Invariance and homogenization of an adaptive time gap car-following model

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June 18, 2013

Abstract

In this paper we consider a microscopic model of traffic flow called the adaptive time gap car-following model. This is a system of ODEs which describes the interactions between cars moving on a single line. The time gap is the time that a car needs to reach the position of the car in front of it (if the car in front of it would not move and if the moving car would not change its velocity). In this model, both the velocity of the car and the time gap satisfy an ODE. We study this model and show that under certain assumptions, there is an invariant set for which the dynamics is defined for all times and for which we have a comparison principle. As a consequence, we show rigorously that after rescaling, this microscopic model converges to a macroscopic model that can be identified as the classical LWR model for traffic.

AMS Classification: 35B27, 35F20, 49L25, 70F45, 70H20.
Keywords: car-following model, homogenization, traffic, particle system, Hamilton-Jacobi equation.

1 Introduction

1.1 The adaptive time gap car-following model

We present the adaptive time gap car-following model that has been introduced in [22].

In the following, one denotes by $n \in \mathbb{Z}$ the index of the vehicle, $x_n(t) \in \mathbb{R}$ its position at time t, and $\dot{x}_n = \dfrac{dx_n}{dt} > 0$ its velocity, and we assume that

$$x_n < x_{n+1}. \tag{1.1}$$

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The distance gap of the vehicle n is by definition its distance to the vehicle in front of it, i.e. $x_{n+1} - x_n$. The time gap of the vehicle n is by definition the time that this vehicle needs to reach the position of the vehicle in front of it (if the vehicle in front of it would not move and if the moving vehicle would not change its velocity). This means that the time gap of the vehicle n is equal to

$$\tau_n = \frac{1}{\dot{x}_n}(x_{n+1} - x_n).$$

Usually, car-following models are defined by a relaxation process applied to the speed or the distance gap, towards a function of equilibrium. In the model that we consider here, this is the vehicle time gap that is relaxed. The adaptive time gap car-following model is defined by the following dynamics for each time $t > 0$ and each vehicle $n \in \mathbb{Z}$,

$$\left\{ \begin{array}{rcl} \dot{x}_n(t) &=& \dfrac{1}{\tau_n(t)}(x_{n+1}(t) - x_n(t)), \\ \\ m\,\dot{\tau}_n(t) &=& g(\dot{x}_n(t)) - \tau_n(t) \end{array} \right. \tag{1.2}$$

where $m > 0$ is a parameter calibrating the relaxation and $g : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is a function called the "targeted time" depending on the velocity. The parameters (even for a more general model) have been calibrated on real data describing vehicles trajectories on a highway and we refer the reader to [22] for more details on the calibration.
Mathematically, one of our motivation to study system (1.2) is to understand how the theory developed in [8] can be adapted here.

The goal of this paper is to obtain homogenization properties for this model, i.e. to prove a hydrodynamic limit towards a macroscopic model. To this end, we first control the long time behaviour of vehicle trajectories. This is done identifying *an invariant set on which we can prove a comparison principle*. Notice that in the limit case $m = 0$, these properties are contained in the work [7] (see Theorem 1.3), since when $m = 0$, system (1.2) can be rewritten as

$$\dot{x}_n(t) = V(x_{n+1}(t) - x_n(t)) \tag{1.3}$$

in the special case where the function V is well defined through its inverse:

$$V^{-1}(v) = vg(v).$$

It is proven (when the function V is continuous monotone increasing) that at large scale, the microscopic positions of vehicles $(x_n(t))_{n\in\mathbb{Z}}$ behave like a continuous macroscopic position $X(t, y)$ of vehicles at time t as a function of the continuous index y. The continuous homogenized macroscopic model (see [7]) is the following Hamilton-Jacobi equation

$$X_t = V(X_y)$$

where $X_\alpha = \dfrac{\partial X}{\partial \alpha}$ for $\alpha = t, y$. Defining the density of vehicles as

$$\rho(t, X(t, y)) = \frac{1}{X_y(t, y)},$$

it is possible to check (at least formally) that $\rho(t,x)$ solves the following LWR model

$$\rho_t + (\rho V(1/\rho))_x = 0.$$

We refer the reader to [14, 19] for the introduction of the classical LWR model for traffic. We will see in the next subsection (see Theorem 1.6) that, under certain assumptions, system (1.2) for $m > 0$ behaves like its limit case $m = 0$.

1.2 Main results

Even, if on the one hand it is easy to find solutions of (1.2) defined for all times (under suitable conditions, see Lemma 4.1), on the other hand, it is not obvious to get a control on the distance $x_{n+1} - x_n$ which is uniform in time. More precisely, in order to insure the following property for all times $t \geq 0$ and $n \in \mathbb{Z}$

$$\left\{ \begin{array}{l} 0 < \alpha \ \leq \ \tau_n(t) \ \leq \ \beta, \\ 0 < a \leq x_{n+1}(t) - x_n(t) \leq b \end{array} \right. \tag{1.4}$$

for certain parameters $0 < \alpha \leq \beta$, and $0 < a < b$, we need to choose initial data satisfying additional assumptions. We also need to impose structural conditions on the dynamics (1.2), which are given in the following assumption (H). This assumption relates the parameter m, the function g of the model and also an extra parameter γ which is used to map system (1.2) onto a monotone system (see later (2.2)).

We first define

$$G(v) := 2vg(v) + v^2 g'(v).$$

Assumption (H)
We assume that there exist $0 < \alpha \leq \beta$, $0 < a < b$ and $\gamma > 0$, $m > 0$ such that $g \in C^1\left(\left[\frac{a}{\beta}, \frac{b}{\alpha}\right]\right)$ and

$$\left\{ \begin{array}{lll} \text{(H0)} & 0 < \alpha \ \leq \ g(v) \ \leq \ \beta & \forall v \in \left[\frac{a}{\beta}, \frac{b}{\alpha}\right], \\ \text{(H1)} & \gamma > \dfrac{b\beta}{a\alpha} > 1, & \\ \text{(H2)} & 0 < m < m_\gamma := \displaystyle\inf_{v \in \left[\frac{a}{\beta}, \frac{b}{\alpha}\right]} h_\gamma(v) & \text{with} \quad h_\gamma(v) := \dfrac{G(v) - b\left(1 + \frac{1}{\gamma}\right)}{v\left(\frac{\gamma}{a} v g(v) - \frac{a}{b}\right)} \end{array} \right.$$

Notice that the denominator of h_γ is always positive under assumptions (H0),(H1), see Lemma 2.2 and its proof. Moreover, we have $m_\gamma > 0$ for γ large enough, if and only if we have

$$\text{(H2')} \qquad \inf_{v \in \left[\frac{a}{\beta}, \frac{b}{\alpha}\right]} G(v) > b.$$

For $\alpha = \beta$, (H0) implies that g is constant and (H2') gives

$$0 < a < b < 2a.$$

For the statistical model of targeted time function proposed in [22]

$$g \;:\; \mathbb{R}^+ \mapsto [\gamma_1, \gamma_1 + \gamma_2/\gamma_3] \tag{1.5}$$
$$v \mapsto \gamma_1 + \frac{\gamma_2}{v} \log(1 + v/\gamma_3),$$

with the time $\gamma_1 \geq 0$, the distance $\gamma_2 \geq 0$, and the speed $\gamma_3 > 0$, we have $g'(v) \leq 0$ and

$$\begin{aligned} G(v) &= 2\gamma_1 v + \gamma_2 \log(1 + v/\gamma_3) + \frac{\gamma_2 v}{\gamma_3 + v} \\ &= v g(v) + \gamma_1 v + \frac{\gamma_2 v}{\gamma_3 + v}, \end{aligned}$$

and, for all $v \in \left[\frac{a}{\beta}, \frac{b}{\alpha}\right]$, $G'(v) = 2\gamma_1 + \frac{\gamma_2}{\gamma_3 + v} + \frac{\gamma_2 \gamma_3}{(\gamma_3 + v)^2} \geq 0$. Therefore assumption (H2') is

$$G(a/\beta) > b.$$

We want to choose α and β and then find a condition on a and b. If we assume that a and β are such that $g(a/\beta) = \beta$, and that b and α are such that $g(b/\alpha) \geq \alpha$ (by taking for instance $\alpha = \gamma_1$), then (H0) holds and moreover assumptions (H0),(H1),(H2') are equivalent to

$$\text{(H')} \qquad 0 < a < b < a + \gamma_1 \frac{a}{\beta} + \frac{\gamma_2 \frac{a}{\beta}}{\gamma_3 + \frac{a}{\beta}}.$$

The condition (H') is presented figure 1. Feasible values of b are plotted for a varying from 0 to 20. The values of the parameters $(\gamma_i)_{i \in 1,\ldots,3}$ are the statistical estimates given in [22] ($\gamma_1 = 0.84$, $\gamma_2 = 0.77$ and $\gamma_3 = 0.02$), while β is the solution of $g(a/\beta) = \beta$.

