\alpha}_i, P') - \Phi(\bar{\alpha}_i, P) \\ &= p(b_i(t',x,\bar{\alpha}_i) - b_i(t,x,\bar{\alpha}_i)) - (\ell_i(t',x,\bar{\alpha}_i) - \ell_i(t,x,\bar{\alpha}_i)) \\ &\leq L|p||t'-t| + \bar{\omega}(|t'-t|) \\ &\leq L\delta^{-1}(C + \max(0, H_i(t,x,p)))|t'-t| + \bar{\omega}(|t'-t|) \end{aligned}$$

where in the fourth line, we have used the fact that b_i is L-Lipschitz continuous (by (6.12)) with respect to t, uniformly with respect to α_i. We have also used the fact that there exists a modulus of continuity $\bar{\omega}$ for ℓ_i with respect to (t,x), uniformly in α_i. In the fifth line, we have used the uniform coercivity estimate (6.34). The previous inequality implies easily (H5).
STEP 6: CHECKING (H6). Recall that H_i is uniformly coercive by (H1), and continuous by (H0). This implies that the map $t \mapsto \min H_i(t,0,\cdot)$ is also continuous. This implies the continuity of

$$A_0^0(t) = \max_{i=1,\ldots,N} \min H_i(t,0,\cdot).$$

STEP 7: CHECKING (A0). The continuity of $A_0(t) = \bar{H}_0(t)$ follows from (6.6).
STEP 8: CHECKING (A1) AND (A2). The bound on $A_0(t)$ and the uniform continuity of $A_0(t)$ are trivial since there is only one vertex.

This ends the proof of the proposition. □

7 Second application: study of Ishii solutions

This section is strongly inspired by the work [10] where one of the main contribution of the authors was to identify the maximal and minimal Ishii solutions (in any dimensions), in the framework of convex Hamiltonians, and using tools of optimal control theory. With our PDE theory in hands, we revisit this problem in dimension one, but for quasi-convex Hamiltonians (in the sense of (1.5)) that can be non-convex. As a by-product of our approach, we give a PDE characterization of both the maximal and the minimal Ishii solutions.

Remark 7.1. Combining results from Subsection 2.4 with the ones from this Section, we can easily see that for one-dimensional problems, the solutions in [9], [10], [38] and [37] fall naturally in our theoretical framework; they coincide with some A-flux-limited solutions for A well chosen.

7.1 The framework

Let us consider two Hamiltonians H_i for $i = 1, 2$ which are level-set convex in the sense of (1.5). In particular H_i is assumed to be minimal at p_i^0.

Ishii solutions on the real line. In [10], Ishii solutions are considered. A function u is said to be a Ishii sub-solution if its upper semi-continuous envelope u^* solves

$$\begin{cases} u_t + H_1(u_x) \leq 0 & \text{for } x < 0, \\ u_t + H_2(u_x) \leq 0 & \text{for } x > 0, \\ u_t + \min(H_1(u_x), H_2(u_x)) \leq 0 & \text{for } x = 0 \end{cases}$$

A function u is said to be a Ishii super-solution if its lower semi-continuous envelope u_* solves

$$\begin{cases} u_t + H_1(u_x) \geq 0 & \text{for } x < 0, \\ u_t + H_2(u_x) \geq 0 & \text{for } x > 0, \\ u_t + \max(H_1(u_x), H_2(u_x)) \geq 0 & \text{for } x = 0. \end{cases}$$

An Ishii solution is a function u which is both an Ishii sub-solution and an Ishii super-solution.

Translation of flux-limited solutions in the real line setting. The notion of solutions $\tilde{u}(t, x)$ from Section 2 on two branches $J_1 \cup J_2$ with two Hamiltonians

$$\tilde{H}_1(q) = H_1(-q) \quad \text{and} \quad \tilde{H}_2(q) = H_2(q)$$

is translated in the framework of the real line into functions u defined for $(t, x) \in [0, +\infty) \times \mathbb{R}$ by

$$u(t, x) = \begin{cases} \tilde{u}(t, x) & \text{for} \quad 0 \leq x \in J_2, \\ \tilde{u}(t, -x) & \text{for} \quad 0 \leq -x \in J_1. \end{cases}$$

Then $\tilde{u}$ solves (1.7) with Hamiltonians $\tilde{H}_i$ if and only if u solves

$$\begin{cases} u_t + H_1(u_x) = 0 & \text{for} \quad (t, x) \in (0, +\infty) \times (-\infty, 0), \\ u_t + H_2(u_x) = 0 & \text{for} \quad (t, x) \in (0, +\infty) \times (0, +\infty), \\ u_t + \check{F}_A(u_x(t, 0^-), u_x(t, 0^+)) = 0 & \text{for} \quad (t, x) \in (0, +\infty) \times \{0\} \end{cases} \tag{7.1}$$

with

$$\check{F}_A(q_1, q_2) = \max(A, H_1^+(q_1), H_2^-(q_2))$$

where

$$H_i^-(q) = \begin{cases} H_i(q) & \text{if} \quad q < p_i^0, \\ H_i(p_i^0) & \text{if} \quad q \le p_i^0 \end{cases} \quad \text{and} \quad H_i^+(q) = \begin{cases} H_i(p_i^0) & \text{if} \quad q \le p_i^0, \\ H_i(q) & \text{if} \quad q > p_i^0. \end{cases}$$

We have the following correspondance

$$\tilde{H}_1^{\pm}(p_1) = H_1^{\mp}(-p_1) \quad \text{and} \quad \tilde{H}_2^{\pm}(p_2) = H_2^{\pm}(p_2).$$

Viscosity inequalities are now naturally written by touching u with test functions $\phi : [0,+\infty) \times \mathbb{R} \to \mathbb{R}$ that are continuous, and C^1 in $[0,+\infty) \times (-\infty, 0]$ and in $[0,+\infty) \times [0,+\infty)$.

Ishii flux-limiters. We recall the quantity

$$A_0 = \max_{i=1,2} \left(\min_{q \in \mathbb{R}} H_i(q) \right) = \max_{i=1,2} H_i(p_i^0).$$

and define

$$A^* = \max_{q \in \mathrm{ch}\left[p_1^0, p_2^0\right]} (\min(H_1(q), H_2(q))).$$

with the chord

$$\mathrm{ch}\left[p_1^0, p_2^0\right] = [\min(p_1^0, p_2^0), \max(p_1^0, p_2^0)].$$

Then we set

$$A_I^+ = \max(A^*, A_0) \tag{7.2}$$

and

$$A_I^- = \begin{cases} A_I^+ & \text{if} \quad p_2^0 < p_1^0, \\ A_0 & \text{if} \quad p_2^0 \ge p_1^0, \end{cases} \tag{7.3}$$

Remark 7.2. Notice that even if the points of minimum p_i^0 of H_i may be not unique, it is easy to see that the quantities $A_I^{\pm}$ are uniquely defined.

These two quantities $A_I^{\pm}$ will play a crucial role here; they have been identified first in [10], in a different way (see below).

7.2 Identification of maximal and minimal Ishii solutions

The main result of this section is the following.

Theorem 7.3 (Identification of maximal and minimal Ishii solutions). *We assume that the Hamiltonians H_i satisfy* (1.5) *for $i = 1, 2$. We have $A_I^- \le A_I^+$ and the following holds.*

i) (Ishii sub-solution) *Every Ishii sub-solution is a $\check{F}_{A_I^-}$-sub-solution.*

ii) (Ishii super-solution) *Every Ishii super-solution is a $\check{F}_{A_I^+}$-super-solution.*

iii) (Particular Ishii solutions) *Every $\check{F}_A$-solution is a Ishii solution if $A \in \left[A_I^-, A_I^+\right]$.*

iv) (Maximal and minimal Ishii solutions) *For a given uniformly continuous initial data, the $\check{F}_{A_I^+}$-solution is the minimal Ishii solution, and the $\check{F}_{A_I^-}$-solution is the maximal Ishii solution. Moreover the Ishii solution is unique if and only if $A_I^+ = A_I^-$.*

We prove successively i)-iv) from Theorem 7.3.

Proof of Theorem 7.3-i). Let u be a Ishii sub-solution. We want to check that u is a $\check{F}_{A_I^-}$-sub-solution. Lemma 2.18 implies the "weak continuity" condition. The only difficulty is on the junction point $x = 0$. If $A_I^- = A_0$, then the result follows from Theorem 2.7 i).

Assume now that

$$A_I^- > A_0.$$

Then $A_I^- = A^*$, and $p_2^0 < p_1^0$. In particular, we can choose $p^* \in [p_2^0, p_1^0]$ such that

$$H_1(p^*) = H_1^+(p^*) = A^* = A_I^- = H_2(p^*) = H_2^-(p^*). \tag{7.4}$$

Now from Theorem 2.7 i), we see that, in order to show that u is a $\check{F}_{A_I^-}$-sub-solution, it is sufficient to consider a test function φ touching u from above at $(t_0, 0)$ for $t_0 > 0$, with

$$\varphi(t, x) = \psi(t) + p^* x$$

with $\psi \in C^1$, and to show that

$$\varphi_t + A_I^- \leq 0 \quad \text{at} \quad (t_0, 0). \tag{7.5}$$

Indeed, such φ is now an admissible test function for Ishii sub-solutions. So we deduce that

$$\varphi_t + \min(H_1^+(\varphi_x(t_0, 0^-)),\ H_2^-(\varphi_x(t_0, 0^+))) \leq 0 \quad \text{at} \quad (t_0, 0)$$

which implies (7.5). We conclude that u is a $\check{F}_{A_I^-}$-sub-solution and this ends the proof. □

Proof of Theorem 7.3-ii). Let u be a Ishii super-solution. We want to show that u is a $\check{F}_{A_I^+}$-super-solution.

