Dirichlet problem and Hölder regularity for non-local fully non-linear elliptic equations

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Outline of the talk

Examples of non-local operators

- A non-singular integral operator

$$J[u](x) = \int u(z)c(x,z)dz$$

with $c \geq 0$ et $\int c(x,z)dz < +\infty$

- The fractional Laplacian

$$-(-\Delta)^{\frac{\alpha}{2}}(x) = \int (u(x+z) - u(x) - Du(x) \cdot z\mathbf{1}_B(z)) \frac{dz}{|z|^{N+\alpha}}$$

with $\alpha \in (0,2)$.

Lévy-Itô operators

$$I_j[u](x) = \int (u(x + j(x,z)) - u(x) - Du(x) \cdot j(x,z)\mathbf{1}_B(z))\mu(dz)$$

with μ singular measure and $j(x, z)$ regular enough.

Singular integral operators have different order.

We focus on order in $(0, 2)$.

Non-local non-linear elliptic equations

- A non-linear diffusion equation from continuum mechanics (dislocations)

$$\partial_t u + \overline{H}(Du, (-\Delta)^{\frac{1}{2}} u) = 0$$

with $\overline{H}(p, l)$ continuous and non-decreasing in l

- A (possibly degenerate) non-linear diffusion equation

$$\inf_{\beta \in \mathcal{B}} \sup_{\alpha \in \mathcal{A}} \left\{ u - b_{\beta,\alpha}(x) \cdot Du - \frac{1}{2} \mathrm{tr}(a_{\beta,\alpha}(x) D^2 u) - I_{j_{\beta,\alpha}}[u] \right\} = 0$$

Bellman-Isaacs equations in stochastic control

These operaters appear in many applications

(biology, continuum mechanics, plasma models, combustion *etc.*)

Lévy processes

- Stochastic processes, generalization of the Brownian motion
- Discontinuous paths

- Small jumps and large jumps : “jump” diffusion?
- Characterized by a drift, a diffusion matrix and a singular measure
- The infinitesimal generator is a Lévy operator

A step further

- Notion of Lévy-Itô jump processes

Viscosity solution theory and non-local operators

Key papers

- Soner (1986) : first definition of viscosity solution
  for a 1st-order integro-diff eq with bounded measures
- Sayah (1991) : theory for a large class of 1st-order equation
  and Perron's method
- Ishii-Koike (1993/1994)
- Alvarez-Tourin (1996) : 2nd-order eq + bounded measures
  and Perron's method
- Jakobsen-Karlsen (2006) : theory for a large class of
  2nd-order eq and singular measures

$$\text{(ENL)} \qquad F(x,u,Du,D^2u,(\mathcal{I}^\alpha_x[u])_\alpha)=0 \qquad \text{dans } \mathbb{R}^N$$

Fundamental assumption: (degenerate) ellipticity

$$X \le Y \quad \& \quad l_\alpha \le m_\alpha \quad \Rightarrow \quad F(\dots,X,l_\alpha) \ge F(\dots,Y,m_\alpha).$$

Notion of sub-jets:

$$\mathcal{J}^{2,-}u(x)=\{(D\phi(x),D^2\phi(x)) : \phi \text{ touchs } u \text{ from below}\}.$$

Notion of limiting sub-jets:

$$\overline{\mathcal{J}}^{2,-}u(x)=\{\lim_n (p_n,A_n)\in \mathcal{J}^{2,-}(x_n) \text{ with } x_n\to x, u(x_n)\to u(x)\}.$$

For local equations, limiting sub-jets can be used to get a viscosity inequality

Notion(s) of viscosity solution

A lsc function u is a super-solution of (ENL) if,
when a smooth test-function ϕ "integrable" for
μ ~~"integrable" for μ~~
that touches u from below x globally ~~globally~~,
and any subjet $X \leq D^2\phi(x)$
then

$$F(x, u(x), D\phi(x), D^2\phi(x) \quad \cancel{D^2\phi(x)} \quad X, I) \geq 0.$$

with

$$\begin{aligned} I &= \int_{|z|\leq r} (\phi(x+z) - \phi(x) - D\phi(x) \cdot z\mathbf{1}_B(z))\mu(dz) \\ &\quad + \int_{|z|\geq r} (\phi \ \cancel{\phi} u(x+z) - \phi \ \cancel{\phi} u(x) - D\phi(x) \cdot z\mathbf{1}_B(z))\mu(dz) \end{aligned}$$