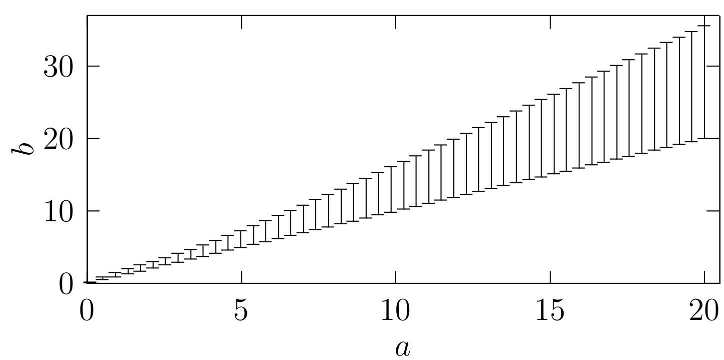

Figure 1: Numerical example for the assumptions (H').

These two examples show that assumption (H) is feasible in the limit case γ large enough. See also Section 5 where we present numerical simulations satisfying condition (H). In order to make assumption (H) less mysterious, let us give some (indirect) consequences of it:

Proposition 1.1 (Consequences of assumption (H): definition of V)
If (H) is satisfied, then for any $p \in [a, b]$, there exists a unique $\lambda \in \left[\frac{a}{\beta}, \frac{b}{\alpha}\right]$ such that

$$f(\lambda) = p \quad \textit{with} \quad f(\lambda) := \lambda g(\lambda).$$

Let us call $\lambda_+, \lambda_- \in \left[\frac{a}{\beta}, \frac{b}{\alpha}\right]$ *the following roots:*

$$f(\lambda_-) = a \quad \textit{and} \quad f(\lambda_+) = b.$$

Then the map $v \mapsto f(v)$ *is increasing on* $[\lambda_-, \lambda_+]$ *and we define a sort of inverse function*

$$V(p) = \begin{cases} \lambda_- & \textit{if} \quad p < a, \\ \lambda \in [\lambda_-, \lambda_+] \quad \textit{with} \quad f(\lambda) = p & \textit{if} \quad p \in [a,b], \\ \lambda_+ & \textit{if} \quad p > b. \end{cases} \tag{1.6}$$

Notice (see Lemma 4.2) that there exists a $\gamma > 1$ such that (H2) is satisfied for $a = b = p$ and $\alpha = \beta = g(\lambda)$ with $\lambda g(\lambda) = p$ if and only if (with the notation of Proposition 1.1):

$$0 < mV'(p) < \frac{1}{Q\left(\dfrac{p}{mV(p)}\right)} \quad \text{with} \quad Q(\tau) := \begin{cases} \tau & \text{if} \quad \tau \le 1, \\ 2\sqrt{\tau} - 1 & \text{if} \quad \tau > 1. \end{cases} \tag{1.7}$$

The function V will appear later in the ergodic and homogenization properties (see respectively Theorems 1.5 and 1.6). Our first main result is

Theorem 1.2 (Existence, uniqueness and invariance)
Assume (H). Let us introduce the notation

$$\xi_n := x_n + \gamma m \dot{x}_n = x_n + \gamma m \left(\frac{x_{n+1} - x_n}{\tau_n}\right).$$

If there exists an initial data $(x_n(0), \xi_n(0))_n$ *satisfying for* $t = 0$ *and for all* $n \in \mathbb{Z}$

$$\begin{cases} \alpha \le \tau_n(t) \le \beta, \\ a \le x_{n+1}(t) - x_n(t) \le b, \\ a \le \xi_{n+1}(t) - \xi_n(t) \le b \end{cases} \tag{1.8}$$

then there exists a unique solution $(x_n, \xi_n)_n$ *of (1.2), satisfying (1.8) for all* $t \ge 0$ *and all* $n \in \mathbb{Z}$.

We can interpret conditions (1.8) as defining an invariant set for the dynamics. More generally, we can wonder when conditions (1.8) can be checked for the initial conditions. The following result answers this question under the restriction (1.9) on $m > 0$.

Proposition 1.3 (Sufficient conditions to check (1.8))
Assume (H) and the following condition

$$0 < m \le \frac{1}{\gamma} \frac{(b-a)}{(b+a)\left(\dfrac{1}{\alpha} - \dfrac{1}{\beta}\right)}. \tag{1.9}$$

If the initial data $(x_n(0), \tau_n(0))_n$ *satisfies the following condition for all* $n \in \mathbb{Z}$

$$\begin{cases} 0 < \alpha \ \le \ \tau_n(0) \ \le \ \beta, \\ 0 < a + c \le x_{n+1}(0) - x_n(0) \le b - c \qquad \textit{with} \quad c = \dfrac{\gamma m \left(\dfrac{b}{\alpha} - \dfrac{a}{\beta}\right)}{1 + \gamma m \left(\dfrac{1}{\alpha} + \dfrac{1}{\beta}\right)} > 0, \end{cases} \tag{1.10}$$

then $(x_n(0), \tau_n(0))_n$ *satisfies (1.8).*

Notice that assumption (1.9) is equivalent to $a+c \leq b-c$ with c given in (1.10). Condition (1.8) shows in particular that the vehicles do not cross each other, which is a natural property that this model does satisfy.

Indeed, we even have the following comparison principle (on the invariant set):

Theorem 1.4 (Comparison principle for solutions)
Assume (H). Let us consider two solutions (given by Theorem 1.2) $(x_n, \tau_n)_{n\in\mathbb{Z}}$ and $(\bar{x}_n, \bar{\tau}_n)_{n\in\mathbb{Z}}$ of (1.2) both satisfying (1.8). We set

$$\xi_n := x_n + \gamma m \dot{x}_n \quad \text{and} \quad \bar{\xi}_n := \bar{x}_n + \gamma m \dot{\bar{x}}_n.$$

If

$$\left\{ \begin{array}{l} x_n(t) \leq \bar{x}_n(t), \\ \xi_n(t) \leq \bar{\xi}_n(t) \end{array} \right| \quad \text{for all} \quad n \in \mathbb{Z}$$

is satisfied for $t = 0$, then it is satisfied for all $t \geq 0$.

This result is derived in the spirit of the results in [8].

We also have the following long time asymptotics (also called ergodicity property in the literature on homogenization):

Theorem 1.5 (Ergodicity property)
Assume (H) and let us consider an initial data satisfying (1.8) for $t = 0$. Let us call $(x_n, \tau_n)_{n\in\mathbb{Z}}$ the corresponding solution of (1.2) given by Theorem 1.2. Let us define

$$\xi_n := x_n + \gamma m \dot{x}_n.$$

If there exists $p > 0$ and $C_0 \geq 0$ such that for all $n \in \mathbb{Z}$

$$\left\{ \begin{array}{l} |x_n(0) - pn| \leq C_0, \\ |\xi_n(0) - pn| \leq C_0, \end{array} \right. \tag{1.11}$$

then there exists a constant $C_1 > 0$ such that for all $n \in \mathbb{Z}$ and for all $t \geq 0$

$$\left\{ \begin{array}{l} |x_n(t) - pn - \lambda t| \leq C_1, \\ |\xi_n(t) - pn - \lambda t| \leq C_1, \end{array} \right. \tag{1.12}$$

where $p \in [a, b]$ and $\lambda = V(p)$ with V defined in Proposition 1.1.

In particular (1.12) shows that $x_n(t)/t \to \lambda$ as $t \to \infty$ and λ is then the mean velocity of the vehicles.

Our last main result is the following homogenization of the microscopic model:

Theorem 1.6 (Homogenization)
Assume (H) and (1.9). Let $u_0 \in \mathrm{Lip}(\mathbb{R})$ be a function satisfying

$$a + c \leq u_0'(y) \leq b - c \qquad \text{with} \quad c = \frac{\gamma m \left(\frac{b}{\alpha} - \frac{a}{\beta}\right)}{1 + \gamma m \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)} > 0.$$

For $\varepsilon > 0$, let us consider the initial data for $n \in \mathbb{Z}$

$$(1.13) \qquad \begin{cases} x_n^\varepsilon(0) = \dfrac{u_0(\varepsilon n)}{\varepsilon}, \\ \\ 0 < \alpha \le \tau_n^\varepsilon(0) \le \beta. \end{cases}$$

We call $(x_n^\varepsilon, \tau_n^\varepsilon)$ the solution of (1.2) given by Theorem 1.2 with initial data satisfying (1.13) (which implies condition (1.8) at the initial time). We denote by $\lfloor x \rfloor$ the floor integer part of a real x. Then we have

$$|\varepsilon x^\varepsilon_{\lfloor y/\varepsilon \rfloor}(t/\varepsilon) - u(t,y)| \to 0 \qquad as \quad \varepsilon \to 0$$

locally uniformly on $\mathbb{R}^+ \times \mathbb{R}$, where u is the unique viscosity solution of

$$(1.14) \qquad \begin{cases} u_t = V(u_y) & on \quad (0,\infty) \times \mathbb{R}, \\ u(0,y) = u_0(y) & on \quad \mathbb{R}. \end{cases}$$

Recall that (1.14) is a reformulation of the classical LWR model (see the end of Subsection 1.1). Concerning the calculus method, we refer the reader to [3, 4] for references on viscosity solutions.

1.3 Brief review of the literature

The paper is devoted to the so-called "car-following" microscopic traffic models, describing coupled interactions between a vehicle and its predecessor. We refer the reader to [18, 9] for nice overviews a several families of microscopic models for traffic.

As an example, a classical car-following model with acceleration is the "optimal velocity" model developed by Bando *et al.* [2]. This model assumes a relaxation of vehicle speed towards an optimal speed that is a function of the distance gap. Notice that in the "adaptive time gap car-following" model [22] that we study in this paper, this is the vehicle time gap that is relaxed.