Step 1: preliminaries. We distinguish two cases.
CASE 1: $A^* \geq A_0$. Then we have $A_I^+ = A^*$. In particular, there exists $p^* \in \text{ch}\,[p_1^0, p_2^0]$ such that

$$A^* = H_1(p^*) = H_2(p^*). \tag{7.6}$$

We set

$$\varphi(t, x) := \psi(t) + p^* x =: \tilde{\varphi}(t, x) \tag{7.7}$$

with $\psi \in C^1$.
CASE 2: $A^* < A_0$. This implies that there is a unique $\alpha \in \{1,2\}$ such that

$$A_I^+ = A_0 = H_\alpha(p_\alpha^0)$$

and for $\bar{\alpha} \in \{1,2\} \setminus \{\alpha\}$ we have

$$H_\alpha(p_\alpha^0) > H_{\bar{\alpha}}(p_\alpha^0).$$

In particular,

$$\max(H_\alpha(p_\alpha^0), H_{\bar{\alpha}}(p_\alpha^0)) = A_I^+. \tag{7.8}$$

If $\alpha = 1$, then we set $(p_1, p_2) = (p_1^0, \pi_2^+(A_0))$; if $\alpha = 2$, then we set $(p_1, p_2) = (\pi_1^-(A_0), p_2^0)$. We remark that we have

$$H_2(p_2) = H_2^+(p_2) = A_0 = A_I^+ = H_1(p_1) = H_1^-(p_1)$$

and

$$p_2 > p_1.$$

We set

$$\varphi(t,x) := \psi(t) + p_1 x 1_{\{x<0\}} + p_2 x 1_{\{x>0\}} \geq \tilde{\varphi}(t,x) := \psi(t) + p_\alpha^0 x \tag{7.9}$$

with $\psi \in C^1$.

Step 2: conclusion. Now from Theorem 2.7 iii), we see that, in order to show that u is a $\check{F}_{A_I^+}$-super-solution, it is sufficient to consider a test function φ (given either in (7.7) in case 1 or (7.9) in case 2) touching u from below at $(t_0, 0)$ for $t_0 > 0$, and to show that

$$\varphi_t + A_I^+ \geq 0 \quad \text{at} \quad (t_0, 0) \tag{7.10}$$

Because we have $\varphi \geq \tilde{\varphi}$ with equality at $(t_0, 0)$, we deduce that $\tilde{\varphi}$ is an admissible test function for the Ishii super-solution u. Therefore, we have

$$\tilde{\varphi}_t + \max(H_1(\tilde{\varphi}_x), H_2(\tilde{\varphi}_x)) \geq 0 \quad \text{at} \quad (t_0, 0)$$

Using either (7.6) in case 1, or (7.8) in case 2, we deduce that

$$\psi_t + A_I^+ \geq 0 \quad \text{at} \quad (t_0, 0)$$

which implies (7.10). This implies that u is a $\check{F}_{A_I^+}$-super-solution and ends the proof. □

We now state and prove a proposition which is more precise than Theorem 7.3-iii).

Proposition 7.4 (Relation between $\check{F}_A$ and Ishii sub/super-solutions)**.** *Under the assumptions of Theorem 7.3, every $\check{F}_A$-subsolution (resp. $\check{F}_A$-super-solution) is a Ishii sub-solution (resp. Ishii super-solution) if $A \geq A_I^-$ (resp. $A \leq A_I^+$).*

Moreover for every $A \in \left[A_0, A_I^-\right)$, there exists a $\check{F}_A$-sub-solution which is not a Ishii sub-solution. For every $A > A_I^+$, there exists a $\check{F}_A$-super-solution which is not a Ishii super-solution.

Proof. We treat successively sub-solutions and super-solutions.

SUB-SOLUTIONS. Let u be a $\check{F}_A$-sub-solution with $A \geq A_I^-$. Consider a C^1 function ϕ touching u from above at $(t, 0)$ for some $t > 0$. Then

$$\lambda + \check{F}_A(q, q) \leq 0$$

where $\lambda = \partial_t \phi(t, 0)$ and $q = \partial_x \phi(t, 0)$. In particular, $\lambda + A \leq 0$. We want to prove that

$$\lambda + \min(H_1(q), H_2(q)) \leq 0.$$

If $q \leq p_2^0$, then

$$\min(H_1(q), H_2(q)) \leq H_2^-(q) \leq \check{F}_A(q, q) \leq -\lambda.$$

Similarly, if $q \geq p_1^0$, then

$$\min(H_1(q), H_2(q)) \leq H_1^+(q) \leq \check{F}_A(q, q) \leq -\lambda.$$

If $p_2^0 < p_1^0$, and $q \in [p_2^0, p_1^0]$, then by definition of A^*, we have

$$\min(H_1(q), H_2(q)) \leq A^* \leq A_I^+ = A_I^- \leq A \leq -\lambda.$$

This shows that u is a Ishii sub-solution.

If $A^* \leq A_0$ or $p_2^0 \geq p_1^0$, there is nothing additional to prove. Assume now that $p_2^0 < p_1^0$ with $A_I^- = A^* > A_0$, and we claim that for any $A \in \left[A_0, A_I^-\right) = [A_0, A^*)$, there exists a $\check{F}_A$-sub-solution which is not an Ishii sub-solution. Indeed, let us consider $p^* \in [p_2^0, p_1^0]$ such that

$$A^* = H_1(p^*) = H_2(p^*).$$

Then there exists $p_2^0 \leq p_2 < p^* < p_1 \leq p_1^0$ such that

$$A = H_1(p_1) = H_2(p_2) = \check{F}_A(p_1, p_2) \tag{7.11}$$

Let us now consider

$$u(t, x) = -At + p_1 x 1_{\{x<0\}} + p_2 x 1_{\{x \geq 0\}}$$

In particular u is $\check{F}_A$-sub-solution because of (7.11). Now the test function $\phi(t, x) = -At + p^* x$ touches u at $(t, 0)$ from above and does not satisfy the inequality

$$\partial_t \phi(t, 0) + \min(H_1(\partial_x \phi(t, 0)), H_2(\partial_x \phi(t, 0))) \leq 0.$$

This shows that u is not a Ishii sub-solution.

Super-Solutions. Let u be a $\check{F}_A$-super-solution with $A \leq A_I^+$. Consider a C^1 function $\phi : \mathbb{R} \to \mathbb{R}$ touching u from below at $(t, 0)$ for some $t > 0$. Then

$$\lambda + F_A(q, q) \geq 0$$

where $\lambda = \partial_t \phi(t, 0)$ and $q = \partial_x \phi(t, 0)$. Without loss of generality, we can assume that $A \geq A_0$. We want to prove that

$$\lambda + \max(H_1(q), H_2(q)) \geq 0.$$

If $F_A(q, q) = A$, then we deduce from Lemma 7.5 below that

$$0 \leq \lambda + A \leq \lambda + A_I^+ \leq \lambda + \max(H_1(q), H_2(q)).$$

If now $F_A(q, q) = H_1^+(q)$, then

$$0 \leq \lambda + F_A(q, q) \leq \lambda + H_1(q) \leq \lambda + \max(H_1(q), H_2(q)).$$

If finally $F_A(q, q) = H_2^-(q)$, then

$$0 \leq \lambda + F_A(q, q) \leq \lambda + H_2(q) \leq \lambda + \max(H_1(q), H_2(q)).$$

This shows that u is a Ishii super-solution.

Assume next that $A > A_I^+$. If $A^* \geq A_0$, let $p^* \in \mathrm{ch}\,[p_1^0, p_2^0]$ such that

$$A^* = H_1(p^*) = H_2(p^*).$$

Let us choose an index $\alpha \in \{1, 2\}$ such that

$$\max_{i=1,2} H_i(p_i^0) = H_\alpha(p_\alpha^0).$$

Then we set

$$\bar{p} = \begin{cases} p^* & \text{if} \quad A^* \geq A_0, \\ p_1^0 & \text{if} \quad A^* < A_0 \quad \text{and} \quad \alpha = 1, \\ p_2^0 & \text{if} \quad A^* < A_0 \quad \text{and} \quad \alpha = 2. \end{cases}$$

In particular we have

$$\max(H_1(\bar{p}), H_2(\bar{p})) = A_I^+. \tag{7.12}$$

Then for $A > A_I^+$, there exist p_1 and p_2 such that

$$p_2 \geq \max(p_1^0, p_2^0) \geq \bar{p} \geq \min(p_1^0, p_2^0) \geq p_1$$

and

$$H_2(p_2) = A = H_1(p_1).$$

Let us now define

$$u(t,x) = -At + p_1 x 1_{\{x<0\}} + p_2 x 1_{\{x\geq 0\}}.$$

Then u is a $\check{F}_A$-super-solution because $\check{F}_A(p_1,p_2) = A$. Now the test function $\phi(t,x) = -At + \bar{p}x$ touches u at $(t,0)$ from below and does not satisfy the inequality

$$\partial_t \phi(t,0) + \max(H_1(\partial_x \phi(t,0)), H_2(\partial_x \phi(t,0))) \geq 0$$

because of (7.12). This shows that u is not a Ishii super-solution. This achieves the proof. □

In the previous proof, we used the following elementary lemma.

Lemma 7.5 (Bound from above for A_I^+)**.** *For all $q \in \mathbb{R}$, $A_I^+ \leq \max(H_1(q), H_2(q))$.*

Proof. We recall that $A_I^+ = \max(A^*, A_0)$. Assume first that $\max(A^*, A_0) = A_0$, then $A_0 = \min H_\alpha$ for some $\alpha \in \{1,2\}$. In particular, for all $q \in \mathbb{R}$, we have $A_I^+ = A_0 \leq H_\alpha(q) \leq \max(H_1(q), H_2(q))$.
If now $\max(A^*, A_0) = A^* > A_0$, then there exists $p^* \in [p_i^0, p_j^0]$ for some $i,j \in \{1,2\}$ $(i \neq j)$, such that

$$A^* = H_i(p^*) = H_j(p^*).$$

Moreover, H_j is non-increasing in $(-\infty, p^*]$ hence

$$H_j(q) \geq A^* \text{ for } q \leq p^*;$$

similarly, H_i is non-decreasing in $[p^*, +\infty)$ hence

$$H_i(q) \geq A^* \text{ for } q \geq p^*.$$

This implies the expected inequality. □

We finally state a proposition which implies Theorem 7.3-iv).

Corollary 7.6 (Conditions for uniqueness of Ishii solution)**.** *We work under the assumptions of Theorem 7.3. Recall that $A_I^+ \geq A_I^-$, and let g be a uniformly continuous initial data.*

- *If $A_I^+ = A_I^-$, then there is uniqueness of the Ishii solution with initial data g.*
- *If $A_I^+ > A_I^-$, then there exists a Lipschitz continuous initial data g such that there are two different Ishii solutions with the same initial data g.*

Proof. If $A_I^+ = A_I^-$, then Theorem 7.3 i) and ii) imply that every Ishii solution u is a $\check{F}_A$-solution for $A = A_I^+$. Given some uniformly continuous initial data, such a solution is then unique.

