HOMOGENIZATION OF THE PEIERLS-NABARRO MODEL FOR DISLOCATION DYNAMICS AND THE OROWAN'S LAW

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ABSTRACT. This paper is concerned with a result of homogenization of an integro-differential equation describing dislocation dynamics. Our model involves both an anisotropic Lévy operator of order 1 and a potential depending periodically on u/ϵ. The limit equation is a non-local Hamilton-Jacobi equation, which is an effective plastic law for densities of dislocations moving in a single slip plane. In dimension 1, we are able to characterize the Hamiltonian of the limit equation close to the origin, recovering a property known in physics as the Orowan's law.

1. INTRODUCTION

In this paper we are interested in homogenization of the Peierls-Nabarro model, which is a phase field model describing dislocations. This model leads to a non-local time dependent PDE with a first order Lévy operator. After a proper rescaling, a macroscopic model describing the evolution of a density of dislocations is obtained.

For a physical introduction to the Peierls-Nabarro model, see for instance [23]; for a recent reference, see [43]; we also refer the reader to the paper of Nabarro [38] which presents an historical tour on the Peierls-Nabarro model.

1.1. **Setting of the problem.** We investigate the limit as $\epsilon \to 0$ of the viscosity solution u^ϵ of the following integro-differential equation (which is an evolution equation associated to the classical Peierls-Nabarro model that has for instance been considered in [36] (see also [13] for a similar model)):

$$\begin{cases} \partial_t u^\epsilon = \mathcal{I}_1[u^\epsilon(t,\cdot)] - W'\left(\frac{u^\epsilon}{\epsilon}\right) + \sigma\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}\right) & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N \\ u^\epsilon(0,x) = u_0(x) & \text{on} \quad \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $\mathcal{I}_1$ is an anisotropic Lévy operator of order 1, defined on bounded C^2- functions for $r > 0$ by

$$\begin{aligned} \mathcal{I}_1[U](x) = & \int_{|z|\le r} (U(x+z) - U(x) - \nabla U(x)\cdot z)\frac{1}{|z|^{N+1}} g\left(\frac{z}{|z|}\right) dz \\ & + \int_{|z|>r} (U(x+z) - U(x))\frac{1}{|z|^{N+1}} g\left(\frac{z}{|z|}\right) dz, \end{aligned} \tag{1.2}$$

where the function g satisfies

(H1) $g \in C(\mathbf{S}^{N-1})$, $g > 0, g$ even.

On the functions W, σ and u_0 we assume:

(H2) $W \in C^{1,1}(\mathbb{R})$ and $W(v+1) = W(v)$ for any $v \in \mathbb{R}$;

(H3) $\sigma \in C^{0,1}(\mathbb{R}^+ \times \mathbb{R}^N)$ and $\sigma(t+1,x) = \sigma(t,x)$, $\sigma(t,x+k) = \sigma(t,x)$ for any $k \in \mathbb{Z}^N$ and $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^N$;

(H4) $u_0 \in W^{2,\infty}(\mathbb{R}^N)$.

When $g \equiv C_N$, with C_N a suitable constant depending on the dimension N, then (1.2) is the integral representation of $-(-\Delta)^{\frac{1}{2}}$ for bounded real smooth functions defined on $\mathbb{R}^N$ (see Theorem 1 in [26]). We recall that $(-\Delta)^{\frac{1}{2}}$ is the fractional operator defined for instance on the Schwartz class $S(\mathbb{R}^N)$ by

$$\widehat{(-\Delta)^{\frac{1}{2}}v}\ (\xi) = |\xi|\ \widehat{v}(\xi), \tag{1.3}$$

where $\widehat{w}$ is the Fourier transform of w.

We prove that the limit u^0 of u^ϵ as $\epsilon \to 0$ exists and is the unique solution of the homogenized problem

$$\begin{cases} \partial_t u = \overline{H}(\nabla_x u, \mathcal{I}_1[u(t,\cdot)]) & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N \\ u(0,x) = u_0(x) & \text{on} \quad \mathbb{R}^N, \end{cases} \tag{1.4}$$

for some continuous function $\overline{H}$ usually called *effective Hamiltonian.*

1.2. **Main results.** As usual in periodic homogenization, the limit equation is determined by a *cell problem.* In our case, such a problem is for any $p \in \mathbb{R}^N$ and $L \in \mathbb{R}$ the following:

$$\begin{cases} \lambda + \partial_\tau v = \mathcal{I}_1[v(\tau,\cdot)] + L - W'(v + \lambda\tau + p\cdot y) + \sigma(\tau,y) & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N \\ v(0,y) = 0 & \text{on} \quad \mathbb{R}^N, \end{cases} \tag{1.5}$$

where $\lambda = \lambda(p, L)$ is the unique number for which there exists a solution of (1.5) which is bounded on $\mathbb{R}^+ \times \mathbb{R}^N$. In order to solve (1.5), we show for any $p \in \mathbb{R}^N$ and $L \in \mathbb{R}$ the existence of a unique solution of

$$\begin{cases} \partial_\tau w = \mathcal{I}_1[w(\tau,\cdot)] + L - W'(w + p\cdot y) + \sigma(\tau,y) & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N \\ w(0,y) = 0 & \text{on} \quad \mathbb{R}^N, \end{cases} \tag{1.6}$$

and we look for some $\lambda \in \mathbb{R}$ for which $w - \lambda\tau$ is bounded. Precisely we have:

Theorem 1.1 (Ergodicity)**.** *Assume (H1)-(H4). For $L \in \mathbb{R}$ and $p \in \mathbb{R}^N$, there exists a unique viscosity solution $w \in C_b(\mathbb{R}^+ \times \mathbb{R}^N)$ of* (1.6) *and there exists a unique $\lambda \in \mathbb{R}$ such that w satisfies: $\frac{w(\tau,y)}{\tau}$ converges towards λ as $\tau \to +\infty$, locally uniformly in y. The real number λ is denoted by $\overline{H}(p, L)$. The function $\overline{H}(p, L)$ is continuous on $\mathbb{R}^N \times \mathbb{R}$ and non-decreasing in L.*

Unfortunately, we cannot directly use the bounded solution of (1.5), usually called *corrector*, in order to prove the convergence of the sequence u^ϵ to the solution of (1.4). Nevertheless we have the following result:

Theorem 1.2 (Convergence)**.** *Assume (H1)-(H4). The solution u^ϵ of* (1.1) *converges towards the solution u^0 of* (1.4) *locally uniformly in (t,x), where $\overline{H}$ is defined in Theorem 1.1.*

The effective Hamiltonian, defined by the cell problem, is usually unknown. We are able to characterize it close to the origin, in dimension $N = 1$, when $\mathcal{I}_1$ is the half-Laplacian (i.e. $g \equiv 1/\pi$) and $\sigma \equiv 0$, assuming moreover the following properties on the potential W:

$$\begin{cases} W \in C^{4,\beta}(\mathbb{R}) & \text{for some } 0 < \beta < 1 \\ W(v+1) = W(v) & \text{for any } v \in \mathbb{R} \\ W = 0 & \text{on } \mathbb{Z} \\ W > 0 & \text{on } \mathbb{R} \setminus \mathbb{Z} \\ \alpha = W''(0) > 0. \end{cases} \tag{1.7}$$

Under assumption (1.7), it is in particular known (see Cabré and Solà-Morales [9]) that there exists a unique function ϕ solution of

$$\begin{cases} \mathcal{I}_1[\phi] = W'(\phi) & \text{in} \quad \mathbb{R} \\ \lim_{x\to-\infty} \phi(x) = 0, \quad \lim_{x\to+\infty} \phi(x) = 1, \quad \phi(0) = \frac{1}{2} & \\ \phi' > 0 & \text{in} \quad \mathbb{R}. \end{cases} \tag{1.8}$$

Indeed, in this special case we have

Theorem 1.3 (Orowan's law)**.** *Assume* (1.7)*,* $N = 1$*,* $g \equiv 1/\pi$*,* $\sigma \equiv 0$ *and let* p_0*,* $L_0 \in \mathbb{R}$*. Then the function* $\overline{H}$ *defined in Theorem 1.1 satisfies*

$$\frac{\overline{H}(\delta p_0, \delta L_0)}{\delta^2} \to c_0|p_0|L_0 \quad \text{as } \delta \to 0^+ \quad \text{with} \quad c_0 = \left(\int_{\mathbb{R}} (\phi')^2\right)^{-1}. \tag{1.9}$$

This result is obtained by a fine comparison of the correctors of the cell problem to explicit solutions, using the phase transition ϕ with some suitable corrections. Property (1.9) is known in physics as the *Orowan's law* (see for instance [42] or p. 3739 in [40]). This law states that the plastic strain velocity is proportional to the product of the dislocation density $|p_0|$ by the effective stress L_0. It is interesting to relate this result to other recent results. To this end, let us consider the following equation (which is equation (1.1) with $N = 1$, $\epsilon = 1$ and a constant prefactor δ in front of σ):

$$\partial_t u = \mathcal{I}_1[u(t,\cdot)] - W'(u) + \delta \cdot \sigma(t,x) \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}$$

Let us set

$$u^{\epsilon,\delta}(t,x) = \epsilon u\left(\frac{t}{\delta^2\epsilon}, \frac{x}{\delta\epsilon}\right)$$

In the present paper, we consider first the homogenization problem as $\epsilon \to 0$, and in a second step look at the Orowan's law when $\delta \to 0$ (only when $\sigma \equiv 0$ and $N = 1$). The inverse procedure has also been studied. In [22], the authors consider first the limit $\delta \to 0$ in dimension $N = 1$ (for general σ), and in [16], the authors consider the second step, i.e. the limit $\epsilon \to 0$ (again for general σ). Choosing again $\sigma \equiv 0$ in the limit model, they also recover the Orowan's law in this special case (see Theorem 2.6 (1.) in [16]).

1.3. **Strategy for proof of the homogenization result.** This non-local equation (1.1) is related to the local equation

$$\begin{cases} \partial_t u^\epsilon = F\left(\frac{x}{\epsilon}, \frac{u^\epsilon}{\epsilon}, \nabla u^\epsilon\right) & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N \\ u^\epsilon(0,x) = u_0(x) & \text{on} \quad \mathbb{R}^N, \end{cases} \tag{1.10}$$

that the first author studied in [31] under the assumption that $F(x,u,p)$ is periodic in (x,u) and coercive in p. The homogenization problem (1.10) when F does not depend on u, has been completely solved by Lions Papanicolaou and Varadhan [35]. After this seminal paper, homogenization of Hamilton-Jacobi equations for coercive Hamiltonians has been treated for a wider class of periodic situations, c.f. Ishii [29], for problems set on bounded domains, c.f. Alvarez [1], Horie and Ishii [24], for equations with different structures, c.f. Alvarez and Ishii [4], for deterministic control problems in L^∞, c.f. Alvarez and Barron [2], for almost periodic Hamiltonians, c.f. Ishii [28], and for Hamiltonians with stochastic dependence, c.f. Souganidis [41]. More recently, inspired by [31], Barles [6] gave an homogenization result for non-coercive Hamiltonians and, as a by-product, obtained a simpler proof of the results [31] of Imbert and Monneau but under slightly more restrictive assumptions on the Hamiltonians. Finally, Imbert, Monneau and Rouy [32] studied homogenization of certain integro-differential equations depending explicitly on u^ϵ/ϵ.

To overcome the additional difficulty raised by the dependence of F on the oscillating variable u^ϵ/ϵ, and solve the homogenization problem (1.10), the authors in [31] imbed the original equations in a higher dimensional space and introduce twisted correctors. Here we have to face a similar difficulty in the much more involved framework of non-local equations and it does not seem possible to apply the approach of Barles [6]. Therefore following the idea in [31], we consider the solution U^ϵ of

$$\begin{cases} \partial_t U^\epsilon = \mathcal{I}_1[U^\epsilon(t,\cdot,x_{N+1})] - W'\left(\frac{U^\epsilon}{\epsilon}\right) + \sigma\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}\right) & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^{N+1} \\ U^\epsilon(0,x,x_{N+1}) = u_0(x) + p_{N+1}x_{N+1} & \text{on} \quad \mathbb{R}^{N+1}, \end{cases} \tag{1.11}$$

where $p_{N+1} \neq 0$. We then consider the following ansatz:

$$U^\epsilon(t,x,x_{N+1}) \simeq U^0(t,x,x_{N+1}) + \epsilon V\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}, \frac{U^0(t,x,x_{N+1}) - \lambda t - p\cdot x}{\epsilon p_{N+1}}\right)$$

where $U^0(t,x,x_{N+1}) = u^0(t,x) + p_{N+1}x_{N+1}$. This ansatz turns out to be the good one, and plugging this expression of U^ϵ into (1.11), we find formally with $\tau = \frac{t}{\epsilon}$, $y = \frac{x}{\epsilon}$, $y_{N+1} = \frac{U^0(t,x,x_{N+1}) - \lambda t - p\cdot x}{p_{N+1}\epsilon}$:

$$\lambda + \partial_\tau V = L + \mathcal{I}_1[V(\tau,\cdot,y_{N+1})] - W'(V + p\cdot y + p_{N+1}y_{N+1} + \lambda\tau) + \sigma(\tau,y), \tag{1.12}$$

where

$$\lambda = \partial_t U^0(t,x,x_{N+1}) = \partial_t u^0(t,x), \quad p = \nabla_x U^0(t,x,x_{N+1}) = \nabla_x u^0(t,x)$$

and

$$L = \mathcal{I}_1[U^0(t,\cdot,x_{N+1})] = \mathcal{I}_1[u^0(t,\cdot)].$$

This heuristic computation, that permits first of all to identify the cell problem in the higher dimensional space, can be made rigorous through the perturbed test function method by Evans [15]. We will prove in Section 4 the convergence of the functions $U^\epsilon(t,x,x_{N+1})$ to $u^0(t,x)+p_{N+1}x_{N+1}$, where u^0 is the solution of (1.4) and, as a consequence we get the proof of Theorem 1.2. In the proof of convergence, in order to control the error terms in the equations, we will need correctors in the higher dimensional space, i.e. bounded solutions of (1.12) in $\mathbb{R}^+ \times \mathbb{R}^{N+1}$, of class $C^{1,\alpha}$ with respect to the additional variable y_{N+1}. Since in (1.12), the quantity $\mathcal{I}_1[V(\tau,\cdot,y_{N+1})]$ is computed only in the y variable, we cannot expect this kind of regularity for the correctors. Nevertheless, we are able to construct regular approximated sub and supercorrectors, i.e., sub and supersolutions of approximate $N+1$-dimensional cell problems, and this is enough to conclude. Recall that in [31], the regular approximate correctors where obtained using a kind of truncation of the Hamiltonian. This is no longer possible to apply this method here because of the problem is non local, and we had to introduce a different method to build such approximate correctors. Finally, this construction works for any $p_{N+1} \neq 0$ and to simplify the presentation we take $p_{N+1} = 1$.

1.4. **Strategy for proof of the Orowan's law.** Let us call p the space derivative of u^0. Remark that if $p \neq 0$, we do not need to increase the dimension in the proof of homogenization. Precisely, let us consider the case $N = 1$. Then a good ansatz is the following

$$u^\epsilon(t,x) \simeq u^0(t,x) + \epsilon v\left(\frac{t}{\epsilon}, \frac{u^0(t,x)-\lambda t}{\epsilon p}\right) = \epsilon h\left(\frac{u^0(t,x)}{\epsilon}\right), \tag{1.13}$$

where

$$\lambda\tau + py + v(\tau,y) =: h(\lambda\tau + py) \tag{1.14}$$

and v is a corrector solution of (1.5) (without the initial conditions). Here the function h is usually called the *hull function*. For the precise definition of such a function we refer to [17] and references therein.

We will not prove the existence of the hull function, we just provide an ansatz of it, $\widetilde{h}$, for small values of p and L, and letting $\lambda = c_0|p|L$. We set (at least formally)

$$\widetilde{h}(x) = \frac{L}{\alpha} + \sum_{i=-\infty}^{+\infty}\left[\phi\left(\frac{x-i}{|p|}\right) - \frac{1}{2}\right] + c_0 L \sum_{i=-\infty}^{+\infty} \psi_1\left(\frac{x-i}{|p|}\right).$$

with ψ_1 solution of

$$\mathcal{I}_1[\psi_1] - W''(\phi)\psi_1 = \phi' + \eta(W''(\phi) - W''(0)) \quad \text{with} \quad \eta = \frac{\int_{\mathbb{R}}(\phi')^2}{W''(0)}.$$

We will show that $\widetilde{h}$ is a good ansatz for p, L small, and then will deduce (1.9) after delicate comparisons.

1.5. **Organization of the paper.** The paper is organized as follows. In Section 2, we give more details about the Peierls-Nabarro model yielding to the study of (1.1) and the mechanical interpretation of the homogenization results. In Section 3, we state various comparison principles, existence and regularity results for solutions of non-local Hamilton-Jacobi equations. In Section 4, we prove the convergence result (Theorem 1.2) by assuming the existence of smooth approximate sub and supercorrectors (Proposition 4.4). In order to show their existence, in Section 5, we first construct Lipschitz continuous sub and supercorrectors (Proposition 5.1). As a byproduct, we prove the ergodicity of the problem (Theorem 1.1) and some properties of the effective Hamiltonian (Proposition 4.3). Proposition 4.4 is then proved in Section 6. Section 7 is devoted to the proof of the Orowan's law (Theorem 1.3). Finally, we put in an appendix (Section 8) the proofs of some technical results that we use in Section 7.

1.6. **Notations.** We denote by $B_r(x)$ the ball of radius r centered at x. The cylinder $(t-\tau, t+\tau)\times B_r(x)$ is denoted by $Q_{\tau,r}(t,x)$.

$\lfloor x \rfloor$ and $\lceil x \rceil$ denote respectively the floor and the ceil integer parts of a real number x.

It is convenient to introduce the singular measure defined on $\mathbb{R}^N \setminus \{0\}$ by

$$\mu(dz) = \frac{1}{|z|^{N+1}} g\left(\frac{z}{|z|}\right) dz = \mu_0(z)dz,$$

and to denote

$$\mathcal{I}_1^{1,r}[U,x] = \int_{|z|\le r} (U(x+z) - U(x) - \nabla U(x)\cdot z)\mu(dz),$$

$$\mathcal{I}_1^{2,r}[U,x] = \int_{|z|>r} (U(x+z) - U(x))\mu(dz).$$

Sometimes when $r=1$ we will omit r and we will write simply $\mathcal{I}_1^1$ and $\mathcal{I}_1^2$.

For a function u defined on $(0,T)\times\mathbb{R}^N$, $0 < T \le +\infty$, for $0<\alpha<1$ we denote by $< u >_x^\alpha$ the seminorm defined by

$$< u >_x^\alpha := \sup_{\substack{(t,x),\,(t,x')\in(0,T)\times\mathbb{R}^N \\ x\neq x'}} \frac{|u(t,x) - u(t,x')|}{|x-x'|^\alpha}$$

and by $C_x^\alpha((0,T)\times\mathbb{R}^N)$ the space of continuous functions defined on $(0,T)\times\mathbb{R}^N$ that are bounded and with bounded seminorm $< u >_x^\alpha$.

Finally, we denote by $USC_b(\mathbb{R}^+\times\mathbb{R}^N)$ (resp., $LSC_b(\mathbb{R}^+\times\mathbb{R}^N)$) the set of upper (resp., lower) semicontinuous functions on $\mathbb{R}^+\times\mathbb{R}^N$ which are bounded on $(0,T)\times\mathbb{R}^N$ for any $T>0$ and we set $C_b(\mathbb{R}^+\times\mathbb{R}^N) := USC_b(\mathbb{R}^+\times\mathbb{R}^N)\cap LSC_b(\mathbb{R}^+\times\mathbb{R}^N)$.

2. Physical modeling and mechanical interpretation of the homogenization results

2.1. **The Peierls-Nabarro model.** Dislocations are line defects in crystals. Their typical length is of the order of $10^{-6}m$ and their thickness of order of $10^{-9}m$. When the material is submitted to shear stress, these lines can move in the crystallographic planes and their dynamics is one of the main explanation of the plastic behavior of metals.

The Peierls-Nabarro model is a phase field model for dislocation dynamics incorporating atomic features into continuum framework. In a phase field approach, the dislocations are represented by transition of a continuous field.

