Strictly convex Hamilton-Jacobi equations: strong trace of the gradient

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Abstract

We consider Lipschitz continuous viscosity solutions to evolutive Hamilton-Jacobi equations. Under a condition of strict convexity of the Hamiltonian, we show that there exists a notion of strong trace of the gradient of the solution. This result is based on a Liouville-type result of classification of global solutions on the half space. Under zero Dirichlet boundary condition, we show that the solution only depends on the normal variable. As a consequence, we show that the existence of a pointwise tangential gradient implies existence of a pointwise normal gradient. For the Liouville-type result, and when the Hamiltonian is not convex, we give a counter-example with a solution which is not one-dimensional.

We give two applications. On the one hand, for the classical stationary Dirichlet problem on a bounded domain, we show the existence of a closed subset of the boundary of the domain, where Taylor expansion of the solution is uniform. On the other hand, for Hamilton-Jacobi equations on a network, we show that the space derivative of the solution has a trace at each node, which satisfies a natural germ condition.

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1 Introduction

1.1 Main results

Let $d \geq 1$ and let us denote the open half space by

$$\Omega := \left\{ x = (x_1, \ldots, x_d) \in \mathbb{R}^d, \quad x_d > 0 \right\}.$$

We consider a function $u : \mathbb{R} \times \overline{\Omega} \to \mathbb{R}$, which is globally Lipschitz continuous. We assume that $u(t,x)$ is a viscosity solution of the following time evolutive Hamilton-Jacobi equation on the half-space

$$\begin{cases} u_t + H(Du) = 0 & \text{on} \quad \mathbb{R} \times \Omega \\ u = 0 & \text{on} \quad \mathbb{R} \times \partial\Omega \end{cases} \tag{1.1}$$

where the zero Dirichlet condition is assumed to be satisfied in the strong sense (i.e. pointwisely).

Our first goal is to classify such solutions u when the Hamiltonian function H is assumed to satisfy the following condition

$$H : \mathbb{R}^d \to \mathbb{R} \text{ is strictly convex, } C^1 \text{ and superlinear (i.e. } \liminf_{|P| \to +\infty} \frac{H(P)}{|P|} = +\infty). \tag{1.2}$$

Recall that strict convexity of H means

$$H(\lambda P + (1-\lambda) Q) < \lambda H(P) + (1-\lambda) H(Q) \quad \text{for all} \quad \lambda \in (0,1), \quad P, Q \in \mathbb{R}^d, \quad P \neq Q.$$

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A very classical theorem of Liouville claims that bounded entire harmonic functions are constant. In this spirit, our core main result is the following.

Theorem 1.1 (A Liouville-type result on the half space)
Assume that the convex function $H : \mathbb{R}^d \to \mathbb{R}$ satisfies (1.2) and that $u : \mathbb{R} \times \overline{\Omega} \to \mathbb{R}$ is a viscosity solution of (1.1), which is globally Lipschitz continuous (say for some Lipschitz constant $L > 0$). Then u is one-dimensional, and we can write

$$u = u(x_d) \quad \text{where} \quad \Omega = \{x_d > 0\}.$$

Notice that the convexity assumption on the Hamiltonian H is necessary, as shows the following result.

Proposition 1.2 (Counter-example to Liouville-type result for non convex H)
Let $d = 1$ with $x = x_1 \in \mathbb{R}$. There exists a C^∞ function $H : \mathbb{R} \to \mathbb{R}$ which is strictly convex on $(0, +\infty)$ and strictly concave on $(-\infty, 0)$, and a globally Lipschitz continuous function $u : \mathbb{R} \times \overline{\Omega} \to \mathbb{R}$ solution of (1.1), such that $u(t, x)$ is not a function of x only, neither one-dimensional (of a linear combination of t, x). Moreover it is homogeneous of degree 1, i.e. satisfies $u(\lambda t, \lambda x) = \lambda u(t, x)$ for all $\lambda \geq 0$.

Consider now the following equation on the cylinder (for the open ball $B_1 = B_1(0) \subset \mathbb{R}_t \times \mathbb{R}^{d-1}_{x'}$ with $x = (x', x_d)$)

$$u_t + H(t, x, Du) = 0 \quad \text{on} \quad C_+ := B_1 \times (0, 1) \quad \subset \mathbb{R} \times \Omega \tag{1.3}$$

with Dirichlet condition on the tangential boundary

$$u = g(t, x) \quad \text{on} \quad \Gamma := B_1 \times \{0\} \quad \subset \mathbb{R} \times \partial\Omega \tag{1.4}$$

where we recall that $\Omega = \mathbb{R}^{d-1}_{x'} \times (0, +\infty)_{x_d}$.

As a corollary of Theorem 1.1, we get the following (at least surprising for the author) existence result of a pointwise normal derivative at the boundary.

Theorem 1.3 (From tangential to normal gradient)
Let $g : \Gamma \to \mathbb{R}$ and $u : C_+ \cup \Gamma \to \mathbb{R}$ be Lipschitz continuous functions with the open cylinder C_+ and its tangential boundary Γ respectively defined in (1.3) and (1.4). Assume that u is a viscosity solution of (1.3)-(1.4) with a continuous Hamiltonian $H : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ such that the map $P \mapsto H(t, x, P)$ satisfies (1.2) for $(t, x) = (0, 0)$. We write $x = (x', x_d) \in \mathbb{R}^d$ with $x' = (x_1, \ldots, x_{d-1}) \in \mathbb{R}^{d-1}$ and $x_d \in \mathbb{R}$.
Assume that u has a pointwise tangential gradient at the origin $0 \in \Gamma$ (with $u(0) = 0$), i.e. there exists $(\lambda, P') \in \mathbb{R} \times \mathbb{R}^{d-1}$ such that

$$g(t, x', 0) = u(t, x', 0) = \lambda t + P' \cdot x' + o(|t| + |x'|) \quad \text{as} \quad (t, x') \to (0, 0).$$

Then u has also a pointwise normal gradient $p_d \in \mathbb{R}$ at the origin, and then a full gradient, i.e. we have with $P = (P', p_d) \in \mathbb{R}^d$

$$u(t, x) = \lambda t + P \cdot x + o(|t| + |x|) \quad \text{as} \quad (t, x) \to (0, 0).$$

Remark 1.4 (Which assumptions are necessary?)
Notice that in Theorem 1.3, strict convexity of the Hamiltonian is necessary, otherwise the case $H \equiv 0$ allows any normal derivative for any functions $u = u(x_d)$.

Conversely, the superlinearity assumption on H is just technical, because we only work with Lipschitz continuous solutions. The superlinearity condition can simply be removed, adding to H the function $P \mapsto (\max\{|P| - L, 0\})^2$, if $L > 0$ is the Lipschitz constant of the solution u.

Similarly for $(t, x) = (0, 0)$, the C^1-regularity of the map $P \mapsto H(t, x, P)$ in Theorem 1.3 is just technical, but simplifies a lot the presentation of the proofs. It is possible to relax this regularity to C^0. Then the subdifferential of H may for instance satisfy $\partial_P H(0, 0, P) \supset [\xi^a, \xi^b]$, and the Legendre-Fenchel transform $\mathcal{L}(0, 0, \cdot)$ of $H(0, 0, \cdot)$ is then affine on $[\xi^a, \xi^b]$. On the one hand, it creates an indeterminacy on some characteristic velocities ξ, but on the other hand, it does not affect the evaluation of the representation formula because the Lagrangian $\mathcal{L}(0, 0, \cdot)$ is then affine along these characteristics.

To simplify the presentation of the proofs, we still keep the superlinearity and C^1-regularity assumptions as in (1.2), in the whole paper.

Remark 1.5 (The boundary data g)
Notice that the function g in Theorem 1.3 has no particular role and can be removed from the statement. Still the introduction of g allows the reader to figure out a classical formulation of the Dirichlet problem.

Theorem 1.6 (A notion of strong trace of the gradient)
Let $u : C_+ \cup \Gamma \to \mathbb{R}$ be a Lipschitz continuous function with the open cylinder C_+ and its tangential boundary Γ respectively defined in (1.3) and (1.4). Assume that u is a viscosity solution of (1.3) with a continuous Hamiltonian $H : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ such that the map $P \mapsto H(t,x,P)$ satisfies (1.2) for all $(t,x) \in \mathbb{R} \times \partial\Omega$. Recall that we write $x = (x', x_d) \in \mathbb{R}^d$ with $x' = (x_1, \dots, x_{d-1}) \in \mathbb{R}^{d-1}$ and $x_d \in \mathbb{R}$. We denote the time-space gradient of u by

$$\hat{D}u := (u_t, Du).$$

Then pointwisely $\hat{D}u(t,x',0)$ exists for almost every $(t,x',0) \in \Gamma$, which means that we have the following Taylor expansion with $Y \in \mathbb{R} \times \overline{\Omega}$ and $X + Y \in C_+ \cup \Gamma$

$$u(X+Y) = u(X) + Y \cdot \hat{D}u(X) + o(|Y|) \quad \textit{for a.e.} \quad X = (t,x',0) \in \Gamma. \tag{1.5}$$

Moreover the value of the time-space gradient on Γ (for $x_d = 0$) is also reached by its values on C_+ (for $x_d > 0$) as follows

$$\lim_{\varepsilon \to 0^+} \int_{(0,1)} \left\{ \int_{B_1} |\hat{D}u(t,x',\varepsilon x_d) - \hat{D}u(t,x',0)| \; dt dx' \right\} \; dx_d = 0. \tag{1.6}$$

This limit is our notion of strong trace of the time-space gradient $\hat{D}u$ on the tangential boundary Γ.

Remark 1.7 (An obvious remark)
Recall that the result of Theorem 1.6 is not true for Lipschitz functions in general, but here is only true because u is also a viscosity solution for a striclty convex Hamiltonian. Otherwise, for $d = 1$ and dropping the time variable, there exist 1-Lipschitz functions on $[0, +\infty)$, with no derivative at $x = 0$. It is also easy to build a 1-Lipschitz function u on $[0,+\infty)$ which has zero derivative at $x = 0$ (and which is piecewise affine in all compact sets of $(0, +\infty)$), but such that $|u_x| = 1$ a.e..

Remark 1.8 (Comparison of our strong trace with other notions)
We have seen that (1.6) gives a notion of strong trace on the boundary $\mathbb{R} \times \partial\Omega$ of the time-space gradient $\hat{D}u$ of solutions to Hamilton-Jacobi equations defined on $\mathbb{R} \times \Omega$. Here the Hamiltonian is strictly convex only at the boundary $\mathbb{R} \times \partial\Omega$ and can be non convex on $\mathbb{R} \times \Omega$. We can see the time-space gradient as a map $(0, +\infty) \ni x_d \mapsto \hat{D}u(\cdot, \cdot, x_d) \in B$ defined for a.e. x_d, with $B := L^1_{loc}(\mathbb{R}_t \times \mathbb{R}^{d-1}_{x'})$, and this map belongs to $L^1_{loc}((0,+\infty)_{x_d}; B)$. Then our notion of strong trace means that $x_d = 0$ is a Lebesgue point of this map, i.e. a point of continuity in L^1(normal) with value in L^1(tangential).

In contrast, classical notion of strong traces (see Vasseur [22] and Panov [19]) for multidimensional scalar conservation laws is a point of (essential) continuity in L^∞(normal) with value in L^1(tangential), and then is a stronger notion.

Notice that if we require that H satisfies (1.2) not only on the boundary $\mathbb{R} \times \partial\Omega$, but also in a neighborhood of it, and under a Dini condition on certain moduli (see condition (7.2)), then we can easily show (by a covering argument) that the time-space gradient converges in the sense of strong traces as in [22, 19], i.e. that

$$ess \lim_{x_d \to 0^+} \int_{B_1} |\hat{D}u(t,x',x_d) - \hat{D}u(t,x',0)| \; dt dx' = 0 \tag{1.7}$$

Nevertheless, even the notion of strong traces as in (1.7) is less strong than property (1.5) which claims the existence of a time-space gradient almost everywhere on the boundary. For more on this topic, we refer to the counter-example given in Proposition 8.4.

Remark 1.9 (Generalization to Lipschitz domains)
From our proof, it is straightforward to generalize the strong trace Theorem 1.6 for HJ equation on a domain $D \subset \mathbb{R}_t \times \mathbb{R}^d_x$, where the boundary ∂D is locally a Lipschitz continuous hypersurface. Then the strong trace can also be considered on any open subset $\Sigma \subset \partial D$, where the boundary ∂D is a Lipschitz continuous graph in a pure space direction, say in direction e_d. See also Corollary 7.3 for local properties of the gradient.

Notice also that Theorem 1.6 can be seen as a sort of BV-like regularity of the time-space gradient of Lipschitz continuous solutions, for strictly convex Hamiltonians $P \mapsto H(t,x,P)$, assuming only continuity in (t,x). We again emphasize that the convexity (and then strict convexity) of H is only assumed on the boundary, but not in the domain.

1.2 Applications

We present two applications.

The first one concerns the classical stationary Dirichlet problem for Hamilton-Jacobi on a bounded open set Ω_0 and is Proposition 6.1. There we show that we can split the boundary in a partition $\partial\Omega_0 = \partial\Omega_- \cup \partial\Omega_+$ where $\partial\Omega_-$ is a closed subset of the boundary where the Taylor expansion of the solution up to order one is uniform.

The second application concerns junction problems, which was our initial motivation (see Cardaliaguet, Forcadel, Monneau [10]). We show in Proposition 8.1 that if u is the viscosity solution to a problem with Hamilton-Jacobi equations on a junction in space dimension 1, then as expected its spatial derivative $v = u_x$ is an entropy solution of the associated conservation laws. Moreover the trace of v at the junction point consists in a vector whose coordinates are associated to each branch, and the trace vector belongs to an explicit germ for almost every time.

1.3 Open questions

We leave open the following questions.

Question 1: Only under assumptions of Theorem 1.6, do we have a strong convergence of the gradient $Du(t,x',x_d) \to Du(t,x',0)$ in $L^1_{loc}(\mathbb{R}_t \times \mathbb{R}^{d-1}_{x'})$ as $E \ni x_d \to 0^+$, where $E \subset (0,+\infty)$ is a set of full Lebesgue measure?[1]

Now consider the following genuine nonlinearity condition on the Hamiltonian

$$\mathcal{L}\left(\{a \in \mathbb{R}, \quad H(P + a\xi) = H(P) + a\xi \cdot DH(P)\}\right) = 0 \quad \text{for all} \quad P \in \mathbb{R}^d, \quad \xi \in \mathbb{R}^d \setminus \{0\} \tag{1.8}$$

where $\mathcal{L}$ is the Lebesgue measure.

Question 2: Do we have a strong trace of the gradient for Lipschitz continuous solutions of evolutive Hamilton-Jacobi equations with C^1 Hamiltonians H satisfying (1.8)?

1.4 Brief review of the literature

We refer to the pioneering work of Lions [16] on viscosity solutions of Hamilton-Jacobi equations and their properties. The reader can also consult the reference books Cannarsa, Sinestrari [8] on semiconcave functions and Bardi, Capuzo-Dolcetta [2] for Hamilton-Jacobi equations related to control problems.

In Jensen, Souganidis [15], the authors study a particular stationary Hamilton-Jacobi equations in dimension one. Assuming that the Hamiltonian (possibly non convex) is genuinely nonlinear (in a certain sense), they show that the left gradient $u_x(x_0^-)$ and the right gradient $u_x(x_0^+)$ do exist at each point x_0. They also get a certain continuity property of these gradients.

In Bianchini, De Lellis, Robyr [5], the authors show that for a uniformly C^2 Hamiltonian H, the time-space gradient of the solution is in SBV_{loc}. This result has been extended to the case of C^3 Hamiltonians depending also on (t,x) in Bianchini, Tonon [6].