On the contrary, if $A_I^+ > A_I^-$, then

$$U^-(t,x) = -At + p_1 x 1_{\{x<0\}} + p_2 x 1_{\{x\geq 0\}}$$

is a $\check{F}_A$-solution with $A = A_I^+$ with initial data $g(x) = U^-(0,x)$ if

$$A_I^+ = A = H_1(p_1) = H_2(p_2), \quad p_2 \geq p_2^0, \quad p_1 \leq p_1^0.$$

On the other hand, U^- is not a $\check{F}_{A_I^-}$-solution because $\check{F}_{A_I^-}(p_1,p_2) = A_I^- < A_I^+$. □

7.3 Link with regional control

In this subsection, we shed light on the consequence of our results in the interpretation of the results from [10] when both frameworks coincide. Roughly speaking, the one-dimensional framework from [10] reduces to our framework with two branches. In this case, the value function U^- defined in [10, Eq. (2.7)] (see also (7.15) in the present paper) and characterized in [10, Theorem 4.4] corresponds to the unique solution of (1.7) for $A = A_I^+$. Similarly, the function U^+ defined in [10, Eq. (2.8)] (see also (7.16) in the present paper) corresponds to the unique solution of (1.7) for $A = A_I^-$. This is shown in this subsection. We also provide the link between our definition of A_I^+ and A_I^- and the tangential Hamiltonians introduced in [10], coming from optimal control theory.

7.3.1 The optimal control framework

The one dimensional framework of [10] corresponds to

$$\Omega_1 = (-\infty, 0), \quad \mathcal{H} = \{0\}, \quad \Omega_2 = (0, +\infty).$$

In this case, $(\mathbf{H}_\Omega)$ in [10] is satisfied. We refer to this framework as *the common framework*.

Hamiltonians. As far as the Hamiltonian is concerned, the (t, x)-dependence is not relevant for what we discuss now; for this reason we consider the simplified case of convex Hamiltonians given for $i = 1, 2$ by

$$H_i(p) = \sup_{\alpha_i \in A_i} \left(-b_i(\alpha_i)p - \ell_i(\alpha_i)\right)$$

for some compact metric space A_i and $b_i, \ell_i : A_i \to \mathbb{R}$. In this simplified framework, $(\mathbf{H}_C)$ reduces to the following assumptions for $i = 1, 2$:

$$\text{(7.13)} \qquad \begin{cases} b_i \text{ and } \ell_i \text{ are continuous and bounded} \\ \{(b_i(\alpha_i), \ell_i(\alpha_i)) : \alpha_i \in A_i\} \text{ is closed and convex} \\ B_i = \{-b_i(\alpha_i) : \alpha_i \in A_i\} \text{ contains } [-\delta, \delta]. \end{cases}$$

In particular, we see that B_i is a compact interval. Introducing the Legendre-Fenchel transform L_i of H_i, it is possible to see that this problem can be reformulated by assuming that for $i = 1, 2$

$$H_i(p) = \sup_{q \in B_i} (qp - L_i(q))$$

where $L_i : B_i \to \mathbb{R}$ is convex where we recall that B_i is a compact interval containing $[-\delta, \delta]$. Indeed the graph of L_i on B_i is the lower boundary of the closed convex set $\{(b_i(\alpha_i), \ell_i(\alpha_i)) : \alpha_i \in A_i\}$ in the plane $\mathbb{R}^2$. In particular, we see that H_i is convex, Lipschitz continuous and $H_i(p) \to +\infty$ as $|p| \to +\infty$. This last fact comes from the fact that $\pm\delta \in B_i$. Moreover H_i reaches its minimum at any convex subgradient p_i^0 of L_i at 0 and satisfies

$$\begin{cases} H_i & \text{is non-increasing on} \quad (-\infty, p_i^0], \\ H_i & \text{is non-decreasing on} \quad [p_i^0, +\infty). \end{cases}$$

Hence, H_i satisfies (1.5).

Tangential Hamiltonians. Using notation similar to the one of [10], we define

$$\hat{A} = A_1 \times A_2 \times [0,1]$$

Now, for $a = (\alpha_1, \alpha_2, \mu) \in \hat{A}$, we define

$$\begin{cases} b_{\mathcal{H}}(a) = \mu b_1(\alpha_1) + (1-\mu) b_2(\alpha_2), \\ \ell_{\mathcal{H}}(a) = \mu \ell_1(\alpha_1) + (1-\mu)\ell_2(\alpha_2) \end{cases}$$

and set

$$\hat{A}_0 = \{a = (\alpha_1, \alpha_2, \mu) \in \hat{A} : 0 = b_{\mathcal{H}}(a)\},$$
$$\hat{A}_0^{\text{reg}} = \{a = (\alpha_1, \alpha_2, \mu) \in \hat{A} : b_1(\alpha_1) \leq 0, b_2(\alpha_2) \geq 0 \text{ and } 0 = b_{\mathcal{H}}(a)\}.$$

In the common framework, the tangential Hamiltonians given in [10] reduce to constants, and we can see that we can write them as follows

$$\begin{cases} H_T & = \sup_{a=(\alpha_1,\alpha_2,\mu)\in \hat{A}_0} (-\ell_{\mathcal{H}}(a)), \\ H_T^{\text{reg}} & = \sup_{a=(\alpha_1,\alpha_2,\mu)\in \hat{A}_0^{\text{reg}}} (-\ell_{\mathcal{H}}(a)). \end{cases} \tag{7.14}$$

The value functions U^- and U^+. We consider the following initial condition

$$u(0,x) = g(x) \quad \text{for} \quad x \in \mathbb{R}$$

with g globally Lipschitz continuous.

For $a = (\alpha_1, \alpha_2, \mu) \in \hat{A}$, and for $x \in \mathbb{R}$, we set

$$b(x,a) = \begin{cases} b_1(\alpha_1) & \text{if} \quad x \in (-\infty, 0) = \Omega_1, \\ b_2(\alpha_2) & \text{if} \quad x \in (0, +\infty) = \Omega_2, \\ b_{\mathcal{H}}(a) & \text{if} \quad x \in \mathcal{H} = \{0\} \end{cases}$$

and

$$\ell(x,a) = \begin{cases} \ell_1(\alpha_1) & \text{if} \quad x \in (-\infty, 0) = \Omega_1, \\ \ell_2(\alpha_2) & \text{if} \quad x \in (0, +\infty) = \Omega_2, \\ \ell_{\mathcal{H}}(a) & \text{if} \quad x \in \mathcal{H} = \{0\}\,. \end{cases}$$

We consider admissible controlled dynamics starting from the point $(0, x)$ and ending at time $t > 0$ defined by

$$\mathcal{T}_{t,x} = \left\{ \begin{array}{l} (X(\cdot), a(\cdot)) \in \mathrm{Lip}(0, t; \mathbb{R}) \times L^\infty(0, t; \hat{A}) \quad \text{such that} \\ \left\{ \begin{array}{l} X(0) = x, \\ \dot{X}(s) = b(X(s), a(s)) \quad \text{for a.e.} \quad s \in (0, t) \end{array} \right. \end{array} \right\}$$

and define the set of regular controlled dynamics as

$$\mathcal{T}_{t,x}^{reg} = \left\{ \begin{array}{l} (X(\cdot), a(\cdot)) \in \mathcal{T}_{t,x} \quad \text{such that} \\ a(s) \in \hat{A}_0^{reg} \quad \text{for a.e.} \quad s \in (0, t) \quad \text{such that} \quad X(s) = 0 \end{array} \right\}.$$

Notice that the definition of $\mathcal{T}_{t,x}$ differs from the one given in (6.7), where now X takes the value x at time 0 instead of at time t. Then we define

$$U^-(x, t) = \inf_{(X(\cdot), a(\cdot)) \in \mathcal{T}_{t,x}} \left\{ g(X(t)) + \int_0^t \ell(X(s), a(s))\, ds \right\} \tag{7.15}$$

and

$$U^+(x, t) = \inf_{(X(\cdot), a(\cdot)) \in \mathcal{T}_{t,x}^{reg}} \left\{ g(X(t)) + \int_0^t \ell(X(s), a(s))\, ds \right\}. \tag{7.16}$$

Then we have the following characterization of U^- and U^+:

Theorem 7.7 (Characterization of U^- and U^+)**.** *Under the previous assumptions, U^- is the unique $\check{F}_A$-solution with initial data g for $A = H_T$. Similarly, U^+ is the unique $\check{F}_A$-solution with initial data g for $A = H_T^{reg}$.*

Proof. Theorem 7.7 is a straightforward application of Theorem 6.4. □

7.3.2 Tangential Hamiltonians and Ishii flux-limiters

In this paragraph, we show that the tangential Hamiltonians from [10] coincide with the Ishii flux-limiters.

We start with defining

$$\begin{aligned} \mathcal{A} &= B_1 \times B_2 \times [0, 1], \\ \mathcal{A}_0 &= \{(v_1, v_2, \mu) \in \mathcal{A} : v_1 v_2 \le 0 \text{ and } 0 = \mu v_1 + (1 - \mu) v_2\}, \\ \mathcal{A}_0^{\mathrm{reg}} &= \{(v_1, v_2, \mu) \in \mathcal{A} : v_1 \le 0, v_2 \ge 0 \text{ and } 0 = \mu v_1 + (1 - \mu) v_2\}. \end{aligned}$$

Then we can see (with $v_i = b_i(\alpha_i)$) that the tangential Hamiltonians given in (7.14) can be written as follows

$$\begin{aligned} H_T &= \sup_{(v_1, v_2, \mu) \in \mathcal{A}_0} (-\mu L_1(v_1) - (1 - \mu) L_2(v_2)), \\ H_T^{\mathrm{reg}} &= \sup_{(v_1, v_2, \mu) \in \mathcal{A}_0^{\mathrm{reg}}} (-\mu L_1(v_1) - (1 - \mu) L_2(v_2)). \end{aligned}$$

Indeed, we use here the construction of L_1 and L_2 explained in the previous Paragraph 7.3.1. In particular, for $-b_i \in B_i$, there exists $\alpha_i \in A_i$ such that $v_i = -b(\alpha_i)$. There are several possible α_i and hence several possible $\ell_i(\alpha_i)$. The construction $L_i(v_i) = \ell_i(\alpha_i^*)$ which is smaller than all the possible $\ell_i(\alpha_i)$.

