Dynamics of dislocation densities in a bounded channel. Part I: smooth solutions to a singular coupled parabolic system.

H. Ibrahim *, M. Jazar [1], R. Monneau *

November 27, 2008

Abstract

We study a coupled system of two parabolic equations in one space dimension. This system is singular because of the presence of one term with the inverse of the gradient of the solution. Our system describes an approximate model of the dynamics of dislocation densities in a bounded channel submitted to an exterior applied stress. The system of equations is written on a bounded interval with Dirichlet conditions and requires a special attention to the boundary. The proof of existence and uniqueness is done under the use of two main tools: a certain comparison principle on the gradient of the solution, and a parabolic Kozono-Taniuchi inequality.

AMS Classification: 35K50, 35K40, 35K55, 42B35, 42B99.
Key words: Boundary value problems, parabolic systems, nonlinear PDE, BMO spaces, logarithmic Sobolev inequality, parabolic Kozono-Taniuchi inequality.

1 Introduction

1.1 Setting of the problem

In this paper, we are concerned in the study of the following singular parabolic system:

$$\begin{cases} \kappa_t = \varepsilon \kappa_{xx} + \dfrac{\rho_x \rho_{xx}}{\kappa_x} - \tau \rho_x & \text{on} \quad I \times (0, \infty) \\ \rho_t = (1+\varepsilon)\rho_{xx} - \tau \kappa_x & \text{on} \quad I \times (0, \infty), \end{cases} \tag{1.1}$$

with the initial conditions:

$$\kappa(x, 0) = \kappa^0(x) \quad \text{and} \quad \rho(x, 0) = \rho^0(x), \tag{1.2}$$

*Cermics, Paris-Est/ENPC, 6 et 8 avenue Blaise Pascal, Cité Descartes Champs-sur-Marne, 77455 Marne-la-Vallée Cedex 2, France. E-mails: ibrahim@cermics.enpc.fr, monneau@cermics.enpc.fr
[1] M. Jazar, Lebanese University, LaMA-Liban, P.O. Box 826, Tripoli Liban, mjazar@ul.edu.lb
M. Jazar is supported by a grant from Lebanese University

and the boundary conditions:

$$\begin{cases} \kappa(0,.) = \kappa^0(0) \quad \text{and} \quad \kappa(1,.) = \kappa^0(1), \\ \rho(0,.) = \rho(1,.) = 0, \end{cases} \tag{1.3}$$

where

$$\varepsilon > 0, \quad \tau \in \mathbb{R},$$

are fixed constants, and

$$I := (0,1)$$

is the open and bounded interval of $\mathbb{R}$.

The goal is to show the long-time existence and uniqueness of a smooth solution of (1.1), (1.2) and (1.3). Our motivation comes from a problem of studying the dynamics of dislocation densities in a constrained channel submitted to an exterior applied stress. In fact, system (1.1) can be seen as an approximate model of the one described in [12]. This approximate model (presented in [12] for $\varepsilon = 0$) reads:

$$\begin{cases} \theta^+_t = \varepsilon\theta^+_{xx} + \left[\left(\dfrac{\theta^+_x - \theta^-_x}{\theta^+ + \theta^-} - \tau\right)\theta^+\right]_x & \text{on} \quad I \times (0,\infty), \\ \theta^-_t = \varepsilon\theta^-_{xx} - \left[\left(\dfrac{\theta^+_x - \theta^-_x}{\theta^+ + \theta^-} - \tau\right)\theta^-\right]_x & \text{on} \quad I \times (0,\infty), \end{cases} \tag{1.4}$$

with τ representing the exterior stress field. System (1.4) can be deduced from (1.1), by spatially differentiating (1.1), and by considering

$$\rho^\pm_x = \theta^\pm, \quad \rho = \rho^+ - \rho^-, \quad \kappa = \rho^+ + \rho^-, \tag{1.5}$$

which explains the presence of the factor $(1+\varepsilon)$ in the second equation of (1.1). Here θ^+ and θ^- represent the densities of the positive and negative dislocations respectively (see [25, 16] for a physical study of dislocations).

The part II of this work will be presented in [18]. There, we will show some kind of convergence of the solution $(\rho^\varepsilon, \kappa^\varepsilon)$ as $\varepsilon \to 0$.

1.2 Statement of the main result

The main result of this paper is:

Theorem 1.1 ***(Existence and uniqueness of a solution).*** *Let ρ^0, κ^0 satisfying:*

$$\rho^0, \kappa^0 \in C^\infty(\bar{I}), \quad \rho^0(0) = \rho^0(1) = \kappa^0(0) = 0, \quad \kappa^0(1) = 1, \tag{1.6}$$

$$\begin{cases} (1+\varepsilon)\rho^0_{xx} = \tau\kappa^0_x & \text{on} \quad \partial I \\ (1+\varepsilon)\kappa^0_{xx} = \tau\rho^0_x & \text{on} \quad \partial I, \end{cases} \tag{1.7}$$

and

$$\kappa^0_x > |\rho^0_x| \quad \text{on} \quad \bar{I}. \tag{1.8}$$

Then there exists a unique global solution (ρ,κ) of system (1.1), (1.2) and (1.3) satisfying

$$(\rho,\kappa)\in C^{3+\alpha,\frac{3+\alpha}{2}}(\bar I\times[0,\infty))\cap C^\infty(\bar I\times(0,\infty)),\quad \forall\alpha\in(0,1). \tag{1.9}$$

Moreover, this solution also satisfies :

$$\kappa_x>|\rho_x|\quad on\quad \bar I\times[0,\infty). \tag{1.10}$$

Remark 1.2 *Conditions (1.7) are natural here. Indeed, the regularity (1.9) of the solution of (1.1) with the boundary conditions (1.2) and (1.3) imply in particular (1.7).*

Remark 1.3 *Remark that the choice $\kappa^0(0)=0$ and $\kappa^0(1)=1$ does not reduce the generality of the problem, because equation (1.1) does not see the constants and has the following invariance: if (ρ,κ) is a solution, then $(\lambda\rho,\lambda\kappa)$ is also a solution for any $\lambda\in\mathbb{R}$.*

1.3 Brief review of the literature

To our knowledge, systems of equations involving the singularity in $1/\kappa_x$ as in (1.1) has not been directly handled elsewhere in the literature. However, parabolic problems involving singular terms have been widely studied in various aspects. Fast diffusion equations:

$$u_t-\Delta u^m=0,\quad 0<m<1,$$

are examined, for instance, in [5, 7, 8]. These equations are singular at points where $u=0$. In dimension 1, setting $u=v_x$ we get, up to a constant of integration:

$$v_t-mv_x^{m-1}v_{xx}=0$$

which makes appear a singularity like $1/v_x$. Other class of singular parabolic equations are for instance of the form:

$$u_t=u_{xx}+\frac{b}{x}u_x, \tag{1.11}$$

where b is a certain constant. Such an equation is related to axially symmetric problems and also occurs in probability theory (see [6, 23]). An important type of equations that can be indirectly related to our system are semilinear parabolic equations:

$$u_t=\Delta u+|u|^{p-1}u,\quad p>1. \tag{1.12}$$

Many authors have studied the blow-up phenomena for solutions of the above equation (see for instance [24, 13]). Equation (1.12) can be somehow related to the first equation of (1.1), but with a singularity of the form $1/\kappa$. This can be formally seen if we first suppose that $u\geq 0$, and then we apply the following change of variables $u=1/v$. In this case, equation (1.12) becomes:

$$v_t=\Delta v-\frac{2|\nabla v|^2}{v}-v^{2-p},$$

and hence if $p=3$, we obtain:

$$v_t=\Delta v-\frac{1}{v}(1+2|\nabla v|^2). \tag{1.13}$$

Since the solution u of (1.12) may blow-up at a finite time $t=T$, then v may vanishes at $t=T$, and therefore equation (1.13) faces similar singularity to that of the first equation of (1.1), but in terms of the solution v instead of v_x.

1.4 Strategy of the proof

The existence and uniqueness is made by using a fixed point argument after a slight artificial modification in the denominator κ_x of the first equation of (1.1) in order to avoid dividing by zero. We will first show the short time existence, proving in particular that

$$\kappa_x(x,t) \geq \sqrt{\gamma^2(t) + \rho_x^2(x,t)} \geq \gamma(t) > 0,$$

for some well chosen initial data and a suitable function $\gamma(t) > 0$. The only, but dangerous, inconvenience is that the function γ depends strongly on $\|\rho_{xxx}(.,t)\|_{L^\infty(I)}$, roughly speaking:

$$\gamma^{'} \simeq -\|\rho_{xxx}\|_{L^\infty(I)}\gamma. \tag{1.14}$$

Let us mention that one of the key points here is that $\left|\frac{\rho_x}{\kappa_x}\right| \leq 1$ which somehow linearize the first equation of (1.1). Nevertheless, standard Sobolev and Hölder estimates for the parabolic system (1.1) are not good enough to bound $\|\rho_{xxx}\|_{L^\infty(I)}$ in order to prevent γ (and as a consequence κ_x) from vanishing. On the contrary, a Sobolev logarithmic estimate (see Section 2, the parabolic Kozono-Taniuchi inequality, Theorem 2.13) can be used in order to obtain a sharp bound of $\|\rho_{xxx}\|_{L^\infty(I)}$ of the form

$$\|\rho_{xxx}\|_{L^\infty(I)} \leq E\left(1 + \log^+ \frac{1}{\gamma}\right),$$

where E is an exponential function in time. This allows, with (1.14), to show that the function $\gamma > 0$ does not vanish in finite time. After that, due to some *a priori* estimates, we can prove the global time existence.

1.5 Organization of the paper

This paper is organized as follows: in Section 2, we present the tools needed throughout this work, this includes a brief recall on the L^p, C^α and the BMO theory for parabolic equations. In Section 3, we show a comparison principle associated to (1.1) that will play a crucial rule in the long time existence of the solution as well as the positivity of κ_x. In Section 4, we present a result of short time existence, uniqueness and regularity of a solution (ρ, κ) of (1.1). Section 5 is devoted to give some exponential bounds on (ρ, κ). In Section 6, we show a control of the $W_2^{2,1}$ norm of ρ_{xxx}. In a similar way, we show a control of the BMO norm of ρ_{xxx} in Section 7. In Section 8, we use a parabolic Kozono-Taniuchi inequality to control the L^∞ norm of ρ_{xxx}. In Section 9, we prove our main result: Theorem 1.1. Finally, Sections 10 and 11 are appendices where we present the proofs of some technical results.

2 Tools: theory of parabolic equations

We start with some basic notations and terminology:

Abridged notation.

- I_T is the cylinder $I \times (0,T)$; $\bar{I}$ is the closure of I; $\overline{I_T}$ is the closure of I_T; ∂I is the boundary of I.

- $\|.\|_{L^p(\Omega)} = \|.\|_{p,\Omega}$, Ω is an open set, $p \geq 1$.

- S_T is the lateral boundary of I_T, or more precisely, $S_T = \partial I \times (0,T)$.

- $\partial^p I_T$ is the parabolic boundary of I_T, i.e. $\partial^p I_T = \overline{S_T} \cup (I \times \{t=0\})$.

- $D_y^s u = \frac{\partial^s u}{\partial y^s}$, u is a function depending on the parameter y, $s \in \mathbb{N}$.

- $[l]$ is the floor part of $l \in \mathbb{R}$.

- $Q_r = Q_r(x_0,t_0)$ is the lower parabolic cylinder given by:

$$Q_r = \{(x,t);\ |x-x_0| < r,\ t_0 - r^2 < t < t_0\},\ r>0,\ (x_0,t_0) \in I_T.$$

- $|\Omega|$ is the n-dimensional Lebesgue measure of the open set $\Omega \subset \mathbb{R}^n$.

- $m_\Omega(u) = \frac{1}{|\Omega|}\int_\Omega u$ is the average integral of the $u \in L^1(\Omega)$ over $\Omega \subset \mathbb{R}^n$.

2.1 L^p and C^α theory of parabolic equations

A major part of this work deals with the following typical problem in parabolic theory:

$$\begin{cases} u_t = \varepsilon u_{xx} + f & \text{on} \quad I_T \\ u(x,0) = \phi & \text{on} \quad I \\ u = \Phi & \text{on} \quad \partial I \times (0,T), \end{cases} \tag{2.1}$$

where $T > 0$ and $\varepsilon > 0$. A wide literature on the existence and uniqueness of solutions of (2.1) in different function spaces could be found for instance in [21], [11] and [22]. We will deal mainly with two types of spaces:

The Sobolev space $W_p^{2,1}(I_T)$, $1 < p < \infty$ which is the Banach space consisting of the elements in $L^p(I_T)$ having generalized derivatives of the form $D_t^r D_x^s u$, with r and s two non-negative integers satisfying the inequality $2r+s \leq 2$, also in $L^p(I_T)$. The norm in this space is defined as $\|u\|_{W_p^{2,1}(I_T)} = \sum_{i=0}^{2} \sum_{2r+s=i} \|D_t^r D_x^s u\|_{p,I_T}$.

The Hölder spaces $C^\ell(\bar{I})$ and $C^{\ell,\ell/2}(\overline{I_T})$, $\ell > 0$ a nonintegral positive number. We do not recall the definition of the space $C^\ell(\bar{I})$ which is very standard. The Hölder space $C^{\ell,\ell/2}(\overline{I_T})$ is the Banach space of functions $v(x,t)$ that are continuous in $\overline{I_T}$, together with all derivatives of the form $D_t^r D_x^s v$ for $2r+s<\ell$, and have a finite norm $|v|_{I_T}^{(\ell)} = \langle v\rangle_{I_T}^{(\ell)} + \sum_{j=0}^{[\ell]} \langle v\rangle_{I_T}^{(j)}$, where

$$\langle v\rangle_{I_T}^{(0)} = |v|_{I_T}^{(0)} = \|v\|_{\infty,I_T}, \quad \langle v\rangle_{I_T}^{(j)} = \sum_{2r+s=j} |D_t^r D_x^s v|_{I_T}^{(0)}, \quad \langle v\rangle_{I_T}^{(\ell)} = \langle v\rangle_{x,I_T}^{(\ell)} + \langle v\rangle_{t,I_T}^{(\ell/2)},$$

and

$$\langle v\rangle_{x,I_T}^{(\ell)} = \sum_{2r+s=[\ell]} \langle D_t^r D_x^s v\rangle_{x,I_T}^{(\ell-[\ell])}, \quad \langle v\rangle_{t,I_T}^{(\ell/2)} = \sum_{0<\ell-2r-s<2} \langle D_t^r D_x^s v\rangle_{t,I_T}^{\left(\frac{\ell-2r-s}{2}\right)},$$

with

$$\langle v\rangle_{x,I_T}^{(\alpha)} = \inf\{c;\ |v(x,t)-v(x',t)| \le c|x-x'|^\alpha,\ (x,t),(x',t)\in \overline{I_T}\}, \quad 0<\alpha<1,$$

$$\langle v\rangle_{t,I_T}^{(\alpha)} = \inf\{c;\ |v(x,t)-v(x,t')| \le c|t-t'|^\alpha,\ (x,t),(x,t')\in \overline{I_T}\}, \quad 0<\alpha<1.$$

The above definitions could be found in details in [21, Section 1]. Now, we write down the compatibility conditions of order 0 and 1. These compatibility conditions concern the given data ϕ, Φ and f of problem (2.1).