We briefly review the model (see [23] for a detailed presentation). As an example, consider an edge dislocation in a crystal with simple cubic lattice. In a Cartesian system of coordinates $x_1x_2x_3$, we assume that the dislocation is located in the slip plane x_1x_2 (where the dislocation can move) and that the Burgers' vector (i.e. a fixed vector associated to the dislocation) is in the direction of the x_1 axis. We write this Burgers' vector as be_1 for a real b. The disregistry of the upper half crystal $\{x_3>0\}$ relative to the lower half $\{x_3<0\}$ in the direction of the Burgers' vector is $\phi(x_1,x_2)$, where ϕ is a phase parameter between 0 and b. Then the dislocation loop can be for instance localized by the level set $\phi = b/2$. For a closed loop, we expect to have $\phi \simeq b$ inside the loop and $\phi\simeq 0$ far outside the loop.

In the Peierls-Nabarro model, the total energy is given by

$$\mathcal{E} = \mathcal{E}^{el} + \mathcal{E}^{mis}. \tag{2.1}$$

In (2.1), $\mathcal{E}^{mis}$ is the so called *misfit energy* due to the nonlinear atomic interaction across the slip plane

$$\mathcal{E}^{mis}(\phi) = \int_{\mathbb{R}^2} W(\phi(x))\, dx \quad \text{with} \quad x = (x_1,x_2),$$

where $W(\phi)$ is the interplanar potential. In the classical Peierls-Nabarro model [39, 37], $W(\phi)$ is approximated by the sinusoidal potential

$$W(\phi) = \frac{\mu b^2}{4\pi^2 d}\left(1 - \cos\left(\frac{2\pi\phi}{b}\right)\right),$$

where d is the lattice spacing perpendicular to the slip plane.

The elastic energy $\mathcal{E}^{el}$ induced by the dislocation is (for $X = (x, x_3)$ with $x = (x_1, x_2)$)

(2.2)
$$\mathcal{E}^{el}(\phi, U) = \frac{1}{2}\int_{\mathbb{R}^3} e : \Lambda : e \, dX \quad \text{with} \quad e = e(U) - \phi(x)\delta_0(x_3)e^0 \quad \text{and} \quad \left\{ \begin{array}{l} e(U) = \frac{1}{2}\left(\nabla U + (\nabla U)^T\right) \\ \\ e^0 = \frac{1}{2}\left(e_1 \otimes e_3 + e_3 \otimes e_1\right) \end{array} \right. ,$$

where $U : \mathbb{R}^3 \to \mathbb{R}^3$ is the displacement and $\Lambda = \{\Lambda_{ijkl}\}$ are the elastic coefficients.
Given the field ϕ, we minimize the energy $\mathcal{E}^{el}(\phi, U)$ with respect to the displacement U and define

$$\mathcal{E}^{el}(\phi) = \inf_U \mathcal{E}^{el}(\phi, U)$$

Following the proof of Proposition 6.1 (iii) in [3], we can see that (at least formally)

$$\mathcal{E}^{el}(\phi) = -\frac{1}{2}\int_{\mathbb{R}^2} (c_0 \star \phi)\phi$$

where c_0 is a certain kernel. In the case of isotropic elasticity, we have

$$\Lambda_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$$

where λ, μ are the Lamé coefficients. Then the kernel c_0 can be written (see Proposition 6.2 in [3], translated in our framework):

$$c_0(x) = \frac{\mu}{4\pi}\left(\partial_{22}\frac{1}{|x|} + \gamma\partial_{11}\frac{1}{|x|}\right) \quad \text{with} \quad \gamma = \frac{1}{1-\nu} \quad \text{and} \quad \nu = \frac{\lambda}{2(\lambda+\mu)}$$

where $\nu \in (-1, 1/2)$ is called the Poisson ratio.

The equilibrium configuration of straight dislocations is obtained by minimizing the total energy with respect to ϕ, under the constraint that far from the dislocation core, the function ϕ tends to 0 in one half plane and to b in the other half plane. In particular, the phase transition ϕ is then solution of the following equation

(2.3)
$$\mathcal{I}_1[\phi] = W'(\phi) \quad \text{on } \mathbb{R}^2,$$

where formally $\mathcal{I}_1[\phi] = c_0 \star \phi$, which is the anisotropic Lévy operator defined in (1.2) for $N = 2$ and $g(z_1, z_2) = \frac{\mu}{4\pi}\left((2\gamma - 1)z_1^2 + (2-\gamma)z_2^2\right)$. Let us now recall the expression of the kernel after a Fourier transform (see paragraph 6.2.2.2 in [3])

$$\widehat{c_0}(\xi) = -\frac{\mu}{2|\xi|}\left(\xi_2^2 + \gamma\xi_1^2\right)$$

Then for $\gamma = 1$ and $\mu = 2$, we see that $\mathcal{I}_1 = -(-\Delta)^{\frac{1}{2}}$. In that special case, we recall that the solution ϕ of (2.3) satisfies $\phi(x) = \tilde{\phi}(x, 0)$ where $\tilde{\phi}(X)$ is the solution of (see [34, 22])

$$\left\{ \begin{array}{lll} \Delta\tilde{\phi} = 0 & \text{in} & \{x_3 > 0\} \\ \\ \dfrac{\partial\tilde{\phi}}{\partial x_3} = W'(\tilde{\phi}) & \text{on} & \{x_3 = 0\} \end{array} \right.$$

Moreover, we have in particular an explicit solution for $b = 1$, $d = 2$ (with $W'(\tilde{\phi}) = \frac{1}{2\pi}\sin(2\pi\tilde{\phi})$)

$$\tilde{\phi}(X) = \frac{1}{2} + \frac{1}{\pi}\arctan\left(\frac{x_1}{x_3 + 1}\right)$$

Then by rescaling, it is easy to check that we can recover the explicit solution found in Nabarro [37]

(2.4)
$$\left\{ \begin{array}{ll} \phi(x) = \dfrac{b}{2} + \dfrac{b}{\pi}\arctan\left(\dfrac{2(1-\nu)x_1}{d}\right) & \text{(edge dislocation)} \\ \\ \phi(x) = \dfrac{b}{2} + \dfrac{b}{\pi}\arctan\left(\dfrac{2x_2}{d}\right) & \text{(screw dislocation)} \end{array} \right.$$

In a more general model, one can consider a potential W satisfying

(i) $W(v + b) = W(u)$ for all $v \in \mathbb{R}$;
(ii) $W(b\mathbb{Z}) = 0 < W(a)$ for all $a \in \mathbb{R} \setminus b\mathbb{Z}$.

The periodicity of W reflects the periodicity of the crystal, while the minimum property is consistent with the fact that the perfect crystal is assumed to minimize the energy.

In the face cubic structured (FCC) observed in many metals and alloys, dislocations move at low temperature on the slip plane. In the present paper we are interested in describing the effective dynamics for a collection of dislocations curves with the same Burgers' vector and all contained in a single slip plane x_1x_2, and moving in a landscape with periodic obstacles (that can be for instance precipitates in the material). These dislocations are represented by a single phase parameter $u(t, x_1, x_2)$ defined on the slip plane x_1x_2. The dynamic of dislocations is then described by the evolutive version of the Peierls-Nabarro model:

$$\partial_t u = \mathcal{I}_1[u(t,\cdot)] - W'(u) + \sigma_{13}^{\text{osbt}}(t,x) \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N \tag{2.5}$$

for $x \in \mathbb{R}^N$ with the physical dimension $N = 2$. In the model, the component $\sigma_{13}^{\text{osbt}}$ of the stress (evaluated on the slip plane) has been introduced to take into account the shear stress not created by the dislocations themselves. This shear stress is created by the presence of the periodic *obstacles* and the possible external applied stress on the material.

We want to identify at *large scale* an evolution model for the dynamics of a density of dislocations. We consider the following rescaling

$$u^\epsilon(t,x) = \epsilon u\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}\right),$$

where ϵ is the ratio between the typical length scale for dislocation (of the order of the micrometer) and the typical macroscopic length scale in mechanics (milimeter or centimeter). With such a rescaling, we see that the number of dislocations is typically of the order of $1/\epsilon$ per unit of macroscopic scale. Moreover, assuming suitable initial data

$$u(0,x) = \frac{1}{\epsilon} u_0(\epsilon x) \quad \text{on } \mathbb{R}^N, \tag{2.6}$$

(where u_0 is a regular bounded function), we see that the functions u^ϵ are solutions of (1.1). This indicates that at the limit $\epsilon \to 0$, we will recover a model for the dynamics of (renormalized) densities of dislocations.

Remark 2.1. Fractional reaction-diffusion equations of the form

$$\partial_t u = \mathcal{I}_1[u] + f(u) \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^N \tag{2.7}$$

where $N \geq 2$ and f is a bistable nonlinearity have been studied by Imbert and Souganidis [27]. In this paper the authors show that solutions of (2.7), after properly rescaling them, exhibit the limit evolution of an interface by (anisotropic) mean curvature motion.

Other results have been obtained by González and Monneau [22] for a rescaling of the evolutive Peierls-Nabarro model in dimension $N = 1$. In the one dimensional space, the limit moving interfaces are points particles interacting with forces as $1/x$. The dynamics of these particles corresponds to the classical discrete dislocation dynamics, in the particular case of parallel straight edge dislocation lines in the same slip plane with the same Burgers' vector. In [16], considering another rescaling of the model of particles obtained in [22], the authors identify at large scale an evolution model for the dynamics of a density of dislocations, that is analoguous to (1.4). In the present paper, we directly deduce the model (1.4) at larger scale from the Peierls-Nabarro model at smaller scale in any dimension $N \geq 1$. That way we remove the limitation to the dimension $N = 1$ that appears in [22].

Finally, let us mention that in [19] and [20] Garroni and Muller study a variational model for dislocations that is the variational formulation of the stationary Peierls-Nabarro equation, where they derive a line tension model.

2.2. **Mechanical interpretation of the homogenization.** Let us briefly explain the meaning of the homogenization result. In the macroscopic model, the function $u^0(t,x)$ can be interpreted as the plastic strain (localized in the slip plane $\{x_3 = 0\}$). Then the three-dimensional displacement $U(t,X)$ is obtained as a minimizer of the elastic energy

$$U(t,\cdot) = \arg\min_{\tilde{U}} \mathcal{E}^{el}(u^0(t,\cdot), \tilde{U})$$

and the stress is

$$\sigma = \Lambda : e \quad \text{with} \quad e = e(U) - u^0(t,x)\delta_0(x_3)e^0$$

Then the resolved shear stress is

$$\mathcal{I}_1[u^0] = \sigma_{13}$$

The homogenized equation (1.4), i.e.

$$\partial_t u^0 = \overline{H}(\nabla_x u^0, \mathcal{I}_1[u^0(t,\cdot)])$$

which is the evolution equation for u^0, can be interpreted as the plastic flow rule in a model for macroscopic crystal plasticity. This is the law giving the plastic strain velocity $\partial_t u^0$ as a function of the resolved shear stress σ_{13} and the dislocation density ∇u^0.

The typical example of such a plastic flow rule is the Orowan's law:

$$\overline{H}(p,\tau) \simeq \tau|p|$$

This is also the law that we recover in dimension $N = 1$ in Theorem 1.3 in the case where there are no obstacles (i.e. $\sigma_{13}^{\text{osbt}} \equiv 0$) and for small stress τ and small density $|p|$. When $\sigma_{13}^{\text{osbt}} \not\equiv 0$, we expect a threshold phenomenon as in [32] (see also Norton's law with threshold in [18]), i.e.

$$\overline{H}(p,\tau) = 0 \quad \text{if} \quad |\tau| \quad \text{is small enough.}$$

This means more generally that our homogenization procedure describes correctly the mechanical behaviour of the stress at large scales, but keeps the memory of the microstructure in the plastic law with possible threshold effects.

3. Results about viscosity solutions for non-local equations

The classical notion of viscosity solution can be adapted for Hamilton-Jacobi equations involving non-local operators, see for instance [5]. In this section we state comparison principles, existence and regularity results for viscosity solutions of (1.1) and (1.4), that will be used later in the proofs.

3.1. **Definition of viscosity solution.** We first recall the definition of viscosity solution for a general first order non-local equation with associated an initial condition:

$$\begin{cases} u_t = F(t,x,u,Du,\mathcal{I}_1[u]) & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N \\ u(0,x) = u_0(x) & \text{on} \quad \mathbb{R}^N, \end{cases} \tag{3.1}$$

where $F(t,x,u,p,L)$ is continuous and non-decreasing in L.

Definition 3.1 (r-viscosity solution)**.** *A function $u \in USC_b(\mathbb{R}^+ \times \mathbb{R}^N)$ (resp., $u \in LSC_b(\mathbb{R}^+ \times \mathbb{R}^N)$) is a r-viscosity subsolution (resp., supersolution) of (3.1) if $u(0,x) \le (u_0)^*(x)$ (resp., $u(0,x) \ge (u_0)_*(x)$) and for any $(t_0,x_0) \in \mathbb{R}^+ \times \mathbb{R}^N$, any $\tau \in (0,t_0)$ and any test function $\phi \in C^2(\mathbb{R}^+ \times \mathbb{R}^N)$ such that $u - \phi$ attains a local maximum (resp., minimum) at the point (t_0,x_0) on $Q_{(\tau,r)}(t_0,x_0)$, then we have*

$$\partial_t\phi(t_0,x_0) - F(t_0,x_0,u(t_0,x_0),\nabla_x\phi(t_0,x_0),\mathcal{I}_1^{1,r}[\phi(t_0,\cdot),x_0] + \mathcal{I}_1^{2,r}[u(t_0,\cdot),x_0]) \le 0$$
$$(\textit{resp.}, \ \ge 0).$$

A function $u \in C_b(\mathbb{R}^+ \times \mathbb{R}^N)$ is a r-viscosity solution of (3.1) if it is a r-viscosity sub and supersolution of (3.1).

It is classical that the maximum in the above definition can be supposed to be global and this will be used later. We have also the following property, see e.g. [5]:

Proposition 3.1 (Equivalence of the definitions)**.** *Assume $F(t,x,u,p,L)$ continuous and non-decreasing in L. Let $r > 0$ and $r' > 0$. A function $u \in USC_b(\mathbb{R}^+ \times \mathbb{R}^N)$ (resp., $u \in LSC_b(\mathbb{R}^+ \times \mathbb{R}^N)$) is a r-viscosity subsolution (resp., supersolution) of (3.1) if and only if it is a r'-viscosity subsolution (resp., supersolution) of (3.1).*

Because of this proposition, if we do not need to emphasize r, we will omit it when calling viscosity sub and supersolutions.

3.2. **Comparison principle and existence results.** In this subsection, we successively give comparison principles and existence results for (1.1) and (1.4). The following comparison theorem is shown in [33] for more general parabolic integro-PDEs.

Proposition 3.2 (Comparison Principle for (1.1))**.** *Consider $u \in USC_b(\mathbb{R}^+ \times \mathbb{R}^N)$ subsolution and $v \in LSC_b(\mathbb{R}^+ \times \mathbb{R}^N)$ supersolution of* (1.1)*, then $u \leq v$ on $\mathbb{R}^+ \times \mathbb{R}^N$.*

Following [33] it can also be proved the comparison principle for (1.1) in bounded domains. Since we deal with a non-local equation, we need to compare the sub and the supersolution everywhere outside the domain.

Proposition 3.3 (Comparison Principle on bounded domains for (1.1))**.** *Let Ω be a bounded domain of $\mathbb{R}^+ \times \mathbb{R}^N$ and let $u \in USC_b(\mathbb{R}^+ \times \mathbb{R}^N)$ and $v \in LSC_b(\mathbb{R}^+ \times \mathbb{R}^N)$ be respectively a sub and a supersolution of*

$$\partial_t u^\epsilon = \mathcal{I}_1[u^\epsilon(t,\cdot)] - W'\left(\frac{u^\epsilon}{\epsilon}\right) + \sigma\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}\right)$$

in Ω. If $u \leq v$ outside Ω, then $u \leq v$ in Ω.

Proposition 3.4 (Existence for (1.1))**.** *For $\epsilon > 0$ there exists $u^\epsilon \in C_b(\mathbb{R}^+ \times \mathbb{R}^N)$ (unique) viscosity solution of* (1.1)*. Moreover, there exists a constant $C > 0$ independent of ϵ such that*

$$|u^\epsilon(t,x) - u_0(x)| \leq Ct. \tag{3.2}$$

Proof. Adapting the argument of [25], we can construct a solution by Perron's method if we construct sub and supersolutions of (1.1). Since $u_0 \in W^{2,\infty}$, the two functions $u^\pm(t,x) := u_0(x) \pm Ct$ are respectively a super and a subsolution of (1.1) for any $\epsilon > 0$, if

$$C \geq D_N \|u_0\|_{2,\infty} + \|W'\|_\infty + \|\sigma\|_\infty,$$

with D_N depending on the dimension N. By comparison we also get the estimate (3.2). □

We next recall the comparison and the existence results for (1.4).

Proposition 3.5 ([32], Proposition 3)**.** *Let $\overline{H} : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be continuous with $\overline{H}(p,\cdot)$ non-decreasing on $\mathbb{R}$ for any $p \in \mathbb{R}^N$. If $u \in USC_b(\mathbb{R}^+ \times \mathbb{R}^N)$ and $v \in LSC_b(\mathbb{R}^+ \times \mathbb{R}^N)$ are respectively a sub and a supersolution of* (1.4)*, then $u \leq v$ on $\mathbb{R}^+ \times \mathbb{R}^N$. Moreover there exists a (unique) viscosity solution of* (1.4)*.*

In the next sections, we will embed the problem in the higher dimensional space $\mathbb{R}^+ \times \mathbb{R}^{N+1}$ by adding a new variable x_{N+1} in the equations. We will need the following proposition showing that sub and supersolutions of the higher dimensional problem are also sub and supersolutions of the lower dimensional one. This in particular implies that the comparison principle between sub and supersolutions remains true increasing the dimension.

Proposition 3.6. *Assume $F(t,x,x_{N+1},U,p,L)$ continuous and non-decreasing in L. Suppose that $U \in LSC_b(\mathbb{R}^+ \times \mathbb{R}^{N+1})$ (resp., $U \in USC_b(\mathbb{R}^+ \times \mathbb{R}^{N+1})$ is a viscosity supersolution (resp., subsolution) of*

$$U_t = F(t,x,x_{N+1},U,D_xU,\mathcal{I}_1[U(t,\cdot,x_{N+1})]) \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^{N+1},$$

then, for any $x_{N+1} \in \mathbb{R}$, U is a viscosity supersolution (resp., subsolution) of

$$U_t = F(t,x,x_{N+1},U,D_xU,\mathcal{I}_1[U(t,\cdot,x_{N+1})]) \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^{N}.$$

Proof. We show the result for supersolutions. Fix $x^0_{N+1} \in \mathbb{R}$. Let us consider a point $(t_0,x_0) \in \mathbb{R}^+ \times \mathbb{R}^N$ and a smooth function $\varphi : \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}$ such that

$$U(t,x,x^0_{N+1}) - \varphi(t,x) \geq U(t_0,x_0,x^0_{N+1}) - \varphi(t_0,x_0) = 0 \quad \text{for } (t,x) \in Q_{\tau,r}(t_0,x_0),$$

with $r = 1$. We have to show that

$$\begin{aligned}\partial_t\varphi(t_0,x_0) \geq F(t_0,x_0,x^0_{N+1},U(t_0,x_0,x^0_{N+1}),D_x\varphi(t_0,x_0),\mathcal{I}^1_1[\varphi(t_0,\cdot),x_0]\\ + \mathcal{I}^2_1[U(t_0,\cdot,x^0_{N+1}),x_0]).\end{aligned}$$

Without loss of generality, we can assume that the minimum is strict. For $\epsilon > 0$ let $\varphi_\epsilon : \mathbb{R}^+ \times \mathbb{R}^{N+1} \to \mathbb{R}$ be defined by

$$\varphi_\epsilon(t,x,x_{N+1}) = \varphi(t,x) - \frac{1}{\epsilon}|x_{N+1} - x^0_{N+1}|^2.$$

Let $(t_\epsilon, x_\epsilon, x^\epsilon_{N+1})$ be a minimum point of $U - \varphi_\epsilon$ in $Q_{\tau,r}(t_0, x_0, x^0_{N+1})$. Standard arguments show that $(t_\epsilon, x_\epsilon, x^\epsilon_{N+1}) \to (t_0, x_0, x^0_{N+1})$ as $\epsilon \to 0$ and that $\lim_{\epsilon\to 0} U(t_\epsilon, x_\epsilon, x^\epsilon_{N+1}) = U(t_0, x_0, x^0_{N+1})$. In particular, $(t_\epsilon, x_\epsilon, x^\epsilon_{N+1})$ is internal to $Q_{\tau,r}(t_0, x_0, x^0_{N+1})$ for ϵ small enough, then we get

$$\partial_t\varphi(t_\epsilon, x_\epsilon) \geq F(t_\epsilon, x_\epsilon, U(t_\epsilon, x_\epsilon, x^\epsilon_{N+1}), D_x\varphi(t_\epsilon, x_\epsilon), \mathcal{I}^1_1[\varphi(t_\epsilon, \cdot), x_\epsilon] + \mathcal{I}^2_1[U(t_\epsilon, \cdot, x^\epsilon_{N+1}), x_\epsilon]). \tag{3.3}$$

By the Dominate Convergence Theorem $\lim_{\epsilon\to 0} \mathcal{I}^1_1[\varphi(t_\epsilon, \cdot), x_\epsilon] = \mathcal{I}^1_1[\varphi(t_0, \cdot), x_0]$; by the Fatou's Lemma and the convergence of $U(t_\epsilon, x_\epsilon, x^\epsilon_{N+1})$ to $U(t_0, x_0, x^0_{N+1})$, we deduce that

$$\mathcal{I}^2_1[U(t_0, \cdot, x^0_{N+1}), x_0] \leq \liminf_{\epsilon\to 0} \mathcal{I}^2_1[U(t_\epsilon, \cdot, x^\epsilon_{N+1}), x_\epsilon].$$

Then, passing to the limit in (3.3) and using the continuity and monotonicity of F, we get the desired inequality. □

3.3. **Hölder regularity.** In this subsection we state and prove a regularity result for sub and supersolutions of semilinear non-local equations.