Proposition 7.8 (Characterization of H_T)**.**

$$H_T = A_I^+.$$

Proof. Reduction. Remark that there exists $p_c \in \mathbb{R}$ such that $A_I^+ = H_{i_c}(p_c)$ for some $i_c \in \{1, 2\}$. We then consider

$$\tilde{H}_i(v_i) = H_i(p_c + v_i) - A_I^+.$$

In this case, using obvious notation, $\tilde{A}_I^+ = 0$ and $\tilde{p}_c = 0$. Remark that

$$\begin{aligned}
\tilde{L}_i(v_i) &= \sup_q (v_i q - \tilde{H}_i(q)) \\
&= \sup_q (v_i q - H_i(p_c + q)) + A_I^+ \\
&= \sup_q (v_i q - H_i(q)) - p_c v_i + A_I^+ \\
&= L_i(v_i) - p_c v_i + A_I^+.
\end{aligned}$$

Then

$$\begin{aligned}
\tilde{H}_T &= \sup_{(v_1, v_2, \mu) \in A_0} (-\mu \tilde{L}_1(v_1) - (1-\mu)\tilde{L}_2(v_2)) \\
&= \sup_{(v_1, v_2, \mu) \in A_0} (-\mu L_1(v_1) - (1-\mu) L_2(v_2)) - A_I^+ \\
&= H_T - A_I^+.
\end{aligned}$$

Hence, it is enough to prove

$$\tilde{H}_T = 0.$$

From now on, we assume that $A_I^+ = 0$ and $p_c = 0$. We distinguish two cases.

First case. Assume first that $0 = A_I^+ = A^* \geq A_0$. Then $0 = A^* = H_1(p^*) = H_2(p^*) = H_{i_c}(p_c)$ with $p^* \in \mathrm{ch}\,[p_1^0, p_2^0]$. Choosing initially $p_c = p^*$, we can assume that $A^* = H_1(0) = H_2(0) = 0$. In particular, $L_1 \geq 0$ and $L_2 \geq 0$. Hence $H_T \leq 0$. To get the reverse inequality, we observe that there exists $v_i^* \in \partial H_i(0)$, $i = 1, 2$, with

$$v_1^* v_2^* \leq 0.$$

Indeed, if this is not true, this implies that for all $v_i \in \partial H_i(0)$,

$$v_1 v_2 > 0$$

which is impossible because the graphs of H_1 and H_2 cross at p^* and p^* lies between p_1^0 and p_2^0 where H_1 and H_2 reach their minimum.

Pick now $\mu \in [0,1]$ such that $\mu v_1^* + (1-\mu)v_2^* = 0$. Then $(v_1^*, v_2^*, \mu) \in \mathcal{A}_0$ and consequently,

$$H_T \geq -\mu L_1(v_1^*) - (1-\mu)L_2(v_2^*) = \mu H_1(0) + (1-\mu)H_2(0) = 0.$$

Hence $H_T = 0$ in the first case, as desired.

Second case. We now assume that $0 = A_I^+ = A_0 > A^*$. In this case, there exists $a \in \{1,2\}$ such that

$$\min H_a = H_a(0) = 0,$$

with the initial choice $p_c = p_a^0$. This implies in particular

$$L_a \geq L_a(0) = 0.$$

Moreover, for $b \neq a$,

$$\min L_b = -H_b(0) \geq 0,$$

where we have used the fact that $A^* < A_0$. Hence, $L_a \geq 0$ and $L_b \geq 0$ and consequently, $H_T \leq 0$. Moreover with $v_i^* \in \partial H_i(0)$, we have, $(0, v_2^*, 1) \in \mathcal{A}_0$ when $a = 1$ and $(v_1^*, 0, 0) \in \mathcal{A}_0$ when $a = 2$. Hence, in both cases,

$$H_T \geq -L_a(0) = 0.$$

Hence $H_T = 0$ in the second case too. The proof is now complete. □

Proposition 7.9 (Characterization of H_T^{reg})**.**

$$H_T^{reg} = A_I^-.$$

Proof. The proof is similar to the proof of Proposition 7.8. We make precise how to adapt it.

Reduction. The reduction to the case $A_I^- = 0$ and $p_c = 0$ is completely analogous. We now have to prove that $H_T^{reg} = 0$.

First case. Assume first that $0 = A_I^- = A^* \geq A_0$. Note that this case only makes sense either when $p_2^0 < p_1^0$ or when $p_2^0 \geq p_1^0$ and $0 = A_I^- = A^* = A_0$. Similarly, we get $H_T^{reg} \leq 0$. To get the reverse inequality, we observe that there exists $v_i^* \in \partial H_i(0)$, $i = 1, 2$, with

$$v_1^* v_2^* \leq 0.$$

We deduce that we can choose $v_2^* \geq 0$ and $v_1^* \leq 0$, both in the case $p_2^0 < p_1^0$ and the case $p_2^0 \geq p_1^0$ and $0 = A_I^- = A^* = A_0$. This implies that we can find $(v_1^*, v_2^*, \mu) \in \mathcal{A}_0^{reg}$ and similarly, we conclude that $H_T^{reg} \geq 0$. Hence $H_T = 0$ in the first case, as desired.

Second case. We now assume that $0 = A_I^- = A_0$. We set again for some $a \in \{1,2\}$:

$$\min H_a = H_a(0) = 0.$$

From our definition of a, we have again

$$L_a \geq L_a(0) = 0 \quad \text{and} \quad p_a^0 = 0.$$

We first prove that $H_T^{reg} \leq 0$. In order to do so, we now distinguish three subcases.

Assume first $p_2^0 < p_1^0$. Then we can assume that $A_0 > A^*$ (otherwise we have $A_0 = A^*$ and we fall into the first case). Then we deduce, as in the proof of Proposition 7.8, that $H_T^{reg} \leq 0$.

Assume now that $p_2^0 \geq p_1^0$ and $a = 1$. We deduce that $0 = p_1^0 \leq p_2^0$. But because H_2 is minimal at p_2^0, we have $0 \in \partial H_2(p_2^0)$, and we deduce that $0 \leq p_2^0 \in \partial L_2(0)$. This implies that $L_2 \geq L_2(0) = -H_2(p_2^0) \geq 0$ on $\mathbb{R}^+$. By definition of H_T^{reg}, this implies that $H_T^{reg} \leq 0$.

Assume finally that $p_2^0 \geq p_1^0$ and $a = 2$. This subcase is symmetric with respect to the previous one. We deduce that $0 = p_2^0 \geq p_1^0$. But because H_1 is minimal at p_1^0, we deduce that $0 \geq p_1^0 \in \partial L_1(0)$. This implies that $L_1 \geq L_1(0) = -H_1(p_1^0) \geq 0$ on $\mathbb{R}^-$. Again, by definition of A_I^-, this implies that $A_I^- \leq 0$.

We now prove that $H_T^{reg} \geq 0$. To do so pick some $(0, v_2, 1) \in \mathcal{A}_0^{reg}$ when $a = 1$ and some $(v_1, 0, 0) \in \mathcal{A}_0^{reg}$ when $a = 2$. Hence, in both cases, we get

$$H_T^{reg} \geq -L_a(0) = 0.$$

Hence $H_T = 0$ in the second case too. The proof is now complete. $\square$

8 Third application: a homogenization result for a network

In this section, we present an application of the comparison principle of viscosity sub- and super-solutions on networks.

8.1 A homogenization problem

We consider the simplest periodic network generated by $\varepsilon\mathbb{Z}^d$. It is in fact a lattice. Hence, the network (or lattice) is naturally embedded in $\mathbb{R}^d$. Let us be more precise now. At scale $\varepsilon = 1$, the edges are the following subsets of $\mathbb{R}^d$: for $k, l \in \mathbb{Z}^d$, $|k - l| = 1$,

$$e_{k,l} = \{\theta k + (1-\theta)l : \theta \in [0,1]\}.$$

If $(e_1, \ldots, e_d)$ denotes the canonical basis of $\mathbb{R}^d$, then for $l = k + e_i$, $e_{k,l}$ is oriented in the direction of e_i. The network $\mathcal{N}_\varepsilon$ at scale $\varepsilon > 0$ is the one corresponding to

$$\begin{cases} \mathcal{E}_\varepsilon = \{\varepsilon e_{k,l}, k, l \in \mathbb{Z}^d, |k-l| = 1\} \\ \mathcal{V}_\varepsilon = \varepsilon\mathbb{Z}^d \end{cases}$$

endowed with the metric induced by the Euclidian norm. We next consider the following "oscillating" Hamilton-Jacobi equation on this network

$$(8.1) \qquad \begin{cases} u_t^\varepsilon + H_{\frac{e}{\varepsilon}}(u_x^\varepsilon) = 0, & t > 0,\ x \in e^*, e \in \mathcal{E}_\varepsilon, \\ u_t^\varepsilon + F_A(\frac{x}{\varepsilon}, u_x^\varepsilon) = 0, & t > 0,\ x \in \mathcal{V}_\varepsilon \end{cases}$$

(for some $A \in \mathbb{R}$) subject to the initial condition

$$(8.2) \qquad u^\varepsilon(0, x) = u_0(x), \qquad x \in \mathcal{N}_\varepsilon.$$

Remark 8.1. In this section, we choose the simplest periodic homogenization problem but much more can be done. For instance, the cell can be larger or have a different shape, Hamiltonians can depend on x etc.

For $m \in \mathbb{Z}^d$, it is convenient to define

$$\varepsilon e_{k,l} + \varepsilon m = \varepsilon e_{k+m,l+m}.$$

Assumptions on H for the homogenization problem For each $e \in \mathcal{N}_1$, we associate a Hamiltonian H_e and we assume

- **(H'0)** (Continuity) For all $e \in \mathcal{E}_1$, $H_e \in C(\mathbb{R})$.
- **(H'1)** (Coercivity) $e \in \mathcal{E}_1$,
$$\liminf_{|q| \to +\infty} H_e(q) = +\infty.$$
- **(H'2)** (Quasi-convexity) For all $e \in \mathcal{E}_1$, there exists a $p_e^0 \in \mathbb{R}$ such that
$$\begin{cases} H_e & \text{is nonincreasing on} \quad (-\infty, p_e^0], \\ H_e & \text{is nondecreasing on} \quad [p_e^0, +\infty). \end{cases}$$
- **(H'3)** (Periodicity) For all $m \in \mathbb{Z}^d$, $H_{e+m}(p) = H_e(p)$.

A homogenization result The goal of this section is to prove the following convergence result for the oscillating Hamilton-Jacobi equation.

