Beyond uniqueness:
Relaxation calculus of junction conditions for coercive Hamilton-Jacobi equations

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Abstract

A junction is a particular network given by the collection of $N \geqslant 1$ half lines $[0, +\infty)$ glued together at the origin. On such a junction, we consider evolutive Hamilton-Jacobi equations with N coercive Hamiltonians. Furthermore, we consider a general desired junction condition at the origin, given by some monotone function $F_0 : \mathbb{R}^N \to \mathbb{R}$. There is existence and uniqueness of solutions which only satisfy weakly the junction condition (at the origin, they satisfy either the desired junction condition or the PDE).

We show that those solutions satisfy strongly a relaxed junction condition $\mathfrak{R}F_0$ (that we can recognize as an effective junction condition). It is remarkable that this relaxed condition can be computed in three different but equivalent ways: 1) using viscosity inequalities, 2) using Godunov fluxes, 3) using Riemann problems. Our result goes beyond uniqueness theory, in the following sense: solutions to two different desired junction conditions F_0 and F_1 do coincide if $\mathfrak{R}F_0 = \mathfrak{R}F_1$.

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Contents

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1 Introduction

In this paper, we consider Hamilton-Jacobi equations of evolution type posed on junctions. The first results concerning these equations have been obtained in the convex case in [1, 9] and are closely related to optimal control. In [8], the authors obtained a strong comparison principle using a PDE approach while the non-convex case were studied in [10, 11] (see also [5, 6] for a new approach for proving comparison principle). Many contributions followed these articles and the reader is referred to the book by G. Barles and E. Chasseigne [3] for an up-to-date state of the art.

It is now well-known that if an equation is posed on a domain, the boundary condition can be in conflict with the equation and the same phenomena naturally appears when considering equations on networks. A classical way to handle this difficulty for Hamilton-Jacobi equations is to impose either the boundary condition (or the junction condition) or the equation at the boundary (or at the junction point), both in the viscosity sense. Such solutions are called weak (viscosity) solutions.

Concerning Hamilton-Jacobi on junctions and for convex and coercive Hamiltonians, C. Imbert and the second author [8] show the existence of weak solutions and proved that these weak solutions satisfy other junction condition in a strong way. These conditions are called relaxed junction conditions and are parametrized by a single real parameter, called the flux limiter (only for convex Hamiltonians).

When the Hamiltonians is still coercive but not necessarily convex, the situation is much more complicated. The first result in that direction was obtained by J. Guerand [7] in the one-dimensional case (a very simple junction with only one branch). She proved that, in this situation, it is still possible to relax the boundary conditions to obtain strong solutions. She showed in particular that the family of relaxed boundary conditions is much more rich and is characterized by a family of limiter points. In [4], with C. Imbert, we revisit this characterization and propose a new formula which can be easily derived from the definition of weak viscosity solutions. We also exhibit a strong connection between this relaxed boundary condition and Godunov's fluxes for conservation laws.

The goal of this paper is to extend these results to the case of general junctions with several branches. In particular we show how to define properly relaxed junction conditions using the definition of weak viscosity solutions. As for the case of one branch, we show that the relaxation can also be defined in terms of Godunov's fluxes (see also [12] for related results). Finally, we also give a third formulation of the relaxation based on Riemann problems.

Let us emphasise the fact that characterization of relaxed junction conditions is important, because several junction conditions can lead to the same relaxed junction condition. Hence, even if at first glance, problems may seem different, they can be exactly the same (see Subsection 5.5). From this point of view, our analysis goes beyond the classical question of uniqueness of solutions.

2 Main results

2.1 The junction problem

We begin to describe what is a (one-dimensional) **junction**. We consider $N \geqslant 1$ copies of the interval $[0,+\infty)$ that we call J_α for $\alpha = 1,\dots,N$. We glue all the branches J_α together at the origin such that

$$J_\alpha \cap J_\beta = \{0\} \quad \text{for all} \quad \alpha \neq \beta$$

We call the junction

$$J = \bigcup_{\alpha=1,\dots,N} J_\alpha .$$

where $x = 0$ is now the **junction point**.

For a function $u : [0,+\infty) \times J \to \mathbb{R}$ whose values are $u(t,x)$, we denote by u_t the time derivative of u, and we define the gradient of u as

$$u_x(t,x) = \begin{cases} \partial_\alpha u(t,x) & \text{if} \quad x \in J_\alpha^* := J_\alpha \backslash \{0\}, \\ (\partial_1 u(t,0^+),\dots,\partial_N u(t,0^+)) & \text{if} \quad x = 0. \end{cases}$$

Throughout the paper, we will consider solutions of the following problem

$$\begin{cases} u_t + H^\alpha(u_x) = 0 & \text{for all} \quad t \in (0,+\infty) \quad \text{and} \quad x \in J_\alpha^*, \quad \text{for} \quad \alpha = 1,\dots,N \\ u_t + F_0(u_x) = 0 & \text{for all} \quad t \in (0,+\infty) \quad \text{and} \quad x = 0. \end{cases} \tag{1}$$

The second line of (1) is the junction condition (JC) and by abuse of terminology, we will also say that the junction condition (or junction function) is F_0. In the particular case of (1), we call the second line, a **desired junction condition** (because as we will see, this condition can not always be truly satisfied). Here the Hamiltonians H^α for $\alpha = 1,\dots,N$ are assumed to satisfy

$$\begin{cases} \textbf{(Continuity)} & H^\alpha \in C(\mathbb{R}^N) \\ \\ \textbf{(Coercivity)} & H^\alpha(p^\alpha) \to +\infty \quad \text{as} \quad |p^\alpha| \to +\infty. \end{cases} \tag{2}$$

When this is the case, also simply say that

$$H = (H^1,\dots,H^N)$$

satisfies (2). We also assume that the function F_0 satisfies

$$\begin{cases} \textbf{(Continuity)} & F_0 \in C(\mathbb{R}^N) \\ \\ \textbf{(Monotonicity)} & F_0 \text{ is nonincreasing in each of its arguments.} \end{cases} \tag{3}$$

For existence results, we will also need an initial condition

$$u(0,x) = u_0(x) \quad \text{for all} \quad x \in J \tag{4}$$

that is assumed to be chosen such that

$$u_0 \text{ is uniformly continuous on } J. \tag{5}$$

2.2 The relaxation formula

Inspired by [4], in order to define the relaxation operator, we define the sub-relaxation operator with respect to H by

$$\underline{R}F_0(p) = \sup_{q \geqslant p} \min \{F_0(q), H_{min}(q)\}$$

and the super-relaxation operator by

$$\overline{R}F_0(p) = \inf_{q \leqslant p} \max \{F_0(q), H_{max}(q)\}$$

where $q \leqslant p$ means $q^\alpha \leqslant p^\alpha$ for each $\alpha = 1, \ldots, N$ and

$$H_{max}(p) = \max_{\alpha=1,\ldots,N} H^\alpha(p^\alpha) \qquad \text{and} \qquad H_{min}(p) = \min_{\alpha=1,\ldots,N} H^\alpha(p^\alpha).$$

We then have the following theorem which enables to define the relaxation operator $\mathfrak{R}$.

Theorem 2.1 (The relaxation operator $\mathfrak{R}$)**.** *Let H satisfy* (2) *and F_0 satisfy* (3)*. Then*

$$\underline{R}\overline{R}F_0 = \overline{R}\underline{R}F_0 \quad (=: \mathfrak{R}F_0).$$

Moreover

$$\mathfrak{R}F_0 = \mathfrak{R}\max(F_0, H_-) \geqslant H_-,$$

where

$$H_-(p) = \max_{\alpha=1,\ldots,N} H^\alpha_-(p^\alpha) \quad \textit{with} \quad H^\alpha_-(p^\alpha) = \inf_{q^\alpha \leqslant p^\alpha} H^\alpha(q^\alpha) \tag{6}$$

i.e. H^α_- is the lower nonincreasing hull of H^α.

In Section 3, we will introduce another relaxation formula using the Godunov's fluxes while in Section 7, we introduce a third equivalent formulation based on Riemann problems.

2.3 Weak and strong viscosity solutions

As mentioned in the introduction, we can consider two types of solutions for Hamilton-Jacobi equations on junctions. The first one, called weak viscosity solution requires that either the junction or the equation is satisfied at the junction point (see Definition 5.1), while the second one, called strong viscosity solutions requires that the junction condition is always satisfied (see Definition 5.3).

The second main result of the paper is to show that a function is a weak viscosity solution of (1) if and only if it is a strong viscosity solution of the same equation but with F_0 replaced by the relaxation of F_0, namely $\mathfrak{R}F_0$. We refer to Proposition 5.9 for the precise result.

Organization of the paper In Section 3, we introduce the Godunov relaxation and prove some properties that will be useful to study the relation of this relaxation with the relaxation operator $\mathfrak{R}$. In Section 4 we show that the Godunov relaxation and the relaxation operator $\mathfrak{R}$ coincide and we prove further properties on the relaxation operator. Section 5 is devoted to the study of viscosity solutions for (1). We show in particular that a function is a F_0-weak solution iff it is a $\mathfrak{R}F_0$-strong solution to the same equation. In Section 6, we state a comparison principle for (1). Finally, in Section 7, we propose a third relaxation formula based on Riemann problems and we show that it coincides with the relaxation operator $\mathfrak{R}$.

3 Godunov relaxation

In this section, we introduce the Godunov relaxation. The precise definition is given in Subsection 3.1, while in Subsection 3.2, we introduce the semi-Godunov relaxation and present some important properties that will be useful in the sequel, in particular in Section 4 for the study of the relation between Godunov relaxation and the relaxation operator $\mathfrak{R}$.

Il all this section, we will assume that the junction condition satisfies the semi-coercivity assumption:

$$F_0(p) \to +\infty \quad \text{as} \quad \min_{\alpha=1,\ldots,N} p^\alpha \to -\infty. \tag{7}$$

3.1 Definition and properties of the Godunov relaxation

Recall that the Godunov flux associated to the Hamiltonian (or flux) H^α is given for $p,q \in \mathbb{R}$ by

$$G^\alpha(p,q) = \begin{cases} \min\limits_{[p,q]} H^\alpha & \text{if} \quad p \leqslant q, \\ \max\limits_{[q,p]} H^\alpha & \text{if} \quad p \geqslant q. \end{cases} \tag{8}$$

In particular, G^α is non-decreasing in the first variable and non-increasing in the second one. Moreover, we have $G^\alpha(p,p) = H^\alpha(p)$. We define next the action of the Godunov flux on a semi-coercive, continuous and non-increasing function F_0.

Proposition 3.1 (Godunov's relaxation)**.** *Let H satisfy* (2) *and consider F_0 satisfying* (3) *and the semi-coercivity property* (7)*. Let $p \in \mathbb{R}^N$, then the following properties hold true.*

(i) There exists at least one $q \in \mathbb{R}^N$ such that $F_0(q) = G^\alpha(q^\alpha, p^\alpha)$ for all $\alpha = 1, \dots, N$. The common value is denoted by λ_q.

(ii) The value λ_q defined above is independent on q. We denote this unique value by

$$\lambda = \lambda(p) =: (F_0 G)(p).$$

Proof. In all the proof, $p \in \mathbb{R}^N$ is fixed.

We begin to prove (i). In order to get an increasing function, we define for $\varepsilon \geqslant 0$

$$K^\alpha_\varepsilon(q^\alpha) = G^\alpha(q^\alpha, p^\alpha) + \varepsilon(1 + \tanh q^\alpha).$$

For $\varepsilon > 0$ and $\lambda > \lambda_\varepsilon := \max\limits_{\alpha \in \{1,\dots,N\}} \inf K^\alpha_\varepsilon \geqslant \max\limits_{\alpha=1,\dots,N} \inf K^\alpha_0$, we define $q^\alpha_{\lambda,\varepsilon}$ as the unique value such that $K^\alpha_\varepsilon(q^\alpha_{\lambda,\varepsilon}) = \lambda$. Since K^α_ε is increasing, we deduce that $\lambda \mapsto q^\alpha_{\lambda,\varepsilon}$ is increasing and continuous. We then define the continuous and increasing function, for $\lambda > \lambda_\varepsilon$

$$N_\varepsilon(\lambda) = \lambda - F_0(q_{\lambda,\varepsilon}).$$

Since $\min_\alpha(q^\alpha_{\lambda,\varepsilon}) \to -\infty$ as $\lambda \to \lambda_\varepsilon$, we deduce, by semi-coercivity of F_0 that $\lim_{\lambda \to \lambda_\varepsilon} N_\varepsilon(\lambda) = -\infty$. In the same way, since $q^\alpha_{\lambda,\varepsilon} \to +\infty$ as $\lambda \to +\infty$ for all $\alpha \in \{1,\dots,N\}$, we deduce that $\lim_{\lambda \to +\infty} N_\varepsilon(\lambda) \geqslant \lim_{\lambda \to +\infty} \lambda - F_0(0) = +\infty$. We can then define λ^*_ε as the unique value such that $N_\varepsilon(\lambda^*_\varepsilon) = 0$. In particular, since $\varepsilon \mapsto q^\alpha_{\lambda,\varepsilon}$ is decreasing, we deduce that $\varepsilon \mapsto N_\varepsilon(\lambda)$ is non-increasing and so $\varepsilon \mapsto \lambda^*_\varepsilon$ is non-decreasing. This implies that λ^*_ε is uniformly bounded for $\varepsilon \leqslant 1$, by a constant depending only on p. We then set

$$q_\varepsilon = q_{\lambda_\varepsilon, \varepsilon},$$

which satisfies

$$\lambda^*_\varepsilon = F_0(q_\varepsilon) = K^\alpha_\varepsilon(q^\alpha_\varepsilon) = G^\alpha(q^\alpha_\varepsilon, p^\alpha) + \varepsilon(1 + \tanh q^\alpha_\varepsilon) \quad \text{for all } \alpha = 1, \dots, N. \tag{9}$$

By semi-coercivity of F_0 and by the fact that $G(q^\alpha, p^\alpha) \to +\infty$ as $q^\alpha \to +\infty$, we deduce, using also that λ^*_ε is uniformly bounded, that q_ε is uniformly bounded. Up to extract a subsequence, we can then assume that $(\lambda^*_\varepsilon, q_\varepsilon) \to (\lambda^*, q)$ as $\varepsilon \to 0$. Passing to the limit in (9), we get, by continuity, that

$$\lambda^* = F_0(q) = G^\alpha(q^\alpha, p^\alpha) \quad \text{for all } \alpha = 1, \dots, N,$$

which proves (i).