Proposition 3.7 (Hölder regularity). *Assume (H1) and let g_1, $g_2 \in \mathbb{R}$. Suppose that $u \in C(\mathbb{R}^+ \times \mathbb{R}^N)$ and bounded on $\mathbb{R}^+ \times \mathbb{R}^N$ is a viscosity subsolution of*

$$\begin{cases} \partial_t u = \mathcal{I}_1[u(t, \cdot)] + g_1 & in \quad \mathbb{R}^+ \times \mathbb{R}^N \\ u(0, x) = 0 & on \quad \mathbb{R}^N, \end{cases}$$

and a viscosity supersolution of

$$\begin{cases} \partial_t u = \mathcal{I}_1[u(t, \cdot)] + g_2 & in \quad \mathbb{R}^+ \times \mathbb{R}^N \\ u(0, x) = 0 & on \quad \mathbb{R}^N. \end{cases}$$

Then, for any $0 < \alpha < 1$, $u \in C^\alpha_x(\mathbb{R}^+ \times \mathbb{R}^N)$ with $< u >^\alpha_x \leq C$, where C depends on $\|u\|_\infty$, g_1 and g_2.

Proof. Suppose by contradiction that u does not belong to $C^\alpha_x(\mathbb{R}^+ \times \mathbb{R}^N)$. Let $u^{\epsilon,\epsilon'}$ and $u_{\epsilon,\epsilon'}$ be respectively the double-parameters sup and inf convolution of u in $\mathbb{R}^+ \times \mathbb{R}^N$, i.e.

$$u^{\epsilon,\epsilon'}(t, x) = \sup_{(s,y)\in\mathbb{R}^+\times\mathbb{R}^N} \left(u(s, y) - \frac{1}{2\epsilon}|x - y|^2 - \frac{1}{2\epsilon'}(t - s)^2 \right),$$

$$u_{\epsilon,\epsilon'}(t, x) = \inf_{(s,y)\in\mathbb{R}^+\times\mathbb{R}^N} \left(u(s, y) + \frac{1}{2\epsilon}|x - y|^2 + \frac{1}{2\epsilon'}(t - s)^2 \right).$$

Then $u^{\epsilon,\epsilon'}$ is semiconvex and is a subsolution of

$$\partial_t u^{\epsilon,\epsilon'} = \mathcal{I}_1[u^{\epsilon,\epsilon'}(t, \cdot)] + g_1 \quad \text{in} \quad (t_{\epsilon'}, +\infty) \times \mathbb{R}^N$$

and $u_{\epsilon,\epsilon'}$ is semiconcave and is a supersolution of

$$\partial_t u_{\epsilon,\epsilon'} = \mathcal{I}_1[u_{\epsilon,\epsilon'}(t, \cdot)] + g_2 \quad \text{in} \quad (t_{\epsilon'}, +\infty) \times \mathbb{R}^N,$$

where $t_{\epsilon'} \to 0$ as $\epsilon' \to 0$, see e.g. Proposition III.2 in [5]. Let us consider smooth functions $\psi_1(t)$ and $\psi_2(x)$ with bounded first and second derivatives such that $\psi_1(t) \to +\infty$ as $t \to +\infty$, $\psi_2(x) \to +\infty$ as $|x| \to +\infty$ and there exists $K_0 > 0$ such that $|\psi_2(x)| \leq K_0(1 + \sqrt{|x|})$. Then, for any $K > 0$ and ϵ, ϵ' and β small enough, the supremum of the function $u^{\epsilon,\epsilon'}(t, x_1) - u_{\epsilon,\epsilon'}(t, x_2) - \phi(t, x_1, x_2)$ on $\mathbb{R}^+ \times \mathbb{R}^{2N}$, where $\phi(t, x_1, x_2) = K|x_1 - x_2|^\alpha + \beta\psi_1(t) + \beta\psi_2(x_1)$, is positive and is attained at some point $(\bar{t}, \overline{x}_1, \overline{x}_2) \in (t_{\epsilon'}, +\infty) \times \mathbb{R}^{2N}$, with $\overline{x}_1 \neq \overline{x}_2$ (because $u \notin C^\alpha_x(\mathbb{R}^+ \times \mathbb{R}^N)$ and $u = 0$ at $t = 0$). Remark that

$$|\overline{x}_1 - \overline{x}_2| \leq \left(\frac{2 \sup_{(t,x)\in\mathbb{R}^+\times\mathbb{R}^N} |u(t, x)|}{K} \right)^{\frac{1}{\alpha}}.$$

In order to apply the Jensen's Lemma, see e.g. Lemma A.3 of [12], we have to transform $(\bar{t}, \overline{x}_1, \overline{x}_2)$ into a strict maximum point. To do so, we consider a smooth bounded function $h : \mathbb{R}^+ \to \mathbb{R}$, with bounded derivatives, such that $h(0) = 0$ and $h(s) > 0$ for $s > 0$ and we set $\theta(t, x_1, x_2) = h((t - \bar{t})^2) + h(|x_1 - \overline{x}_1|^2) + h(|x_2 - \overline{x}_2|^2)$. Next we consider a smooth function $\chi : \mathbb{R}^N \to \mathbb{R}$ such that $\chi(x) = 1$ if $|x| \leq 1/2$ and $\chi(x) = 0$ for $|x| \geq 1$. Clearly $(\bar{t}, \overline{x}_1, \overline{x}_2)$ is a strict maximum point of

$u^{\epsilon,\epsilon'}(t,x_1) - u_{\epsilon,\epsilon'}(t,x_2) - \phi(t,x_1,x_2) - \theta(t,x_1,x_2)$ and by Jensen's Lemma, for every small and positive δ there exist $t^\delta \in \mathbb{R}$, q_1^δ, $q_2^\delta \in \mathbb{R}^N$ with $|t^\delta|$, $|q_1^\delta|, |q_2^\delta| \le \delta$ such that the function

$$u^{\epsilon,\epsilon'}(t,x_1) - u_{\epsilon,\epsilon'}(t,x_2) - K|x_1 - x_2|^\alpha - \varphi_1(t,x_1) - \varphi_2(x_2), \tag{3.4}$$

where

$$\varphi_1(t,x_1) = \beta\psi_1(t) + \beta\psi_2(x_1) + h((t-\bar{t})^2) + h(|x_1 - \overline{x}_1|^2) + t^\delta t + \chi(x_1 - \overline{x}_1)q_1^\delta \cdot x_1,$$
$$\varphi_2(x_2) = h(|x_2 - \overline{x}_2|^2) + \chi(x_2 - \overline{x}_2)q_2^\delta \cdot x_2,$$

has a maximum at $(t^\delta, x_1^\delta, x_2^\delta)$ with $|t_\delta - \bar{t}|$, $|x_1^\delta - \overline{x}_1|$, $|x_2^\delta - \overline{x}_2| \le \delta$ and $u^{\epsilon,\epsilon'}(t,x_1) - u_{\epsilon,\epsilon'}(t,x_2)$ is twice differentiable at $(t^\delta, x_1^\delta, x_2^\delta)$. In particular $u^{\epsilon,\epsilon'}$ is twice differentiable w.r.t. x_1 at (t^δ, x_1^δ) and $u_{\epsilon,\epsilon'}$ is twice differentiable w.r.t. x_2 at (t^δ, x_2^δ). For δ small enough, we can assume $x_1^\delta \neq x_2^\delta$. The fact that $(t^\delta, x_1^\delta, x_2^\delta)$ is a maximum point implies that

$$\nabla_{x_1} u^{\epsilon,\epsilon'}(t^\delta, x_1^\delta) = \nabla_{x_1}\varphi_1(t^\delta, x_1^\delta) + \alpha K |x_1^\delta - x_2^\delta|^{\alpha-2}(x_1^\delta - x_2^\delta),$$
$$\nabla_{x_2} u_{\epsilon,\epsilon'}(t^\delta, x_2^\delta) = -\nabla_{x_2}\varphi_2(x_2^\delta) + \alpha K |x_1^\delta - x_2^\delta|^{\alpha-2}(x_1^\delta - x_2^\delta),$$

and for any $z \in \mathbb{R}^N$, $r > 0$

$$\begin{aligned} &u^{\epsilon,\epsilon'}(t^\delta, x_1^\delta + z) - u^{\epsilon,\epsilon'}(t^\delta, x_1^\delta) - \nabla_{x_1} u^{\epsilon,\epsilon'}(t^\delta, x_1^\delta) \cdot z\mathbf{1}_{B_r}(z) \\ &\le u_{\epsilon,\epsilon'}(t^\delta, x_2^\delta + z) - u_{\epsilon,\epsilon'}(t^\delta, x_2^\delta) - \nabla_{x_2} u_{\epsilon,\epsilon'}(t^\delta, x_2^\delta) \cdot z\mathbf{1}_{B_r}(z) \\ &+ \varphi_1(t^\delta, x_1^\delta + z) - \varphi_1(t^\delta, x_1^\delta) - \nabla_{x_1}\varphi_1(t^\delta, x_1^\delta) \cdot z\mathbf{1}_{B_r}(z) \\ &+ \varphi_2(x_2^\delta + z) - \varphi_2(x_2^\delta) - \nabla_{x_2}\varphi_2(x_2^\delta) \cdot z\mathbf{1}_{B_r}(z), \end{aligned} \tag{3.5}$$

where $B_r = B_r(0)$. The last inequality, in particular implies that

$$\mathcal{I}_1^{2,r}[u^{\epsilon,\epsilon'}(t^\delta, \cdot), x_1^\delta] \le \mathcal{I}_1^{2,r}[u_{\epsilon,\epsilon'}(t^\delta, \cdot), x_2^\delta] + \mathcal{I}_1^{2,r}[\varphi_1(t^\delta, \cdot), x_1^\delta] + \mathcal{I}_1^{2,r}[\varphi_2, x_2^\delta]. \tag{3.6}$$

Since $(t^\delta, x_1^\delta, x_2^\delta)$ is a maximum point, we have in addition

$$\begin{aligned} &u^{\epsilon,\epsilon'}(t^\delta, x_1^\delta + z) - u^{\epsilon,\epsilon'}(t^\delta, x_1^\delta) - \nabla_{x_1} u^{\epsilon,\epsilon'}(t^\delta, x_1^\delta) \cdot z \\ &\le \varphi_1(t^\delta, x_1^\delta + z) - \varphi_1(t^\delta, x_1^\delta) - \nabla_{x_1}\varphi_1(t^\delta, x_1^\delta) \cdot z \\ &+ K|x_1^\delta + z - x_2^\delta|^\alpha - K|x_1^\delta - x_2^\delta|^\alpha - \alpha K|x_1^\delta - x_2^\delta|^{\alpha-2}(x_1^\delta - x_2^\delta) \cdot z, \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} &- (u_{\epsilon,\epsilon'}(t^\delta, x_2^\delta + z) - u_{\epsilon,\epsilon'}(t^\delta, x_2^\delta) - \nabla_{x_2} u_{\epsilon,\epsilon'}(t^\delta, x_2^\delta) \cdot z) \\ &\le \varphi_2(x_2^\delta + z) - \varphi_2(x_2^\delta) - \nabla_{x_2}\varphi_2(x_2^\delta) \cdot z \\ &+ K|x_1^\delta - z - x_2^\delta|^\alpha - K|x_1^\delta - x_2^\delta|^\alpha + \alpha K|x_1^\delta - x_2^\delta|^{\alpha-2}(x_1^\delta - x_2^\delta) \cdot z. \end{aligned} \tag{3.8}$$

In order to test, we need to double the time variables. Hence, for $j > 0$, let us consider the maximum point (t^j, x_1^j, s^j, x_2^j) of the function

$$u^{\epsilon,\epsilon'}(t,x_1) - u_{\epsilon,\epsilon'}(s,x_2) - \Psi(t,x_1,x_2) - \frac{j}{2}|t-s|^2,$$

where

$$\Psi(t,x_1,x_2) = K|x_1 - x_2|^\alpha + \varphi_1(t,x_1) + \varphi_2(x_2) + |t - t^\delta|^2 + |x_1 - x_1^\delta|^2 + |x_2 - x_2^\delta|^2,$$

on $Q_{\overline{\rho},\overline{\rho}}(t^\delta, x_1^\delta) \times Q_{\overline{\rho},\overline{\rho}}(t^\delta, x_2^\delta)$, for $\overline{\rho} > 0$ sufficiently small. Standard arguments show that $(t^j, x_1^j, s^j, x_2^j) \to (t^\delta, x_1^\delta, t^\delta, x_2^\delta)$ as $j \to +\infty$. Hence for j large enough there exists $\rho > 0$ such that $Q_{\rho,\rho}(t^j, x_1^j) \times Q_{\rho,\rho}(s^j, x_2^j) \subset Q_{\overline{\rho},\overline{\rho}}(t^\delta, x_1^\delta) \times Q_{\overline{\rho},\overline{\rho}}(t^\delta, x_2^\delta)$ and $x_1^j \neq x_2^j$. Testing, we get

$$j(t^j - s^j) + 2(t^j - t^\delta) + \partial_t\varphi_1(t^j, x_1^j) \le \mathcal{I}_1^{1,\rho}[\Psi(t^j, \cdot, x_2^j), x_1^j] + \mathcal{I}_1^{2,\rho}[u^{\epsilon,\epsilon'}(t^j, \cdot), x_1^j] + g_1,$$
$$j(t^j - s^j) \ge -\mathcal{I}_1^{1,\rho}[\Psi(t^j, x_1^j, \cdot), x_2^j] + \mathcal{I}_1^{2,\rho}[u_{\epsilon,\epsilon'}(s^j, \cdot), x_2^j] + g_2.$$

Subtracting the two last inequalities, and then letting $j \to +\infty$, we have

$$\begin{aligned} \partial_t\varphi_1(t^\delta, x_1^\delta) &\le \mathcal{I}_1^{1,\rho}[\Psi(t^\delta, \cdot, x_2^\delta), x_1^\delta] + \mathcal{I}_1^{1,\rho}[\Psi(t^\delta, x_1^\delta, \cdot), x_2^\delta] \\ &+ \mathcal{I}_1^{2,\rho}[u^{\epsilon,\epsilon'}(t^\delta, \cdot), x_1^\delta] - \mathcal{I}_1^{2,\rho}[u_{\epsilon,\epsilon'}(t^\delta, \cdot), x_2^\delta] + g_1 - g_2. \end{aligned}$$

Since $u^{\epsilon,\epsilon'}(t^\delta,\cdot)$ and $u_{\epsilon,\epsilon'}(t^\delta,\cdot)$ are twice differentiable respectively at $x_1 = x_1^\delta$ and $x_2 = x_2^\delta$, we can pass to the limit as $\rho \to 0^+$ and finally obtain

$$\partial_t \varphi_1(t^\delta, x_1^\delta) \le \mathcal{I}_1[u^{\epsilon,\epsilon'}(t^\delta,\cdot), x_1^\delta] - \mathcal{I}_1[u_{\epsilon,\epsilon'}(t^\delta,\cdot), x_2^\delta] + g_1 - g_2.$$

Next, using (3.6), we get

$$\begin{aligned} \partial_t \varphi_1(t^\delta, x_1^\delta) &\le \mathcal{I}_1^{1,r}[u^{\epsilon,\epsilon'}(t^\delta,\cdot), x_1^\delta] - \mathcal{I}_1^{1,r}[u_{\epsilon,\epsilon'}(t^\delta,\cdot), x_2^\delta] \\ &\quad + \mathcal{I}_1^{2,r}[\varphi_1(t^\delta,\cdot), x_1^\delta] + \mathcal{I}_1^{2,r}[\varphi_2, x_2^\delta] + g_1 - g_2. \end{aligned} \tag{3.9}$$

Now, let us estimate the term $\mathcal{I}_1^{1,r}[u^{\epsilon,\epsilon'}(t^\delta,\cdot), x_1^\delta] - \mathcal{I}_1^{1,r}[u_{\epsilon,\epsilon'}(t^\delta,\cdot), x_2^\delta]$ and show that it contains a main negative part. For $0 < \nu_0 < 1$, let us denote

$$A_r := \left\{ z \in B_r(0)\,,\ |z \cdot (x_1^\delta - x_2^\delta)| \ge \nu_0 |z||x_1^\delta - x_2^\delta| \right\}.$$

Then

$$\begin{aligned} &\mathcal{I}_1^{1,r}[u^{\epsilon,\epsilon'}(t^\delta,\cdot), x_1^\delta] - \mathcal{I}_1^{1,r}[u_{\epsilon,\epsilon'}(t^\delta,\cdot), x_2^\delta] \\ &= \int_{A_r} [u^{\epsilon,\epsilon'}(t^\delta, x_1^\delta + z) - u^{\epsilon,\epsilon'}(t^\delta, x_1^\delta) - \nabla_{x_1} u^{\epsilon,\epsilon'}(t^\delta, x_1^\delta)\cdot z \\ &- (u_{\epsilon,\epsilon'}(t^\delta, x_2^\delta + z) - u_{\epsilon,\epsilon'}(t^\delta, x_2^\delta) - \nabla_{x_2} u_{\epsilon,\epsilon'}(t^\delta, x_2^\delta)\cdot z)]\mu(dz) \\ &+ \int_{B_r \setminus A_r} [...]\mu(dz) = T_1 + T_2. \end{aligned}$$

From (3.5) we have

$$T_2 \le C.$$

Here and henceforth C denotes various positive constants independent of the parameters. Let us estimate T_1. From (3.7) and (3.8) it follows that

$$\begin{aligned} T_1 &\le \int_{A_r} [K|x_1^\delta + z - x_2^\delta|^\alpha - K|x_1^\delta - x_2^\delta|^\alpha - \alpha K |x_1^\delta - x_2^\delta|^{\alpha-2}(x_1^\delta - x_2^\delta)\cdot z]\mu(dz) + C \\ &+ \int_{A_r} [K|x_1^\delta - z - x_2^\delta|^\alpha - K|x_1^\delta - x_2^\delta|^\alpha + \alpha K |x_1^\delta - x_2^\delta|^{\alpha-2}(x_1^\delta - x_2^\delta)\cdot z]\mu(dz) \\ &= 2\int_{A_r} [K|x_1^\delta + z - x_2^\delta|^\alpha - K|x_1^\delta - x_2^\delta|^\alpha - \alpha K |x_1^\delta - x_2^\delta|^{\alpha-2}(x_1^\delta - x_2^\delta)\cdot z]\mu(dz) + C \\ &\le \alpha K \int_{A_r} \sup_{|t|\le 1} \{ |x_1^\delta - x_2^\delta + tz|^{\alpha-4}(|x_1^\delta - x_2^\delta + tz|^2 |z|^2 \\ &- (2-\alpha)[(x_1^\delta - x_2^\delta + tz)\cdot z]^2 \}\mu(dz) + C. \end{aligned}$$