In the context of homogeneous scalar conservation laws, a notion of strong trace on a Lipschitz boundary of a domain has been introduced by Vasseur [22] under a condition of genuine nonlinearity of the C^3 flux function. This result has been generalized by Panov [19] to the case of C^0 homogeneous fluxes, and C^1 boundary (the case of Lipschitz boundary is also claimed to remain valid with the same proof).

1.5 Organization of the paper

In Section 2, we prepare the work to show later the Liouville-type result. We show that global solutions to Hamilton-Jacobi equations on the half space have to be sandwiched in between two linear solutions $u_\pm$. This is the barrier's result Lemma 2.1. Section 3 recalls quite standard results about characteritics $\xi_\pm$ (associated to the solutions $u_\pm$) and the representation formula of the solution to convex Hamilton-Jacobi equations, in the spirit of optimal control theory and/or Lax-Hopf formula.

In Section 4, the proof of the Liouville-type result (Theorem 1.1) is done.

[1] This question seems delicate because we do not assume the convexity of $H(t,x,P)$ in P, except for $(t,x) \in \mathbb{R} \times \partial\Omega$, contrarily to the argument proposed in Remark 1.8.

In Section 5, as a corollary of the Liouville-type result, we show Theorem 1.3, i.e. existence of a tangential gradient implies the existence of a normal gradient. In Section 6, we give a direct application of Section 5, which is an analysis of the standard stationary Dirichlet problem, namely Proposition 6.1.

In Section 7, we give the proof of existence of a notion of strong trace, namely Theorem 1.6. In Section 8, we show Proposition 8.1 as an application of the notion of strong trace to a junction problem. We explain there the relation between the junction condition for Hamilton-Jacobi equations and the natural germ condition satisfied by the trace of its space derivative.

In Section 9, we show a counter-example to the Liouville-type result when the Hamiltonian is not convex (see Proposition 1.2).

Finally Section 10 is an appendix where we recall useful results used throughout the paper.

1.6 Main notations

$\Omega = \mathbb{R}^{d-1} \times (0,+\infty)$	= half space
$x = (x_1,\dots,x_d) \in \mathbb{R}^d$	= space coordinate
$x' = (x_1,\dots,x_{d-1})$	= tangential space coordinate
$(e_1,\dots,e_d)$	= standard basis of $\mathbb{R}^d$
$P \cdot Q$	= standard euclidean scalar product of vectors $P,Q \in \mathbb{R}^d$
H	= the Hamiltonian
$\mathcal{L} = H^*$	= the Legendre-Fenchel transform of H
$P_\pm = (p_\pm)e_d$	= special roots of $H(P_\pm) = 0$
$u_\pm(t,x) = P_\pm \cdot x$	= barrier solutions
$\xi_\pm = DH(P_\pm)$	= characteristic velocities
$\gamma^{\xi}_{t_0,x_0}(t)$	$= \left\{ \begin{array}{l} \text{trajectory in } \Omega \text{ parametrized by the time } t \in (-\infty,t_0], \\ \text{of terminal point } x_0 \text{ at time } t_0, \text{ and of velocity } \xi(t) \end{array} \right.$
$\mathcal{E}^{t_1}_{t_0,x_0}$	= set of trajectories $\gamma^{\xi}_{t_0,x_0}$ defined on time interval $[t_1,t_0]$
$C_+ = B_1(0) \times (0,1)$	= cylinder included in $(\mathbb{R}_t \times \mathbb{R}^{d-1}_{x'}) \times (0,+\infty)_{x_d}$
$\Gamma = B_1(0) \times \{0\}$	= tangential boundary of the cylinder
$\hat{D}u = (u_t, Du)$	= time-space gradient
$X = (t,x) = (t,x',x_d)$	= time-space coordinate
$X' = (t,x')$	= tangential time-space coordinate
$\nu_X(P)$	= for a.e. $X \in \mathbb{R}\times\Omega$, probability measure with support on $\mathbb{R}^d \ni P$

2 Existence of barriers

With the standard orthonormal basis $(e_1,\dots,e_d)$ of $\mathbb{R}^d$ with $e_d = (0,\dots,0,1) \in \mathbb{R}^d$, and given a function $H : \mathbb{R}^d \to \mathbb{R}$, we set for $p \in \mathbb{R}$, the reduced Hamiltonian function $h : \mathbb{R} \to \mathbb{R}$ defined by

$$h(p) := H(pe_d). \tag{2.1}$$

We start with the following result.

Lemma 2.1 (Existence of linear barriers $u_\pm$)
Assume that $H : \mathbb{R}^d \to \mathbb{R}$ is continuous and let h be the associated function defined in (2.1). Let $u : \mathbb{R}\times\overline{\Omega} :\to \mathbb{R}$ be a globally Lipschitz continuous viscosity solution of (1.1). We define

$$\left.\begin{array}{l} \overline{p} := \inf\left\{p \in \mathbb{R}, \quad u \le \ell_p \quad \text{on} \quad \mathbb{R}\times\overline{\Omega}\right\} \\ \underline{p} := \sup\left\{p \in \mathbb{R}, \quad u \ge \ell_p \quad \text{on} \quad \mathbb{R}\times\overline{\Omega}\right\} \end{array}\right\} \quad \textit{with the linear function} \quad \ell_p(t,x) := px_d.$$

i) (Basic result)
Then we have $h(\overline{p}) \le 0 \le h(\underline{p})$.
ii) (Improved result when H is convex)
Assume that H is convex. Then we have $h(\underline{p}) = 0$.
Furthermore, if h is coercive (i.e. satisfies $\liminf_{|p|\to+\infty} h(p) = +\infty$), then there exists a unique couple $(p_-,p_+) \in \mathbb{R}^2$ such that $p_- \le p_+$ and

$$[p_-,p_+] = \{h \le 0\} \quad \textit{with} \quad h(p_\pm) = 0 \tag{2.2}$$

and we have

$$u_- \le u \le u_+ \quad \textit{with} \quad u_\pm(t,x) = P_\pm \cdot x \quad \textit{with} \quad P_\pm := (p_\pm)e_d \in \mathbb{R}^d \tag{2.3}$$

where $u_\pm$ are both solutions of (1.1).
Moreover if $p_+ = p_-$, then $u = u_+ = u_-$.

Remark 2.2 *Even if the functions $u_\pm$ do not depend on the variable t, our presentation is simplified allowing this time dependence.*

Proof of Lemma 2.1
We start the proof assuming only that H is continuous.
Step 1: bound from above
We define

$$\overline{p} := \inf \left\{ p \in \mathbb{R}, \quad u \le \ell_p \quad \text{on} \quad \mathbb{R} \times \overline{\Omega} \right\} \quad \text{with the linear function} \quad \ell_p(t,x) := p x_d.$$

The number $\overline{p}$ is a well-defined finite quantity, because u is globally Lipschitz continuous and $u = 0$ on $\mathbb{R} \times \partial\Omega$. It is also classical that $\overline{p}$ is a critical slope from above and then satisfies the subsolution viscosity inequality (2.4) given below and associated to the PDE.

For sake of completness, we present a direct proof. By assumption, for any $\varepsilon > 0$, there exists $X^\varepsilon := (t^\varepsilon, x^\varepsilon) \in \mathbb{R} \times \overline{\Omega}$ such that

$$u \le \ell_{\overline{p}} \text{ on } \mathbb{R} \times \overline{\Omega} \qquad \text{and} \qquad u > \ell_{\overline{p}-\varepsilon} \text{ at } X^\varepsilon.$$

Then we can rescale the function u as follows, setting for $x = (x', x_d)$

$$u^\varepsilon(t,x) := \eta^{-1} u(t^\varepsilon + \eta t, x^{\varepsilon\prime} + \eta x', \eta x_d) \quad \text{for} \quad \eta := x_d^\varepsilon$$

which is still a globally Lipschitz continuous viscosity solution of (1.1), and satisfies furthermore

$$\ell_{\overline{p}} \ge u^\varepsilon \text{ on } \mathbb{R} \times \overline{\Omega} \qquad \text{and} \qquad u^\varepsilon > \ell_{\overline{p}-\varepsilon} \text{ at } X^0 := (0,0,\ldots,0,1) \in \mathbb{R} \times \Omega.$$

Using Ascoli-Arzela theorem, we can extract a convergent subsequence (still denoted by $(u^\varepsilon)_\varepsilon$) such that $u^\varepsilon \to u^0$ locally uniformly on compact sets of $\mathbb{R} \times \overline{\Omega}$, which satisfies furthermore

$$\ell_{\overline{p}} \ge u^0 \text{ on } \mathbb{R} \times \overline{\Omega} \qquad \text{and} \qquad u^0 \ge \ell_{\overline{p}} \text{ at } X^0 := (0,0,\ldots,0,1) \in \mathbb{R} \times \Omega.$$

This shows that $\ell_{\overline{p}}$ is a test function touching u^0 from above at X^0. Because by stability of viscosity solutions, we know that u^0 is still a viscosity solution of (1.1), we deduce the subsolution viscosity inequality

$$0 + H(\overline{p} e_d) \le 0 \tag{2.4}$$

which shows that $h(\overline{p}) \le 0$.
Step 2: bound from below
The symmetric argument to Step 1, shows that

$$0 + H(\underline{p} e_d) \ge 0 \tag{2.5}$$

i.e. $h(\underline{p}) \ge 0$, which shows point i).
Step 3: proof of ii)
We now assume that H is convex. Then the argument in Step 2 (similar to Step 1) shows that $\ell_{\underline{p}}$ is a test function from below to the Lipschitz solution u^0. Now from Barron, Jensen characterization of Lipschitz continuous solutions for convex Hamiltonians (see Lemma 10.1 in the appendix), we know that we have equality in inequality (2.5), which shows that

$$h(\underline{p}) = 0.$$

Recall that $h(\overline{p}) \le 0$. Now if the convex function h is coercive, there exists unique couple satisfying $p_- \le p_+$ such that $[p_-, p_+] = \{h \le 0\}$ with $h(p_\pm) = 0$ and moreover $p_- \le \underline{p} \le \overline{p} \le p_+$. This ends the proof of the lemma.

3 Characteristics and the representation formula

3.1 Characteristic velocities

In the remaining part of the paper, we assume that

$$P_+ \neq P_- \quad \text{with } P_\pm \text{ defined in (2.3)} \tag{3.1}$$

which means $p_- < p_+$. We associate the fundamental characteristic velocities

$$\xi_\pm := DH(P_\pm) \in \mathbb{R}^d. \tag{3.2}$$

For the Hamiltonian H satisfying (1.2), we define its Legendre-Fenchel transform as the Lagrangian $\mathcal{L}$ given by

$$\mathcal{L}(\xi) := \sup_{P\in\mathbb{R}^d} \{\xi \cdot P - H(P)\}. \tag{3.3}$$

Then $\mathcal{L}$ satisfies again (1.2), namely it is strictly convex, C^1 and superlinear (see Lemma 10.2 in the appendix).

Lemma 3.1 (Sign of the fundamental characteristic velocities)
Under assumption (1.2) on H, the fundamental characteristic velocities $\xi_\pm$ defined in (3.2) satisfy

$$\mathcal{L}(\xi) \geq P_\pm \cdot \xi \quad \textit{with equality at} \quad \xi = \xi_\pm \quad \textit{and} \quad P_\pm = D\mathcal{L}(\xi_\pm). \tag{3.4}$$

Assuming moreover (3.1), we have

$$\xi_- \cdot e_d < 0 < \xi_+ \cdot e_d. \tag{3.5}$$

Proof of Lemma 3.1
Definition (3.2) implies $P_\pm = D\mathcal{L}(\xi_\pm)$ by convex duality (see Lemma 10.2 in the appendix). Finally, we have $\mathcal{L}(\xi_\pm) = \xi_\pm \cdot P_\pm - H(P_\pm) = \xi_\pm \cdot P_\pm$ and by convexity we get $\mathcal{L}(\xi) \geq \mathcal{L}(\xi_\pm) + (\xi - \xi_\pm) \cdot D\mathcal{L}(\xi_\pm) = P_\pm \cdot \xi$ which shows (3.4). On the other hand, the function $h(p) := H(pe_d)$ is strictly convex and satisfies $h(p_\pm) = 0$ with $p_- < p_+$ when we assume (3.1). Hence $\pm h'(p_\pm) > 0$, which implies (3.5). This ends the proof of the lemma.

3.2 Representation formula

Given $(t, x) \in \mathbb{R} \times \Omega$ and $\xi(\cdot) \in L^1_{loc}((-\infty, t]; \mathbb{R}^d)$, we consider the following backward trajectory

$$\begin{cases} \dfrac{d}{d\sigma}\gamma^\xi_{t,x}(\sigma) = \xi(\sigma) \quad \text{for} \quad \sigma \leq t \\ \text{with terminal data} \quad \gamma^\xi_{t,x}(t) = x \end{cases}$$

and call for all $t_0 < t$

$$\mathcal{E}^{t_0}_{t,x} := \left\{ \begin{array}{l} (s,\xi) \in [t_0, t) \times L^1_{loc}((-\infty, t]; \mathbb{R}^d), \\ \gamma^\xi_{t,x}(\sigma) \in \Omega, \quad \text{for all} \quad \sigma \in (s, t], \end{array} \right| \text{ with } \left| \begin{array}{ll} \gamma^\xi_{t,x}(s) \in \partial\Omega & \text{if} \quad s \in (t_0, t) \\ \gamma^\xi_{t,x}(s) \in \overline{\Omega} & \text{if} \quad s = t_0 \end{array} \right\}$$

which is the set of parameters such that the backward trajectory stays in the set $\Omega = \mathbb{R}^{d-1} \times (0, +\infty)$ and in a time interval contained in $[t_0, t]$.

We recall the following standard result for convex Hamiltonians (which can be seen as a generalization of Lax-Hopf formula).

Lemma 3.2 (Representation formula)
Assume that $H : \mathbb{R}^d \to \mathbb{R}$ satisfies (1.2), and let $\mathcal{L}$ be the Legendre-Fenchel transform of H given in (3.3).

Assume that $u : \mathbb{R} \times \overline{\Omega} \to \mathbb{R}$ is a globally Lipschitz continuous function satisfying $u = 0$ on $\mathbb{R} \times \partial\Omega$. Then u satisfies for all $(t,x) \in \mathbb{R} \times \Omega$ and all $t_0 \in (-\infty, t)$

$$(3.6) \qquad u(t,x) = \inf_{(s,\xi)\in\mathcal{E}^{t_0}_{t,x}} G(s,t;\gamma^\xi_{t,x}) \quad \text{with} \quad G(s,t;\gamma^\xi_{t,x}) := u(s,\gamma^\xi_{t,x}(s)) + \int_s^t \mathcal{L}\left(\frac{d}{d\sigma}\gamma^\xi_{t,x}(\sigma)\right) d\sigma$$

if and only if u solves (1.1).

Representation formula (3.6) means that $u(t,x)$ is the infimum of some cost function over all trajectories with terminal point (t,x) and initial point on the part of the boundary $([t_0,t) \times \partial\Omega) \cup (\{t_0\} \times \Omega)$.

Sketch of the proof
The standard proof first shows the dynamic programming principle which implies (by variations/comparison) the viscosity inequalities on the time interval $(t_0, +\infty)$ (see for instance [14] for a result of the same flavour). Conversely, the comparison principle implies that every solution of (1.1) on the time interval $[t_0, +\infty)$ coincides with the unique solution given by the representation formula (3.6). This ends the sketch of the proof.