We now turn to (ii). Assume that there exists q_1, q_2 such that $\lambda_{q_1} < \lambda_{q_2}$. We then have for all $\alpha \in \{1, \dots, N\}$

$$F_0(q_1) = G^\alpha(q^\alpha_1, p^\alpha) = \lambda_{q_1} < \lambda_{q_2} = F_0(q_2) = G^\alpha(q^\alpha_2, p^\alpha).$$

By monotonicity of the Godunov fluxes, we deduce that $q^\alpha_1 < q^\alpha_2$ for $\alpha \in \{1, \dots, N\}$. Using the monotonicity of F_0, this implies that

$$\lambda_{q_1} = F_0(q_1) \geqslant F_0(q_2) = \lambda_{q_2}$$

which is a contradiction. This ends the proof of the proposition. □

In order to study the relation between the relaxation operator $\mathfrak{R}$ and the Godunov relaxation, we need to introduce the Godunov semi-fluxes.

3.2 Godunov semi-fluxes

For $\alpha = 1, \dots, N$, we introduce the Godunov semi-fluxes, $\underline{G}^\alpha$ and $\overline{G}^\alpha$, which are set-valued applications defined by

$$\underline{G}^\alpha(q^\alpha, p^\alpha) = \begin{cases} \{-\infty\} & \text{if } q^\alpha < p^\alpha \\ [-\infty, H^\alpha(p^\alpha)] & \text{if } q^\alpha = p^\alpha, \\ \left\{ \max\limits_{[p^\alpha, q^\alpha]} H^\alpha \right\} & \text{if } q^\alpha > p^\alpha \end{cases}$$

and

$$\overline{G}^\alpha(q^\alpha, p^\alpha) = \begin{cases} \left\{ \min\limits_{[q^\alpha, p^\alpha]} H^\alpha \right\} & \text{if } q^\alpha < p^\alpha, \\ [H^\alpha(p), +\infty] & \text{if } q^\alpha = p^\alpha, \\ \{+\infty\} & \text{if } q^\alpha > p^\alpha. \end{cases}$$

As before, we can define the action of these semi-fluxes on non-increasing semi-coercive continuous functions.

Proposition 3.2 (Lower Godunov's relaxation)**.** *Let H satisfy* (2) *and F_0 satisfy* (3) *and the semi-coercivity property* (7)*. Let $p \in \mathbb{R}^N$, then the following properties hold true.*

(i) There exists at least one $q \in \mathbb{R}^N$ such that $F_0(q) \in \underline{G}^\alpha(q^\alpha, p^\alpha)$ for all $\alpha = 1, \dots, N$. The value $F_0(q)$ is denoted by λ_q. Moreover, all the q satisfying the previous property satisfy $q \geqslant p$.

(ii) The value λ_q defined above is independent on q. We denote this unique value by

$$\lambda = \lambda(p) =: (F_0\underline{G})(p).$$

Proof. Let $p \in \mathbb{R}^N$. If $F_0(p) \leqslant \min\limits_{\alpha=1,\dots,N} H^\alpha(p^\alpha)$, then we can choose $q^\alpha = p^\alpha$ for all $\alpha = 1, \dots, N$ and we get $F_0(q) = F_0(p) \in \underline{G}^\alpha(q^\alpha, p^\alpha) = [-\infty, H^\alpha(p^\alpha)]$. If $F_0(p) > \min\limits_{\alpha=1,\dots,N} H^\alpha(p^\alpha)$, we denote by $I^- := \{\alpha,\ F_0(p) \leqslant H^\alpha(p^\alpha)\}$ and by $I^+ := \{\alpha,\ F_0(p) > H^\alpha(p^\alpha)\}$. For $\alpha \in I^-$, we set $q^\alpha = p^\alpha$. In particular, $F_0(p) \in \underline{G}^\alpha(q^\alpha, p^\alpha)$ for $\alpha \in I^-$. Arguing as in the proof of Proposition 3.1, but working only on the indices $\alpha \in I^+$, we can construct $q^\alpha \geqslant p^\alpha$ such that $F_0(q) = G^\alpha(q^\alpha, p^\alpha) = \max\limits_{[p^\alpha, q^\alpha]} H^\alpha$ for all $\alpha \in I^+$. Since we also have $F_0(q) \leqslant F_0(p) \leqslant H^\alpha(p^\alpha)$ for all $\alpha \in I^-$, we get (i).

The proof of (ii) is similar to the one of Proposition 3.1 (ii).

□

In the same way, we have the following result

Proposition 3.3 (Upper Godunov's relaxation)**.** *Let H satisfy* (2) *and F_0 satisfy* (3) *and the semi-coercivity property* (7)*. Let $p \in \mathbb{R}^N$, then the following properties hold true.*

(i) There exists at least one $q \in \mathbb{R}^N$ such that $F_0(q) \in \overline{G}^\alpha(q^\alpha, p^\alpha)$ for all $\alpha = 1, \dots, N$. The value $F_0(q)$ is denoted by λ_q. Moreover, all the q satisfying the previous property satisfy $q \leqslant p$.

(ii) The value λ_q defined above is independent on q. We denote this unique value by

$$\lambda = \lambda(p) =: (F_0\overline{G})(p).$$

In order to compose the Godunov semi-fluxes, we have to show that $F_0\underline{G}$ and $F_0\overline{G}$ satisfy the same assumptions as F_0.

Proposition 3.4 (Properties of $F_0\underline{G}$ and $F_0\overline{G}$)**.** *Under the same assumptions, $F_0\underline{G}$ and $F_0\overline{G}$ are continuous, non-increasing and semi-coercive in the sense of* (7)*.*

Proof. We only do the proof for $F_0\underline{G}$ since it is similar for $F_0\overline{G}$.

We begin to show that $F_0\underline{G}$ is non-increasing. Let p_1 and p_2 such that $p_1 \geqslant p_2$ and assume by contradiction that $(F_0\underline{G})(p_1) > (F_0\underline{G})(p_2)$. Let $q_1,\ q_2$ be such that $F_0(q_i) \in \underline{G}^\alpha(q_i^\alpha, p_i^\alpha)$ for all $\alpha \in \{1, \dots, N\}$. In particular, we have $q_1 \geqslant p_1$, $q_2 \geqslant p_2$ and $F_0(q_1) > F_0(q_2)$. We claim that $q_1 \geqslant q_2$. Indeed, if for some α, we have $p_2^\alpha \leqslant p_1^\alpha \leqslant q_1^\alpha < q_2^\alpha$, then

$$G^\alpha(q_2^\alpha, p_2^\alpha) = F_0(q_2) < F_0(q_1) \leqslant G^\alpha(q_1^\alpha, p_1^\alpha) \leqslant G^\alpha(q_2^\alpha, p_2^\alpha),$$

which is a contradiction. This implies that $q_1 \geqslant q_2$ and so

$$(F_0\underline{G})(p_1) = F_0(q_1) \leqslant F_0(q_2) = (F_0\underline{G})(p_2),$$

which is a contradiction. This implies that $F_0\underline{G}$ is non-increasing.

We now show that $F_0\underline{G}$ is continuous. Let $p_n \to p$ and $q_n \geqslant p_n$ such that

$$(F_0\underline{G})(p_n) = F_0(q_n) \in \underline{G}^\alpha(q_n^\alpha, p_n^\alpha) \qquad \text{for all } \alpha \in \{1, \dots, N\}.$$

We claim that q_n is bounded. Indeed, if (up to a subsequence) $q_n^\beta \to +\infty$ for a certain $\beta \in \{1, \dots, N\}$, then $q_n^\beta > p_n^\beta$ and

$$F_0(p_n) \geqslant F_0(q_n) = G^\beta(q_n^\beta, p_n^\beta) \geqslant G^\beta(q_n^\beta, q_n^\beta) = H^\beta(q_n^\beta).$$

Passing to the limit, we then have $F_0(p) = +\infty$, which is absurd. Then q_n is bounded, and up to extract a subsequence, we can assume that $q_n \to q_0 \geqslant p$.

We claim that $F_0(q_0) \in \underline{G}^\alpha(q_0^\alpha, p^\alpha)$ for all $\alpha \in \{1, \dots, N\}$. Indeed, fix $\alpha \in \{1, \dots, N\}$ and assume first that $q_0^\alpha > p^\alpha$. We then have $q_n^\alpha > p_n^\alpha$ for n large enough and so $F_0(q_n) = G^\alpha(q_n^\alpha, p_n^\alpha)$. Passing to the limit $n \to +\infty$, we get $F_0(q_0) = G^\alpha(q_0^\alpha, p^\alpha) \in \underline{G}^\alpha(q_0^\alpha, p^\alpha)$. Assume now that $q_0^\alpha = p^\alpha$. We distinguish two cases. On the one hand, if there exists a subsequence n_j such that $q_{n_j}^\alpha = p_{n_j}^\alpha$, then $F_0(q_{n_j}) \leqslant H^\alpha(q_{n_j}^\alpha) = H^\alpha(p_{n_j}^\alpha)$. Passing to the limit, this implies that $F_0(q_0) \leqslant H^\alpha(q_0^\alpha) = H^\alpha(p^\alpha)$ and so $F_0(q_0) \in [-\infty, H^\alpha(p^\alpha)] = \underline{G}(q_0, p_0)$. On the other hand, if $q_n^\alpha > p_n^\alpha$ for n large enough, then $F_0(q_n) = G^\alpha(q_n^\alpha, p_n^\alpha)$. Again, passing to the limit $n \to +\infty$, we get $F_0(q_0) = G^\alpha(q_0^\alpha, p^\alpha) = H^\alpha(p^\alpha)$ and so $F_0(q_0) \in \underline{G}^\alpha(q_0^\alpha, p^\alpha)$.

In all the cases, we then have $F_0(q_0) \in \underline{G}^\alpha(q_0^\alpha, p^\alpha)$. This implies that $(F_0\underline{G})(p) = F_0(q_0)$. Since $(F_0\underline{G})(p_n) = F_0(q_n)$ and $F_0(q_n) \to F_0(q_0)$, we recover that $(F_0\underline{G})(p_n) \to (F_0\underline{G})(p)$ and so $F_0\underline{G}$ is continuous.

We end the proof showing that $F_0\underline{G}$ is semi-coercive. We fix $\alpha \in \{1, \dots, N\}$ and we want to show that if $p^\alpha \to -\infty$, then $(F_0\underline{G})(p) \to +\infty$. Let $M > 0$. By coercivity of H^α and semi-coercivity of F_0, there exists p_0^α such that for all $p \in \mathbb{R}^N$ such that $p^\alpha \leqslant p_0^\alpha$, we have

$$H^\alpha(p^\alpha) \geqslant M \quad \text{and} \quad F_0(p) \geqslant M.$$

Let p be such that $p^\alpha \leqslant p_0^\alpha$. Then there exists $q \geqslant p$ such that $(F_0\underline{G})(p) = F_0(q) \in \underline{G}^\alpha(q^\alpha, p^\alpha)$. We distinguish two cases. If $q^\alpha > p^\alpha$, then

$$(F_0\underline{G})(p) = F_0(q) = G^\alpha(q^\alpha, p^\alpha) \geqslant H^\alpha(p^\alpha) \geqslant M.$$

If $q^\alpha = p^\alpha$, then, since $q^\alpha \leqslant p_0^\alpha$, we have $(F_0\underline{G})(p) = F_0(q) \geqslant M$. Then, in all the cases, we have

$$(F_0\underline{G})(p) \geqslant M$$

and so $(F_0\underline{G})$ is semi-coercive. ☐

The result of the previous proposition allows us to compose the action of $\overline{G}$ with the action of $\underline{G}$. We now want to prove that the action of G on F_0 is in fact the action of $\underline{G}$ on the action of $\overline{G}$ on F_0. More precisely, we have the following result

Proposition 3.5 (Composition of Godunov semi-fluxes)**.** *Under the same assumptions, we have*

$$(F_0\overline{G})\underline{G} = F_0 G = (F_0\underline{G})\overline{G}.$$

In order to prove this proposition, the following lemma is needed. The proof can be found in [4, Lemma 5.7], where the result holds independently for each component α.

Lemma 3.6 (Key composition result). *(i) For all $q, p \in \mathbb{R}^N$, there exists $\tilde{q} \in \mathbb{R}^N$ such that for all $\alpha \in \{1, \dots, N\}$, $\overline{G}^\alpha(q^\alpha, \tilde{q}^\alpha) \cap \underline{G}^\alpha(\tilde{q}^\alpha, p^\alpha) \neq \emptyset$. Moreover, for such a vector $\tilde{q}$, we have*

$$\overline{G}^\alpha(q^\alpha, \tilde{q}^\alpha) \cap \underline{G}^\alpha(\tilde{q}^\alpha, p^\alpha) = \{G^\alpha(q^\alpha, p^\alpha)\} \quad \text{for all } \alpha \in \{1, \dots, N\}.$$

(ii) For all $q, p \in \mathbb{R}^N$, there exists $\tilde{q} \in \mathbb{R}^N$ such that for all $\alpha \in \{1, \dots, N\}$, $\underline{G}^\alpha(q^\alpha, \tilde{q}^\alpha) \cap \overline{G}^\alpha(\tilde{q}^\alpha, p^\alpha) \neq \emptyset$. Moreover, for such a vector $\tilde{q}$, we have

$$\underline{G}^\alpha(q^\alpha, \tilde{q}^\alpha) \cap \overline{G}^\alpha(\tilde{q}^\alpha, p^\alpha) = \{G^\alpha(q^\alpha, p^\alpha)\} \quad \text{for all } \alpha \in \{1, \dots, N\}.$$

We are now turn to the proof of Proposition 3.5.