Let us fix $r = \sigma |x_1^\delta - x_2^\delta|$, $\sigma > 0$, then for $z \in A_r$

$$|x_1^\delta - x_2^\delta + tz| \le (1+\sigma)|x_1^\delta - x_2^\delta|,$$

$$|(x_1^\delta - x_2^\delta + tz)\cdot z| \ge |(x_1^\delta - x_2^\delta)\cdot z| - |z|^2 \ge (\nu_0 - \sigma)\,|x_1^\delta - x_2^\delta||z|.$$

Let us choose ν_0 and σ such that

$$C_0 := -(1+\sigma)^2 + (2-\alpha)(\nu_0 - \sigma)^2 > 0,$$

then

$$T_1 \le -C C_0 K |x_1^\delta - x_2^\delta|^{\alpha-2} \int_{A_r} |z|^2 \mu(dz) + C.$$

By homogeneity

$$\int_{A_r} |z|^2 \mu(dz) = Cr.$$

Then, we conclude

$$T_1 \le -C C_0 K |x_1^\delta - x_2^\delta|^{\alpha-2} r + C \le -C C_0 K |x_1^\delta - x_2^\delta|^{\alpha-1} + C.$$

Finally, from (3.9), we obtain

$$\begin{aligned} CC_0K|x_1^\delta - x_2^\delta|^{\alpha-1} &\le -\partial_t\varphi_1(t^\delta, x_1^\delta) + g_1 - g_2 + C \\ &+ \mathcal{I}_1^{2,r}[\varphi_1(t^\delta,\cdot), x_1^\delta] + \mathcal{I}_1^{2,r}[\varphi_2, x_2^\delta] \\ &\le g_1 - g_2 + C, \end{aligned}$$

which is a contradiction for K large enough. Hence $u \in C_x^\alpha(\mathbb{R}^+ \times \mathbb{R}^N)$. □

4. The proof of convergence

This section is dedicated to the proof of Theorem 1.2. Before presenting it, we first imbed the problem in a higher dimensional one. Precisely, we consider U^ϵ solution of

$$\begin{cases} \partial_t U^\epsilon = \mathcal{I}_1[U^\epsilon(t,\cdot,x_{N+1})] - W'\left(\frac{U^\epsilon}{\epsilon}\right) + \sigma\left(\frac{t}{\epsilon},\frac{x}{\epsilon}\right) & \text{in} \quad \mathbb{R}^+\times\mathbb{R}^{N+1} \\ U^\epsilon(0,x,x_{N+1}) = u_0(x) + x_{N+1} & \text{on} \quad \mathbb{R}^{N+1}. \end{cases} \tag{4.1}$$

By Proposition 3.6 and Proposition 3.2, the comparison principle holds true for (4.1). Then, as in the proof of Proposition 3.4, by Perron's method we have:

Proposition 4.1 (Existence for (4.1))**.** *For $\epsilon > 0$ there exists $U^\epsilon \in C_b(\mathbb{R}^+\times\mathbb{R}^{N+1})$ (unique) viscosity solution of* (4.1)*. Moreover, there exists a constant $C > 0$ independent of ϵ such that*

$$|U^\epsilon(t,x,x_{N+1}) - u_0(x) - x_{N+1}| \le Ct. \tag{4.2}$$

Let us exhibit the link between the problem in $\mathbb{R}^N$ and the problem in $\mathbb{R}^{N+1}$.

Lemma 4.2 (Link between the problems on $\mathbb{R}^N$ and on $\mathbb{R}^{N+1}$)**.** *If u^ϵ and U^ϵ denote respectively the solution of* (1.1) *and* (4.1)*, then we have*

$$\left| U^\epsilon(t,x,x_{N+1}) - u^\epsilon(t,x) - \epsilon\left\lfloor \frac{x_{N+1}}{\epsilon}\right\rfloor\right| \le \epsilon,$$

$$U^\epsilon\left(t,x,x_{N+1} + \epsilon\left\lfloor\frac{a}{\epsilon}\right\rfloor\right) = U^\epsilon(t,x,x_{N+1}) + \epsilon\left\lfloor\frac{a}{\epsilon}\right\rfloor \quad \textit{for any } a\in\mathbb{R}. \tag{4.3}$$

This lemma is a consequence of comparison principle for (4.1), of invariance by ϵ-translations w.r.t. x_{N+1} and the monotonicity of U^ϵ w.r.t. x_{N+1}.

We need to make more precise the dependence of the real number λ given by Theorem 1.1 on its variables. The following properties will be shown in the next section.

Proposition 4.3 (Properties of the effective Hamiltonian)**.** *Let $p \in \mathbb{R}^N$ and $L \in \mathbb{R}$. Let $\overline{H}(p,L)$ be the constant defined by Theorem 1.1, then $\overline{H}: \mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ is a continuous function with the following properties:*

(i) *$\overline{H}(p,L)\to\pm\infty$ as $L\to\pm\infty$ for any $p\in\mathbb{R}^N$;*
(ii) *$\overline{H}(p,\cdot)$ is non-decreasing on $\mathbb{R}$ for any $p\in\mathbb{R}^N$;*
(iii) *If $\sigma(\tau,y)=\sigma(\tau,-y)$ then*
$$\overline{H}(p,L) = \overline{H}(-p,L);$$
(iv) *If $W'(-s) = -W'(s)$ and $\sigma(\tau,-y) = -\sigma(\tau,y)$ then*
$$\overline{H}(p,-L) = -\overline{H}(p,L).$$

In the proof of convergence, we will use smooth approximate sub and super-correctors on $\mathbb{R}^+\times\mathbb{R}^{N+1}$. More precisely, we consider for $P=(p,1)\in\mathbb{R}^{N+1}$ and $L\in\mathbb{R}$:

$$\begin{cases} \lambda + \partial_\tau V = L + \mathcal{I}_1[V(\tau,\cdot,y_{N+1})] - W'(V + P\cdot Y + \lambda\tau) + \sigma(\tau,y) & \text{in } \mathbb{R}^+\times\mathbb{R}^{N+1} \\ V(0,Y) = 0 & \text{on } \mathbb{R}^{N+1}. \end{cases} \tag{4.4}$$

Here and in what follows, we denote $Y = (y, y_{N+1})$. We will use also the notation $X = (x, x_{N+1})$.

Then, we have the following proposition.

Proposition 4.4 (Smooth approximate correctors)**.** *Let λ be the constant defined by Theorem 1.1. For any fixed $p \in \mathbb{R}^N$, $P = (p,1)$, $L \in \mathbb{R}$ and $\eta > 0$ small enough, there exist real numbers $\lambda^+_\eta(p,L)$, $\lambda^-_\eta(p,L)$, a constant $C > 0$ (independent of η, p and L) and bounded super and subcorrectors V^+_η, V^-_η, i.e. respectively a super and a subsolution of*

$$\tag{4.5} \begin{cases} \lambda^\pm_\eta + \partial_\tau V^\pm_\eta = L + \mathcal{I}_1[V^\pm_\eta(\tau,\cdot,y_{N+1})] & \\ \qquad\qquad -W'(V^\pm_\eta + P\cdot Y + \lambda^\pm_\eta \tau) + \sigma(\tau,y) \mp o_\eta(1) & \text{in } \mathbb{R}^+\times\mathbb{R}^{N+1} \\ V^\pm_\eta(0,Y) = 0 & \text{on } \mathbb{R}^{N+1}, \end{cases}$$

where $0 \le o_\eta(1) \to 0$ as $\eta \to 0^+$, such that

$$\tag{4.6} \lim_{\eta\to 0^+} \lambda^+_\eta(p,L) = \lim_{\eta\to 0^+} \lambda^-_\eta(p,L) = \lambda(p,L),$$

locally uniformly in (p,L), $\lambda^\pm_\eta$ satisfy (i) and (ii) of Proposition 4.3 and for any $(\tau,Y) \in \mathbb{R}^+\times\mathbb{R}^{N+1}$

$$\tag{4.7} |V^\pm_\eta(\tau,Y)| \le C.$$

Moreover $V^\pm_\eta$ are of class C^2 w.r.t. y_{N+1}, and for any $0<\alpha<1$

$$\tag{4.8} -1 \le \partial_{y_{N+1}} V^\pm_\eta \le \frac{\|W''\|_\infty}{\eta},$$

$$\tag{4.9} < \partial_{y_{N+1}} V^\pm_\eta >^\alpha_y,\ \|\partial^2_{y_{N+1}y_{N+1}} V^\pm_\eta\|_\infty \le C_\eta.$$

4.1. **Proof of Theorem 1.2.** By (4.2), we know that the family of functions $\{U^\epsilon\}_{\epsilon>0}$ is locally bounded, then $U^+ := \limsup^*_{\epsilon\to 0} U^\epsilon$ is everywhere finite. Classically we prove that U^+ is a subsolution of

$$\tag{4.10} \begin{cases} \partial_t U = \overline{H}(\nabla_x U, \mathcal{I}_1[U(t,\cdot,x_{N+1})]) & \text{in} \quad \mathbb{R}^+\times\mathbb{R}^{N+1} \\ U(0,x,x_{N+1}) = u_0(x) + x_{N+1} & \text{on} \quad \mathbb{R}^{N+1}. \end{cases}$$

Similarly, we can prove that $U^- = \liminf_{*\epsilon\to 0} U^\epsilon$ is a supersolution of (4.10). Moreover $U^+(0,x,x_{N+1}) = U^-(0,x,x_{N+1}) = u_0(x)+x_{N+1}$. The comparison principle for (4.10), which is an immediate consequence of Propositions 3.5 and 3.6, then implies that $U^+ \le U^-$. Since the reverse inequality $U^- \le U^+$ always holds true, we conclude that the two functions coincide with U^0, the unique viscosity solution of (4.10).

The link between problems (1.4) and (4.10) is given by the following lemma.

Lemma 4.5. *Let u^0 and U^0 be respectively the solutions of* (1.4) *and* (4.10)*. Then, we have*

$$U^0(t,x,x_{N+1}) = u^0(t,x) + x_{N+1},$$

$$U^0(t,x,x_{N+1}+a) = U^0(t,x,x_{N+1}) + a.$$

Lemma 4.5 is a consequence of comparison principle for (4.10) and the invariance by translations w.r.t. y_{N+1}.

By Lemmata 4.2 and 4.5, the convergence of U^ϵ to U^0 proves in particular that u^ϵ converges towards u^0 viscosity solution of (1.4).

We argue by contradiction. We consider a test function ϕ such that $U^+ - \phi$ attains a zero maximum at (t_0,X_0) with $t_0>0$ and $X_0 = (x_0, x^0_{N+1})$. Without loss of generality we may assume that the maximum is strict and global. Suppose that there exists $\theta > 0$ such that

$$\partial_t\phi(t_0,X_0) = \overline{H}(\nabla_x\phi(t_0,X_0), L_0) + \theta,$$

where

$$\tag{4.11} \begin{aligned} L_0 = &\int_{|x|\le 1} (\phi(t_0,x_0+x,x^0_{N+1}) - \phi(t_0,X_0) - \nabla_x\phi(t_0,X_0)\cdot x)\mu(dx) \\ &+ \int_{|x|>1} (U^+(t_0,x_0+x,x^0_{N+1}) - U^+(t_0,X_0))\mu(dx). \end{aligned}$$

By Proposition 4.3, we know that there exists $L_1 > 0$ such that

$$\overline{H}(\nabla_x\phi(t_0,X_0), L_0) + \theta = \overline{H}(\nabla_x\phi(t_0,X_0), L_0 + L_1).$$

By Propositions 4.4 and 4.3, we can consider a sequence $L_\eta \to L_1$ as $\eta \to 0^+$, such that $\lambda_\eta^+(\nabla_x\phi(t_0,X_0), L_0 + L_\eta) = \lambda(\nabla_x\phi(t_0,X_0), L_0+L_1)$. We choose η so small that $L_\eta - o_\eta(1) \geq L_1/2 > 0$, where $o_\eta(1)$ is defined in Proposition 4.4. Let V_η^+ be the approximate supercorrector given by Proposition 4.4 with

$$p = \nabla_x\phi(t_0,X_0), \quad L = L_0 + L_\eta$$

and

$$\lambda_\eta^+ = \lambda_\eta^+(p, L_0+L_\eta) = \partial_t\phi(t_0,X_0).$$

For simplicity of notations, in the following we denote $V = V_\eta^+$. We consider the function $F(t,X) = \phi(t,X) - p\cdot x - \lambda t$, and as in [31] and [32] we introduce the "x_{N+1}-twisted perturbed test function" ϕ^ϵ defined by:

$$\phi^\epsilon(t,X) := \begin{cases} \phi(t,X) + \epsilon V\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}, \frac{F(t,X)}{\epsilon}\right) + \epsilon k_\epsilon & \text{in} \quad (\frac{t_0}{2}, 2t_0)\times B_{\frac12}(X_0)\\ U^\epsilon(t,X) & \text{outside,}\end{cases} \tag{4.12}$$

where $k_\epsilon \in \mathbb{Z}$ will be chosen later. We are going to prove that ϕ^ϵ is a supersolution of (4.1) in $Q_{r,r}(t_0,X_0)$ for some $r < \frac12$ properly chosen and such that $Q_{r,r}(t_0,X_0) \subset (\frac{t_0}{2}, 2t_0)\times B_{\frac12}(X_0)$. First, remark that since $U^+ - \phi$ attains a strict maximum at (t_0,X_0) with $U^+ - \phi = 0$ at (t_0,X_0) and V is bounded, we can ensure that there exists $\epsilon_0 = \epsilon_0(r) > 0$ such that for $\epsilon \leq \epsilon_0$

$$U^\epsilon(t,X) \leq \phi(t,X) + \epsilon V\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}, \frac{F(t,X)}{\epsilon}\right) - \gamma_r, \quad \text{in} \ \left(\frac{t_0}{3}, 3t_0\right)\times B_1(x_0)\setminus Q_{r,r}(t_0,x_0) \tag{4.13}$$

for some $\gamma_r = o_r(1) > 0$. Hence choosing $k_\epsilon = \lceil \frac{-\gamma_r}{\epsilon}\rceil$ we get $U^\epsilon \leq \phi^\epsilon$ outside $Q_{r,r}(t_0,X_0)$.

Let us next study the equation. From (4.3), we deduce that $U^+(t,x,x_{N+1}+a) = U^+(t,x,x_{N+1}) + a$ for any $a\in\mathbb{R}$, from which we derive that $\partial_{x_{N+1}}F(t_0,X_0) = \partial_{x_{N+1}}\phi(t_0,X_0) = 1$. Then, there exists $r_0 > 0$ such that the map

$$\begin{array}{rccc} Id\times F: & Q_{r_0,r_0}(t_0,X_0) & \longrightarrow & \mathcal{U}_{r_0}\\ & (t,x,x_{N+1}) & \longmapsto & (t,x,F(t,x,x_{N+1}))\end{array}$$

is a C^1-diffeomorphism from $Q_{r_0,r_0}(t_0,X_0)$ onto its range $\mathcal{U}_{r_0}$. Let $G:\mathcal{U}_{r_0}\to\mathbb{R}$ be the map such that

$$\begin{array}{rccc} Id\times G: & \mathcal{U}_{r_0} & \longrightarrow & Q_{r_0,r_0}(t_0,X_0)\\ & (t,x,\xi_{N+1}) & \longmapsto & (t,x,G(t,x,\xi_{N+1}))\end{array}$$

is the inverse of $Id\times F$. Let us introduce the variables $\tau = t/\epsilon$, $Y = (y, y_{N+1})$ with $y = x/\epsilon$ and $y_{N+1} = F(t,X)/\epsilon$. Let us consider a test function ψ such that $\phi^\epsilon - \psi$ attains a global zero minimum at $(\bar t,\overline{X}) \in Q_{r_0,r_0}(t_0,X_0)$ and define

$$\Gamma^\epsilon(\tau,Y) = \frac{1}{\epsilon}[\psi(\epsilon\tau,\epsilon y, G(\epsilon\tau,\epsilon y,\epsilon y_{N+1})) - \phi(\epsilon\tau,\epsilon y, G(\epsilon\tau,\epsilon y,\epsilon y_{N+1}))] - k_\epsilon.$$

Then

$$\Gamma^\epsilon(\overline{\tau},\overline{Y}) = V(\overline{\tau},\overline{Y}) \quad \text{and} \quad \Gamma^\epsilon(\tau,Y) \leq V(\tau,Y) \quad \text{for all } (\epsilon\tau,\epsilon Y)\in Q_{r_0,r_0}(t_0,X_0), \tag{4.14}$$

where $\overline{\tau} = \bar t/\epsilon$, $\overline{y} = \overline{x}/\epsilon$, $\overline{y}_{N+1} = F(\bar t,\overline{X})/\epsilon$, $\overline{Y} = (\overline{y},\overline{y}_{N+1})$. From Proposition 4.4, we know that V is Lipschitz continuous w.r.t. y_{N+1} with Lipschitz constant M_η depending on η. This implies that

$$|\partial_{y_{N+1}}\Gamma^\epsilon(\overline{\tau},\overline{Y})| \leq M_\eta. \tag{4.15}$$

Simple computations yield with $P = (p,1)\in\mathbb{R}^{N+1}$:

$$\begin{cases} \lambda_\eta^+ + \partial_\tau\Gamma^\epsilon(\overline{\tau},\overline{Y}) = \partial_t\psi(\bar t,\overline{X}) + \left(1+\partial_{y_{N+1}}\Gamma^\epsilon(\overline{\tau},\overline{Y})\right)(\partial_t\phi(t_0,X_0) - \partial_t\phi(\bar t,\overline{X})),\\ p + \nabla_y\Gamma^\epsilon(\overline{\tau},\overline{Y}) = \nabla_x\psi(\bar t,\overline{X}) + \left(1+\partial_{y_{N+1}}\Gamma^\epsilon(\overline{\tau},\overline{Y})\right)(\nabla_x\phi(t_0,X_0) - \nabla_x\phi(\bar t,\overline{X})),\\ \lambda_\eta^+\overline{\tau} + P\cdot\overline{Y} + V(\overline{\tau},\overline{Y}) = \frac{\phi^\epsilon(\bar t,\overline{X})}{\epsilon} - k_\epsilon.\end{cases} \tag{4.16}$$

Using (4.16) and (4.15), Equation (4.5) yields for any $\rho > 0$

$$\begin{aligned}\partial_t\psi(\bar t,\overline{X}) + o_r(1) \geq{}& L_0 + L_\eta + \mathcal{I}_1^{1,\rho}[\Gamma^\epsilon(\overline{\tau},\cdot,\overline{y}_{N+1}),\overline{y}] + \mathcal{I}_1^{2,\rho}[V(\overline{\tau},\cdot,\overline{y}_{N+1}),\overline{y}]\\ &- W'\left(\frac{\phi^\epsilon(\bar t,\overline{X})}{\epsilon}\right) + \sigma\left(\frac{\bar t}{\epsilon},\frac{\overline{x}}{\epsilon}\right) - o_\eta(1).\end{aligned} \tag{4.17}$$

We now use the following lemma whose proof is postponed:

Lemma 4.6. *For $\epsilon \le \epsilon_0(r) < r \le r_0$, we have*

$$\partial_t\psi(\bar t,\overline{X}) \ge \mathcal{I}_1^{1,1}\left[\psi(\bar t,\cdot,\overline{x}_{N+1}),\overline{x}\right] + \mathcal{I}_1^{2,1}\left[\phi^\epsilon(\bar t,\cdot,\overline{x}_{N+1}),\overline{x}\right] - W'\left(\frac{\phi^\epsilon(\bar t,\overline{X})}{\epsilon}\right) + \sigma\left(\frac{\bar t}{\epsilon},\frac{\overline{x}}{\epsilon}\right) - o_\eta(1) + o_r(1) + L_\eta.$$

Let $r \le r_0$ be so small that $o_r(1) \ge -L_1/4$. Then, recalling that $L_\eta - o_\eta(1) \ge L_1/2$, for $\epsilon \le \epsilon_0(r)$ we have

$$\partial_t\psi(\bar t,\overline{X}) \ge \mathcal{I}_1^{1,1}\left[\psi(\bar t,\cdot,\overline{x}_{N+1}),\overline{x}\right] + \mathcal{I}_1^{2,1}\left[\phi^\epsilon(\bar t,\cdot,\overline{x}_{N+1}),\overline{x}\right] - W'\left(\frac{\phi^\epsilon(\bar t,\overline{X})}{\epsilon}\right) + \sigma\left(\frac{\bar t}{\epsilon},\frac{\overline{x}}{\epsilon}\right) + \frac{L_1}{4},$$

and therefore ϕ^ϵ is a supersolution of (4.1) in $Q_{r,r}(t_0,X_0)$. Since $U^\epsilon \le \phi^\epsilon$ outside $Q_{r,r}(t_0,X_0)$, by the comparison principle, Proposition 3.3, we conclude that $U^\epsilon(t,X) \le \phi(t,X) + \epsilon V\left(\frac{t}{\epsilon},\frac{x}{\epsilon},\frac{F(t,X)}{\epsilon}\right) + \epsilon k_\epsilon$ in $Q_{r,r}(t_0,X_0)$ and we obtain the desired contradiction by passing to the upper limit as $\epsilon \to 0$ at (t_0,X_0) using the fact that $U^+(t_0,X_0) = \phi(t_0,X_0)$: $0 \le -\gamma_r$.