4 Proof of Theorem 1.1: the Liouville-type result

In this section, we show that the value of the solution at one point can be u_+ if its characteristic is ξ_+, i.e. if the information comes from the fixed boundary $\partial\Omega$. The only other possibility is that the value of the solution is computed along the characteristic of direction ξ_- (which comes from infinity, i.e. far away from the fixed boundary $\partial\Omega$). In this last case, we show that for two points on such a characteristic line, the values of the solution are explicitly related. This rigidity will imply the Liouville-type result.

We start with the following result which claims that for long time *optimal* trajectories, the foot of the trajectory never belongs to the Dirichlet boundary $\mathbb{R} \times \partial\Omega$, if the head of the trajectory satisfies $u < u_+$.

Lemma 4.1 (Solution along an optimal trajectory)
Assume that H satisfies (1.2) and (3.1). Let u be a global Lipschitz solution of (1.1) satisfying $u_- \le u \le u_+$. Let $X_0 := (t_0, x_0) \in \mathbb{R} \times \Omega$ be such that

$$u(X_0) < u_+(X_0).$$

Then for any $\tau > 0$, there exists $X_1 := (t_0 - \tau, y^\tau) \in \mathbb{R} \times \Omega$ such that

$$u(X_0) = u(X_1) + \tau\mathcal{L}(\xi^\tau) \quad \text{with} \quad \xi^\tau := \frac{x_0 - y^\tau}{\tau} \quad \text{and} \quad u(X_1) < u_+(X_1).$$

Remark 4.2 *This result holds because*
(4.1)

$$u_+(X_0) = \inf_{\tau>0,\ y\in\partial\Omega} \left\{0 + \tau\mathcal{L}\left(\frac{x_0 - y}{\tau}\right)\right\} = \bar\tau_0\mathcal{L}(\xi_+) \quad \text{for} \quad \bar\tau_0 > 0 \quad \text{defined by} \quad \xi_+ = \frac{x_0 - y_0}{\bar\tau_0} \quad \text{with} \quad y_0 \in \partial\Omega$$

and because condition $u(X_0) < u_+(X_0)$ somehow prevents the information to propagate from the fixed boundary $\mathbb{R} \times \partial\Omega$ with characteristic velocity ξ_+ as it does otherwise.

Proof of Lemma 4.1
Step 1: splitting the representation formula in two parts
Recall from (3.6) that for all $X_0 := (t_0, x_0) \in \mathbb{R} \times \Omega$, we have for all $t \in (-\infty, t_0)$

$$u(X_0) = \inf_{(s,\xi)\in\mathcal{E}^t_{X_0}} G(s,t_0;\gamma^\xi_{X_0}) \quad \text{with} \quad G(s,t_0;\gamma^\xi_{X_0}) := u(s,\gamma^\xi_{X_0}(s)) + \int_s^{t_0} \mathcal{L}\left(\frac{d}{d\sigma}\gamma^\xi_{X_0}(\sigma)\right) d\sigma.$$

We split this formula in two parts. The first part is generated by the boundary $\partial\Omega$

$$u^t_b(X_0) := \inf_{s_0\in[t,t_0)} \bar u^{s_0}_b(X_0) \quad \text{with} \quad \bar u^{s_0}_b(X_0) := \inf_{(s,\xi)\in\mathcal{E}^t_{X_0},\ s\in\{s_0\},\ \gamma^\xi_{X_0}(s)\in\partial\Omega} G(s,t_0;\gamma^\xi_{X_0}) \quad \text{for} \quad s_0 \in [t,t_0)$$

while the second part is generated by the domain Ω

$$u_d^t(X_0) := \inf_{(s,\xi)\in\mathcal{E}_{X_0},\ s\in\{t\},\ \gamma^\xi_{X_0}(s)\in\Omega} G(s,t_0;\gamma^\xi_{X_0}).$$

Here for u_b^t, each trajectory has its foot on the boundary $[t,t_0)\times\partial\Omega$, and for u_d^t, each trajectory has its foot inside the domain $\{t\}\times\Omega$. Hence we have

$$u(X_0) = \min\left\{u_b^t(X_0), u_d^t(X_0)\right\}. \tag{4.2}$$

Step 2: boundary contribution
Using the fact that straight trajectories are always more competitive than other trajectories because $\mathcal{L}$ is strictly convex, for $s_0 := t_0-\tau$ with $\tau>0$, we get

$$\bar{u}_b^{s_0}(X_0) = \inf_{y\in\partial\Omega}\left\{0+\tau\mathcal{L}\left(\frac{x_0-y}{\tau}\right)\right\} = \tau\mathcal{L}\left(\xi^\tau\right) \quad \text{for} \quad \xi^\tau := \frac{x_0-y^\tau}{\tau} \quad \text{and some} \quad y^\tau\in\partial\Omega$$

and recall that by convexity we have

$$\mathcal{L}\left(\xi^\tau\right) \geq \mathcal{L}(\xi_+) + (\xi^\tau-\xi_+)\cdot D\mathcal{L}(\xi_+) = P_+\cdot\xi^\tau. \tag{4.3}$$

with $\bar\tau_0$ defined in (4.1). We see that for $\tau\neq\bar\tau_0$, we have $\xi^\tau\neq\xi_+$ because their component along e_d are then different. Hence the strict convexity of $\mathcal{L}$ gives $\mathcal{L}\left(\xi^\tau\right) > P_+\cdot\xi^\tau$ for $\tau\neq\bar\tau_0$, i.e.

$$\bar{u}_b^{t_0-\tau}(X_0) = \inf_{y\in\partial\Omega}\left\{0+\tau\mathcal{L}\left(\frac{x_0-y}{\tau}\right)\right\} > u_+(X_0) \quad \text{for} \quad \tau\neq\bar\tau_0, \quad \text{with equality for } \tau=\bar\tau_0 \tag{4.4}$$

and then considering the infimum of those $\bar{u}_b^{s_0}(X_0)$ with $s_0=t_0-\tau$, we get

$$u_b^t(X_0) = u_+(X_0) \quad \text{if} \quad t\leq t_0-\bar\tau_0$$

and

$$u_b^t(X_0) > u_+(X_0) \quad \text{if} \quad t\in(t_0-\bar\tau_0, t_0)$$

Step 3: domain contribution
Notice that (4.2) and $u(X_0)<u_+(X_0)$ imply that

$$u(X_0) = u_d^t(X_0) \quad \text{for} \quad t<t_0. \tag{4.5}$$

Then we have for $\tau := t_0-t>0$ and for $G(y) := u(t,y)+\tau\mathcal{L}\left(\dfrac{x_0-y}{\tau}\right)$

$$u_+(X_0) > u(X_0) = u_d^t(X_0) := \inf_{y\in\Omega} G(y) = \inf_{y\in\overline{\Omega}} G(y)$$

where the infimum is reached for some $y^\tau\in\overline{\Omega}$. Here we have used the superlinearity of $\mathcal{L}$ and the global Lipschitz regularity of u. Notice that (4.4) shows that $y^\tau\notin\partial\Omega$, i.e. $y^\tau\in\Omega$. We get

$$P_+\cdot x_0 = u_+(X_0) > u(X_0) = u(t,y^\tau)+\tau\mathcal{L}(\xi^\tau) \geq u(t,y^\tau)+\tau P_+\cdot\xi^\tau \quad \text{setting} \quad \xi^\tau := \frac{x_0-y^\tau}{\tau}$$

where we have used (4.3) in the last inequality. This implies

$$u_+(t,y^\tau) > u(t,y^\tau)$$

which shows the desired result. This ends the proof of the lemma.

Lemma 4.3 (Key equality along the characteristic ξ_-)
Assume that H satisfies (1.2) and (3.1). Let u be a global Lipschitz solution of (1.1) satisfying $u_-\leq u\leq u_+$. Let $X_0 := (t_0,x_0)\in\mathbb{R}\times\Omega$ be such that $u(X_0)<u_+(X_0)$.
Then for any $\tau>0$ we have

$$u(X_0) = u(X_0-\tau(1,\xi_-))+\tau\mathcal{L}(\xi_-). \tag{4.6}$$

Proof of Lemma 4.3

Step 1: the direction ξ_2 remains fixed for $\tau_2 > \tau_0$

We apply two times Lemma 4.1. The first time from X_0 shows for any $\tau_0 > 0$, the existence of some $X_1 := (t_0 - \tau_0, y_{X_0}^{\tau_0}) \in \mathbb{R} \times \Omega$ such that

$$u(X_0) = u(X_1) + \tau_0 \mathcal{L}(\xi_{X_0}^{\tau_0}) \quad \text{with} \quad \xi_{X_0}^{\tau_0} := \frac{x_0 - y_{X_0}^{\tau_0}}{\tau_0} \quad \text{and} \quad u(X_1) < u_+(X_1).$$

Applying a second time Lemma 4.1 from $X_1 = (t_1, x_1)$, shows for any $\tau_1 > 0$, the existence of some $X_2 := (t_1 - \tau_1, y_{X_1}^{\tau_1}) \in \mathbb{R} \times \Omega$ such that

$$u(X_1) = u(X_2) + \tau_1 \mathcal{L}(\xi_{X_1}^{\tau_1}) \quad \text{with} \quad \xi_{X_1}^{\tau_1} := \frac{x_1 - y_{X_1}^{\tau_1}}{\tau_1} \quad \text{and} \quad u(X_2) < u_+(X_2).$$

Hence we get for $\tau_2 := \tau_0 + \tau_1$

$$u(X_0) = u(X_2) + \tau_1 \mathcal{L}(\xi_{X_1}^{\tau_1}) + \tau_0 \mathcal{L}(\xi_{X_0}^{\tau_0}) \geq u(X_2) + \tau_2 \mathcal{L}(\xi_2) \quad \text{with} \quad \xi_2 := \tau_2^{-1} \left\{ \tau_1 \xi_{X_1}^{\tau_1} + \tau_0 \xi_{X_0}^{\tau_0} \right\} \tag{4.7}$$

where the inequality remains strict if $\xi_{X_0}^{\tau_0} \neq \xi_{X_1}^{\tau_1}$. For $X_2 = (t_2, x_2)$ with $t_2 = t_0 - \tau_2$, we have $\xi_2 = \dfrac{x_0 - x_2}{\tau_2}$, $x_2 \in \Omega$ and we get

$$u(X_2) + \tau_2 \mathcal{L}(\xi_2) \leq u(X_0) = u_d^{t_2}(X_0) = \inf_{y \in \Omega} \{ u(t_0 - \tau_2, y) + \tau_2 \mathcal{L}(\xi) \} \quad \text{with} \quad \xi := \frac{x_0 - y}{\tau_2}$$

where we have used (4.5) for the first equality. Hence the infimum is reached for $\xi = \xi_2$ and we have equality in (4.7). This implies $\xi_{X_0}^{\tau_0} = \xi_{X_1}^{\tau_1} = \xi_2$. This also shows that we can choose $\xi_{X_0}^{\tau_2} = \xi_2$, $y^{\tau_2} = x_2$, i.e. for all $\tau_2 > \tau_0$, there exists $x_2 \in \Omega$ such that $X_2 := (t_0 - \tau_2, x_2)$ satisfies

$$u(X_0) = u(X_2) + \tau_2 \mathcal{L}(\xi_2) \quad \text{with} \quad \xi_2 = \frac{x_0 - x_2}{\tau_2} = \xi_{X_0}^{\tau_0}.$$

Step 2: proof that $\xi_2 = \xi_-$

By assumption, we have

$$u_+(X_0) > u(X_0) = u(X_2) + \tau_2 \mathcal{L}(\xi_2) \geq u_-(X_2) + \tau_2 \mathcal{L}(\xi_2)$$

and then $u_+(X_0) - u_-(X_0) > u_-(X_2 - X_0) + \tau_2 \mathcal{L}(\xi_2)$, i.e.

$$\mathcal{L}(\xi_2) < P_- \cdot \xi_2 + \frac{A}{\tau_2} \quad \text{with} \quad A := u_+(X_0) - u_-(X_0) > 0.$$

We set

$$S_\tau^A := \left\{ \xi \in \mathbb{R}^d, \quad \mathcal{L}(\xi) < P_- \cdot \xi + \frac{A}{\tau} \right\}$$

and recall (see 3.4) that $P_- \cdot \xi \leq \mathcal{L}(\xi)$ with equality at $\xi = \xi_-$. The strict convexity of $\mathcal{L}$ then implies

$$\operatorname{dist}(\{\xi_-\}, S_\tau^A) \to 0 \quad \text{as} \quad \tau \to +\infty.$$

Hence $\xi_2 \in S_{\tau_2}^A$ satisfies $\xi_2 \to \xi_-$ as $\tau_2 \to +\infty$. As ξ_2 is constant, we deduce that $\xi_2 = \xi_-$. This implies (4.6) and ends the proof of the lemma.

Lemma 4.4 (Property of global solutions)

Assume that H satisfies (1.2) and (3.1). Let u be a global Lipschitz solution of (1.1) satisfying $u_- \leq u \leq u_+$. Let $X_0 := (t_0, x_0), X_1 := (t_0, x_1) \in \mathbb{R} \times \Omega$ be such that $u(X_0) < u_+(X_0)$, $u(X_1) < u_+(X_1)$. Then we have

$$u(X_1) - u(X_0) = u_-(X_1 - X_0).$$

Proof of Lemma 4.4
The equality is proven establishing two inequalities.
From Lemma 4.3 applied to X_1, we know that for $\tau > 0$ large enough, we have the key equality along the characteristic ξ_-

$$u(X_1) = u(Y_1) + \tau\mathcal{L}(\xi_-) \quad \text{with} \quad Y_1 = (t_1, y_1) := (t_0 - \tau, x_1 - \tau\xi_-).$$

From representation formula (3.6), we also have

$$u(X_0) \le u(Y_1) + \tau\mathcal{L}(\xi_1) \quad \text{with} \quad \xi_1 := \frac{x_0 - y_1}{\tau} = \xi_- + \bar{\xi}, \quad \bar{\xi} := \frac{x_0 - x_1}{\tau}$$

Hence we get

$$\begin{array}{rcl} u(X_1) - u(X_0) & = & u(Y_1) + \tau\mathcal{L}(\xi_-) - u(X_0) \\ & \ge & \tau\mathcal{L}(\xi_-) - \tau\mathcal{L}(\xi_1) \\ & = & -\tau \displaystyle\int_0^1 d\sigma\ \bar{\xi} \cdot D\mathcal{L}(\xi_- + \sigma\bar{\xi}) \\ & \to & (x_1 - x_0) \cdot D\mathcal{L}(\xi_-) \quad \text{as} \quad \tau \to +\infty \\ & = & P_- \cdot (x_1 - x_0) \\ & = & u_-(X_1 - X_0) \end{array}$$

where in the fifth line we have used that $D\mathcal{L}(\xi_-) = P_-$. Hence $u(X_1) - u(X_0) \ge u_-(X_1 - X_0)$ and exchanging the roles of X_1 and X_0 gives the reverse inequality and then the equality. This ends the proof of the lemma.

Corollary 4.5 (Characterization of solutions)
Assume that H satisfies (1.2) and (3.1). Let u be a global Lipschitz solution of (1.1) satisfying $u_- \le u \le u_+$. Then $u = u(x_d)$.