Compatibility condition of order 0. Let $\phi \in C(\bar{I})$ and $\Phi \in C(\overline{S_T})$. We say that the compatibility condition of order 0 is satisfied if

$$\phi\big|_{\partial I} = \Phi\big|_{t=0}. \tag{2.2}$$

Compatibility condition of order 1. Let $\phi \in C^2(\bar{I})$, $\Phi \in C^1(\overline{S_T})$ and $f \in C(\overline{I_T})$. We say that the compatibility condition of order 1 is satisfied if (2.2) is satisfied and in addition we have:

$$(\varepsilon\phi_{xx}+f)\big|_{\partial I} = \frac{\partial\Phi}{\partial t}\Big|_{t=0}. \tag{2.3}$$

We state two results of existence and uniqueness adapted to our special problem. We begin by presenting the solvability of parabolic equations in Hölder spaces.

Theorem 2.1 ***(Solvability in Hölder spaces, [21, Theorem 5.2]).*** *Suppose $0<\alpha<2$, a non-integral number. Then for any*

$$\phi \in C^{2+\alpha}(\bar{I}), \quad \Phi \in C^{1+\alpha/2}(\overline{S_T}) \quad \text{and} \quad f \in C^{\alpha,\alpha/2}(\overline{I_T})$$

satisfying the compatibility condition of order 1 (see (2.2) and (2.3)), problem (2.1) has a unique solution $u \in C^{2+\alpha,1+\alpha/2}(\overline{I_T})$ satisfying the following inequality:

$$|u|_{I_T}^{(2+\alpha)} \le c^H \left(|f|_{I_T}^{(\alpha)} + |\phi|_I^{(2+\alpha)} + |\Phi|_{S_T}^{(1+\alpha/2)}\right), \tag{2.4}$$

for some $c^H = c^H(\varepsilon,\alpha,T) > 0$.

Remark 2.2 ***(Estimating the term c^H of (2.4)).*** *The constant appearing in the above Hölder estimate (2.4) can be estimated, using some time iteration, as $c^H(\varepsilon,\alpha,T) \le e^{c(T+1)}$, where $c = c(\varepsilon,\alpha) > 0$ is a positive constant.*

We now present the solvability in Sobolev spaces. Recall the norm of fractional Sobolev spaces. If $f \in W_p^s(a,b)$, $s>0$ and $1<p<\infty$, then

$$\|f\|_{W_p^s(a,b)} = \|f\|_{W_p^{[s]}(a,b)} + \left(\int_a^b\int_a^b \frac{|f^{([s])}(x)-f^{([s])}(y)|^p}{|x-y|^{1+(s-[s])p}}\right)^{1/p}. \tag{2.5}$$

Theorem 2.3 ***(Solvability in Sobolev spaces, [21, Theorem 9.1]).*** *Let $p > 1$, $\varepsilon > 0$ and $T > 0$. For any $f \in L^p(I_T)$, $\phi \in W_p^{2-2/p}(I)$ and $\Phi \in W_p^{1-1/2p}(S_T)$, with $p \neq 3/2$ ($p = 3/2$ is called the* ***singular*** *index) satisfying in the case $p > 3/2$ the compatibility condition of order zero (see (2.2)), there exists a unique solution $u \in W_p^{2,1}(I_T)$ of (2.1) satisfying the following estimate:*

$$\|u\|_{W_p^{2,1}(I_T)} \leq c \left(\|f\|_{p,I_T} + \|\phi\|_{W_p^{2-2/p}(I)} + \|\Phi\|_{W_p^{1-1/2p}(S_T)} \right), \tag{2.6}$$

for some $c = c(\varepsilon, p, T) > 0$.

For a better understanding of the spaces stated in the above two theorems, especially fractional Sobolev spaces, we send the reader to [1] or [21]. The dependence of the constant c of Theorem 2.3 on the variable T will be of notable importance and this what is emphasized by the next lemma.

Lemma 2.4 ***(The constant c given by (2.6): case $\phi = 0$ and $\Phi = 0$).*** *Under the same hypothesis of Theorem 2.3, with $\phi = 0$ and $\Phi = 0$, the estimate (2.6) can be written as:*

$$\frac{\|u\|_{p,I_T}}{T} + \frac{\|u_x\|_{p,I_T}}{\sqrt{T}} + \|u_{xx}\|_{p,I_T} + \|u_t\|_{p,I_T} \leq c\|f\|_{p,I_T}, \tag{2.7}$$

where $c = c(\varepsilon, p) > 0$ is a positive constant depending only on p and ε.

The proof of this lemma will be done in Appendix A. Moreover, We will frequently make use of the following two lemmas also depicted from [21].

Lemma 2.5 ***(Sobolev embedding in Hölder spaces, [21, Lemma 3.3]).***
(i) (Case $p > 3$). For any function $u \in W_p^{2,1}(I_T)$, if $\alpha = 1 - 3/p > 0$, i.e. $p > 3$, then $u \in C^{1+\alpha, \frac{1+\alpha}{2}}(\overline{I_T})$ with $|u|_{I_T}^{(1+\alpha)} \leq c\|u\|_{W_p^{2,1}(I_T)}$, $c = c(p, T) > 0$. However, in terms of u_x, we have that $u_x \in C^{\alpha,\alpha/2}(\overline{I_T})$ satisfies the following estimates:

$$\|u_x\|_{\infty,I_T} \leq c\left\{\delta^\alpha(\|u_t\|_{p,I_T} + \|u_{xx}\|_{p,I_T}) + \delta^{\alpha-2}\|u\|_{p,I_T}\right\}, \quad c = c(p) > 0, \tag{2.8}$$

$$\langle u_x \rangle_{I_T}^{(\alpha)} \leq c\left\{\|u_t\|_{p,I_T} + \|u_{xx}\|_{p,I_T} + \frac{1}{\delta^2}\|u\|_{p,I_T}\right\}, \quad c = c(p) > 0.$$

(ii) (Case $p > 3/2$). If $u \in W_p^{2,1}(I_T)$ with $p > 3/2$, then $u \in C(\overline{I_T})$, and we have the following estimate:

$$\|u\|_{\infty,I_T} \leq c\left\{\delta^{2-3/p}(\|u_t\|_{p,I_T} + \|u_{xx}\|_{p,I_T}) + \delta^{-3/p}\|u\|_{p,I_T}\right\}, \quad c = c(p) > 0. \tag{2.9}$$

In the above two cases $\delta = \min\{1/2, \sqrt{T}\}$.

Lemma 2.6 ***(Trace of functions in $W_p^{2,1}(I_T)$, [21, Lemma 3.4]).*** *If $u \in W_p^{2,1}(I_T)$, $p > 1$, then for $2r + s < 2 - 2/p$, we have $D_t^r D_x^s u\big|_{t=0} \in W_p^{2-2r-s-2/p}(I)$ with*

$$\|u\|_{W_p^{2-2r-s-2/p}(I)} \leq c(T)\|u\|_{W_p^{2,1}(I_T)}.$$

In addition, for $2r + s < 2 - 1/p$, we have $D_t^r D_x^s u\big|_{\overline{S_T}} \in W_p^{1-r-s/2-1/2p}(\overline{S_T})$ with

$$\|u\|_{W_p^{1-r-s/2-1/2p}(\overline{S_T})} \leq c(T)\|u\|_{W_p^{2,1}(I_T)}.$$

A useful technical lemma will now be presented. The proof of this lemma will be done in Appendix A.

Lemma 2.7 ***(L^∞ control of the spatial derivative).*** *Let $p > 3$ and let $0 < T \le 1/4$ (this condition is taken for simplification). Then for every $u \in W_p^{2,1}(I_T)$ with $u = 0$ on $\partial^p(I_T)$ in the trace sense (see Lemma (2.6)), there exists a constant $c(T,p) > 0$ such that*

$$\|u_x\|_{\infty,I_T} \le c(T,p)\|u\|_{W_p^{2,1}(I_T)}, \quad \textit{with} \quad c(T,p) = c(p)T^{\frac{p-3}{2p}} \to 0 \textit{ as } T \to 0.$$

2.2 BMO theory for parabolic equation

A very useful tool in this paper is the limit case of the L^p theory, $1 < p < \infty$, for parabolic equations, which is the BMO theory. Roughly speaking, if the function f appearing in (2.1) is in the L^p space for some $1 < p < \infty$, then we expect our solution u to have u_t and u_{xx} also in L^p. This is no longer valid in the limit case, i.e. when $p = \infty$. In this case, it is shown that the solution u of the parabolic equation have u_t and u_{xx} in the parabolic/anisotropic BMO (bounded mean oscillation) space that is convenient to present some of its related theories.

Definition 2.8 ***(Parabolic/Anisotropic BMO spaces).*** *A function $u \in L^1_{loc}(I_T)$ is said to be of bounded mean oscillation, $u \in BMO(I_T)$, if the quantity*

$$\sup_{Q_r \subset I_T} \left(\frac{1}{|Q_r|} \int_{Q_r} |u - m_{Q_r}(u)| \right)$$

is finite. Here the supremum is taken over all parabolic lower cylinders Q_r.

Remark 2.9 *The parabolic $BMO(I_T)$ space, which will be refereed, for simplicity, as the $BMO(I_T)$ space, and sometimes, where there is no confusion, as BMO space, is a Banach space (whose elements are defined up to an additive constant) equipped with the norm*

$$\|u\|_{BMO(I_T)} = \sup_{Q_r \subset I_T} \left(\frac{1}{|Q_r|} \int_{Q_r} |u - m_{Q_r}(u)| \right).$$

We move now to the two main theorems of this subsection, the BMO theory for parabolic equations, and the Kozono-Taniuchi parabolic type inequality. To be more precise, we have the following:

Theorem 2.10 ***(BMO theory for parabolic equations in the periodic case).*** *Take $0 < T_1 \le T$. Consider the following Cauchy problem:*

$$\begin{cases} u_t = \varepsilon u_{xx} + f & \textit{on} \quad \mathbb{R} \times (0,T), \\ u(x,0) = 0. \end{cases} \tag{2.10}$$

If $f \in L^\infty(\mathbb{R} \times (0,T))$ and f is a $2I$-periodic function in space, i.e. $f(x+2,t) = f(x,t)$, then there exists a unique solution $u \in BMO(\mathbb{R} \times (0,T))$ of (2.10) with u_t, $u_{xx} \in BMO(\mathbb{R} \times (0,T))$. Moreover, there exists $c > 0$ that may depend on T_1 but independent of T such that:

$$\|u_t\|_{BMO(\mathbb{R}\times(0,T))} + \|u_{xx}\|_{BMO(\mathbb{R}\times(0,T))} \le c\big[\|f\|_{BMO(\mathbb{R}\times(0,T))} + m_{2I\times(0,T)}(|f|)\big]. \tag{2.11}$$

The proof of this theorem will be presented in Appendix B. Our next tool (see Theorem 2.13) shows an estimate involving parabolic BMO spaces. This estimate is a control of the L^∞ norm of a given function by its BMO norm and the logarithm of its norm in a certain Sobolev space. It can also be considered as the parabolic version on a bounded domain I_T of the Kozono-Taniuchi inequality (see [20]) that we recall here.

Theorem 2.11 ***(The Kozono-Taniuchi inequality in the elliptic case, [20, Theorem 1]).*** *Let $1 < p < \infty$ and let $s > n/p$. There is a constant $C = C(n,p,s)$ such that, for all $f \in W_p^s(\mathbb{R}^n)$, the following estimate holds:*

$$\|f\|_{\infty,\mathbb{R}^n} \leq C\left(1 + \|f\|_{BMO_e(\mathbb{R}^n)}\left(1 + \log^+ \|f\|_{W_p^s(\mathbb{R}^n)}\right)\right), \ \log^+ = \max(0, \log). \quad (2.12)$$

Remark 2.12 *It is worth mentioning that the BMO_e norm appearing in (2.12) is the elliptic BMO_e norm, i.e. the one where the supremum is taken over ordinary balls*

$$B_r(X_0) = \{X \in \mathbb{R}^n;\ |X - X_0| < r\}.$$

The original type of the logarithmic Sobolev inequality was found in [3, 4] (see also [9]), where the authors investigated the relation between L^∞, W_r^k and W_p^s and proved that there holds the embedding

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq C\left(1 + \log^{\frac{r-1}{r}}\left(1 + \|u\|_{W_p^s(\mathbb{R}^n)}\right)\right), \quad sp > n$$

provided $\|u\|_{W_r^k} \leq 1$ for $kr = n$. This estimate was applied to prove existence of global solutions to the nonlinear Schrödinger equation (see [3, 14]).

In our work, we need to have an estimate similar to (2.12), but for the parabolic BMO space and on the bounded domain I_T. This will be essential, on one hand, to show a suitable positive lower bound of κ_x (κ given by Theorem 1.1), and on the other hand, to show the long time existence of our solution. Indeed, there is a similar inequality and this is what will be illustrated by the next theorem.

Theorem 2.13 ***(A parabolic Kozono-Taniuchi inequality, [17, Appendix B2], [19]).*** *Let $v \in W_2^{2,1}(I_T)$, then there exists a constant $c = c(T) > 0$ such that the estimate holds*

$$\|v\|_{\infty,I_T} \leq c\|v\|_{\overline{BMO}(I_T)}\left(1 + \log^+ \|v\|_{W_2^{2,1}(I_T)}\right), \quad (2.13)$$

where $\overline{BMO}(I_T) = BMO(I_T) \cap L^1(I_T)$, and for $v \in \overline{BMO}(I_T)$,

$$\|v\|_{\overline{BMO}(I_T)} = \|v\|_{BMO(I_T)} + \|v\|_{L^1(I_T)}.$$

This inequality is first shown over $\mathbb{R}_x \times \mathbb{R}_t$, then it is deduced over I_T.