Theorem 8.2 (Homogenization of a network)**.** *Assume* (H'0)-(H'3). *Let u_0 be Lipschitz continuous and u^ε be the solution of* (8.1)-(8.2). *There exists a continuous function $\bar{H} : \mathbb{R}^d \to \mathbb{R}$ such that u^ε converges locally uniformly towards the unique solution u^0 of*

$$(8.3) \qquad u_t^0 + \bar{H}(\nabla_x u^0) = 0, \qquad t > 0, x \in \mathbb{R}^d$$

$$(8.4) \qquad u^0(0, x) = u_0(x), \qquad x \in \mathbb{R}^d.$$

Remark 8.3. The meaning of the convergence u^ε towards u^0 is

$$\lim_{\substack{(s,y) \to (t,x) \\ y \in \mathcal{N}_\varepsilon}} u^\varepsilon(s, y) = u^0(t, x).$$

8.2 The cell problem

Keeping in mind the definitions of networks and derivatives of functions defined on networks, solving the cell problem consists in finding specific global solutions of (8.1) for $\varepsilon = 1$, *i.e.*

$$\begin{cases} w_t + H_e(w_y) = 0, & t \in \mathbb{R},\ y \in e^*, e \in \mathcal{E}_1, \\ w_t + F_A(y, w_y) = 0, & t \in \mathbb{R},\ y \in \mathcal{V}_1. \end{cases} \tag{8.5}$$

Precisely, for some $P \in \mathbb{R}^d$, we look for solutions $w(t, y) = \lambda t + P \cdot y + v(y)$ with a $\mathbb{Z}^d$-periodic function v; in other words, we look for (λ, v) such that

$$\begin{cases} \lambda + H_e((P \cdot y + v)_y) = 0, & y \in e^*, e \in \mathcal{E}_1, \\ \lambda + F_A(y, (P \cdot y + v)_y) = 0, & y \in \mathcal{V}_1. \end{cases} \tag{8.6}$$

Theorem 8.4. *For all $P \in \mathbb{R}^d$ there exists a unique $\lambda \in \mathbb{R}$ for which there exists a $\mathbb{Z}^d$-periodic solution v of* (8.6)*. Moreover, the function $\bar{H}$ which maps P to $-\lambda$ is continuous.*

Proof. We consider the following $\mathbb{Z}^d$-periodic stationary problem

$$\begin{cases} \alpha v^\alpha + H_e((P \cdot y + v^\alpha)_y) = 0, & y \in e^*, e \in \mathcal{E}_1, \\ \alpha v^\alpha + F_A(y, (P \cdot y + v^\alpha)_y) = 0, & y \in \mathcal{V}_1. \end{cases} \tag{8.7}$$

We consider

$$C = \max_{e \in \mathcal{E}_1} |H_e((P \cdot y)_y)|.$$

Then the existence result and the comparison principle for the stationary equation (see Appendix B) imply that there exists a (unique) $\mathbb{Z}^d$-periodic solution v^α of (8.7) such that

$$|\alpha v^\alpha| \le C.$$

Since H_e is coercive, this implies that there exists a constant $\tilde{C}$ such that for all $\alpha > 0$, v_α is Lipschitz-continuous and

$$|v^\alpha_y| \le \tilde{C};$$

in other words, the family $(v^\alpha)_{\alpha > 0}$ is equi-Lipschitz continuous. We then consider

$$\tilde{v}_\alpha = v_\alpha - v_\alpha(0).$$

By Arzelà-Ascoli theorem, there exists $\alpha_n \to 0$ such that $\tilde{v}^n := \tilde{v}_{\alpha_n}$ converges uniformly towards v. Moreover, we can also assume that

$$\alpha_n v_{\alpha_n}(0) \to \lambda.$$

Passing to the limit into the equation yields that (λ, v) solves the cell problem (8.6).

The continuity of λ is completely classical too. Consider $P_n \to P_\infty$ as $n \to \infty$ and consider (λ_n, v_n) solving (8.6). We proved above that

$$|\lambda_n| \le C.$$

Hence, arguing as above, we can extract a subsequence from (λ_n, v_n) converging towards $(\lambda_\infty, v_\infty)$. Passing to the limit into the equation implies that $(\lambda_\infty, v_\infty)$ solves the cell problem (8.6). The uniqueness of λ yields the continuity of $\bar{H}$. The proof is now complete. □

8.3 Proof of convergence

Before proving the convergence, we state without proof the following elementary lemma.

Lemma 8.5 (Barriers)**.** *There exists $C > 0$ such that for all $\varepsilon > 0$,*

$$|u^\varepsilon(t,x) - u_0(x)| \le Ct.$$

We can now turn to the proof of convergence.

Proof of Theorem 8.2. We classically consider the relaxed semi-limits

$$\begin{cases} \overline{u}(t,x) = \limsup_{\substack{\varepsilon\to 0,(s,y)\to(t,x)\\ y\in\mathcal{N}_\varepsilon}} u^\varepsilon(s,y), \\ \underline{u}(t,x) = \liminf_{\substack{\varepsilon\to 0,(s,y)\to(t,x)\\ y\in\mathcal{N}_\varepsilon}} u^\varepsilon(s,y). \end{cases}$$

In order to prove convergence of u^ε towards u^0, it is enough to prove that $\overline{u}$ is a sub-solution of (8.3) and $\underline{u}$ is a super-solution of (8.3). We only prove that $\overline{u}$ is a sub-solution since the proof for $\underline{u}$ is very similar.

We consider a test function φ touching (strictly) $\overline{u}$ from above at (t_0, x_0): there exists $r_0 > 0$ such that for all $(t,x) \in B_{r_0}(t_0,x_0)$, $(t,x) \ne (t_0,x_0)$,

$$\varphi(t,x) > \overline{u}(t,x)$$

and $\varphi(t_0,x_0) = \overline{u}(t_0,x_0)$. We argue by contradiction by assuming that there exists $\theta > 0$ such that

$$\partial_t\varphi(t_0,x_0) - \lambda = \partial_t\varphi(t_0,x_0) + \bar{H}(\nabla_x\varphi(t_0,x_0)) = \theta > 0. \tag{8.8}$$

We then consider the following "perturbed test" function $\varphi^\varepsilon \colon \mathbb{R}^+ \times \mathcal{N}_\varepsilon \to \mathbb{R}$ [20],

$$\varphi^\varepsilon(t,x) = \varphi(t,x) + \varepsilon v(\varepsilon^{-1}x)$$

where (λ, v) solves the cell problem (8.6) for $P = \nabla_x\varphi(t_0,x_0)$.

Lemma 8.6. *For $r \le r_0$ small enough, the function φ^ε is a super-solution of* (8.1) *in $B((t_0,x_0),r) \subset (0,T)\times\mathcal{N}_\varepsilon$ and $\varphi^\varepsilon \ge u^\varepsilon + \eta_r$ in $\partial B((t_0,x_0),r)$ for some $\eta_r > 0$.*

Proof. Consider a test function ψ touching φ^ε from below at $(t,x) \in]0,+\infty[\times\mathcal{N}_\varepsilon$. Then the function

$$\psi_\varepsilon(s,y) = \varepsilon^{-1}(\psi(s,\varepsilon y) - \varphi(s,\varepsilon y))$$

touches v from below at $y = \frac{x}{\varepsilon} \in e$. In particular,

$$\partial_t\psi(t,x) = \partial_t\varphi(t,x), \tag{8.9}$$

$$\lambda + H_{\mathcal{N}_1}(y, \varphi_x(t_0,x_0) + \psi_x(t,x) - \varphi_x(t,x)) \ge 0. \tag{8.10}$$

Combine now (8.8), (8.9) and (8.10) and get

$$\partial_t \psi(t,x) + H_{\mathcal{N}_1}(y, \psi_x(t,x)) \geq \theta + E$$

where

$$E = (\varphi_t(t,x) - \varphi_t(t_0,x_0)) + (H_{\mathcal{N}_1}(y, \psi_x(t,x)) - H_{\mathcal{N}_1}(y, \psi_x(t,x) + \varphi_x(t_0,x_0) - \varphi_x(t,x))).$$

The fact that φ is C^1 implies that we can choose $r > 0$ small enough so that for all $(t,x) \in B((t_0,x_0),r)$,

$$E \geq -\theta.$$

Moreover, since φ is strictly above $\overline{u}$, we conclude that $\varphi^\varepsilon \geq u^\varepsilon + \eta_r$ on $\partial B((t_0,x_0),r)$ for some $\eta_r > 0$. This achieves the proof of the lemma. □

From the lemma, we deduce thanks to the (localized) comparison principle that

$$\varphi^\varepsilon(t,x) \geq u^\varepsilon(t,x) + \eta_r.$$

In particular, this implies

$$u(t_0,x_0) = \varphi(t_0,x_0) \geq u(t_0,x_0) + \eta_r > u(t_0,x_0)$$

which is the desired contradiction. □

8.4 Characterization of the effective Hamiltonian

We remark that, in view of **(H'3)**, there are exactly d different Hamiltonians $H_1, \ldots, H_d$ corresponding to e_{0,b_i} where $(b_i)_i$ denotes the canonical basis of $\mathbb{R}^d$. With such a remark in hand, we can know give the explicit form of the effective Hamiltonian $\bar{H}$.

Proposition 8.7 (Characterization of the effective Hamiltonian)**.** *Under assumptions of Theorem 8.2, for all $P = (p_1, \ldots, p_d) \in \mathbb{R}^d$,*

$$\bar{H}(P) = \max(A, \max_{i=1,\ldots,d} H_i(p_i)).$$

Proof. Let $\bar{\mu}$ denote $\max(A, \max_{i=1,\ldots,d} H_i(p_i))$ and μ denote $\bar{H}(P)$. We prove successively that $\mu \leq \bar{\mu}$ and $\bar{\mu} \leq \mu$.

Step 1: bound from above. Consider the following sub-solution of (8.5)

$$\bar{w}(t,y) = -\bar{\mu}t + P \cdot y.$$

By comparison with

$$w(t,y) = -\mu t + P \cdot y + v(y)$$

where the bounded corrector v is a solution of (8.6) with $\lambda = -\mu$, we deduce that

$$\bar{H}(P) = \mu \leq \bar{\mu}$$

by letting $t \to +\infty$.

