Infinite Laplacian diffusion equations by stochastic time homogenization of a coupled system of first order equations

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1 Introduction

In this work, we consider solutions $u^{\pm,\varepsilon}(t,x,\omega)$ to the following system in dimension $N \geq 1$

$$(1.1) \quad \left\{ \begin{aligned} u_t^{+,\varepsilon} &= -\varepsilon^{-1}\left(\varepsilon^{-1}(u^{+,\varepsilon}-u^{-,\varepsilon}) + a\left(\varepsilon^{-2}t,\omega\right)\right)|Du^{+,\varepsilon}| \\ u_t^{-,\varepsilon} &= +\varepsilon^{-1}\left(\varepsilon^{-1}(u^{+,\varepsilon}-u^{-,\varepsilon}) + a\left(\varepsilon^{-2}t,\omega\right)\right)|Du^{-,\varepsilon}| \end{aligned} \right. \quad \Bigg| \quad \text{on} \quad (0,+\infty)_t \times \mathbb{R}^N_x$$

with initial data for each $\omega \in \Omega$

$$(1.2) \qquad u^{+,\varepsilon}(0,x,\omega) = u^{-,\varepsilon}(0,x,\omega) = u_0(x) \quad \text{for} \quad x \in \mathbb{R}^N$$

Our goal is to study the stochastic homogenization of this system as ε goes to zero. This model is strongly inspired from the modeling of the dynamics of the population of densities of two types of dislocations $+$ and $-$. In dimension $N = 1$, this model is a simplification of a physical model for dislocations given in Groma, Balogh [6], inspired from [5] for a derivation of several models of dislocations dynamics. The periodic homogenization in dimension $N = 1$ has been done in [1]. See also [4] for an analysis for $\varepsilon = 1$ of a similar deterministic system in dimension $N = 1$ with a further non-local term which has been dropped in our model here.

Probabilistic setting
We assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a given probability space and that for each $s \in \mathbb{R}$, $\tau_s : \Omega \to \Omega$ there is a measure preserving transformation which is a group, i.e. satisfies $\tau_0 = Id_\Omega$ and

$$\tau_s \circ \tau_{s'} = \tau_{s+s'} \quad \text{for all} \quad s, s' \in \mathbb{R}$$

Here we assume that

$$a : \mathbb{R} \times \Omega \to \mathbb{R}$$

*In the absence of further informations, the first author thinks it is fair to cosign this work, which was mainly done in collaboration during an invitation in 2007 at Austin University, Texas.

is *stationary*, i.e. for all $t \in \mathbb{R}$ and $s \in \mathbb{R}$ we have

$$a(t+s,\omega) = a(t,\tau_s\omega) \quad \text{for all} \quad \omega \in \Omega$$

We also assume that the family of transformations $\{\tau_s\}_{s\in\mathbb{R}}$ is *ergodic*, i.e. if there exists a set $A \in \mathcal{F}$ such that $\tau_s A = A$ for all $s \in \mathbb{R}$, then $\mathbb{P}(A) = 0$ or 1.

We make the following assumption:

(A) there exists a constant C_0 such that for each $\omega \in \Omega$ we have

$$a(\cdot,\omega) \quad \text{is continuous and} \quad |a(t,\omega)| \le C_0 \quad \text{for all} \quad t \in \mathbb{R}.$$

It can be checked that system (1.1) is quasi-monotone in the sense of Ishii, Koike [7], which will ensure the existence and uniqueness of the solution. Then our main result is

Theorem 1.1 (Time homogenization)
Assume (A) and that the initial data satisfies $u_0 \in W^{2,\infty}_{loc}(\mathbb{R}^N)$ with Du_0 and D^2u_0 bounded on $\mathbb{R}^N$. Assume moreover that there exists $\delta > 0$ such that the initial data satisfies with $x' = (x_1, ..., x_{N-1})$

$$(1.3) \qquad u_0(x', x_N + h) - u_0(x', x_N) \ge \delta h \quad \text{for all} \quad h \in [0,+\infty), \quad (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$$

Then for each $\omega \in \Omega$, there exists a unique solution $(u^{+,\varepsilon}, u^{-,\varepsilon})$ of (1.1)-(1.2). Moreover there exists a continuous and non-negative function $\overline{H} : (0,+\infty) \to [0,+\infty)$ such that almost surely in $\omega \in \Omega$, we have

$$u^{\pm,\varepsilon}(\cdot,\cdot,\omega) \to u^0 \quad in \quad L^\infty_{loc}([0,+\infty) \times \mathbb{R}^N)$$

where u^0 is the unique (viscosity) solution of the infinite Laplacian diffusion equation

$$(1.4) \qquad \begin{cases} u^0_t = \overline{H}(|Du^0|) \dfrac{Du^0}{|Du^0|} \cdot D^2u^0 \cdot \dfrac{Du^0}{|Du^0|} & on \quad (0,+\infty)_t \times \mathbb{R}^N_x \\ u^0 = u_0 \quad on \quad \{0\}_t \times \mathbb{R}^N_x \end{cases}$$

We have

$$\overline{H}(q) = 2q\, \mathbb{E}\left(\{z(t,q,\cdot)\}^2\right) \quad \textit{which is independent on} \quad t$$

with

$$(1.5) \qquad \begin{cases} v(t,q,\omega) = -q \displaystyle\int_{-\infty}^t e^{-2q(t-s)} a(s,\omega) ds \\ z(t,q,\omega) = -\displaystyle\int_{-\infty}^t e^{-2q(t-s)} \left(2v(s,q,\omega) + a(s,\omega)\right) ds \end{cases}$$

Moreover $\bar{H}$ is Lispchitz continuous locally in $(0,+\infty)$.
Finally $\overline{H} > 0$ if and only if almost surely a is not constant.

Notice that the theory for limit equation (1.4) is covered by [2], and can be applied to show the existence and uniqueness of the solution u^0.

One possible meaning of this result is the following. When $a = 0$, the two components of the solution are at the equilibrium when they are equal. When a is a constant, the two components of the solution are basically translated in opposite directions of some quantity depending on a. When a oscillates in time with mean value zero, the effective behaviour at the first order is just an oscillation around the equilibrium position. Now, because the two components do not come back exactly at the same position when their gradient is not constant, this creates a diffucion effect at the second order, which is exactly the result of the theorem, with the correct rescaling in ε, in order to detect this effect.

Let us mention that stochastic homogenization for similar (but different) Hamilton-Jacobi equations have been done. We can cite the original work [11], and for instance the question of correctors is discussed in [8]. Let us also cite [10] for homogenization of Hamilton-Jacobi equations where stochasticity in time is considered. In our case, we will see that we can construct true correctors for our homogenization problem, which simplifies the approach.