Proof of Proposition 3.5. The proof is similar to the one of [4, Proposition 5.6], but for the reader's convenience we give the details. Let $F_1 = F_0\overline{G}$. We use successively the definition of F_0G, (i) from Lemma 3.6, the definitions of $F_0\overline{G}$ and of $F_1\underline{G}$ to write,

$$\begin{aligned}
\{F_0G(p)\} &= \{F_0(q) \text{ for some } q \text{ s.t. } F_0(q) \in G^\alpha(q^\alpha, p^\alpha) \text{ for all } \alpha \in \{1, \dots, N\}\} \\
&= \{F_0(q) \text{ for some } q \text{ and } \tilde{q} \text{ s.t. } F_0(q) \in \overline{G}^\alpha(q^\alpha, \tilde{q}^\alpha) \cap \underline{G}^\alpha(\tilde{q}^\alpha, p^\alpha) \text{ for all } \alpha \in \{1, \dots, N\}\} \\
\{F_1(\tilde{q})\} &= \{F_0\overline{G}(\tilde{q})\} = \{F_0(q) \text{ for some } q \text{ s.t. } F_0(q) \in \overline{G}^\alpha(q^\alpha, \tilde{q}^\alpha) \text{ for all } \alpha \in \{1, \dots, N\}\} \\
\{F_1\underline{G}(p)\} &= \{F_1(\tilde{q}) \text{ for some } \tilde{q} \text{ s.t. } F_1(\tilde{q}) \in \underline{G}^\alpha(\tilde{q}^\alpha, p^\alpha) \text{ for all } \alpha \in \{1, \dots, N\}\} \\
&= \{F_0(q) \text{ for some } q \text{ and } \tilde{q} \text{ s.t. } F_0(q) \in \overline{G}^\alpha(q^\alpha, \tilde{q}^\alpha) \cap \underline{G}^\alpha(\tilde{q}^\alpha, p^\alpha) \text{ for all } \alpha \in \{1, \dots, N\}\}.
\end{aligned}$$

This implies that $F_0G(p) = F_1\underline{G}(p) = (F_0\overline{G})\underline{G}(p)$.

Using (ii) from Lemma 3.6, we can follow the same reasoning and get $F_0G(p) = (F_0\underline{G})\overline{G}(p)$. □

4 Relaxation operator

This section is devoted to the proof of Theorem 2.1. Conversely to the case of a single branch (see [4]), we are not able to show directly that $\underline{R}\overline{R}F_0 = \overline{R}\underline{R}F_0$. To prove this, we will use the link between the relaxation operators and the Godunov's semi-fluxes. This is done in Subsection 4.1 in the case where F_0 is semi-coercive. In Subsection 4.2 we will give some useful properties of the relaxation operators that will be used to prove Theorem 2.1 in the general case. Finally, in Subsection 4.3, we give further properties of the relaxation operators that will be used in the rest of the paper.

We recall that the sub-relaxation operator with respect to H is defined by

$$\underline{R}F_0(p) = \sup_{q \geqslant p} \min \{F_0(q), H_{min}(q)\}$$

and the super-relaxation operator by

$$\overline{R}F_0(p) = \inf_{q \leqslant p} \max \{F_0(q), H_{max}(q)\}$$

where $q \leqslant p$ means $q^\alpha \leqslant p^\alpha$ for each $\alpha = 1, \dots, N$ and

$$H_{max}(p) = \max_{\alpha=1,\dots,N} H^\alpha(p^\alpha) \qquad \text{and} \qquad H_{min}(p) = \min_{\alpha=1,\dots,N} H^\alpha(p^\alpha).$$

4.1 Link between Relaxation and Godunov fluxes

The goal of this subsection is to show that the relaxation operator $\mathfrak{R}$ and the Godunov relaxation coincide when F_0 is semi-coercive. We begin by the following proposition.

Proposition 4.1 (Semi-relaxations and Godunov's semi-fluxes)**.** *Let H satisfy* (2) *and F_0 satisfy* (3) *and the semi-coercivity property* (7)*. Then*

$$\underline{R}F_0 = F_0\underline{G} \quad \textit{and} \quad \overline{R}F_0 = F_0\overline{G}.$$

Proof. We only prove that $\underline{R}F_0 = F_0\underline{G}$, the proof of $\overline{R}F_0 = F_0\overline{G}$ being similar. The proof is decomposed into two steps.

Step 1: $\underline{R}F_0 \geqslant F_0\underline{G}$. Let $p \in \mathbb{R}^N$. By Proposition 3.2, there exists $\tilde{q}$ such that $F_0\underline{G}(p) = F_0(\tilde{q}) \in \underline{G}^\alpha(\tilde{q}^\alpha, p^\alpha)$ for all $\alpha \in \{1, \ldots, N\}$. If $\tilde{q}^\alpha > p^\alpha$, we have $F_0(\tilde{q}) = G^\alpha(\tilde{q}^\alpha, p^\alpha) = \max_{[p^\alpha, \tilde{q}^\alpha]} H^\alpha$ and we define $\bar{q}^\alpha \in [p^\alpha, \tilde{q}^\alpha]$ such that

$$H^\alpha(\bar{q}^\alpha) = \max_{[p^\alpha, \tilde{q}^\alpha]} H^\alpha = F_0(\tilde{q}).$$

If $\tilde{q}^\alpha = p^\alpha$, we set $\bar{q}^\alpha = p^\alpha$, so that, using $F_0(\tilde{q}) \in \underline{G}^\alpha(p^\alpha, p^\alpha)$, $F_0(\tilde{q}) \leqslant H^\alpha(p^\alpha) = H^\alpha(\bar{q}^\alpha)$.

Using that $\bar{q} \geqslant p$, we then have

$$\underline{R}F_0(p) = \sup_{q \geqslant p} \min(F_0(q), H_{\min}(q)) \geqslant \min(F_0(\bar{q}), H_{\min}(\bar{q})). \tag{10}$$

Since $\bar{q} \leqslant \tilde{q}$, we have $F_0(\bar{q}) \geqslant F_0(\tilde{q})$. Moreover, by definition of $\bar{q}$, we have $H^\alpha(\bar{q}^\alpha) \geqslant F_0(\tilde{q})$. Injecting these two estimates in (10), we get

$$\underline{R}F_0(p) \geqslant F_0(\tilde{q}) = F_0\underline{G}(p).$$

Step 2: $\underline{R}F_0 \leqslant F_0\underline{G}$. With the same notation as in Step 1, we have, for all $q \geqslant \tilde{q}$

$$\min(F_0(q), H_{\min}(q)) \leqslant F_0(q) \leqslant F_0(\tilde{q}). \tag{11}$$

If $\tilde{q} = p$, we directly get the result, just taking the supremum on $q \geqslant \tilde{q} = p$ on the left hand side. We then assume that $\tilde{q} \neq p$. Hence there exists α such that $\tilde{q}^\alpha > p^\alpha$. Let $q \geqslant p$ be such that there exists $\alpha \in \{1, \ldots, N\}$ such that $q^\alpha < \tilde{q}^\alpha$. Then

$$\min\left(F_0(q), H_{\min}(q)\right) \leqslant H_{\min}(q) \leqslant H^\alpha(q^\alpha) \leqslant \max_{[p^\alpha, \tilde{q}^\alpha]} H^\alpha = G^\alpha(\tilde{q}^\alpha, p^\alpha) = F_0(\tilde{q}).$$

Combining this with (11), we finally get

$$\underline{R}F_0(p) = \sup_{q \geqslant p} \min(F_0(q), H_{\min}(q)) \leqslant F_0(\tilde{q}) = F_0\underline{G}(p).$$

This ends the proof of the proposition.

□

A direct consequence of this result, combining with Proposition 3.5, is the following result.

Proposition 4.2 (Definition of $\mathfrak{R}F_0$ with F_0 semi-coercive)**.** *Let H satisfy* (2) *and F_0 satisfy* (3) *and the semi-coercivity property* (7)*. Then*

$$\overline{R}\underline{R}F_0 = \underline{R}\overline{R}F_0.$$

The goal of the next section is to generalize this result to the case where F_0 doesn't satisfy the semi-coercivity assumption. To do that, we will need some additional properties on the semi-relaxations.

4.2 First properties of the semi-relaxations and proof of Theorem 2.1

We begin by the following lemma whose proof is an immediate consequence of the definition.

Lemma 4.3 (Monotonicity of the semi-relaxations)**.** *Let H, H' satisfy* (2) *and F_0, F_0' satisfy* (3)*. If $F_0' \leqslant F_0$, then $\underline{R}F_0' \leqslant \underline{R}F_0$ and $\overline{R}F_0' \leqslant \overline{R}F_0$. Moreover, if we denote by $\underline{R}_H F_0$ (resp. $\overline{R}_H F_0$) the sub- (resp. super-) relaxation of F_0 with respect to H, then if $H \leqslant H'$, then*

$$\underline{R}_H F_0 \leqslant \underline{R}_{H'} F_0 \quad \text{and} \quad \overline{R}_H F_0 \leqslant \overline{R}_{H'} F_0.$$

Lemma 4.4 ($\underline{R}$ and $\overline{R}$ as projectors)**.** *Let H satisfy* (2) *and F_0 satisfy* (3)*. Then*

$$\underline{R}(\underline{R}F_0) = \underline{R}F_0 \quad \text{and} \quad \overline{R}(\overline{R}F_0) = \overline{R}F_0.$$

Proof. We do only the proof for $\underline{R}$. We set $F_1 = \underline{R}F_0$. On the one hand, by definition of $\underline{R}$ and by monotonicity of F_0, we have

$$\underline{R}F_0(p) = \sup_{q \geqslant p} \min(F_0(q), H_{\min}(q)) \leqslant \sup_{q \geqslant p} F_0(q) = F_0(p). \tag{12}$$

Similarly, we get $\underline{R}F_1(p) \leqslant F_1(p)$. On the other hand, we have $F_1(p) = \sup_{q \geqslant p} \min(F_0(q), H_{\min}(q))$ and we assume for simplicity that there exists $q^* \geqslant p$ such that $F_1(p) = \min(F_0(q^*), H_{\min}(q^*))$. If such point does not exist, we simply consider ε-maximizer, which leads classically to the same conclusion in the limit $\varepsilon \to 0$.

By definition and monotonicity of F_1, we have

$$F_1(p) \geqslant F_1(q^*) \geqslant \min(F_0(q^*), H_{\min}(q^*)) = F_1(p).$$

Using (12), we get $F_1(q^*) = \underline{R}F_0(q^*) \leqslant F_0(q^*)$. Since $\underline{R}F_0(q^*) \geqslant H_{\min}(q^*)$, we get that

$$\min(F_0(q^*), H_{\min}(q^*)) = H_{\min}(q^*).$$

Therefore $H_{\min}(q^*) = F_1(p) = F_1(q^*)$. We then deduce that

$$\underline{R}F_1(p) \geqslant \min(F_1(q^*), H_{\min}(q^*)) = F_1(p).$$

which implies that $\underline{R}F_1 = F_1$ and ends the proof. □

The following lemma will enable us to replace F_0 by a semi-coercive function.

Lemma 4.5 (Relaxation of $\max(F_0, H_-)$)**.** *Let H satisfy* (2) *and F_0 satisfy* (3)*. Then*

$$\overline{R}(\max(F_0, H_-)) = \overline{R}F_0$$

with H_- defined in (6)*.*

Proof. By definition, we have

$$\overline{R}(\max(F_0, H_-)) = \inf_{q \leqslant p} \max(\max(F_0(q), H_-(q)), H_{\max}(q)) = \inf_{q \leqslant p} \max(F_0(q), H_-(q), H_{\max}(q)) = \overline{R}F_0,$$

where we use that $H_- \leqslant H_{\max}$ for the last equality. This ends the proof. □

We are now ready to give the proof of Theorem 2.1

Proof of Theorem 2.1. We first claim that

$$\overline{R}\underline{R}F_0 = \overline{R}\underline{R}\max(F_0, H_-). \tag{13}$$

Indeed, by definition of $\underline{R}F_0$, we have

$$\begin{aligned}
\underline{R}\max(F_0, H_-)(p) &= \sup_{q \geqslant p} \min\{\max(F_0(q), H_-(q)), H_{\min}(q)\} \\
&\leqslant \sup_{q \geqslant p} \max\{\min(F_0(q), H_{\min}(q)), \min(H_-(q), H_{\min}(q))\} \\
&\leqslant \max\left\{\sup_{q \geqslant p} \min(F_0(q), H_{\min}(q)), \sup_{q \geqslant p} \min(H_-(q), H_{\min}(q))\right\} \\
&\leqslant \max(\underline{R}F_0(p), \underline{R}H_-(p)).
\end{aligned}$$

Since

$$\underline{R}H_-(p) = \sup_{q \geqslant p} \min(H_-(q), H_{\min}(q)) \leqslant \sup_{q \geqslant p} H_-(q) = H_-(p),$$

we finally get

$$\underline{R}\max(F_0,H_-)(p)\leqslant \max(\underline{R}F_0(p),H_-(p)).$$

Taking the super-relaxation operation, we get (using Lemma 4.3)

$$\overline{R}\underline{R}F_0\leqslant \overline{R}\underline{R}\max(F_0,H_-)\leqslant \overline{R}\max(\underline{R}F_0(p),H_-(p))=\overline{R}\underline{R}F_0, \tag{14}$$

where we use Lemma 4.5 for the last equality. This implies (13). Hence, we have

$$\overline{R}\underline{R}F_0=\overline{R}\underline{R}\max(F_0,H_-)=\underline{R}\overline{R}\max(F_0,H_-)=\underline{R}\overline{R}F_0,$$

where for the second equality we used Proposition 4.2 and for the last one, we used Lemma 4.5.

It just remains to show that $\mathfrak{R}F_0\geqslant H_-$. Indeed, by (14), we have

$$\mathfrak{R}F_0=\overline{R}\underline{R}F_0=\overline{R}(\max(\underline{R}F_0,H_-))\geqslant \overline{R}H_-\geqslant H_-,$$

since

$$\overline{R}H_-(p)=\inf_{q\leqslant p}\max(H_-(q),H_{\max}(q))\geqslant \inf_{q\leqslant p}H_-(q)=H_-(p).$$

This ends the proof of the theorem.