Step 2: bound from below. To deduce the reverse inequality, we first notice that the periodic corrector v is Lipschitz continuous (by coercivity of the Hamiltonians), which implies

$$-\mu + H_e(p_e + v_y) = 0 \quad \text{for a.e.} \quad y \in e \in \mathcal{E}_1.$$

If H_e is convex, we deduce that

$$\int_0^1 \mu \; dy \geq H_e(\int_0^1 (p_e + v_y(y)) \; dy)$$

which implies

$$\mu \geq H_e(p_e). \tag{8.11}$$

When H_e is only quasi-convex, we still get the same inequality, because for any $\varepsilon > 0$, we can find a Hamiltonian $\tilde{H}_e^\varepsilon$ such that $|\tilde{H}_e^\varepsilon - H_e| \leq \varepsilon$ with $\tilde{H}_e$ satisfying (4.8). By Lemma 4.4, we know that there exists a convex increasing function β_ε such that $\beta_\varepsilon \circ \tilde{H}_e^\varepsilon$ is convex for all $e \in \mathcal{E}_1$, which implies again

$$\beta_\varepsilon(\mu + \varepsilon) \geq \beta_\varepsilon \circ \tilde{H}_e^\varepsilon(p_e).$$

Composing by β_ε^{-1} and letting ε go to zero, we recover (8.11).

Let us now consider what happens at the junction point $y = 0$. Since $w(t,0) = v(t,0) - \mu t$, Theorem 2.11 implies

$$-\mu + A \leq 0.$$

Together with (8.11), this implies

$$\bar{H}(P) = \mu \geq \bar{\mu}. \quad \square$$

A Appendix: proofs of some technical results

A.1 Technical results on a junction

In order to prove Lemma 3.4, we need the following one.

Lemma A.1 (A priori control at the same time)**.** *Let $T > 0$ and let u be a sub-solution and w be a super-solution as in Theorem 1.4. Then there exists a constant $C_T > 0$ such that for all $t \in [0,T), x, y \in J$, we have*

$$u(t,x) \leq w(t,y) + C_T(1 + d(x,y)). \tag{A.1}$$

We first derive Lemma 3.4 from Lemma A.1.

Proof of Lemma 3.4. Let us fix some $\varepsilon > 0$ and let us consider the sub-solution u_ε^- and super-solutions u_ε^+ defined in (2.33). Using (2.32), we see that we have for all $(t,x),(s,y) \in [0,T) \times J$

$$u_\varepsilon^+(t,x) - u_\varepsilon^-(s,y) \le 2C_\varepsilon T + 2\varepsilon + L_\varepsilon d(x,y) \tag{A.2}$$

We first apply Lemma A.1 to control $u(t,x) - u_\varepsilon^+(t,x)$, and then apply Lemma A.1 to control $u_\varepsilon^-(s,y) - w(s,y)$. Finally we get the control on $u(t,x) - w(s,y)$, using (A.2). □

We now turn to the proof of Lemma A.1.

Proof of Lemma A.1. Let us define

$$\varphi(x,y) = \sqrt{1 + d^2(x,y)}.$$

Then $\varphi \in C^1(J^2)$ and satisfies

$$|\varphi_x(x,y)|,\ |\varphi_y(x,y)| \le 1. \tag{A.3}$$

For constants $C_1, C_2 > 0$ to be chosen, let us consider

$$M = \sup_{t\in[0,T),\ x,y\in J} \left(u(t,x) - w(t,y) - C_2 t - C_1\varphi(x,y) \right).$$

The result follows if we show that M is non-positive for C_1 and C_2 large enough. Assume by contradiction that $M > 0$ for any C_1 and C_2. Then for $\eta, \alpha > 0$ small enough, we have $M_{\alpha,\eta} \ge M/2 > 0$ with

$$M_{\eta,\alpha} = \sup_{t\in[0,T),\ x,y\in J} \left(u(t,x) - w(t,y) - C_2 t - C_1\varphi(x,y) - \frac{\eta}{T-t} - \alpha\frac{d^2(x_0,x)}{2} \right). \tag{A.4}$$

From (1.10), we have

$$u(t,x) - w(t,y) \le C_T(2 + d(0,x) + d(0,y))$$

which shows that the supremum in (A.4) is reached at a point (t,x,y), assuming $C_1 > C_T$. Moreover, we have (for $0 < \alpha \le 1$)

$$\alpha d(0,x) \le C = C(C_T). \tag{A.5}$$

From the uniform continuity of the initial data u_0, there exists a constant $C_0 > 0$ such that

$$u_0(x) - u_0(y) \le C_0\varphi(x,y)$$

and therefore $t > 0$, assuming $C_1 > C_0$. Then the classical time penalization (or doubling variable technique) implies the existence of $a, b \in \mathbb{R}$ (that play the role of u_t and v_t) such that we have the following viscosity inequalities

$$\begin{cases} a + H\left(x, C_1\varphi_x(x,y) + \alpha d(x_0,x)\right) \le 0, \\ b + H(y, -C_1\varphi_y(x,y)) \ge 0 \end{cases}$$

(using the shorthand notation (3.1) and writing $\alpha d(x_0, x)$ for $\alpha\left(d^2(x_0,x)/2\right)_x$ for the purposes of notation) with $a - b = C_2 + \eta(T-t)^{-2}$. Substracting these inequalities yields

$$C_2 + \frac{\eta}{(T-t)^2} \leq H(y, -C_1\varphi_y(x,y)) - H\left(x, C_1\varphi_x(x,y) + \alpha d(0,x)\right). \tag{A.6}$$

Using bounds (A.3) and (A.5), this yields a contradiction in (A.6) for C_2 large enough. □

A.2 Technical results on a network

Proof of Lemma 5.2

Proof of Lemma 5.2. (H1) and (H2) imply the uniform boundedness of the $p_e^0(t,x)$, *i.e.* (5.8). We also notice that because of (5.8), there exists a constant $C_0 > 0$ such that for all $t \in [0,T]$, $e \in \mathcal{E}$ and $n \in \partial e$,

$$|H_e(t,n,p_e^0(t,n))| \leq C_0 \tag{A.7}$$

from which (5.9) is easily derived.

We now turn to the proof of (5.10). In view of the definition of F_A and (A2), (H5), we see that it is enough to prove that for all for $n \in \mathcal{V}$, $t, s \in [0,T]$, $p = (p_e)_{e\in\mathcal{E}_n} \in \mathbb{R}^{\text{Card } \mathcal{E}_n}$, $x \in \mathcal{V}$,

$$A_n^0(t,p) - A_n^0(s,p) \leq \tilde{\omega}_T\Big(|t-s|(1+\max(0, A_n^0(s,p)))\Big). \tag{A.8}$$

where

$$A_n^0(t,p) := \max_{e\in\mathcal{E}_n^-} H_e^-(t,n,p_e) \geq A_n^0(t)$$

or

$$A_n^0(t,p) := \max_{e\in\mathcal{E}_n^+} H_e^+(t,n,p_e) \geq A_n^0(t).$$

We only treat the first case, since the second case reduces to the first one by a simple change of orientation of the network.

We have

$$A_n^0(a,p) = H_{e_a}^-(a,n,p_{e_a}) \quad \text{for} \quad a = t,s.$$

Let us assume that we have (otherwise there is nothing to prove)

$$0 \leq I(t,s) := A_n^0(t,p) - A_n^0(s,p).$$

We also have

$$H_{e_s}^-(t,n,p_{e_s}) \leq A_n^0(t,p) = H_{e_t}^-(t,n,p_{e_t})$$

and

$$H_{e_t}^-(s,n,p_{e_t}) \leq A_n^0(s,p) = H_{e_s}^-(s,n,p_{e_s}).$$

We now distinguish three cases.

Case 1: $H^-_{e_t}(s,n,p_{e_t}) < H_{e_t}(s,n,p_{e_t})$**.** We first note that

$$0 \le I(t,s) \le A^0_n(t,p) - A^0_n(s). \tag{A.9}$$

Let us define

$$\tau = \begin{cases} \inf\left\{\sigma \in [t,s], \quad H^-_{e_t}(\sigma,n,p_{e_t}) < H_{e_t}(\sigma,n,p_{e_t})\right\} & \text{if} \quad t < s, \\ \sup\left\{\sigma \in [s,t], \quad H^-_{e_t}(\sigma,n,p_{e_t}) < H_{e_t}(\sigma,n,p_{e_t})\right\} & \text{if} \quad t \ge s. \end{cases}$$

Let us consider a optimizing sequence $\sigma_k \to \tau$ such that

$$H^-_{e_t}(\sigma_k,n,p_{e_t}) < H_{e_t}(\sigma_k,n,p_{e_t}).$$

Then we have

$$H^-_{e_t}(\sigma_k,n,p_{e_t}) = H_{e_t}(\sigma_k,n,p^0_{e_t}(\sigma_k,n)) \le A^0_n(\sigma_k) \le A^0_n(\sigma_k,p).$$

Then passing to the limit $k \to +\infty$, we get (by convergence of the minimum values of the Hamiltonians, even if the map $\bar{t} \mapsto p^0_e(\bar{t},n)$ is discontinuous)

$$H^-_{e_t}(\tau,n,p_{e_t}) = H_{e_t}(\tau,n,p^0_{e_t}(\tau,n)) \le A^0_n(\tau) \le A^0_n(\tau,p). \tag{A.10}$$

If $\tau = t$, then (A.10) implies that $A^0_n(t,p) = A^0_n(t)$ (keeping in mind the definition of p_{e_t}).