1.1 Questions and extensions

It would be interesting to get generalization of this result under weaker assumptions, in particular without assuming the bound from below on the gradient of the initial data in the direction x_N, up to add some other assumptions on a. It would also be interesting to try to weaken as much as possible the assumptions on a.

We could consider the system:

$$\left\{\begin{aligned} u_t^{+,\varepsilon} &= -\varepsilon^{-1}\left(\varepsilon^{-\alpha}(u^{+,\varepsilon}-u^{-,\varepsilon}) + a\left(\varepsilon^{-2}t,\omega\right)\right)|Du^{+,\varepsilon}| \\ u_t^{-,\varepsilon} &= +\varepsilon^{-1}\left(\varepsilon^{-\alpha}(u^{+,\varepsilon}-u^{-,\varepsilon}) + a\left(\varepsilon^{-2}t,\omega\right)\right)|Du^{-,\varepsilon}| \end{aligned}\right. \quad \Bigg| \quad \text{on} \quad (0,+\infty)_t \times \mathbb{R}^N_x$$

supplemented with initial data (1.2).
Our case corresponds to the value $\alpha = 1$. It would be interesting to consider the whole range $(\alpha - \infty, 1]$ (and also the limit $\alpha \to -\infty$).

1.2 Organization of the paper

In Section 2, we recall some useful material for system (1.1), and on the ergodic theorem and its generalizations. In Section 3, we present the heuristic computation which shows formally, but quite simply why our homogenization result is expected. In Section 4, we perform the study of the cell problem. We build correctors and give some properties on the effective Hamiltonian. In Section 5, we make the proof of our main result of convergence, i.e. Theorem 1.1.

2 Useful tools

2.1 Known results for the system

As already mentioned in the Introduction, system (1.1) is quasi-monotone in the sense of Ishii, Koike [7]. In this subsection, we drop the dependence on ω which is not relevant, and set

$$a(t) = a(t, \omega)$$

We recall the definition of a viscosity subsolution for system (1.1), which is an extension of the one in [2].

Definition 2.1 (Viscosity subsolution for the system)
Given a continuous initial data u_0, we say that a couple of upper semi-continuous functions $u = (u^+, u^-)$ is a subsolution of (1.1)-(1.2) with $\varepsilon = 1$, if and olny if for any test function $\varphi \in C^1$, if $u^+ - \varphi$ reaches its maximum at $P_+ = (t_+, x_+)$ (resp. $u^- - \varphi$ reaches its maximum at $P_- = (t_-, x_-)$), then we have

$$\varphi_t \leq -\left(u^+ - u^- + a\left(t_+\right)\right)|D\varphi| \quad at \quad P_+$$

$$(resp. \quad \varphi_t \leq +\left(u^+ - u^- + a\left(t_-\right)\right)|D\varphi| \quad at \quad P_-)$$

Moreover, we require the comparison of the initial data to the initial conditions for each sign

$$u^+(0, \cdot) \leq u_0, \quad u^-(0, \cdot) \leq u_0$$

Similarly, we define a subsolution for general ε. Similarly, we also define the notion of viscosity supersolution for lower semi-continuous functions. A viscosity solution u is then defined as a function such that its upper semi-continuous envelope is a subsolution and its lower semi-continuous envelope is a supersolution.

Given two couples of functions $u = (u^+, u^-)$ and $v = (v^+, v^-)$, we write

$$u \leq v \quad \Longleftrightarrow \quad \left(u^+ \leq v^+ \quad \text{and} \quad u^- \leq v^-\right)$$

For the system, we have the following result

Theorem 2.2 (Comparison principle)
Assume the initial data u_0 as in Theorem 1.1. Let $u = (u^+, u^-)$ be a subsolution and $v = (v^+, v^-)$ be a supersolution for the system (1.1)-(1.2), satisfying for some constant $C > 0$:

$$u_0(x) - Ct \leq u^+(t, x), u^-(t, x), v^+(t, x), v^-(t, x) \leq u_0(x) + Ct \quad for\ all \quad (t, x) \in [0, T) \times \mathbb{R}^N$$

Then

$$u(0, \cdot) \leq v(0, \cdot) \quad \Longrightarrow \quad \left(u \leq v \quad on \quad [0, T) \times \mathbb{R}^N\right)$$

A version of this Theorem is proven in dimension $N = 1$ by El Hajj, Forcadel [4]. The proof of Theorem 2.2 is an easy adaptation of the result of [4], and can be also obtained directly using classical results of Ishii, Koike [7].

The existence of a solution follows by Perron's method as it is classical (see also [4]).Then we deduce the

Corollary 2.3 (Bound from below on the gradient)
Assume the initial data u_0 as in Theorem 1.1 and that the function a satisfies (A). Then there exists a unique solution $(u^{+,\varepsilon}, u^{-,\varepsilon})$ of (1.1)-(1.2). This solution satisfies
(2.6)
$$u^{\pm,\varepsilon}(t,x',x_N+h)-u^{\pm,\varepsilon}(t,x',x_N) \geq \delta h \quad \text{for all} \quad h\in[0,+\infty), \quad (t,x',x_N)\in[0,+\infty)\times\mathbb{R}^{N-1}\times\mathbb{R}$$
Moreover we have
$$\left.\begin{array}{ll} \max_{\pm}|Du^{\pm,\varepsilon}(t,\cdot)|_{L^\infty(\mathbb{R}^N)} & \leq |Du_0|_{L^\infty(\mathbb{R}^N)} =: B_0 \\ & \\ |u^{\pm,\varepsilon}(t,\cdot)-u_0|_{L^\infty(\mathbb{R}^N)} & \leq C_0B_0t \\ & \\ |u_t^{\pm,\varepsilon}(t,\cdot)| & \leq C_0B_0(2tB_0+1) \end{array}\right\} \quad \text{for all} \quad t\in[0,+\infty)$$
Here the constant C_0 is defined in assumption (A).

Proof of Corollary 2.3
The existence follows from the Perron's method and the uniqueness from the comparison principle (Theorem 2.2). Let us consider the solution $u^{\pm,\varepsilon,h}$ of (1.1) with initial data u_0^h defined by
$$u_0^h(x) := u_0(x',x_N+h)-\delta h$$
From (1.3), we know that
$$u_0^h \geq u_0$$
From the comparison principle, we deduce that
$$u^{\pm,\varepsilon,h} \geq u^{\pm,\varepsilon} \tag{2.7}$$
On the other hand, from the invariance by translation of the system, and the fact that for any constant $K\in\mathbb{R}$, $(u^{+,\varepsilon}+K, u^{-,\varepsilon}+K)$ is still a solution, we deduce that
$$u^{\pm,\varepsilon,h}(t,x) = u^{\pm,\varepsilon}(t,x',x_N+h)-\delta h$$
Therefore estimate (2.6) follows from (2.7).
Similarly, we get the bound from above on the gradient. The bound on $u^{\pm,\varepsilon}$ follows from the fact that $(u_0+C_0B_0t, u_0+C_0B_0t)$ is a supersolution, and $(u_0-C_0B_0t, u_0-C_0B_0t)$ is a subsolution.
Finally, the last estimate on $u_t^{\pm,\varepsilon}$ follows from the equation itself. This ends the proof of the Corollary.