3 A comparison principle

Proposition 3.1 ***(A comparison principle for system (1.1)).*** *Let*

$$(\rho, \kappa) \in \left(C^{3+\alpha, \frac{3+\alpha}{2}}\left(\overline{I_T}\right)\right)^2 \quad \textit{for some} \quad 0 < \alpha < 1,$$

be a solution of (1.1), (1.2) and (1.3) with $\kappa_x > 0$, *and the initial conditions* ρ^0, κ^0 *satisfying:*

$$\kappa_x^0 \geq \sqrt{\gamma_0^2 + (\rho_x^0)^2} \quad on \quad I, \quad \gamma_0 \in (0,1). \tag{3.1}$$

Choose $\beta = \beta(\varepsilon, \tau) > 0$ *large enough. Let the function* $\gamma(t)$ *satisfies:*

$$\begin{cases} \dfrac{\gamma'(t)}{\gamma(t)} \leq -\left(c_0 + \|\rho_{xxx}(.,t)\|_{L^\infty(I)}\right), \quad c_0 = c_0(\varepsilon, \beta, \tau), \\ \gamma(0) = \gamma_0/2. \end{cases} \tag{3.2}$$

Define $\overline{M}(x,t) := \cosh(\beta x)\{\kappa_x(x,t) - \sqrt{\gamma^2(t) + (\rho_x(x,t))^2}\}$ *for* $(x,t) \in \overline{I_T}$ *and* $T > 0$. *Then* $\overline{m}(t) := \min_{x \in I} \overline{M}(x,t)$ *satisfies* $\overline{m}(t) \geq \gamma^2(t)$ *for all* $t \in [0,T]$. *In particular, we have*

$$\kappa_x(x,t) \geq \sqrt{\gamma^2(t) + \rho_x^2(x,t)}, \quad in \quad \overline{I_T}. \tag{3.3}$$

Proof. Throughout the proof, we will extensively use the following notation:

$$G_a(y) = \sqrt{a^2 + y^2}, \quad a, y \in \mathbb{R}.$$

Without loss of generality (up to a change of variables in (x,t) and a re-definition of τ), assume in the proof that

$$I = (-1, 1).$$

Define the quantity M by:

$$M(x,t) = \kappa_x(x,t) - G_{\gamma(t)}(\rho_x(x,t)), \quad (x,t) \in \overline{I_T},$$

$\gamma(t) > 0$ is a function to be determined. The proof could be divided into five steps.

Step 1. (Partial differential inequality satisfied by M)

We first do the following computations on I_T:

$$M_t = \kappa_{xt} - G'_\gamma(\rho_x)\rho_{xt} - \frac{\gamma\gamma'}{\sqrt{\gamma^2 + \rho_x^2}}, \tag{3.4}$$

$$M_x = \kappa_{xx} - G'_\gamma(\rho_x)\rho_{xx}, \quad M_{xx} = \kappa_{xxx} - G''_\gamma(\rho_x)\rho_{xx}^2 - G'_\gamma(\rho_x)\rho_{xxx}. \tag{3.5}$$

Deriving (1.1) with respect to x, we deduce that

$$\begin{cases} \kappa_{xt} = \varepsilon\kappa_{xxx} + \dfrac{\rho_{xx}^2}{\kappa_x} + \dfrac{\rho_x\rho_{xxx}}{\kappa_x} - \dfrac{\rho_x\rho_{xx}\kappa_{xx}}{\kappa_x^2} - \tau\rho_{xx}, \\ \rho_{xt} = (1+\varepsilon)\rho_{xxx} - \tau\kappa_{xx}. \end{cases} \tag{3.6}$$

We set

$$\Gamma = \frac{\gamma\gamma'}{\sqrt{\gamma^2 + \rho_x^2}}, \quad F_\gamma(y) = y - \gamma\arctan(y/\gamma).$$

Doing again some direct computations, and using (3.4), (3.5) and (3.6), we obtain

$$\begin{aligned} M_t = \varepsilon M_{xx} + \left(\tau G_\gamma^{'}(\rho_x) - \frac{\rho_x\rho_{xx}}{\kappa_x^2}\right) M_x + \left(\frac{\rho_{xx}^2}{\kappa_x^2} - \frac{\rho_{xxx} G_\gamma^{'}(\rho_x)}{\kappa_x}\right) M \\ + \varepsilon G_\gamma^{''}(\rho_x)\rho_{xx}^2 + \frac{\rho_{xx}^2}{\kappa_x^2}[G_\gamma(\rho_x) - G_\gamma^{'}(\rho_x)\rho_x] - \tau(1 - F_\gamma^{'}(\rho_x))\rho_{xx} - \Gamma. \end{aligned} \tag{3.7}$$

Using Young's inequality $2ab \leq a^2 + b^2$, we have:

$$\frac{\tau\gamma^2|\rho_{xx}|}{\gamma^2 + \rho_x^2} \leq \frac{\varepsilon\gamma^2\rho_{xx}^2}{(\gamma^2 + \rho_x^2)^{3/2}} + \frac{\gamma^2\tau^2}{4\varepsilon\sqrt{\gamma^2 + \rho_x^2}}. \tag{3.8}$$

Plugging (3.8) into (3.7), and using some properties of G_γ and F_γ, we get:

$$\begin{aligned} M_t \geq \varepsilon M_{xx} + \left(\tau G_\gamma^{'}(\rho_x) - \frac{\rho_x\rho_{xx}}{\kappa_x^2}\right) M_x + \left(\frac{\rho_{xx}^2}{\kappa_x^2} - \frac{\rho_{xxx} G_\gamma^{'}(\rho_x)}{\kappa_x}\right) M \\ - \frac{\gamma^2\tau^2}{4\varepsilon\sqrt{\gamma^2 + \rho_x^2}} - \frac{\gamma\gamma^{'}}{\sqrt{\gamma^2 + \rho_x^2}}. \end{aligned}$$

Step 2. (The boundary conditions for M)

The boundary conditions (1.3), and the PDEs of system (1.1) imply the following equalities on the boundary (using the smoothness of the solution up to the boundary),

$$\begin{cases} \varepsilon\kappa_{xx} + \dfrac{\rho_x\rho_{xx}}{\kappa_x} - \tau\rho_x = 0 & \text{on} \quad \partial I \times [0, T] \\ (1 + \varepsilon)\rho_{xx} - \tau\kappa_x = 0 & \text{on} \quad \partial I \times [0, T]. \end{cases} \tag{3.9}$$

In particular (3.9) implies

$$M_x = -\frac{\tau}{1 + \varepsilon} G_\gamma^{'}(\rho_x) M \quad \text{on} \quad \partial I \times [0, T]. \tag{3.10}$$

To deal with the boundary condition (3.10), we now introduce the following change of unknown function:

$$\overline{M}(x, t) = \cosh(\beta x) M(x, t), \quad (x, t) \in \overline{I_T}.$$

We calculate $\overline{M}$ on the boundary of I to get:

$$\overline{M}_x = \left(\beta\tanh(\beta x) - \frac{\tau}{1 + \varepsilon} G_\gamma^{'}(\rho_x)\right) \overline{M} \quad \text{on} \quad \partial I \times [0, T]. \tag{3.11}$$

We claim that, for any fixed time t, it is impossible for $\overline{M}$ to have a positive minimum at the boundary of I. Indeed we have the following two cases:

$$\overline{M} \text{ has a positive minimum at } x = 1 \quad \Rightarrow \quad \overline{M}_x \leq 0;$$

$$\overline{M} \text{ has a positive minimum at } x = -1 \quad \Rightarrow \quad \overline{M}_x \geq 0.$$

Both cases violate the equation (3.11) in the case of the choice of $\beta = \beta(\varepsilon, \tau)$ large enough, and hence the minimum of $\overline{M}$ is attained inside the interval I. Direct computations give:

$$\overline{M}_t \geq \varepsilon \overline{M}_{xx} + \left[\tau G_\gamma^{'}(\rho_x) - \frac{\rho_x \rho_{xx}}{\kappa_x^2} - 2\beta\varepsilon \tanh(\beta x)\right] \overline{M}_x - \frac{\cosh(\beta x)\gamma^2\tau^2}{4\varepsilon\sqrt{\gamma^2 + \rho_x^2}} - \frac{\cosh(\beta x)\gamma\gamma^{'}}{\sqrt{\gamma^2+\rho_x^2}}$$
$$+ \left[\frac{\rho_{xx}^2}{\kappa_x^2} - \frac{\rho_{xxx} G_\gamma^{'}(\rho_x)}{\kappa_x} - \beta \tanh(\beta x)\left(\tau G_\gamma^{'}(\rho_x) - \frac{\rho_x\rho_{xx}}{\kappa_x^2}\right) + \varepsilon\beta^2(2\tanh^2(\beta x) - 1)\right]\overline{M}. \tag{3.12}$$

Step 3. (The inequality satisfied by the minimum of $\overline{M}$)

Let

$$\overline{m}(t) = \min_{x \in I} \overline{M}(x,t).$$

Since the minimum is attained inside I, and since $\overline{M}$ is regular, there exists $x_0(t) \in I$ such that $\overline{m}(t) = \overline{M}(x_0(t), t)$. We remark that we have:

$$\overline{M}_x(x_0(t), t) = 0, \quad \text{and} \quad \overline{M}_{xx}(x_0(t), t) \geq 0,$$

and hence, using (3.12), we can write down the equation satisfied by $\overline{m}$, we get (indeed in the viscosity sense):

$$\overline{m}_t \geq \overbrace{\left(\frac{\rho_{xx}^2}{\kappa_x^2} - \frac{\rho_{xxx} G_\gamma^{'}(\rho_x)}{\kappa_x} - \beta\tanh(\beta x)\left(\tau G_\gamma^{'}(\rho_x) - \frac{\rho_x\rho_{xx}}{\kappa_x^2}\right) + \varepsilon\beta^2(2\tanh^2(\beta x) - 1)\right)}^{\mathrm{R}}\overline{m}$$
$$- \frac{\cosh(\beta x)\gamma^2\tau^2}{4\varepsilon\sqrt{\gamma^2+\rho_x^2}} - \frac{\cosh(\beta x)\gamma\gamma^{'}}{\sqrt{\gamma^2+\rho_x^2}} \quad \text{at} \quad x = x_0(t). \tag{3.13}$$

Step 4. (Estimate of the term R)

We turn our attention now to the term R from (3.13). Using elementary identities, we get

$$R \geq -\frac{\rho_{xxx} G_\gamma^{'}(\rho_x)}{\kappa_x} - \frac{\beta^2\tanh^2(\beta x)}{4}\frac{\rho_x^2}{\kappa_x^2} - \frac{\tau^2}{8\varepsilon}(G_\gamma^{'}(\rho_x))^2 - \varepsilon\beta^2. \tag{3.14}$$

By (3.1), we know that

$$\overline{m}(0) \geq \gamma^2(0),$$

and the continuity of $\overline{m}$ preserves its positivity at least for short time. Then, as long as $\overline{m}$ is positive, we have

$$\kappa_x \geq \sqrt{\gamma^2 + \rho_x^2}. \tag{3.15}$$

Let

$$\widetilde{c}(t) = \|\rho_{xxx}(.,t)\|_{\infty, I}.$$

By using (3.15) and some basic identities, inequality (3.14) implies:

$$R \geq -\frac{\widetilde{c}}{\sqrt{\gamma^2+\rho_x^2}} - c_1, \quad c_1 = \frac{\beta^2}{4} + \frac{\tau^2}{8\varepsilon} + \varepsilon\beta^2. \tag{3.16}$$

Step 5. (The choice of γ and conclusion)

When $\gamma' \leq 0$, we deduce from (3.13) and (3.16) that the function $\overline{m}$ is a viscosity super-solution of:

$$\overline{m}_t = -\left(\frac{\widetilde{c}}{\sqrt{\gamma^2+\rho_x^2}} + c_1\right)\overline{m} - \frac{c_2\gamma^2}{\sqrt{\gamma^2+\rho_x^2}} - \frac{\gamma\gamma'}{\sqrt{\gamma^2+\rho_x^2}}, \quad c_2 = \left(\frac{\tau^2\cosh\beta}{4\varepsilon}\right). \tag{3.17}$$

We remind the reader that $\rho_x = \rho_x(x_0(t), t)$. Take the function γ satisfying:

$$\begin{cases} \dfrac{\gamma'}{\gamma} \leq -(c_0 + \widetilde{c}), \quad c_0 = \min(c_1, c_2), \\ \gamma(0) = \gamma_0/2 \end{cases}$$

Plug $\overline{m} = \gamma^2$ into (3.17), we directly deduce that γ^2 is a viscosity sub-solution of (3.17), and the result follows by comparison. □

4 Short time existence, uniqueness, and regularity

In this section, we will prove a result of short time existence, uniqueness and regularity of a solution of problem (1.1), (1.2) and (1.3).

4.1 Short-time existence and uniqueness of a truncated system

We denote

$$I_{a,b} := I \times (a, a+b), \quad a, b \geq 0.$$

Fix $T_0 \geq 0$. Consider the following system defined on $I_{T_0,T}$ by:

$$\begin{cases} \kappa_t = \varepsilon\kappa_{xx} + \dfrac{\rho_x\rho_{xx}}{\kappa_x} - \tau\rho_x & \text{on} \quad I_{T_0,T} \\ \rho_t = (1+\varepsilon)\rho_{xx} - \tau\kappa_x & \text{on} \quad I_{T_0,T}, \end{cases} \tag{4.1}$$

with the initial conditions:

$$\kappa(x, T_0) = \kappa^{T_0}(x) \quad \text{and} \quad \rho(x, T_0) = \rho^{T_0}(x), \tag{4.2}$$

and the boundary conditions:

$$\begin{cases} \kappa(0,.) = 0 \quad \text{and} \quad \kappa(1,.) = 1 & \text{for} \quad T_0 < t < T_0 + T \\ \rho(0,.) = \rho(1,.) = 0, & \text{for} \quad T_0 < t < T_0 + T. \end{cases} \tag{4.3}$$

Remark 4.1 ***(The terms*** p ***and*** α***).*** *In all what follows, and unless otherwise precised, the terms p and $\alpha \in (0,1)$ are two fixed positive real numbers such that*

$$p > 3 \quad \textit{and} \quad \alpha = 1 - 3/p.$$

Concerning system (4.1), (4.2) and (4.3), we have the following existence and uniqueness result.