□

Remark 4.6. *Note that the fact that $\mathfrak{R}F_0\geqslant H_-$ implies in particular that $\mathfrak{R}F_0$ is semi-coercive.*

4.3 Further properties of the relaxation operator and characteristic points

In this subsection, we give some properties of the relaxation that will be useful in the rest of the paper. We begin by a characterization of the sub-relaxation.

Proposition 4.7 (Characterization of sub-relaxation)**.** *Let H satisfy (2) and F_0 satisfy (3). For $p\in\mathbb{R}^N$, we define $\bar p\in\mathbb{R}^N$ by*

$$\bar p^\alpha=\begin{cases} p^\alpha & \text{if}\quad H^\alpha(p^\alpha)\geqslant F_0(p)\\ \sup\{q^\alpha\geqslant p^\alpha, H^\alpha(\tilde q^\alpha)<F_0(p)\quad \text{for all}\quad \tilde q^\alpha\in[p^\alpha,q^\alpha)\} & \text{if}\quad H^\alpha(p^\alpha)<F_0(p).\end{cases}$$

Then F_0 is sub-relaxed, i.e. $F_0=\underline{R}F_0$, if, and only if, for all $p\in\mathbb{R}^N$, we have

$$F_0=const=F_0(p)\quad\text{in}\quad[p,\bar p], \tag{15}$$

where $[p,\bar p]=\prod_{\alpha=1}^N[p^\alpha,\bar p^\alpha]$.

Proof. We begin to show that $\underline{R}F_0=F_0$ implies (15). Let $p\in\mathbb{R}^N$ be such that

$$\lambda=F_0(p)>H_{\min}(p)$$

(otherwise $\bar p=p$ and the result is trivial). We claim that

$$\underline{R}F_0(p)=\sup_{q\geqslant\bar p}\{\min(F_0(q),H_{\min}(q))\}. \tag{16}$$

Indeed, let $q\geqslant p$ such that there exists $\alpha\in\{1,\dots,N\}$ such that $q^\alpha<\bar p^\alpha$. Then

$$\min(F_0(q),H_{\min}(q))\leqslant\min(F_0(q),H^\alpha(q^\alpha))=H^\alpha(q^\alpha)<\lambda=F_0(p)=\underline{R}F_0(p)=\sup_{\tilde q\geqslant p}\{\min(F_0(\tilde q),H_{\min}(\tilde q))\}$$

which proves (16). We then have

$$\lambda=F_0(p)=\underline{R}F_0(p)=\sup_{q\geqslant\bar p}\{\min(F_0(q),H_{\min}(q))\}\leqslant\sup_{q\geqslant\bar p}F_0(q)=F_0(\bar p)\leqslant F_0(p).$$

Therefore $F_0(\bar p)=F_0(p)$ and since F_0 is non-increasing, this implies (15).

Conversely, assume that (15) is satisfied for every $p\in\mathbb{R}^N$. We then have

$$F_0(p)\geqslant\underline{R}F_0(p)=\sup_{q\geqslant p}\{\min(F_0(q),H_{\min}(q))\}\geqslant\min(F_0(\bar p),H_{\min}(\bar p))=F_0(\bar p)=F_0(p)$$

which implies that $F_0=\underline{R}F_0$ and ends the proof of the proposition. □

A direct consequence of this proposition is the following corollary:

Corollary 4.8 (Lower dimension restriction of sub-relaxed F_0 are sub-relaxed)**.** *Assume that H satisfy* (2) *and that F_0 satisfy* (3)*. For $p \in \mathbb{R}^N$ and $\alpha \in \{1, \dots, N\}$, we define $F_0^{\alpha,p} : \mathbb{R} \mapsto \mathbb{R}$ by*

$$F_0^{\alpha,p}(q^\alpha) = F_0(p^1, \dots, p^{\alpha-1}, q^\alpha, p^{\alpha+1}, \dots, p^N).$$

If F_0 is sub-relaxed with respect to H, then $F_0^{\alpha,p}$ is sub-relaxed with respect to H^α.

We now give the characterization for super-relaxation.

Proposition 4.9 (Characterization of super-relaxation)**.** *Let H satisfy* (2) *and F_0 satisfy* (3)*. For $p \in \mathbb{R}^N$, we define $\underline{p} \in \mathbb{R}^N$ by*

$$\underline{p}^\alpha = \begin{cases} p^\alpha & \text{if} \quad H^\alpha(p^\alpha) \leqslant F(p) \\ \inf\{q^\alpha \leqslant p^\alpha, H^\alpha(\tilde{q}^\alpha) > F_0(p) \quad \textit{for all} \quad \tilde{q}^\alpha \in (q^\alpha, p^\alpha]\} & \text{if} \quad H^\alpha(p^\alpha) > F_0(p). \end{cases}$$

Then F_0 is super-relaxed, i.e. $F_0 = \overline{R}F_0$, if, and only if, for all $p \in \mathbb{R}^N$, we have

$$F = const = F(p) \quad \text{in} \quad [\underline{p}, p], \tag{17}$$

and

$$\underline{p}^\alpha > -\infty \quad \text{if} \quad H^\alpha(p^\alpha) > F_0(p). \tag{18}$$

Proof. We begin to show that $\overline{R}F_0 = F_0$ implies (17) and (18). Let $p \in \mathbb{R}^N$ be such that

$$\lambda = F_0(p) < H_{\max}(p)$$

(otherwise $\underline{p} = p$ and the result is trivial). We first show (18). By contradiction, assume that there exists α such that $\underline{p}^\alpha = -\infty$ and $H^\alpha(p^\alpha) > F_0(p)$. By definition of $\underline{p}^\alpha$, we have $H^\alpha(q^\alpha) > F_0(p)$ for all $q^\alpha \leqslant p^\alpha$.

By coercivity of H^α, there exists $\delta > 0$ such that

$$H^\alpha(q^\alpha) \geqslant \delta + \lambda \qquad \text{for all} \;\; q^\alpha \leqslant p^\alpha.$$

We then have

$$\begin{aligned} \lambda = F_0(p) = \overline{R}F_0(p) &= \inf_{q \leqslant p}\{\max(F_0(q), H_{\max}(q))\} \\ &\geqslant \inf_{q \leqslant p} H^\alpha(q^\alpha) \geqslant \delta + \lambda, \end{aligned}$$

which is absurd. Then (18) is satisfied. We now prove (17). For all $\tilde{q} \leqslant p$ such that there exists $\alpha \in \{1, \dots, N\}$ such that $\tilde{q}^\alpha > \underline{p}^\alpha$, we have

$$\max(F_0(\tilde{q}), H_{\max}(\tilde{q})) \geqslant H^\alpha(\tilde{q}^\alpha) > \lambda = F_0(p) = \overline{R}F_0(p) = \inf_{q \leqslant p} \max(F_0(q), H_{\max}(q)).$$

Hence

$$\lambda = F_0(p) = \overline{R}F_0(p) = \inf_{q \leqslant \underline{p}}\{\max(F_0(q), H_{\max}(q))\} \geqslant \inf_{q \leqslant \underline{p}} F_0(q) = F_0(\underline{p}) \geqslant F_0(p).$$

Therefore $F_0(\underline{p}) = F_0(p)$ and since F_0 is non-increasing, this implies (17).

Conversely, assume that (17) and (18) are satisfied for every $p \in \mathbb{R}^N$. We then have

$$F_0(p) \leqslant \overline{R}F_0(p) = \inf_{q \leqslant p}\{\max(F_0(q), H_{\max}(q))\} \leqslant \max(F_0(\underline{p}), H_{\max}(\underline{p})) = F_0(\underline{p}) = F_0(p)$$

which implies that $F_0 = \overline{R}F_0$ and ends the proof of the proposition. □

A direct consequence of this proposition is the following corollary:

Corollary 4.10 (Lower dimension restriction of super-relaxed F_0 are super-relaxed)**.** *Assume that H satisfy* (2) *and that F_0 satisfy* (3)*. For $p \in \mathbb{R}^N$ and $\alpha \in \{1, \dots, N\}$, we define $F_0^{\alpha,p} : \mathbb{R} \mapsto \mathbb{R}$ by*

$$F_0^{\alpha,p}(q^\alpha) = F_0(p^1, \dots, p^{\alpha-1}, q^\alpha, p^{\alpha+1}, \dots, p^N).$$

If F_0 is super-relaxed with respect to H, then $F_0^{\alpha,p}$ is super-relaxed with respect to H^α.

We now give an important result concerning the relaxation at crossing points.

Proposition 4.11 (Relaxation at crossing points)**.** *Let H satisfy* (2) *and F_0 satisfy* (3)*. For $p \in \mathbb{R}^N$, assume that*

$$H^\alpha(p^\alpha) = F_0(p) \quad \text{for all } \alpha \in \{1, \dots, N\}.$$

Then

$$\mathfrak{R}F_0(p) = \underline{R}F_0(p) = \overline{R}F_0(p) = F_0(p).$$

Proof. We only prove that $\overline{R}F_0(p) = F_0(p)$, the other one being similar. First, by definition of $\overline{R}$, we have

$$\overline{R}F_0(p) = \inf_{q \leqslant p} \max\{F_0, H_{\max}\}(q) \leqslant \max\{F_0, H_{\max}\}(p) = F_0(p)$$

since $H_{\max}(p) = F_0(p)$. On the other hand, we have

$$\overline{R}F_0(p) = \inf_{q \leqslant p} \max(F_0(q), H_{\max}(q)) \geqslant \inf_{q \leqslant p} F_0(q) = F_0(p).$$

This ends the proof of the Proposition. □

We now introduce the notion of characteristic points which play an important role in the sequel.

Definition 4.12 (Characteristic points)**.** *(i) $p \in \mathbb{R}^N$ is a* super-characteristic point *of F_0 if there exists $\varepsilon > 0$ such that $H^\alpha(p^\alpha) = F_0(p)$ and $H^\alpha(q^\alpha) > H^\alpha(p^\alpha)$ for all $q^\alpha \in (p^\alpha, p^\alpha + \varepsilon)$ and for all $\alpha = 1, \dots, N$. We denote by $\overline{\chi}(F_0)$ the set of super-characteristic points.*

(ii) $p \in \mathbb{R}^N$ is a sub-characteristic point *of F_0 if there exists $\varepsilon > 0$ such that $H^\alpha(p^\alpha) = F_0(p)$ and $H^\alpha(q^\alpha) < H^\alpha(p^\alpha)$ for all $q^\alpha \in (p^\alpha - \varepsilon, p^\alpha)$ and for all $\alpha = 1, \dots, N$. We denote by $\underline{\chi}(F_0)$ the set of sub-characteristic points.*

(iii) The set of all characteristic points *is denoted by $\chi(F_0)$, i.e. $\chi(F_0) := \overline{\chi}(F_0) \cup \underline{\chi}(F_0)$.*

The following proposition will be useful for the reduction of test functions.

Proposition 4.13 (Properties of relaxation)**.** *Let H satisfy* (2) *and F_0 satisfy* (3)*. Then*

$$\mathfrak{R}F_0 \leqslant \overline{R}F_0 \leqslant F_0 \quad \text{on } \underline{\chi}(\mathfrak{R}F_0) \tag{19}$$

and

$$\mathfrak{R}F_0 \geqslant \underline{R}F_0 \geqslant F_0 \quad \text{on } \overline{\chi}(\mathfrak{R}F_0). \tag{20}$$

Proof. We only prove (19), the proof of (20) being similar. Let $p \in \underline{\chi}(\mathfrak{R}F_0)$. By Theorem 2.1, we have

$$\mathfrak{R}F_0(p) = \underline{R}\,\overline{R}F_0(p) = \sup_{q \geqslant p} \min(\overline{R}F_0(q), H_{\min}(q)) \leqslant \sup_{q \geqslant p} \overline{R}F_0(q) = \overline{R}F_0(p)$$

where we used the monotonicity of $\overline{R}F_0$ for the last equality. Note also that this inequality is true for all $p \in \mathbb{R}^N$, and not only for $p \in \underline{\chi}(\mathfrak{R}F_0)$.