We also have a localized version of Theorem 2.2.

Theorem 2.4 (Local comparison principle)
Let $u=(u^+,u^-)$ be a subsolution and $v=(v^+,v^-)$ be a supersolution for the system (1.1) in a half cylinder $Q_r^-(P_0)$ with $P_0=(t_0,x_0)$. Then
$$\sup_{\overline{Q_r^-(P_0)}} \max_{\pm}(u^\pm - v^\pm) \leq \sup_{\partial^- Q_r^-(P_0)} \max_{\pm}(u^\pm - v^\pm)$$
where
$$Q_r^-(P_0) := (t_0-r^2,t_0)\times B_r(x_0), \quad \partial^- Q_r^-(P_0) := (\overline{B_r(x_0)}\times\{t_0-r^2\}) \cup (\partial B_r(x_0)\times[t_0-r^2,t_0])$$

Proof of Theorem 2.4
This result follows from an adaptation of the comparison principle named Theorem 4.7 in Ishii, Koike [7].

2.2 On the ergodic theorem

We recall the classical ergodic theorem in its standard form (Theorem 1.14 on page 34 in [12]; see also Theorem 1.1 on page 89 in [9]).

Theorem 2.5 (Birkhoff's ergodic theorem)
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $T : \Omega \to \Omega$ be a measurable and measure preserving map, i.e. satisfying

$$\mathbb{P}(T^{-1}(A)) = \mathbb{P}(A) \quad \text{for all} \quad A \in \mathcal{F}$$

i) (Convergence of averages)
Then for any $f \in L^1(\Omega, \mathbb{P})$, there exists $\bar{f} \in L^1(\Omega, \mathbb{P})$ such that

$$f_n(\omega) := \frac{1}{n}\sum_{i=0}^{n-1} f(T^i(\omega)) \underset{n\to+\infty}{\to} \bar{f}(\omega) \quad \text{for } \mathbb{P}\text{-almost every } \omega \in \Omega$$

and

$$f_n \to \bar{f} \quad \text{in} \quad L^1(\Omega, \mathbb{P})$$

and

$$\bar{f}(T(\omega)) = \bar{f}(\omega) \quad \text{for } \mathbb{P}\text{-almost every } \omega \in \Omega$$

ii) (Identification of the limit in the ergodic case)
Assume moreover that T is ergodic, ie satisfies

$$T(A) = A \quad \text{for} \quad A \in \mathcal{F} \quad \text{implies} \quad \mathbb{P}(A) = 0 \quad \text{or} \quad 1$$

Then $\bar{f}$ is equal to a constant, ie

$$\bar{f}(\omega) = \int_\Omega f d\mathbb{P} =: \mathbb{E}_\mathbb{P}(f) \quad \text{for } \mathbb{P}\text{-almost every } \omega \in \Omega$$

Using the probabilistic setting of the Introduction, we can then deduce easily from Birkhoff's ergodic theorem, the following version, which is more suitable for our framework.

Theorem 2.6 (Ergodic theorem)
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and assume that $b(t, \omega)$ satisfies assumption (A) and is stationary with respect to a measure preserving group $\{\tau_t\}_{t\in\mathbb{R}}$ which is ergodic. Then there exists $\mathcal{N}_1 \in \mathcal{F}$ with $\mathbb{P}(\mathcal{N}_1) = 0$ such that for any $\omega \in \Omega\backslash\mathcal{N}_1$, we have

$$\frac{1}{t}\int_0^t b(s, \omega) ds \to \mathbb{E}(b) \quad \text{as} \quad |t| \to +\infty \tag{2.8}$$

where

$$\mathbb{E}(b) = \mathbb{E}_\mathbb{P}(b(s, \cdot)) \quad \text{is independent on} \quad s \in \mathbb{R}$$

Remark 2.7 *We have no rate of convergence in (2.8), and this rate may depend on ω.*

Precisely, we will use the following technical extension:

Corollary 2.8 (A variant of the ergodic theorem)
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and assume that $b : \mathbb{R} \times (0, +\infty) \times \Omega \to \mathbb{R}$ is stationary, ie

$$b(t + s, q, \omega) = b(t, q, \tau_s \omega)$$

where the measure preserving group $\{\tau_t\}_{t\in\mathbb{R}}$ is ergodic. We also assume that for any $q \in (0, +\infty)$, the function $b(\cdot, q, \cdot)$ satisfies assumption (A).

Then there exists $\mathcal{N} \in \mathcal{F}$ with $\mathbb{P}(\mathcal{N}) = 0$ such that for any $\omega \in \Omega\backslash\mathcal{N}$ and for any $q \in \mathbb{Q} \cap (0, +\infty)$ and any $\gamma \in \mathbb{Q}$, we have

$$\frac{1}{t}\int_0^t b(s + \gamma t, q, \omega)ds \to \mathbb{E}(b(s, q, \cdot)) \quad \text{as} \quad |t| \to +\infty \tag{2.9}$$

where

$$\mathbb{E}(b(s, q, \cdot)) \quad \text{is independent on} \quad s \in \mathbb{R}$$

Proof of Corollary 2.8
We use a countable set argument. For any fixed $q \in \mathbb{Q} \cap (0, +\infty)$ and $\gamma \in \mathbb{Q}$, we can apply Theorem 2.6 and deduce (after some simple computations splitting the time integral in several integrals of the type in (2.8)) the convergence for any $\omega \in \Omega\backslash\mathcal{N}^{q,\gamma}$ for some set $\mathcal{N}^{q,\gamma}$ satisfying

$$\mathbb{P}(\mathcal{N}^{q,\gamma}) = 0$$

Then defining

$$\mathcal{N} = \bigcup_{q\in\mathbb{Q}\cap(0,+\infty),\ \gamma\in\mathbb{Q}} \mathcal{N}^{q,\gamma}$$

we have $\mathbb{P}(\mathcal{N}) = 0$ and we get the result. This ends the proof of the corollary.