Proposition 4.2 ***(Short time existence and uniqueness).*** *Let $p > 3$, and $T_0 \geq 0$. Let*

$$\rho^{T_0}, \kappa^{T_0} \in C^\infty(\bar{I} \times \{T_0\})$$

be two given functions such that $\rho^{T_0}(0) = \rho^{T_0}(1) = \kappa^{T_0}(0) = 0$, and $\kappa^{T_0}(1) = 1$. Suppose furthermore that

$$\kappa_x^{T_0} \geq \gamma_0 \quad on \quad I \times \{t = T_0\},$$

and

$$\|(D_x^s \rho^{T_0}, D_x^s \kappa^{T_0})\|_{\infty,I} \leq M_0 \quad on \quad I \times \{t = T_0\}, \quad s = 1, 2,$$

where $\gamma_0 > 0$ and $M_0 > 0$ are two given positive real numbers. Then there exists

$$T = T^* = T^*(M_0, \gamma_0, \varepsilon, \tau, p) > 0, \tag{4.4}$$

such that the system (4.1), (4.2) and (4.3) admits a unique solution

$$(\rho, \kappa) \in (W_p^{2,1}(I_{T_0,T}))^2.$$

Moreover, this solution satisfies

$$\kappa_x \geq \gamma_0/2 \quad on \quad \overline{I_{T_0,T}}, \tag{4.5}$$

and

$$|\rho_x| \leq 2M_0 \quad on \quad \overline{I_{T_0,T}}. \tag{4.6}$$

Proof. The short time existence is done by using a fixed point argument. Since we are looking for solutions satisfying (4.5) and (4.6), we artificially modify (4.1), and look for a solution of

$$\begin{cases} \kappa_t = \varepsilon \kappa_{xx} + \dfrac{\rho_{xx} T_{2M_0}(\rho_x)}{(\gamma_0/2) + (\kappa_x - \gamma_0/2)^+} - \tau \rho_x & \text{in} \quad I_{T_0,T} \\ \rho_t = (1 + \varepsilon)\rho_{xx} - \tau \kappa_x & \text{in} \quad I_{T_0,T}, \end{cases} \tag{4.7}$$

with the truncation function $T_\zeta(x) = x 1\!\!1_{(-\zeta,\zeta)} + \zeta 1\!\!1_{\{x \geq \zeta\}} - \zeta 1\!\!1_{\{x \leq -\zeta\}}$, $\zeta > 0$, and satisfying the same initial and boundary data (4.2), (4.3). Denote

$$Y = W_p^{2,1}(I_{T_0,T}).$$

For any constant $\lambda > 0$, let us define D_λ^ρ and D_λ^κ as the two closed subsets of Y given by:

$$D_\lambda^\rho = \{u \in Y;\ \|u_x\|_{p, I_{T_0,T}} \leq \lambda,\ \ u = \rho^{T_0} \text{ on } \partial^p I_{T_0,T}\}$$

and

$$D_\lambda^\kappa = \{v \in Y;\ \|v_x\|_{p, I_{T_0,T}} \leq \lambda,\ \ v = \kappa^{T_0} \text{ on } \partial^p I_{T_0,T}\}.$$

We choose λ large enough such that these sets are nonempty. Define the application Ψ by:

$$\begin{aligned} \Psi : D_\lambda^\rho \times D_\lambda^\kappa &\longmapsto \quad D_\lambda^\rho \times D_\lambda^\kappa \\ (\hat{\rho}, \hat{\kappa}) &\longmapsto \quad \Psi(\hat{\rho}, \hat{\kappa}) = (\rho, \kappa), \end{aligned}$$

where (ρ,κ) is a solution of the following system:

$$\begin{cases} \kappa_t = \varepsilon\kappa_{xx} + \dfrac{\rho_{xx}T_{2M_0}(\hat\rho_x)}{(\gamma_0/2)+(\hat\kappa_x-\gamma_0/2)^+} - \tau\hat\rho_x & \text{in} \quad I_{T_0,T}, \\ \rho_t = (1+\varepsilon)\rho_{xx} - \tau\hat\kappa_x & \text{in} \quad I_{T_0,T}, \end{cases} \tag{4.8}$$

with the same initial and boundary conditions given by (4.2) and (4.3) respectively. The existence of the solution of (4.8), (4.2) and (4.3) is a direct consequence of Theorem 2.3. Taking $\bar\rho(x,t) = \rho(x,t) - \rho^{T_0}(x)$ and $\bar\kappa(x,t) = \kappa(x,t) - \kappa^{T_0}(x)$, we can easily check that $(\bar\rho,\bar\kappa)$ satisfies a parabolic system similar to (4.8) with $(\bar\rho,\bar\kappa) = 0$ on $\partial^p I_{T_0,T}$. Using Sobolev estimates for parabolic equations to the system satisfied by $(\bar\rho,\bar\kappa)$, particularly (2.7), we deduce that for sufficiently small $T>0$, we have $\|\rho_x\|_{p,I_{T_0,T}} \le \lambda$, $\|\kappa_x\|_{p,I_{T_0,T}} \le \lambda$, and hence the application Ψ is well defined.

The application Ψ is a contraction map. Let $\Psi(\hat\rho,\hat\kappa) = (\rho,\kappa)$ and $\Psi(\hat\rho',\hat\kappa') = (\rho',\kappa')$. Direct computations, using in particular (2.7), give:

$$\|\rho-\rho'\|_Y \le c\sqrt{T}\|\hat\kappa-\hat\kappa'\|_Y, \tag{4.9}$$

and

$$\|\kappa-\kappa'\|_Y \le c\|F\|_{p,I_{T_0,T}}, \tag{4.10}$$

with the function F satisfying:

$$\begin{aligned} F + \tau(\hat\rho-\hat\rho')_x = {} & \overbrace{\frac{T_{2M_0}(\hat\rho_x)}{(\gamma_0/2)+(\hat\kappa_x-\gamma_0/2)^+}(\rho_{xx}-\rho'_{xx})}^{A_1} + \overbrace{\frac{\rho'_{xx}(T_{2M_0}(\hat\rho_x)-T_{2M_0}(\hat\rho'_x))}{(\gamma_0/2)+(\hat\kappa_x-\gamma_0/2)^+}}^{A_2} \\ & + \overbrace{\rho'_{xx}T_{2M_0}(\hat\rho'_x)\left(\frac{1}{(\gamma_0/2)+(\hat\kappa_x-\gamma_0/2)^+} - \frac{1}{(\gamma_0/2)+(\hat\kappa'_x-\gamma_0/2)^+}\right)}^{A_3}. \end{aligned} \tag{4.11}$$

In order to prove the contraction for some small $T>0$, we need to estimate all the terms appearing in (4.11). The term A_1 can be easily handled. However, for the term A_2, we proceed as follows. We apply the L^∞ control of the spatial derivative (see Lemma 2.7) to the function $\hat\rho-\hat\rho'$, we get:

$$\|(\hat\rho-\hat\rho')_x\|_{\infty,I_{T_0,T}} \le cT^{\frac{p-3}{2p}}\|\hat\rho-\hat\rho'\|_Y. \tag{4.12}$$

For the term ρ'_{xx}, we apply (2.7), and hence we deduce that

$$\|\rho'_{xx}\|_{p,I_{T_0,T}} \le c(M_0+\lambda). \tag{4.13}$$

From (4.12) and (4.13), we deduce that

$$\|A_2\|_{p,I_{T_0,T}} \le c\frac{(M_0+\lambda)}{\gamma_0}T^{\frac{p-3}{2p}}\|\hat\rho-\hat\rho'\|_Y.$$

The term A_3 could be treated in a similar way as the term A_2. The above arguments, particularly (4.9) and (4.10), give the contraction of Ψ for small time $T = T^*(M_0,\gamma_0,\varepsilon,\tau,p) > 0$. Finally, inequalities (4.5) and (4.6) directly follow using the Sobolev embedding in Hölder spaces (Lemma 2.5). $\square$

4.2 Regularity of the solution

This subsection is devoted to show that the solution of (4.1), (4.2) and (4.3) enjoys more regularity than the one indicated in Proposition 4.2. This will be done using a special bootstrap argument, together with the Hölder regularity of solutions of parabolic equations.

Proposition 4.3 ***(Regularity of the solution: bootstrap argument).*** *Under the same hypothesis of Proposition 4.2, let ρ^{T_0} and κ^{T_0} satisfy:*

$$\begin{cases} (1+\varepsilon)\rho^{T_0}_{xx} = \tau\kappa^{T_0}_x & \text{at} \quad \partial I, \\ (1+\varepsilon)\kappa^{T_0}_{xx} = \tau\rho^{T_0}_x & \text{at} \quad \partial I. \end{cases} \tag{4.14}$$

Then the unique solution (ρ, κ) given by Proposition 4.2 is in fact more regular. Precisely, it satisfies for $\alpha = 1 - 3/p$:

$$\rho, \kappa \in C^{3+\alpha, \frac{3+\alpha}{2}}(\overline{I_{T_0,T}}) \cap C^\infty(\bar{I} \times (T_0, T_0 + T)), \tag{4.15}$$

where T is the time given by Proposition 4.2.

Proof. For the sake of simplicity, let us suppose that $T_0 = 0$.

The Hölder regularity. Since $\kappa \in W^{2,1}_p(I_T)$, we use Lemma 2.5 to deduce that $\kappa_x \in C^{\alpha,\alpha/2}(\overline{I_T})$. We apply the Hölder theory for parabolic equations Theorem 2.1, to the second equation of (4.1) (using in particular the regularity of the initial data ρ^0), we deduce that:

$$\rho \in C^{2+\alpha,1+\alpha/2}(\overline{I_T}). \tag{4.16}$$

Here the compatibility condition is satisfied by (4.14). Using (4.16) and (4.5), we deduce that $\frac{\rho_x\rho_{xx}}{\kappa_x} - \tau\rho_x \in C^{\alpha,\alpha/2}(\overline{I_T})$ and similar arguments as above give that:

$$\kappa \in C^{2+\alpha,1+\alpha/2}(\overline{I_T}). \tag{4.17}$$

Repeating the above arguments, using this time (see (4.17)) that $\kappa_x \in C^{1+\alpha,\frac{1+\alpha}{2}}(\overline{I_T})$, and hence

$$\rho \in C^{3+\alpha,\frac{3+\alpha}{2}}(\overline{I_T}), \tag{4.18}$$

where (4.18) directly implies that $\frac{\rho_x\rho_{xx}}{\kappa_x} - \tau\rho_x \in C^{1+\alpha,\frac{1+\alpha}{2}}(\overline{I_T})$, and therefore

$$\kappa \in C^{3+\alpha,\frac{3+\alpha}{2}}(\overline{I_T}). \tag{4.19}$$

The compatibility condition of order 1 which is needed to apply Theorem 2.1 is always satisfied by (4.14). The Hölder regularity of (ρ, κ) directly follows from (4.18) and (4.19).

The C^∞ regularity. In order to get the C^∞ regularity, we argue as in the case of the Hölder regularity (bootstrap argument). In this case the compatibility condition is replaced by multiplying by a test function that vanishes near $t = 0$. $\square$.

5 Exponential bounds

In this section, we will give some exponential bounds of the solution given by Proposition 4.2, and having the regularity shown by Proposition 4.3. It is very important, throughout all this section, to precise our notation concerning the constants that may certainly vary from line to line. Let us mention that a constant depending on time will be denoted by $c(T)$. Those which do not depend on T will be simply denoted by c. In all other cases, we will follow the changing of the constants in a precise manner.

Proposition 5.1 ***(Exponential bound in time for ρ_x and κ_x).*** *Let*

$$\rho, \kappa \in C^{3+\alpha, \frac{3+\alpha}{2}}(\bar{I} \times [0,\infty)) \cap C^{\infty}(\bar{I} \times (0,\infty)),$$

be a solution of (1.1), (1.2) and (1.3), with $\rho^0(0) = \rho^0(1) = 0$, $\kappa^0(0) = 0$ and $\kappa^0(1) = 1$. Suppose furthermore that the function

$$B = \frac{\rho_x}{\kappa_x} \quad \textit{satisfies} \quad \|B\|_{L^\infty(I\times(0,\infty))} \leq 1.$$

Then, for small $T^* = T^*(\varepsilon, \tau, p) > 0$, and $A = 1 + \|\rho^0\|_{W_p^{2-2/p}(I)} + \|\kappa^0\|_{W_p^{2-2/p}(I)}$, we have for all $t \geq 0$:

$$|\rho_x|^{(\alpha)}_{I_{t,T^*}}, |\kappa_x|^{(\alpha)}_{I_{t,T^*}} \leq cAe^{ct}, \tag{5.1}$$

and c is a fixed constant independent of the initial data.

Proof. We use the special coupling of the system (1.1) to find our *a priori* estimate. Roughly speaking, the fact that κ_x appears as a source term in the second equation of system (1.1) permits, by the L^p theory for parabolic equations, to have L^p bounds, in terms of $\|\kappa_x\|_{p,I_T}$, on ρ_x and ρ_{xx} which in their turn appear in the source terms of the first equation of (1.1) satisfied by κ. All this permit to deduce our estimates. To be more precise, let $T > 0$ an arbitrarily fixed time, the proof is divided into four steps:

Step 1. (estimating κ_x in the L^p norm)

Let κ' be the solution of the following equation:

$$\begin{cases} \kappa'_t = \kappa'_{xx} & \text{on} \quad I_T \\ \kappa' = \kappa & \text{on} \quad \partial^p I_T. \end{cases} \tag{5.2}$$

As a solution of a parabolic equation, we use the L^p parabolic estimate (2.6) to the function κ' to deduce that:

$$\|\kappa'\|_{W_p^{2,1}(I_T)} \leq c(T)\left(\|\kappa^0\|_{W_p^{2-2/p}(I)} + 1\right), \tag{5.3}$$

where the term 1 comes from the value of $\kappa' = \kappa$ on S_T. Take

$$\bar{\kappa} = \kappa - \kappa', \tag{5.4}$$

then the system satisfied by $\bar{\kappa}$ reads:

$$\begin{cases} \bar{\kappa}_t = \bar{\kappa}_{xx} - (\kappa_t^{'} - \varepsilon\kappa_{xx}^{'}) + \dfrac{\rho_x\rho_{xx}}{\kappa_x} - \tau\rho_x \quad \text{on} \quad I_T \\ \bar{\kappa} = 0 \quad \text{on} \quad \partial^p I_T. \end{cases}$$

Using the special version (2.7) of the parabolic L^p estimate to the function $\bar{\kappa}$, we obtain:

$$\|\bar{\kappa}_x\|_{p,I_T} \leq c\sqrt{T}\left(\|\kappa_t^{'}\|_{p,I_T} + \|\kappa_{xx}^{'}\|_{p,I_T} + \|\rho_{xx}\|_{p,I_T} + \|\rho_x\|_{p,I_T}\right), \tag{5.5}$$

where we have plugged into the constant c the terms ε, τ, p and $\|B\|_\infty$. Combining (5.3), (5.4) and (5.5), we get:

$$\|\kappa_x\|_{p,I_T} \leq c(T)\left(\|\kappa^0\|_{W_p^{2-2/p}(I)} + 1\right) + c\sqrt{T}\|\rho\|_{W_p^{2,1}(I_T)}. \tag{5.6}$$

The term $\|\rho\|_{W_p^{2,1}(I_T)}$ appearing in the previous inequality is going to be estimated in the next step.

Step 2. (estimating ρ in the $W_p^{2,1}$ norm)

As in Step 1, let $\rho^{'}$, $\bar{\rho}$ be the two functions defined similarly as $\kappa^{'}$, $\bar{\kappa}$ respectively (see (5.2) and (5.4)). The function $\rho^{'}$ satisfies an inequality similar to (5.3) that reads:

$$\|\rho^{'}\|_{W_p^{2,1}(I_T)} \leq c(T)\|\rho^0\|_{W_p^{2-2/p}(I)}. \tag{5.7}$$

The term 1 disappeared here because $\rho^{'} = \rho = 0$ on $\overline{S_T}$. We write the system satisfied by $\bar{\rho}$, we obtain:

$$\begin{cases} \bar{\rho}_t = (1+\varepsilon)\bar{\rho}_{xx} + ((1+\varepsilon)\rho_{xx}^{'} - \rho_t^{'}) - \tau\kappa_x \quad \text{on} \quad I_T \\ \bar{\rho}(x,0) = 0 \quad \text{on} \quad \partial^p I_T, \end{cases}$$

hence the following estimate on $\bar{\rho}$, due to the special L^p interior estimate (2.7), holds:

$$\|\bar{\rho}\|_{W_p^{2,1}(I_T)} \leq c\left(\|\rho_t^{'}\|_{p,I_T} + \|\rho_{xx}^{'}\|_{p,I_T} + \|\kappa_x\|_{p,I_T}\right). \tag{5.8}$$

Again, we have plugged ε, τ and p into the constant c, and we have assumed that $T \leq 1$. Combining (5.7) and (5.8), we get in terms of ρ:

$$\|\rho\|_{W_p^{2,1}(I_T)} \leq c(T)\|\rho^0\|_{W_p^{2-2/p}(I)} + c\|\kappa_x\|_{p,I_T}. \tag{5.9}$$

We will use this estimate in order to have a control on $\|\kappa_x\|_{p,I_T}$ for sufficiently small time.