Proof of Lemma 4.6. We call

$$L_0^1 = \int_{|x|\le 1}(\phi(t_0,x_0+x,x_{N+1}^0) - \phi(t_0,X_0) - \nabla\phi(t_0,X_0)\cdot x)\mu(dx),$$

$$L_0^2 = \int_{|x|>1}(U^+(t_0,x_0+x,x_{N+1}^0) - U^+(t_0,X_0))\mu(dx).$$

Then

$$L_0 = L_0^1 + L_0^2. \tag{4.18}$$

Keep in mind that $\overline{y}_{N+1} = \frac{F(\bar t,\overline{X})}{\epsilon}$. Since $\psi(t,X) = \phi(t,X) + \epsilon\Gamma^\epsilon\left(\frac{t}{\epsilon},\frac{x}{\epsilon},\frac{F(t,X)}{\epsilon}\right) + \epsilon k_\epsilon$, we have

$$\mathcal{I}_1^{1,1}\left[\psi^\epsilon(\bar t,\cdot,\overline{x}_{N+1}),\overline{x}\right] = I_1 + I_2, \tag{4.19}$$

where

$$I_1 = \int_{|x|\le 1}\epsilon\left(\Gamma^\epsilon\left(\frac{\bar t}{\epsilon},\frac{\overline{x}+x}{\epsilon},\frac{F(\bar t,\overline{x}+x,\overline{x}_{N+1})}{\epsilon}\right) - \Gamma^\epsilon(\overline{\tau},\overline{Y}) - \nabla_y\Gamma^\epsilon(\overline{\tau},\overline{Y})\cdot\frac{x}{\epsilon} - \partial_{y_{N+1}}\Gamma^\epsilon(\overline{\tau},\overline{Y})\nabla_x F(\bar t,\overline{X})\cdot\frac{x}{\epsilon}\right)\mu(dx),$$

$$I_2 = \int_{|x|\le 1}\left(\phi(\bar t,\overline{x}+x,\overline{x}_{N+1}) - \phi(\bar t,\overline{X}) - \nabla\phi(\bar t,\overline{X})\cdot x\right)\mu(dx).$$

To show the result, we proceed in several steps. In what follows, we denote by C various positive constants independent of ϵ.

Step 1: We can choose ϵ_0 so small that for any $\epsilon \le \epsilon_0$ and any $\rho > 0$ small enough

$$I_1 \le \mathcal{I}_1^{1,\rho}[\Gamma^\epsilon(\overline{\tau},\cdot,\overline{y}_{N+1}),\overline{y}] + \mathcal{I}_1^{2,\rho}[V(\overline{\tau},\cdot,\overline{y}_{N+1}),\overline{y}] + o_r(1) + C_\epsilon\rho.$$

Take $\rho > 0$, $\delta > \rho$ small and $R > 0$ large and such that $\epsilon R < 1$. Since g is even, we can write

$$I_1 = I_1^0 + I_1^1 + I_1^2 + I_1^3,$$

where

$$I_1^0 = \int_{|x|\le\epsilon\rho}\epsilon\left(\Gamma^\epsilon\left(\frac{\bar t}{\epsilon},\frac{\overline{x}+x}{\epsilon},\frac{F(\bar t,\overline{x}+x,\overline{x}_{N+1})}{\epsilon}\right) - \Gamma^\epsilon(\overline{\tau},\overline{Y}) - \nabla_y\Gamma^\epsilon(\overline{\tau},\overline{Y})\cdot\frac{x}{\epsilon} - \partial_{y_{N+1}}\Gamma^\epsilon(\overline{\tau},\overline{Y})\nabla_x F(\bar t,\overline{X})\cdot\frac{x}{\epsilon}\right)\mu(dx),$$

$$I_1^1 = \int_{\epsilon\rho\le|x|\le\epsilon\delta} \epsilon\left(\Gamma^\epsilon\left(\frac{\bar t}{\epsilon}, \frac{\overline{x}+x}{\epsilon}, \frac{F(\bar t,\overline{x}+x,\overline{x}_{N+1})}{\epsilon}\right) - \Gamma^\epsilon(\overline{\tau},\overline{Y})\right)\mu(dx),$$

$$I_1^2 = \int_{\epsilon\delta\le|x|\le\epsilon R} \epsilon\left(\Gamma^\epsilon\left(\frac{\bar t}{\epsilon}, \frac{\overline{x}+x}{\epsilon}, \frac{F(\bar t,\overline{x}+x,\overline{x}_{N+1})}{\epsilon}\right) - \Gamma^\epsilon(\overline{\tau},\overline{Y})\right)\mu(dx),$$

$$I_1^3 = \int_{\epsilon R\le|x|\le 1} \epsilon\left(\Gamma^\epsilon\left(\frac{\bar t}{\epsilon}, \frac{\overline{x}+x}{\epsilon}, \frac{F(\bar t,\overline{x}+x,\overline{x}_{N+1})}{\epsilon}\right) - \Gamma^\epsilon(\overline{\tau},\overline{Y})\right)\mu(dx),$$

and

$$\mathcal{I}_1^{2,\rho}[V(\overline{\tau},\cdot,\overline{y}_{N+1}),\overline{y}] = J_1 + J_2 + J_3,$$

where

$$J_1 = \int_{\rho<|z|\le\delta} (V(\overline{\tau},\overline{y}+z,\overline{y}_{N+1}) - V(\overline{\tau},\overline{Y}))\mu(dz),$$

$$J_2 = \int_{\delta<|z|\le R} (V(\overline{\tau},\overline{y}+z,\overline{y}_{N+1}) - V(\overline{\tau},\overline{Y}))\mu(dz),$$

$$J_3 = \int_{|z|>R} (V(\overline{\tau},\overline{y}+z,\overline{y}_{N+1}) - V(\overline{\tau},\overline{Y}))\mu(dz).$$

STEP 1.1: *Estimate of I_1^0 and $\mathcal{I}_1^{1,\rho}[\Gamma^\epsilon(\overline{\tau},\cdot,\overline{y}_{N+1}),\overline{y}]$.*

Since Γ^ϵ is of class C^2, we have

$$|I_1^0|,\ |\mathcal{I}_1^{1,\rho}[\Gamma^\epsilon(\overline{\tau},\cdot,\overline{y}_{N+1}),\overline{y}]| \le C_\epsilon\rho, \tag{4.20}$$

where C_ϵ depends on the second derivatives of Γ^ϵ.

STEP 1.2 *Estimate of $I_1^1 - J_1$.*

Using (4.14) and the fact that g is even, we can estimate $I_1^1 - J_1$ as follows

$$\begin{aligned}
I_1^1 - J_1 &\le \int_{\rho<|z|\le\delta}\left[V\left(\overline{\tau},\overline{y}+z,\frac{F(\bar t,\overline{x}+\epsilon z,\overline{x}_{N+1})}{\epsilon}\right) - V\left(\overline{\tau},\overline{y}+z,\frac{F(\bar t,\overline{x})}{\epsilon}\right)\right]\mu(dz)\\
&= \int_{\rho<|z|\le\delta}\left\{\left[V\left(\overline{\tau},\overline{y}+z,\frac{F(\bar t,\overline{x}+\epsilon z,\overline{x}_{N+1})}{\epsilon}\right) - V\left(\overline{\tau},\overline{y}+z,\frac{F(\bar t,\overline{x})}{\epsilon}\right)\right.\right.\\
&\left.-\partial_{y_{N+1}}V\left(\overline{\tau},\overline{y}+z,\frac{F(\bar t,\overline{X})}{\epsilon}\right)\nabla_x F(\bar t,\overline{X})\cdot z\right]\\
&\left.+\left[\partial_{y_{N+1}}V(\overline{\tau},\overline{y}+z,\overline{y}_{N+1}) - \partial_{y_{N+1}}V(\overline{\tau},\overline{Y})\right]\nabla_x F(\bar t,\overline{X})\cdot z\right\}\mu(dz).
\end{aligned}$$

Next, using (4.8) and (4.9), we get

$$I_1^1 - J_1 \le C\int_{|z|\le\delta}(|z|^2 + |z|^{1+\alpha})\mu(dz) \le C\delta^\alpha. \tag{4.21}$$

STEP 1.3 *Estimate of $I_1^2 - J_2$.*

If M_η is the Lipschitz constant of V w.r.t. y_{N+1}, then

$$\begin{aligned}
I_1^2 - J_2 &\le \int_{\delta<|z|\le R}\left(V\left(\overline{\tau},\overline{y}+z,\frac{F(\bar t,\overline{x}+\epsilon z,\overline{x}_{N+1})}{\epsilon}\right) - V\left(\overline{\tau},\overline{y}+z,\frac{F(\bar t,\overline{X})}{\epsilon}\right)\right)\mu(dz)\\
&\le M_\eta\int_{\delta<|z|\le R}\left|\frac{F(\bar t,\overline{x}+\epsilon z,\overline{x}_{N+1})}{\epsilon} - \frac{F(\bar t,\overline{X})}{\epsilon}\right|\mu(dz)\\
&\le M_\eta\int_{\delta<|z|\le R}\sup_{|z|\le R}|\nabla_x F(\bar t,\overline{x}+\epsilon z,\overline{x}_{N+1})||z|\mu(dz).
\end{aligned}$$

Then

$$I_1^2 - J_2 \le C\sup_{|z|\le R}|\nabla_x F(\bar t,\overline{x}+\epsilon z,\overline{x}_{N+1})|\log(R/\delta) \tag{4.22}$$

STEP 1.4: *Estimate of I_1^3 and J_3.*

Since V is uniformly bounded on $\mathbb{R}^+ \times \mathbb{R}^{N+1}$, we have

$$\begin{aligned} I_1^3 &\le \int_{R<|z|\le\frac{1}{\epsilon}} \left(V\left(\overline{\tau}, \overline{y}+z, \frac{F(\overline{t},\overline{x}+\epsilon z, \overline{x}_{N+1})}{\epsilon}\right) - V(\overline{\tau},\overline{Y})\right)\mu(dz) \\ &\le \int_{|z|>R} 2\|v\|_\infty \mu(dz) \le \frac{C}{R}. \end{aligned} \tag{4.23}$$

Similarly

$$|J_3| \le \frac{C}{R}. \tag{4.24}$$

Now, from (4.20), (4.21), (4.22), (4.23) and (4.24), we infer that

$$\begin{aligned} I_1 &\le \mathcal{I}_1^{1,\rho}[\Gamma^\epsilon(\overline{\tau},\cdot,\overline{y}_{N+1}),\overline{y}] + \mathcal{I}_1^{2,\rho}[V(\overline{\tau},\cdot,\overline{y}_{N+1}),\overline{y}] + 2C_\epsilon\rho + C\delta^\alpha \\ &\quad + C\sup_{|z|\le R} |\nabla_x F(\overline{t},\overline{x}+\epsilon z,\overline{x}_{N+1})|\log\left(\frac{R}{\delta}\right) + \frac{C}{R}. \end{aligned}$$

We choose $R = R(r)$ such $R \to +\infty$ as $r \to 0^+$, $\epsilon_0 = \epsilon_0(r)$ such that $R\epsilon_0(r) \le r$ and $\delta = \delta(r) > 0$ such that $\delta \to 0$ as $r \to 0^+$ and $r\log(R/\delta) \to 0$ as $r \to 0^+$. With this choice, for any $\epsilon \le \epsilon_0$ and any $\rho < \delta$

$$C\delta^\alpha + C\sup_{|z|\le R} |\nabla_x F(\overline{t},\overline{x}+\epsilon z,\overline{x}_{N+1})|\log\left(\frac{R}{\delta}\right) + \frac{C}{R} = o_r(1) \quad \text{as } r \to 0^+,$$

and Step 1 is proved.

Step 2: $I_2 \le L_0^1 + o_r(1)$**.**

For $0 < \nu < 1$ we can split I_2 and L_0^1 as follows

$$\begin{aligned} I_2 &= \int_{|x|\le\nu} (\phi(\overline{t},\overline{x}+x,\overline{x}_{N+1}) - \phi(\overline{t},\overline{X}) - \nabla\phi(\overline{t},\overline{X})\cdot x)\mu(dx) \\ &\quad + \int_{\nu\le|x|\le1} (\phi(\overline{t},\overline{x}+x,\overline{x}_{N+1}) - \phi(\overline{t},\overline{X}))\mu(dx) = I_2^1 + I_2^2, \end{aligned}$$

$$\begin{aligned} L_0^1 &= \int_{|x|\le\nu} (\phi(t_0,x_0+x,x_{N+1}^0) - \phi(t_0,X_0) - \nabla\phi(t_0,X_0)\cdot x)\mu(dx) \\ &\quad + \int_{\nu\le|x|\le1} (\phi(t_0,x_0+x,x_{N+1}^0) - \phi(t_0,X_0))\mu(dx) = T_1 + T_2. \end{aligned}$$

Since ϕ is of class C^2 we have

$$I_2^1, T_1 \le C\nu.$$

Using the Lipschitz continuity of ϕ we get

$$I_2^2 - T_2 = \int_{\nu<|x|\le1} Cr\mu(dx) \le C\frac{r}{\nu}.$$

Hence, Step 2 follows choosing $\nu = \nu(r)$ such that $\nu \to 0$ and $r/\nu \to 0$ as $r \to 0^+$.

Step 3: $\mathcal{I}_1^{2,1}\left[\phi^\epsilon(\overline{t},\cdot,\overline{x}_{N+1}),\overline{x}\right] \le L_0^2 + o_r(1)$**.**

Remark that

$$U^\epsilon(\overline{t},\overline{x}+x,\overline{x}_{N+1}) - \phi(\overline{t},\overline{X}) - \epsilon V(\overline{\tau},\overline{Y}) - \epsilon k_\epsilon \le U^+(t_0,x_0+x,x_{N+1}^0) - \phi(t_0,X_0) + o_\epsilon(1) + o_r(1).$$

Then, recalling that $\phi(t_0,X_0) = U^+(t_0,X_0)$, for $\epsilon \le \epsilon_0$ we get

$$\mathcal{I}_1^{2,1}\left[\phi^\epsilon(\overline{t},\cdot,\overline{x}_{N+1}),\overline{x}\right] - L_0^2 \le o_r(1)$$

and Step 3 is proved.

Finally (4.18), (7.19), Steps 1, 2 and 3 give

$$\begin{aligned} \mathcal{I}_1^{1,1}\left[\psi(\overline{t},\cdot,\overline{x}_{N+1}),\overline{x}\right] + \mathcal{I}_1^{2,1}\left[\phi^\epsilon(\overline{t},\cdot,\overline{x}_{N+1}),\overline{x}\right] &\le \mathcal{I}_1^{1,\rho}[\Gamma^\epsilon(\overline{\tau},\cdot,\overline{y}_{N+1}),\overline{y}] + \mathcal{I}_1^{2,\rho}[V(\overline{\tau},\cdot,\overline{y}_{N+1}),\overline{y}] \\ &\quad + L_0 + o_r(1) + C_\epsilon\rho. \end{aligned}$$

from which, using inequality (4.17) and letting $\rho \to 0^+$, we get for $\epsilon \le \epsilon_0$

$$\partial_t \psi(\bar t, \overline{X}) \ge \mathcal{I}_1^{1,1}\left[\psi(\bar t, \cdot, \overline{x}_{N+1}), \overline{x}\right] + \mathcal{I}_1^{2,1}\left[\phi^\epsilon(\bar t, \cdot, \overline{x}_{N+1}), \overline{x}\right] - W'\left(\frac{\phi^\epsilon(\bar t, \overline{X})}{\epsilon}\right) + \sigma\left(\frac{\bar t}{\epsilon}, \frac{\overline{x}}{\epsilon}\right)$$
$$- o_\eta(1) + o_r(1) + L_\eta$$

and this concludes the proof of the lemma. □

5. Building of Lipschitz sub and supercorrectors

In this section we construct sub and supersolutions of (4.4) that are Lipschitz w.r.t. y_{N+1}. As a byproduct, we will prove Theorem 1.1 and Proposition 4.3.

Proposition 5.1 (Lipschitz continuous sub and supercorrectors)**.** *Let λ be the quantity defined by Theorem 1.1. Then, for any fixed $p \in \mathbb{R}^N$, $P = (p,1)$, $L \in \mathbb{R}$ and $\eta > 0$ small enough, there exist real numbers $\lambda_\eta^+(p,L)$, $\lambda_\eta^-(p,L)$, a constant $C > 0$ (independent of η, p and L) and bounded super and subcorrectors W_η^+, W_η^- i.e. respectively a super and a subsolution of (4.4) (with respectively λ_η^+ and λ_η^- in place of λ) such that*

$$\lim_{\eta\to 0^+} \lambda_\eta^+(p,L) = \lim_{\eta \to 0^+} \lambda_\eta^-(p,L) = \lambda(p,L),$$

$\lambda_\eta^\pm$ satisfy (i) and (ii) of Proposition 4.3 and for any $(\tau, Y) \in \mathbb{R}^+ \times \mathbb{R}^{N+1}$

$$|W_\eta^\pm(\tau, Y)| \le C. \tag{5.1}$$

Moreover $W_\eta^\pm$ are Lipschitz continuous w.r.t. y_{N+1} and α-Hölder continuous w.r.t. y for any $0 < \alpha < 1$, with

$$-1 \le \partial_{y_{N+1}} W_\eta^\pm \le \frac{\|W''\|_\infty}{\eta}, \tag{5.2}$$

$$< W_\eta^\pm >_y^\alpha \le C_\eta. \tag{5.3}$$

In order to prove the proposition, for $\eta \ge 0$, $L \in \mathbb{R}$, $p \in \mathbb{R}^N$ and $P = (p,1)$, we introduce the problem

$$\begin{cases} \partial_\tau U = L + \mathcal{I}_1[U(\tau, \cdot, y_{N+1})] - W'(U + P\cdot Y) + \sigma(\tau, y) & \\ \qquad + \eta[a_0 + \inf_{Y'} U(\tau, Y') - U(\tau, Y)]|\partial_{y_{N+1}} U + 1| & \text{in } \mathbb{R}^+ \times \mathbb{R}^{N+1} \\ U(0,Y) = 0 & \text{on } \mathbb{R}^{N+1}. \end{cases} \tag{5.4}$$

5.1. Comparison principle.

Proposition 5.2 (Comparison principle for (5.4))**.** *Let $U_1 \in USC_b(\mathbb{R}^+ \times \mathbb{R}^{N+1})$ and $U_2 \in LSC_b(\mathbb{R}^+ \times \mathbb{R}^{N+1})$ be respectively a viscosity subsolution and supersolution of (5.4), then $U_1 \le U_2$ on $\mathbb{R}^+ \times \mathbb{R}^{N+1}$.*

Proof. Let us define the functions $V_1(\tau, Y) := e^{-k\tau} U_1(\tau, Y)$ and $V_2(\tau, Y) := e^{-k\tau} U_2(\tau, Y)$, where $k := \|W''\|_\infty + 1$. It is easy to see that V_1 and V_2 are respectively sub and supersolution of

$$\begin{cases} \partial_\tau V = Le^{-k\tau} + \mathcal{I}_1[V(\tau, \cdot, y_{N+1})] + g(\tau, Y, V) & \\ \qquad + \eta[a_0 + e^{k\tau}(\inf_{Y'} V(\tau, Y') - V(\tau, Y))]|\partial_{y_{N+1}} V + e^{-k\tau}| & \text{in } \mathbb{R}^+ \times \mathbb{R}^{N+1} \\ V(0,Y) = 0 & \text{on } \mathbb{R}^{N+1}, \end{cases} \tag{5.5}$$

where $g(\tau, Y, V) = -e^{-k\tau} W'(e^{k\tau} V + P \cdot Y) - kV + e^{-k\tau}\sigma(\tau, y)$. Remark that, by the choice of k,

$$g(\tau, Y, V_1) - g(\tau, Z, V_2) \le -(V_1 - V_2) + e^{-k\tau}(\|W''\|_\infty |P| + \|\sigma'\|_\infty)|Y - Z|. \tag{5.6}$$

To prove the comparison between U_1 and U_2, it suffices to show that $V_1(\tau, Y) \le V_2(\tau, Y)$ for all $(\tau, Y) \in (0,T) \times \mathbb{R}^{N+1}$ and for any $T > 0$.