3 Heuristics

In this section we perform some formal computations to understand the homogenization procedure, without any technicalities.
Step 1: Deriving the cell problem
Let us consider the following ansatz in a neighborhood of a point (t_0, x_0) with $t_0 > 0$:

$$u^{\pm,\varepsilon}(t, x, \omega) \simeq u^0(t, x) \pm \varepsilon u^1(\varepsilon^{-2}(t - t_0), x, \omega) + \varepsilon^2 u^2(\varepsilon^{-2}(t - t_0), x, \omega)$$

where u^1 is a corrector at the first order in ε and u^2 is a corrector at the second order. We will see that this second order corrector is fundamental here to recover the diffusion by homogenization.
Calling s the variable $\varepsilon^{-2}(t - t_0)$, we get formally with $a^\varepsilon(s, \omega) = a(s + \varepsilon^{-2}t_0, \omega)$:

$$u^0_t \pm \varepsilon^{-1}u^1_s + u^2_s \simeq \mp\varepsilon^{-1}\left(2u^1 + a^\varepsilon)\right)|Du^0 \pm \varepsilon Du^1 + ...|$$

Calling $\overline{a}$ a formal limit of a^ε as ε goes to zero (with $\overline{a}$ also assummed stationary), this gives at the order ε^{-1}

$$u^1_s = -\left(2u^1 + \overline{a}\right)|Du^0| \tag{3.10}$$

and at the order ε^0

$$u_t^0 + u_s^2 = -\left(2u^1 + \overline{a}\right) \frac{Du^0}{|Du^0|} \cdot Du^1$$

Let us set

$$z = \frac{Du^0}{|Du^0|} \cdot Du^1$$

Taking the derivative of (3.10) with respect to x, we see that z formally satisfies

$$z_s = -2z|Du^0| - \left(2u^1 + \overline{a}\right) \frac{Du^0}{|Du^0|} \cdot D|Du^0|$$

where we have used the fact that the expression $Du^0(t,x)$ is asymptotically independent on s as ε goes to zero. Setting now $\lambda = u_t^0$, $p = Du^0$, $M = D^2u^0$, we get the *cell problem*

$$\text{(3.11)} \qquad \begin{cases} u_s^1 = -\left(2u^1 + \overline{a}\right) q \\ z_s = -2qz - \left(2u^1 + \overline{a}\right) \ \Lambda \\ \lambda + u_s^2 = -\left(2u^1 + \overline{a}\right) z \end{cases}$$

where we have denoted

$$\Lambda = \frac{p}{|p|} \cdot M \cdot \frac{p}{|p|} \quad \text{and} \quad q = |p|$$

Step 2: First computation of the effective Hamiltonian

System (3.11) is now independent on x, and we can solve it for $u^1(s,\omega)$, $z(s,\omega)$ and $u^2(s,\omega)$, dropping the x-dependence. We now remark that in the expansion of $u^{\pm,\varepsilon}$, we have terms like $\varepsilon u^1(\varepsilon^{-2}(t-t_0),\omega)$ and $\varepsilon^2 u^1(\varepsilon^{-2}(t-t_0),\omega)$. Therefore to ensure the homogenization of the system, we expect the following behaviour of the solutions:

$$\text{(3.12)} \qquad \left.\begin{array}{l} \dfrac{u^1(s,\omega)}{\sqrt{s}} \to 0 \\[2ex] \dfrac{z(s,\omega)}{\sqrt{s}} \to 0 \\[2ex] \dfrac{u^2(s,\omega)}{s} \to 0 \end{array}\right\} \quad \text{as} \quad |s| \to +\infty, \qquad \text{a.s. in} \quad \omega$$

We will see in Section 4 that, under our assumptions, we can even choose u^1 and z bounded. On the contrary the best behaviour of $u^2(s,\omega)$ for large s will be the one indicated in (3.12).

Using the linearity of λ in z and of z in Λ, we see in particular that we have

$$\lambda = \overline{H}(q)\ \Lambda$$

for some coefficient $\overline{H}(q)$ which is obtained solving (3.11) for $\Lambda = 1$.

The precise computation of the effective Hamiltonian $\overline{H}(q)$ is done in the next section.

4 Existence of correctors and computation of the effective Hamiltonian

Keeping in mind problem (3.11) for $\Lambda = 1$, we are now considering a function $\overline{a} : \mathbb{R} \times \Omega \to \mathbb{R}$ satisfying (A) which is assumed stationary. We then consider the associated cell problem, i.e. for any $q > 0$, we are looking for a constant $\overline{H}(q) := \lambda$ and functions v, z, w of the variables (s, q, ω) which are solutions on $\mathbb{R}$ of

$$(4.13) \qquad \left\{ \begin{array}{l} v_s = -\left(2v + \overline{a}\right) q \\ \\ z_s = -2qz - \left(2v + \overline{a}\right) \quad \text{with} \quad z = v_q \\ \\ \lambda + w_s = -\left(2v + \overline{a}\right) z \end{array} \right.$$

satisfying

$$(4.14) \qquad \left. \begin{array}{l} \dfrac{v(s,q,\omega)}{\sqrt{s}} \to 0 \\ \\ \dfrac{z(s,q,\omega)}{\sqrt{s}} \to 0 \\ \\ \dfrac{w(s,q,\omega)}{s} \to 0 \end{array} \right\} \quad \text{as} \quad |s| \to +\infty, \qquad \text{a.s. in} \quad \omega$$

We start with the following result

Lemma 4.1 *Let $\overline{a} : \mathbb{R} \times \Omega \to \mathbb{R}$ satisfying (A) and stationary and let $q > 0$. Then the unique bounded solution of*

$$v_s = -(2v + \overline{a})q \quad \textit{on} \quad \mathbb{R}$$

is

$$v(t, q, \omega) = -q \int_{-\infty}^{t} e^{-2q(t-s)} \overline{a}(s, \omega) ds$$

Moreover v is stationary and satisfies

$$|v(\cdot, q, \omega)| \le C_0/2 \quad \textit{and} \quad \mathbb{E}(v) = -\frac{1}{2}\, \mathbb{E}(\overline{a})$$

Proof of Lemma 4.1
The proof of the first part of the Lemma is straightforward and is left to the reader. We now define

$$V(t) = \mathbb{E}(v(t, q, \omega))$$

which satisfies

$$V_s = -(2V + \mathbb{E}(\overline{a}))q$$

From the stationarity of $\bar{a}$ and then of v, we deduce that $V(t)$ is independent on t, and then $2V + \mathbb{E}(\overline{a}) = 0$ which proves the last assertion of the lemma. This ends the proof of the Lemma.

The following main result of this section shows that, for the stochastic homogenization problem we are looking at, there exist true correctors (contrarily to usual problems in stochastic homogenization, see [8]).