Although the microscopic car-following approach is frequently used, rigorous analysis of the models are rarely done. According to our knowledge, only the speed model by Newell [16] has been completely described mathematically, see [23]. However, several approaches are developed to analyse partially the models. Let us mention some of them.

Models are investigated using rescaling and non linear analysis. Relations are established between microscopic models and macroscopic ones, see [13] for formal computations, [1] for partial rigorous results and [7, 8] for rigorous results.

In the same spirit, studies derived the Korteweg-de Vries or modified Korteweg-de Vries equation to explore traffic waves as kink and anti-kink solitons [11, 15, 25].

Models are also described using stability analysis of homogeneous configuration. See [17, 12] for such an analysis performed on a ring, or [2, 24] for the case of an infinite line.

Note that probability theory for interacting particles system have been used to analyse mathematically traffic models, see [10] and [5]. While cellular automata models can be described using mean-field theory or cluster-approximations [20, 21].

In this paper, we rigorously show the existence, invariance, ergodicity and homogenization properties of the adaptive time gap car-following model, using invariance principle, rescaling and hydrodynamical limit. In doing so, we obtain explicit relations between the microscopic model and a macroscopic one.

1.4 Organization of the paper

The rest of the paper gives the proves of the theorems and propositions stated in the introduction. In Section 2 we give the proof of Theorem 1.2 of existence and uniqueness of a solution (satisfying an invariance property) and the proof of Proposition 1.3. In Section 3, we prove a general comparison principle (Theorem 3.1) for sub and supersolutions, which implies Theorem 1.4. In Section 4, we prove Theorem 1.5, Proposition 1.1 and Theorem 1.6. Finally in section 5, numerical simulations are presented to illustrate the results.

2 Existence, uniqueness and invariance

This section is subdivided in four subsections. In the first subsection, we reformulate the original system in (x_n, τ_n) in an equivalent system for (x_n, ξ_n) for which we show certain monotonicity properties of the dynamics. In a second subsection, we consider a truncated system to apply Cauchy-Lipschitz theorem for existence of a solution. In the third subsection, we show a crucial property of invariance of the dynamics that allows in the fourth subsection to get a global existence and uniqueness result.

2.1 Equivalent system

We first show that the original system in (x_n, τ_n) can be reformulated in a system on the unknown (x_n, ξ_n) (which will enjoy good properties for the comparison principle).

Proposition 2.1 (Equivalent system)
We assume $m, \gamma > 0$. Let us set

$$\xi_n(t) := \ x_n(t) + \gamma m \dot{x}_n(t). \tag{2.1}$$

If $(x_n, \tau_n)_n$ solves (1.2) with $x_{n+1} - x_n > 0$ and $\tau_n > 0$, then $(x_n, \xi_n)_n$ solves the following system:

$$\begin{cases} \dot{x}_n = \dfrac{1}{\gamma m}\left\{\xi_n - x_n\right\}, \\ \\ \dot{\xi}_n = F(x_n, x_{n+1}, \xi_n, \xi_{n+1}), \end{cases} \tag{2.2}$$

with

$$F(x_n, x_{n+1}, \xi_n, \xi_{n+1}) := \frac{1}{m}(\xi_n - x_n)\left(1 + \frac{1}{\gamma}\left\{\frac{\xi_{n+1} - \xi_n}{x_{n+1} - x_n} - \frac{(\xi_n - x_n)\ g\left(\frac{\xi_n - x_n}{\gamma m}\right)}{m(x_{n+1} - x_n)}\right\}\right).$$

Reciprocally, if $(x_n, \xi_n)_n$ solves (2.2) with $x_{n+1} - x_n > 0$ and $\xi_n - x_n > 0$, then $(x_n, \tau_n)_n$ solves (1.2) with

$$\tau_n = \gamma m \frac{(x_{n+1} - x_n)}{(\xi_n - x_n)}. \tag{2.3}$$

All the solutions here are assumed to be C^1 in time.

Proof of Proposition 2.1
Step 1: Preliminary
We notice that we have the following equivalence

$$
\text{(2.4)} \qquad \left\{\begin{array}{l} \xi_n = \ x_n + \gamma m \dot{x}_n, \\ \\ \dot{x}_n = \dfrac{1}{\tau_n}(x_{n+1} - x_n), \\ \\ \tau_n > 0, \quad x_{n+1} - x_n > 0, \end{array}\right. \qquad \Longleftrightarrow \left\{\begin{array}{l} \dot{x}_n = \dfrac{1}{\gamma m}\left\{\xi_n - x_n\right\}, \\ \\ \tau_n = \gamma m \dfrac{(x_{n+1} - x_n)}{(\xi_n - x_n)}, \\ \\ \xi_n - x_n > 0, \quad x_{n+1} - x_n > 0. \end{array}\right.
$$

Step 2: derivation of the ODE on ξ_n
We compute

$$
\begin{aligned}
\dot{\xi}_n &= \dot{x}_n + \gamma m \left\{ -\frac{\dot{\tau}_n}{\tau_n}\dot{x}_n + \frac{1}{\tau_n}(\dot{x}_{n+1} - \dot{x}_n) \right\} \\
&= \dot{x}_n - \gamma \frac{m\dot{\tau}_n}{\tau_n}\dot{x}_n + \frac{1}{\tau_n}\left\{(\xi_{n+1} - \xi_n) - (x_{n+1} - x_n)\right\} \\
&= -\gamma \frac{m\dot{\tau}_n}{\tau_n}\dot{x}_n + \frac{1}{\tau_n}(\xi_{n+1} - \xi_n) \\
&= \frac{(\xi_n - x_n)}{m} - \gamma \frac{(m\dot{\tau}_n + \tau_n)}{\tau_n}\dot{x}_n + \frac{1}{\tau_n}(\xi_{n+1} - \xi_n)
\end{aligned}
$$

where in the first line we have used the first line of (1.2); in the second line we have also used the first line of (1.2); in the last line we have used (2.1). In the last line, we can replace $\dot{x}_n$ by its value in (2.4) (without time derivative), and we check that

$$
\dot{\xi}_n = F(x_n, x_{n+1}, \xi_n, \xi_{n+1})
$$

if and only if

$$
m\dot{\tau}_n = g(\dot{x}_n) - \tau_n.
$$

Step 3: Conclusion
We have shown the equivalence between (2.2) and (1.2).
This ends the proof of the proposition.

We now present some monotonicity properties of $F(x_n, x_{n+1}, \xi_n, \xi_{n+1}) = F(y)$ that will be used later and set the notation

$$
\text{(2.5)} \qquad \left\{\begin{array}{l} y = (y_1, y_2, y_3, y_4) = (x_n, x_{n+1}, \xi_n, \xi_{n+1}), \\ \\ F_i'(y) = \dfrac{\partial F}{\partial y_i}(y) \quad \text{for} \quad i = 1, ..., 4. \end{array}\right.
$$

Lemma 2.2 (Equivalent condition for (H2))
Under assumptions (H0),(H1), the assumption (H2) is equivalent to the following assumption

$$
\text{(2.6)} \qquad \inf_{v \in \left[\frac{a}{\beta}, \frac{b}{\alpha}\right]} G_{a,b}(v) > b\left(1 + \frac{1}{\gamma}\right) \quad \textit{with} \quad G_{a,b}(v) := v g(v)\left(2 - \frac{\gamma m}{a} v\right) + \frac{ma}{b} v + v^2 g'(v).
$$

Proof of Lemma 2.2
We first remark that
$$\frac{\gamma}{a} v g(v) - \frac{a}{b} \geq \frac{\gamma}{\beta}\alpha - \frac{a}{b} > \frac{b}{a} - \frac{a}{b} > 0,$$
by using successively (H0), (H1) and the fact that $0 < a < b$. Then assumption (H2) holds if and only if
$$m < m_\gamma = \inf_{[a/\beta, b/\alpha]} h_\gamma(v) \qquad \text{with} \qquad h_\gamma(v) = \frac{G(v) - b\left(1 + \frac{1}{\gamma}\right)}{v\left(\frac{\gamma}{a} v g(v) - \frac{a}{b}\right)}.$$
This ends the proof of the lemma.