Suppose by contradiction that $M = \sup_{(\tau,Y)\in(0,T)\times\mathbb{R}^{N+1}}(V_1(\tau, Y) - V_2(\tau, Y)) > 0$ for some $T > 0$. Define for small $\nu_1, \nu_2, \beta, \delta > 0$ the function $\phi \in C^2((\mathbb{R}^+ \times \mathbb{R}^{N+1})^2)$ by

$$\phi(\tau, Y, s, Z) = \frac{1}{2\nu_1}|\tau - s|^2 + \frac{1}{2\nu_2}|Y - Z|^2 + \beta\psi(Y) + \frac{\delta}{T - \tau},$$

where ψ is defined as the function ψ_2 in the proof of Proposition 3.7. The supremum of $V_1(\tau,Y)-V_2(s,Z)-\phi(\tau,Y,s,Z)$ is attained at some point $(\overline{\tau},\overline{Y},\overline{s},\overline{Z})\in((0,T)\times\mathbb{R}^{N+1})^2$. Standard arguments show that, because U_1 and U_2 are assumed bounded

$$(\overline{\tau},\overline{Y},\overline{s},\overline{Z})\to(\widehat{\tau},\widehat{\tau},\widehat{Y},\widehat{Z})\quad\text{as }\nu_1\to 0,$$

$$V_1(\overline{\tau},\overline{Y})\to V_1(\widehat{\tau},\widehat{Y}),\ V_2(\overline{s},\overline{Z})\to V_2(\widehat{\tau},\widehat{Z})\quad\text{as }\nu_1\to 0,$$

where $(\widehat{\tau},\widehat{Y},\widehat{Z})$ is a maximum point of $V_1(\tau,Y)-V_2(\tau,Z)-\frac{1}{2\nu_2}|Y-Z|^2-\beta\psi(Y)-\frac{\eta}{T-\tau}$. Moreover, it is easy to see that

$$\limsup_{\nu_1\to 0}\inf_{Y'}V_1(\overline{\tau},Y')\le\inf_{Y'}V_1(\widehat{\tau},Y'),\ \liminf_{\nu_1\to 0}\inf_{Y'}V_2(\overline{s},Y')\ge\inf_{Y'}V_2(\widehat{\tau},Y').$$

Since V_1 and V_2 are respectively sub and supersolution of (5.5), for any $r>0$ we have

$$\begin{aligned}&\frac{\delta}{(T-\overline{\tau})^2}+\frac{\overline{\tau}-\overline{s}}{\nu_1}\\ &\le Le^{-k\overline{\tau}}+\frac{C_Nr}{\nu_2}+\beta\mathcal{I}_1^{1,r}[\psi(\cdot,\overline{y}_{N+1}),\overline{y}]+\mathcal{I}_1^{2,r}[V_1(\overline{\tau},\cdot,\overline{y}_{N+1}),\overline{y}]+g(\overline{\tau},\overline{Y},V_1(\overline{\tau},\overline{Y}))\\ &\quad+\eta[a_0+e^{k\overline{\tau}}(\inf_{Y'}V_1(\overline{\tau},Y')-V_1(\overline{\tau},\overline{Y}))]\left|\frac{\overline{y}_{N+1}-\overline{z}_{N+1}}{\nu_2}+\beta\partial_{y_{N+1}}\psi(\overline{Y})+e^{-k\overline{\tau}}\right|\end{aligned}\tag{5.7}$$

and

$$\begin{aligned}\frac{\overline{\tau}-\overline{s}}{\nu_1}&\ge Le^{-k\overline{s}}-\frac{C_Nr}{\nu_2}+\mathcal{I}_1^{2,r}[V_2(\overline{s},\cdot,\overline{z}_{N+1}),\overline{z}]+g(\overline{s},\overline{Z},V_2(\overline{s},\overline{Z}))\\ &\quad+\eta[a_0+e^{k\overline{s}}(\inf_{Y'}V_2(\overline{s},Y')-V_2(\overline{s},\overline{Z}))]\left|\frac{\overline{y}_{N+1}-\overline{z}_{N+1}}{\nu_2}+e^{-k\overline{s}}\right|,\end{aligned}\tag{5.8}$$

where C_N is a constant depending on the dimension N. Since $(\overline{\tau},\overline{Y},\overline{s},\overline{Z})$ is a maximum point, we have

$$V_1(\overline{\tau},\overline{y}+x,\overline{y}_{N+1})-V_1(\overline{\tau},\overline{Y})\le V_2(\overline{s},\overline{z}+x,\overline{z}_{N+1})-V_2(\overline{s},\overline{Z})+\beta[\psi(\overline{y}+x,\overline{y}_{N+1})-\psi(\overline{Y})],$$

for any $x\in\mathbb{R}^N$, which implies that for any $r>0$

$$\mathcal{I}_1^{2,r}[V_1(\overline{\tau},\cdot,\overline{y}_{N+1}),\overline{y}]\le\mathcal{I}_1^{2,r}[V_2(\overline{s},\cdot,\overline{z}_{N+1}),\overline{z}]+\beta\mathcal{I}_1^{2,r}[\psi(\cdot,\overline{y}_{N+1}),\overline{y}].$$

Then, subtracting (5.7) with (5.8) and letting $r\to 0^+$, we get

$$\begin{aligned}\frac{\delta}{(T-\overline{\tau})^2}&\le L(e^{-k\overline{\tau}}-e^{-k\overline{s}})+\beta\mathcal{I}_1[\psi(\cdot,\overline{y}_{N+1}),\overline{y}]+g(\overline{\tau},\overline{Y},V_1(\overline{\tau},\overline{Y}))-g(\overline{s},\overline{Z},V_2(\overline{s},\overline{Z}))\\ &\quad+\eta[a_0+e^{k\overline{\tau}}(\inf_{Y'}V_1(\overline{\tau},Y')-V_1(\overline{\tau},\overline{Y}))]\left|\frac{\overline{y}_{N+1}-\overline{z}_{N+1}}{\nu_2}+\beta\partial_{y_{N+1}}\psi(\overline{Y})+e^{-k\overline{\tau}}\right|\\ &\quad-\eta[a_0+e^{k\overline{s}}(\inf_{Y'}V_2(\overline{s},Y')-V_2(\overline{s},\overline{Z}))]\left|\frac{\overline{y}_{N+1}-\overline{z}_{N+1}}{\nu_2}+e^{-k\overline{s}}\right|.\end{aligned}$$

Next, letting $\nu_1\to 0$ and using (5.6), we obtain

$$\begin{aligned}&\frac{\delta}{(T-\widehat{\tau})^2}\\ &\le\beta\mathcal{I}_1[\psi(\cdot,\widehat{y}_{N+1}),\widehat{y}]-(V_1(\widehat{\tau},\widehat{Y})-V_2(\widehat{\tau},\widehat{Z}))+e^{-k\widehat{\tau}}(\|W''\|_\infty|P|+\|\sigma'\|_\infty)|\widehat{Y}-\widehat{Z}|+C\beta\\ &\quad+\eta e^{k\widehat{\tau}}[\inf_{Y'}V_1(\widehat{\tau},Y')-\inf_{Y'}V_2(\widehat{\tau},Y')-(V_1(\widehat{\tau},\widehat{Y})-V_2(\widehat{\tau},\widehat{Z}))]\left|\frac{\widehat{y}_{N+1}-\widehat{z}_{N+1}}{\nu_2}+e^{-k\widehat{\tau}}\right|.\end{aligned}\tag{5.9}$$

It is easy to prove that

$$\liminf_{(\beta,\delta)\to(0,0)}(V_1(\widehat{\tau},\widehat{Y})-V_2(\widehat{\tau},\widehat{Z}))\ge M\tag{5.10}$$

and

$$\frac{|\widehat{Y}-\widehat{Z}|^2}{\nu_2}\le C,$$

where C is independent of β and δ. Up to subsequence, $\widehat{\tau} \to \tau_0 \in [0,T]$ as $(\beta,\delta) \to (0,0)$ and by (5.10), we have

$$\begin{aligned}
&\limsup_{(\beta,\delta)\to(0,0)} [\inf_{Y'} V_1(\widehat{\tau}, Y') - \inf_{Y'} V_2(\widehat{\tau}, Y') - (V_1(\widehat{\tau}, \widehat{Y}) - V_2(\widehat{\tau}, \widehat{Z}))] \\
&\le \inf_{Y'} V(\tau_0, Y') - \inf_{Y'} V_2(\tau_0, Y') - \sup_{Y'}(V_1(\tau_0, Y') - V_2(\tau_0, Y')) \\
&\le 0.
\end{aligned}$$

Then, passing to the limit first as $(\beta, \delta) \to (0,0)$ and then as $\nu_2 \to 0$ in (5.9) we finally get the contradiction:

$$M \le 0,$$

and this concludes the proof of the comparison theorem. □

5.2. **Lipschitz regularity.**

Proposition 5.3 (Lipschitz continuity in y_{N+1}). *Suppose $\eta > 0$. Let $U_\eta \in C_b(\mathbb{R}^+ \times \mathbb{R}^{N+1})$ be the viscosity solution of* (5.4). *Then U_η is Lipschitz continuous w.r.t. y_{N+1} and for almost every $(\tau, Y) \in \mathbb{R}^+ \times \mathbb{R}^{N+1}$*

$$-1 \le \partial_{y_{N+1}} U_\eta(\tau, Y) \le \frac{\|W''\|_\infty}{\eta}. \tag{5.11}$$

Proof. Let us define $\widehat{U}(\tau, Y) = U(\tau, Y) + y_{N+1}$, then $\widehat{U}$ satisfies

$$\begin{cases} \partial_\tau \widehat{U} = L + \mathcal{I}_1[\widehat{U}(\tau, \cdot, y_{N+1})] - W'(\widehat{U} + p\cdot y) + \sigma(\tau, y) & \\ \qquad +\eta[a_0 + \inf_{Y'}(\widehat{U}(\tau, Y') - y'_{N+1}) - (\widehat{U}(\tau, Y) - y_{N+1})]|\partial_{y_{N+1}}\widehat{U}| & \text{in } \mathbb{R}^+ \times \mathbb{R}^{N+1} \\ \widehat{U}(0, Y) = y_{N+1} & \text{on } \mathbb{R}^{N+1}. \end{cases} \tag{5.12}$$

We are going to prove that $\widehat{U}$ is Lipschitz continuous w.r.t. y_{N+1} with

$$0 \le \partial_{y_{N+1}} \widehat{U}(\tau, Y) \le 1 + \frac{\|W''\|_\infty}{\eta}.$$

By comparison, $\widehat{U}(t, y, y_{N+1}) \le \widehat{U}(t, y, y_{N+1} + h)$ for $h \ge 0$, from which immediately follows that $\partial_{y_{N+1}}\widehat{U} \ge 0$. In particular we can replace $|\partial_{y_{N+1}}\widehat{U}|$ by $\partial_{y_{N+1}}\widehat{U}$ in (5.12).

Let us now show that $\partial_{y_{N+1}}\widehat{U} \le 1 + \frac{\|W''\|_\infty}{\eta}$. We argue by contradiction by assuming that for some $T > 0$ the supremum of the function $\widehat{U}(\tau, y, y_{N+1}) - \widehat{U}(\tau, y, z_{N+1}) - K|y_{N+1} - z_{N+1}|$ on $[0,T] \times \mathbb{R}^{N+1}$ is strictly positive as soon as $K > 1 + \frac{\|W''\|_\infty}{\eta}$. Then for $\delta, \beta > 0$ small enough, M defined by

$$M = \max_{\substack{(\tau,y)\in[0,T]\times\mathbb{R}^N \\ y_{N+1}, z_{N+1}\in\mathbb{R}}} \left(\widehat{U}(\tau, y, y_{N+1}) - \widehat{U}(\tau, y, z_{N+1}) - K|y_{N+1} - z_{N+1}| - \beta\psi(Y) - \frac{\delta}{T-\tau} \right),$$

where ψ is defined as the function ψ_2 in the proof of Proposition 3.7, is positive. For $j > 0$ let

$$\begin{aligned}
M_j = \max_{\substack{\tau,s\in[0,T], y,z\in\mathbb{R}^N \\ y_{N+1}, z_{N+1}\in\mathbb{R}}} &\Big(\widehat{U}(\tau, y, y_{N+1}) - \widehat{U}(s, z, z_{N+1}) - K|y_{N+1} - z_{N+1}| - \beta\psi(Y) \\
&- \frac{\delta}{T-\tau} - j|\tau - s|^2 - j|y - z|^2 \Big),
\end{aligned}$$

and let $(\tau^j, y^j, y^j_{N+1}, s^j, z^j, z^j_{N+1}) \in ([0,T] \times \mathbb{R}^{N+1})^2$ be a point where M_j is attained. Classical arguments show that $M_j \to M$, $(\tau^j, y^j, y^j_{N+1}, s^j, z^j, z^j_{N+1}) \to (\overline{\tau}, \overline{y}, \overline{y}_{N+1}, \overline{\tau}, \overline{y}, \overline{z}_{N+1})$ as $j \to +\infty$, where $(\overline{\tau}, \overline{y}, \overline{y}_{N+1}, \overline{z}_{N+1})$ is a point where M is attained.

Remark that $0 < \overline{\tau} < T$, moreover, since $\widehat{U}(\overline{\tau}, \overline{y}, \overline{y}_{N+1}) > \widehat{U}(\overline{\tau}, \overline{y}, \overline{z}_{N+1})$ and $\widehat{U}$ is nondecreasing in y_{N+1}, it is

$$\overline{y}_{N+1} > \overline{z}_{N+1}. \tag{5.13}$$

In particular $y^j_{N+1} \neq z^j_{N+1}$ and $0 < s_j, \tau_j < T$ for j large enough. Hence, for $r > 0$, we obtain the following viscosity inequalities

$$\begin{aligned}
&\frac{\delta}{(T-\tau_j)^2} + j(t_j - s_j)\\
&\le L + C_N jr + \beta \mathcal{I}_1^{1,r}[\psi(\cdot, y^j_{N+1}), y^j] + \mathcal{I}_1^{2,r}[\widehat{U}(\tau^j, \cdot, y^j_{N+1}), y^j]\\
&- W'(\widehat{U}(\tau^j, y^j, y^j_{N+1}) + p\cdot y^j) + \sigma(\tau^j, y^j) + \eta[a_0 + \inf_{Y'}(\widehat{U}(\tau_j, Y') - y'_{N+1})\\
&- (\widehat{U}(\tau^j, y^j, y^j_{N+1}) - y^j_{N+1})] \left(K \frac{y^j_{N+1} - z^j_{N+1}}{|y^j_{N+1} - z^j_{N+1}|} + \beta \partial_{y_{N+1}} \psi(y^j, y^j_{N+1}) \right),
\end{aligned} \tag{5.14}$$

and

$$\begin{aligned}
&j(t_j - s_j)\\
&\ge L - C_N jr + \mathcal{I}_1^{2,r}[\widehat{U}(s^j, \cdot, z^j_{N+1}), z^j] - W'(\widehat{U}(s^j, z^j, z^j_{N+1}) + p\cdot z^j) + \sigma(s^j, z^j)\\
&+ \eta[a_0 + \inf_{Y'}(\widehat{U}(s_j, Y') - y'_{N+1}) - (\widehat{U}(s^j, z^j, z^j_{N+1}) - z^j_{N+1})] K \frac{y^j_{N+1} - z^j_{N+1}}{|y^j_{N+1} - z^j_{N+1}|},
\end{aligned} \tag{5.15}$$

where C_N is a constant depending on N. Since $(\tau^j, y^j, y^j_{N+1}, s^j, z^j, z^j_{N+1})$ is a maximum point, we have

$$\begin{aligned}
\widehat{U}(\tau^j, y^j + x, y^j_{N+1}) - \widehat{U}(\tau^j, y^j, y^j_{N+1}) &\le \widehat{U}(s^j, z^j + x, z^j_{N+1}) - \widehat{U}(s^j, z^j, z^j_{N+1})\\
&\quad + \beta[\psi(y^j + x, y^j_{N+1}) - \psi(y^j, y^j_{N+1})]
\end{aligned}$$

for any $x \in \mathbb{R}^N$, which implies that for any $r > 0$

$$\mathcal{I}_1^{2,r}[\widehat{U}(\tau^j, \cdot, y^j_{N+1}), y^j] \le \mathcal{I}_1^{2,r}[\widehat{U}(s^j, \cdot, z^j_{N+1}), z^j] + \beta \mathcal{I}_1^{2,r}[\psi(\cdot, y^j_{N+1}), y^j].$$

Hence, subtracting (5.14) with (5.15), sending $r \to 0^+$ and then $j \to +\infty$, we get

$$\begin{aligned}
\frac{\delta}{(T-\overline{\tau})^2} &\le \beta \mathcal{I}_1[\psi(\cdot, \overline{y}_{N+1}), \overline{y}] + W'(\widehat{U}(\overline{\tau}, \overline{y}, \overline{z}_{N+1}) + p\cdot \overline{y}) - W'(\widehat{U}(\overline{\tau}, \overline{y}, \overline{y}_{N+1}) + p\cdot \overline{y})\\
&\quad - \eta[\widehat{U}(\overline{\tau}, \overline{y}, \overline{y}_{N+1}) - \widehat{U}(\overline{\tau}, \overline{y}, \overline{z}_{N+1}) - (\overline{y}_{N+1} - \overline{z}_{N+1})] K \frac{\overline{y}_{N+1} - \overline{z}_{N+1}}{|\overline{y}_{N+1} - \overline{z}_{N+1}|}\\
&\quad + \beta \partial_{y_{N+1}} \psi(\overline{y}, \overline{y}_{N+1}) \eta[a_0 + \inf_{Y'}(\widehat{U}(\overline{\tau}, Y') - y'_{N+1}) - (\widehat{U}(\overline{\tau}, \overline{y}, \overline{y}_{N+1}) - \overline{y}_{N+1})]\\
&\le \|W''\|_\infty |\widehat{U}(\overline{\tau}, \overline{y}, \overline{y}_{N+1}) - \widehat{U}(\overline{\tau}, \overline{y}, \overline{z}_{N+1})|\\
&\quad - K\eta[\widehat{U}(\overline{\tau}, \overline{y}, \overline{y}_{N+1}) - \widehat{U}(\overline{\tau}, \overline{y}, \overline{z}_{N+1}) - (\overline{y}_{N+1} - \overline{z}_{N+1})] \frac{\overline{y}_{N+1} - \overline{z}_{N+1}}{|\overline{y}_{N+1} - \overline{z}_{N+1}|} + \beta C.
\end{aligned}$$

Then, using (5.13) and that $K|\overline{y}_{N+1} - \overline{z}_{N+1}| < \widehat{U}(\overline{\tau}, \overline{y}, \overline{y}_{N+1}) - \widehat{U}(\overline{\tau}, \overline{y}, \overline{z}_{N+1})$, for β small enough, we finally obtain

$$(\|W''\|_\infty + \eta - \eta K)(\widehat{U}(\overline{\tau}, \overline{y}, \overline{y}_{N+1}) - \widehat{U}(\overline{\tau}, \overline{y}, \overline{z}_{N+1})) \ge 0,$$

which is a contradiction for $K > 1 + \frac{\|W''\|_\infty}{\eta}$. □

5.3. **Ergodicity.**

Proposition 5.4 (Ergodic properties)**.** *There exists a unique $\lambda_\eta = \lambda_\eta(p, L)$ such that the viscosity solution $U_\eta \in C_b(\mathbb{R}^+ \times \mathbb{R}^{N+1})$ of (5.4) satisfies:*

$$|U_\eta(\tau, Y) - \lambda_\eta \tau| \le C_3 \text{ for all } \tau > 0,\ Y \in \mathbb{R}^{N+1}, \tag{5.16}$$

with C_3 independent of η. Moreover

$$L - \|W'\|_\infty - \|\sigma\|_\infty + \eta a_0 \le \lambda_\eta \le L + \|W'\|_\infty + \|\sigma\|_\infty + \eta a_0. \tag{5.17}$$

Proof. For simplicity of notations, in what follows we denote $U = U_\eta$ and $\lambda = \lambda_\eta$.