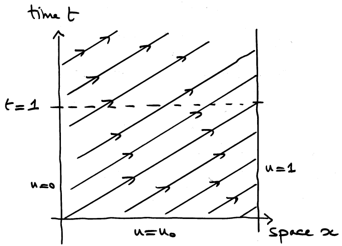
Existence and uniqueness of solutions

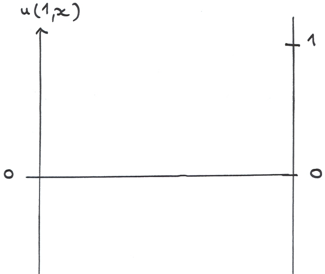
- Uniqueness *via* comparison principles
  - Compare sub-solutions with super-solutions.
  - To prove comparison principle
    1. The dedoubling variable technique
    2. Jensen-Ishii's lemma: a famous and useful block box (Jakobsen-Karlsen'06, Barles-I.'07) adapted with care!!

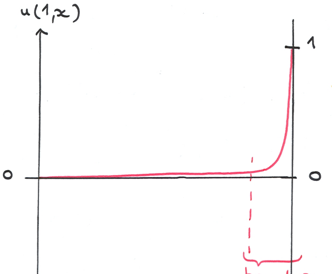
- Existence *via* Perron's method
  - $\left.\begin{array}{r}\text{Comparison principle}\\ +\ \text{existence}\end{array}\right| \Rightarrow$ *the* solution = maximal subsolution
  - To get existence
    1. Consider the maximal subsolution
    2. prove it is a supersolution by contradiction (bump construction ... with care!!)

Local equations and transport effect

- First order linear equation $\partial_t u + v\partial_x u = 0$
- Second order term can save you $\partial_t u + v\partial_x u = = \varepsilon\partial_{xx}^2 u$







- How to prescribe the boundary datum?

Think of the exit time problem for a jump process.

$$\begin{cases} (-\Delta)^{\alpha/2} u = 0 & \text{in } \Omega \\ u = g & \text{where ??outside } \Omega \end{cases}$$

Viscosity solutions for non-local equations on domains

- The non-local operator

$$\int_{x+z\in\Omega} \Big[u(x+z) - u(x)\Big]\mu(dz) + \int_{x+z\notin\Omega} \Big[g(x+z) - u(x)\Big]\mu(dz)$$

In particular, if $g \equiv 0$

$$\int_{x+z\in\Omega} \Big[u(x+z) - u(x)\Big]\mu(dz) - \boxed{u(x)\int_{x+z\notin\Omega}\mu(dz)}$$

- Notion of solution at the boundary?

At the boundary, either the Boundary Condition (BC) or the equation is satisfied

- **Fact**
At $x \in \partial\Omega$ either the equation or the (BC) is satisfied
((BC) in the viscosity solution sense)

- **Consequence**
A solution does not necessarily satisfy the (BC) at $x \in \partial\Omega$

- **Question**
Can we find Structure Conditions on the singular measure μ ensuring that the (BC) is satisfied (in the classical sense)?

Second order equation and curvature effect

- Linear diffusion equation

$$-\frac{1}{2}\mathrm{tr}(a(x)D^2u) - b(x)\cdot Du(x) + u = s$$

Boundary condition (BC) is satisfied if:

1. either $a(x)Dd(x) \neq 0$ (non-degeneracy wrt normal)
2. or $\mathrm{tr}(a(x)D^2d(x)) + b(x)\cdot Dd(x) < 0$ (curvature/transport)

- PDE's proofs: Barles-Burdeau'95, Da Lio'02

- Different scales compete

Choose $\phi\Big(d(x)/\eta\Big)$ as a test-function

and play with $\phi'(0)$ and $\phi''(0)$

The fractional Laplacian case

Non-local linear diffusion equation

$$-\frac{1}{2}\mathrm{tr}(a(x)D^2u) - b(x)\cdot Du(x) + (-\Delta)^{\frac{\alpha}{2}}u + u = s$$

How does the non-local term interfer with others?