Lemma 2.3 (Monotonicity of F)
Let us consider a function $g \in C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ such that assumption (H) is satisfied. We define
$$f(v) := v g(v).$$
i) Perturbation of assumptions (H1) and (H2)
Then there exists $\rho > 0$ (small enough) such that setting
$$(2.7) \qquad \begin{cases} \bar{a} := a - \rho, \\ \bar{b} := b + \rho, \\ \underline{a} := \dfrac{a-\rho}{\beta+\rho}, \\ \underline{b} := \dfrac{b+\rho}{\alpha-\rho}, \end{cases}$$
we have
$$(2.8) \qquad \gamma\left(\inf_{v\in[\underline{a},\underline{b}]} f(v)\right) > \bar{b}$$
and
$$(2.9) \qquad \inf_{v\in[\underline{a},\underline{b}]} G_{\bar{a},\bar{b}}(v) > \bar{b}\left(1 + \frac{1}{\gamma}\right)$$
where the function $G_{\bar{a},\bar{b}}$ is defined in (2.6).
ii) Monotonicity of F
Then (with notation (2.5)) we have
$$F_1'(y) \geq 0, \quad F_2'(y) \geq 0, \quad F_4'(y) \geq 0,$$

if $y \in co(K_\rho)$, *where* $co(K_\rho)$ *is the convex hull of the set* K_ρ *defined by*

(2.10)

$$K_\rho := \left\{ y = (y_1, y_2, y_3, y_4) \in \mathbb{R}^4, \quad \middle| \begin{array}{l} 0 < a - \rho \le y_2 - y_1 \le b + \rho, \\ \\ 0 < a - \rho \le y_4 - y_3 \le b + \rho, \\ \\ 0 < \alpha - \rho \le \tau \le \beta + \rho \quad \text{with} \quad \tau = \gamma m \left(\dfrac{y_2 - y_1}{y_3 - y_1} \right) \end{array} \right\}.$$

Proof of Lemma 2.3

We have

$$F(y) = \frac{1}{m}(y_3 - y_1)\left(1 + \frac{1}{\gamma(y_2 - y_1)}\left\{(y_4 - y_3) - \gamma f\left(\frac{y_3 - y_1}{\gamma m}\right)\right\}\right). \tag{2.11}$$

Step 0: proof of i)

We notice that for $v \in \left[\dfrac{a}{\beta}, \dfrac{b}{\alpha}\right]$, assumption (H0) implies

$$\gamma f(v) \ge \gamma \frac{a}{\beta}\alpha$$

and then

$$\gamma f(v) > b \tag{2.12}$$

by assumption (H1). Then (2.8) and (2.9) follow respectively by perturbation of (2.12) and (H2).

Step 1: proof of $F_4' \ge 0$

We have

$$F_4'(y) = \frac{1}{\gamma m}\frac{y_3 - y_1}{y_2 - y_1} > 0.$$

Step 2: proof of $F_2' \ge 0$

We have

$$F_2'(y) = -\frac{1}{\gamma m}\frac{(y_3 - y_1)}{(y_2 - y_1)^2}J \quad \text{with} \quad J := (y_4 - y_3) - \gamma f\left(\frac{y_3 - y_1}{\gamma m}\right).$$

If $y \in K_\rho$, we set $v = \dfrac{y_3 - y_1}{\gamma m} = \dfrac{y_2 - y_1}{\tau} \in \left[\dfrac{a - \rho}{\beta + \rho}, \dfrac{b + \rho}{\alpha - \rho}\right] = [\underline{a}, \underline{b}]$. If now $y \in \text{co}(K_\rho)$, then $v = \dfrac{y_3 - y_1}{\gamma m} \in \text{co}([\underline{a}, \underline{b}]) = [\underline{a}, \underline{b}]$ and we can compute

$$\begin{aligned} -J &= \gamma f(v) - (y_4 - y_3) \\ &\ge \gamma f(v) - (b + \rho) \\ &\ge 0 \end{aligned}$$

where in the second line we have used the definition of K_ρ, and we have used (2.8) in the last line. This implies $F_2' \ge 0$ on $\text{co}(K_\rho)$.

Step 3: proof of $F_1' \geq 0$
Setting $z = y_2 - y_1$ and $\bar{z} = y_4 - y_3$, we write

$$F(y) = \frac{1}{m}(y_3 - y_1)\left(1 + \frac{1}{\gamma z}\left\{\bar{z} - \gamma f\left(\frac{y_3 - y_1}{\gamma m}\right)\right\}\right)$$

and we compute with the previous notation

$$\begin{aligned}
F_1'(y) &= -\frac{1}{m}\left(1 + \frac{1}{\gamma z}\{\bar{z} - \gamma f(v)\}\right) + \frac{1}{\gamma m}\frac{(y_3 - y_1)}{z^2}\{\bar{z} - \gamma f(v)\} + \frac{f'(v)}{\gamma m^2}\left(\frac{y_3 - y_1}{z}\right) \\
&= -\frac{1}{m}\left(1 + \frac{1}{\gamma z}\{\bar{z} - \gamma f(v)\}\right) + \frac{v}{z^2}\{\bar{z} - \gamma f(v)\} + \frac{vf'v)}{mz} \\
&= \frac{1}{mz}\left\{f(v) + vf'(v) + \frac{mv}{z}\{\bar{z} - \gamma f(v)\} - \left(z + \frac{\bar{z}}{\gamma}\right)\right\} \\
&\geq \frac{1}{mz}\left\{f(v) + mv\left(\frac{\bar{a}}{\bar{\bar{b}}} - \frac{\gamma f(v)}{\bar{a}} + \frac{f'(v)}{m}\right) - \bar{b}\left(1 + \frac{1}{\gamma}\right)\right\} \\
&\geq 0
\end{aligned}$$

where in the fourth line we have used the fact that $y \in K_\rho$, and the last line follows from (2.9) and the fact that $v \in [\underline{a}, \underline{b}]$ when $y \in \mathrm{co}(K_\rho)$.
This ends the proof of the lemma.

2.2 Truncated system

Notice that the dynamics of system (2.2) is not globally Lipschitz. This is the reason to introduce a new truncated system. To this end, for any $r \leq s$, we define the truncation function

$$T_{r,s}(x) = \begin{cases} r & \text{if} \quad x < r, \\ x & \text{if} \quad x \in [r, s], \\ s & \text{if} \quad x > s. \end{cases}$$

We want our new truncated system to be equivalent to the original system (without truncation) when $(x_n, x_{n+1}, \xi_n, \xi_{n+1}) \in K_\rho$ for all $n \in \mathbb{Z}$, for K_ρ defined in (2.10) with $\rho > 0$. For this reason we consider the following truncated system

$$(2.13) \qquad \begin{cases} \dot{x}_n &= T_{\underline{a},\underline{b}}\left(\frac{\xi_n - x_n}{\gamma m}\right), \\ \dot{\xi}_n &= \bar{F}(x_n, x_{n+1}, \xi_n, \xi_{n+1}) = \bar{F} \end{cases}$$

with (keeping in mind the expression (2.11) of F)

$$\bar{F} = \gamma T_{\underline{a},\underline{b}}\left(\frac{\xi_n - x_n}{\gamma m}\right)\left(1 + \frac{1}{\gamma T_{\bar{a},\bar{b}}(x_{n+1} - x_n)}\left\{T_{\bar{a},\bar{b}}(\xi_{n+1} - \xi_n) - \gamma f\left(T_{\underline{a},\underline{b}}\left(\frac{\xi_n - x_n}{\gamma m}\right)\right)\right\}\right)$$

where $0 < \bar{a}, \bar{b}$, $0 < \underline{a}, \underline{b}$ are defined in (2.7).

Then we have

Proposition 2.4 (Existence and uniqueness for the truncated system)
Assume $0 < a < b$, $0 < \alpha \leq \beta$, $m, \gamma > 0$ *and that* $g \in C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$.
i) (Existence and uniqueness)
Given any initial condition $(x_n(0), \xi_n(0))_n$, *there exists a unique solution* $(x_n(t), \xi_n(t))_n$ *of (2.13) defined for all times* $t \geq 0$.
ii) (Time estimate)
Assume (H) and let ρ *given by Lemma 2.3. There exists* $\delta > 0$ *such that if*

$$(x_n(0), x_{n+1}(0), \xi_n(0), \xi_{n+1}(0)) \in K_0 \quad \text{for all } n \in \mathbb{Z} \tag{2.14}$$

then

$$(x_n(t), x_{n+1}(t), \xi_n(t), \xi_{n+1}(t)) \in K_\rho \quad \text{for all } n \in \mathbb{Z} \quad \text{for all } t \in [0, \delta] \tag{2.15}$$

where K_ρ *is defined in (2.10).*

Proof of Proposition 2.4
Proof of i)
The result is a simple consequence of the classical Cauchy-Lipschitz theorem in Banach spaces (see for instance the proof of theorem 5.1 in [6], where a similar argument is given).
Proof of ii)
We notice that we have

$$|\dot{x}_n|, \ |\dot{\xi}_n| \leq M_\rho.$$

Therefore there exists $\delta > 0$ (only depending on ρ, but not on the initial data satisfying (2.14)) such that (2.15) holds.
This ends the proof of the proposition.