To prove the proposition we follow the proof of the analogue result in [32]. We proceed in three steps.

Step 1: existence The functions $W^+(\tau, Y) = C^+\tau$ and $W^-(\tau, Y) = C^-\tau$, where

$$C^\pm = L \pm \|W'\|_\infty \pm \|\sigma\|_\infty + \eta a_0,$$

are respectively sub and supersolution of (5.4). Then the existence of a unique solution of (5.4) follows from Perron's method.

Step 2: control of the oscillations w.r.t. space.

We want to prove that there exists $C_1 > 0$ such that

$$|U(\tau, Y) - U(\tau, Z)| \le C_1 \quad \text{for all } \tau \ge 0,\, Y, Z \in \mathbb{R}^{N+1}. \tag{5.18}$$

STEP 2.1. For a given $k \in \mathbb{Z}^{N+1}$, we set $P \cdot k = l + \alpha$, with $l \in \mathbb{Z}$ and $\alpha \in [0, 1)$. The function $\widetilde{U}(\tau, Y) = U(\tau, Y + k) + \alpha$ is still a solution of (5.4), with $\widetilde{U}(0, Y) = \alpha$ Moreover

$$U(0, Y) = 0 \le \widetilde{U}(0, Y) = \alpha \le 1 = U(0, Y) + 1.$$

Then from the comparison principle for (5.4) and invariance by integer translations we deduce for all $\tau \ge 0$:

$$|U(\tau, Y + k) - U(\tau, Y)| \le 1. \tag{5.19}$$

STEP 2.2. We proceed as in [32] by considering the functions

$$M(\tau) := \sup_{Y \in \mathbb{R}^{N+1}} U(\tau, Y), \quad m(\tau) := \inf_{Y \in \mathbb{R}^{N+1}} U(\tau, Y),$$

$$q(\tau) := M(\tau) - m(\tau) = \operatorname{osc} U(\tau, \cdot).$$

Let us assume that the extrema defining these functions are attained: $M(\tau) = U(\tau, Y^\tau)$, $m(\tau) = U(\tau, Z^\tau)$.

It is easy to see that $M(\tau)$ and $m(\tau)$ satisfy in the viscosity sense

$$\partial_\tau M \le L + \mathcal{I}_1^2[U(\tau, \cdot, y^\tau_{N+1}), y^\tau] - W'(M + P \cdot Y^\tau) + \sigma(\tau, y^\tau) + \eta[a_0 + m(\tau) - M(\tau)],$$

$$\partial_\tau m \ge L + \mathcal{I}_1^2[U(\tau, \cdot, z^\tau_{N+1}), z^\tau] - W'(m + P \cdot Z^\tau) + \sigma(\tau, z^\tau) + \eta a_0.$$

Then q satisfies in the viscosity sense

$$\begin{aligned}
\partial_\tau q &\le \mathcal{I}_1^2[U(\tau, \cdot, y^\tau_{N+1}), y^\tau] - \mathcal{I}_1^2[U(\tau, \cdot, z^\tau_{N+1}), z^\tau] - W'(M + P \cdot Y^\tau) \\
&\quad + W'(m + P \cdot Z^\tau) + \sigma(\tau, y^\tau) - \sigma(\tau, z^\tau) \\
&\le \mathcal{I}_1^2[U(\tau, \cdot, y^\tau_{N+1}), y^\tau] - \mathcal{I}_1^2[U(\tau, \cdot, z^\tau_{N+1}), z^\tau] + 2\|W'\|_\infty + 2\|\sigma\|_\infty.
\end{aligned}$$

Let us estimate the quantity $\mathcal{L}(\tau) := \mathcal{I}_1^2[U(\tau, \cdot, y^\tau_{N+1}), y^\tau] - \mathcal{I}_1^2[U(\tau, \cdot, z^\tau_{N+1}), z^\tau]$ from above by a function of q. Let us define $k^\tau \in \mathbb{Z}^{N+1}$ such that $Y^\tau - (Z^\tau + k^\tau) \in [0, 1)^{N+1}$ and let $\widetilde{Z}^\tau := Z^\tau + k^\tau$. Using successively (5.19) and the first inequality in (5.11), we obtain:

$$\begin{aligned}
\mathcal{L}(\tau) &\le \int_{|z|>1} (U(\tau, y^\tau + z, y^\tau_{N+1}) - U(\tau, Y^\tau))\mu(dz) \\
&\quad - \int_{|z|>1} (U(\tau, \widetilde{z}^\tau + z, \widetilde{z}^\tau_{N+1}) - U(\tau, Z^\tau))\mu(dz) + \overline{\mu} \\
&\le \int_{|z|>1} (U(\tau, y^\tau + z, y^\tau_{N+1}) - U(\tau, Y^\tau))\mu(dz) \\
&\quad - \int_{|z|>1} (U(\tau, \widetilde{z}^\tau + z, y^\tau_{N+1}) - U(\tau, Z^\tau))\mu(dz) + 2\overline{\mu},
\end{aligned}$$

where $\overline{\mu} = \|\mu_0\|_{L^1(\mathbb{R}^N\setminus B_1(0))}$. Now, let us introduce $c^\tau = \frac{y^\tau+\widetilde{z}^\tau}{2}$ and $\delta^\tau = \frac{y^\tau-\widetilde{z}^\tau}{2} \in [0,\frac{1}{2})^N$ so that $y^\tau = c^\tau + \delta^\tau$ and $\widetilde{z}^\tau = c^\tau - \delta^\tau$. Hence

$$\begin{aligned}
\mathcal{L}(\tau) &\le 2\overline{\mu} + \int_{|z|>1} (U(\tau, c^\tau + z + \delta^\tau, y^\tau_{N+1}) - U(\tau, Y^\tau))\mu(dz)\\
&\quad - \int_{|z|>1} (U(\tau, c^\tau + z - \delta^\tau, y^\tau_{N+1}) - U(\tau, Z^\tau))\mu(dz)\\
&\le 2\overline{\mu} + \int_{|z-\delta^\tau|>1} (U(\tau, c^\tau + z, y^\tau_{N+1}) - U(\tau, Y^\tau))\mu_0(z-\delta^\tau)dz\\
&\quad - \int_{|z+\delta^\tau|>1} (U(\tau, c^\tau + z, y^\tau_{N+1}) - U(\tau, Z^\tau))\mu_0(z+\delta^\tau)dz\\
&\le 2\overline{\mu} - \int_{\{|z-\delta^\tau|>1\}\cap\{|z+\delta^\tau|>1\}} (U(\tau, Y^\tau) - U(\tau, Z^\tau)) \min\{\mu_0(z-\delta^\tau), \mu_0(z+\delta^\tau)\}dz\\
&\le 2\overline{\mu} - c_0 q(\tau)
\end{aligned}$$

where $c_0 > 0$. We conclude that q satisfies in the viscosity sense

$$\partial_\tau q(\tau) \le 2\|W'\|_\infty + 2\|\sigma\|_\infty + 2\overline{\mu} - c_0 q(\tau),$$

with $q(0) = 0$, from which we obtain (5.18).

If the extrema are not attained, it suffices to consider for $\beta > 0$, $M_\beta(\tau) := \sup_{Y\in\mathbb{R}^{N+1}}(U(\tau,Y) - \beta\psi(Y))$, $m_\beta(\tau) := \inf_{Y\in\mathbb{R}^{N+1}}(U(\tau,Y) + \beta\psi(Y))$, and $q_\beta(\tau) := M_\beta(\tau) - m_\beta(\tau)$, where ψ is defined as the function ψ_2 in the proof of Proposition 3.7. By the properties of ψ, $M_\beta(\tau)$ and $m_\beta(\tau)$ are attained. Then, the previous argument shows that

$$q_\beta \le C_1 + C\beta,$$

and passing to the limit as $\beta \to 0^+$ we get (5.18).

Step 3: control of the oscillations in time. We follow [32] by introducing the two quantities:

$$\lambda^+(T) := \sup_{\tau\ge 0} \frac{U(\tau+T,0) - U(\tau,0)}{T} \quad \text{and} \quad \lambda^-(T) := \inf_{\tau\ge 0} \frac{U(\tau+T,0) - U(\tau,0)}{T},$$

and proving that they have a common limit as $T \to +\infty$. First let us estimate $\lambda^+(T)$ from above. The function $U^+(t,Y) := U(\tau,0) + C_1 + C^+ t$, is a supersolution of (5.4) if $C^+ = L + \|W'\|_\infty + \|\sigma\|_\infty + \eta a_0$. Since $U^+(0,Y) \ge U(\tau,Y)$ if C_1 is as in (5.18), by the comparison principle for (5.4) in the time interval $[\tau, \tau+\tau_0]$, for any $\tau_0 > 0$ and $t \in [0,\tau_0]$ we get

$$U(\tau+t, Y) \le U(\tau,0) + C_1 + C^+ t. \tag{5.20}$$

Similarly

$$U(\tau+t, Y) \ge U(\tau,0) - C_1 + C^- t, \tag{5.21}$$

where $C^- = L - \|W'\|_\infty - \|\sigma\|_\infty + \eta a_0$. We then obtain for $\tau_0 = t = T$ and $y = 0$:

$$L - \|W'\|_\infty - \|\sigma\|_\infty + \eta a_0 - \frac{C_1}{T} \le \lambda^-(T) \le \lambda^+(T) \le L + \|W'\|_\infty + \|\sigma\|_\infty + \eta a_0 + \frac{C_1}{T}. \tag{5.22}$$

By definition of $\lambda^\pm(T)$, for any $\delta > 0$, there exist $\tau^\pm \ge 0$ such that

$$\left|\lambda^\pm(T) - \frac{U(\tau^\pm + T, 0) - U(\tau^\pm, 0)}{T}\right| \le \delta.$$

Let us consider $\alpha, \beta \in [0,1)$ such that $\tau^+ - \tau^- - \beta = k \in \mathbb{Z}$, and $U(\tau^+,0) - U(\tau^+ - k, 0) + \alpha \in \mathbb{Z}$. From (5.18) we have

$$\begin{aligned}
U(\tau^+, Y) &\le U(\tau^+, 0) + C_1 \le U(\tau^+ - k, Y) + 2C_1 + (U(\tau^+,0) - U(\tau^+ - k, 0))\\
&\le U(\tau^+ - k, Y) + 2\lceil C_1\rceil + (U(\tau^+,0) - U(\tau^+ - k, 0) + \alpha).
\end{aligned}$$

Since $\sigma(\cdot, y)$ and $W'(\cdot)$ are $\mathbb{Z}$-periodic, the comparison principle for (5.4) on the time interval $[\tau^+, \tau^+ + T]$ implies that:

$$U(\tau^+ + T, Y) \le U(\tau^+ - k + T, Y) + 2\lceil C_1\rceil + U(\tau^+,0) - U(\tau^+ - k, 0) + 1.$$

Choosing $Y=0$ in the previous inequality we get

$$\begin{aligned} U(\tau^++T,0)-U(\tau^+,0) &\le U(\tau^+-k+T,0)-U(\tau^+-k,0)+2\lceil C_1\rceil+1\\ &= U(\tau^-+\beta+T,0)-U(\tau^-+\beta,0)+2\lceil C_1\rceil+1, \end{aligned}$$

and setting $t=\beta$ and $\tau=\tau^-+T$ in (5.20) and $\tau=\tau^-$ in (5.21) we finally obtain:

$$T\lambda^+(T)\le T\lambda^-(T)+4\lceil C_1\rceil+1+2\|W'\|_\infty+2\|\sigma\|_\infty+2\delta T.$$

Since this is true for any $\delta>0$, we conclude that:

$$|\lambda^+(T)-\lambda^-(T)|\le \frac{4\lceil C_1\rceil+1+2\|W'\|_\infty+2\|\sigma\|_\infty}{T}.$$

Now arguing as in [31] and [32], we conclude that there exist $\lim_{T\to+\infty}\lambda^\pm(T)=:\lambda$ and

$$|\lambda^\pm(T)-\lambda|\le \frac{4\lceil C_1\rceil+1+2\|W'\|_\infty+2\|\sigma\|_\infty}{T},$$

which implies that

$$|U(T,0)-\lambda T|\le 4\lceil C_1\rceil+1+2\|W'\|_\infty+2\|\sigma\|_\infty,$$

and then, using (5.18) we get (5.16). The uniqueness of λ follows from (5.16). Finally, (5.17) is obtained from (5.22) as $T\to+\infty$. □

5.4. **Proof of Theorem 1.1.** Let us consider the viscosity solution of (5.4) for $\eta=0$. By Proposition 5.4 we know that there exists a unique λ such that $U(\tau,Y)/\tau$ converges to λ as τ goes to $+\infty$ for any $Y\in\mathbb{R}^{N+1}$. Moreover, by Proposition 3.6, $U(\tau,y,0)$ is viscosity solution of (1.6). Hence, the theorem follows immediately from the uniqueness of the viscosity solution of (1.6).

5.5. **Proof of Proposition 5.1.** Let us denote by U_η^+ the solution of (5.4) with $a_0=C_1$, where C_1 is defined as in (5.18), and by U_η^- the solution of (5.4) with $a_0=0$. Let $\lambda_\eta^+=\lim_{\tau\to+\infty}\frac{U_\eta^+(\tau,Y)}{\tau}$ and $\lambda_\eta^-=\lim_{\tau\to+\infty}\frac{U_\eta^-(\tau,Y)}{\tau}$; the existence of λ_η^+ and λ_η^- is guaranteed by Proposition 5.4. By stability (see e.g. [8]), for $\eta\to0^+$ the sequence $(U_\eta^+)_\eta$ converges to U solution of (5.4) with $\eta=0$. Moreover by (5.17) the sequence $(\lambda_\eta^+)_\eta$ is bounded. Take a subsequence $\eta_n\to0$ as $n\to+\infty$ such that $\lambda_{\eta_n}^+\to\lambda_\infty$ as $n\to+\infty$. We want to show that $\lambda_\infty=\lambda$, where $\lambda=\lim_{\tau\to+\infty}\frac{U(\tau,Y)}{\tau}$. By the proof of Theorem 1.1, we know that λ is the same quantity defined in Theorem 1.1. Using (5.16), we get

$$\begin{aligned} |\lambda-\lambda_\infty| &\le \left|\lambda-\frac{U(\tau,0)}{\tau}\right|+\left|\frac{U(\tau,0)}{\tau}-\frac{U_{\eta_n}^+(\tau,0)}{\tau}\right|+\left|\frac{U_{\eta_n}^+(\tau,0)}{\tau}-\lambda_{\eta_n}^+\right|+|\lambda_{\eta_n}^+-\lambda_\infty|\\ &\le \left|\lambda-\frac{U(\tau,0)}{\tau}\right|+\left|\frac{U(\tau,0)}{\tau}-\frac{U_{\eta_n}^+(\tau,0)}{\tau}\right|+\frac{C_3}{\tau}+|\lambda_{\eta_n}^+-\lambda_\infty| \end{aligned}$$

where C_3 does not depend on n. Then, passing to the limit first as $n\to+\infty$ and then as $\tau\to+\infty$, we obtain that $\lambda=\lambda_\infty$. This implies that $\lambda_\eta^+\to\lambda$ as $\eta\to0$.

The same argument shows that $\lambda_\eta^-\to\lambda$ as $\eta\to0$.

Now, we set

$$W_\eta^+(\tau,Y):=U_\eta^+(\tau,Y)-\lambda_\eta^+\tau$$

and

$$W_\eta^-(\tau,Y):=U_\eta^-(\tau,Y)-\lambda_\eta^-\tau.$$

Then, W_η^+ and W_η^- are respectively the desired super and subsolution.

Indeed, since by (5.18), $C_0+\inf_{Y'}U_\eta^+(\tau,Y')-U_\eta^+(\tau,Y)\ge0$, W_η^+ is supersolution of (4.4) with $\lambda=\lambda_\eta^+$. Moreover, by (5.16), W_η^+ is bounded on $\mathbb{R}^+\times\mathbb{R}^{N+1}$ uniformly w.r.t. η: $|W_\eta^+(\tau,Y)|\le C_3$ for all $(\tau,Y)\in\mathbb{R}^+\times\mathbb{R}^{N+1}$.

By (5.11), W_η^+ is Lipschitz continuous w.r.t. y_{N+1} and $-1\le\partial_{y_{N+1}}W_\eta^+\le\frac{\|W''\|_\infty}{\eta}$. This implies that W_η^+ is also a viscosity subsolution of

$$\text{(5.23)}\qquad \begin{cases} \lambda_\eta^++\partial_\tau V=L+\mathcal{I}_1[V(\tau,\cdot,y_{N+1})]-W'(V+\lambda_\eta^+\tau+P\cdot Y)+\sigma(\tau,y) & \\ \qquad +C_1(\|W''\|_\infty+\eta) & \text{in } \mathbb{R}^+\times\mathbb{R}^{N+1}\\ V(0,Y)=0 & \text{on } \mathbb{R}^{N+1}. \end{cases}$$

By Proposition 3.6, W_η^+ is supersolution of (4.4) and subsolution of (5.23) in $\mathbb{R}^+ \times \mathbb{R}^N$ for any $y_{N+1} \in \mathbb{R}$. Then by Proposition 3.7, W_η^+ is of class C^α w.r.t. y uniformly in y_{N+1} and η, for any $0 < \alpha < 1$.

Similar arguments show that W_η^- is subsolution of (4.4) with $\lambda = \lambda_\eta^-$, is bounded on $\mathbb{R}^+ \times \mathbb{R}^{N+1}$, Lipschitz continuous w.r.t. y_{N+1} with $-1 \leq \partial_{y_{N+1}} W_\eta^+ \leq \frac{\|W''\|_\infty}{\eta}$ and Hölder continuous w.r.t. y. This concludes the proof of Proposition 5.1.

5.6. **Proof of Proposition 4.3.** The continuity of $\overline{H}(p, L)$ follows from stability of viscosity solutions of (1.6) (see e.g. [8]) and from (5.16). Indeed, let (p_n, L_n) be a sequence converging to (p_0, L_0) as $n \to +\infty$ and set $\lambda_n = \lambda(p_n, L_n)$, $n \geq 0$. By (5.16), we have for any $\tau > 0$

$$\left| \lambda_n - \frac{w_n(\tau, y)}{\tau} \right| \leq \frac{C_3}{\tau}.$$

Stability of viscosity solutions of (1.6) implies that w_n converges locally uniformly in (τ, y) to a function w_0 which is a solution of (1.6) with $(p, L) = (p_0, L_0)$. This implies that $\limsup_{n\to+\infty} |\lambda_n - \lambda_0| \leq \frac{2C_3}{\tau}$ for any $\tau > 0$. Hence, we conclude that $\lim_{n\to+\infty} \lambda_n = \lambda_0$.

Property (i) is an immediate consequence of (5.17).

The monotonicity in L of $\overline{H}(p, L)$ comes from the comparison principle.

Let us show (iii). Let v be the solution of (1.5) and $\lambda = \lambda(p, L)$. Set $\widetilde{v}(\tau, y) := v(\tau, -y)$. Remark that $\mathcal{I}_1[\widetilde{v}(\tau, \cdot), y] = \mathcal{I}_1[v(\tau, \cdot), -y]$. If $\sigma(\tau, \cdot)$ is even then $\widetilde{v}$ satisfies

$$\begin{cases} \lambda + \partial_\tau \widetilde{v} = \mathcal{I}_1[\widetilde{v}(\tau, \cdot), y] + L - W'(\widetilde{v} + \lambda t - p \cdot y) + \sigma(\tau, y) & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N \\ \widetilde{v}(0, y) = 0 & \text{on} \quad \mathbb{R}^N. \end{cases}$$

By the uniqueness of λ we deduce that $\lambda(L, p) = \lambda(L, -p)$, i.e. (iii).

Finally let us turn to (iv). Define $\widetilde{v}(\tau, y) := -v(\tau, -y)$. If $W'(\cdot)$ and $\sigma(\tau, \cdot)$ are odd functions, $\widetilde{v}$ satisfies

$$\begin{cases} -\lambda + \partial_\tau \widetilde{v} = \mathcal{I}_1[\widetilde{v}(\tau, \cdot), y] - L - W'(\widetilde{v} - \lambda t + p \cdot y) + \sigma(\tau, y) & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N \\ \widetilde{v}(0, y) = 0 & \text{on} \quad \mathbb{R}^N. \end{cases}$$

As before, we conclude that $\lambda(-L, p) = -\lambda(L, p)$, i.e. (iv).