Step 3. (Estimate on a small time interval)

From (5.6) and (5.9), we deduce that:

$$\|\kappa_x\|_{p,I_T} \leq c(T)\left(\|\kappa^0\|_{W_p^{2-2/p}(I)} + \|\rho^0\|_{W_p^{2-2/p}(I)} + 1\right) + c\sqrt{T}\|\kappa_x\|_{p,I_T}. \tag{5.10}$$

Let us remind the reader that all constants c and $c(T)$ have been changing from line to line. In fact, the important thing is whether they depend on T or not. Let

$$T^* = \frac{1}{2c^2}, \quad c \text{ is the constant appearing in (5.10)},$$

we deduce, from (5.10), that

$$\|\kappa_x\|_{p,I_{T^*}} \leq c_3 \left(\|\kappa^0\|_{W_p^{2-2/p}(I)} + \|\rho^0\|_{W_p^{2-2/p}(I)} + 1 \right),$$

where $c_3 = c_3(T^*) > 0$ is a positive constant which depends on T^*. Recall the special coupling of system (1.1), together with the above estimate, we can deduce that:

$$\|(\rho, \kappa)\|_{W_p^{2,1}(I_{T^*})} \leq c_4 \left(\|\kappa^0\|_{W_p^{2-2/p}(I)} + \|\rho^0\|_{W_p^{2-2/p}(I)} + 1 \right), \tag{5.11}$$

with $c_4 = c_4(T^*) > 0$ is also a positive constant depending on T^* but independent of the initial data.

Step 4. (The exponential estimate by iteration)

Now we move to show the exponential bound. Set

$$f(t) = \|(\rho, \kappa)\|_{W_p^{2,1}(I\times(t,t+T^*))}, \quad \text{and} \quad g(t) = \|\kappa(\cdot,t)\|_{W_p^{2-2/p}(I)} + \|\rho(\cdot,t)\|_{W_p^{2-2/p}(I)}.$$

Using estimate (5.11) of Lemma 2.6, together with estimate (5.11) of Step 3, we get

$$g(T^*) \leq c_5 f(0) \leq c_5 c_4 (g(0) + 1), \quad c_5 = c_5(T^*).$$

In this case, the Sobolev embedding in Hölder spaces (see Lemma 2.6), and the time iteration give immediately the result. □

Proposition 5.2 ***(Exponential bound in time for ρ_{xx}).*** *Under the same hypothesis of Proposition 5.1, and for some $T^* = T^*(\varepsilon, \tau, p) > 0$, we have:*

$$|\rho|_{I_{t,T^*}}^{(2+\alpha)} \leq cAe^{ct}, \quad t \geq 0, \tag{5.12}$$

where $A = 1 + \|\rho^0\|_{W_p^{2-2/p}(I)} + \|\kappa^0\|_{W_p^{2-2/p}(I)} + |\rho^0|_I^{(2+\alpha)}$, and $c > 0$ is a fixed positive constant independent of the initial data.

Proof. The proof is very similar to the proof of Proposition 5.1. It uses in particular the Hölder estimate for parabolic equations (namely (2.4)), the Hölder embedding in Sobolev spaces (Lemma 2.5), and finally the iteration in time. □

Remark 5.3 *We can not obtain, using similar arguments as in the proof of Proposition 5.2, a similar exponential bound (5.12) for the term $|\kappa|_{I_{t,T^*}}^{(2+\alpha)}$. This is due to the presence of the term $1/\kappa_x$ in the equation involving κ.*

6 An upper bound for the $W_2^{2,1}$ norm of ρ_{xxx}

This section is devoted to give a suitable upper bound for the $W_2^{2,1}$ norm of ρ_{xxx}. This result will be a consequence of the control of the $W_2^{2,1}$ norm of κ_t and κ_{xx}. The goal is to use this upper bound in the parabolic Kozono-Taniuchi inequality (see inequality (2.13) of Theorem 2.13) in order to control the L^∞ norm of ρ_{xxx}.

Let us fix $T_1 > 0$. In this section (Section 6) and in the following section (Section 7), we will obtain some estimates on the solution (ρ, κ) on the time interval $(0, T)$, with

$$T > T_1 > 0. \tag{6.1}$$

In these estimates, we will precise the dependence on T which involves some constants depending on T_1 that may blow up as T_1 goes to zero. Consider the following hypothesis:

(H1) The function κ_x satisfies:

$$\kappa_x(x,t) \geq \gamma(t) > 0,$$

where $\gamma(t)$ is a positive decreasing function with $\gamma(0) = \gamma_0/2$, $\gamma_0 \in (0,1)$.

Let

$$\mathcal{D} = I_T,$$

we start with the following lemma.

Lemma 6.1 ***($W_2^{2,1}$ bound for κ_t and κ_{xx}).*** *Under hypothesis (H1), and under the same hypothesis of Proposition 5.1, we have:*

$$\|\kappa_t\|_{W_2^{2,1}(\mathcal{D})}, \|\kappa_{xx}\|_{W_2^{2,1}(\mathcal{D})} \leq \frac{E}{\gamma^4},$$

where

$$\gamma := \gamma(T) \quad \textit{and} \quad E := de^{dT},$$

with $d \geq 1$ is a positive constant depending on the initial conditions but independent of T, and will be given at the end of the proof.

Remark 6.2 ***(The constant E depending on time).*** *Let us stress on the fact that, throughout the proof and in the rest of the paper, the term $E = de^{dT}$ of Lemma 6.1 might vary from line to line. In other words, the term d in the expression of E might certainly vary from line to line, but always satisfying the fact of just being dependent on the initial data of the problem. The different E's appearing in different estimates can be made the same by simply taking the maximum between them. Therefore they will all be denoted by the same letter E.*

Proof of Lemma 6.1. Define the functions u and v by:

$$u(x,t) = \rho_t(x,t) \quad \text{and} \quad v(x,t) = \kappa_t(x,t).$$

We write down the equations satisfied by u and v respectively:

$$\begin{cases} u_t = (1+\varepsilon)u_{xx} - \tau v_x & \text{on} \quad \mathcal{D}, \\ u|_{S_T} = 0, \\ u|_{t=0} = u^0 := (1+\varepsilon)\rho^0_{xx} - \tau\kappa^0_x & \text{on} \quad I, \end{cases} \tag{6.2}$$

$$\begin{cases} v_t = \varepsilon v_{xx} + \dfrac{\rho_{xx}}{\kappa_x}u_x + Bu_{xx} - B\dfrac{\rho_{xx}}{\kappa_x}v_x - \tau u_x & \text{on} \quad \mathcal{D}, \\ v|_{S_T} = 0, \\ v|_{t=0} = v^0 := \varepsilon\kappa^0_{xx} + \dfrac{\rho^0_x\rho^0_{xx}}{\kappa^0_x} - \tau\rho^0_x & \text{on} \quad I. \end{cases} \tag{6.3}$$

The proof could be divided into three steps. As a first step, we will estimate the $L^\infty(\mathcal{D})$ norm of the term $v_x = \kappa_{tx}$. In the second step, we will control the $W^{2,1}_2(\mathcal{D})$ norm of $v = \kappa_t$. Finally, in the third step, we will show how to deduce a similar control on the $W^{2,1}_2(\mathcal{D})$ norm of κ_{xx}.

Step 1. (Estimating $\|\kappa_{xt}\|_{\infty,\mathcal{D}}$)

It is worth recalling the equation satisfied by κ:

$$\kappa_t = \varepsilon\kappa_{xx} + \frac{\rho_x\rho_{xx}}{\kappa_x} - \tau\rho_x.$$

In Proposition 4.3, we have shown that $\kappa \in C^{3+\alpha,\frac{3+\alpha}{2}}$. Therefore, writing the parabolic Hölder estimate (see (2.4)), we obtain:

$$\|\kappa_{tx}\|_{\infty,\mathcal{D}} \le |\kappa|^{(3+\alpha)}_{\mathcal{D}} \le c^H\left(1 + \left|\frac{\rho_x\rho_{xx}}{\kappa_x}\right|^{(1+\alpha)}_{\mathcal{D}} + |\rho_x|^{(1+\alpha)}_{\mathcal{D}}\right), \tag{6.4}$$

where the term 1 comes from the boundary conditions, and $c^H > 0$ is a positive constant that can be estimated as $c^H \le E$ (see Remark 2.2). We use the elementary identity

$$|fg|^{(1+\alpha)}_{\mathcal{D}} \le \|f\|_{\infty,\mathcal{D}}|g|^{(1+\alpha)}_{\mathcal{D}} + \|g\|_{\infty,\mathcal{D}}|f|^{(1+\alpha)}_{\mathcal{D}} + \|f_x\|_{\infty,\mathcal{D}}|g|^{(\alpha)}_{\mathcal{D}} + \|g_x\|_{\infty,\mathcal{D}}|f|^{(\alpha)}_{\mathcal{D}},$$

to the term $\left|\frac{\rho_x\rho_{xx}}{\kappa_x}\right|^{(1+\alpha)}_{\mathcal{D}}$ with $f = \frac{\rho_x}{\kappa_x}$ and $g = \rho_{xx}$, we get:

$$\begin{aligned} \left|\frac{\rho_x\rho_{xx}}{\kappa_x}\right|^{(1+\alpha)}_{\mathcal{D}} &\le 3|\rho|^{(3+\alpha)}_{\mathcal{D}} + \|\rho_{xx}\|_{\infty,\mathcal{D}}\left\langle\frac{\rho_x}{\kappa_x}\right\rangle^{(1+\alpha)}_{\mathcal{D}} + \|\rho_{xxx}\|_{\infty,\mathcal{D}}\left\langle\frac{\rho_x}{\kappa_x}\right\rangle^{(\alpha)}_{\mathcal{D}} \\ &+ \frac{2|\rho|^{(2+\alpha)}_{\mathcal{D}}}{\gamma}(\|\rho_{xx}\|_{\infty,\mathcal{D}} + \|\kappa_{xx}\|_{\infty,\mathcal{D}}), \end{aligned} \tag{6.5}$$

where we have used the fact that $\kappa_x \ge \gamma$ and $\kappa_x \ge |\rho_x|$. We plug (6.5) in (6.4), we obtain:

$$\|\kappa_{tx}\|_{\infty,\mathcal{D}} \le E\left(1 + |\rho|^{(3+\alpha)}_{\mathcal{D}} + \left\langle\frac{\rho_x}{\kappa_x}\right\rangle^{(1+\alpha)}_{\mathcal{D}} + |\rho|^{(3+\alpha)}_{\mathcal{D}}\left\langle\frac{\rho_x}{\kappa_x}\right\rangle^{(\alpha)}_{\mathcal{D}} + \frac{1}{\gamma}\left(1 + |\kappa|^{(2+\alpha)}_{\mathcal{D}}\right)\right), \tag{6.6}$$

where we have used used the fact that the term $|\rho|_{\mathcal{D}}^{(2+\alpha)}$ has an exponential bound (see Proposition 5.2) of the form $|\rho|_{\mathcal{D}}^{(2+\alpha)} \le E$.

Step 1.1. $\left(\text{Estimating } \left\langle \frac{\rho_x}{\kappa_x} \right\rangle_{\mathcal{D}}^{(1+\alpha)}\right)$

From the definition of the Hölder norm, we see that in order to control $\left\langle \frac{\rho_x}{\kappa_x} \right\rangle_{\mathcal{D}}^{(1+\alpha)}$, it suffices to control the three quantities:

$$\left\langle \frac{\rho_x}{\kappa_x} \right\rangle_{t,\mathcal{D}}^{\left(\frac{1+\alpha}{2}\right)}, \quad \left\langle \left(\frac{\rho_x}{\kappa_x}\right)_x \right\rangle_{x,\mathcal{D}}^{(\alpha)}, \quad \text{and} \quad \left\langle \left(\frac{\rho_x}{\kappa_x}\right)_x \right\rangle_{t,\mathcal{D}}^{\left(\frac{\alpha}{2}\right)}.$$

We use the the following identity:

$$\left\langle \frac{f}{g} \right\rangle_{t,\mathcal{D}}^{(\alpha)} \le \left\| \frac{f}{g} \right\|_{\infty,\mathcal{D}} \left\| \frac{1}{g} \right\|_{\infty,\mathcal{D}} \langle g \rangle_{t,\mathcal{D}}^{(\alpha)} + \left\| \frac{1}{g} \right\|_{\infty,\mathcal{D}} \langle f \rangle_{t,\mathcal{D}}^{(\alpha)},$$

with $f = \rho_x$ and $g = \kappa_x$, we get

$$\left\langle \frac{\rho_x}{\kappa_x} \right\rangle_{t,\mathcal{D}}^{\left(\frac{1+\alpha}{2}\right)} \le \frac{1}{\gamma} \left(\langle \rho_x \rangle_{t,\mathcal{D}}^{\left(\frac{1+\alpha}{2}\right)} + \langle \kappa_x \rangle_{t,\mathcal{D}}^{\left(\frac{1+\alpha}{2}\right)} \right). \tag{6.7}$$

Similarly, we obtain:

$$\left\langle \frac{\rho_{xx}}{\kappa_x} \right\rangle_{x,\mathcal{D}}^{(\alpha)} \le \frac{\|\rho_{xx}\|_{\infty,\mathcal{D}}}{\gamma^2} \langle \kappa_x \rangle_{x,\mathcal{D}}^{(\alpha)} + \frac{\langle \rho_{xx} \rangle_{x,\mathcal{D}}^{(\alpha)}}{\gamma}. \tag{6.8}$$

We also use the inequality:

$$\langle fg \rangle_{x,\mathcal{D}}^{(\alpha)} \le \|f\|_{\infty,\mathcal{D}} \langle g \rangle_{x,\mathcal{D}}^{(\alpha)} + \|g\|_{\infty,\mathcal{D}} \langle f \rangle_{x,\mathcal{D}}^{(\alpha)},$$

with $f = \frac{\kappa_{xx}}{\kappa_x}$ and $g = \frac{\rho_x}{\kappa_x}$, we get:

$$\left\langle \frac{\kappa_{xx}\rho_x}{\kappa_x^2} \right\rangle_{x,\mathcal{D}}^{(\alpha)} \le \frac{\langle \kappa_{xx} \rangle_{x,\mathcal{D}}^{(\alpha)}}{\gamma} + \frac{\|\kappa_{xx}\|_{\infty,\mathcal{D}}}{\gamma^2} \langle \rho_x \rangle_{x,\mathcal{D}}^{(\alpha)} + \frac{\|\kappa_{xx}\|_{\infty,\mathcal{D}}}{\gamma^2} \langle \kappa_x \rangle_{x,\mathcal{D}}^{(\alpha)}. \tag{6.9}$$