Proposition 4.2 (Existence of correctors for $q > 0$)
Given $\overline{a} : \mathbb{R} \times \Omega \to \mathbb{R}$ satisfying (A) and stationary and ergodic, and for any fixed $q > 0$, the functions v, z given in (1.5) and

$$\lambda = 2q\ \mathbb{E}(z^2(t, q, \cdot)) \quad \textit{is independent on } t$$

and

$$w(t, q, \omega) = \frac{1}{2}\left(z^2(t, q, \omega) - z^2(0, q, \omega)\right) - \lambda t + \int_0^t 2qz^2(s, q, \omega)ds$$

are solutions of (4.13)-(4.14), with

$$|v(t, q, \omega)| \le C_0/2 \quad \textit{and} \quad |z(t, q, \omega)| \le C_0/q \quad \textit{for all} \quad t \in \mathbb{R}$$

Moreover $\lambda > 0$ if and only if almost surely $\overline{a}$ is not constant.

Proof of Proposition 4.2
Step 1: bounds on v, z
Applying Lemma 4.1 first to v, and then to z satisfying

$$z_s = -(2z + b)\, q \quad \text{with} \quad b = \frac{2v + \overline{a}}{q}$$

we get the result for v and z.
Step 2: checking that $z = v_q$
We then notice that v_q solves the equation satisfied by z. Hence we get (at least in the sense of distributions in q)

$$(z - v_q)(t, q, \omega) = K(q, \omega)e^{-2qs} \tag{4.15}$$

for some constant $K(q, \omega)$. Approximating v_q by finite differences allows to see that v_q has also to be stationary. Because z is stationary, we see that the ergodic theorem implies that $K(q, \omega)$ has to be zero, in order to avoid a contradiction. This shows that

$$v_q = z$$

Step 3: checking (4.13)
For λ and w, we simply remark from the second line of (4.13), that the third line of (4.13) is equivalent to

$$\lambda + \left(w - \frac{z^2}{2}\right)_s = 2qz^2$$

Because $(t, \omega) \mapsto (z(t, q, \omega)$ is stationary ergodic, this implies the result for w. Indeed from the ergodic theorem (Theorem 2.6), we know that

$$\frac{1}{t}\int_0^t z^2(s, q, \omega)ds \to \mathbb{E}(z^2) \quad \text{as} \quad |t| \to +\infty, \quad \text{a.s. in} \quad \omega$$

Moreover, because z is bounded, we deduce from the expression of w that

$$\frac{w(s, q, \omega)}{s} \to 0 \quad \text{as} \quad |s| \to +\infty, \quad \text{a.s. in} \quad \omega$$

Step 4: characterizing $\lambda > 0$
Finally, let us deal with the condition $\lambda > 0$. If $\lambda = 0$, then $\mathbb{E}(z^2(t,q,\cdot)) = 0$ for each $t \in \mathbb{R}$. This implies in particular that $z(0,q,\omega) = 0$ almost surely. Therefore we deduce that almost surely $z(t,q,\omega) = 0$ for every $t \in \mathbb{Q}$. Now because z is continuous in t (see (1.5)), we deduce that almost surely

$$z(t,q,\omega) = 0 \quad \text{for every} \quad t \in \mathbb{R}$$

Therefore we get from (4.13) that almost surely

$$\overline{a}(s,\omega) = -2v(\omega)$$

Now from the ergodic theorem (Theorem 2.6), we deduce that almost surely

$$v(\omega) = \mathbb{E}(v)$$

and then almost surely

$$\overline{a}(s,\omega) = \mathbb{E}(\overline{a}) \quad \text{for any} \quad s \in \mathbb{R}$$

Reciprocically, if almost surely $\overline{a}$ is constant, then $v = -\overline{a}/2$, $z = 0$ and $\lambda = 0$.
This ends the proof of the proposition.

Later we will use the following properties.

Proposition 4.3 (Lipschitz-continuity of $\overline{H}$ and of v, z)
Under the notation and the assumptions of Proposition 4.2, we have for $\overline{H}(q) := \lambda$ and $q > 0$:

$$|\overline{H}(q)| \le 2C_0^2/q \quad \text{and} \quad |\overline{H}'(q)| \le 10C_0^2/q^2$$

Moreover we have

$$|v_q(t,q,\omega)| \le C_0/q \quad \text{and} \quad |z_q(t,q,\omega)| \le 2C_0/q^2$$

Proof of Proposition 4.3
The first estimate follows from $|z| \le C_0/q$ given in Proposition 4.2. Recall that

$$z = v_q$$

For the second estimate, we set

$$\zeta = z_q$$

and taking the derivative with respect to q in the equation satisfied by z (second line in (4.13)), we see that ζ satisfies

$$\zeta_s = -2q\zeta - 4z$$

Then we have (the unique bounded solution)

$$\zeta(s,q,\omega) = -4\int_{-\infty}^{t} e^{-2q(t-s)} z(s,q,\omega)ds$$

ans then $|\zeta| \le 2|z|_\infty/q \le 2C_0/q^2$. Then

$$\overline{H}'(q) = 2\mathbb{E}(z^2) + 4q\mathbb{E}(zz_q)$$

which implies the result. This ends the proof of the proposition.

Corollary 4.4 (Effect of a time shift on the original a)
For some $h \in \mathbb{R}$, let us define

$$a_h(t,\omega) = a(t+h,\omega)$$

Given $q > 0$, let us call v, z, w, λ the quantities given in Proposition 4.2 associated to $\overline{a} = a$, and $\overline{v}, \overline{z}, \overline{w}, \overline{\lambda}$ associated to $\overline{a} = a_h$. Then we have

$$\overline{v}(t,q,\omega) = v(t+h,q,\omega), \quad \overline{z}(t,q,\omega) = z(t+h,q,\omega)$$

and

$$\overline{\lambda} = \lambda = \overline{H}(q)$$

Proof of Corollary 4.4
The result comes from the fact that the bounded solutions v, z of the first two equations of (4.13) are unique.

5 Proof of Theorem 1.1 and convergence

In this section we will make the proof of Theorem 1.1, but first start with a preliminary result to control the initial conditions uniformly in ε, before to pass to the limit.