It just remains to show that

$$\overline{R}F_0(p) \leqslant F_0(p). \tag{21}$$

Assume by contradiction that $F_0(p) < \overline{R}F_0(p)$. Since $p \in \underline{\chi}(\mathfrak{R}F_0)$, we have $H_{\max}(p) = H_{\min}(p) = \mathfrak{R}F_0(p)$ and

$$H_{\max}(p) = \mathfrak{R}F_0(p) \leqslant \overline{R}F_0(p) \leqslant \max(F_0(p), H_{\max}(p)) = H_{\max}(p).$$

Hence $\mathfrak{R}F_0(p) = \overline{R}F_0(p)$ and $p \in \underline{\chi}(\overline{R}F_0)$. This implies that there exists $\varepsilon > 0$ such that

$$H_{\max}(q) < \overline{R}F_0(p) \quad \text{for all } q \in \prod_{\alpha=1,\dots,N} (p^\alpha - \varepsilon, p^\alpha).$$

By continuity of F_0, we can find $\bar{q} \in \prod_{\alpha=1,\dots,N}(p^\alpha - \varepsilon, p^\alpha)$ such that $F_0(\bar{q}) < \overline{R}F_0(p)$. We then have

$$F_0(\bar{q}) < \overline{R}F_0(p) \quad \text{and} \quad H_{\max}(\bar{q}) < \overline{R}F_0(p).$$

Using the definition of $\overline{R}F_0$, we get

$$\overline{R}F_0(\bar{q}) = \inf_{q \leqslant \bar{q}} \max(F_0(q), H_{\max}(q)) \leqslant \max(F_0(\bar{q}), H_{\max}(\bar{q})) < \overline{R}F_0(p),$$

which contradicts the fact that $\overline{R}F_0$ is non-increasing. This proves (21) and ends the proof of the proposition. □

Lemma 4.14 (Constant sub-relaxed and super-relaxed F_0 in a box)**.** *Let H satisfy* (2) *and F_0 satisfy* (3)*. If $\underline{R}F_0(\underline{p}_0) > H_{\max}(\underline{p}_0)$ for some $\underline{p}_0 \in \mathbb{R}^N$, then there exists $\underline{p}_1 > \underline{p}_0$ (i.e. such that $\underline{p}_1^\alpha > \underline{p}_0^\alpha$ for all $\alpha \in \{1, \dots, N\}$), such that*

$$\underline{p}_1 \in \underline{\chi}(\underline{R}F_0) \quad \text{and} \quad \underline{R}F_0(\underline{p}_0) = \underline{R}F_0(\underline{p}_1).$$

In the same way, if $\overline{R}F_0(\overline{p}_0) < H_{\min}(\overline{p}_0)$ for some $\overline{p}_0 \in \mathbb{R}^N$, then there exists $\overline{p}_1 < \overline{p}_0$ (i.e. such that $\overline{p}_1^\alpha < \overline{p}_0^\alpha$ for all $\alpha \in \{1, \dots, N\}$), such that

$$\overline{p}_1 \in \overline{\chi}(\overline{R}F_0) \quad \text{and} \quad \overline{R}F_0(\overline{p}_0) = \overline{R}F_0(\overline{p}_1).$$

Proof. We just show the first part of the lemma, since the second one can be prove in a similar way (using moreover the general fact that $\overline{R}F_0 \geqslant H_-$). For $\alpha \in \{1, \dots, N\}$, we set

$$\underline{p}_1^\alpha = \sup\{p^\alpha \geqslant \underline{p}_0^\alpha,\ H^\alpha(q^\alpha) < \underline{R}F_0(\underline{p}_0)\ \forall q^\alpha \in [\underline{p}_0^\alpha, p^\alpha]\}.$$

Since H^α is coercive, $\underline{p}_1^\alpha$ is well defined. Moreover, by continuity, we have

$$H_{\min}(\underline{p}_1) = H^\alpha(\underline{p}_1^\alpha) = \underline{R}F_0(\underline{p}_0) \quad \text{and} \quad H^\alpha < \underline{R}F_0(\underline{p}_0) \text{ on } [\underline{p}_0^\alpha, \underline{p}_1^\alpha).$$

Since $\underline{R}F_0(\underline{p}_0) > H_{\max}(\underline{p}_0) \geqslant H^\alpha(\underline{p}_0^\alpha)$ for all α, we also have $\underline{p}_1^\alpha > \underline{p}_0^\alpha$.

Using that $\underline{R}(\underline{R}F_0) = \underline{R}F_0$, we get

$$\underline{R}F_0(\underline{p}_0) = \sup_{q \geqslant \underline{p}_0} \min(\underline{R}F_0(q), H_{\min}(q)). \tag{22}$$

Now let $q \geqslant \underline{p}_0$ such that $q^\alpha < \underline{p}_1^\alpha$ for some α. We then have

$$H_{\min}(q) \leqslant H^\alpha(q^\alpha) < \underline{R}F_0(\underline{p}_0) = H_{\min}(\underline{p}_1).$$

This implies, for all such q, that $\min(\underline{R}F_0(q), H_{\min}(q)) < H_{\min}(\underline{p}_1) = \underline{R}F_0(\underline{p}_0)$. Hence, by (22), we get

$$\underline{R}F_0(\underline{p}_0) = \sup_{q \geqslant \underline{p}_1} \min(\underline{R}F_0(q), H_{\min}(q)) = \underline{R}F_0(\underline{p}_1).$$

In particular, $\underline{p}_1 \in \underline{\chi}(\underline{R}F_0)$. This ends the proof of the lemma. □

5 Viscosity solutions and relaxation

5.1 Weak and strong solutions

We now introduce our two notions of viscosity solutions: weak and strong viscosity solutions. Every strong solution will be a weak solution. So, if not specified, any solution can be understood as a weak solution.

For $T > 0$, we set $J_T = (0,T) \times J$ and we consider the class of test functions on J_T

$$C^1(J_T) = \left\{\varphi \in C(J_T), \quad \text{the restriction of } \varphi \text{ to } (0,T) \times J_\alpha \text{ is } C^1 \text{ for } \alpha = 1, \dots, N\right\}.$$

We recall the definition of upper and lower semi-continuous envelopes u^* and u_* of a function u defined on $[0,T) \times J$,

$$u^*(t,x) = \limsup_{(s,y)\to(t,x)} u(s,y) \quad \text{and} \quad u_*(t,x) = \liminf_{(s,y)\to(t,x)} u(s,y).$$

Definition 5.1 (Weak viscosity solution)**.** *Assume that H, F_0 and u_0 satisfy respectively* (2), (3) *and* (5)*, and let $u : [0,+\infty) \times J \to \mathbb{R}$. We say that u is a weak F_0-subsolution (resp. weak F_0-supersolution) of* (1) *in $(0,T) \times J$ if u^* is locally bounded from above (resp. u_* is locally bounded from below) and for all test function $\varphi \in C^1(J_T)$ touching u^* from above (resp. u_* from below) at $(t_0, x_0) \in J_T$, we have*

$$\varphi_t + H_\alpha(\varphi_x) \leqslant 0 \quad (\textit{resp.} \quad \geqslant 0) \quad \textit{at} \quad (t_0,x_0) \quad \text{if } x_0 \in J_\alpha^*$$

and

$$\begin{array}{rll} & \varphi_t + \min\{F_0(\varphi_x), H_{min}(\varphi_x)\} & \leqslant 0 \qquad \textit{at} \quad (t_0,x_0), \\ (\textit{resp.} & \varphi_t + \max\{F_0(\varphi_x), H_{max}(\varphi_x)\} & \geqslant 0 \qquad \textit{at} \quad (t_0,x_0)), \end{array} \tag{23}$$

if $x_0 = 0$.

We say that u is a weak F_0-subsolution (resp. weak F_0-supersolution) of (1), (4) *on $[0,T) \times J$ if additionally*

$$u^*(0,x) \leqslant u_0(x) \quad (\textit{resp.} \quad u_*(0,x) \geqslant u_0(x)) \quad \textit{for all} \quad x \in J.$$

We say that u is a weak F_0-solution if u is both a weak F_0-subsolution and a weak F_0-supersolution.

Notice that the first line of (23) means for subsolutions that either the desired junction subsolution inequality is satisfied, or the PDE inequality is satisfied for at least one branch J_α. The remark is similar for supersolutions. A good property of this notion of relaxed solutions is its stability. For instance the limit of a sequence of weak F_0-subsolutions (resp. weak F_0-supersolutions) u_ε is still a weak F_0-subsolution (resp. weak F_0-supersolution).

Another application of the notion of relaxed solution is an existence result, easily obtained adapting Perron's method, so we skip the proof (indeed the interested reader may look at [8, Theorem 2.14] where the proof only uses the monotonicity of F_0 and the continuity of the Hamiltonians):

Theorem 5.2 (Existence of relaxed solutions)**.** *Assume that H, F_0, u_0 satisfy respectively* (2), (3) *and* (5)*. Then there exists a relaxed solution u to* (1) *with initial data* (4)*.*

We now give a second definition which requires more on the solution.

Definition 5.3 (Strong viscosity solution)**.** *The definition of strong F_0-subsolutions, strong F_0-supersolutions and strong F_0-solutions is exactly the same word by word as in Definition 5.1, except that we replace condition* (23) *by the following one*

$$\begin{array}{rll} & \varphi_t + F_0(\varphi_x) & \leqslant 0 \qquad \textit{at} \quad (t_0,x_0), \\ (\textit{resp.} & \varphi_t + F_0(\varphi_x) & \leqslant 0 \qquad \textit{at} \quad (t_0,x_0)). \end{array} \tag{24}$$

From the definition itself, we see that any strong solution is a weak solution, but the converse is false in general. We will show in the next subsections that the reverse is true if the junction condition F_0 is self-relaxed, i.e. satisfies $F_0 = \mathfrak{R}F_0$.

5.2 Weak continuity condition at the junction

We now introduce the weak continuity condition that will play an important role for reducing the set of test function. We say that u satisfies the weak continuity condition if

$$u(t,0) = \limsup_{(s,y)\to(t,0),\ y\in J_\alpha^*} u(s,y) \quad \text{for all } t>0 \ \text{ and for each } \alpha = 1,\dots,N. \tag{25}$$

Here the choice of the lim sup (instead of the lim inf) is due to the fact that the Hamiltonians H^α are coercive.

We now state the following result whose proof is done in [8] (see there Lemma 2.3, where the proof does not use other properties than the coercivity of the Hamiltonians to bound the gradient term, and the semi-coercivity of the junction condition to get a contradiction with a possible discontinuity of the subsolution at the junction):

Lemma 5.4 ("Weak continuity" condition at the junction point; [8])**.** *Let H satisfy condition* (2)*, and let F_0 satisfy* (3) *and the semi-coercivity condition* (7)*. Let u be a weak subsolution to* (1)*. Then u satisfies for all $t>0$*

$$u^*(t,0) = \limsup_{(s,y)\to(t,0),\ y\in J_\alpha^*} u(s,y) \quad \textit{for each} \quad \alpha\in\{1,\dots,N\}\,.$$

5.3 Reducing the set of test functions

We consider functions satisfying a Hamilton-Jacobi equation on $J\setminus\{0\}$:

$$u_t + H^\alpha(u_x) = 0 \quad \text{for} \quad (t,x)\in(0,T)\times J_\alpha^*. \tag{26}$$

Proposition 5.5 (Reducing the set of test functions for subsolutions)**.** *Assume that H satisfies* (2) *and that F_0 satisfies* (3)*. For any $p\in \underline{\chi}(\underline{R}F_0)$, let us fix a time independent test function $\phi_p(x)$ satisfying $\partial_\alpha\phi_p(0) = p^\alpha$. We then consider the class of test functions of the form*

$$\varphi(t,x) = \psi(t) + \phi_p(x) \tag{27}$$

with ϕ_p fixed for each p as above and ψ a C^1 function of time. Let u be a function $u:(0,T)\times J\to\mathbb{R}$, upper semi-continuous which is a subsolution of (26)*. Given $t_0\in(0,T)$, we assume that*

$$u(t_0,0) = \limsup_{(s,y)\to(t_0,0),\ y\in J_\alpha^*} u(s,y) \quad \textit{for each} \quad \alpha = 1,\dots,N.$$

If for any test function of the class (27)*, touching u from above at $(t_0,0)$ we have*

$$\varphi_t + \underline{R}F_0(\varphi_x) \leqslant 0 \tag{28}$$

then u is a strong $\underline{R}F_0$-subsolution at $(t_0,0)$.

We will use the following result whose proof is done in [8, Lemma 2.10] (where the proof does not use other properties than the continuity of the Hamiltonians and the coercivity of the Hamiltonians to bound the gradient term):

Lemma 5.6 (Subsolution property for the critical slopes on each branch; [8])**.** *Let $u:(0,T)\times J_\alpha\to\mathbb{R}$ be an upper semi-continuous subsolution of* (26) *for some $\alpha\in\{1,\dots,N\}$. Let ϕ be a test function touching u from above at some point $(t_0,0)$ with $t_0\in(0,T)$. Consider the following critical slope*

$$\underline{p}^\alpha = \inf\{p^\alpha\in\mathbb{R},\ \exists r>0,\ \phi(t,x)+p^\alpha x \geqslant u(t,x) \ \textit{for}\ (t,x)\in(t_0-r,t_0+r)\times[0,r) \ \textit{with}\ x\in J_\alpha\}$$

If

$$u(t_0,0) = \limsup_{(s,y)\to(t_0,0),\ y\in J_\alpha^*} u(s,y)$$

then $\underline{p}^\alpha > -\infty$, and we have

$$\phi_t + H^\alpha(\partial_\alpha\phi + \underline{p}^\alpha) \leqslant 0 \quad \textit{at} \quad (t_0,0) \quad \textit{with} \quad \underline{p}^\alpha\leqslant 0.$$

Proof of Proposition 5.5. Let u be a subsolution of (26) such that for any test function of the class (27), touching u from above at $(t_0, 0)$, (28) holds. Let ϕ be a test function touching u from above at $(t_0, 0)$. We want to show that

$$\phi_t(t_0, 0) + \underline{R}F_0(\phi_x(t_0, 0)) \leqslant 0. \tag{29}$$

Notice that, by Lemma 5.6, there exists $-\infty < \underline{p}^\alpha \leqslant 0$ for each $\alpha = 1, \ldots, N$ such that

$$\phi_t(t_0, 0) + H^\alpha(\partial_\alpha \phi + \underline{p}^\alpha) \leqslant 0 \quad \text{at} \quad (t_0, 0). \tag{30}$$

We set $p_0 = \phi_x(t_0, 0)$ and $\underline{p}_0 = p_0 + \underline{p} \leqslant p_0$. If $\underline{R}F(p_0) \leqslant H_{\max}(\underline{p}_0)$, then (30) implies

$$\phi_t(t_0, 0) + \underline{R}F_0(p_0) \leqslant \phi_t(t_0, 0) + H_{\max}(\underline{p}_0) \leqslant 0,$$

which gives the desired result.

We then assume that $\underline{R}F_0(p_0) > H_{\max}(\underline{p}_0)$. We then have $\underline{R}F_0(\underline{p}_0) \geqslant \underline{R}F_0(p_0) > H_{\max}(\underline{p}_0)$, and Lemma 4.14 implies the existence of some $\underline{p}_1 \geqslant \underline{p}_0$, with $\underline{p}_1^\alpha > \underline{p}_0^\alpha$ such that $\underline{R}F_0(\underline{p}_0) = \underline{R}F_0(\underline{p}_1)$ and $\underline{p}_1 \in \underline{\chi}(\underline{R}F_0)$. Since $\underline{p}_1^\alpha > \underline{p}_0^\alpha = p_0^\alpha + \underline{p}^\alpha$ for all α, we have by definition of the critical slope $\underline{p}^\alpha$ that

$$\varphi(t, x) := \phi(t, 0) + \phi_{\underline{p}_1}(x) \geqslant u(t, x)$$

in a neighborhood of $(t_0, 0)$ with equality at $(t_0, 0)$ and with $\phi_{\underline{p}_1}$ satisfying $\partial_\alpha \phi_{\underline{p}_1}(0) = \underline{p}_1^\alpha$. Then φ is a test function of the class (27), touching u from above at $(t_0, 0)$. By assumption, we then have

$$0 \geqslant \varphi_t(t_0, 0) + \underline{R}F_0(\underline{p}_1) = \phi_t(t_0, 0) + \underline{R}F_0(\underline{p}_0) \geqslant \phi_t(t_0, 0) + \underline{R}F_0(p_0)$$

which shows (29). This ends the proof of the proposition. □

As far as strong supersolutions are concerned, it is not necessary to impose a weak continuity assumption, and we show similarly the following result.