2.3 Invariance

Proposition 2.5 (Invariance property)
Under the assumptions of Proposition 2.4, assuming in particular that the initial data satisfies (2.14), we have

$$(x_n(t), x_{n+1}(t), \xi_n(t), \xi_{n+1}(t)) \in K_0 \quad \text{for all } n \in \mathbb{Z} \quad \text{for all } t \in [0, \delta]. \tag{2.16}$$

Proof of Proposition 2.5
Because of Proposition 2.4 ii), we deduce that our solution $(x_n, \xi_n)_n$ solves the original system (without truncation) (2.2) on the time interval $[0, \delta]$, and then by Proposition 2.1, we have equivalence with system (1.2).
Let us define

$$\begin{cases} z_n^1(t) = x_{n+1}(t) - x_n(t) - b, \\ \\ z_n^2(t) = a - (x_{n+1}(t) - x_n(t)), \\ \\ z_n^3(t) = \xi_{n+1}(t) - \xi_n(t) - b, \\ \\ z_n^4(t) = a - (\xi_{n+1}(t) - \xi_n(t)), \\ \\ z_n^5(t) = \tau_n(t) - \beta, \\ \\ z_n^6(t) = \alpha - \tau_n(t) \end{cases}$$

where τ_n is given in (2.3). We set

$$M(t) = \sup_{n\in\mathbb{Z}} \max_{j=1,\dots,6} z_n^j(t).$$

Let us consider $t_* \in (0,\delta)$ and let us assume that

$$M(t_*) = \sup_{n\in Z} z_n^j(t_*) \quad \text{for some } j \in \{1,...,6\}.$$

Part I: first (formal) analysis
In a first analysis, we assume that

$$M(t_*) = z_{n_*}^j(t_*) \quad \text{for some } j \in \{1,...,6\} \text{ and for some } n_* \in \mathbb{Z}.$$

Step 1: $j = 1$
We have for $t = t_*$, using the dynamics

$$\dot z_{n_*}^1 = \frac{1}{\gamma m}\left(z_{n_*}^3 - z_{n_*}^1\right) \le 0$$

where the last inequality follows from the fact that $z_{n_*}^j(t_*)$ is maximal for $j = 1$.
Step 2: $j = 2$
Similarly, we have for $t = t_*$

$$\dot z_{n_*}^2 = \frac{1}{\gamma m}\left(z_{n_*}^4 - z_{n_*}^2\right) \le 0.$$

Step 3: $j = 3$
We set

$$Y_n = ((Y_n)_1, (Y_n)_2, (Y_n)_3, (Y_n)_4) = (x_n, x_{n+1}, \xi_n, \xi_{n+1}).$$

We have for $t = t_*$, $n = n_*$ and $B = (b, b, b, b)$

$$\begin{aligned}
\dot z_{n_*}^3 &= F(Y_{n+1}) - F(Y_n)\\
&= F(Y_{n+1} - B) - F(Y_n)\\
&= \sum_{i=1,\dots,4} ((Y_{n+1})_i - (Y_n)_i - b)F_i'(Z) \quad \text{with} \quad Z = \theta(Y_{n+1} - B) + (1-\theta)Y_n\\
&= z_n^1 F_1'(Z) + z_{n+1}^1 F_2'(Z) + z_n^3 F_3'(Z) + z_{n+1}^3 F_4'(Z)\\
&\le z_n^3 \left(\sum_{i=1,\dots,4} F_i'(Z)\right)\\
&= 0.
\end{aligned}$$

In the third line we have introduced $\theta \in [0,1]$, in the fifth line, we have used the monotonicity $F_1', F_2', F_4' \ge 0$ on $\mathrm{co}(K_\rho)$, and in the last line we have used the fact that $\sum_{i=1,\dots,4} F_i' = 0$ because $F(Y + (z,z,z,z)) = F(Y)$ for all admissible real z.

Step 4: $j = 4$
Similarly, we get with the same notation for $t = t_*$, $n = n_*$ and $A = (a, a, a, a)$

$$\begin{aligned} \dot{z}^4_{n_*} &= -(F(Y_{n+1} - A) - F(Y_n)) \\ &= z_n^2 F_1'(\bar{Z}) + z_{n+1}^2 F_2'(\bar{Z}) + z_n^4 F_3'(\bar{Z}) + z_{n+1}^4 F_4'(\bar{Z}) \quad \text{with} \quad \bar{Z} = \theta(Y_{n+1} - A) + (1-\theta) Y_n \\ &\le z_n^4 \left(\sum_{i=1,\dots,4} F_i'(\bar{Z}) \right) = 0. \end{aligned}$$

Step 5: $j = 5$
We recall that τ_n satisfies the second equation of (1.2). Then we have for $t = t_*$ and $n = n_*$

$$\dot{z}^5_{n_*} = \frac{1}{m}\left(g(\dot{x}_n) - \tau_n\right) = \frac{1}{m}\left(g(\dot{x}_n) - \beta - z_n^5\right) \tag{2.17}$$

and with L_g the Lipschitz constant of g:

$$g(\dot{x}_n) - \beta \quad \le \quad \begin{cases} g(\dot{x}_n) - g\left(\dfrac{a}{\beta}\right) \le L_g\left(\dfrac{a}{\beta} - \dot{x}_n\right) & \text{if} \quad \dot{x}_n < \dfrac{a}{\beta}, \\ 0 & \text{if} \quad \dot{x}_n \in \left[\dfrac{a}{\beta}, \dfrac{b}{\alpha}\right], \\ g(\dot{x}_n) - g\left(\dfrac{b}{\alpha}\right) \le L_g\left(\dot{x}_n - \dfrac{b}{\alpha}\right) & \text{if} \quad \dot{x}_n > \dfrac{b}{\alpha}. \end{cases}$$

We also have

$$\dot{x}_n - \frac{a}{\beta} = \frac{x_{n+1} - x_n}{\tau_n} - \frac{a}{\beta} = \frac{\beta(x_{n+1} - x_n - a) - a(\tau_n - \beta)}{\beta \tau_n} = \frac{-\beta z_n^2 - a z_n^5}{\beta \tau_n}$$

and similarly

$$\dot{x}_n - \frac{b}{\alpha} = \frac{\alpha z_n^1 + b z_n^6}{\alpha \tau_n}$$

which implies (using the fact that the z_n^j are maximal for $j = 5$)

$$g(\dot{x}_n) - \beta \le L_g\left(1 + \frac{b}{\alpha}\right)\frac{|z_n^5|}{\tau_n}.$$

Using the fact that $\tau_n \ge \alpha - \rho > 0$, we deduce from (2.17) that there exists a constant L_5 such that

$$\dot{z}^5_{n_*} \le L_5 |z_n^5|.$$

Step 6: $j = 6$
Similarly we have for $t = t_*$ and $n = n_*$

$$\dot{z}^6_{n_*} = -\frac{1}{m}\left(g(\dot{x}_n) - \tau_n\right) = \frac{1}{m}\left(\alpha - z_n^6 - g(\dot{x}_n)\right) \tag{2.18}$$

and we get the existence of a constant L_6 such that

$$\dot{z}^6_{n_*} \le L_6 |z_n^6|.$$

Step 7: Conclusion
Under the previous assumptions, we have (formally)

$$\dot{M}(t_*) \le L|M(t_*)|. \tag{2.19}$$

Part II: the (rigorous) proof
In general we may also have

$$M(t_*) = \sup_{n\in Z} z_n^j(t_*) = \lim_{k\to\infty} z_{n_k}^j(t_*)$$

for some unbounded sequence $(n_k)_k$. In that case, up to redefine the sequences, we have

$$\left\{ \begin{array}{lcl} x_n^k(t) = x_{n+n_k}(t) - x_{n_k}(0) & \to & x_n^\infty(t), \\ \xi_n^k(t) = \xi_{n+n_k}(t) - x_{n_k}(0) & \to & \xi_n^\infty(t) \end{array} \right| \quad \text{as} \quad k \to +\infty$$

and we can apply the reasoning of part I (Steps 1 to 6) to the solution $(x_n^\infty, \xi_n^\infty)_n$. And then this implies that (2.19) holds true, but this has to be understood in the viscosity sense (see [3, 4]). We now notice that

$$M(0) \le 0$$

because the initial data satisfies (2.14). Then a Gronwall type argument (which, in the viscosity setting, is simply the comparison principle applied with the zero solution) implies that

$$M(t) \le 0 \quad \text{for all} \quad t \in [0, \delta]$$

which means exactly (2.16). This ends the proof of the Proposition.

2.4 Proofs of Theorem 1.2 and Proposition 1.3

Proof of Theorem 1.2
Propositions 2.4 and 2.5 imply the existence and uniqueness of a solution $(x_n, \xi_n)_n$ satisfying condition (2.16) on the time interval $[0, \delta]$. This implies (1.8) on the same time interval. Iterating on new intervals $[k\delta, (k+1)\delta]$, we get the result for all times $t \ge 0$. This ends the proof of the theorem.

Proof of Proposition 1.3
We have at time $t = 0$:

$$\xi_n = x_n + \gamma m \frac{(x_{n+1} - x_n)}{\tau_n}$$

and then

$$\xi_{n+1} - \xi_n = x_{n+1} - x_n + \gamma m \left\{ \frac{(x_{n+2} - x_{n+1})}{\tau_{n+1}} - \frac{(x_{n+1} - x_n)}{\tau_n} \right\}.$$

Notice that

$$\frac{(a+c)}{\beta} \le \frac{(x_{n+1} - x_n)}{\tau_n} \le \frac{(b-c)}{\alpha}.$$

This implies that $\xi_{n+1} - \xi_n \in [a, b]$ (and then condition (1.8) is checked at time $t = 0$) if

$$c \ge \gamma m \left\{ \frac{(b-c)}{\alpha} - \frac{(a+c)}{\beta} \right\}$$

i.e.

$$c \geq \frac{\gamma m \left(\frac{b}{\alpha} - \frac{a}{\beta}\right)}{1 + \gamma m \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)} \tag{2.20}$$

which is true because our choice (1.10) of c means equality in (2.20). Finally the (necessary) condition that $a + c \leq b - c$ is satisfied if and only if condition (1.9) is satisfied. This ends the proof of the proposition.

3 Comparison principle

This section is devoted to the proof of the comparison principle.