Similarly, we get

$$\left\langle \frac{\rho_{xx}}{\kappa_x} \right\rangle_{t,\mathcal{D}}^{\left(\frac{\alpha}{2}\right)} \le \frac{\|\rho_{xx}\|_{\infty,\mathcal{D}}}{\gamma^2} \langle \kappa_x \rangle_{t,\mathcal{D}}^{\left(\frac{\alpha}{2}\right)} + \frac{\langle \rho_{xx} \rangle_{t,\mathcal{D}}^{\left(\frac{\alpha}{2}\right)}}{\gamma}, \tag{6.10}$$

and

$$\left\langle \frac{\kappa_{xx}\rho_x}{\kappa_x^2} \right\rangle_{t,\mathcal{D}}^{\left(\frac{\alpha}{2}\right)} \le \frac{\langle \kappa_{xx} \rangle_{t,\mathcal{D}}^{\left(\frac{\alpha}{2}\right)}}{\gamma} + \frac{\|\kappa_{xx}\|_{\infty,\mathcal{D}}}{\gamma^2} \langle \rho_x \rangle_{t,\mathcal{D}}^{\left(\frac{\alpha}{2}\right)} + \frac{\|\kappa_{xx}\|_{\infty,\mathcal{D}}}{\gamma^2} \langle \kappa_x \rangle_{t,\mathcal{D}}^{\left(\frac{\alpha}{2}\right)}. \tag{6.11}$$

Collecting the above inequalities (6.7), (6.8), (6.9), (6.10), and (6.11) yield:

$$\left\langle \frac{\rho_x}{\kappa_x} \right\rangle_{\mathcal{D}}^{(1+\alpha)} \le \frac{E}{\gamma^2} \left(1 + |\kappa|_{\mathcal{D}}^{(2+\alpha)} + \|\kappa_{xx}\|_{\infty,\mathcal{D}} \langle \kappa_x \rangle_{\mathcal{D}}^{(\alpha)} \right), \tag{6.12}$$

where we have used the fact that $1 \leq \frac{E}{\gamma}$, $\gamma \leq 1$ and $|\rho|_{\mathcal{D}}^{(2+\alpha)} \leq E$ (see Proposition 5.2).

Step 1.2. $\Big($Estimating $|\rho|_{\mathcal{D}}^{(3+\alpha)}$ and $|\kappa|_{\mathcal{D}}^{(2+\alpha)}\Big)$

Using Hölder estimate for parabolic equations (estimate (2.4) of Proposition 2.1), and similar computations to that of the previous step, we deduce that:

$$|\kappa|_{\mathcal{D}}^{(2+\alpha)} \leq \frac{E}{\gamma}\left(1 + |\kappa_x|_{\mathcal{D}}^{(\alpha)}\right), \tag{6.13}$$

and

$$|\rho|_{\mathcal{D}}^{(3+\alpha)} \leq \frac{E}{\gamma}\left(1 + |\kappa_x|_{\mathcal{D}}^{(\alpha)}\right). \tag{6.14}$$

Step 1.3. (The estimate for $\|\kappa_{tx}\|_{\infty,\mathcal{D}}$)

By combining (6.6), (6.12), (6.13), (6.14), and by using the fact that $|\kappa_x|_{\mathcal{D}}^{(\alpha)}$ has an exponential estimate (see estimate (5.1) of Proposition 5.1), we deduce that:

$$|\kappa|_{\mathcal{D}}^{(3+\alpha)} \leq \frac{E}{\gamma^3}, \tag{6.15}$$

which will be useful later, and as a particular subcase, we have:

$$\|\kappa_{tx}\|_{\infty,\mathcal{D}} \leq \frac{E}{\gamma^3}, \tag{6.16}$$

where we have frequently used that $\gamma \leq 1$, and we have always taken the maximum of all the exponential bounds of the $E = de^{dT}$ form.

Step 2. (Estimating $\|\kappa_t\|_{W_2^{2,1}(\mathcal{D})}$)

Step 2.1. (Estimating $\|u\|_{W_2^{2,1}(\mathcal{D})}$)

We use the L^2 estimates for parabolic equations (Theorem 2.3) to the function u satisfying (6.2), we obtain:

$$\|u\|_{W_2^{2,1}(\mathcal{D})} \leq E(1 + \|v_x\|_{2,\mathcal{D}}). \tag{6.17}$$

The term 1 in (6.17) comes from estimating the initial data u^0. Since $v_x = \kappa_{tx}$, we plug the estimate (6.16) obtained in Step 1.3 into (6.17), we get

$$\|u\|_{W_2^{2,1}(\mathcal{D})} \leq \frac{E}{\gamma^3}. \tag{6.18}$$

Step 2.2. (Estimating $\|v\|_{W_2^{2,1}(\mathcal{D})}$)

Arguing in a similar manner as in the previous step, we obtain the following estimate for the function v, the solution of the parabolic equation (6.3):

$$\begin{aligned}\|v\|_{W_2^{2,1}(\mathcal{D})} \leq E\Bigg(1 + \left\|\tfrac{\rho_{xx}}{\kappa_x}\right\|_{\infty,\mathcal{D}} \|u_x\|_{2,\mathcal{D}} + \|B\|_{\infty,\mathcal{D}}\|u_{xx}\|_{2,\mathcal{D}} \\ + \|B\|_{\infty,\mathcal{D}} \left\|\tfrac{\rho_{xx}}{\kappa_x}\right\|_{\infty,\mathcal{D}} \|v_x\|_{2,\mathcal{D}} + \|u_x\|_{2,\mathcal{D}}\Bigg),\end{aligned} \tag{6.19}$$

and hence, from (6.16), (6.18), and doing some computations, we deduce from (6.19) that:

$$\|v\|_{W_2^{2,1}(\mathcal{D})} \leq \frac{E}{\gamma^4}. \tag{6.20}$$

The goal of Step 2 follows since $v = \kappa_t$.

Step 3. (Estimating $\|\kappa_{xx}\|_{W_2^{2,1}(\mathcal{D})}$)

The estimate of $\|\kappa_{xx}\|_{W_2^{2,1}(\mathcal{D})}$ requires a special attention. We will mainly use the equations on ρ and κ. The four parts $\|\kappa_{xx}\|_{2,\mathcal{D}}$, $\|\kappa_{xxt}\|_{2,\mathcal{D}}$, $\|\kappa_{xxx}\|_{2,\mathcal{D}}$ and $\|\kappa_{xxxx}\|_{2,\mathcal{D}}$ of the above norm will be estimated separately.

Step 3.1. (Estimate of $\|\kappa_{xx}\|_{2,\mathcal{D}}$)

Inequality (6.13) directly implies that

$$\|\kappa_{xx}\|_{\infty,\mathcal{D}} \leq \frac{E}{\gamma}, \tag{6.21}$$

hence $\|\kappa_{xx}\|_{2,\mathcal{D}} \leq \frac{E}{\gamma}$.

Step 3.2. (Estimate of $\|\kappa_{xxxx}\|_{2,\mathcal{D}}$)

We first derive the equation on ρ two times in x, we deduce (using (6.18)) that $\|\rho_{xxxx}\|_{2,\mathcal{D}}$ has the same upper bound as $\|\kappa_{xxx}\|_{2,\mathcal{D}}$, i.e.

$$\|\rho_{xxxx}\|_{2,\mathcal{D}} \leq \frac{E}{\gamma^3}. \tag{6.22}$$

We derive the equation on κ two times with respect to the variable x, we obtain:

$$\begin{aligned}
\kappa_{txx} \quad = \quad & \varepsilon\kappa_{xxxx} + \frac{2\rho_{xx}\rho_{xxx}}{\kappa_x} - \frac{\kappa_{xx}\rho_{xx}^2}{\kappa_x^2} + \frac{\rho_x\rho_{xxxx}}{\kappa_x} - \frac{\rho_x\rho_{xxx}\kappa_{xx}}{\kappa_x^2} \\
- & \frac{\rho_{xx}^2\kappa_{xx}}{\kappa_x^2} - \frac{\rho_x\rho_{xx}\kappa_{xxx}}{\kappa_x^2} - \frac{\rho_x\kappa_{xx}\rho_{xxx}}{\kappa_x^2} + \frac{2\kappa_{xx}^2\rho_x\rho_{xx}}{\kappa_x^3} - \tau\rho_{xxx},
\end{aligned}$$

and we use (6.22) and our controls obtained in the previous steps, in order to deduce that:

$$\|\kappa_{xxxx}\|_{2,\mathcal{D}} \leq \frac{E}{\gamma^4}.$$

In fact, the highest power comes from estimating the following term:

$$\left\|\frac{\kappa_{xx}^2\rho_x\rho_{xx}}{\kappa_x^3}\right\|_{2,\mathcal{D}} \leq \left\|\frac{\kappa_{xx}^2\rho_{xx}}{\kappa_x^2}\right\|_{\infty,\mathcal{D}} \sqrt{T} \leq \frac{E}{\gamma^4},$$

where we have used the L^∞ estimate of $\|\kappa_{xx}\|_{\infty,\mathcal{D}}$. All other estimates are easily deduced. Let us just state how to estimate the other term were $\|\kappa_{xx}\|_{\infty,\mathcal{D}}$ interferes. In fact, we have:

$$\left\|\frac{\rho_x\rho_{xxx}\kappa_{xx}}{\kappa_x^2}\right\|_{2,\mathcal{D}} \leq \left\|\frac{\kappa_{xx}}{\kappa_x}\right\|_{\infty,\mathcal{D}} \|\rho_{xxx}\|_{2,\mathcal{D}} \leq \frac{E}{\gamma^3}.$$

Step 3.3. (Estimate of $\|\kappa_{xxt}\|_{2,\mathcal{D}}$ and $\|\kappa_{xxx}\|_{2,\mathcal{D}}$)

As an immediate consequence of (6.20), we get

$$\|\kappa_{xxt}\|_{2,\mathcal{D}} \leq \frac{E}{\gamma^4}.$$

Deriving the equation on κ with respect to x, we obtain:

$$\kappa_{tx} = \varepsilon\kappa_{xxx} + \frac{\rho_{xx}^2}{\kappa_x} + \frac{\rho_x\rho_{xxx}}{\kappa_x} - \frac{\rho_x\kappa_{xx}\rho_{xx}}{\kappa_x^2} - \tau\rho_{xx}. \tag{6.23}$$

The estimate (6.14) gives

$$\|\rho_{xxx}\|_{2,\mathcal{D}} \leq \frac{E}{\gamma},$$

which, together with (6.21) and (6.23), give $\|\kappa_{xxx}\|_{2,\mathcal{D}} \leq \frac{E}{\gamma^3}$. We deduce as a conclusion that:

$$\|\kappa_{xx}\|_{W_2^{2,1}(\mathcal{D})} \leq \frac{E}{\gamma^4},$$

and this terminates the proof. □

We move now to the main result of this section.

Lemma 6.3 *($W_2^{2,1}$ **bound for** ρ_{xxx})*
Under the same hypothesis of Lemma 6.1, we have:

$$\|\rho_{xxx}\|_{W_2^{2,1}(\mathcal{D})} \leq \frac{E}{\gamma^4}. \tag{6.24}$$

Proof. Set

$$\bar{\kappa} = \frac{\tau}{(1+\varepsilon)^2}\kappa_t + \frac{\tau}{1+\varepsilon}\kappa_{xx}$$

and

$$w = \rho_{xxx} - \bar{\kappa}.$$

We write down, after doing some computations, the equation satisfied by w:

$$\begin{cases} w_t = (1+\varepsilon)w_{xx} - \dfrac{\tau}{(1+\varepsilon)^2}\kappa_{tt} \quad \text{on} \quad \mathcal{D} \\ w_x|_{S_T} = 0 \quad \text{on} \quad S_T \\ w|_{t=0} := w^0 = \rho_{xxx}^0 - \dfrac{\tau(1+2\varepsilon)}{(1+\varepsilon)^2}\kappa_{xx}^0 - \dfrac{\tau}{(1+\varepsilon)^2}\dfrac{\rho_x^0\rho_{xx}^0}{\kappa_x^0} + \dfrac{\tau^2}{(1+\varepsilon)^2}\rho_x^0. \end{cases} \tag{6.25}$$

Here $w_x|_{S_T} = 0$ can be checked by deriving the equation satisfied by ρ with respect to x and then with respect to t, and by using the equality $(1+\varepsilon)\rho_{xx} = \tau\kappa_x$ satisfied on the boundary ∂I (which is a consequence of the compatibility conditions). Applying the L^2 theory with Neumann conditions (see for instance [21, Chapter 4, Section 10]) to (6.25), we get that $\rho_{xxx} = w + \bar{\kappa}$ satisfies

$$\|\rho_{xxx}\|_{W_2^{2,1}(\mathcal{D})} \leq E\left(1 + \|\kappa_{tt}\|_{2,\mathcal{D}} + \|\kappa_t\|_{W_2^{2,1}(\mathcal{D})} + \|\kappa_{xx}\|_{W_2^{2,1}(\mathcal{D})}\right), \tag{6.26}$$

and eventually (6.26) with Lemma 6.1 gives immediately the result. □

7 An upper bound for the BMO norm of ρ_{xxx}

This section is devoted to give a suitable upper bound for the BMO norm of ρ_{xxx}. This result will be a consequence of the control of the BMO norm of a suitable extension of κ_{xx}. The goal is to use this upper bound in the Kozono-Taniuchi inequality (see inequality (2.13) of Theorem 2.13) in order to control the L^∞ norm of ρ_{xxx}. We first give some useful definitions.

Definition 7.1 ***(The "symmetric and periodic" extension of a function).*** *Let $f \in C(\overline{I_T})$, we define f^{sym} (constructed out of f) over $\mathbb{R}\times(0,T)$, first by the symmetry of f with respect to the line $x=0$ over the interval $(-1,0)$, and then by spatial periodicity.*

Definition 7.2 ***(The "antisymmetric and periodic" extension of a function).*** *We define the function f^{asym} in a similar manner as f^{sym}, where we take the antisymmetry of f instead of the symmetry.*

We start with the following lemma that reflects a useful relation between the BMO norm of f^{sym} and f^{asym}.

Lemma 7.3 ***(A relation between f^{sym} and f^{asym}).*** *Let $f \in C(\overline{I_T})$, then:*

$$\|f^{sym}\|_{BMO(\mathbb{R}\times(0,T))} \leq c\left(\|f^{asym}\|_{BMO(\mathbb{R}\times(0,T))} + m_{2I\times(0,T)}\left(|f^{sym}|\right)\right),$$

where $c>0$ is a universal constant.