Subcase 1.1: $\tau \ne t$. This shows that

$$H_{e_t}(\tau,n,p_{e_t}) \le A^0_n(\tau) \quad \text{and} \quad H_{e_t}(t,n,p_{e_t}) \ge A^0_n(t).$$

We now choose some $\bar{\tau}$ in between t and τ such that

$$H_{e_t}(\bar{\tau},n,p_{e_t}) = A^0_n(\bar{\tau})$$

and estimate, using (A.9) and (A.7) and (H5)-(H6),

$$\begin{aligned} 0 \le I(t,s) &\le \{A^0_n(t,p) - H_{e_t}(\bar{\tau},n,p_{e_t})\} + \{A^0_n(\bar{\tau}) - A^0_n(s)\} \\ &\le \{H_{e_t}(t,n,p_{e_t}) - H_{e_t}(\bar{\tau},n,p_{e_t})\} + \{A^0_n(\bar{\tau}) - A^0_n(s)\} \\ &\le \bar{\omega}_T(|t-\bar{\tau}|(1+\max(A^0_n(\bar{\tau}),0))) + \bar{\omega}_T(|\bar{\tau}-s|) \\ &\le \bar{\omega}_T(|t-s|(1+C_0)) + \bar{\omega}_T(|t-s|). \end{aligned}$$

Subcase 1.2: $\tau = t$. Then $A^0_n(t,p) = A^0_n(t)$. Using (A.9), this gives directly

$$0 \le I(t,s) \le A^0_n(t) - A^0_n(s) \le \bar{\omega}_T(|t-s|).$$

Case 2: $H^-_{e_t}(s,n,p_{e_t}) = H_{e_t}(s,n,p_{e_t})$ **and** $H^-_{e_t}(t,n,p_{e_t}) = H_{e_t}(t,n,p_{e_t})$**.** We have

$$\begin{aligned} 0 \le I(t,s) &= H^-_{e_t}(t,n,p_{e_t}) - A^0_n(s,p) \\ &\le H^-_{e_t}(t,n,p_{e_t}) - H^-_{e_t}(s,n,p_{e_t}) \\ &= H_{e_t}(t,n,p_{e_t}) - H_{e_t}(s,n,p_{e_t}) \\ &\le \bar{\omega}_T(|t-s|(1+\max(H_{e_t}(s,n,p_{e_t}),0))) \\ &\le \bar{\omega}_T(|t-s|(1+\max(H^-_{e_t}(s,n,p_{e_t}),0))) \\ &\le \bar{\omega}_T(|t-s|(1+\max(A^n_0(s,p),0))). \end{aligned}$$

Case 3: $H^-_{e_t}(s,n,p_{e_t}) = H_{e_t}(s,n,p_{e_t})$ **and** $H^-_{e_t}(t,n,p_{e_t}) < H_{e_t}(t,n,p_{e_t})$**.** Then

$$p^0_{e_t}(t,n) < p_{e_t} \le p^0_{e_t}(s,n).$$

Because of (A.7) and the uniform bound on the Hamiltonians for bounded gradients, (H2), we deduce that

$$H_{e_t}(s,n,p_{e_t}) \le C_1$$

for some constant $C_1 > 0$ only depending on our assumptions. Therefore, we have

$$\begin{aligned} 0 \le I(t,s) \quad &= H^-_{e_t}(t,n,p_{e_t}) - A^0_n(s,p) \\ &\le H^-_{e_t}(t,n,p_{e_t}) - H^-_{e_t}(s,n,p_{e_t}) \\ &< H_{e_t}(t,n,p_{e_t}) - H_{e_t}(s,n,p_{e_t}) \\ &\le \bar{\omega}_T(|t-s|(1+C_1)). \end{aligned}$$

The proof is now complete. □

Semi-concavity of the distance

In order to prove Lemmas 5.10 and 5.11, we need the following one.

Lemma A.2 (Semi-concavity of φ and d^2)**.** *Let $\mathcal{N}$ be a network defined in* (5.2) *with edges $\mathcal{E}$ and vertices $\mathcal{V}$. Let*

$$\varphi(x,y) = \sqrt{1+d^2(x,y)}$$

where d is the distance function on the network $\mathcal{N}$. Then $\varphi(x,\cdot)$ and $\varphi(\cdot,y)$ are 1*-Lipschitz for all $x,y \in \mathcal{N}$. Moreover φ and d^2 are semi-concave on $e_a \times e_b$ for all $e_a, e_b \in \mathcal{E}$.*

Proof. The Lipschitz properties of φ are trivial. Since $r \mapsto r^2$ and $r \mapsto \sqrt{1+r^2}$ are smooth increasing functions in $\mathbb{R}^+$, the result follows from the fact that the distance function d itself is semi-concave; it is even the minimum of a finite number of smooth functions.

If $e_a = e_b$, then $d^2(x,y) = (x-y)^2$ which implies that $\varphi \in C^1(e_a \times e_a)$. Then we only consider the cases where $e_a \neq e_b$.

Case 1: e_a **and** e_b **isometric to** $[0,+\infty)$**.** Then for $(x,y) \in e_a \times e_b$, we have

$$d(x,y) = x + y + d(e^0_a, e^0_b)$$

which implies that $\varphi \in C^1(e_a \times e_b)$.

Case 2: e_a **isometric to** $[0,+\infty)$ **and** e_b **isometric to** $[0,l_b]$**.** Reversing the orientation of e_b if necessary, we can assume that

$$d_0 := d(e^0_a, e^0_b) \le d(e^0_a, e^1_b) =: d_1$$

and then for $(x,y) \in e_a \times e_b$, we have

$$d(x,y) = x + \min(d_0 + y, d_1 + (l_b - y)) = \min(d_0 + x + y, d_1 + x + (l_b - y)).$$

Then φ is the minimum of two C^1 functions, it is semi-concave.

Case 3: e_a and e_b isometric to $[0, l_a]$ and $[0, l_b]$. Changing the orientations of both e_a and e_b if necessary, we can assume that

$$d(e_a^0, e_b^0) = \min_{i,j=0,1} d_{ij} \quad \text{with} \quad d_{ij} = d(e_a^i, e_b^j).$$

Therefore

$$d(x,y) = \min(d_{00} + x + y, d_{01} + x + (l_b - y), d_{10} + (l_a - x) + y, d_{11} + (l_a - x) + (l_b - y))$$

and again φ is the minimum of four C^1 functions, it is therefore semi-concave. □

Proof of Lemma 5.10

Proof of Lemma 5.10. We first prove (5.14) for $t = s$ by adapting in a straightforward way the proof of Lemma A.1. The only difference is that for any $e_a, e_b \in \mathcal{E}$, the function

$$\varphi(x,y) = \sqrt{1 + d^2(x,y)}$$

may not be $C^1(e_a \times e_b)$. But Lemma A.2 and Remark 5.6 ensure that this is harmless. The remaining of the proof of Lemma A.1 is unchanged. In particular the uniform bound on the Hamiltonians for bounded gradients is used, see (H2).

Now (5.14) is obtained for $t \neq s$ by following the proof of Lemma 3.4 and using the barriers given in the proof of Theorem 5.7. □

Proof of Lemma 5.11

Proof of Lemma 5.11. We do the proof for sub-solutions (the proof for super-solutions being similar). We consider the following barrier (similar to the ones in the proof of Theorem 5.7)

$$u_\varepsilon^+(t,x) = u_0^\varepsilon(x) + K_\varepsilon t + \varepsilon$$

with

$$|u_0^\varepsilon - u_0| \leq \varepsilon \quad \text{and} \quad |(u_0^\varepsilon)_x| \leq L_\varepsilon$$

and $K_\varepsilon \geq C_\varepsilon$ with C_ε given in (5.12). It is enough to prove that for all $(t,x) \in [0,T) \times \mathcal{N}$,

$$u(t,x) \leq u_\varepsilon^+(t,x)$$

for a suitable choice of $K_\varepsilon \geq C_\varepsilon$ in order to conclude. Indeed, this implies

$$u(t,x) \leq u_0(x) + f(t)$$

with

$$f(t) = \min_{\varepsilon > 0}(K_\varepsilon t + \varepsilon)$$

which is non-negative, non-decreasing, concave and $f(0) = 0$.

We consider for $0 < \tau \le T$,

$$M = \sup_{(t,x)\in[0,\tau)\times\mathcal{N}} (u - u_\varepsilon^+)(t,x)$$

and assume by contradiction that $M > 0$. We know by Lemma 5.10 that M is finite. Then for any $\alpha, \eta > 0$ small enough, we have $M_\alpha \ge M/2 > 0$ with

$$M_\alpha = \sup_{(t,x)\in[0,\tau)\times\mathcal{N}} \left\{ u(t,x) - u_\varepsilon^+(t,x) - \frac{\eta}{\tau - t} - \alpha\psi(x) \right\}.$$

(we recall that $\psi = d^2(x_0,\cdot)/2$). By the sublinearity of u and u_ε^+, we know that this supremum is reached at some point (t,x). Moreover $t > 0$ since $u(0,x) \le u_0(x) \le u_\varepsilon^+(0,x)$.

This implies in particular that

$$\begin{aligned} 0 < M/2 \le M_\alpha &= u(t,x) - u_\varepsilon^+(t,x) - \frac{\eta}{\tau - t} - \alpha\frac{d^2(x_0,x)}{2} \\ &\le C_T(1 + d(x_0,x)) - u_0^\varepsilon(x_0) + L_\varepsilon d(x,x_0) - \alpha\frac{d^2(x_0,x)}{2} \\ &\le C_T(1 + d(x_0,x)) + |u_0(x_0)| + \varepsilon + L_\varepsilon d(x,x_0) - \alpha\frac{d^2(x_0,x)}{2} \\ &\le R_\varepsilon(1 + d(x_0,x)) - \alpha\frac{d^2(x_0,x)}{2} \end{aligned}$$

with

$$R_\varepsilon = C_T + \max(L_\varepsilon, |u_0(x_0)| + \varepsilon).$$

Then $z = \alpha d(x_0,x)$ satisfies

$$\frac{z^2}{2} \le R_\varepsilon\alpha + R_\varepsilon z \le R_\varepsilon\alpha + R_\varepsilon^2 + \frac{z^2}{4}$$

which implies that for $\alpha \le 1$,

$$\alpha d(x_0,x) \le 2\sqrt{R_\varepsilon + R_\varepsilon^2}. \tag{A.11}$$

Writing the sub-solution viscosity inequality, we get

$$K_\varepsilon + H_{\mathcal{N}}(t,x,(u_0^\varepsilon)_x(x) + \alpha\psi_x(x)) \le 0$$

We get a contradiction for the choice

$$K_\varepsilon = 1 + \max\left(\sup_{t\in[0,T]} \sup_{n\in\mathcal{V}} |\max(A_n(t), A_n^0(t))|, \sup_{t\in[0,T]} \sup_{e\in\mathcal{E}} \sup_{x\in e} \sup_{|p_e|\le L_\varepsilon + 2\sqrt{R_\varepsilon + R_\varepsilon^2}} |H_e(t,x,p_e)| \right).$$

□

B Appendix: stationary results for networks

This short section is devoted to the statement of an existence and uniqueness result for the following stationary HJ equation posed on a network $\mathcal{N}$ satisfying (5.1),

$$u + H_{\mathcal{N}}(x, u_x) = 0 \quad \text{for all} \quad x \in \mathcal{N}. \tag{B.1}$$

For each $e \in \mathcal{E}$, we consider a Hamiltonian $H_e : e \times \mathbb{R} \to \mathbb{R}$ satisfying

- **(H0-s)** (Continuity) $H_e \in C(e \times \mathbb{R})$.
- **(H1-s)** (Uniform coercivity)

$$\liminf_{|q| \to +\infty} H_e(x, q) = +\infty$$

  uniformly with respect to $x \in e$, $e \in \mathcal{E}$.