We say that $(x_n, \xi_n)_n$ is a subsolution of (2.2) if for all $n \in \mathbb{Z}$ and all $t > 0$

$$\begin{cases} \dot{x}_n \leq \dfrac{1}{\gamma m} \{\xi_n - x_n\}, \\ \\ \dot{\xi}_n \leq F(x_n, x_{n+1}, \xi_n, \xi_{n+1}). \end{cases}$$

Similarly, we say that $(\bar{x}_n, \bar{\xi}_n)_n$ is a supersolution of (2.2) if for all $n \in \mathbb{Z}$ and all $t > 0$

$$\begin{cases} \dot{\bar{x}}_n \geq \dfrac{1}{\gamma m} \{\bar{\xi}_n - \bar{x}_n\}, \\ \\ \dot{\bar{\xi}}_n \geq F(\bar{x}_n, \bar{x}_{n+1}, \bar{\xi}_n, \bar{\xi}_{n+1}). \end{cases}$$

We have the following generalization which implies Theorem 1.4.

Theorem 3.1 (Comparison principle for Lipschitz sub and supersolutions) *Assume (H). Let us consider a subsolution $(x_n, \xi_n)_{n \in \mathbb{Z}}$ and a supersolution $(\bar{x}_n, \bar{\xi}_n)_{n \in \mathbb{Z}}$ of (2.2) both satisfying (1.8) with*

$$\tau_n := \gamma m \frac{(x_{n+1} - x_n)}{(\xi_n - x_n)}, \quad \bar{\tau}_n := \gamma m \frac{(\bar{x}_{n+1} - \bar{x}_n)}{(\bar{\xi}_n - \bar{x}_n)}.$$

We also assume that there exists a constant $C > 0$ such that

$$|\dot{x}_n(t)|, \quad |\dot{\xi}_n(t)|, \quad |\dot{\bar{x}}_n(t)|, \quad |\dot{\bar{\xi}}_n(t)| \leq C \quad \text{for all } t \geq 0 \text{ and for all } n \in \mathbb{Z}. \tag{3.1}$$

If

$$\left\{ \begin{array}{l} x_n(t) \leq \bar{x}_n(t), \\ \xi_n(t) \leq \bar{\xi}_n(t) \end{array} \right| \quad \text{for all} \quad n \in \mathbb{Z} \tag{3.2}$$

is satisfied for $t = 0$, then it is satisfied for all $t \geq 0$.

Proof of Theorem 3.1
We proceed similarly as in the proof of Proposition 2.5. We define

$$\begin{cases} z_n^1 = x_n - \bar{x}_n, \\ z_n^2 = \xi_n - \bar{\xi}_n \end{cases}$$

and we set

$$M(t) = \sup_{n \in \mathbb{Z}} \max_{j=1,2} \ z_n^j(t).$$

Let us consider $t_* > 0$ and let us assume that

$$M(t_*) = \sup_{n \in Z} z_n^j(t_*) \quad \text{for some } j \in \{1, 2\}.$$

Part I: first (formal) analysis
In a first analysis, we assume that

$$M(t_*) = z_{n_*}^j(t_*) \quad \text{for some } j \in \{1, 2\} \text{ and for some } n_* \in \mathbb{Z}$$

and that all the functions are derivable at time t_*.
Step 1: $j = 1$
We have for $t = t_*$, using the dynamics

$$\dot{z}_{n_*}^1 \leq \frac{1}{\gamma m} \left(z_{n_*}^2 - z_{n_*}^1 \right) \leq 0$$

where the last inequality follows from the fact that $z_{n_*}^j(t_*)$ is maximal for $j = 1$.
Step 2: $j = 2$
We proceed similarly to Step 3 of the proof of Proposition 2.5. We set

$$Y_n = ((Y_n)_1, (Y_n)_2, (Y_n)_3, (Y_n)_4) = (x_n, x_{n+1}, \xi_n, \xi_{n+1})$$

and

$$\bar{Y}_n = ((\bar{Y}_n)_1, (\bar{Y}_n)_2, (\bar{Y}_n)_3, (\bar{Y}_n)_4) = (\bar{x}_n, \bar{x}_{n+1}, \bar{\xi}_n, \bar{\xi}_{n+1}).$$

We have for $t = t_*$ and $n = n_*$

$$\begin{aligned} \dot{z}_{n_*}^2 &\leq F(Y_n) - F(\bar{Y}_n) \\ &\leq z_n^2 \left(\sum_{i=1,\dots,4} F_i'(Z) \right) \quad \text{with} \quad Z = \theta Y_n + (1-\theta)\bar{Y}_n \\ &= 0. \end{aligned}$$

Step 3: Conclusion
Under the previous assumptions, we have (formally)

$$\dot{M}(t_*) \leq 0. \tag{3.3}$$

Part II: the (rigorous) proof
Using (3.1), we proceed as in the proof of Proposition 2.5 and then conclude from $M(0) \leq 0$ that $M(t) \leq 0$ for all times $t \geq 0$. This shows (3.2) and ends the proof of the Theorem.

4 Ergodicity and homogenization

This section is composed of three subsections. In the first subsection we establish the ergodicity properties of the dynamics, while in the second short subsection we indicate how the homogenization result can be obtained. Finally in the third subsection, we prove miscellaneous results of independent interest.

4.1 Ergodicity property

Proof of Theorem 1.5
Because the initial data satisfies both (1.11) and (1.8), we deduce that $p \in [a, b]$.
Step 1: Construction of a root of $\lambda g(\lambda) = p$
Step 1.1: First proof
The existence of a root λ of the equation $\lambda g(\lambda) = p$, can be obtained following the lines of proof of [8] (see Definition 1.8 and Theorem 1.10 in [8]). Indeed, given $p \in [a, b]$, we can show the existence of an effective Hamiltonian $\lambda = \bar{H}(p)$ such that there exists hull functions h_1, h_2 satisfying $h_1' = 1 = h_2'$ (because of the invariance by translation of the problem) such that $(x(t, y), \xi(t, y)) = (h_1(py + \lambda t), h_2(py + \lambda t))$ is solution of the system

$$
(4.1) \qquad \begin{cases} x_t = \dfrac{1}{\gamma m}(\xi - x), \\ \\ \xi_t = F(x(t, y), x(t, y + 1), \xi(t, y), \xi(t, y + 1)). \end{cases}
$$

Up to translation, we can then show that we can choose

$$
x(t, y) = py + \lambda t, \quad \xi(t, y) = py + \lambda t + d
$$

where

$$
p = \lambda g(\lambda) \quad \text{and} \quad d = \gamma m \lambda.
$$

Step 1.2: Second proof
It is usefull to give a second self-contained proof of the existence of a root λ, that will be also useful later for showing the monotonicity of λ in terms of p. We consider initial data for $p \in [a, b]$

$$
x_n(0) = pn, \quad \xi_n(0) = pn + d_0
$$

with $d_0 > 0$ that we choose such that

$$
\tau_n(0) := \gamma m \left(\frac{x_{n+1}(0) - x_n(0)}{\xi_n(0) - x_n(0)} \right) = \frac{\gamma m p}{d_0} \in [\alpha, \beta].
$$

Our initial data satisfies (1.8) and then Theorem 1.2 gives us the existence of $(x_n, \tau_n)_n$ solution of (1.2) (or equivalently $(x_n, \xi_n)_n$ solution of (2.2)) such that (1.8) is satisfied for all times $t \geq 0$. In particular, this implies that

$$
(4.2) \qquad \frac{p}{\dot{x}_n(0)} \in [\alpha, \beta] \quad \text{and} \quad \dot{x}_n(t) \in \left[\frac{a}{\beta}, \frac{b}{\alpha} \right].
$$

Notice that

$$
(4.3) \qquad x_{n+1}(t) = x_n(t) + p, \quad \xi_{n+1}(t) = \xi_n(t) + p
$$

holds for $t = 0$, and then (using a comparison principle for (4.1), or alternatively the uniqueness result of Theorem 1.2), we get that (4.3) holds true for all times $t \geq 0$. This shows that

$$x_n(t) = pn + x_0(t), \quad \xi_n(t) = pn + \xi_0(t)$$

and (x_0, ξ_0) solves

$$\begin{cases} \dot{x}_0 = \dfrac{1}{\gamma m}(\xi_0 - x_0), \\ \\ \dot{\xi}_0 = F(x_0, x_0 + p, \xi_0, \xi_0 + p) = \dfrac{(\xi_0 - x_0)}{m}\left(1 + \dfrac{1}{\gamma p}\left\{p - \gamma f\left(\dfrac{\xi_0 - x_0}{\gamma m}\right)\right\}\right) \end{cases}$$

where $f(v) = vg(v)$. Setting

$$\lambda_0 = \dot{x}_0$$

we deduce that λ_0 solves

$$\text{(4.4)} \qquad \begin{cases} m\dot{\lambda}_0 = \lambda_0 \left\{1 - \dfrac{1}{p} f(\lambda_0)\right\}, \\ \\ \dfrac{p}{\lambda_0(0)} \in [\alpha, \beta] \end{cases}$$

where $\lambda_0(t) \in \left[\dfrac{a}{\beta}, \dfrac{b}{\alpha}\right]$ because of (4.2).

We choose $\lambda_0(0) = \dfrac{a}{\beta}$, and then realize that $\lambda_0(t)$ is non decreasing and bounded, and then converges as $t \to +\infty$ to a value λ which is a root of $p = f(\lambda)$ which establishes the desired result.