The proof of this lemma will be presented in Appendix B. The next lemma gives a control of the BMO norm of $(\kappa_{xx})^{asym}$.

Lemma 7.4 ***(BMO bound for $(\kappa_{xx})^{asym}$).*** *Under hypothesis (H1), and under the same hypothesis of Proposition 5.1, we have:*

$$\|(\kappa_{xx})^{asym}\|_{BMO(\mathbb{R}\times(0,T))} \leq ce^{cT}, \tag{7.1}$$

where $c>0$ is a constant depending on the initial conditions (but independent of T). The function $(\kappa_{xx})^{asym}$ is given via Definition 7.2.

Proof. Let $\bar{\kappa}(x,t) = \kappa(x,t) - \kappa^0(x)$. We notice that $\bar{\kappa}|_{S_T} = 0$, therefore $\bar{\kappa}^{asym}$ satisfies:

$$\begin{cases} \bar{\kappa}_t^{asym} - \varepsilon\bar{\kappa}_{xx}^{asym} = \dfrac{(\rho_x)^{asym}\rho_{xx}^{asym}}{(\kappa_x)^{asym}} - \tau(\rho_x)^{asym} + \varepsilon(\kappa_{xx}^0)^{asym} \quad \text{on} \quad \mathbb{R}\times(0,T) \\ \bar{\kappa}^{asym}(x,0) = 0. \end{cases} \tag{7.2}$$

We already know that the right hand side of (7.2) is bounded in L^∞ by $E = ce^{cT}$, and hence (using Theorem 2.10) the result follows. □

We now present the principal result of this section.

Lemma 7.5 ***(BMO bound for ρ_{xxx}).*** *Under the same hypothesis of Lemma 7.4, we have:*

$$\|\rho_{xxx}\|_{BMO(\mathcal{D})} \leq E. \tag{7.3}$$

Proof. Take

$$v = \rho_x - \frac{\tau\kappa}{1+\varepsilon},$$

the equation satisfied by v reads:

$$\begin{cases} v_t = (1+\varepsilon)v_{xx} - \dfrac{\varepsilon\tau}{1+\varepsilon}\kappa_{xx} - \dfrac{\tau}{1+\varepsilon}\dfrac{\rho_x\rho_{xx}}{\kappa_x} + \dfrac{\tau^2}{1+\varepsilon}\rho_x \quad \text{on} \quad \mathcal{D} \\ v|_{t=0} = v^0 := \rho_x^0 - \dfrac{\tau}{1+\varepsilon}\kappa^0 \quad \text{on} \quad I \\ v_x|_{S_T} = 0, \end{cases}$$

where we have used the compatibility conditions to check that $v_x|_{S_T} = 0$. We can assume, without loss of generality, that the initial condition $v^0 = 0$. This is because being non-zero just adds a constant depending on the initial conditions in the final estimate that we are looking for. From the fact that $v_x|_{S_T} = 0$, we can easily deduce that the function v^{sym} satisfies:

$$\begin{cases} v_t^{sym} = (1+\varepsilon)v_{xx}^{sym} + \overbrace{\dfrac{\tau^2}{1+\varepsilon}(\rho_x)^{sym} - \dfrac{\tau}{1+\varepsilon}\dfrac{(\rho_x)^{sym}(\rho_{xx})^{sym}}{(\kappa_x)^{sym}} - \dfrac{\varepsilon\tau}{1+\varepsilon}(\kappa_{xx})^{sym}}^{g} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on} \quad \mathbb{R}\times(0,T) \\ v^{sym}(x,0) = 0 \quad \text{on} \quad \mathbb{R}, \end{cases}$$

therefore, using the BMO estimate (2.11) for parabolic equations, to the function v, one gets:

$$\|v_{xx}^{sym}\|_{BMO(\mathbb{R}\times(0,T))} \le c\left[\|g\|_{BMO(\mathbb{R}\times(0,T))} + m_{2I\times(0,T)}(|g|)\right],$$

where $c = c(T_1) > 0$ with $0 < T_1 \le T$. From Propositions 5.1, 5.2, we deduce that

$$\|g\|_{BMO(\mathbb{R}\times(0,T))} \le E + \|(\kappa_{xx})^{sym}\|_{BMO(\mathbb{R}\times(0,T))},$$

and

$$m_{2I\times(0,T)}(|g|) \le E + m_{2I\times(0,T)}(|(\kappa_{xx})^{sym}|).$$

Recall the definition of the term E from Remark 6.2. At this stage, we write the following estimate:

$$\|(\kappa_{xx})^{sym}\|_{BMO(\mathbb{R}\times(0,T))} \le c\left[\|(\kappa_{xx})^{asym}\|_{BMO(\mathbb{R}\times(0,T))} + m_{2I\times(0,T)}(|(\kappa_{xx})^{sym}|)\right], \quad (7.4)$$

which can be deduced using Lemma 7.3. The constant $c > 0$ appearing in (7.4) is independent of T. Finally, we deduce that:

$$\|v_{xx}^{sym}\|_{BMO(\mathbb{R}\times(0,T))} \le c\left(E + T^{-1/p}\|\kappa_{xx}\|_{p,\mathcal{D}}\right).$$

From (6.1), (5.11), we know that $T^{-1/p}\|\kappa_{xx}\|_{p,\mathcal{D}} \le T_1^{-1/p}E$. From the previous two inequalities, and since $v_{xx} = \rho_{xxx} - \frac{\tau\kappa_{xx}}{1+\varepsilon}$, we easily arrive to our result. □

8 L^∞ bound for ρ_{xxx}

In this section, we use the results of Sections 5, 6 and 7, in order to give an L^∞ bound for ρ_{xxx} via the Kozono-Taniuchi inequality.

Proposition 8.1 ***(L^∞ bound for ρ_{xxx}).*** *Under hypothesis (H1), and under the same hypothesis of Proposition 5.1, we have $\forall T > 0$:*

$$\|\rho_{xxx}\|_{\infty,I_T} \leq E\left(1 + \log^+ \frac{1}{\gamma}\right). \tag{8.1}$$

Proof. For $T < T_1 = T^*$, where T^* is the short time existence result (see Theorem 4.2) given by (4.4), inequality (8.1) directly follows. In the other case where $T \geq T_1$, we apply the parabolic Kozono-Taniuchi estimate (2.13) to the function ρ_{xxx}, together with (7.3) and (6.24). Remark that the $\|\rho_{xxx}\|_{L^1(I_T)}$ can be easily estimated by the term E. □

Proposition 8.2 ***(A priori estimates).*** *Under the same hypothesis of Proposition 3.1, the solution $(\rho,\kappa) \in C^{3+\alpha,\frac{3+\alpha}{2}}(\overline{I_T})$ satisfies for every $0 \leq t \leq T$:*

$$\kappa_x(.,t) \geq e^{-e^{e^{b(t+1)}}} > 0, \tag{8.2}$$

$$|\rho(.,t)|_I^{(3+\alpha)} \leq e^{e^{e^{b(t+1)}}} \quad \text{and} \quad |\kappa(.,t)|_I^{(3+\alpha)} \leq e^{e^{e^{b(t+1)}}}. \tag{8.3}$$

Here $b > 0$ is a positive constant depending on the initial conditions and the fixed terms of the problem, but independent of time.

Proof. Remark that if we consider a function γ satisfying (3.2), then the right hand side of (3.2) can be estimated using (8.1) as follows:

$$-(c_0 + \|\rho_{xxx}(.,t)\|_{L^\infty(I)}) \geq -E(1 + |\log\gamma(t)|), \quad E = E(T) = de^{dT}. \tag{8.4}$$

This is the motivation to consider the solution γ_T of the following ordinary differential equation:

$$\begin{cases} \gamma_T' = -E(1 + |\log\gamma_T|)\gamma_T, & t \in (0,T) \\ \gamma_T(0) = \gamma_0/2, \end{cases} \tag{8.5}$$

where γ_0 is given by (3.1). Then, using a continuity argument, joint to the fact that $\overline{m}(t) \geq \gamma^2(t)$ (see Proposition 3.1), it is easy to check that both (8.4) and (3.2) are satisfied with $\gamma = \gamma_T$. Let us now define

$$\tilde{\gamma}(T) := \gamma_T(T) \leq \gamma_T(t) \quad \text{for} \quad t \in [0,T].$$

Then we deduce from (3.3) that

$$\kappa_x(.,T) \geq \sqrt{\tilde{\gamma}^2(T) + (\rho_x(.,T))^2} \geq \tilde{\gamma}(T), \quad \forall T > 0,$$

and finally, solving (8.5) explicitly, inequality (8.2) directly follows. Therefore, from (6.14) and (6.15), we easily deduce (8.3). □

9 Long time existence and uniqueness

Now we are ready to show the main result of this paper, namely Theorem 1.1.

Proof of Theorem 1.1. Define the set $\mathcal{B}$ by:

$$\mathcal{B} = \left\{ \begin{array}{l} T > 0;\ \exists !\ \text{ solution } (\rho, \kappa) \in C^{3+\alpha, \frac{3+\alpha}{2}}(\overline{I_T}) \text{ of} \\ (1.1), (1.2) \text{ and } (1.3), \text{ satisfying } (1.10) \end{array} \right\}.$$

This set is non empty by the short time existence result (Theorem 4.2). Set

$$T_\infty = \sup \mathcal{B}.$$

We claim that $T_\infty = \infty$. Assume, by contradiction that $T_\infty < \infty$. In this case, let $\delta > 0$ be an arbitrary small positive constant, and apply the short time existence result (Theorem 4.2) with $T_0 = T_\infty - \delta$. Indeed, by the tri-exponential bounds (8.2) and (8.3), we deduce that the time of existence T^* given by (4.4) is in fact independent of δ. Hence, choosing δ small enough, we obtain $T_0 + T^* \in \mathcal{B}$ with $T_0 + T^* > T_\infty$ and hence a contradiction. □

10 Appendix A: miscellaneous parabolic estimates

A1. Proof of Lemma 2.4 (L^p estimate for parabolic equations)

As a first step, we will prove the result in the case where $\varepsilon = 1$, and in a second step, we will move to the case $\varepsilon > 0$. It is worth noticing that the term c may take several values only depending on p.

Step 1. (The estimate: case $\varepsilon = 1$)

Suppose $\varepsilon = 1$. Since $u = 0$ on $\partial I \times [0, T]$, we take $\tilde{u} = u^{asym}$ (see Definition 7.2). Also consider the function $\tilde{f} = f^{asym}$. Define $\bar{u}$ by

$$\bar{u} = \tilde{u}\phi^n,$$

with

$$\begin{cases} \phi^n(x) = 1 & \text{if } x \in (0, 2n) \\ \phi^n(x) = 0 & \text{if } x \geq 2n+1 \text{ or } x \leq -1. \end{cases}$$

This function satisfies

$$\begin{cases} \bar{u}_t = \bar{u}_{xx} + \bar{f}, & \text{on } \mathbb{R} \times (0, T) \\ \bar{u}(x, 0) = 0, & \text{on } \mathbb{R}, \end{cases}$$

with

$$\bar{f} = \tilde{f}\phi^n - \tilde{u}\phi^n_{xx} - 2\tilde{u}_x\phi^n_x.$$

The proof that

$$\|u_t\|_{p,I_T} + \|u_{xx}\|_{p,I_T} \le c\|f\|_{p,I_T} \tag{10.1}$$

can be easily deduced by applying the Calderon-Zygmund estimates to the function $\bar{u}$ satisfying the above equation, and passing to the limit $n \to \infty$. Now, since $u \in W_p^{2,1}(I_T)$ with $u|_{t=0} = 0$, we use [21, Lemma 4.5, page 305] to get

$$\|u\|_{p,I_T} \le cT(\|u_t\|_{p,I_T} + \|u_{xx}\|_{p,I_T}) \tag{10.2}$$

and

$$\|u_x\|_{p,I_T} \le c\sqrt{T}(\|u_t\|_{p,I_T} + \|u_{xx}\|_{p,I_T}). \tag{10.3}$$

Combining (10.1), (10.2) and (10.3), we deduce that

$$\frac{1}{T}\|u\|_{p,I_T} + \frac{1}{\sqrt{T}}\|u_x\|_{p,I_T} + \|u_{xx}\|_{p,I_T} + \|u_t\|_{p,I_T} \le c\|f\|_{p,I_T}.$$

Step 2. (The estimate: general case $\varepsilon > 0$)

To get the general inequality, we consider the following rescaling of the function u:

$$\hat{u}(x,t) = u(x, t/\varepsilon), \quad (x,t) \in I_{\varepsilon T},$$

which allows to get the desired result. □

A2. Proof of Lemma 2.7 (L^∞ control of the spatial derivative)

Since $u \in W_p^{2,1}(I_T)$ for $p > 3$, we know from Lemma 2.5 that $u_x \in C^{\alpha,\alpha/2}(\overline{I_T})$ for $\alpha = 1 - \frac{3}{p}$. In this case, we use the estimate (2.8) with $\delta = \sqrt{T}$, we obtain

$$\|u_x\|_{\infty,I_T} \le c(p)\{T^{\frac{\alpha}{2}}(\|u_t\|_{p,I_T} + \|u_{xx}\|_{p,I_T}) + T^{\frac{\alpha}{2}-1}\|u\|_{p,I_T}\}. \tag{10.4}$$

Remark that the fact that $u = 0$ on the parabolic boundary $\partial^p I_T$, and that it obviously satisfies the equation:

$$\begin{cases} u_t = u_{xx} + f, \quad \text{with} \quad f = u_t - u_{xx} \\ u = 0 \quad \text{on} \quad \partial^p I_T, \end{cases}$$

then we can apply estimate (2.7) to bound the term $\|u\|_{p,I_T}$. Hence (10.4) becomes (with a different constant $c(p)$):

$$\begin{aligned} \|u_x\|_{\infty,I_T} &\le c(p)\{T^{\frac{\alpha}{2}}\|u_t - u_{xx}\|_{p,I_T} + T^{\frac{\alpha}{2}-1}T\|u_t - u_{xx}\|_{p,I_T}\} \\ &\le c(p)T^{\frac{\alpha}{2}}\|u\|_{W_p^{2,1}(I_T)} \\ &\le c(p)T^{\frac{p-3}{2p}}\|u\|_{W_p^{2,1}(I_T)}, \end{aligned}$$

and the result follows. □

11 Appendix B: parabolic BMO theory

B1. Proof of Theorem 2.10 (A BMO estimate in the periodic case)

Let f be a bounded function defined on $\mathbb{R}\times(0,T)$ satisfying $f(x+2,t)=f(x,t)$. We extend the function f to $\mathbb{R}\times\mathbb{R}_+$, first by symmetry with respect to the line $\{t=T\}$ and after that by time periodicity of period $2T$. Call this function $\tilde{f}$. Set $\bar{u}$ as the solution of the following equation:

$$\begin{cases} \bar{u}_t = \varepsilon \bar{u}_{xx} + \tilde{f} \quad \text{on} \quad \mathbb{R}\times\mathbb{R}_+ \\ \bar{u}(x,0)=0. \end{cases} \tag{11.1}$$