Proposition 5.7 (Reducing the set of test functions for supersolutions)**.** *Assume that H satisfies* (2) *and that F_0 satisfies* (3)*. For any $p \in \overline{\chi}(\overline{R}F_0)$, let us fix a time independent test function $\phi_p(x)$ satisfying $\partial_\alpha \phi_p(0) = p^\alpha$. We then consider the class of test functions of the form*

$$\varphi(t, x) = \psi(t) + \phi_p(x) \tag{31}$$

with ϕ_p fixed for each p as above and ψ a C^1 function of time. Let u be a function $u : (0, T) \times J \to \mathbb{R}$, lower semi-continuous which is a supersolution of (26)*. Given $t_0 \in (0, T)$, if for any test function of the class* (31)*, touching u from below at $(t_0, 0)$ we have*

$$\varphi_t + \overline{R}F_0(\varphi_x) \geqslant 0 \tag{32}$$

then u is a strong $\overline{R}F_0$-supersolution at $(t_0, 0)$.

5.4 Weak F_0-solutions are strong $\mathfrak{R}F_0$-solutions

In this section, we will show that u is a weak F_0-solution if and only if u is a strong $\mathfrak{R}F_0$-solution. This result justifies the introduction of the relaxation operator. We begin by the following lemma.

Lemma 5.8 (Weak F_0 sub/supersolutions are strong $\underline{R}F_0/\overline{R}F_0$ sub/supersolutions)**.** *Assume that H satisfies* (2) *and that F_0 satisfies* (3)*.*

(i) Let u be upper semi-continuous. Then u is a weak F_0-subsolution of (1) *if and only if u is a strong $\underline{R}F_0$-subsolution of* (1)*.*

(ii) Let u be lower semi-continuous. Then u is a weak F_0-supersolution of (1) *if and only if u is a strong $\overline{R}F_0$-supersolution of* (1)*.*

Proof. We only do the proof for subsolutions since the case of supersolutions is treated similarly.

Step 1: weak implies strong. Let u be a weak F_0-subsolution of (1) and ϕ be a test function touching u from above at $(t_0, 0)$. Let $\overline{q} \in \mathbb{R}^N$, $\overline{q} \geqslant 0$. We set

$$\psi(t,x) = \phi(t,x) + \overline{q}^\alpha x \quad \text{if } x \in J_\alpha.$$

In particular ψ is a test function touching u from above at $(t_0, 0)$. Hence

$$\psi_t(t_0,0) + \min\{F_0, H_{\min}\}(\psi_x(t_0,0)) \leqslant 0$$

i.e., for all $\overline{q} \geqslant 0$, we get

$$\phi_t(t_0,0) + \min\{F_0, H_{\min}\}(\phi_x(t_0,0) + \overline{q}) \leqslant 0.$$

Taking the supremum over $q := \phi_x(t_0,0) + \overline{q}$, we finally get

$$\phi_t(t_0,0) + \sup_{q \geqslant \phi_x(t_0,0)} \min\{F_0, H_{\min}\}(q) \leqslant 0$$

and so u is a strong $\underline{R}F_0$-subsolution.

Step 2: strong implies weak. Let u be a strong $\underline{R}F_0$-subsolution of (1) and ϕ be a test function touching u from above at $(t_0, 0)$. Hence

$$\phi_t(t_0,0) + \min\{F_0, H_{\min}\}(\phi_x(t_0,0)) \leqslant \phi_t(t_0,0) + \sup_{q \geqslant \phi_x(t_0,0)} \min\{F_0, H_{\min}\}(q) \leqslant 0,$$

which implies that u is a weak F_0-subsolution. This ends the proof of the lemma. □

Since we may have $\underline{R}F_0 < \overline{R}F_0$, Lemma 5.8 is not completely satisfactory, because we would like to have the same boundary function for sub- and supersolutions. This is achieved in the following results where the common boundary function is $\mathfrak{R}F_0$.

Proposition 5.9 (Weak F_0 sub/supersolutions are strong $\mathfrak{R}F_0$ sub/supersolutions)**.** *Assume that H satisfies* (2) *and that F_0 satisfies* (3)*. Let u be an upper semi-continuous function.*

(i) If u is a weak F_0 subsolution of (1) *and if u satisfy the weak continuity condition* (25)*, then u is a strong $\mathfrak{R}F_0$-subsolution of* (1)*.*

(ii) If u is a strong $\mathfrak{R}F_0$-subsolution of (1)*, then u is a weak F_0-subsolution of* (1)*.*

Let v be a lower semi-continuous function.

(i) If v is a weak F_0 supersolution of (1)*, then v is a strong $\mathfrak{R}F_0$-supersolution of* (1)*.*

(ii) If v is a strong $\mathfrak{R}F_0$-supersolution of (1)*, then v is a weak F_0-supersolution of* (1)*.*

Proof. We only do the proof for the subsolution. We set $F := \mathfrak{R}F_0$. Let u be a weak F_0 subsolution of (1)satisfying the weak continuity condition (25) and let ϕ be a test function touching u from above at $(t_0, 0)$ with $t_0 > 0$. We set $\lambda = \phi_t(t_0, 0)$ and $p = \phi_x(t_0, 0)$. We then have

$$\lambda + \min(F_0(p), H_{\min}(p)) \leqslant 0.$$

Since, by Lemma 4.4, we have $F = \underline{R}F$, we know from Proposition 5.5 that we can assume that $p \in \underline{\chi}F$. By Proposition 4.13, we then have $H_{\min}(p) = F(p) = \mathfrak{R}F_0(p) \leqslant F_0(p)$ and so $\lambda + \mathfrak{R}F_0(p) \leqslant 0$. This implies that u is a strong $\mathfrak{R}F_0$-subsolution.

We now assume that u is a strong F-subsolution. Since $F = \overline{R}\underline{R}F_0 \geqslant \underline{R}F_0$, we deduce that u is also a strong $\underline{R}F_0$-subsolution. By Lemma 5.8 (i), we then deduce that u is a weak F_0-subsolution. □

5.5 Different junction conditions leading to the same problem

In this short subsection, we give a simple example limited to a single branch ($N = 1$), in order to simplify the presentation. We will introduce different boundary conditions (i.e. junction conditions here). We consider the convex Hamiltonian $H : \mathbb{R} \to \mathbb{R}$ given by $H(p) := |p|$, and define its nonincreasing envelope $H^-(p) := \max(0, -p)$. We consider the following equation

$$\begin{cases} u_t + H(u_x) = 0 & \text{on} \quad (0,+\infty)_t \times (0,+\infty)_x \\ u = u_0 & \text{on} \quad \{0\}_t \times [0,+\infty)_x \end{cases} \tag{33}$$

where the initial data u_0 is assumed to be Lipschitz continuous.

Given a flux limiter, i.e. a parameter $A \geqslant 0 = \min H$, we consider the following boundary condition

$$u_t + \max\left\{A, H^-(u_x)\right\} = 0 \quad \text{on} \quad (0,+\infty)_t \times \{0\}_x \,. \tag{34}$$

Now we consider the following three other boundary conditions

$$u_t + A = 0 \quad \text{on} \quad (0,+\infty)_t \times \{0\}_x \tag{35}$$

or

$$u_t + 2A - u_x = 0 \quad \text{on} \quad (0,+\infty)_t \times \{0\}_x \tag{36}$$

or

$$-u_x = -A \quad \text{on} \quad (0,+\infty)_t \times \{0\}_x \,. \tag{37}$$

Then we have the following result.

Lemma 5.10. (Same effective boundary condition)
Under the previous assumptions and say, for continuous weak solutions $u : [0,+\infty)^2 \to \mathbb{R}$, the four following problems are equivalent: problem (33),(35)*, problem* (33),(35)*, problem* (33),(36) *and problem* (33),(37)*.*

Remark 5.11. *At the boundary $x = 0$, recall that weak subsolutions satisfy either the boundary subsolution inequality or the PDE subsolution inequality (and similarly for supersolutions).*

Proof. Each desired boundary condition (34), (35), (36) writes

$$u_t + F_0(u_x) = 0. \tag{38}$$

Moreover the point A satisfies $\chi(F_0) = \{A\}$. It is then easy to check that $\mathfrak{R}F_0 = \min\{A, H^-\}$ and to conclude to the equivalence.

Condition (37) is slightly different and requires more attention. Let us define the maximal monotone (nonincreasing) graph

$$F_0(p) := \begin{cases} \{+\infty\} & \text{if} \quad p < A \\ [-\infty,+\infty] & \text{if} \quad p = A \\ \{-\infty\} & \text{if} \quad p > A \end{cases}$$

Then subsolutions and supersolutions of (37) can be rewritten at the junction point $x = 0$ as

$$\begin{cases} \text{there exists } \overline{\mu} \in F_0(u_x) \text{ such that } u_t + \overline{\mu} \geqslant 0 & \textbf{(supersolution)} \\ \text{there exists } \underline{\mu} \in F_0(u_x) \text{ such that } u_t + \underline{\mu} \leqslant 0 & \textbf{(subsolution).} \end{cases}$$

This formulation is now closer to formulation (38). Notice also that such F_0 can be seen as a certain limit as $\varepsilon \to 0$ of $F_\varepsilon(p) = A - \varepsilon^{-1}(p - A)$. Moreover the point A satisfies $\{p \in \mathbb{R},\ H(p) \in F_0(p)\} = \{A\}$ (and $\chi(F_\varepsilon) = \{A\}$). It is then easy to check that $\mathfrak{R}F_\varepsilon = \min\{A, H^-\}$. With a certain abuse of definition, we can also write $\chi(F_0) = \{A\}$ and $\mathfrak{R}F_0 = \min\{A, H^-\}$. Then along the same lines (and skipping the technicalities specific to that case), it is also possible to check that problem (33),(37) is equivalent to problem (33),(34). This ends the proof of the lemma. ☐

6 Comparison principle

Theorem 6.1 (Comparison principle)**.** *Assume that H satisfies* (2) *and that F_0 satisfies* (3) *and* (7)*. Assume also that u_0 is bounded and Lipschitz continuous. Let $u : [0,T) \times J \to \mathbb{R}$ (resp. v) be a bounded upper semi-continuous weak viscosity F_0-subsolution (resp. bounded lower semi-continuous weak viscosity F_0-supersolution) of* (1)-(4)*. If $u(0,\cdot) \leqslant u_0 \leqslant v(0,)$ in J, then*

$$u \leqslant v \quad \text{in } [0,T) \times J.$$

Remark 6.2. *Notice that in Theorem 6.1, semi-coercivity of F_0 in condition* (7) *can be replaced by the weak continuity of the subsolution u at the junction, using Proposition 5.9 and replacing F_0 by $\mathfrak{R}F_0$.*

The proof of the theorem is very closed to the one of [5, Theorem 1] (see also [6]) and we skip it. The key point is to replace [5, Corollary 1] by the following proposition which proof is a very simple extension.

Proposition 6.3 (Junction viscosity inequalities)**.** *We consider two sets of functions $u, v : J \to \mathbb{R} \cup \{-\infty, +\infty\}$ with u upper semi-continuous and v lower semicontinous satisfying*

$$u(0) = 0 = v(0) \quad \textit{with} \quad u \leqslant v \quad \textit{on} \quad J. \tag{39}$$

For $\alpha = 1, \ldots, N$, we define

$$\overline{p}^\alpha := \limsup_{J_\alpha^* \ni x \to 0} \frac{u(x)}{x}, \quad \underline{p}^\alpha := \liminf_{J_\alpha^* \ni x \to 0} \frac{v(x)}{x}. \tag{40}$$

We also set $a^\alpha := \min\{\underline{p}^\alpha, \overline{p}^\alpha\}$, $b^\alpha := \max\{\underline{p}^\alpha, \overline{p}^\alpha\}$ and

$$[a,b] \cap \mathbb{R}^N := \prod_{\alpha=1,\ldots,N} \left([a^\alpha, b^\alpha] \cap \mathbb{R}\right).$$

For $\gamma = 1,2$, consider continuous functions $H_\gamma^\alpha : \mathbb{R} \to \mathbb{R}$ and $F_\gamma : \mathbb{R}^N \to \mathbb{R}$ with H_1^α coercive and F_1 semi-coercive. For $p = (p^1, \ldots, p^N) \in \mathbb{R}^N$, we set

$$H_{\gamma;\min}(p) = \min_{\alpha=1,\ldots,N} H_\gamma^\alpha(p^\alpha), \quad H_{\gamma;\max}(p) = \max_{\alpha=1,\ldots,N} H_\gamma^\alpha(p^\alpha).$$

We also assume that we have the following viscosity inequalities for some $\eta > 0$

$$\begin{cases} H_1^\alpha(u_x) \leqslant 0 & \textit{on} \quad J_\alpha \cap \{|u| < +\infty\} \quad \textit{for} \quad \alpha = 1, \ldots, N \\ \min\{F_1, H_{1;\min}\}(u_x) \leqslant 0 & \textit{on} \quad \{0\} \cap \{|u| < +\infty\} \\ & \\ H_2^\alpha(v_x) \geqslant \eta & \textit{on} \quad J_\alpha \cap \{|v| < +\infty\} \quad \textit{for} \quad \alpha = 1, \ldots, N \\ \max\{F_2, H_{2;\max}\}(v_x) \geqslant \eta & \textit{on} \quad \{0\} \cap \{|v| < +\infty\}. \end{cases} \tag{41}$$

Then there exists $p = (p^1, \ldots, p^N) \in [a,b] \cap \mathbb{R}^N \neq \varnothing$ such that

$$\begin{cases} \textit{either} & H_1^\alpha(p^\alpha) \leqslant 0 < \eta \leqslant (H_2^\alpha - H_1^\alpha)(p^\alpha) \quad \textit{for some } \alpha \in \{1, \ldots, N\} \\ \textit{or} & \max(F_1, H_{1;\max})(p) \leqslant 0 < \eta \leqslant (F_2 - F_1)(p). \end{cases} \tag{42}$$

7 A third relaxation formula using Riemann problem

The goal of this section is to give a third relaxation formula. This formula is based on Riemann problem, defined for $p \in \mathbb{R}^N$ by

$$\begin{cases} u_t + H^\alpha(u_x) = 0 & \text{on } (0,+\infty) \times J_\alpha^*, \quad \text{for} \quad \alpha = 1, \ldots, N \\ u_t + F_0(\partial_1 u(t,0^+), \ldots, \partial_N u(t,0^+)) = 0 & \text{on } (0,+\infty) \times \{0\}, \\ u(0,x) = u_0(x) = px & \text{on } \{0\} \times J, \end{cases} \tag{43}$$

where the initial condition $u_0(x) = px$ means that

$$u_0(x) = p^\alpha x \quad \text{for all } x \in J_\alpha \quad \text{and} \quad \alpha = 1, \ldots, N.$$

We then have the following result concerning the properties of the solution to the Riemann problem.