Step 2: Uniqueness of λ

Assume that we have two candidates $\lambda < \bar{\lambda}$ with $\lambda, \bar{\lambda} \in \left[\dfrac{a}{\beta}, \dfrac{b}{\alpha}\right]$ satisfying

$$p = \lambda g(\lambda) = \bar{\lambda} g(\bar{\lambda})$$

and let us set

$$d = \gamma m \lambda, \quad \text{and} \quad \bar{d} = \gamma m \bar{\lambda}.$$

We set

$$\text{(4.5)} \qquad x_n(t) = pn + \lambda t, \quad \xi_n(t) = pn + \lambda t + d$$

and

$$\bar{x}_n(t) = pn + \bar{\lambda} t - C, \quad \bar{\xi}_n(t) = pn + \bar{\lambda} t + \bar{d} - C.$$

We choose $C > 0$ large enough such that

$$\text{(4.6)} \qquad \bar{x}_n(t) < x_n(t) \quad \text{and} \quad \bar{\xi}_n(t) < \xi_n(t)$$

holds at $t = 0$, and then the comparison principle implies that (4.6) holds true for all $t \geq 0$, which implies that $\bar{\lambda} \leq \lambda$. Contradiction. This shows the uniqueness of the parameter λ.

Step 3: proof of (1.12)

The result follows from the comparison to the exact solution given in (4.5).
This ends the proof of the theorem.

Proof of Proposition 1.1
Because of our proof of Theorem 1.5, it only remains to show that $f(v) = vg(v)$ is increasing on $\left[\frac{a}{\beta}, \frac{b}{\alpha}\right]$. To this end, we notice that if $p < \bar{p}$ and λ_0 solves (4.4) and $\bar{\lambda}_0$ solves

$$m\dot{\bar{\lambda}}_0 = \bar{\lambda}_0 \left\{1 - \frac{1}{\bar{p}} f(\bar{\lambda}_0)\right\}$$

with the same initial data $\lambda_0(0) = \bar{\lambda}_0(0) = \dfrac{a}{\beta}$, then we have

$$\bar{\lambda}_0(t) \geq \lambda_0(t)$$

and then

$$\bar{\lambda} = \bar{\lambda}_0(+\infty) \geq \lambda_0(+\infty) = \lambda$$

where

$$\bar{p} = f(\bar{\lambda}) \quad \text{and} \quad p = f(\lambda).$$

This implies that f is non decreasing on $\left[\dfrac{a}{\beta}, \dfrac{b}{\alpha}\right]$, and then increasing on the same interval, by uniqueness of the λ. This allows us to define the inverse V given in (1.6). This ends the proof of the proposition.

4.2 Homogenization

Proof of Theorem 1.6
The proof follows closely the lines of the proof of Theorem 1.5 in [8] with no further difficulties. For this reason, we skip the details of the proof.

4.3 Proofs of miscellaneous results

In this subsection we first prove that the dynamics (1.2) is well defined for all time under poor assumptions (Lemma 4.1). We next characterize assumption (H2) restricted on homogeneous states.

Lemma 4.1 (Existence and uniqueness under poor assumptions)
Let $0 < \alpha \leq \beta$, $0 < a \leq b$. Let us assume that $g \in C^1([0, +\infty); [\alpha, \beta])$ and that the initial data $(x_n(0), \tau_n(0))_n$ satisfies (1.4) at $t = 0$. Then there exists a unique solution $(x_n(t), \tau_n(t))_n$ of (1.2) satisfying for all times $t \geq 0$ and all $n \in \mathbb{Z}$

$$0 < ae^{-\frac{t}{\alpha}} \leq x_{n+1}(t) - x_n(t) \leq be^{\frac{t}{\alpha}}, \quad 0 < \alpha \leq \tau_n(t) \leq \beta \tag{4.7}$$

Sketch of the proof of Lemma 4.1
The proof of existence (and uniqueness) can be done similarly to the proof of Theorem 1.2. We only focus on the proof of estimate (4.7).
Step 1: control on τ_n
We recall that τ_n solves

$$m\dot{\tau}_n = g(\dot{x}_n) - \tau_n.$$

Because of the range $[\alpha, \beta]$ of g and $\tau_n(0) \in [\alpha, \beta]$, we immediately deduce that $\tau_n(t) \in [\alpha, \beta]$ for all times.
Step 2: control on $x_{n+1} - x_n$
Let us call

$$d_n = x_{n+1} - x_n$$

and

$$\overline{M}(t) = \sup_{n\in\mathbb{Z}} d_n(t) \quad \text{and} \quad \underline{M}(t) = \inf_{n\in\mathbb{Z}} d_n(t).$$

We have

$$\dot{d}_n = \frac{d_{n+1}}{\tau_{n+1}} - \frac{d_n}{\tau_n}.$$

Therefore we deduce (while $\underline{M} \geq 0$)

$$\dot{\underline{M}} \geq -\frac{\underline{M}}{\alpha} \quad \text{and} \quad \dot{\overline{M}} \leq \frac{\overline{M}}{\alpha}.$$

This implies (4.7) and ends the proof of the lemma.

Lemma 4.2 (Equivalent condition for (H2) on a homogeneous state)
There exists a $\gamma > 1$ such that (H2) is satisfied for $a = b = p > 0$ and $\alpha = \beta = g(\lambda)$ with $f(\lambda) = p$ if and only if

$$\frac{1}{m} f'(\lambda) > Q\left(\frac{1}{m} g(\lambda)\right) > 0 \tag{4.8}$$

with

$$Q(\tau) := \begin{cases} \tau & \text{if} \quad \tau \leq 1, \\ 2\sqrt{\tau} - 1 & \text{if} \quad \tau > 1 \end{cases} \tag{4.9}$$

of equivalently

$$0 < mV'(p) < \frac{1}{Q\left(\dfrac{p}{mV(p)}\right)}. \tag{4.10}$$

Proof of Lemma 4.2
Step 1: reduction
We first notice that if $(x_n(t), \tau_n(t))_n$ satisfies (1.2), then

$$\begin{cases} \bar{x}_n(\bar{t}) = x_n(t), \\ \bar{\tau}_n(\bar{t}) = \dfrac{\tau_n(t)}{m}, \end{cases} \quad \text{with} \quad t = m\bar{t}$$

and $(\bar{x}_n, \bar{\tau}_n)_n$ solves

$$(4.11) \qquad \left\{ \begin{array}{rcl} \dot{\bar{x}}_n & = & \dfrac{1}{\bar{\tau}_n}(\bar{x}_{n+1} - \bar{x}_n), \\ \\ \dot{\bar{\tau}}_n & = & \bar{g}(\dot{\bar{x}}_n) - \bar{\tau}_n \end{array} \right.$$

which is similar to system (1.2) with m replaced by 1 and g replaced by

$$\bar{g}(\bar{v}) = \frac{1}{m} g\left(\frac{\bar{v}}{m}\right).$$

Step 2: checking (H2)
By construction, condition (H2) for (1.2) for $v = \lambda$ is equivalent to (H2) for system (4.11) for $\bar{v} = \bar{\lambda} = m\lambda$, i.e. with $p = \lambda g(\lambda) = \bar{\lambda}\bar{g}(\bar{\lambda})$:

$$(4.12) \qquad \bar{g}(\bar{\lambda}) + 1 + \bar{\lambda}\bar{g}'(\bar{\lambda}) > \gamma + \frac{\bar{g}(\bar{\lambda})}{\gamma} =: H(\gamma).$$

If is easy to check that

$$\inf_{\gamma \in (1,+\infty)} H(\gamma) = H\left(\max\left(1, \sqrt{\bar{g}(\bar{\lambda})}\right)\right)$$

and then there exists a $\gamma > 1$ such that (4.12) holds if and only if

$$\bar{g}(\bar{\lambda}) + 1 + \bar{\lambda}\bar{g}'(\bar{\lambda}) > H\left(\max\left(1, \sqrt{\bar{g}(\bar{\lambda})}\right)\right)$$

i.e. with $\bar{f}(\bar{\lambda}) = \bar{\lambda}\bar{g}(\bar{\lambda})$:

$$(4.13) \qquad \bar{f}'(\bar{\lambda}) > Q(\bar{g}(\bar{\lambda})) > 0.$$

Setting $\bar{V} = (\bar{f})^{-1}$, we see that (4.13) is equivalent to

$$(4.14) \qquad 0 < \bar{V}'(p) < \frac{1}{Q\left(\dfrac{p}{\bar{V}(p)}\right)}.$$

Step 3: conclusion
Finally we notice that $\bar{f}(\bar{\lambda}) = f\left(\dfrac{\bar{\lambda}}{m}\right)$ and then $\bar{V}(p) = mV(p)$, and conclude that (4.13) and (4.14) are respectively equivalent to (4.8) and (4.10).
This ends the proof of the lemma.

5 Numerical results

We propose here some numerical simulations of the car-following model, with 10 vehicles on a 200 long ring (with periodic boundary conditions). The values of the parameters and the initial conditions are the followings :

– The targeted time function g is the model (1.5), depending on the parameters γ_1, γ_2 and γ_3 calibrated as in [22] ($\gamma_1 = 0.84$, $\gamma_2 = 0.77$ and $\gamma_3 = 0.02$);

– We fix $a = 18$ and $b = 22$, $\alpha = 1.1\underline{0}$ and $\beta = 1.1\underline{7}$ are respectively solution of $g(b/\alpha) = \alpha$ and $g(a/\beta) = \beta$;

– We set $\gamma = 10$;

– The initial configuration is

$$\begin{cases} x_{n+1}(0) - x_n(0) = a, & n = 1, \dots, 5, \\ x_{n+1}(0) - x_n(0) = b, & n = 6, \dots, 10, \end{cases} \qquad \tau_n(0) = g(v^\star), \quad n = 1, \dots, 10,$$

where $v^\star = 17.6\underline{8}$ is the solution of $v^\star = s^\star / g(v^\star)$, with $s^\star = 200/10$ the mean spacing.