Look at its order $\alpha \in (0,2)$!!

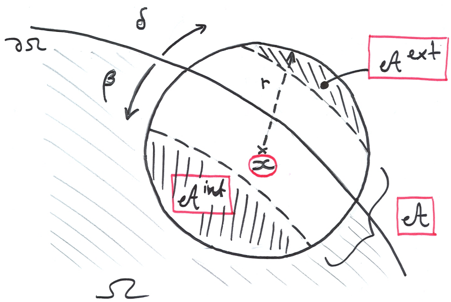
- Boundary condition (BC) is satisfied if:
  1. either $a(x)Dd(x) \neq 0$ (second order always wins)
  2. or $\alpha \geq 1$ (α order wins)
  3. or $\mathrm{tr}(a(x)D^2d(x)) + b(x)\cdot Dd(x) < 0$ (1st order does the job)
- Counter-example if $\alpha < 1$

Decomposition of the neighbourhood of $x \in \partial\Omega$

Given parameters r and δ, β,

▶ The neighbourhood of a point $x \in \partial\Omega$ is decomposed into three pieces.

$$B_r = \mathcal{A}^{\text{int}} \cup \mathcal{A}^{\text{ext}} \cup \mathcal{A}$$



Decomposition of integral operators

- # of jumps / size of inner/outer normal jps / size of all i/o jps

$$\begin{cases} I^{\mathrm{int/ext},1} &= \int_{\mathcal{A}^{\mathrm{ext}}} 1 \, d\mu_x(z) \\ I^{\mathrm{int/ext},2} &= \int_{\mathcal{A}^{\mathrm{ext}}} Dd(x) \cdot z \, d\mu_x(z) \\ I^{\mathrm{int/ext},3} &= \int_{\mathcal{A}^{\mathrm{ext}}} |z| \, d\mu_x(z) \end{cases}$$

- Second moment of the measure / Non-local transport term

$$\begin{cases} I^{4} &= \frac{1}{2} \int_{\mathcal{A}} |z|^2 \, d\mu_x(z) \\ I^{\mathrm{tr}} &= \displaystyle\int_{r<|z|<1} Dd(x) \cdot z \, d\mu_x(z) \end{cases}$$

Structure conditions

1. The expectation of the size of outer jumps is $O(\delta)$

$$|I^{\text{ext},2}| \leq |I^{\text{ext},3}| \leq O(\delta) I^{\text{ext},1}$$

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2. Either the # of inner normal jumps is infinite
   or inner jumps are controlled by outer ones

$$\begin{cases} \textit{either} \quad I^{\text{int},2} \to +\infty \quad \beta(\eta) I^{\text{int},1}, I^{\text{int},3} \leq O(1) I^{\text{int},2} \\ \textit{or} \quad I^{\text{int},1}, \frac{1}{\eta} I^{\text{int},2}, \frac{1}{\varepsilon(\eta)} I^{\text{int},3} = o(1) I^{\text{ext},1} \end{cases}$$

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3. Second moment of the measure
   controlled by inner or outer jumps

$$I^4 = o(1) I^{\text{int},2} + o(1) I^{\text{ext},1}$$

Sufficient conditions for (BC)

Remark

Structure Conditions can be relaxed for weakly singular measures.
$\int |z|\mu_x(dz) < +\infty$

Under these structure conditions on the measure, the (BC) is satisfied if

1. either $a(x)Dd(x) \neq 0$ (non-degeneracy wrt normal)
2. or $\int |Dd(x) \cdot z|\mu_x(dz) = +\infty$ (jumps do the job alone)
3. or $\mathrm{tr}(a(x)D^2d(x)) + b(x) \cdot Dd(x) + \limsup_{r\to 0} I^{\mathrm{tr}}(x) < 0$ (curvature/local transport/nl transport)

- The uniformly elliptic case
  De Giorgi, Krylov-Safonov, . . . , Caffarelli, . . .