Theorem 7.1 (Properties of the Riemann problem)**.** *Assume that H satisfies* (2) *and that F_0 satisfies* (3)*. Then for every $p \in \mathbb{R}^N$ there exists a unique weak F_0-solution u to problem* (43) *satisfying the weak continuity condition* (25) *and such that, for any $T > 0$ there exists $C_T > 0$ such that*

$$|u(t,x) - u_0(x)| \leqslant C_T \quad \text{for all } (t,x) \in \{0,T) \times J.$$

Moreover, u is self-similar, i.e. it satisfies

$$u(\mu t, \mu x) = \mu u(t,x) \quad \text{for all } \mu > 0 \text{ and } (t,x) \in [0,+\infty) \times J$$

and u is globally Lipschitz continuous and convex or concave in (t,x) on each branch $[0,+\infty) \times J_\alpha$, the convexity/concavity depending on the branch $\alpha = 1,\dots,N$.

Proof. We set $F = \mathfrak{R}F_0$, which is semi-coercive because of Remark 4.6.
Step 1: Existence of a solution to (43)**.** Let

$$C \geqslant \max(\max_\alpha |H_\alpha(p^\alpha)|, |F(p)|).$$

Then $(t,x) \mapsto u_0(x) + Ct$ and $(t,x) \mapsto u_0(x) - Ct$ are respectively super and subsolution to (43) with F_0 replaced by F. Using Perron's method, we deduce that there exists a weak F-solution u to (43), satisfying the following barriers

$$u_0 - Ct \leqslant u \leqslant u_0 + Ct. \tag{44}$$

Then the comparison principle implies that $u^* \leqslant u_*$ which shows that u is continuous. Then, using $\underline{R}F = F = \overline{R}F$, Lemma 5.8 implies that u is a strong F-solution. Now Proposition 5.9 implies that u is a weak F_0-solution.

By Proposition 5.9, we deduce that u is also a F_0-strong solution and a F_0-weak solution.

Step 2: u is globally Lipschitz continuous. For $h > 0$, we set $u^h(t,x) = u(t+h,x) - Ch$. From (44), we deduce that u^h is solution to the same equation and satisfies

$$u^h(0,x) = u(h,x) - Ch \leqslant u_0(x).$$

The comparison principle then implies that

$$u(t+h,x) - Ch = u^h(t,x) \leqslant u(t,x).$$

In the same way, we have that

$$u(t+h,x) + Ch \geqslant u(t,x),$$

which implies that u is Lipschitz continuous in time. Since H^α is coercive, the equation satisfied by u implies that u is also globally Lipschitz continuous on each open branch $(0,+\infty) \times J_\alpha^*$, and then on the whole $[0,+\infty) \times J$.

Step 3: u is self-similar and convexe or concave. Let $\mu > 0$ and set $u^\mu(t,x) = \frac{1}{\mu} u(\mu t, \mu x)$. Then u^μ is solution to the same equation and satisfies

$$u^\mu(0,x) = \frac{1}{\mu} u(0,\mu x) = px = u(0,x).$$

Then, by uniqueness, we get that

$$u = u^\mu = \frac{1}{\mu} u(\mu t, \mu x) \quad \text{for all } \mu > 0,$$

which show that u is self-similar.

We now fix $\alpha \in \{1,\dots,N\}$. Since u is self-similar, there exists $W : J_\alpha \to \mathbb{R}$ such that

$$u(t,x) = tW\left(\frac{x}{t}\right).$$

We want to prove that W is either convex or concave on J_α. We have that W is a viscosity solution of

$$W - \xi\partial_\alpha W(\xi) + H^\alpha(\partial_\alpha W(\xi)) = 0 \quad \text{if } \xi \in J_\alpha^*. \tag{45}$$

For $0 < a < b$ with $a, b \in J_\alpha^*$, we define

$$S_{a,b}^\alpha := \left\{(t,x) \in (0,+\infty) \times J_\alpha;\ a < \frac{x}{t} < b\right\}.$$

We then set

$$\bar{p} := \frac{W(b) - W(a)}{b-a}, \quad \bar{\lambda} := \frac{bW(a) - aW(b)}{b-a}$$

and

$$v(t,x) = \bar{\lambda}t + \bar{p}x$$

so that

$$v(t,x) = u(t,x) \quad \text{for any } (t,x) \in \partial S_{a,b}^\alpha.$$

Since v is self-similar, there exists $W_{a,b}$ such that $v(t,x) = tW_{a,b}\left(\frac{x}{t}\right)$. We define $\tilde{W} := W - W_{a,b}$ which satisfies $\tilde{W} = 0$ on $\partial[a,b]$. Using that $\bar{\lambda} = v_t(t,x) = W_{a,b}(\xi) - \xi\partial_\alpha W_{a,b}(\xi)$ for $\xi := \frac{x}{t}$, we deduce that

$$\tilde{W} - \xi\partial_\alpha\tilde{W} = W - \xi\partial_\alpha W - \{W_{a,b} - \xi\partial_\alpha W_{a,b}\} = -H^\alpha(\partial_\alpha W) - \bar{\lambda} = -\tilde{H}^\alpha(\partial_\alpha\tilde{W})$$

with $\tilde{H}^\alpha(p) := H^\alpha(p + \bar{p}) + \bar{\lambda}$.

We now distinguish two cases. If $\tilde{H}(0) \geqslant 0$, we assume by contradiction that

$$\max_{\xi \in J_\alpha,\ \xi \in [a,b]} \tilde{W} = \tilde{W}(\bar{\xi}) > 0 \quad \text{with } \bar{\xi} \in (a,b).$$

Then the equation satisfied by $\tilde{W}$ implies that

$$\tilde{W}(\bar{\xi}) + \tilde{H}^\alpha(0) \leqslant 0$$

which is absurd. Therefore

$$\tilde{W} = W - W_{a,b} \leqslant 0 \quad \text{on } (a,b) \subset J_\alpha^*,$$

i.e.

$$\begin{aligned} W(\xi) \leqslant & W_{a,b}(\xi) = \bar{\lambda} + \bar{p}\xi \\ = & \frac{bW(a) - aW(b)}{b-a} + |\xi|\frac{W(b) - W(a)}{b-a} \\ = & W(a) + (\xi - a)\left(\frac{W(b) - W(a)}{b-a}\right) := c_{a,b}(\xi) \end{aligned} \tag{46}$$

In the same way, if $\tilde{H}(0) \leqslant 0$, we can show that

$$W \geqslant c_{a,b}(\xi) \quad \text{on } (a,b) \subset J_\alpha^* \tag{47}$$

We can easily check that this implies that we have either (46) for all choices $b > a > 0$ or we have (47) for all choices $b > a > 0$. This means that W is either convex or concave on J_α and so u is convex or concave on each branch. □

We are now ready to give the third relaxation formula:

Theorem 7.2 (Relaxation through Riemann problem)**.** *Assume that H satisfies* (2) *and that F_0 satisfies* (3)*. For $p \in \mathbb{R}^N$, let u be the unique F_0-relaxed solution to problem* (43) *given by Theorem 7.1. Then the relaxation operator is given by*

$$\mathfrak{R}F_0(p) = -u_t(t,0) \quad \textit{for all } t > 0.$$

Moreover the restriction of u *on each branch has a derivative at* $x = 0$ *and satisfies*

$$u_x(t, 0^+) = \hat{p} \quad \textit{for all} \quad t > 0$$

where $\hat{p} \in \mathbb{R}^N$ *and is given by (with* $F = \mathfrak{R}F_0$*)*

$$\hat{p}^\alpha = \begin{cases} p^\alpha & \textit{if} \quad H^\alpha(p^\alpha) = F(p), \\ \sup\{q^\alpha \geqslant p^\alpha, \quad H^\alpha(q'^\alpha) < F(p) \quad \textit{for all} \quad q'^\alpha \in [p^\alpha, q^\alpha)\} & \textit{if} \quad H^\alpha(p^\alpha) < F(p), \\ \inf\{q^\alpha \leqslant p^\alpha, \quad H^\alpha(q'^\alpha) > F(p) \quad \textit{for all} \quad q'^\alpha \in (q^\alpha, p^\alpha]\} & \textit{if} \quad H^\alpha(p^\alpha) > F(p), \end{cases}$$

Moreover $F(\hat{p}) = F(p)$ *and then in particular* $(F(\hat{p}), \hat{p})$ *belongs to the associated germ*

$$\mathcal{G} = \left\{(\lambda, q) \in \mathbb{R} \times \mathbb{R}^N, \quad \lambda = F(q) = H^\alpha(q^\alpha) \quad \textit{for all} \quad \alpha = 1, \ldots, N\right\}$$

Therefore, we have

$$F(p) = F(\hat{p}) = G^\alpha(\hat{p}^\alpha, p^\alpha) = H^\alpha(\hat{p}^\alpha) \quad \textit{for all} \quad \alpha = 1, \ldots, N$$

Proof. The idea of the proof is to give another construction of the solution to the Riemann problem. This solution is constructed on each branch separately (Step 1) and glued at the junction (Step 2). By uniqueness, we will get that this solution is in fact equal to the solution u given by Theorem 7.1.

We fix $p \in \mathbb{R}^N$ and we set $F := \mathfrak{R}F_0$.

Step 1: Construction on each branch separately. Given $p \in \mathbb{R}^N$, we fix $\alpha \in \{1, \ldots, N\}$ and we want to construct a solution w to

$$\begin{cases} w_t + H^\alpha(w_x) = 0 & \text{on} \quad (0, +\infty) \times J_\alpha^* \\ w_t + F_p^\alpha(w_x) = 0 & \text{on} \quad (0, +\infty) \times \{0\}, \\ w(0, x) = p^\alpha x & \text{for all} \quad x \in J_\alpha \end{cases} \tag{48}$$

where $F_p^\alpha(q^\alpha) = F(p^1, \ldots, , p^{\alpha-1}, q^\alpha, p^{\alpha+1}, \ldots, p^N)$. Using the same arguments as in the proof of Theorem 7.1, we deduce that there exists a unique solution w. For $\beta = 1, \ldots, N$, we define

$$\overline{p}^\beta = \max(p^\beta, \hat{p}^\beta) \quad \text{and} \quad \underline{p}^\beta = \min(p^\beta, \hat{p}^\beta).$$

Recall that F is relaxed with respect to H. From the characterization of relaxed junction conditions (Proposition 4.7 and 4.9), we deduce that F is constant on the box $[\underline{p}, \overline{p}]$. In particular, we get

$$F_p^\alpha(\hat{p}^\alpha) = F_p^\alpha(p^\alpha) = F(p).$$

Notice that it can also be seen from Corollaries 4.8 and 4.10 for F_p^α which is then also relaxed with respect to H^α. We now distinguish two cases.

Case 1: $\hat{p}^\alpha \leqslant p^\alpha$**.** We then have

$$H^\alpha(p^\alpha) \geqslant F(p) = F_p^\alpha(p^\alpha) = F_p^\alpha(\hat{p}^\alpha) = F(\hat{p}) = H^\alpha(\hat{p}^\alpha). \tag{49}$$

We claim that

$$\hat{p}^\alpha \leqslant w_x \leqslant p^\alpha. \tag{50}$$

Since the arguments are rather classical, we juste give the sketch of the proof. Let us prove the first inequality. By contradiction, we assume that

$$M := \sup_{\{t \in [0,T],\ x,y \in J_\alpha,\ x \geqslant y\}} \{w(t, y) - w(t, x) - \hat{p}^\alpha(y - x)\} > 0.$$

Hence, for every $\delta > 0$ small enough, we have

$$M_\delta := \sup_{\{t \in [0,T],\ x,y \in J_\alpha,\ x > y\}} \left\{w(t, y) - w(t, x) - \hat{p}^\alpha(y - x) - \frac{\delta}{2}(x^2 + y^2)\right\} > 0.$$

Note that all maximizers $(\bar{t}, \bar{x}, \bar{y})$ satisfy $\bar{x} > \bar{y}$ (otherwise the maximum is negative) and, by classical arguments, $\delta\bar{x}, \delta\bar{y} \to 0$ as $\delta \to 0$. We now duplicate the time variable and consider, for $\eta, \varepsilon > 0$,

$$M_{\varepsilon,\delta} := \sup_{\{t,s\in[0,T],\ x,y\in J_\alpha,\ x>y\}} \left\{ w(t,y) - w(s,x) - \hat{p}^\alpha (y-x) - \frac{\delta}{2}(x^2+y^2) - \frac{|t-s|^2}{2\varepsilon} - \frac{\eta}{T-t} \right\} > 0.$$

This maximum is reached at some point denoted by $(t_\varepsilon, s_\varepsilon, x_\varepsilon, y_\varepsilon)$ which satisfies $t_\varepsilon, s_\varepsilon \to \bar{t}$, $x_\varepsilon \to \bar{x}$ and $y_\varepsilon \to \bar{y}$, where $(\bar{t}, \bar{x}, \bar{y})$ is a maximum point of M_δ. In particular, $x_\varepsilon > y_\varepsilon$ for ε small enough and we assume that $x_\varepsilon > 0 = y_\varepsilon$, the other cases being treated in a similar (and even simpler) way. Using the viscosity inequality satisfied by w (which is a strong F-solution to (48)), we then have

$$\frac{t_\varepsilon - s_\varepsilon}{\varepsilon} + \frac{\eta}{(T-t_\varepsilon)^2} + F_p^\alpha(\hat{p}^\alpha + \delta y_\varepsilon) \leqslant 0$$

and

$$\frac{t_\varepsilon - s_\varepsilon}{\varepsilon} + H^\alpha(\hat{p}^\alpha - \delta x_\varepsilon) \leqslant 0.$$

Subtracting these two inequalities, we then get

$$\frac{\eta}{T^2} \leqslant H^\alpha(\hat{p}^\alpha - \delta x_\varepsilon) - F_p^\alpha(\hat{p}^\alpha + \delta y_\varepsilon).$$

Passing to the limit $\varepsilon \to 0$ and then $\delta \to 0$, we then obtain

$$\frac{\eta}{T^2} \leqslant H^\alpha(\hat{p}^\alpha) - F_p^\alpha(\hat{p}^\alpha) \leqslant 0,$$

where for the last inequality, we used (49). Contradiction. Hence $M \leqslant 0$ and then $w_x \geqslant \hat{p}$.