We find numerically that here $m_\gamma = 0.05\underline{3}$ with m_γ defined in (H2).

We focus on the evolution of the variables

$$\text{(5.1)} \qquad \left| \begin{array}{l} m_x(t) = \min_n\{x_{n+1}(t) - x_n(t)\} \\ m_\xi(t) = \min_n\{\xi_{n+1}(t) - \xi_n(t)\} \\ m_\tau(t) = \min_n\{\tau_n(t)\} \end{array} \right. \qquad \left| \begin{array}{l} M_x(t) = \max_n\{x_{n+1}(t) - x_n(t)\} \\ M_\xi(t) = \max_n\{\xi_{n+1}(t) - \xi_n(t)\} \\ M_\tau(t) = \max_n\{\tau_n(t)\}, \end{array} \right.$$

by taking

$$m = 0.05 < m_\gamma \qquad \text{and} \qquad m = 0.09 > m_\gamma.$$

If assumptions (H) hold, then we have the invariance property (theorem 1.2)

$$\text{(5.2)} \quad \forall t \geq 0 \quad a \leq m_x(t) \leq M_x(t) \leq b, \quad a \leq m_\xi(t) \leq M_\xi(t) \leq b, \quad \alpha \leq m_\tau(t) \leq M_\tau(t) \leq \beta.$$

The numerical results are obtained by using an explicit Euler scheme. The evolution of the variables (5.1) are presented figure 2 on the time interval $[0, 2]$, for $m = 0.05$ and for $m = 0.09$. As expected, the invariance property (5.2) holds for $m = 0.05$. It is not the case when $m = 0.09$, where it exists t such that $m_\xi(t) < a$ and $M_\xi(t) > b$.

By doing more simulations, we observe that the invariance (5.2) occurs as soon as $m < 0.08\underline{6}$, and invariance (5.2) is broken above this threshold. Similar experiences have been done with more vehicles and longer rings (keeping constant the density level). The same threshold value has been obtained.

The values for m previously used are far from the mean statistical estimate of this parameter $m \approx 5$, which is given in [22]. For this parameter value, we can observe instability of homogeneous configurations, if enough vehicles are implemented (see [22]). On the figure 3, the evolution of variables (5.1) are plotted for 50 vehicles on a 1000 long ring, with $m = 5$, and the previous parameter setting. The flow obtained is not homogeneous, with propagations of “stop-and-go” waves and limit-cycle stationary states. The invariance conditions (5.2) clearly do not hold. But for this value of $m \approx 5$, numerically we did not find collisions.

Acknowledgements

The authors wish to acknowledge Helena Frankowska for an idea used in the proof of Proposition 2.5 and indications on the literature. The third author acknowledges support from VINCI ParisTech Grant No. 8L1142, *Éco-conception des ensembles bâtis et des infrastructures.*

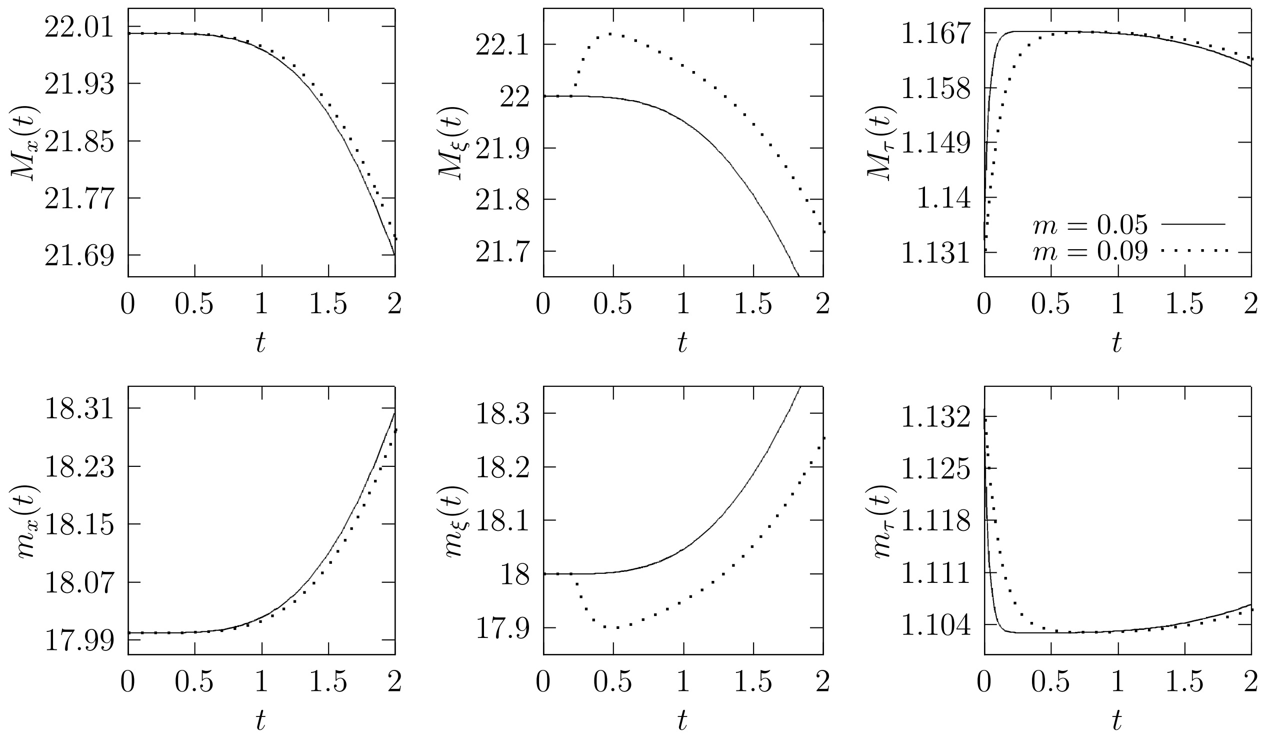

Figure 2: Evolution of the variables (5.1) for $m = 0.05$ and $m = 0.09$.

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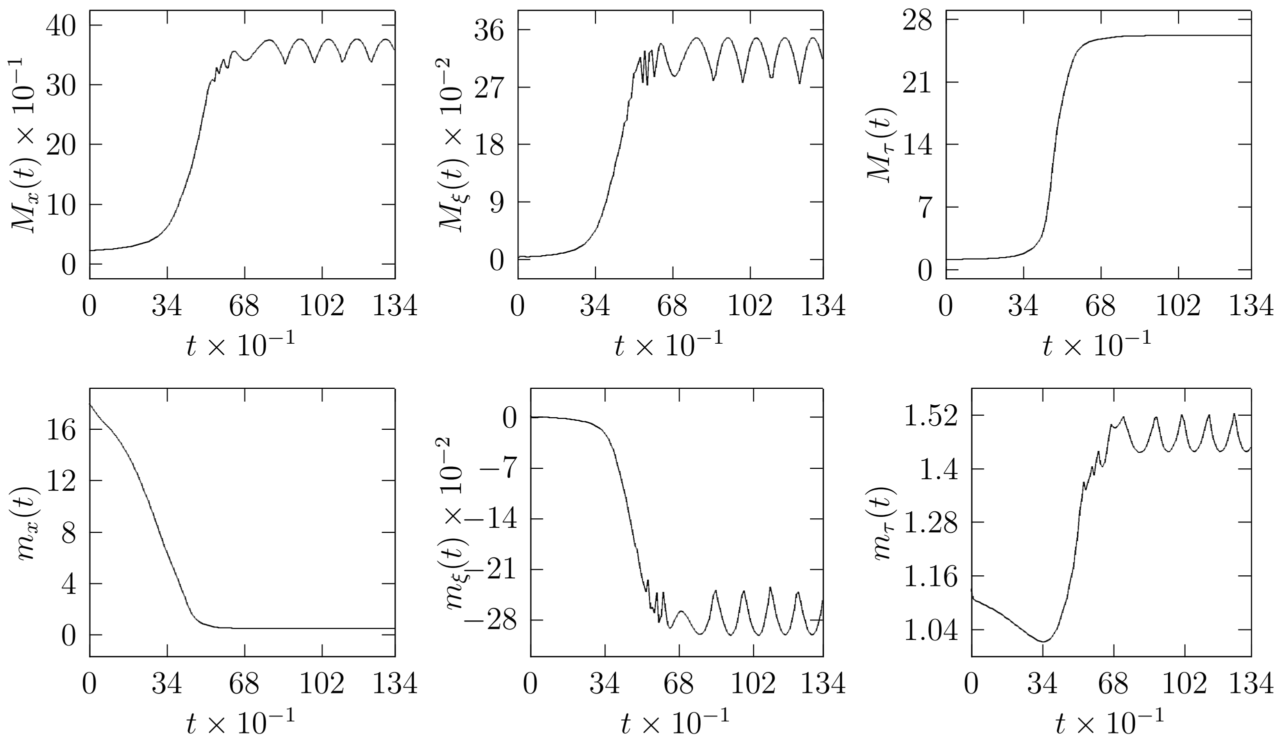

Figure 3: Evolution of the variables (5.1) for $m = 5$.

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