Proposition 5.1 (Barriers for the initial data)
Under the assumptions of Theorem 1.1, let $(u^{+,\varepsilon}, u^{-,\varepsilon})$ be the solution to (1.1)-(1.2). For any fixed $(x,\omega) \in \mathbb{R}^N \times \Omega$, let $b(\tau,x,\omega)$ be the solution to the following equation

$$(5.16) \qquad \begin{cases} b_\tau(\tau,x,\omega) = -(2b(\tau,x,\omega) + a(\tau,\omega))|\nabla u_0(x)| & \text{for} \quad \tau \in (0,+\infty) \\ b(0,x,\omega) = 0 \end{cases}$$

Then we have

$$(5.17) \qquad u_0(x) \pm \varepsilon b\left(\varepsilon^{-2}t,x,\omega\right) - \mu t \quad \leq \quad u^{\pm,\varepsilon} \quad \leq \quad u_0(x) \pm \varepsilon b\left(\varepsilon^{-2}t,x,\omega\right) + \mu t$$

with

$$|b(\tau,x,\omega)| \leq C_0/2 \quad \text{and} \quad \mu = 2\frac{C_0^2}{\delta}|D^2 u_0|_{L^\infty(\mathbb{R}^N)}$$

Proof of Proposition 5.1
We perform the proof in four steps, first estimating some quatities useful in the construction of a supersolution.
Step 1: Estimate on b
Let $q(x) = |\nabla u_0(x)|$. Then from (1.3), we have

$$q(x) \geq \left|\frac{\partial u_0}{\partial x_N}(x)\right| \geq \delta > 0$$

and we can compute

$$b(\tau,x,\omega) = -q(x)\int_0^\tau e^{-2q(x)(\tau-s)}a(s,\omega)ds$$

which implies

$$|b|_\infty \le |a|_\infty/2 \le C_0/2$$

Step 2: Estimate on $\nabla_x b$
We now take a derivative of equation (5.16) with respect to x, and get for $c = \nabla_x b$ which satisfies (at least for a.e. $x \in \mathbb{R}^N$ when $\nabla_x u_0$ is only Lipschitz, and for every $x \in \mathbb{R}^N$ when we assume moreover that $\nabla_x u_0$ is C^1)

$$c_\tau = -2c|\nabla u_0(x)| - (2b+a)\nabla_x|\nabla u_0(x)| \tag{5.18}$$

Setting

$$K_0 = (2|b|_\infty + |a|_\infty)|\nabla_x|\nabla u_0(x)||_\infty$$

we get

$$\begin{cases} |c|_\tau \le -2|c||\nabla u_0(x)| + K_0 \quad \text{for} \quad \tau \in (0,+\infty) \\ \\ |c| = 0 \quad \text{at} \quad \tau = 0 \end{cases}$$

We deduce similarly that

$$|c|_\infty \le \frac{K_0}{2\delta} \le \frac{C_0}{\delta}|D^2 u_0|_{L^\infty(\mathbb{R}^N)}$$

which is then justified for $\nabla_x u_0$ only Lipschitz, using some approximation argument.
Step 3: Construction of a supersolution
We now consider $\underline{u} = (\underline{u}^+, \underline{u}^-)$ with

$$\underline{u}^\pm(t,x,\omega) = u_0(x) \pm \varepsilon b\left(\varepsilon^{-2}t, x, \omega\right) + \mu t$$

If $u_0 \in C^2$, then from (5.18) we see that $\nabla_x b$ is continuous in (τ, x) and then $\underline{u}$ is an admissible test function. In that case we compute

$$\begin{aligned} &\underline{u}^\pm_t \pm \varepsilon^{-1}\left(\varepsilon^{-1}(\underline{u}^+ - \underline{u}^-) + a\left(\varepsilon^{-2}t, \omega\right)\right)|D\underline{u}^\pm| \\ &= \pm\varepsilon^{-1} b_\tau + \mu \pm \varepsilon^{-1}(2b+a)|\nabla u_0(x) \pm \varepsilon\nabla_x b| \\ &\ge \mu - (2|b|_\infty + |a|_\infty)|\nabla_x b|_\infty \\ &\ge 0 \end{aligned}$$

where the last inequality is true for our particular choice of μ. Therefore from the comparison principle (Theorem 2.2), we deduce that

$$u^{\pm,\varepsilon} \le \underline{u}^\pm \quad \text{on} \quad [0,+\infty) \times \mathbb{R}^N \tag{5.19}$$

In the case where the initial data u_0 only belongs to $W^{2,\infty}(\mathbb{R}^N)$ without being C^2, we simply approximate this initial data by a smooth one, for which we have estimate (5.19), which stays true passing to the limit.
Step 4: Construction of a subsolution and conclusion
Similarly, we show that

$$\overline{u}^\pm(t,x,\omega) = u_0(x) \pm \varepsilon b\left(\varepsilon^{-2}t, x, \omega\right) - \mu t$$

is a subsolution and get the other part of the inequality (5.17). This ends the proof of the proposition.

Proof of Theorem 1.1
Step 1: Construction of an exceptional set $\mathcal{N}$ of probability zero
Let us consider a sequence $(\gamma_k)_{k\in\mathbb{N}}$ such that

$$\gamma_{k'} \neq \gamma_k \quad \text{if} \quad k' \neq k$$

and

$$\mathbb{Q} = \bigcup_{k\in\mathbb{N}} \{\gamma_k\}$$

We first apply a variant of the ergodic theorem (i.e. Corollary 2.8) to $b(t,q,\omega) = z^2(t,q,\omega)$ where $z(t,q,\omega)$ is a solution of (4.13), and get the existence of a set $\mathcal{N}$ with

$$\mathbb{P}(\mathcal{N}) = 0$$

such that for any $\omega \in \Omega\backslash\mathcal{N}$ and for any $q \in \mathbb{Q}_+^*$, we have for any $\gamma \in \mathbb{Q}$

$$\frac{1}{s}\int_0^s z^2(\overline{s}+\gamma s, q,\omega)d\overline{s} \to \mathbb{E}(z^2(t,q,\cdot)) \quad \text{as} \quad |s| \to +\infty \tag{5.20}$$

where

$$\mathbb{E}(z^2(t,q,\cdot)) \quad \text{is independent on} \quad t\in\mathbb{R}$$

Step 2: Construction of the half-relaxed limits on $\Omega\backslash\mathcal{N}$
For any fixed $\omega \in \Omega\backslash\mathcal{N}$, we define

$$\overline{u}(t,x,\omega) = \limsup_{\varepsilon\to 0}{}^* \left(\max_{\pm} u^{\pm,\varepsilon}(t,x,\omega)\right)$$

and

$$\underline{u}(t,x,\omega) = \liminf_{\varepsilon\to 0}{}_* \left(\min_{\pm} u^{\pm,\varepsilon}(t,x,\omega)\right)$$

We want to show that for any $\omega \in \Omega\backslash\mathcal{N}$, the function $\overline{u}$ is a subsolution of the limit equation (1.4) (the proof is the same to show that $\underline{u}$ is a supersolution of (1.4)).
By Proposition 5.1, we already get that

$$|\overline{u}(t,x,\omega) - u_0(x)| \le \mu t$$

and therefore, we only have to check the viscosity inequality at an interior point (i.e. for positive time).
Let us now fix some $\omega \in \Omega\backslash\mathcal{N}$ and assume that $\overline{u}(\cdot,\cdot,\omega)$ is not a subsolution of (1.4). Because ω is fixed, we can now *drop the dependence on ω everywhere*, in order to simplify the notation, setting:

$$\overline{u}(t,x) = \overline{u}(t,x,\omega), \quad a(s) = a(s,\omega)$$

$$v(s,q) = v(s,q,\omega), \quad z(s,q) = z(s,q,\omega)$$

for v,z given in (1.5), and for $q = |p|$ which will be precised below.
In particular, reminding (2.6), we have
(5.21)

$$\overline{u}(t,x',x_N+h) - \overline{u}(t,x',x_N) \ge \delta h \quad \text{for all} \quad h\in[0,+\infty), \quad (t,x',x_N)\in[0,T)\times\mathbb{R}^{N-1}\times\mathbb{R}$$