In the same way (using that $H^\alpha(p^\alpha) \geqslant F_p^\alpha(p^\alpha)$), we can prove the second inequality in (50).

We now want to prove that

$$w_x(t,0) = \hat{p}^\alpha \quad \text{and} \quad -w_t(t,0) = F(p). \tag{51}$$

Note that since w is convex or concave, these derivatives exist. For all $t_0 > 0$, we then set

$$w_x(t_0,0) = \check{p}^\alpha \quad \text{with} \quad \hat{p}^\alpha \leqslant \check{p}^\alpha \leqslant p^\alpha$$

and

$$\lambda = -w_t(t_0,0) = F_p^\alpha(\check{p}^\alpha) = F(p) = F(\hat{p}).$$

Making a blow up at $(t_0,0)$, we see that

$$\varepsilon^{-1}\{w(t_0+\varepsilon t, \varepsilon x) - w(t_0,0)\} \to W(t,x) := -\lambda t + \check{p}^\alpha x \quad \text{as} \quad \varepsilon \to 0.$$

By stability of viscosity solutions, we get that W is still a solution and then satisfies for $x \in J_\alpha^*$

$$W_t + H^\alpha(W_x) = 0$$

i.e.

$$-\lambda + H^\alpha(\check{p}^\alpha) = 0.$$

We conclude that

$$F_p^\alpha(\check{p}^\alpha) = H^\alpha(\check{p}^\alpha) = F(p) \quad \text{with} \quad \check{p}^\alpha \in [\hat{p}^\alpha, p^\alpha]$$

which implies that $\check{p}^\alpha = \hat{p}^\alpha$ and so (51).

Case 2: $H^\alpha(p^\alpha) \leqslant F(p)$. A proof similar to the one of Case 1 gives again (51).

Step 2: Gluing all together. In step 1, the function w has been constructed on each branch separately, but satisfies $-w_t(t,0) = \lambda = F(p)$, i.e.

$$w(t,0) = -\lambda t$$

Therefore w is continuous at $x = 0$, as defined on each branch J_α for each $\alpha = 1, \ldots, N$. We simply have to check that w satisfies the junction condition at $x = 0$, i.e.

$$w_t + F(\partial_1 w(t,0), \ldots, \partial_N w(t,0)) = 0.$$

Using (51), this is equivalent to show that

$$F(p) = F(\hat{p})$$

which is true since F is constant on the box $[\underline{p}, \overline{p}]$ and $p, \hat{p} \in [\underline{p}, \overline{p}]$. Therefore w is solution to Riemann problem, and by uniqueness of this solution, we conclude that $w = u$. This ends the proof of the theorem.
□

A Appendices

In these appendices, we give some interesting results concerning relaxation, which are not necessary for our paper. Since these results seem important, we only give the statement without giving the proof.

A.1 Further properties of the relaxation

We begin by a simple characterization of the relaxation at the characteristic points.

Proposition A.1 (Characterization of effective junction conditions)). *Let H satisfying* (2) *and F_0, F satisfying* (3). *Then $F = \mathfrak{R}F_0$ is characterized by the following properties*

$$\begin{cases} F = \underline{R}F = \overline{R}F, \\ F \geqslant F_0 \quad \text{on} \quad \overline{\chi}(F), \\ F \leqslant F_0 \quad \text{on} \quad \underline{\chi}(F), \end{cases}$$

As an application of Godunov relaxation formula to standard Hamilton-Jacobi equations, we can easily prove the following result. We consider here a standard Hamilton-Jacobi equation for a single Hamiltonian $\tilde{H} : \mathbb{R} \to \mathbb{R}$

$$\tilde{u}_t + \tilde{H}(\tilde{u}_x) = 0 \quad \text{for all} \quad (t,x) \in (0,T) \times \Omega \quad \text{with} \quad \Omega = (a,b) \ni 0 \tag{52}$$

Then we have the following result

Theorem A.2 (Classical viscosity solutions are solutions at a single point). *Assume that $\tilde{H}$ satisfies condition* (2), *and let $G_{\tilde{H}}$ be the Godunov flux associated to $\tilde{H}$, defined as in* (8).
i) (Subsolutions) *Let $\tilde{u} : (0,T) \times \Omega \to \mathbb{R}$ be a subsolution to* (52). *Then $\tilde{u}$ satisfies for all $t \in (0,T)$*

$$\tilde{u}_t(t,0) + G_{\tilde{H}}(\tilde{u}_x(t,0^-), \tilde{u}_x(t,0^+)) \leqslant 0 \tag{53}$$

ii) (Supersolutions) *Let $\tilde{u} : (0,T) \times \Omega \to \mathbb{R}$ be a supersolution of* (52). *Then $\tilde{u}$ satisfies for all $t \in (0,T)$*

$$\tilde{u}_t(t,0) + G_{\tilde{H}}(\tilde{u}_x(t,0^-), \tilde{u}_x(t,0^+)) \geqslant 0 \tag{54}$$

A.2 Refined gradient estimates on a junction

In this subsection, we state a result concerning Lipschitz estimates for the solution on the junction.

Theorem A.3 (Refined gradient estimates on a junction). *Assume that H satisfies condition* (2) *and that F_0 satisfies* (3) *on the box*

$$Q = \prod_{\alpha=1,\ldots,N} [m^\alpha, M^\alpha]$$

Let the initial data u_0 be Lipschitz continuous and satisfy the following gradient estimate

$$m^\alpha \leqslant (u_0)_x \leqslant M^\alpha \quad a.e.\ on\ J_\alpha^* \quad for\ each \quad \alpha = 1, \ldots, N \tag{55}$$

with $m^\alpha < M^\alpha$. If these bounds satisfy the following inequalities

$$\left\{ \begin{array}{l} H^\alpha(M^\alpha) \geqslant F_0(m^1, \ldots, m^{\alpha-1}, M^\alpha, m^{\alpha+1}, \ldots, m^N), \\ H^\alpha(m^\alpha) \leqslant F_0(M^1, \ldots, M^{\alpha-1}, m^\alpha, M^{\alpha+1}, \ldots, M^N), \end{array} \right| \quad for\ all \quad \alpha = 1, \ldots, N \tag{56}$$

then there exists the solution u to (1), (4) is globally Lipschitz continuous, with $u(t,\cdot)$ satisfying moreover the following gradient estimate for all $t \geqslant 0$

$$m^\alpha \leqslant u_x(t,\cdot) \leqslant M^\alpha \quad a.e.\ on\ J_\alpha^* \quad for\ each \quad \alpha = 1, \ldots, N$$

The result is particularly interesting and the proof follows easily from the construction using an implicit scheme which mimics the PDE. We do not know how to recover (even heuristically) conditions (56) directly from the PDE, without using the scheme.

A.3 Explicit relaxation: the finite dimensional manifold of limiters

There is a good case, where the relaxation formula can be constructed explicitly, and the set of effective junction conditions is parametrized by a finite-dimensional set $\mathcal{A}$. To this end, we assume that H satisfies (2) and we assume moreover that each Hamiltonian H^α has a finite number of minima and maxima. Then we show how we can parametrize explicitly all effective boundary conditions by a finite dimensional manifold $\mathcal{A}$ of limiters.

For $\alpha = 1, \ldots, N$, we call m_k^α for $k = 1, \ldots n_\alpha$, the increasing sequence of points of minima of H^α and M_k^α for $k = 1, \ldots, n_\alpha - 1$ the increasing sequence of points of maxima of H^α, such that

$$m_0^\alpha = M_0^\alpha := -\infty < m_1^\alpha < M_1^\alpha < m_2^\alpha < \cdots < M_{n_\alpha-1}^\alpha < m_{n_\alpha}^\alpha < +\infty =: M_{n_\alpha}^\alpha = m_{n_\alpha+1}^\alpha.$$

For each index $I = (i_1, \ldots, i_N)$, we assume that

$$\max_{\alpha=1,\ldots,N} H^\alpha(m_{i_\alpha}^\alpha) =: \quad A_I^0 \leqslant B_I^0 \quad := \min_{\alpha=1,\ldots,N} H^\alpha(M_{i_\alpha}^\alpha)$$

with the convention that $H^\alpha(\pm\infty) = +\infty$ for $\alpha = 1, \ldots, N$. We also denote by $(e_1, \ldots, e_N)$ the orthonormal basis of $\mathbb{R}^N$.

Definition of limiters associated to F_0.
We now give the recipe to construct flux limiters A given a general function $F_0 : \mathbb{R}^N \to \mathbb{R}$ satisfying (3). Notice that $m_I < M_I$ (if none of the indices i^α is equal to zero), and then $F_0(M_I) \leqslant F_0(m_I)$.

Given F_0, we now define a tensor $A = (A_I)_I \in \mathbb{R}^{n_1 \times \cdots \times n_N}$ where the A_I are the associated limiters given by

$$A_I := \left\{ \begin{array}{lll} \min(A_I^0, \min_{\alpha=1,\ldots,N} A_{I-e_\alpha}) & & \text{if} \quad F_0(m_I) < A_I^0, \\ \max(B_I^0, \max_{\alpha=1,\ldots,N} A_{I+e_\alpha}) & & \text{if} \quad F_0(M_I) > B_I^0, \\ F_0(p_I) = H^\alpha(p_I^\alpha) \quad \text{for all} \quad \alpha = 1, \ldots, N & & \text{if} \quad A_I^0 \leqslant F_0(m_I) \quad \text{and} \quad F_0(M_I) \leqslant B_I^0, \end{array} \right.$$

where $p_I = (p_I^1, \ldots, p_I^N)$ is the unique solution satisfying $p_I^\alpha \in [m_I^\alpha, M_I^\alpha]$. Here we use the convention that the tensor A is assumed to be extended for all $I \in \prod_{\alpha=1,\ldots,N} \{0, \ldots, n_\alpha\}$ by $A_I = +\infty$ if there exists some α such that $i_\alpha = 0$, and for all $I \in \prod_{\alpha=1,\ldots,N} \{1, \ldots, n_\alpha + 1\}$ by $A_I = -\infty$ if there exists some α such that $i_\alpha = n_\alpha + 1$. They satisfy in particular $A_I \geqslant A_{I'}$ if $I \leqslant I'$.

Parametrization of the level sets of $F := \mathfrak{R}F_0$.
We define $p_I^-(\mu) = (p_{i_1}^{1-}(\mu), \ldots, p_{i_N}^{N-}(\mu))$ with

$$p_k^{\alpha-}(\mu) := \left\{ \begin{array}{lll} M_{k-1}^\alpha & & \text{if} \quad \mu > H^\alpha(M_{k-1}^\alpha), \\ m_k^\alpha & & \text{if} \quad \mu < H^\alpha(m_k^\alpha), \\ p_k^{\alpha-} \in \left[M_{k-1}^\alpha, m_k^\alpha\right] \quad \text{with} \quad H^\alpha(p_k^{\alpha-}) = \mu & & \text{if} \quad \mu \in \left[H^\alpha(m_k^\alpha), H^\alpha(M_{k-1}^\alpha)\right]. \end{array} \right.$$

We use the convention that $p_1^{\alpha-}(+\infty) = -\infty$ and $p_{n_\alpha+1}^{\alpha-}(+\infty) = +\infty$. For $g \in \{-1,0\}^N$, we define the quadrant

$$Q_g := \prod_{\alpha=1,\ldots,N} \Delta^\alpha \quad \text{with} \quad \Delta^\alpha := \begin{cases} (-\infty, 0] & \text{if} \quad g^\alpha = -1, \\ [0, +\infty) & \text{if} \quad g^\alpha = 0. \end{cases}$$

Theorem A.4 (Explicit relaxation for Hamiltonians H^α with n_α minima)**.** *Under the previous assumptions, we associate to each function F_0, a tensor of limiters $A = (A_I)_I \in \mathbb{R}^{n_1\times\cdots\times n_N}$ which is generically parametrized by a manifold $\mathcal{A}$ of dimension at most equal to $n_1 \times \cdots \times n_N$.*

Then the relaxed junction function $F := \mathfrak{R}F_0$ only depends on the tensor A and is given as follows. For all $p \in \prod_{\alpha=1,\ldots,N} \left[M_{i_\alpha-1}^\alpha, M_{i_\alpha}^\alpha\right]$, we have

$$F(p) = \inf\left\{\mu \in \mathbb{R}, \quad p \in p_I^-(\mu) + S_I(\mu)\right\} \quad \textit{with} \quad S_I(\mu) := \bigcup_{g\in\{-1,0\}^N,\ A_{I+g}\leqslant\mu} Q_g.$$

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