Proof of Corollary 4.5
From Lemma 4.4, we deduce that

$$Du = P_- \quad \text{a.e. in} \quad \{u < u_+\}.$$

Because $H(P_-) = 0$, the PDE itself implies that

$$u_t = 0 \quad \text{a.e. in} \quad \{u < u_+\}.$$

Then either $u \equiv u_+$, or there exists some point $X_0 \in \mathbb{R} \times \Omega$ such that $u(X_0) < u_+(X_0)$. Let $\mathcal{C}$ be the connected component of $\{u < u_+\}$ containing X_0. Then there exists some open set $\omega \subset \Omega$ such that $\mathcal{C} = \mathbb{R} \times \omega$. Moreover there exists a constant $c \in \mathbb{R}$ such that

$$u(t, x) = c + P_- \cdot x \quad \text{on} \quad \mathbb{R} \times \omega.$$

Then either $\omega = \Omega$ and $u = u_-$, or $\Omega \cap \partial\omega \ne \emptyset$. By definition, on the boundary $\partial\omega$, we have

$$u(t, x) = u_+(t, x) = P_+ \cdot x$$

with P_+ parallel to P_- because $P_\pm = (p_\pm)e_d$. This forces the boundary $\partial\omega$ to be flat, i.e. precisely to have $\omega = \{x_d > \lambda\}$ for some $\lambda > 0$. This implies the uniqueness of the connected components of $\{u < u_+\}$, i.e.

$$u(t, x) = \min\{P_+ \cdot x, c + P_- \cdot x\} = \min\{(p_+)x_d, c + (p_-)x_d\}. \tag{4.8}$$

This ends the proof of the corollary.

Proof of Theorem 1.1
From Lemma 2.1, we have either $p_- = p_+$ and then $u = u_- = u_+$. The other possibility is $p_- < p_+$, and then Corollary 4.5 and Lemma 2.1 give the result in this case. This ends the proof of the theorem.

5 Proof of Theorem 1.3: the normal gradient

Proof of Theorem 1.3
Step 1: preliminaries
Because the map $(t,x') \mapsto u(t,x',0)$ has a derivative $(\lambda, P') \in \mathbb{R}\times\mathbb{R}^{d-1}$, it is a classical fact (in viscosity theory) that there exists a C^1 function $\phi : B_1 \to \mathbb{R}$ tangential to u at the origin, with ϕ from above and $-\phi$ from below, as follows

$$\begin{cases} -\phi(t,x') < u(t,x',0) < \phi(t,x') \quad \text{for all} \quad (t,x') \in B_1 \backslash \{(0,0)\} \\ \\ \phi(0,0) = u(0,0,0), \quad (\partial_t \phi^{\pm}, D_{x'}\phi)(0,0) = (\lambda, P'). \end{cases}$$

Up to substract $u(0,0,0) + \lambda t + P'\cdot x'$ to u, and redefine both u and H, we can assume that $(\lambda, P') = (0,0)$, and $u(0,0,0) = 0$. For $\varepsilon > 0$, we consider the blow-up functions

$$u^\varepsilon(t,x) = \varepsilon^{-1} u(\varepsilon t, \varepsilon x) \tag{5.1}$$

which are Lipschitz continuous, uniformly with respect to ε, with the same Lipschitz constant. By Ascoli-Arzela theorem, from any sequence $\varepsilon \to 0$, we can extract a subsequence (still denoted by ε) such that $u^\varepsilon \to u^0$ locally uniformly on compact sets of $\mathbb{R}\times\overline{\Omega}$. Moreover by stability of viscosity solutions, the limit u^0 solves the whole half space problem

$$\begin{cases} u^0_t + H(0,0,Du^0) = 0 & \text{on} \quad \mathbb{R}\times\Omega & \text{(in the viscosity sense)} \\ u^0 = 0 & \text{on} \quad \mathbb{R}\times\partial\Omega & \text{(in the strong sense).} \end{cases} \tag{5.2}$$

From Theorem 1.1 we know that $u^0 = u^0(x_d)$ and from (4.8), we even know that

$$u^0(t,x) = \min\{(p_+)x_d, c + (p_-)x_d\} \tag{5.3}$$

for some constant $c \in \mathbb{R}$. Hence x_d-derivatives of u^0 are well-defined both on $\mathbb{R}\times\partial\Omega$ and at an infinite distance from it. Precisely, the following two quantities $p_0 := \partial_{x_d} u^0(0,0,0)$ and $p_\infty := \partial_{x_d} u^0(0,0,+\infty)$ are well defined (and can be equal or different). One way to recover them is to consider the blow-up/blow-down

$$(u^0)^\mu(t,x) := \mu^{-1} u^0(\mu t, \mu x) \to \begin{cases} p_0 x_d & \text{if} \quad \mu \to 0^+ \\ p_\infty x_d & \text{if} \quad \mu \to +\infty. \end{cases}$$

In the next step, we will use in a suitable way this idea in order to show that the limit u^0 has to be independent on the sequence ε, and then has to be linear.
Step 2: setting of the problem
Consider now two sequences $\varepsilon^i = \varepsilon^i_k \to 0$ for $i = 1,2$, such that for rescaling (5.1), we have $u^{\varepsilon^i} \to u^i$ locally uniformly on compact sets of $\mathbb{R}\times\overline{\Omega}$. Notice that each limit u^i has a shape as in (5.3). Then by a diagonal extraction argument, we can always find sequences $a^{\varepsilon^i} \to +\infty$ which go to infinity sufficiently slowly such that $a^{\varepsilon^i}\varepsilon^i \to 0$ and

$$u^{a^{\varepsilon^i}\varepsilon^i}(t,x) \to p^i_\infty x_d \quad \text{with} \quad p^i_\infty := \partial_{x_d} u^i(0,0,+\infty)$$

and we can similarly find sequences $b^{\varepsilon^i} \to 0^+$ which go to zero sufficiently slowly such that

$$u^{b^{\varepsilon^i}\varepsilon^i}(t,x) \to p^i_0 x_d \quad \text{with} \quad p^i_0 := \partial_{x_d} u^i(0,0,0).$$

Hence up to redefine the sequences ε^i (by $a^{\varepsilon^i}\varepsilon^i \to 0$ or $b^{\varepsilon^i}\varepsilon^i \to 0$), and redefine the limit u^i, we can assume that for $i = 1,2$

$$u^{\varepsilon^i}(t,x) \to u^i(t,x) = p^i x_d \quad \text{as} \quad \varepsilon^i \to 0. \tag{5.4}$$

For $\varepsilon > 0$, we set

$$\phi^\varepsilon(t,x') := \varepsilon^{-1}\phi(\varepsilon t, \varepsilon x').$$

Then for any $\eta > 0$, there exists $\varepsilon_\eta > 0$ such that for all $\varepsilon^i < \varepsilon_\eta$, we have

$$\begin{cases} |u^{\varepsilon^i}(t,x',x_d) - p^i x_d| \le \eta + \phi^{\varepsilon^i}(t,x') & \text{for all} \quad (t,x) \in \overline{B}_1 \times [0,1] =: Q_1 \\ |u^{\varepsilon^i}(t,x',x_d)| \le \phi^{\varepsilon^i}(t,x') & \text{for all} \quad (t,x) \in \overline{B}_1 \times \{0\} \\ \phi(0,0) = \partial_t\phi(0,0) = 0 = D_{x'}\phi(0,0). \end{cases} \tag{5.5}$$

We now choose such elements of the sequence ε^i for $i=1,2$.
Step 3: proof by contradiction
Up to exchange the indices $i=1,2$, assume by contradiction that

$$p^1 < p^2 \quad \text{and let us choose any} \quad p \in (p^1,p^2).$$

Step 3.1: case $\varepsilon^1 \le \varepsilon^2$
For $\varepsilon := \varepsilon^2$, we consider the flat function (which is almost affine on Q_1)

$$\ell_0(t,x) := -\phi^\varepsilon(t,x') + px_d - \eta(t^2+|x'|^2)$$

that we expect to behave like a test function from below for u^ε.
Then (5.5) implies $-\phi^\varepsilon \le u^\varepsilon - p^2 x_d + \eta$. Hence we get

$$\ell_0 \le u^\varepsilon + (p-p^2)x_d + \eta\left\{1-(t^2+|x'|^2)\right\} \quad \text{on} \quad Q_1.$$

Together with (5.5) on $\overline{B}_1 \times \{0\}$, and for $\eta \in (0, p^2-p)$ on $\overline{B}_1 \times \{1\}$, this gives

$$\ell_0 \le u^\varepsilon \quad \text{on} \quad \partial Q_1. \tag{5.6}$$

On the other hand, for $\varepsilon' = \varepsilon^1 = \mu\varepsilon$ with $\mu = \dfrac{\varepsilon^1}{\varepsilon} \in (0,1]$, we deduce from (5.5) that $u^{\mu\varepsilon} \le p^1 x_d + \phi^{\mu\varepsilon} + \eta$ on Q_1, i.e. by a change of variables

$$u^\varepsilon \le p^1 x_d + \phi^\varepsilon + \mu\eta \quad \text{on} \quad \overline{B}_\mu \times [0,\mu] =: Q_\mu$$

and in particular

$$u^\varepsilon(0,\mu) \le \mu(p^1+\eta) < \mu p = \ell_0(0,\mu)$$

where the strict inequality arises for $\eta \in (0, p-p^1)$, which also implies $\mu \in (0,1)$ from (5.6). Hence

$$\sup_{\partial Q_1}(\ell_0 - u^\varepsilon) \le 0 < \sup_{Q_1}(\ell_0 - u^\varepsilon) = (\ell_0 - u^\varepsilon)(\bar X) \quad \text{with} \quad \bar X = (\bar t, \bar x) \in Q_1 \backslash \partial Q_1.$$

Hence the viscosity supersolution inequality gives $\partial_t \ell_0 + H(\cdot, D\ell_0) \ge 0$ at $\varepsilon\bar X$, i.e. for $\bar x = (\bar x', \bar x_d)$

$$-\phi_t(\varepsilon\bar X) - 2\eta\bar t + H(\varepsilon\bar X, -D_{x'}\phi(\varepsilon\bar X) - 2\eta\bar x', p) \ge 0.$$

In the limit $\varepsilon \to 0$ and then $\eta \to 0$, we get

$$H(0,p) \ge 0 \quad \text{for all} \quad p \in (p^1,p^2). \tag{5.7}$$

Step 3.2: case $\varepsilon^1 \ge \varepsilon^2$
We proceed similarly to Step 3.1, and testing from above, we get

$$H(0,p) \le 0 \quad \text{for all} \quad p \in (p^1,p^2).$$

Step 3.3: conclusion
Choosing alternatively elements of the sequences ε^i such that $\varepsilon^1 \le \varepsilon^2$ and then $\varepsilon^1 \ge \varepsilon^2$, we conclude from Steps 3.1 and 3.2 that

$$H(0,p) = 0 \quad \text{for all} \quad p \in (p^1,p^2)$$

which is in contradiction with the strict convexity of $H(0,\cdot)$.
This implies that for any blow-up limit u^0, we have

$$\partial_{x_d} u^0(0,0) = \partial_{x_d} u^0(0,+\infty) \quad \text{and} \quad u^0(t,x) = p^0 x_d$$

and moreover that p^0 is unique, independent of the chosen subsequence. We conclude that $u^\varepsilon \to u^0$ locally uniformly on compact sets for the whole sequence $\varepsilon \to 0$. This shows the desired result and ends the proof of the theorem.

Remark 5.1 (Shortcut in Step 3 of the proof of Theorem 1.3)
Notice that the limits $u^i(x) = p^i x_d$ in (5.4) have to satisfy the PDE which implies $H(0,p^i)=0$. Now the strict convexity of H implies that

$$H(0,p) < 0 \quad \textit{for all} \quad p \in (p^1,p^2).$$

Hence inequality (5.7) and then Step 3.1 is sufficient to conclude to a contradiction. Nevertheless, even if Step 3.2 is not strictly necessary, it is useful to present the proof in a more natural (and symmetric) way.

6 Application to the Dirichlet problem

Consider now the following stationary Dirichlet problem

$$(6.1)\qquad \begin{cases} H(x,Du) &= 0 \qquad \text{on} \quad \Omega_0 \subset \mathbb{R}^d \\ \qquad u &= g(x) \qquad \text{on} \quad \partial\Omega_0. \end{cases}$$

As a corollary of our results we get.

Proposition 6.1 (Continuity of the normal derivative for Dirichlet problem)
Let $\Omega_0 \subset \mathbb{R}^d$ be a C^1 bounded open set with outward unit normal n to Ω_0. Assume that $H : \overline{\Omega}_0 \times \mathbb{R}^d \to \mathbb{R}$ is a continuous function such that the convex map $P \mapsto H(x,P)$ satisfies (1.2) for all $x \in \partial\Omega_0$.

Assume that $g : \partial\Omega_0 \to \mathbb{R}$ is a C^1 function, and let us call $D_\tau g(x)$ the tangential gradient of g along the boundary $\partial\Omega_0$ at x. Then the real roots p of the equation

$$H(x, D_\tau g(x) - pn(x)) = 0 \quad \textit{for} \quad x \in \partial\Omega_0$$

are given exactly by two continuous functions $p_\pm : \partial\Omega_0 \to \mathbb{R}$ satisfying $p_-(x) \le p_+(x)$.

Let $u : \overline{\Omega}_0 \to \mathbb{R}$ be a Lipschitz continuous viscosity solution of (6.1) where the Dirichlet boundary condition is satisfied in the strong sense.

i) (existence of a normal derivative)
Then u has a normal derivative $\dfrac{\partial u}{\partial n} : \partial\Omega_0 \to \mathbb{R}$. Moreover we have

$$-\frac{\partial u}{\partial n} \in \{p_-, p_+\} \quad \textit{and set} \quad \partial\Omega_\pm := \left\{ x \in \partial\Omega_0, \quad -\frac{\partial u}{\partial n} = p_\pm \quad \textit{at} \quad x \right\}.$$

ii) (uniform modulus of continuity on the closed set $\partial\Omega_-$)
Then there exists a modulus of continuity $\varepsilon_0(r) \to 0$ as $r \to 0$ such that for $P_{x_0} := D_\tau g(x_0) - p_-(x_0)n(x_0)$ we have

$$(6.2)\qquad |u(x_0 + x) - \{u(x_0) + P_{x_0} \cdot x\}| \le |x|\varepsilon_0(|x|) \quad \textit{for all} \quad x \in \overline{\Omega}_0 \quad \textit{and all} \quad x_0 \in \partial\Omega_-.$$

In particular $\partial\Omega_-$ is a closed set and $\partial\Omega_+$ is an open subset of $\partial\Omega_0$.

Notice here that the case $p_+(x_0)$ arises for characteristic trajectories coming immediately from the fixed boundary $\partial\Omega_0$ at x_0, while the case $p_-(x_0)$ corresponds to characteristic trajectories coming far away from x_0, through the interior of the domain Ω_0.