Then there exists a point $P_0 = (t_0, x_0)$ and a test function $\varphi \in C^2$ such that

$$(5.22) \qquad \begin{cases} \overline{u} \le \varphi \quad \text{and} \quad \overline{u}(P_0) = \varphi(P_0) \\ \varphi - \overline{u} > 0 \quad \text{on} \quad \partial^- Q_r^-(P_0) \quad \text{for any} \quad r > 0 \quad \text{small enough} \\ \varphi_t(P_0) = \alpha + F(D\varphi(P_0), D^2\varphi(P_0)) \quad \text{with} \quad \alpha > 0 \end{cases}$$

with

$$Q_r^-(P_0) := (t_0 - r^2, t_0) \times B_r(x_0), \quad \partial^- Q_r^-(P_0) := (\overline{B_r(x_0)} \times \{t_0 - r^2\}) \cup (\partial B_r(x_0) \times [t_0 - r^2, t_0])$$

where F is the function defined by

$$F(p, M) = \overline{H}(|p|) \, \frac{p}{|p|} \cdot M \cdot \frac{p}{|p|}$$

which is continuous for $|p| \neq 0$, by Proposition 4.3.
Notice that by (5.21), we necessarily have

$$|D\varphi(P_0)| \ge \frac{\partial \varphi}{\partial x_N}(P_0) \ge \delta > 0$$

which gives a meaning to the last equation of (5.22).

Step 3: Construction of the perturbed test function
We set

$$a_\varepsilon(s) := a(s + \varepsilon^{-2} t_0), \quad v_\varepsilon(s, q) := v(s + \varepsilon^{-2} t_0, q), \quad z_\varepsilon(s, q) := z(s + \varepsilon^{-2} t_0, q)$$

Reminding Corollary 4.4, and using also the definition of w given in Proposition 4.2, we set

$$(5.23) \qquad w_\varepsilon(s, q) := \frac{1}{2}\left(z_\varepsilon^2(s, q) - z_\varepsilon^2(0, q)\right) - s\overline{H}(q) + \int_0^s 2q z_\varepsilon^2(\overline{s}, q) d\overline{s}$$

which satisfies

$$(5.24) \qquad (w_\varepsilon)_s(s, q) = -\overline{H}(q) - (2v_\varepsilon(s, q) + a_\varepsilon(s)) z_\varepsilon(s, q)$$

For $P = (t, x)$, using the fact that $q = |D\varphi(P)| \ge \delta/2$ for $P \in B_{2r}(P_0)$ with r small enough, we define the perturbed test function (in the spirit of [3])

$$(5.25) \qquad \varphi^{\pm,\varepsilon}(t, x) = \varphi(t, x) \pm \varepsilon v_\varepsilon\left(\varepsilon^{-2}(t - t_0), |D\varphi(t, x)|\right) + \varepsilon^2 \Lambda_0 w_\varepsilon\left(\varepsilon^{-2}(t - t_0), q_k\right)$$

where

$$\Lambda_0 = \Lambda(P_0) \quad \text{with} \quad \Lambda(P) = \frac{D\varphi(P)}{|D\varphi(P)|} \cdot D^2\varphi(P) \cdot \frac{D\varphi(P)}{|D\varphi(P)|}$$

and where $q_k = q_k(\omega)$ is chosen as a rational perturbation of $D\varphi(P_0)$

$$q_k \in \mathbb{Q} \cap (0, +\infty) \quad \text{with} \quad |q_k - |D\varphi(P_0)|| \le r$$

in order to avoid any x-dependence of the term of order ε^2 in the definition of $\varphi^{\pm,\varepsilon}$.

Step 4: $\varphi^{\pm,\varepsilon}$ is a supersolution of (1.1) in a neighborhood of P_0
Because the functions $\varphi^{\pm,\varepsilon}$ are smooth enough, it is sufficient to plug this expression into the equation to check that it is a supersolution. We have with $s = \varepsilon^{-2}(t-t_0)$, $q = |D\varphi(t,x)|$:

$$\varphi_t^{\pm,\varepsilon} = \varphi_t \pm \varepsilon^{-1}(v_\varepsilon)_s(s,q) + \Lambda_0(w_\varepsilon)_s(s,q_k)$$

and

$$\begin{aligned}
&\mp\varepsilon^{-1}\left(\varepsilon^{-1}(\varphi^{+,\varepsilon} - \varphi^{-,\varepsilon}) + a(\varepsilon^{-2}t)\right)|D\varphi^{\pm,\varepsilon}| \\
&= \mp\varepsilon^{-1}(2v_\varepsilon(s,q) + a_\varepsilon(s))\,|D\varphi(t,x) \pm \varepsilon(v_\varepsilon)_q(s,q)(D|D\varphi(t,x)|)| \\
&= \mp\varepsilon^{-1}(2v_\varepsilon(s,q) + a_\varepsilon(s))\,(q \pm \varepsilon\Lambda(t,x)z_\varepsilon(s,q) + O(\varepsilon^2))
\end{aligned}$$

where $O(\varepsilon^2)$ is uniform for $(t,x) \in Q_r^-(P_0)$, because of the bound

$$|(v_\varepsilon)_q| = |z_\varepsilon| \le C_0/q \le 2C_0/\delta \tag{5.26}$$

given in Proposition 4.2. For the last line we have also used the fact that $(v_\varepsilon)_q = z_\varepsilon$. Therefore we get

$$\begin{aligned}
&\varphi_t^{\pm,\varepsilon} \pm \varepsilon^{-1}\left(\varepsilon^{-1}(\varphi^{+,\varepsilon} - \varphi^{-,\varepsilon}) + a(\varepsilon^{-2}t)\right)|D\varphi^{\pm,\varepsilon}| \\
&= \varphi_t(t,x) - \overline{H}(q_k)\Lambda_0 - \Lambda_0(2v_\varepsilon(s,q_k) + a^\varepsilon(s))z_\varepsilon(s,q_k) + \Lambda(t,x)(2v_\varepsilon(s,q) + a^\varepsilon(s))z_\varepsilon(s,q) + O(\varepsilon) \\
&= \alpha + o_r(1) + O(\varepsilon)
\end{aligned}$$