We apply the standard result of BMO theory for parabolic equations. Since $f \in L^\infty(\mathbb{R}\times(0,T))$, then $\tilde{f}\in BMO(\mathbb{R}\times\mathbb{R}_+)$, and hence we obtain that $\bar{u}_t, \bar{u}_{xx}\in BMO(\mathbb{R}\times\mathbb{R}_+)$, with the following estimate:

$$\|\bar{u}_t\|_{BMO(\mathbb{R}\times\mathbb{R}_+)} + \|\bar{u}_{xx}\|_{BMO(\mathbb{R}\times\mathbb{R}_+)} \le c\|\tilde{f}\|_{BMO(\mathbb{R}\times\mathbb{R}_+)}, \tag{11.2}$$

and hence (from the definition of the BMO space),

$$\|\bar{u}_t\|_{BMO(\mathbb{R}\times(0,T)} + \|\bar{u}_{xx}\|_{BMO(\mathbb{R}\times(0,T))} \le c\|\tilde{f}\|_{BMO(\mathbb{R}\times\mathbb{R}_+)}. \tag{11.3}$$

The BMO theory for parabolic equations, particularly estimate (11.2) is rather classical. This is due to the fact that the solution of (11.1) can be expressed in terms of the heat kernel Γ defined by:

$$\Gamma(x,t)=\begin{cases} (4\pi\varepsilon t)^{-1/2}e^{-\frac{x^2}{4\varepsilon t}}, & \text{for} \quad t>0 \\ 0 & \text{for} \quad t\le 0, \end{cases}$$

in the following way:

$$\bar{u}(x,t)=\int_{\mathbb{R}\times\mathbb{R}^+}\Gamma(x-\xi,t-s)\tilde{f}(\xi,s)\,d\xi\,ds.$$

As a matter of fact, it is shown in [10] that Γ_{xx} is a parabolic Calderon-Zygmund kernel (here we are working in nonhomogeneous metric spaces in which the variable t accounts for twice the variable x). Therefore $\Gamma_{xx}: BMO \to BMO$ is a bounded linear operator. This result is quite technical and can be adapted from its elliptic version (see [2, Theorem 3.4]). It is less difficult to show that $\Gamma_{xx}: L^\infty \to BMO$, a bounded linear operator (see for instance [15, Lemma 3.3]).

Having (11.3) in hands, it remains to show that

$$\|\tilde{f}\|_{BMO(\mathbb{R}\times\mathbb{R}_+)} \le c\left(\|f\|_{BMO(\mathbb{R}\times(0,T))} + m_{2I\times(0,T)}(|f|)\right),$$

with $c>0$ independent of T. This can be divided into two steps:

Step 1. (Treatment of small parabolic cubes)

We consider parabolic cubes $Q_r = Q_r(x_0, t_0)$, $(x_0, t_0) \in \mathbb{R} \times \mathbb{R}_+$, with $r \leq \sqrt{T}$. Let us estimate the term

$$\frac{1}{|Q_r|} \int_{Q_r} |\tilde{f} - m_{Q_r} \tilde{f}|.$$

Assume, without loss of generality, that $T \leq t_0 < 2T$. In fact, any other case can be done in a similar way because of the time symmetry of the function $\tilde{f}$. Two cases can be considered. If $r^2 < t_0 - T$ then the cube Q_r lies in the strip $\mathbb{R} \times (T, 2T)$ and in this case

$$\frac{1}{|Q_r|} \int_{Q_r} |\tilde{f} - m_{Q_r} \tilde{f}| \leq \|f\|_{BMO(\mathbb{R} \times (0,T))}.$$

The other case is when $r^2 \geq t_0 - T$. In this case, define Q_r^a and Q_r^b, the above and the below parabolic cubes, as follows:

$$Q_r^a = Q_r(x_0, T + r^2) \quad \text{and} \quad Q_r^b = Q_r(x_0, T).$$

Since

$$T - r^2 < t_0 - r^2 \leq T < t_0 \leq T + r^2,$$

then $Q_r \subset (Q_r^a \cup Q_r^b)$. Moreover, we have $|Q_r| = |Q_r^a| = |Q_r^b|$. We compute:

$$\begin{aligned}
\frac{1}{|Q_r|} \int_{Q_r} |\tilde{f} - m_{Q_r} \tilde{f}| &\leq \frac{2}{|Q_r|} \int_{Q_r} |\tilde{f} - 2m_{Q_r^b} \tilde{f} + m_{Q_r^a} \tilde{f}| \\
&\leq \frac{4}{|Q_r|} \int_{Q_r^a} |\tilde{f} - m_{Q_r^b} \tilde{f}| + \frac{4}{|Q_r|} \int_{Q_r^b} |\tilde{f} - m_{Q_r^b} \tilde{f}| \\
&\quad + \frac{2}{|Q_r|} \int_{Q_r^a} |\tilde{f} - m_{Q_r^a} \tilde{f}| + \frac{2}{|Q_r|} \int_{Q_r^b} |\tilde{f} - m_{Q_r^a} \tilde{f}|.
\end{aligned}$$

We remark (from the symmetry-in-time of the function $\tilde{f}$) that $m_{Q_r^a} \tilde{f} = m_{Q_r^b} f$, and

$$\int_{Q_r^a} |\tilde{f} - c| = \int_{Q_r^b} |f - c|, \quad \forall c \in \mathbb{R}.$$

Therefore the above inequalities give:

$$\frac{1}{|Q_r|} \int_{Q_r} |\tilde{f} - m_{Q_r} \tilde{f}| \leq 16 \|f\|_{BMO(\mathbb{R} \times (0,T))}.$$

Step 2. (Treatment of big parabolic cubes)

Consider now parabolic cubes $Q_r \subset \mathbb{R} \times \mathbb{R}_+$, $r > \sqrt{T}$. Suppose first that $r > 1$. Because of the symmetry-in-time of the function $\tilde{f}$, and its spatial periodicity, we compute:

$$\frac{1}{|Q_r|} \int_{Q_r} |\tilde{f} - m_{Q_r} \tilde{f}| \leq \frac{2}{|Q_r|} \int_{Q_r} |\tilde{f}| \leq \frac{2N}{|Q_r|} \int_{2I \times (0,T)} |f|,$$

where N is the minimum number of domains D of the form $D = (k, k+2) \times (nT, (n+1)T)$, $k \in \mathbb{Z}$ and $n \in N$, that cover Q_r. Here

$$|Q_r| \sim N \times |2I \times (0,T)|, \quad N > 1.$$

Therefore, the above inequalities give:

$$\frac{1}{|Q_r|}\int_{Q_r}|\tilde{f}-m_{Q_r}\tilde{f}|\le c\, m_{2I\times(0,T)}(|f|).$$

Now suppose that $\sqrt{T}<r\le 1$. In this case we use the fact that $0<T_1\le T$, we compute:

$$\frac{1}{|Q_r|}\int_{Q_r}|\tilde{f}-m_{Q_r}\tilde{f}|\le\frac{2}{|Q_r|}\int_{Q_r}|\tilde{f}|\le\frac{2N}{|Q_r|}\int_{2I\times(0,T)}|f|\le\frac{N}{T_1^{3/2}}\int_{2I\times(0,T)}|f|.$$

Here $N\lesssim\frac{1}{T}$, and hence

$$\frac{1}{|Q_r|}\int_{Q_r}|\tilde{f}-m_{Q_r}\tilde{f}|\le c(T_1)\, m_{2I\times(0,T)}(|f|).$$

Steps 1 and 2 give the required result. □

B2. Proof of Lemma 7.3. We divide the proof into two steps.

Step 1. (Treatment of small parabolic cubes)

Let us consider parabolic cubes $Q=Q_r(x_0,t_0)\subset\mathbb{R}\times(0,T)$ with $0<r\le\frac{1}{2}$. Assume, without loss of generality, that $1<x_0<2$ (the other cases can be treated similarly). Define the left and the right neighbor cubes of $Q_r(x_0,t_0)$ by $Q^-=Q_r^-(1-r,t_0)$, and $Q^+=Q_r^+(1+r,t_0)$ respectively. Since $2r\le 1$, then

$$Q^-\subset(0,1)\times(0,T)\quad\text{and}\quad Q^+\subset(1,2)\times(0,T).$$

Using the fact that for any function $g\in L^1(\Omega)$:

$$\int_\Omega|g-m_\Omega(g)|\le 2\int_\Omega|g-c|,\quad\forall c\in\mathbb{R},$$

We compute:

$$\begin{aligned}\frac{1}{|Q|}\int_Q|f^{sym}-m_Q(f^{sym})| &\le \frac{2}{|Q|}\int_Q|f^{sym}+m_{Q^+}(f^{asym})|\\ &\le \frac{2}{|Q^-|}\int_{Q^-}|f^{sym}+m_{Q^+}(f^{asym})|\\ &+ \frac{2}{|Q^+|}\int_{Q^+}|f^{sym}+m_{Q^+}(f^{asym})|. \end{aligned}\tag{11.4}$$

We know that from the properties of f^{sym} and f^{asym} that $m_{Q^+}(f^{asym})=-m_{Q^-}(f^{sym})$, and

$$f^{sym}=-f^{asym}\quad\text{on}\quad Q^+,\quad\text{and}\quad f^{sym}=f^{asym}\quad\text{on}\quad Q^-.$$

Using the above two inequalities in (11.4), we get:

$$\frac{1}{|Q|}\int_Q|f^{sym}-m_Q(f^{sym})|\le 4\|f^{asym}\|_{BMO(\mathbb{R}\times(0,T))}.$$

Step 2. (Treatment of big parabolic cubes)

Consider parabolic cubes $Q = Q_r \subset \mathbb{R} \times (0,T)$ such that $r > \frac{1}{2}$. In this case, we compute:

$$\frac{1}{|Q|}\int_Q |f^{sym} - m_Q(f^{sym})| \leq \frac{2}{|Q|}\int_Q |f^{sym}| \leq \frac{2N}{|Q|}\int_{2I\times(0,T)} |f^{sym}|,$$

with

$$|Q| \sim N \times |2I \times (0,T)|,$$

therefore

$$\frac{1}{|Q|}\int_Q |f^{sym} - m_Q(f^{sym})| \leq c\, m_{2I\times(0,T)}(|f^{sym}|),$$

where c is a universal constant. Steps 1 and 2 directly implies the result. □

Acknowledgments. This work was supported by the contract ANR MICA (2006-2009). The authors would like to thank Jérôme Droniou for his valuable remarks while reading the manuscript of the paper.

References

[1] R. A. Adams, *Sobolev spaces*, Academic Press, New York-London, 1975. Pure and Applied Mathematics, Vol. 65.

[2] M. Bramanti and L. Brandolini, *Estimates of BMO type for singular integrals on spaces of homogeneous type and applications to hypoelliptic PDEs*, Rev. Mat. Iberoamericana, 21 (2005), pp. 511–556.

[3] H. Brézis and T. Gallouët, *Nonlinear Schrödinger evolution equations*, Nonlinear Anal., 4 (1980), pp. 677–681.

[4] H. Brézis and S. Wainger, *A note on limiting cases of Sobolev embeddings and convolution inequalities*, Comm. Partial Differential Equations, 5 (1980), pp. 773–789.

[5] J. A. Carrillo and J. L. Vázquez, *Fine asymptotics for fast diffusion equations*, Comm. Partial Differential Equations, 28 (2003), pp. 1023–1056.

[6] C. Y. Chan and H. G. Kaper, *Quenching for semilinear singular parabolic problems*, SIAM J. Math. Anal., 20 (1989), pp. 558–566.

[7] E. Chasseigne and J. L. Vázquez, *Theory of extended solutions for fast-diffusion equations in optimal class of data. Radiation from singularities*, Arch. Ration. Mech. Anal., 164 (2002), pp. 133–187.

[8] E. DiBenedetto, *Degenerate parabolic equations*, Universitext, Springer-Verlag, New York, 1993.

[9] H. Engler, *An alternative proof of the Brezis-Wainger inequality*, Comm. Partial Differential Equations, 14 (1989), pp. 541–544.

[10] E. B. Fabes and N. M. Rivière, *Singular intervals with mixed homogeneity*, Studia Math., 27 (1966), pp. 19–38.

[11] A. Friedman, *Partial differential equations of parabolic type*, Prentice-Hall Inc., Englewood Cliffs, N.J., 1964.

[12] I. Groma, F. F. Czikor, and M. Zaiser, *Spatial correlations and higher-order gradient terms in a continuum description of dislocation dynamics*, Acta Mater, 51 (2003), pp. 1271–1281.

[13] J.-S. Guo and P. Souplet, *Fast rate of formation of dead-core for the heat equation with strong absorption and applications to fast blow-up*, Math. Ann., 331 (2005), pp. 651–667.

[14] N. Hayashi and W. von Wahl, *On the global strong solutions of coupled Klein-Gordon-Schrödinger equations*, J. Math. Soc. Japan, 39 (1987), pp. 489–497.

[15] M. A. Herrero, A. A. Lacey, and J. J. L. Velázquez, *Global existence for reaction-diffusion systems modelling ignition*, Arch. Rational Mech. Anal., 142 (1998), pp. 219–251.

[16] J. R. Hirth and L. Lothe, *Theory of dislocations*, Second edition, Kreiger publishing company, Florida 32950, 1982.

[17] H. Ibrahim, M. Jazar, and R. Monneau, *Dynamics of dislocation densities in a bounded channel. Part I: smooth solutions to a singular coupled parabolic system*, preprint hal-00281487, 65 pages.

[18] ——, *Dynamics of dislocation densities in a bounded channel. Part II: existence of weak solutions to a singular Hamilton-Jacobi/parabolic strongly coupled system*, preprint hal-00281859, 33 pages.

[19] H. Ibrahim and R. Monneau, *A parabolic version of the Kozono-Taniuchi inequality*, in preparation.

[20] H. Kozono and Y. Taniuchi, *Limiting case of the Sobolev inequality in BMO, with application to the Euler equations*, Comm. Math. Phys., 214 (2000), pp. 191–200.

[21] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural′ceva, *Linear and quasilinear equations of parabolic type*, Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23, American Mathematical Society, Providence, R.I., 1967.

[22] G. M. Lieberman, *Second order parabolic differential equations*, World Scientific Publishing Co. Inc., River Edge, NJ, 1996.

[23] A. Maugeri, *A boundary value problem for a class of singular parabolic equations*, Boll. Un. Mat. Ital. B (5), 17 (1980), pp. 325–339.

[24] F. Merle and H. Zaag, *Optimal estimates for blowup rate and behavior for nonlinear heat equations*, Comm. Pure Appl. Math., 51 (1998), pp. 139–196.

[25] F. R. N. Nabarro, *Theory of crystal dislocations*, Oxford, Clarendon Press, 1969.