Proof of Proposition 6.1
Step 1: Existence of pointwise normal derivative
Consider a point $x_0 \in \partial\Omega_0$ with outward unit normal $n = n(x_0)$ to Ω_0. Up to change the coordinates, we can assume that $x_0 = 0$ and $n = -e_d$. Then we have locally with $h \in C^1$

$$\Omega_0 = \{x_d > h(x')\}, \quad h(0) = 0 = D_{x'}h(0).$$

We then rectify locally Ω_0, setting $y = (y', y_d)$, $y' = x'$, $y_d = x_d - h(x')$, $u(x) = v(y)$. Since u solves (6.1), we see that v solves

$$\begin{cases} \tilde{H}(y, Dv) = 0 & \text{locally on} \quad \{y_d > 0\} \\ v = \tilde{g} & \text{locally on} \quad \{y_d = 0\} \end{cases}$$

with $\tilde{g}(y', 0) := g(y', h(y'))$ and $\tilde{H}(y,P) = H(y', y_d + h(y'), P' - p_d D_{y'}h(y'), p_d)$ for $P = (P', p_d)$. The function v inherits its Lipschitz continuity from the one of u and the fact that $h \in C^1$. Consider now its blow-up $v^\varepsilon(y) := \varepsilon^{-1}\{v(\varepsilon y) - v(0)\}$. From Theorem 1.3, we know that $v^\varepsilon(y) \to v^0(y) = P' \cdot y' + p_d y_d$ as $\varepsilon \to 0$, with $P' := D_{y'}\tilde{g}(0,0)$ and $p_d = p^0(x_0) \in \mathbb{R}$ satisfying $\tilde{H}(0, P', p^0) = 0$, which is equivalent as $\varepsilon \to 0$ to

$$u^\varepsilon(x) := \varepsilon^{-1}\{u(x_0 + \varepsilon x) - u(x_0)\} \to u^{x_0}(x) = \left(P_\tau - p^0 n\right) \cdot x$$

where $P_\tau := D_\tau g(x_0)$ is the tangent gradient of g along the boundary $\partial\Omega_0$ at x_0. Moreover, we have

$$H(x_0, D_\tau g(x_0) - p^0(x_0)n(x_0)) = 0.$$

By strict convexity of $H(x_0,P)$ in P, we know that there exist two functions $p_\pm : \partial\Omega_0 \to \mathbb{R}$ with $p_- \le p_+$ and which are roots of equation $H(\cdot, D_\tau g - p_\pm n) = 0$ on $\partial\Omega_0$. Hence we have

$$-\frac{\partial u}{\partial n} = p^0 \in \{p_-, p_+\} \quad \text{on} \quad \partial\Omega_0.$$

Step 2: Modulus of continuity on $\partial\Omega_-$
Step 2.1: setting of the problem
Now the proof consists in a variant of the proof of Theorem 1.3. To simplify the presentation, we will already assume that $\partial\Omega_0 = \partial\Omega = \mathbb{R}^{d-1}$ is flat with $n = -e_d$ and then $\tilde{H} = H$, $\tilde{g} = g =: \phi$, $v = u$ (and there is no time variable). We consider a Lipschitz continuous solution u to the problem

$$\left\{\begin{array}{rlllll} H(x,Du) = 0 & \text{on} & B_1(0)\times(0,1) & \subset \Omega & = \{x_d > 0\} \\ u = \phi & \text{on} & B_1(0)\times\{0\} & \subset \partial\Omega & = \{x_d = 0\} \end{array}\right.$$

with $\phi \in C^1$. We set $\partial\Omega_\pm := \left\{-\dfrac{\partial u}{\partial n} = p_\pm\right\}$.

Step 2.2 proof by contradiction and first statements
Assume by contradiction that (6.2) is wrong. Then there exists some constant $\kappa > 0$ and a sequence of points

$$x_\varepsilon \in \partial\Omega_- \cap \overline{B}_{1/2}(0)\times\{0\}$$

and $y_\varepsilon \in \overline{\Omega}$, normalized such that $\varepsilon := |y_\varepsilon| \to 0$ with

$$|u(x_\varepsilon + y_\varepsilon) - \{u(x_\varepsilon) + P_{x_\varepsilon}\cdot y_\varepsilon\}\,| \ge \kappa|y_\varepsilon| \quad \text{with} \quad P_{x_\varepsilon} := D_{x'}\phi(x_\varepsilon) + p_-(x_\varepsilon)e_d, \quad \varepsilon := |y_\varepsilon|. \tag{6.3}$$

We then consider the blow-up with moving center x_ε:

$$u^\varepsilon_{x_\varepsilon}(x) := \varepsilon^{-1}\{u(x_\varepsilon + \varepsilon x) - u(x_\varepsilon)\}, \quad \phi^\varepsilon_{x_\varepsilon}(x) := \varepsilon^{-1}\{\phi(x_\varepsilon+\varepsilon x) - \phi(x_\varepsilon)\}.$$

Up to extract a subsequence, we have $x_\varepsilon \to x_0 \in \partial\Omega$, and $z_\varepsilon := \varepsilon^{-1}y_\varepsilon \to z_0 \in \partial B_1 \cap \overline{\Omega}$. Up to redefine once ϕ and u and then H, we can assume that $D_{x'}\phi(x_0) = 0$. Then $\phi^\varepsilon_{x_\varepsilon} \to \phi^0 = 0$, $P_{x_\varepsilon} \to P_{x_0} := p_-(x_0)e_d$. Moreover, by Liouville-type Theorem 1.1 and in particular (4.8), we get that

$$u^\varepsilon_{x_\varepsilon}(x) \to u^0(x) = w(x_d) \quad \text{with} \quad w(x_d) := \min\left\{p^2x_d, c^0 + p^1x_d\right\}, \quad w(0) = 0, \quad c^0 \in \mathbb{R}$$

with $p^2 := p_+(x^0)$, $p^1 := p_-(x_0)$. Passing to the limit in (6.3), we get

$$|w(a) - p^1a| \ge \kappa > 0 \quad \text{with} \quad a := (z_0)_d.$$

This implies that $c^0 > 0$ in the definition of u^0, and $p^1 \ne p^2$. Hence there exists a fixed factor $\beta \in (0,1)$ (independent on ε) such that

$$u^{\beta\varepsilon}_{x_\varepsilon}(x) \to \beta^{-1}u^0(\beta x) = p^2x_d \quad \text{on} \quad \overline{B}_1\times[0,1] \subset \overline{\Omega}.$$

On the other hand, from Theorem 1.3, for $x_\varepsilon \in \partial\Omega_-$ fixed, and for any sequence $\beta > \alpha_k \to 0$, we have

$$u^{\alpha_k\varepsilon}_{x_\varepsilon} \to P_{x_\varepsilon}\cdot x \quad \text{on} \quad \overline{\Omega}, \quad \text{as} \quad \alpha_k \to 0.$$

Step 2.3 end of the proof by contradiction
Hence for any $\eta > 0$, there exists $\varepsilon_\eta > 0$ such that for all $\varepsilon < \varepsilon_\eta$, there exists $\alpha_{\eta,\varepsilon} \in (0,\beta)$ such that for all $\alpha_k < \alpha_{\eta,\varepsilon}$ we have

$$\left\{\begin{array}{lll} |P_{x_\varepsilon} - p^1| < \eta/3 & \text{with} & P_{x_\varepsilon} := D_{x'}\phi(x_\varepsilon) + p_-(x_\varepsilon)e_d, \\ |u^{\alpha_k\varepsilon}_{x_\varepsilon}(x) - P_{x_\varepsilon}\cdot x| \le \eta/3 + \phi^{\alpha_k\varepsilon}_{x_\varepsilon}(x) & \text{for all} & x \in \overline{B}_1\times[0,1] \\ |u^{\beta\varepsilon}_{x_\varepsilon}(x',x_d) - p^2x_d| \le \eta + \phi^{\beta\varepsilon}_{x_\varepsilon}(x) & \text{for all} & x \in \overline{B}_1\times[0,1] \end{array}\right.$$

and then

$$\left\{\begin{array}{llll} |u^{\alpha_k\varepsilon}_{x_\varepsilon}(x',x_d) - p^1x_d| \le \eta + \phi^{\alpha_k\varepsilon}_{x_\varepsilon}(x) & \text{for all} & x \in \overline{B}_1\times[0,1] =: Q_1 & \\ |u^{\beta\varepsilon}_{x_\varepsilon}(x',x_d) - p^2x_d| \le \eta + \phi^{\beta\varepsilon}_{x_\varepsilon}(x) & \text{for all} & x \in \overline{B}_1\times[0,1] & \\ u^{\theta\varepsilon}_{x_\varepsilon} = \phi^{\theta\varepsilon}_{x_\varepsilon} & \text{for all} & x \in \overline{B}_1\times\{0\}, & \theta = \beta, \alpha_k. \end{array}\right. \tag{6.4}$$

Now let us choose any $p \in (p^1, p^2)$, and because $\alpha_k \varepsilon < \beta\varepsilon$, let us consider the flat function

$$\ell_0(x) := \phi^{\beta\varepsilon}_{x_\varepsilon}(x', 0) + p x_d - \eta |x'|^2.$$

Proceeding exactly as in the proof of Theorem 1.3, we get the viscosity inequality $H(x_\varepsilon + \cdot, D\ell_0) \geq 0$ at $\beta\varepsilon\bar{x}$ with $\bar{x} \in Q_1 \backslash \partial Q_1$, which shows in the limit $H(x_0, 0, p) \geq 0$. This is in contradiction with $H(x_0, 0, p) < 0$ which arises because $H(x_0, \cdot)$ is strictly convex and $H(x_0, 0, p^i) = 0$ for $i = 1, 2$. Hence we conclude that (6.2) is true and this ends the proof of the proposition.

Remark 6.2 *Notice that the function $u(x', x_d) = \min(p_+ x_d, p_- x_d)$ on $\Omega_0 = \{x_d > h(x')\}$ with h concave C^2 and $h(0) = 0 = D_{x'}h(0)$ gives an example where $\partial\Omega_- = \{0\}$, showing that (6.2) does not hold uniformly*[2] *for $x_0 \in \partial\Omega_+$.*

We can also map this example locally onto a half space $\Omega = \{x_d > 0\}$, using the map $\Phi(x', x_d) := (x', x_d - h(x'))$, with $\Phi : \Omega_0 \to \Omega$ and define $v = u \circ \Phi^{-1}$ which solves an equation $\tilde{H}(x, Dv) = 0$ locally on Ω with $v(x', 0) = p_+ h(x')$ for $(x', 0) \in \partial\Omega$. In particular, we have $Dv(\varepsilon x) \to (p_-)e_d$ in $L^1_{loc}(\overline{\Omega})$ as $\varepsilon \to 0$, but the convergence does not hold in $L^\infty_{loc}(\overline{\Omega})$.

7 Proof of Theorem 1.6: a notion of strong trace

We start with the following result.

Proposition 7.1 (Strong convergence of the blow-up gradient at the boundary)
We work under the assumptions of Theorem 1.3 with (λ, P) replaced by (λ^0, P^0). In particular, there exists $(\lambda^0, P^0) \in \mathbb{R} \times \mathbb{R}^d$ such that for $X = (t, x)$, we have

$$u(X) = u^0(X) + o(|X|) \quad \text{as} \quad X \to (0,0) \quad \text{in} \quad \mathbb{R} \times \overline{\Omega}, \quad \text{with} \quad u^0(t,x) := \lambda^0 t + P^0 \cdot x$$

and u is a Lipschitz continuous viscosity solution of $u_t + H(t, x, Du) = 0$ in a neighborhood of $(0,0)$ in $\mathbb{R} \times \Omega$, with $H(0, 0, \cdot)$ strictly convex.
Then for $\varepsilon > 0$, the blow-up $u^\varepsilon(t,x) := \varepsilon^{-1}\{u(\varepsilon t, \varepsilon x) - u(0,0)\}$ enjoys the following strong convergence of its time-space gradient

$$(7.1) \qquad (u^\varepsilon_t, Du^\varepsilon) \to (\lambda^0, P^0) \quad \text{in} \quad L^1_{loc}(\mathbb{R} \times \Omega; \mathbb{R}^{1+d}) \quad \text{as} \quad \varepsilon \to 0.$$

Proof of Proposition 7.1
Step 1: preliminaries
Because Du^ε is uniformly bounded, applying Lemma 10.3 on Young measures in the Appendix, we can extract a subsequence (still denoted by ε) and find a family of probability measures ν_X on $\mathbb{R}^d$ for $X = (t, x) \in \mathbb{R} \times \overline{\Omega}$ such that for any continuous function $F : \mathbb{R}^d \to \mathbb{R}$, we have

$$F(Du^\varepsilon) \to \bar{F} := \langle \nu_X, F \rangle = \int_{\mathbb{R}^d} F(P) d\nu_X(P) \quad \text{in} \quad L^\infty_{loc}(\mathbb{R} \times \overline{\Omega}) \quad \text{weak} - *.$$

Because u is Lipschitz continuous, we have in particular almost everywhere with $0 := (0,0)$

$$u^\varepsilon_t + H(\varepsilon X, Du^\varepsilon) = 0, \quad u^0_t + H(0, Du^0) = 0.$$

Step 2: limit of a nonnegative integral
We set

$$0 \leq \Psi(P) := H(0, P) - H(0, P^0) - (P - P^0) \cdot DH(0, P^0) \quad \text{with} \quad P^0 = Du^0$$

where the nonnegativity of Ψ follows from the convexity of $H(0, \cdot)$. Now for any test function $0 \leq \varphi \in C^\infty_c(\mathbb{R} \times \overline{\Omega})$, we consider the following integral

$$0 \leq I^\varepsilon := \int_{\mathbb{R}\times\Omega} \varphi(X) \; \Psi(Du^\varepsilon(X)) \; dX.$$

[2]Indeed, in Step 2 of the proof of Proposition 6.1, notice that if $x_\varepsilon \in \partial\Omega_+$, then we still have a sequence $\alpha_k \to 0$ but such that $u^{\alpha_k \varepsilon} \to (p_+)x_d$. On the contrary, we have to choose a sequence $\beta_{\varepsilon,k} \to +\infty$ such that $(\beta_{\varepsilon,k})\varepsilon \to 0$ such that $u^{\beta_{\varepsilon,k}\varepsilon} \to (p_-)x_d$. We still have $p_- = p^1 < p^2 = p_+$. Now the condition to get a contradiction would be $\alpha_k \varepsilon > \beta_{\varepsilon,k}\varepsilon$ which is not the case.

On the other hand, setting $A^\varepsilon := -\{H(\varepsilon X, Du^\varepsilon) - H(0, Du^\varepsilon)\}$, $B^\varepsilon := -(Du^\varepsilon - Du^0)\cdot DH(0,Du^0)$, we get

$$\begin{aligned} I^\varepsilon &= \int_{\mathbb{R}\times\Omega} \varphi\left\{A^\varepsilon + B^\varepsilon + H(\varepsilon X, Du^\varepsilon) - H(0,Du^0)\right\}\ dX \\ &= \int_{\mathbb{R}\times\Omega} \left\{\varphi\left(A^\varepsilon + B^\varepsilon\right) + \varphi_t(u^\varepsilon - u^0)\right\}\ dX \end{aligned}$$

where we have used the PDE for the last line. On the one hand, from the local uniform continuity of H, we get $A^\varepsilon \to 0$ locally uniformly and then $\int\ \varphi A^\varepsilon\ dX \to 0$. From the strong uniform convergence of u^ε towards u^0, we also get $\int\ \varphi_t(u^\varepsilon - u^0)\ dX \to 0$. On the other hand, we have

$$\int_{\mathbb{R}\times\Omega} \varphi B^\varepsilon\ dX = \int_{\mathbb{R}\times\Omega} (u^\varepsilon - u^0)\ D\varphi\cdot DH(0,Du^0)\ dX - \int_{\mathbb{R}\times\partial\Omega} \varphi(u^\varepsilon - u^0)\ n\cdot DH(0,Du^0)$$

where $n = -e_d$ is the outward unit normal to Ω. This shows that we also get $\int \varphi B^\varepsilon\ dX \to 0$. Therefore we get

$$I^\varepsilon \to 0 = I^0 := \int_{\mathbb{R}\times\Omega} \varphi(X)\overline{\Psi}(X)\ dX$$

with $0 \le \overline{\Psi}(X) = \int_{\mathbb{R}^d}\ \Psi(P)\ d\nu_X(P)$ for a.e. $X \in \mathbb{R}\times\Omega$, where the nonnegativity of Ψ follows again from the convexity of $H(0,\cdot)$.