In the second line, we have used equation (5.24) and the first equation of (4.13) satisfied by v_ε with a replaced by a_ε. In the third line, we have set for $(t,x) \in Q_r^-(P_0)$:

$$\begin{aligned}
o_r(1) \quad = \quad & \varphi_t(t,x) - \varphi_t(P_0) \\
+ \quad & \overline{H}(|D\varphi(P_0)|)\Lambda(P_0) - \overline{H}(q_k)\Lambda_0 \\
+ \quad & \Lambda_0\left\{(2v_\varepsilon(s,q) + a^\varepsilon(s))z_\varepsilon(s,q) - (2v_\varepsilon(s,q_k) + a^\varepsilon(s))z_\varepsilon(s,q_k)\right\} \\
+ \quad & (\Lambda(t,x) - \Lambda_0)(2v_\varepsilon(s,q) + a^\varepsilon(s))z_\varepsilon(s,q)
\end{aligned}$$

To justify the notation $o_r(1)$, we use in particular the fact that v_ε and z_ε are Lipschitz-continuous with respect to q, uniformly in ε (see Proposition 4.3). We deduce that for $(t,x) \in Q_r^-(P_0)$ and ε small enough.

$$\varphi_t^{\pm,\varepsilon} \pm \varepsilon^{-1}\left(\varepsilon^{-1}(\varphi^{+,\varepsilon} - \varphi^{-,\varepsilon}) + a(\varepsilon^{-2}t)\right)|D\varphi^{\pm,\varepsilon}| \quad \ge \alpha/2 > 0$$

and therefore $\varphi^{\pm,\varepsilon}$ is a supersolution of (1.1) on $Q_r^-(P_0)$ for r and ε small enough.
Step 5: Consequences of the local comparison principle
Now from the local comparison principle (Theorem 2.4), we deduce with the notation

$$u^{\pm,\varepsilon}(t,x) = u^{\pm,\varepsilon}(t,x,\omega)$$

that

$$\sup_{Q_r^-(P_0)} \max_{\pm}(u^{\pm,\varepsilon} - \varphi^{\pm,\varepsilon}) \le \sup_{\partial^- Q_r^-(P_0)} \max_{\pm}(u^{\pm,\varepsilon} - \varphi^{\pm,\varepsilon})$$

We now make the
Claim : $\lim_{\varepsilon\to 0} \sup_{Q_r^-(P_0)} |\varphi^{\pm,\varepsilon} - \varphi| = 0$ **for each** $\omega \in \Omega\backslash\mathcal{N}$
Let us first assume that the claim is true (this will be proven in the next step). Then we deduce that

$$0 = \overline{u}(P_0) - \varphi(P_0) \leq \sup_{\partial^- Q_r^-(P_0)} \max_{\pm}(\overline{u} - \varphi)$$

This is in contradiction with (5.22).
Step 6: Proof of the claim
From the expression (5.25) of $\varphi^{\pm,\varepsilon}$ and the fact that $|v_\varepsilon| \leq C_0/2$ (see Proposition 4.2), it is sufficient to show that

$$\varepsilon^2 w_\varepsilon(\varepsilon^{-2}(t-t_0), q_k) \to 0 \quad \text{as} \quad \varepsilon \to 0 \quad \text{uniformly for} \quad t \in [t_0 + r, t_0]$$

From the expression (5.23), we recall that

$$\varepsilon^2 w_\varepsilon(s, q_k) = \frac{\varepsilon^2}{2}\left(z_\varepsilon^2(s, q_k) - z_\varepsilon^2(0, q_k)\right) + 2q_k\psi_\varepsilon(s)$$

where

$$\psi_\varepsilon(s) := \frac{1}{2q_k}\varepsilon^2 \left\{ -s\overline{H}(q_k) + \int_0^s 2q_k z_\varepsilon^2(\overline{s}, q_k)d\overline{s} \right\} \tag{5.27}$$

From the bound (5.26) on $|z_\varepsilon|$, we see that it is sufficient to show that with $s = \varepsilon^{-2}(t - t_0)$

$$\psi_\varepsilon(\varepsilon^{-2}(t-t_0)) \to 0 \quad \text{as} \quad \varepsilon \to 0 \quad \text{uniformly for} \quad t \in [t_0 + r, t_0] \tag{5.28}$$

Moreover, we see from (5.27) that there exists a constant $C > 0$ (independent on ε) such that

$$|\psi_\varepsilon(s') - \psi_\varepsilon(s)| \leq C\varepsilon^2|s' - s| \quad \text{and} \quad \psi_\varepsilon(0) = 0 \tag{5.29}$$

We remark that we can rewrite

$$\psi_\varepsilon(s) = \overline{\psi}_\gamma(s) \quad \text{with} \quad \gamma = \frac{t_0}{\varepsilon^2 s}$$

where

$$\overline{\psi}_\gamma(s) = \frac{t_0}{\gamma}\left\{ -\mathbb{E}(z^2(0, q_k, \cdot)) + \frac{1}{s}\int_0^s z^2(\overline{s} + \gamma s, q_k)d\overline{s} \right\}$$

Now from the ergodic property (5.20), we know that for any $\gamma \in \mathbb{Q}$, the term in the bracket goes to zero as $|s| \to +\infty$. Therefore we have

$$\frac{1}{\gamma} = \frac{t - t_0}{t_0} \in \mathbb{Q} \Longrightarrow \left(\psi_\varepsilon(\varepsilon^{-2}(t - t_0)) \to 0 \quad \text{as} \quad \varepsilon \to 0\right) \tag{5.30}$$

Defining

$$\Psi_\varepsilon(t) = \psi_\varepsilon(\varepsilon^{-2}(t - t_0))$$

we can rephrase (5.29)-(5.30) as

$$|\Psi_\varepsilon(t') - \Psi_\varepsilon(t)| \leq C|t' - t| \quad \text{and} \quad \Psi_\varepsilon(t_0) = 0$$

and

$$\frac{t-t_0}{t_0} \in \mathbb{Q} \Longrightarrow (\Psi_\varepsilon(t) \to 0 \quad \text{as} \quad \varepsilon \to 0)$$

This implies that

$$\Psi_\varepsilon \to 0 \quad \text{in} \quad L^\infty_{loc}(\mathbb{R})$$

which gives in particular (5.28). This ends the proof of the claim.

Step 7: Conclusion

From the contradiction obtained in Step 5, we deduce that $\overline{u}$ is a subsolution of the limit equation (1.4). Proceeding similarly, we can show that $\underline{u}$ is a subsolution. Then the comparison principle for the limit equation implies that

$$\overline{u} \leq \underline{u}$$

But by construction, we have the reverse inequality. This implies that

$$\overline{u} = \underline{u} = u^0$$

where u^0 is the solution of (1.4). This ends the proof of the Theorem.

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