- **(H2-s)** (Uniform bound on the Hamiltonians for bounded gradients) For all $L > 0$, there exists $C_L > 0$ such that

$$\sup_{p \in [-L, L], x \in \mathcal{N} \setminus \mathcal{V}} |H_{\mathcal{N}}(x, p)| \leq C_L.$$

- **(H3-s)** (Uniform modulus of continuity for bounded gradients) For all $L > 0$, there exists a modulus of continuity ω_L such that for all $|p|, |q| \leq L$ and $x \in e \in \mathcal{E}$,

$$|H_e(x, p) - H_e(x, q)| \leq \omega_L(|p - q|).$$

- **(H4-s)** (Quasi-convexity) For all $n \in \mathcal{V}$, there exists a $p_e^0(n)$ such that

$$\begin{cases} H_e(n, \cdot) & \text{is nonincreasing on} \quad (-\infty, p_e^0(n)], \\ H_e(n, \cdot) & \text{is nondecreasing on} \quad [p_e^0(n), +\infty). \end{cases}$$

As far as flux limiters are concerned, the following assumptions will be used.

- **(A1-s)** (Uniform bound on A) There exists a constant $C > 0$ such that for all $n \in \mathcal{V}$,

$$|A_n| \leq C.$$

The following result is a straightforward adaptation of Corollary 5.9. Proofs are even simpler since the time dependance was an issue when proving the comparison principle in the general case.

Theorem B.1 (Existence and uniqueness – stationary case)**.** *Assume* (H0-s)-(H4-s) *and* (A1-s). *Then there exists a unique sublinear viscosity solution u of* (B.1) *in $\mathcal{N}$.*

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References

[1] Yves Achdou, Fabio Camilli, Alessandra Cutrì, and Nicoletta Tchou. Hamilton-Jacobi equations constrained on networks. *NoDEA Nonlinear Differential Equations Appl.*, 20(3):413–445, 2013.

[2] Yves Achdou, Salomé Oudet, and Nicoletta Tchou. Hamilton-Jacobi equations for optimal control on junctions and networks. *ESAIM Control Optim. Calc. Var.*, 21(3):876–899, 2015.

[3] Yves Achdou and Nicoletta Tchou. Hamilton-Jacobi equations on networks as limits of singularly perturbed problems in optimal control: dimension reduction. *Comm. Partial Differential Equations*, 40(4):652–693, 2015.

[4] Olivier Alvarez, Emmanuel N. Barron, and Hitoshi Ishii. Hopf-Lax formulas for semicontinuous data. *Indiana Univ. Math. J.*, 48(3):993–1035, 1999.

[5] Luigi Ambrosio and Jin Feng. On a class of first order Hamilton-Jacobi equations in metric spaces. *J. Differential Equations*, 256(7):2194–2245, 2014.

[6] B. Andreianov and C. Cancès. Vanishing capillarity solutions of Buckley-Leverett equation with gravity in two-rocks medium. *Comput. Geosci.*, 17 (3):551–572, 2013.

[7] Boris Andreianov, Kenneth Hvistendahl Karlsen, and Nils Henrik Risebro. A theory of L^1-dissipative solvers for scalar conservation laws with discontinuous flux. *Arch. Ration. Mech. Anal.*, 201(1):27–86, 2011.

[8] Martino Bardi and Lawrence C. Evans. On Hopf's formulas for solutions of Hamilton-Jacobi equations. *Nonlinear Anal.*, 8(11):1373–1381, 1984.

[9] G. Barles, A. Briani, and E. Chasseigne. A Bellman approach for two-domains optimal control problems in $\mathbb{R}^N$. *ESAIM Control Optim. Calc. Var.*, 19(3):710–739, 2013.

[10] G. Barles, A. Briani, and E. Chasseigne. A Bellman approach for regional optimal control problems in $\mathbb{R}^N$. *SIAM J. Control Optim.*, 52(3):1712–1744, 2014.

[11] Guy Barles and Emmanuel Chasseigne. (Almost) everything you always wanted to know about deterministic control problems in stratified domains. *Netw. Heterog. Media*, 10(4):809–836, 2015.

[12] Alberto Bressan and Yunho Hong. Optimal control problems on stratified domains. *Netw. Heterog. Media*, 2(2):313–331 (electronic), 2007.

[13] R. Bürger, K.H. Karlsen, and J. Towers. An Engquist-Osher type scheme for conservation laws with discontinuous flux adapted to flux connections. *SIAM J. Numer. Anal.*, 47:1684–1712, 2009.

[14] Fabio Camilli and Claudio Marchi. A comparison among various notions of viscosity solution for Hamilton-Jacobi equations on networks. *J. Math. Anal. Appl.*, 407(1):112–118, 2013.

[15] Yun Gang Chen, Yoshikazu Giga, and Shun'ichi Goto. Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations. *J. Differential Geom.*, 33(3):749–786, 1991.

[16] Giuseppe Maria Coclite and Nils Henrik Risebro. Viscosity solutions of Hamilton-Jacobi equations with discontinuous coefficients. *J. Hyperbolic Differ. Equ.*, 4(4):771–795, 2007.

[17] Cecilia De Zan and Pierpaolo Soravia. Cauchy problems for noncoercive Hamilton-Jacobi-Isaacs equations with discontinuous coefficients. *Interfaces Free Bound.*, 12(3):347–368, 2010.

[18] Klaus Deckelnick and Charles M. Elliott. Uniqueness and error analysis for Hamilton-Jacobi equations with discontinuities. *Interfaces Free Bound.*, 6(3):329–349, 2004.

[19] Paul Dupuis. A numerical method for a calculus of variations problem with discontinuous integrand. In *Applied stochastic analysis (New Brunswick, NJ, 1991)*, volume 177 of *Lecture Notes in Control and Inform. Sci.*, pages 90–107. Springer, Berlin, 1992.

[20] Lawrence C. Evans. The perturbed test function method for viscosity solutions of nonlinear PDE. *Proc. Roy. Soc. Edinburgh Sect. A*, 111(3-4):359–375, 1989.

[21] Hélène Frankowska and Sławomir Plaskacz. Hamilton-Jacobi equations for infinite horizon control problems with state constraints. In *Calculus of variations and optimal control (Haifa, 1998)*, volume 411 of *Chapman & Hall/CRC Res. Notes Math.*, pages 97–116. Chapman & Hall/CRC, Boca Raton, FL, 2000.

[22] Hélène Frankowska and Sławomir Plaskacz. Semicontinuous solutions of Hamilton-Jacobi-Bellman equations with degenerate state constraints. *J. Math. Anal. Appl.*, 251(2):818–838, 2000.

[23] Giulio Galise, Cyril Imbert, and Régis Monneau. A junction condition by specified homogenization and application to traffic lights. *Anal. PDE*, 8(8):1891–1929, 2015.

[24] Wilfrid Gangbo and Andrzej Święch. Metric viscosity solutions of Hamilton-Jacobi equations depending on local slopes. *Calc. Var. Partial Differential Equations*, 54(1):1183–1218, 2015.

[25] M. Garavello and P. Soravia. Representation formulas for solutions of the HJI equations with discontinuous coefficients and existence of value in differential games. *J. Optim. Theory Appl.*, 130(2):209–229, 2006.

[26] Mauro Garavello, Roberto Natalini, Benedetto Piccoli, and Andrea Terracina. Conservation laws with discontinuous flux. *Netw. Heterog. Media*, 2(1):159–179, 2007.

[27] Yoshikazu Giga and Nao Hamamuki. Hamilton-Jacobi equations with discontinuous source terms. *Comm. Partial Differential Equations*, 38(2):199–243, 2013.

[28] Yoshikazu Giga, Nao Hamamuki, and Atsushi Nakayasu. Eikonal equations in metric spaces. *Trans. Amer. Math. Soc.*, 367(1):49–66, 2015.

[29] Cyril Imbert and Régis Monneau. Quasi-convex hamilton-jacobi equations posed on junctions: the multi-dimensional case. HAL 01073954.

[30] Cyril Imbert, Régis Monneau, and Hasnaa Zidani. A Hamilton-Jacobi approach to junction problems and application to traffic flows. *ESAIM: Control, Optimisation and Calculus of Variations*, 19:129–166, 2013.

[31] Cyril Imbert and Vinh Duc Nguyen. Generalized junction conditions for degenerate parabolic equations. 26 pages, HAL preprint, 2016.

[32] Hitoshi Ishii. Perron's method for Hamilton-Jacobi equations. *Duke Math. J.*, 55(2):369–384, 1987.

[33] Hitoshi Ishii and Shigeaki Koike. A new formulation of state constraint problems for first-order PDEs. *SIAM J. Control Optim.*, 34(2):554–571, 1996.

[34] Pierre-Louis Lions. Lectures at Collège de France, 2013-2014.

[35] Pierre-Louis Lions and Panagiotis E. Souganidis. Differential games, optimal control and directional derivatives of viscosity solutions of Bellman's and Isaacs' equations. *SIAM J. Control Optim.*, 23(4):566–583, 1985.

[36] Atsushi Nakayasu. Metric viscosity solutions for Hamilton-Jacobi equations of evolution type. *Adv. Math. Sci. Appl.*, 24(2):333–351, 2014.

[37] Z. Rao, A. Siconolfi, and H. Zidani. Transmission conditions on interfaces for Hamilton-Jacobi-Bellman equations. *J. Differential Equations*, 257(11):3978–4014, 2014.

[38] Zhiping Rao and Hasnaa Zidani. Hamilton-Jacobi-Bellman equations on multi-domains. *Control and Optimization with PDE Constraints, International Series of Numerical Mathematics*, 164, 2013.

[39] R. Tyrrell Rockafellar. Existence theorems for general control problems of Bolza and Lagrange. *Advances in Math.*, 15:312–333, 1975.

[40] Dirk Schieborn. *Viscosity Solutions of Hamilton-Jacobi Equations of Eikonal Type on Ramified Spaces*. PhD thesis, Eberhard-Karls-Universitat Tubingen, 2006.

[41] Dirk Schieborn and Fabio Camilli. Viscosity solutions of Eikonal equations on topological networks. *Calc. Var. Partial Differential Equations*, 46(3-4):671–686, 2013.

[42] Pierpaolo Soravia. Optimality principles and representation formulas for viscosity solutions of Hamilton-Jacobi equations. I. Equations of unbounded and degenerate control problems without uniqueness. *Adv. Differential Equations*, 4(2):275–296, 1999.

[43] Pierpaolo Soravia. Optimality principles and representation formulas for viscosity solutions of Hamilton-Jacobi equations. II. Equations of control problems with state constraints. *Differential Integral Equations*, 12(2):275–293, 1999.