Step 3: consequence

Step 1 implies $\varphi\overline{\Psi} = 0$ a.e. for all test function $\varphi \ge 0$. Therefore we get $\overline{\Psi} = 0$ a.e. on $\mathbb{R}\times\Omega$. Now the strict convexity of $H(0,\cdot)$ implies that $\mathrm{supp}(\nu_X) \subset \left\{P^0\right\}$ and then

$$\nu_X(P) = \delta_0(P - P^0) \quad \text{for a.e.} \quad X \in \mathbb{R}\times\Omega.$$

From point iii) of Lemma 10.3, we deduce that

$$Du^\varepsilon \to P^0 = Du^0 \quad \text{in} \quad L^1_{loc}(\mathbb{R}\times\Omega;\mathbb{R}^d)$$

not only for the subsequence, but also for the full sequence $\varepsilon \to 0$, because any limit Young measure is a unique Dirac mass. Finally, writing again $u^\varepsilon_t - u^0_t = A^\varepsilon + \left\{H(0,Du^\varepsilon) - H(0,Du^0)\right\}$ and using the fact that $H(0,\cdot)$ is locally Lipschitz, we get the convergence $u^\varepsilon_t \to u^0_t = \lambda^0$ in L^1_{loc}. This ends the proof of the lemma.

Remark 7.2 (Do we have convergence of the gradient in L^∞?)
In Proposition 7.1, notice that the convergence of the time-space gradient does not hold in $L^\infty_{loc}(\mathbb{R}\times\overline{\Omega})$ in general. See for instance the example in Remark 6.2.

There is still a natural situation where we can say more. Assume moreover that H satisfies (1.2) not only at $X = (t,x) = (0,0)$, but in a neighborhood. Then consider the Legendre-Fenchel transform $\mathcal{L}(X,\xi)$ of $H(X,P)$. Let ω be the modulus of continuity of $\mathcal{L}$ in X and let $\hat{\eta}$ be its (total) modulus of strict convexity $\mathcal{L}(X,P+h) + \mathcal{L}(X,P-h) - 2\mathcal{L}(X,P) \ge \hat{\eta}(|h|)$ (where $\eta(r) = r^{-1}\hat{\eta}(r)$ is a standard modulus of continuity). Then under the following Dini condition

$$\bar{\omega}(\rho) := \int_0^\rho \frac{dr}{r}\ \hat{\eta}^{-1}\circ\omega(r) < +\infty, \tag{7.2}$$

we claim that we have

$$(u^\varepsilon_t, Du^\varepsilon) \to (\lambda^0, P^0) \quad in \quad L^\infty_{loc}(\mathbb{R}\times\Omega;\mathbb{R}^{1+d}) \quad as \quad \varepsilon \to 0.$$

This is in particular the case if H is β-Hölder in X for some $\beta \in (0,1]$ and if H is C^2 in P with $\delta^{-1} \ge D^2_{PP}H \ge \delta > 0$ for some constant $\delta > 0$.

Indeed, if $X_1 = (t_1,x_1) \in \mathbb{R}\times\Omega$ is a point where u has a time-space gradient $(u_t,Du)(X_1) = (\lambda_1,P_1)$, then it is possible to show that there exists some backward characteristic $\gamma^\xi_{X_1}$ with terminal point X_1, with velocity $\frac{d\gamma^\xi_{X_1}}{ds}(s) = \xi(s)$ such that

$$\xi(t_1) := DH(X_1,P_1) \quad \textit{in some formal sense, with} \quad \lambda_1 + H(P_1) = 0 \tag{7.3}$$

This formal sense can be made rigorous (see for instance [17]) with help of the Dini condition which controls the characteristic velocity ξ *and implies that* $\left|\xi(t_1) - \tau^{-1}\int_{t_1-\tau}^{t_1}\xi(s)ds\right| \le C\bar{\omega}(C\tau)$*, with a constant* $C > 0$ *uniform with respect to* X_1 *in a neighborhood of* $(0,0)$*. From the representation formula, we also have* $u(X_1) = u(t_2, \gamma^{\xi}_{X_1}(t_2)) + \int_{t_2}^{t_1} \mathcal{L}(\gamma^{\xi}_{X_1}(s), \dot{\gamma}^{\xi}_{X_1}(s))ds$ *with* $t_2 := t_1 - \tau$*, while the point* $(t_2, \gamma^{\xi}_{X_1}(t_2))$ *does not touch the boundary* $\mathbb{R}\times\partial\Omega$*. This is in particular true for* $\tau > 0$ *small enough depending on* $dist(x_1, \partial\Omega) > 0$ *(and other constants like the Lipschitz constant of* u*).*

All together, if we now consider a compact set $K \subset \mathbb{R}\times\Omega$*, it is easy to show that the uniform convergence of* $u^\varepsilon(X) = \varepsilon^{-1}u(\varepsilon X)$ *towards* u^0 *on* K*, and the uniform control of the characteritic velocities imply the convergence of the characteristics with constant velocity* $\xi^0 = DH(0, Du^0) = \lim_{\varepsilon\to 0} DH(\varepsilon X_1^\varepsilon, Du^\varepsilon(X_1^\varepsilon))$ *for any sequence of points* $X_1^\varepsilon \in K$ *where* u^ε *has a time-space derivative. This implies* $|Du^\varepsilon - Du^0|_{L^\infty(K)} \to 0$ *as* $\varepsilon \to 0$*. The convergence of the time derivatives then follows from the PDE.*

In the spirit of BV-regularity (see theorem 5.7 on page 208 in Evans, Gariepy [12]), we give the following result of independent interest.

Corollary 7.3 (Local property of the gradient)
Assume that u *is a Lipschitz continuous viscosity solution of the Hamilton-Jacobi equation* $u_t + H(t,x,Du) = 0$ *on a domain* $D \subset \mathbb{R}\times\mathbb{R}^d$ *with a continuous Hamiltonian* $H : \overline{D}\times\mathbb{R}^d \to \mathbb{R}$ *such that the convex map* $P \mapsto H(t,x,P)$ *satisfies (1.2) for all* $(t,x) \in \partial D$*.*
Consider an mesurable subset $\Gamma \subset \partial D$ *which is a Lipschitz continuous graph in the pure space direction* e_d*. Then we have for* $\hat{D}u := (u_t, Du)$

$$\hat{D}u(X) = \lim_{\varepsilon\to 0} \frac{1}{|B_\varepsilon(X)\cap D|}\int_{B_\varepsilon(X)\cap D} \hat{D}u(Y)\ dY \quad \text{for } \mathcal{H}^d\text{- a.e. } X = (t,x) \in \Gamma \subset \partial D \tag{7.4}$$

where $\mathcal{H}^d$ *is the Hausdorff d-dimensional measure on* $\mathbb{R}\times\mathbb{R}^d$*.*

Proof of Corollary 7.3
Because Γ is Lipschitz continuous, it has a tangential hyperplane for $\mathcal{H}^d$- a.e. $X \in \Gamma$. Given such point $X \in \Gamma$, up to a change of variables, we can also assume that the tangential hyperplane at X is orthogonal to e_d. Then it is straightforward to adapt the proof of Proposition 7.1 to still deduce the convergence (7.1) of the blow-up at X. Because the time-space gradient is bounded, this convergence implies the one of (7.4) at such X, and this ends the proof.

Proof of Theorem 1.6
Step 1: preliminaries
For $X' = (t,x') \in B_1 \subset \mathbb{R}\times\mathbb{R}^{d-1}$ and $(X',0) \in \Gamma = B_1 \times \{0\}$, we consider the tangential gradient $(u_t, D_{x'}u)(X',0)$ of the Lipschitz continuous function $X' \mapsto u(X',0)$. From Rademacher's theorem, we know that the tangential gradient exists a.e.. Now from Theorem 1.3 we deduce that $(u_t, Du)(X',0) = (\lambda, P)$ is well defined for a.e. X', as a right derivative on the set $\{x_d \ge 0\}$, i.e. that we have for $Y = (s,y)$

$$u((X',0)+Y) - u(X',0) = \lambda s + P\cdot y + o(|Y|) \quad \text{for all} \quad Y \in \mathbb{R}\times\overline{\Omega}, \quad (X',0), (X',0)+Y \in C_+ \cup \Gamma$$

which shows (1.5).
Similarly, $(u_t, Du)(X', y_d)$ is well defined for a.e. $X' \in B_1$ and for all $y_d \in (0,1)$ as a right derivative on the set $\{x_d \ge y_d\}$ (and also well defined for a.e. $X' \in B_1$ and for a.e. $y_d \in (0,1)$ as a standard derivative). We now set the (right) space gradients

$$P^0(X') := Du(X',0), \quad P(X',x_d) := Du(X',x_d) \quad \text{for a.e.} \quad X' \in B_1 \quad \text{and all} \quad x_d \in (0,1).$$

Step 2: rescaling and extraction of Young measure
For $\varepsilon > 0$, we consider the anisotropic rescaling

$$P^\varepsilon(X',x_d) := P(X', \varepsilon x_d)$$

Because Du is bounded, applying Lemma 10.3 on Young measures in the Appendix, we can extract a subsequence (still denoted by ε) and find a family of probability measures ν_X on $\mathbb{R}^d$ for $X = (X',x_d) \in C_+ = B_1 \times (0,1)$ such that for any continuous function $F : \mathbb{R}^d \to \mathbb{R}$, we have

$$F(P^\varepsilon) \to \bar{F} := \langle \nu_X, F\rangle = \int_{\mathbb{R}^d} F(P)d\nu_X(P) \quad \text{in} \quad L^\infty_{loc}(C_+) \quad \text{weak} - *.$$

Step 3: characterization of the Young measure
Our goal is to show that the limit Young measure ν_X is a Dirac mass of center $P^0(X')$.
Let us consider a test function $0 \le \varphi \in C_c^\infty(B_1)$, and let us consider the following integral which is well defined for $\varepsilon > 0$ small enough (because φ has compact support in the unit ball) for $Y = (Y', y_d)$

$$J^\varepsilon := |B_1|^{-1} \int_{B_1} \varphi(X') \left\{ \int_{B_1\times(0,1)} |P((X',0)+\varepsilon Y) - P^0(X')| \; dY \right\} \; dX'.$$

From Proposition 7.1, we have for the special case $X = (X', 0)$ for almost every $X' \in B_1$

$$P(X+\varepsilon Y) = Du_X^\varepsilon(Y) \to Du(X) = P^0(X') \quad \text{in} \quad L^1_{loc}(\mathbb{R}\times\overline{\Omega}) \quad \text{with} \quad u_X^\varepsilon(Y) := \varepsilon^{-1}\left\{u(X+\varepsilon Y) - u(X)\right\}.$$

Hence on the one hand, from the Lebesgue dominated convergence theorem, we get $J^\varepsilon \to 0$. On the other hand, consider the change of variable $Z' = X' + \varepsilon Y'$. We get

$$J^\varepsilon = |B_1|^{-1} \int_{B_1\times(0,1)} \left\{ \int_{B_1} \varphi(Z'-\varepsilon Y') \; |P^\varepsilon(Z', y_d) - P^0(Z'-\varepsilon Y')| \; dZ' \right\} \; dY.$$

We now introduce

$$\hat{J}^\varepsilon := |B_1|^{-1} \int_{B_1\times(0,1)} \left\{ \int_{B_1} \varphi(Z') \; |P^\varepsilon(Z', y_d) - P^0(Z')| \; dZ' \right\} \; dY$$

which satisfies

$$\hat{J}^\varepsilon - J^\varepsilon \to 0$$

from the continuity of translations in L^1 for the term P^0 and from the uniform continuity for the factor φ. Hence $\hat{J}^\varepsilon \to 0$ with (for $z_d = y_d$ and $Z = (Z', z_d)$)

$$\hat{J}^\varepsilon = \int_{B_1\times(0,1)} \varphi(Z') \; |P^\varepsilon(Z) - P^0(Z')| \; dZ.$$

By density of continuous functions in $L^1(B_1)$, it is easy to justify by aproximations (of P^0) that we have (for a subsequence still denoted by ε) the following limit

$$\hat{J}^\varepsilon \to 0 = \hat{J}^0 := \int_{B_1\times(0,1)} \varphi(Z') \left\{ \int_{\mathbb{R}^d} |P - P^0(Z')| \; d\nu_Z(P) \right\} \; dZ.$$

Because $\varphi \ge 0$, this implies $\text{supp}(\nu_Z) \subset \left\{P^0(Z')\right\}$, and then

$$\nu_Z(P) = \delta_0(P - P^0(Z')) \quad \text{for a.e.} \quad Z \in B_1 \times (0,1).$$

Step 4: conclusion
From iii) of Lemma 10.3 on Young measures, we deduce from the uniqueness and the expression of ν_Z, that we have

$$P^\varepsilon \to P^0 \quad \text{in} \quad L^1(B_1 \times (0,1))$$

not only for the extracted subsequence, but also for the whole sequence ε (even for a continuous parameter $\varepsilon \to 0$). Finally, the convergence of $u_t(X', \varepsilon x_d)$ follows from the PDE, the uniform bounds on the gradient, the L^1 convergence of the gradient P^ε, and the local uniform continuity of H. This shows convergence (1.6) of the time-space gradient. This ends the proof of the theorem.

8 Application to junctions

We now work in dimension $d = 1$ and consider a junction. A junction can be viewed as the set of N distinct copies ($N \ge 1$) of the half-line which are glued at the origin. For $\alpha = 1, ..., N$, each branch J_α is assumed to be isometric to $[0, +\infty)$ and

(8.1) $$J = \bigcup_{\alpha=1,\dots,N} J_\alpha \quad \text{with} \quad J_\alpha \cap J_\beta = \{0\} \quad \text{for} \quad \alpha \ne \beta, \quad J^* := J\backslash\{0\}$$

where the origin 0 is called the junction point. For points $x, y \in J$, $d(x,y)$ denotes the geodesic distance on J defined as

$$d(x,y) := \begin{cases} |x-y| & \text{if } x, y \text{ belong to the same branch,} \\ |x|+|y| & \text{if } x, y \text{ belong to different branches.} \end{cases}$$

For a smooth real-valued function u defined on J, the quantity $\partial_\alpha u(x)$ denotes the (spatial) derivative of u at $x \in J_\alpha$. Then the "gradient" of u is defined as follows

$$(8.2) \qquad u_x(x) := \begin{cases} \partial_\alpha u(x) & \text{if} \quad x \in J_\alpha^* := J_\alpha \setminus \{0\} \\ (\partial_1 u(0), \dots, \partial_N u(0)) & \text{if} \quad x = 0. \end{cases}$$

To each branch $\alpha = 1, \dots, N$, is associated a concave Hamiltonian $\hat{H}_\alpha : \mathbb{R} \to \mathbb{R}$ such that the convex function $H_\alpha(p) := -\hat{H}_\alpha(-p)$ satisfies (1.2) for $d = 1$. For $T > 0$ and a given function $A : (0,T) \to \mathbb{R}$, we now consider the following junction problem

$$(8.3) \qquad \begin{cases} u_t + \hat{H}_\alpha(u_x) = 0 & \text{on} \quad (0,T) \times J_\alpha^*, \qquad \text{for} \quad \alpha = 1, \dots, N \\ u_t + \hat{F}_A(u_x) = 0 & \text{on} \quad (0,T) \times \{0\} \end{cases}$$

with
(8.4)

$$\hat{F}_A(p_1, \dots, p_N) := \min \left\{ A, \min_{\alpha=1,\dots,N} \hat{H}_\alpha^-(p_\alpha) \right\} \quad \text{and the nonincreasing function} \quad \hat{H}_\alpha^-(p) := \sup_{q \ge p} \hat{H}_\alpha(q)$$

where the boundary condition means explicitly

$$u_t(t,0) + \min \left\{ A(t), \min_{\alpha=1,\dots,N} \hat{H}_\alpha^-(\partial_\alpha u(t,0)) \right\} = 0 \quad \text{for} \quad t \in (0,T).$$

Then we have the following result.