  No regularity assumptions made the coefficients

- The strictly elliptic case
  Ishii-Lions'90, Barles-Souganidis'01, Barles-Da Lio'06 . . .

  Working with (continuous) viscosity solutions
  "require" at least continuity of the coefficients

  Main application for us: stochastic control
  uniqueness is essential

Existing results for non-local equations

- Silvestre (Indiana Univ MJ'06)

$$\int [u(x+z) - u(x)]K(x,y)dy = 0 \quad \text{in} \quad B_{2r}$$

  - No regularity assumption on $x \mapsto K(x,y)$
  - $\int |z|^\beta \mu_x < +\infty$ for β small
  - Specific non-linear equations

- Caffarelli, Silvestre ...

Main idea of the proof

To be proven

$$|u(x) - u(y)| \leq L|x - y|^\alpha \tag{1}$$

- Suppose it is false: for any L, $\exists \bar{x}, \bar{y}$ s.t. (1) is false
- Write two viscosity inequalities and combine them
- Get a contradiction $\left|\begin{array}{l}\text{either from second-order terms}\\ \text{or from non-local terms}\end{array}\right.$

$\hookrightarrow$ a Structure Condition ensuring Hölder continuity.

$\rightarrow \left|\begin{array}{l}\text{either locally strictly elliptic}\\ \text{or non-locally “strictly elliptic”}\end{array}\right.$

The Bellman-Isaacs equation

Assumptions

- I_{j_α} with common μ s.t. $\frac{c_\mu}{|z|^{N+\beta}} \leq \mu(dz) \leq \frac{C_\mu}{|z|^{N+\beta}}$
- The family $j_\alpha(z)$ are s.t. for common $r, \tilde{\theta} > 0$

$$\begin{cases} D_z j_\alpha(x,z) \text{ cont. } (x,z) + \text{ not singular in } B_r(x_0,0) \\ |j_\alpha(x,z) - j_\alpha(y,z)| \leq C_0 |z| |x-y|^{\tilde{\theta}} \end{cases}$$

- Coeff $\sigma_\alpha, b_\alpha, f_\alpha$ s.t. for a common θ

$$\|\sigma_\alpha\|_{0,\theta} + \|b_\alpha\|_{0,\theta} + \|f_\alpha\|_{0,\theta} \leq C_0$$

Theorem

If $\theta, \tilde{\theta} > \frac{1}{2}(2-\beta)$,

then the value function is $\left| \begin{array}{ll} \beta\text{-Hölder} & \text{if } \beta < 1 \\ \alpha\text{-Hölder for any } \alpha < 1 & \text{if } \beta \geq 1 \end{array} \right.$

Conclusion

1. Dirichlet problem
   - For the fractional Laplacian,
     classical results are naturally extended
   - For general operators,
     structure conditions on inner jumps and outer jumps
   - Jumps can enforce the boundary condition, without (local) diffusion
2. Hölder regularity
   - Ishii-Lions technique extends to non-local equation
   - The Bellman-Isaacs equation can be treated

Future works and references

To be done now

- Boundary Hölder regularity, Lipschitz continuity
- $C^{1,\alpha}$ regularity
- Ergodicity

▶ **with G. Barles and E. Chasseigne.** *The Dirichlet problem for second-order elliptic integro-differential equations.* Indiana Univ MJ

▶ **with G. Barles and E. Chasseigne.** *Hölder continuity of solutions of second-order elliptic integro-differential equations*

See also

▶ with G. Barles. *Second-Order Elliptic Integro-Differential Equations: Viscosity Solutions' Theory Revisited.* Annales IHP

Papers are available (or soon) here

`http://www.ceremade.dauphine.fr/∼imbert`