Proposition 8.1 (Junction: from Hamilton-Jacobi to Conservation Law)
Let $N \ge 1$ and for each $\alpha = 1, \dots, N$, let us consider a concave function $\hat{H}_\alpha$ with the map $p \mapsto -\hat{H}_\alpha(-p)$ satisfying (1.2) for $d = 1$. Let $T > 0$ and let J be the junction defined in (8.1). Assume that $A \in C((0,T);\mathbb{R})$. With notation just above, if u is a Lipschitz continuous viscosity solution of (8.3) (in the sense of Imbert, Monneau [14]), then $v := u_x \in L^\infty((0,T) \times J^*;\mathbb{R})$ defined in (8.2), is such that $v(t,0) = (v^1(t), \dots, v^N(t)) \in L^1((0,T);\mathbb{R}^N)$ with the strong trace property

$$(8.5) \qquad \lim_{\varepsilon \to 0^+} \sum_{\alpha=1,\dots,N} \int_{(0,T)\times(0,1)_\alpha} |v(t,\varepsilon x) - v^\alpha(t)| \, dtdx = 0 \quad \textit{with} \quad (0,1)_\alpha := J_\alpha^* \cap (0,1)$$

and v is an entropy solution of

$$(8.6) \qquad \begin{cases} v_t + \partial_x(\hat{H}_\alpha(v)) = 0 & \textit{on} \quad (0,T) \times J_\alpha^* \qquad \textit{for} \quad \alpha = 1, \dots, N \\ v(t,0) \in \mathcal{G}_{A(t)} & \textit{for a.e. } t \in (0,T) \end{cases}$$

with

$$\mathcal{G}_A = \left\{ P = (p_1, \dots, p_N) \in \mathbb{R}^N, \quad \hat{F}_A(P) = \hat{H}_\alpha(p_\alpha) \quad \textit{for all} \quad \alpha = 1, \dots, N \right\}$$

where $\hat{F}_A$ is defined in (8.4).

Remark 8.2 (When uniqueness of the solution is known?)
Notice that for initial data u_0 uniformly continuous on J, the uniqueness of the viscosity solution u of (8.3) is known (see [14]). On the contrary, even if the initial data for v is $v_0 := \partial_x u_0$ and belongs to $L^1(J)$, up to our knowledge, the uniqueness of the entropy solution v to (8.6) is not known for continuous functions A which are not constant, or if $N \ge 3$. In the case $N = 1, 2$, then the Hamilton-Jacobi germ $\mathcal{G}_A$ is also a L^1-contraction germ, which is not the case for $N \ge 3$. Notice that uniqueness of v is proven in Andreianov, Karlsen, Risebro [1] in the special case $N = 1, 2$ with $A = const$ (see also Musch, Fjordholm, Risebro [18], where the work [1] has been generalized to a theory of L^1-contraction germs in case of junctions with $N \ge 1$ branches).

Remark 8.3 *Notice that even if our notion of strong trace of v in (8.5) is less strong than the one of Panov [19], it is sufficient to make sense (for $N = 1, 2$ and $A = const$) of a weak formulation of (8.6) as in [18], which allows to recover L^1-contraction and uniqueness properties of the solutions v, given some suitable initial data.*

Moreover, if necessary, using Remark 1.8 in the case where the Hamiltonian is independent on (t, x), we can also recover a stronger convergence as in (1.7).

Proof of Proposition 8.1
Step 1: sketch the proof that $v = u_x$ is an entropy solution
Part of the following argument is due to P. Cardaliaguet. We only sketch the proof that $v := u_x$ is an entropy solution of the conservation law, because this is not the main contribution of our result. The key point is the finite speed of propagation both for the Hamilton-Jacobi equation (HJ) and for the conservation law (CL), which makes the result only local in space and time. We can then extend the PDEs from one branch to the whole line $\mathbb{R}$ and add ε-viscosity terms both in HJ equation and in CL equation, with new respective solutions $\tilde{u}^\varepsilon$ and $\tilde{v}^\varepsilon$ which satisfy $\tilde{v}^\varepsilon = \tilde{u}^\varepsilon_x$. It is then classical that $\tilde{u}^\varepsilon \to \tilde{u}$, and $\tilde{v}^\varepsilon \to \tilde{v}$, where $\tilde{u}$ and $\tilde{v}$ are respectively viscosity solution of the HJ equation and entropy solution of the CL equation. Finally, the finite speed of propagation implies that $u = \tilde{u}$ and $v = \tilde{v}$ on a cone of dependence for some suitable initial data. This can be done for any BV initial data for v. The case of L^∞ initial data then follows by L^1 approximation and localization of the L^1-contraction.

Because this reasoning can be done at any time-space scale (for small or large cones, possibly translated and included in $(0, T) \times J^*_\alpha$), we deduce that $v := u_x$ is an entropy solution on each branch $(0, T) \times J^*_\alpha$.
The reader can also consult Cardaliaguet, Forcadel, Girard, Monneau [9] for a proof of this result where the vanishing viscosity approximation is replaced by a numerical scheme approximation (under the stronger assumption $(\hat{H}_\alpha)'' \le -\delta < 0$).
Step 2: proof that $u_x(t, 0) \in \mathcal{G}_{A(t)}$ for a.e. t
We know from Theorem 1.3 that $(\lambda, P) = (u_t, u_x)(t, 0) \in \mathbb{R} \times \mathbb{R}^N$ is defined for a.e. time $t \in (0, T)$, with $P = (p_1, \dots, p_N)$. This implies $\lambda + \hat{F}_{A(t)}(P) = 0$. On the other hand, from Lemma 2.9 for supersolutions (resp. Lemma 2.10 for subsolutions) in [14], we get for the critical slope p_α that $\lambda + \hat{H}_\alpha(p_\alpha) \ge 0$ (resp. $\lambda + \hat{H}_\alpha(p_\alpha) \le 0$, i.e. $\lambda + \hat{H}_\alpha(p_\alpha) = 0$. This implies that $u_x(t, 0) \in \mathcal{G}_{A(t)}$ for a.e. $t \in (0, T)$.
Step 3: conclusion
Now setting $p_\alpha = p_\alpha(t)$ with $u_x(t, 0) = P = (p_1, \dots, p_N)$, and using Theorem 1.6, we see that

$$\lim_{\varepsilon \to 0^+} \sum_{\alpha=1,\dots,N} \int_{(0,T)\times(0,1)_\alpha} |v(t, \varepsilon x) - p_\alpha(t)| \, dtdx = 0 \tag{8.7}$$

which shows (8.5). Hence the trace $v(t, 0)$ of v at the junction point $(t, 0)$ is $u_x(t, 0)$ for a.e. time $t \in (0, T)$. From Step 2, we then deduce the second line of (8.6). This ends the proof.

Recall that Panov in [19] proves under the assumptions of Proposition 8.1, and with notation $p_\alpha(t)$ in Step 3 of its proof, that bounded entropy solutions v of (8.6) have strong traces in the following sense for each $\alpha = 1, \dots, n$

$$\text{ess} \lim_{J^{\alpha *} \ni x \to 0^+} \int_0^T |v(t, x) - p_\alpha(t)| \, dt = 0$$

which is a stronger notion of convergence than (8.7). Still this notion alone does not imply the existence of a normal derivative $u_x(t, 0)$ for almost every time $t \in (0, T)$, as shows the following counter-example.

Proposition 8.4 (Counter-example: strong trace of the gradient is weaker than normal derivative) *Consider the torus $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. Then there exists a 1-Lipschitz continuous map $u : \mathbb{T} \times [0, 1) \to [0, 1]$ with*

$$u = 0 \quad \text{on} \quad \mathbb{T} \times \{0\}, \quad \text{and} \quad \text{ess} \lim_{(0,1) \ni x \to 0^+} \int_{\mathbb{T}} |u_x(t, x)| \, dt = 0 \tag{8.8}$$

such that

$$\limsup_{(0,1) \ni x \to 0^+} \frac{u(t, x) - u(t, 0)}{x} = \frac{1}{3} \quad \text{for all} \quad t \in \mathbb{T}$$

Proof of Proposition 8.4
Step 1: preliminaries with simplifications
For $\varepsilon := x \in (0,1)$, we set

$$v^\varepsilon(t) := u_x(t,\varepsilon)$$

with $v^\varepsilon \in L^\infty(\mathbb{T})$ which is well defined for almost every $\varepsilon \in (0,1)$. If we forget that v^ε is associated to the gradient of Lipschitz function u, we can find a simple example of a sequence of functions satisfying precisely

$$|v^\varepsilon|_{L^1(\mathbb{T})} \to 0 \quad \text{for all} \quad \varepsilon \to 0.$$

If we choose $v^\varepsilon(t) := \psi_\varepsilon(t-\varepsilon^{-1})$, with $\psi_\varepsilon(t) := \begin{cases} 1 & \text{for} \quad t \in [0,\varepsilon] \mod \mathbb{Z}, \\ 0 & \text{otherwise,} \end{cases}$ then for all $t \in \mathbb{T}$, the quantity $v^\varepsilon(t)$ has no limit as $\varepsilon \to 0^+$.
Step 2: an auxiliary function
Now we want to build a similar sequence, but clearly associated to a Lipschitz continuous function $u(t,x)$, which is Lipschitz in both variables. We will build u as follows

$$u(t,x) := \sum_{k\in\mathbb{Z}} \tilde{u}(t+k,x) \tag{8.9}$$

where $\tilde{u} : \mathbb{R}^2 \to \mathbb{R}$ is an auxiliary function that we design in the present step.
To this end, we first consider a decreasing sequence of positive numbers $\delta_k \to 0$ as $k \to +\infty$ such that $\sum_{k\geq 1} \delta_k = +\infty$ and set $S_j := \sum_{k=1}^{j} \delta_k$ for $j \geq 1$, and $S_0 := 0$. We set $r_j := 2^{-j}$ for $j \geq 1$. For $X = (t,x)$ and $Y = (s,y)$ with $X, Y \in \mathbb{R}^2$, we define for $\eta > 0$

$$\tilde{u}(X) := \max\left\{0, \sup_Y \{w(Y) - |X-Y|\}\right\} \quad \text{with} \quad w(t,x) := \eta \sum_{j\geq 1} r_j \cdot 1_{[S_{j-1},S_j]\times\{r_j\}}(t,x).$$

For the closed ball $\bar{B}_{r_j} = \bar{B}_{r_j}(0) \subset \mathbb{R}^2$, the support satisfies $\text{supp}(\tilde{u}) \supset \bigcup_{j\geq 1} \mathcal{N}_j$ with $\mathcal{N}_j := \bar{B}_{\eta r_j} + [S_{j-1}, S_j] \times \{r_j\}$ with equality when we have $\mathcal{N}_j \cap \mathcal{N}_k = \emptyset$ for all $j \neq k$. This arises for

$$\begin{aligned} 0 &< \{r_j - \eta r_j\} - \{r_{j+1} + \eta r_{j+1}\} \\ &= (1-\eta) r_j - \tfrac{r_j}{2}(1+\eta) \\ &= \tfrac{r_j}{2}(1-3\eta) \end{aligned}$$

i.e. for $\eta < \frac{1}{3}$. By construction, the Lipschitz constant of $\tilde{u}$ is 1, and $\tilde{u} = 0$ on $\mathbb{R} \times \{0\}$. Moreover

$$\frac{\tilde{u}(t,r_j) - \tilde{u}(t,0)}{r_j} = \eta > 0 \quad \text{for all} \quad t \in [S_{j-1}, S_j], \quad j \geq 1 \tag{8.10}$$

Moreover, we have $|\tilde{u}_x| = 1$ a.e. on $\text{supp}(\tilde{u})$, and then

$$\int_{\mathbb{R}} |\tilde{u}_x(t,x)| dt \leq \begin{cases} 2\eta r_j + \delta_j & \text{for a.e.} \quad x \in [(1-\eta)r_j), (1+\eta)r_j], \quad j \geq 1 \\ 0 & \text{otherwise} \end{cases} \tag{8.11}$$

Step 3: properties of u
Recall that the function u is defined in (8.9), and we want to check that it satisfies all the conditions for a counter-example. For δ_1 small enough, we have $\text{diam}(\mathcal{N}_k) \leq \delta_k + \eta r_k \leq 1$ (indeed $\delta_1 \leq \frac{2}{3}$ is admissible). For $\mathbb{Z}^* = \mathbb{Z} \backslash \{0\}$, we then deduce that $\mathcal{N}_k \cap (\mathbb{Z}^*(0,1) + \mathcal{N}_j) = \emptyset$ for all $k, j \geq 1$, which implies that the Lipschitz constant of u is still 1. We also have $u = 0$ on $\mathbb{T} \times \{0\}$, and (8.10) implies that

$$\frac{u(t,r_j) - u(t,0)}{r_j} = \eta > 0 \quad \text{for all} \quad t \in [S_{j-1}, S_j] \mod \mathbb{Z}$$

and (8.11) implies that $\int_{\mathbb{T}} |u_x(t,x)| dt = \int_{\mathbb{R}} |\tilde{u}_x(t,x)| dt \to 0$ for a.e. $x \to 0^+$.

We can finally pass to the limit $\eta = \frac{1}{3}$, and preserve all desired properties of u. This ends the proof of the proposition.

9 Proof of Proposition 1.2: a counter-example

Recall that the Liouville-type result (Theorem 1.1) works for suitable convex Hamiltonians. Equivalently, we can formulate it with concave Hamiltonians (using the change of variable $H(P) \to -H(-P) =: f(P)$). Our preference for this change originates from our intuition/knowledge from traffic flow theory with concave fluxes. In this section we work in dimension $d = 1$ and consider HJ equation for $x = x_1$

$$\begin{cases} u_t + f(u_x) = 0 & \text{on} \quad \mathbb{R} \times (0, +\infty) \\ u = 0 & \text{on} \quad \mathbb{R} \times \{0\}\,. \end{cases} \tag{9.1}$$

The reference case where the Liouville-type result applies is now when f is suitably concave.

In this section, we present a case where f is not concave and we will build a globally Lipschitz continuous solution $u(t, x)$ which is not one-dimensional (as a function of a linear combination of t and x). This provides a counter-example in dimension $d = 1$ to Theorem 1.1. The counter-example is obtained on the half-space just by restriction of a solution constructed on the whole space. The proof of Proposition 1.2 is given at the very end of the section.

The solution u to HJ equation is obtained by integration in space of a solution v to the associated conservation law equation

$$v_t + f(v)_x = 0 \quad \text{on} \quad \mathbb{R} \times \mathbb{R} \tag{9.2}$$

where as usual $f(v)_x = \partial_x(f(v))$.

We introduce the letters S, R, L for Shock, Right, Left, and assume the following condition on f

$$\begin{cases} \text{there exists } \rho_S < \rho_R < \rho_L & \text{and a constant } \delta > 0 & \text{such that} \\ f : [\rho_S, \rho_L] \to \mathbb{R} & \text{is } C^2 & \\ \dfrac{f(\rho_S) - f(\rho_L)}{\rho_S - \rho_L} = f'(\rho_S) & =: \xi_S < 0 & \textbf{(tangential line)} \\ f'' \leq -\delta & \text{on} \quad [\rho_S, \rho_R] & \textbf{(concavity)} \\ f \leq f(\rho_R) = f(\rho_L) & \text{on} \quad [\rho_R, \rho_L] & \textbf{(horizontal chord)}. \end{cases} \tag{9.3}$$

Remark 9.1 *Notice that Lemmata 9.2 and 9.3 below do work also if the C^2 regularity of f on $[\rho_S, \rho_L]$ is replaced by the following. We only assume f to be C^1 on $[\rho_S, \rho_L]$ with independently both restrictions $f_{|[\rho_S,\rho_R]}$ and $f_{|[\rho_R,\rho_L]}$ assumed to be C^2 with furthermore $(f_{|[\rho_S,\rho_R]})'' \leq -\delta$ on $[\rho_S, \rho_R]$ and $(f_{|[\rho_R,\rho_L]})'' \geq \delta$ on $[\rho_S, \rho_L]$, with then f not C^2 at ρ_R.*

Then we have the following result.

Lemma 9.2 (Existence of a global solution to the conservation law)
Assume that f satisfies (9.3). Then there exists a bounded entropy solution v of (9.2), homogeneous of degree zero, i.e. satisfying

$$v(t, x) = V(\varepsilon t, \varepsilon x) \quad \text{on} \quad \mathbb{R}^2, \quad \text{for all} \quad \varepsilon > 0$$

given by

$$v(t, x) = \begin{cases} \rho_L & \text{on} & \{x \leq 0\} \cup \{\, 0 < x < \xi_S t \quad \text{with} \quad t < 0 \,\} \\ \rho(\dfrac{x}{t}) & \text{on} & \{\, \xi_S t \leq x \leq \xi_R t \quad \text{with} \quad t < 0, \quad x > 0 \,\} \\ \rho_R & \text{on} & \{x > 0, \quad t \geq 0\} \cup \{\, \xi_R t < x \quad \text{with} \quad t < 0 \,\} \end{cases}$$

where $\xi_\alpha = f'(\rho_\alpha) < 0$ for $\alpha = R, S$ and the function $\rho : [\xi_R, \xi_S] \to [\rho_S, \rho_R]$ is C^1 and defined by $\rho := (f'_{|[\rho_R,\rho_S]})^{-1}$.

Proof of Lemma 9.2
The result can be deduced from Lemma 9.3 below, because space derivative of global viscosity solutions of HJ equations are known to be entropy solutions of the associated conservation law. For sake of completness and also because it is probably more natural, we provide a detailed direct proof.
Let us define

$$\left\{\begin{array}{l} L_0 := \{x=0, \quad t>0\} \\ L_S := \{t<0, \quad x=\xi_S t\} \\ L_R := \{t<0, \quad x=\xi_R t\}\,. \end{array}\right. \tag{9.4}$$

The function v is homogeneous of degree zero, bounded and C^1 except on the set $\Sigma := \{(0,0)\}\cup L_0\cup L_S\cup L_R$. Where it is C^1, it is straightforward to check that v is a solution. Moreover, it is easy to check that v is continuous on the half line L_R, and discontinuous across L_0 and L_S which are two shock half lines, and v is also discontinuous at the origin $(0,0) = \overline{L}_0 \cap \overline{L}_S$. In such a case, except for the origin, each shock arises along a C^1 curve. Then, in order to check that v is an entropy solution (even for non concave flux f), it is known (see (2.11) on page 41 in Serre [20]) and it is easy to verify that it is sufficient for each shock of Rankine-Hugoniot velocity c (defined below), of left value v_l and right value v_r, to satisfy the classical Oleinik entropy condition[3]

$$\frac{f(p)-f(v_l)}{p-v_l} \ge c = \frac{f(v_r)-f(v_l)}{v_r-v_l} \ge \frac{f(v_r)-f(p)}{v_r-p} \quad \left\{\begin{array}{lll} \text{for all} & p\in(v_l,v_r) & \text{if} \quad v_l<v_r \\ & & \\ \text{for all} & p\in(v_r,v_l) & \text{if} \quad v_l>v_r. \end{array}\right. \tag{9.5}$$

Notice here that the two inequalities in (9.5) are equivalent because of Rankine-Hugoniot relation. On L_0 we have $v_l=\rho_L$, $v_r=\rho_R$, while on L_S we have $v_l=\rho_L$, $v_r=\rho_S$. In each case, it is easy to check (9.5). We conclude that v is an entropy solution and this ends the proof of the lemma.

We are now interested in the associated Hamilton-Jacobi equation

$$u_t + f(u_x) = 0 \quad \text{on} \quad \mathbb{R}\times\mathbb{R}. \tag{9.6}$$

Lemma 9.3 (Snowdrift on the roof)
Assume that f satisfies (9.3) with zero common value $f(\rho_R)=f(\rho_L)=0$. Then there exists a viscosity solution u of (9.6), homogeneous of degree one, i.e. satisfying

$$u(t,x) = \varepsilon^{-1}U(\varepsilon t,\varepsilon x) \quad \textit{on} \quad \mathbb{R}^2, \quad \textit{for all} \quad \varepsilon>0$$

which is globally Lipschitz continuous and given by

$$u(t,x) = \left\{\begin{array}{lllll} \rho_L x & \textit{on} & \{x\le 0\}\cup & \{\,0 \quad < x < \xi_S t & \textit{with} \quad t<0 \qquad\qquad \} \\ t\bar{R}(\frac{x}{t}) & \textit{on} & & \{\,\xi_S t \quad \le x \le \xi_R t & \textit{with} \quad t<0, \quad x>0 \ \} \\ \rho_R x & \textit{on} & \{x>0, \quad t\ge 0\}\cup & \{\,\xi_R t \quad < x & \textit{with} \quad t<0 \qquad\qquad \} \end{array}\right.$$

where $\xi_\alpha = f'(\rho_\alpha)<0$ for $\alpha=R,S$ and the function $\bar{R}:[\xi_R,\xi_S]\to\mathbb{R}$ is C^2 and defined by $\bar{R}' := (f'_{|[\rho_R,\rho_S]})^{-1}$ and $\bar{R}(\xi_R) := \rho_R\xi_R$. Additionally it then satisfies $\bar{R}(\xi_S)=\rho_L\xi_S$.
Moreover we have

$$u(t,0)=0 \quad \textit{for all} \quad t\in\mathbb{R}.$$

Proof of Lemma 9.3
Even if the result could maybe be deduced from Lemma 9.2 with some suitable argument, we find simple and useful to propose a direct proof.
We first notice that the function $\bar{R}(\xi)$ satisfies $\bar{R}'=\rho := (f'_{|[\rho_R,\rho_S]})^{-1}$ and

$$\bar{R} - \xi\bar{R}' + f(\bar{R}') = const \quad \text{on} \quad [\xi_R,\xi_S] \tag{9.7}$$

as it can be checked computing the derivative of the left hand side. The evaluation of the constant at $\xi=\xi_S,\xi_R$ gives

$$\bar{R}(\xi_S) - \rho_S\xi_S + f(\rho_S) = \bar{R}(\xi_R) - \rho_R\xi_R + f(\rho_R) = 0$$

[3] We recover for traffic applications with concave flux f the well-known fact that a shock is entropic if and only if $v_l \le v_r$.

where we have used the definition of $\bar{R}(\xi_R) := \rho_R\xi_R$ in the last equality. Now (9.3) implies $\bar{R}(\xi_S) = \rho_L\xi_S$. Notice that u is C^2 outside $\Sigma := \{(0,0)\} \cup L_0 \cup L_S \cup L_R$ with L_0, L_S, L_R defined in (9.4). Moreover relation (9.7) with $const = 0$ implies that HJ equation is satisfied outside Σ. It is also satisfied on L_R because u is C^1 there. It is also easy to check that $\bar{R}'' < 0$ and that $u(t,x)$ is concave in x for every $t \in \mathbb{R}$. Then any test function φ touching u on $\{(0,0)\} \cup L_0 \cup L_S$ can only touch it from above. We then have to check the viscosity inequalities along the lines (L_0 and L_S) of velocity c and of left gradient v_l and right gradient v_r

$$\varphi_t + f(\varphi_x) \le 0 \quad \text{for all} \quad p_r \le p := \varphi_x \le p_l \quad \text{satisfying} \quad (\partial_t + c\partial_x)(\varphi - u) = 0. \tag{9.8}$$

On L_0 we have $v_l = \rho_L$, $v_r = \rho_R$ and $(\partial_t + c\partial_x)u = 0$ with $c = 0$, while on L_S we have $v_l = \rho_L$, $v_r = \rho_S$ and and $(\partial_t + c\partial_x)u = \bar{R}(\xi_S)$ with $c = \xi_S$. In each case it is easy to check (9.8), using in particular the properties of the graph of f in the second case.
Finally it remains to check the viscosity inequality at the origin $(0,0)$. The property $u(t,0) = 0$ implies $\varphi_t = 0$. On the other hand, fact that $u_x = \rho_L$ on a cone containing $\{x \le 0\}$ and of positive intersection with $\{x > 0\}$ implies that $\varphi_x = \rho_L$. This implies again the desired viscosity inequality. Hence u is a viscosity solution on $\mathbb{R} \times \mathbb{R}$ and this ends the proof of the lemma.

Proof of Proposition 1.2
We can first consider a function f satisfying (9.3) with zero common value $f(\rho_R) = f(\rho_L) = 0$. Additionally, we can require that f is C^∞ instead of C^2 only, and is moreover stricly concave on (ρ_S, ρ_0) and strictly convex on (ρ_0, ρ_L) with $\rho_0 \in (\rho_R, \rho_L)$. We can extend f from $[\rho_S, \rho_L]$ to $\mathbb{R}$, in a C^∞ strictly concave function on $(-\infty, \rho_0)$ and strictly convex function on $(\rho_0, +\infty)$. Finally, up to shift f, we can also assume that $\rho_0 = 0$. Then the result is a straightforward corollary of Lemma 9.3, restricting the global Lipschitz continuous viscosity solution u to the half space $\mathbb{R} \times [0, +\infty)$. This ends the proof of the proposition.

10 Appendix: reminder of some useful results

Consider some open set $\omega \subset \mathbb{R}^d$ and $x_0 \in \omega$ and some lower semi-continuous function $u : \omega \to \mathbb{R}$. We define the subdifferential of u at x_0 on ω as

$$D^- u(x_0) := \sup \{P \in \mathbb{R}^n, \quad u(x) - u(x_0) \ge P \cdot (x - x_0) + o(x - x_0) \quad \text{on} \quad \omega\}$$

which is a compact convex set.

We now consider the following equation:

$$H(Du) = 0 \quad \text{on} \quad \omega. \tag{10.1}$$

Lemma 10.1 (The Barron, Jensen result, [4])
Let $u : \omega \to \mathbb{R}$ be a Lipschitz continuous function, and let $H : \mathbb{R}^n \to \mathbb{R}$ be a convex (continuous) function.
i) (Barron, Jensen subsolutions)
Then u is a standard viscosity subsolution of (10.1) if and only if **we can test it from below***, i.e.*

$$H(P) \le 0 \quad \text{for all } P \in D^- v(x_0) \text{ and all } x_0 \in \omega.$$

ii) (Barron, Jensen solutions)
Then u is a standard viscosity solution of (10.1) if and only if **we can test it from below***, i.e.*

$$H(P) = 0 \quad \text{for all } P \in D^- v(x_0) \text{ and all } x_0 \in \omega.$$

iii) (minimum of solutions)
Let $u, v : \omega \to \mathbb{R}$ be two Lipschitz continuous viscosity solutions of (10.1). Then $\min(u,v)$ *is also a viscosity solution of (10.1).*

Notice that a proof of this result is also given by Theorem 9.2 in Barles [3].

The following result is classical (see Theorem 4.1.1, and Corollary 4.1.3 in Hiriart-Urruty, Lemaréchal [13]).

Lemma 10.2 (Properties of Legendre-Fenchel transform)
Let $H : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be convex, proper (i.e. $H \not\equiv +\infty$) and lower semi-continuous.
i) (Duality)
Then we define its Legendre-Fenchel transform

$$H^*(\xi) := \sup_{P \in \mathbb{R}^d} \{\xi \cdot P - H(P)\}.$$

Then H^* is also convex, proper and lower semi-continuous, and we have the duality $H^{**} = H$.
ii) (Subdifferential characterization)
Define the subdifferential of H at P by

$$\partial H(P) := \left\{\xi \in \mathbb{R}^d, \quad H(Q) \geq H(P) + \xi \cdot (Q - P) \quad \text{for all} \quad Q \in \mathbb{R}^d\right\}.$$

Then we have

$$\xi \in \partial H(P) \Longleftrightarrow P \in \partial H^*(\xi) \Longleftrightarrow H(P) + H^*(Q) = P \cdot Q.$$

iii) (C^1 characterization)
The function H is real valued and C^1 if and only if H^* is strictly convex and superlinear (i.e. satisfying $\lim_{|P| \to +\infty} \dfrac{H(P)}{|P|} = +\infty$).

The following result is quite classical.

Lemma 10.3 (Young measures)
Let $n, m \geq 1$. Let $Q \subset \mathbb{R}^n$ be a bounded open set, and $K \subset \mathbb{R}^m$ be a compact set. For $k \in \mathbb{N}$, let us consider a sequence of functions $v_k : Q \to \mathbb{R}^m$ such that $v_k(x) \in K$ for a.e. $x \in Q$.
i) (Extraction of Young measures)
Then there exists a subsequence $(v_{k_j})_{j \in \mathbb{N}}$ and a family of Borel probability measures $(\nu_x)_{x \in Q}$ on $\mathbb{R}^m$ (depending measurably on x) with supp$(\nu_x) \subset K$ such that if $f : \mathbb{R}^m \to \mathbb{R}$ is continuous, then we have

$$f(v_{k_j}) \rightharpoonup \langle \nu_x, f\rangle := \int_{\mathbb{R}^m} f(\lambda) d\nu_x(\lambda) \quad \text{in} \quad L^\infty(Q) \quad \text{weak} - *, \quad \text{as} \quad j \to +\infty.$$

The family ν_x is called the Young measure associated to the subsequence $(v_{k_j})_j$.
ii) (Strong convergence of the subsequence)
Assume furthermore that there exists a function $v : Q \to \mathbb{R}^m$ such that $v(x) \in K$ for a.e. $x \in Q$, and that the Young measure is a Dirac mass

$$\nu_x(\lambda) = \delta_0(\lambda - v(x)) \quad \text{for a.e. } x \in Q. \tag{10.2}$$

Then we have

$$v_{k_j} \to v \quad \text{in} \quad L^1(Q; \mathbb{R}^m), \quad \text{as} \quad j \to +\infty.$$

iii) (Strong convergence of the sequence)
Assume furthermore that all Young measures associated to any subsequence of $(v_k)_k$ coincide with the same Dirac mass ν_x given in (10.2) for a unique function v. Then we have convergence of the full sequence

$$v_k \to v \quad \text{in} \quad L^1(Q; \mathbb{R}^m), \quad \text{as} \quad k \to +\infty.$$

References for the proof of Lemma 10.3
Point i) is Theorem 11 on page 16 in Evans [11]. As a complement for K only compact, see Theorem 5 on page 147 in Tartar [21]. For point ii), notice that the proof of (1.29) on page 17 in [11], shows for $f(w) = w^2$ that $|v_{k_j}|^2_{L^2(Q)} \to |v|^2_{L^2(Q)}$, which implies $v_{k_j} \to v$ in $L^2(Q)$ (for instance by Proposition 3.32 on page 78 in Brézis [7]). Because Q is bounded this also implies the convergence in $L^1(Q)$. Point iii) is a standard consequence of the uniqueness of the limit points in separated spaces (Hausdorff T2 space), that we apply in two steps. For the first step, the space is the space of measures $M(Q \times \mathbb{R}^m)$ with its natural weak-$*$ topology $\sigma(M, C_c)$ (agains test functions which are continuous with compact support $C_c(Q \times \mathbb{R}^m)$). This implies the full convergence towards its unique limit (Young) measure. For the second step, the space is $L^1(Q, \mathbb{R}^m)$. Then point ii) implies the convergence of subsequences in L^1 towards a unique limit, and again we get the convergence of the whole sequence in L^1. This ends the comment for the proof.

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