6. Smooth approximate correctors

In this section, we prove the existence of approximate correctors that are smooth w.r.t. y_{N+1}, namely Proposition 4.4. We first need the following lemma:

Lemma 6.1. *Let $u_1, u_2 \in C_b(\mathbb{R}^+ \times \mathbb{R}^N)$ be viscosity subsolutions (resp. supersolutions) of* (4.4) *in $\mathbb{R}^+ \times \mathbb{R}^N$, then $u_1 + u_2$ is viscosity subsolution (resp., supersolution) of*

$$\begin{cases} 2\lambda + \partial_\tau v = 2L + \mathcal{I}_1[v] - W'(u_1 + P \cdot Y + \lambda\tau) & \\ \qquad\qquad -W'(u_2 + P \cdot Y + \lambda\tau) + 2\sigma(\tau, y) & \textit{in } \mathbb{R}^+ \times \mathbb{R}^N \\ v(0, y) = 0 & \textit{on } \mathbb{R}^N. \end{cases}$$

For the proof see Lemma 5.8 in [11].

Next, let us consider a positive smooth function $\rho : \mathbb{R} \to \mathbb{R}$, with support in $B_1(0)$ and mass 1. We define a sequence of mollifiers $(\rho_\delta)_\delta$ by $\rho_\delta(s) = \frac{1}{\delta}\rho\left(\frac{s}{\delta}\right)$, $s \in \mathbb{R}$. Let W_η^+ (resp. W_η^-) be the Lipschitz supersolution (resp. subsolution) of (4.4) with $\lambda = \lambda_\eta^+$ (resp. $\lambda = \lambda_\eta^-$), whose existence is guaranteed by Proposition 5.1. We define

$$V_{\eta,\delta}^\pm(t, y, y_{N+1}) := W_\eta^\pm(t, y, \cdot) \star \rho_\delta(\cdot) = \int_{\mathbb{R}} W_\eta^\pm(t, y, z)\rho_\delta(y_{N+1} - z)dz. \tag{6.1}$$

Lemma 6.2. *The functions $V_{\eta,\delta}^+$ and $V_{\eta,\delta}^-$ are respectively super and subsolution of*

$$\begin{cases} \lambda_\eta^\pm + \partial_\tau V_{\eta,\delta}^\pm = L + \mathcal{I}_1[V_{\eta,\delta}^\pm(\tau, \cdot, y_{N+1})] + \sigma(\tau, y) & \\ \qquad\qquad - \int_{\mathbb{R}} W'(W_\eta^\pm(\tau, y, z) + p \cdot y + z + \lambda_\eta^\pm \tau)\rho_\delta(y_{N+1} - z)dz & \textit{in } \mathbb{R}^+ \times \mathbb{R}^{N+1} \\ V_\eta^\pm(0, Y) = 0 & \textit{on } \mathbb{R}^{N+1}. \end{cases} \tag{6.2}$$

Proof. We prove the lemma for supersolutions. Let $Q_h^e = e + [-h/2, h/2)$, $\overline{\rho}_\delta(e,h) = \int_{Q_h^e} \rho_\delta(y)dy$ and

$$I_h(\tau, y, y_{N+1}) = \sum_{e \in h\mathbb{Z}} W_\eta^+(\tau, y, y_{N+1} - e)\overline{\rho}_\delta(e,h).$$

The function I_h is a discretization of the convolution integral and by classical results, converges uniformly to $V_{\eta,\delta}^+$ as $h \to 0$. By Proposition 3.6, W_η^+ is a viscosity supersolution of (4.4) also in $\mathbb{R}^+ \times \mathbb{R}^N$. Then, by Lemma 6.1, for any $y_{N+1} \in \mathbb{R}$, $I_h(\tau, y, y_{N+1})$ is a supersolution of

$$\begin{cases} \lambda_\eta^+ + \partial_\tau V = L + \mathcal{I}_1[V(\tau, \cdot, y_{N+1})] + \sigma(\tau, y) \sum_{e \in h\mathbb{Z}} \overline{\rho}_\delta(e,h) & \\ \qquad - \sum_{e \in h\mathbb{Z}} W'(W_\eta^+(\tau, y, y_{N+1} - e) & \\ \qquad + p \cdot y + (y_{N+1} - e) + \lambda_\eta^+ \tau)\overline{\rho}_\delta(e,h) & \text{in } \mathbb{R}^+ \times \mathbb{R}^N \\ V(0,y) = 0 & \text{on } \mathbb{R}^N. \end{cases}$$

Using the stability result for viscosity solution of non-local equations, see [8], we conclude that $V_{\eta,\delta}^+$ is supersolution of (6.2) in $\mathbb{R}^+ \times \mathbb{R}^N$ and hence also in $\mathbb{R}^+ \times \mathbb{R}^{N+1}$. □

6.1. **Proof of Proposition 4.4.** We first show that the functions $V_{\eta,\delta}^+$ and $V_{\eta,\delta}^-$, defined in (6.1), are respectively super and subsolution of

$$\begin{cases} \lambda_\eta^\pm + \partial_\tau V_{\eta,\delta}^\pm = L + \mathcal{I}_1[V_{\eta,\delta}^\pm(\tau, \cdot, y_{N+1})] - W'(V_{\eta,\delta}^\pm + P \cdot Y + \lambda_\eta^\pm \tau) & \\ \qquad + \sigma(\tau, y) \mp C_{\eta,\delta} & \text{in } \mathbb{R}^+ \times \mathbb{R}^{N+1} \\ V_\eta^\pm(0, Y) = 0 & \text{on } \mathbb{R}^{N+1}, \end{cases} \tag{6.3}$$

where $C_{\eta,\delta} = \|W''\|_\infty (2\delta \|W''\|_\infty/\eta + \delta)$. Using (5.2) and the properties of the mollifiers, we get

$$\begin{aligned}
&\Big| W'(V_{\eta,\delta}^\pm(\tau, y, y_{N+1}) + p \cdot y + y_{N+1} + \lambda_\eta^\pm \tau) \\
&- \int_{\mathbb{R}} W'(W_\eta^\pm(\tau, y, z) + p \cdot y + z + \lambda_\eta^\pm \tau)\rho_\delta(y_{N+1} - z)dz \Big| \\
&\le \int_{\mathbb{R}} \Big| W'(V_{\eta,\delta}^\pm(\tau, y, y_{N+1}) + p \cdot y + y_{N+1} + \lambda_\eta^\pm \tau) \\
&- W'(W_\eta^\pm(\tau, y, z) + p \cdot y + z + \lambda_\eta^\pm \tau) \Big| \rho_\delta(y_{N+1} - z)dz \\
&\le \|W''\|_\infty \int_{\mathbb{R}} \left[\left| V_{\eta,\delta}^\pm(\tau, y, y_{N+1}) - W_\eta^\pm(\tau, y, z) \right| + |y_{N+1} - z| \right] \rho_\delta(y_{N+1} - z)dz \\
&\le \|W''\|_\infty \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \left| W_\eta^\pm(\tau, y, r) - W_\eta^\pm(\tau, y, z) \right| \rho_\delta(y_{N+1} - r)dr + |y_{N+1} - z| \right] \rho_\delta(y_{N+1} - z)dz \\
&\le \|W''\|_\infty \int_{\mathbb{R}} \left[\int_{|y_{N+1} - r| \le \delta} \frac{\|W''\|_\infty}{\eta} |r - z| \rho_\delta(y_{N+1} - r)dr + |y_{N+1} - z| \right] \rho_\delta(y_{N+1} - z)dz \\
&\le \|W''\|_\infty \int_{|y_{N+1} - z| \le \delta} \left[\frac{\|W''\|_\infty}{\eta} (|y_{N+1} - z| + \delta) + |y_{N+1} - z| \right] \rho_\delta(y_{N+1} - z)dz \\
&\le \|W''\|_\infty \left(2\delta \frac{\|W''\|_\infty}{\eta} + \delta \right)
\end{aligned}$$

From this estimate and Lemma 6.2, we deduce that $V_{\eta,\delta}^+$ and $V_{\eta,\delta}^-$ are respectively super and subsolution of (6.3). Now, we choose $\delta = \delta(\eta)$ such that $\|W''\|_\infty (2\delta \|W''\|_\infty/\eta + \delta) = o_\eta(1)$ as $\eta \to 0$ and define

$$V_\eta^\pm(\tau, Y) := V_{\eta,\delta(\eta)}^\pm(\tau, Y).$$

Then the functions $V_\eta^\pm$ are the desired super and subcorrectors. Indeed, we have already shown that they are super and subsolution of (4.5) with λ_η^+ and λ_η^- satisfying (4.6). Properties (i) and (ii) of Proposition 4.3 can be shown as in the proof of the proposition. Finally, (4.7), (4.8) and (4.9) easily follow from (5.1), (5.2), (5.3) and the properties of the mollifiers. □

7. The Orowan's law

In this section we want to prove (1.9) in Theorem 1.3. In order to prove it, let us introduce the so called *hull function*. For $p \neq 0$ and $L \in \mathbb{R}$, let w be the solution of (1.6) without assuming the initial condition and let $u(\tau, y) = w(\tau, y) + py$. Let $h(x)$ be such that $u(\tau, y) = h(\lambda\tau + py)$. We see that h is formally a solution of

$$\lambda h' = |p|\mathcal{I}_1[h] - W'(h) + L \quad \text{in } \mathbb{R}. \tag{7.1}$$

Moreover, we also expect that h satisfies

$$|h(x) - x| \leq C_3 \quad \text{for any } x \in \mathbb{R}. \tag{7.2}$$

As we have already pointed out in the introduction, we will not prove the existence of the hull function, we will just give an ansatz of it for small values of p and L. Precisely, let us fix $p_0 \in \mathbb{R} \setminus \{0\}$, $L_0 \in \mathbb{R}$ and let $p = \delta p_0$ and $L = \delta L_0$, where δ is a small parameter. The main idea to prove (1.9) is to approximate h, for such p and L, by the following ansatz

$$h(x) \simeq \frac{L_0 \delta}{\alpha} + \sum_{i=-\infty}^{+\infty} \left[\phi\left(\frac{x-i}{\delta|p_0|}\right) - \frac{1}{2}\right] + \delta \sum_{i=-\infty}^{+\infty} \psi\left(\frac{x-i}{\delta|p_0|}\right), \tag{7.3}$$

where $\alpha = W''(0) > 0$ (defined in (1.7)) and the functions ϕ and ψ are respectively the solutions of the following problems

$$\begin{cases} \mathcal{I}_1[\phi] = W'(\phi) & \text{in} \quad \mathbb{R} \\ \lim_{x \to -\infty} \phi(x) = 0, \quad \lim_{x \to +\infty} \phi(x) = 1, \quad \phi(0) = \frac{1}{2} & \\ \phi' > 0 & \text{in} \quad \mathbb{R}, \end{cases} \tag{7.4}$$

and

$$\begin{cases} \mathcal{I}_1[\psi] = W''(\phi)\psi + \frac{L_0}{W''(0)}(W''(\phi) - W''(0)) + c\phi' \quad \text{in} \quad \mathbb{R} \\ \lim_{x \to \pm\infty} \psi(x) = 0 \\ c = \frac{L_0}{\int_{\mathbb{R}} (\phi')^2}. \end{cases} \tag{7.5}$$

Here and in what follows, $\mathcal{I}_1$ denotes the half-Laplacian in dimension 1, i.e., $\mu(dy) = dy/(\pi|y|^2)$. On the function W, we assume (1.7). Then there exists a unique solution of (7.4) which is of class $C^{2,\beta}$, as shown by Cabré and Solà-Morales in [9]. Under (1.7), the existence of a solution of class $C^{1,\beta}$ of the problem (7.5) is proved by Gonzáles and Monneau in [22]. Actually, the regularity of W implies, that $\phi \in C^{4,\beta}(\mathbb{R})$ and $\psi \in C^{3,\beta}(\mathbb{R})$, see Lemma 2.3 in [9].

We will show that the ansatz defined in (7.3) satisfies, up to small errors, the equation (7.1) with $\lambda = c_0|\delta p_0|\delta L_0$, where $c_0 = (\int_{\mathbb{R}} (\phi')^2)^{-1}$, and has the ergodic property (7.2). This implies, by comparison, that $\overline{H}(\delta p_0, \delta L_0) \sim c_0|\delta p_0|\delta L_0$ as $\delta \to 0^+$, and then will show Theorem 1.3.

7.1. **Proof of Theorem 1.3.**

Suppose $p_0 \neq 0$. For $L \in \mathbb{R}$, $\delta > 0$ and $n \in \mathbb{N}$ we define the sequence $\{s^L_{\delta,n}(x)\}_n$ by

$$s^L_{\delta,n}(x) = \frac{L\delta}{\alpha} - n + \sum_{i=-n}^{n} \phi\left(\frac{x-i}{\delta|p_0|}\right) + \delta \sum_{i=-n}^{n} \psi\left(\frac{x-i}{\delta|p_0|}\right)$$

where ϕ is a solution of (7.4) and ψ is a solution of (7.5) with L_0 replaced by L. We consider the differential operator $NL_L^{\overline{\lambda}^L_\delta}$, defined on smooth functions as follows

$$NL_L^{\overline{\lambda}^L_\delta}[h] = \overline{\lambda}^L_\delta h' - \delta|p_0|\mathcal{I}_1[h] + W'(h) - \delta L,$$

where

$$\overline{\lambda}^L_\delta = \delta^2 c_0 |p_0| L.$$

Then we have

Proposition 7.1. *Assume* (1.7). *For any $x \in \mathbb{R}$ there exists the finite limit*

$$h_\delta^L(x) = \lim_{n\to+\infty} s_{\delta,n}^L(x).$$

Moreover h_δ^L has the following properties:

(i) $h_\delta^L \in C^2(\mathbb{R})$ *and satisfies*

$$NL_L^{\overline{\lambda}_\delta^L}[h_\delta^L](x) = o(\delta),$$

where $\lim_{\delta\to 0}\frac{o(\delta)}{\delta} = 0$, uniformly for $x \in \mathbb{R}$ and locally uniformly in $L \in \mathbb{R}$;

(ii) *There exists a constant $C > 0$ such that $|h_\delta^L(x) - x| \le C$ for any $x \in \mathbb{R}$.*

The proof of Proposition 7.1 is postponed.

Remark 7.2. From (ii) of Proposition 7.1, we see that the function $h_\delta^L(x)$ goes to infinity like the power x. Then the integral $\mathcal{I}_1^{2,r}[h_\delta^L, x] = \int_{|y|>r}(h_\delta^L(x+y) - h_\delta^L(x))\mu(dy)$, $r > 0$, is not well defined in the sense of the Lesbegue integration. By $\mathcal{I}_1^{2,r}[h_\delta^L, x]$, we mean

$$\mathcal{I}_1^{2,r}[h_\delta^L, x] = \lim_{a\to+\infty}\int_{r<|y|<a}(h_\delta^L(x+y) - h_\delta^L(x))\mu(dy),$$

and this allows us to define $NL_L^{\overline{\lambda}_\delta^L}[h_\delta^L]$ in Proposition 7.1. This definition coincides with the standard Lesbegue integral for integrable functions. In what follows we will consider the function

$$\widetilde{w}(\tau, y) = h_\delta^L(\delta p_0 y + \overline{\lambda}_\delta^L\tau) - \delta p_0 y \tag{7.6}$$

which belongs to $C_b(\mathbb{R}^+ \times \mathbb{R})$ and then for which $\mathcal{I}_1^2$ is well defined and

$$\mathcal{I}_1^2[\widetilde{w}(\tau,\cdot), y] = \lim_{a\to+\infty}\int_{1<|z|<a}(\overline{w}(\tau, y+z) - \overline{w}(\tau, y))\mu(dz) = \delta|p_0|\mathcal{I}_1^{2,r}[h_\delta^L, \delta p_0 y + \overline{\lambda}_\delta^L\tau],$$

with $r = \delta|p_0|$.

Fix $\eta > 0$ and let $L = L_0 - \eta$. By (i) of Proposition 7.1, there exists $\delta_0 = \delta_0(\eta) > 0$ such that for any $\delta \in (0, \delta_0)$ we have

$$NL_{L_0}^{\overline{\lambda}_\delta^L}[h_\delta^L] = NL_L^{\overline{\lambda}_\delta^L}[h_\delta^L] - \delta\eta < 0 \quad \text{in } \mathbb{R}. \tag{7.7}$$

Let us consider the function $\widetilde{w}(\tau, y)$, defined by (7.6). By (ii) of Proposition 7.1

$$|\widetilde{w}(\tau, y) - \overline{\lambda}_\delta^L\tau| \le \lceil C\rceil, \tag{7.8}$$

then, $\widetilde{w} \in C_b(\mathbb{R}^+ \times \mathbb{R})$. Moreover, by (7.7) and (7.8) $\widetilde{w}$ satisfies

$$\begin{cases} \widetilde{w}_\tau \le \mathcal{I}_1[\widetilde{w}] - W'(\overline{w} + \delta p_0 y) + \delta L_0 & \text{in } \mathbb{R}^+ \times \mathbb{R} \\ \widetilde{w}(0, y) \le \lceil C\rceil & \text{on } \mathbb{R}. \end{cases}$$

Let $w(\tau, y)$ be the solution of (1.6) with $N = 1$, $p = \delta p_0$, $L = \delta L_0$ and $\sigma \equiv 0$, whose existence is ensured by Theorem 1.1, then from the comparison principle and the periodicity of W, we deduce that

$$\widetilde{w}(\tau, y) \le w(\tau, y) + \lceil C\rceil.$$

By the previous inequality and (7.8), we get

$$\overline{\lambda}_\delta^L\tau \le w(\tau, y) + 2\lceil C\rceil,$$

and dividing by τ and letting τ go to $+\infty$, we finally obtain

$$\delta^2 c_0|p_0|(L_0 - \eta) = \overline{\lambda}_\delta^L \le \overline{H}(\delta p_0, \delta L_0).$$

Similarly, it is possible to show that

$$\overline{H}(\delta p_0, \delta L_0) \le \delta^2 c_0|p_0|(L_0 + \eta).$$

We have proved that for any $\eta > 0$ there exists $\delta_0 = \delta_0(\eta) > 0$ such that for any $\delta \in (0, \delta_0)$ we have

$$\left|\frac{\overline{H}(\delta p_0, \delta L_0)}{\delta^2} - c_0|p_0|L_0\right| \le c_0|p_0|\eta,$$

i.e. (1.9), as desired. $\square$

7.2. **Preliminary results.**

To prove Proposition 7.1 we need several preliminary results. We first state the following two lemmas about the behavior of the functions ϕ and ψ at infinity. We denote by $H(x)$ the Heaviside function defined by

$$H(x) = \begin{cases} 1 & \text{for } x \geq 0 \\ 0 & \text{for } x < 0. \end{cases}$$

Then we have

Lemma 7.3 (Behavior of ϕ)**.** *Assume* (1.7)*. Let ϕ be the solution of* (7.4)*, then there exist constants $K_0, K_1 > 0$ such that*

$$\left|\phi(x) - H(x) + \frac{1}{\alpha\pi x}\right| \leq \frac{K_1}{x^2}, \quad \text{for } |x| \geq 1, \tag{7.9}$$

and for any $x \in \mathbb{R}$

$$0 < \frac{K_0}{1+x^2} \leq \phi'(x) \leq \frac{K_1}{1+x^2}, \tag{7.10}$$

$$-\frac{K_1}{1+x^2} \leq \phi''(x) \leq \frac{K_1}{1+x^2}, \tag{7.11}$$

$$-\frac{K_1}{1+x^2} \leq \phi'''(x) \leq \frac{K_1}{1+x^2}. \tag{7.12}$$

Lemma 7.4 (Behavior of ψ)**.** *Assume* (1.7)*. Let ψ be the solution of* (7.5)*, then for any $L \in \mathbb{R}$ there exist constants K_2 and K_3, with $K_3 > 0$, depending on L such that*

$$\left|\psi(x) - \frac{K_2}{x}\right| \leq \frac{K_3}{x^2}, \quad \text{for } |x| \geq 1, \tag{7.13}$$

and for any $x \in \mathbb{R}$

$$-\frac{K_3}{1+x^2} \leq \psi'(x) \leq \frac{K_3}{1+x^2}, \tag{7.14}$$

$$-\frac{K_3}{1+x^2} \leq \psi''(x) \leq \frac{K_3}{1+x^2}. \tag{7.15}$$

We postpone the proof of the two lemmas in the appendix (Section 8).

For simplicity of notation we denote (for the rest of the paper)

$$x_i = \frac{x-i}{\delta|p_0|}, \quad \widetilde{\phi}(z) = \phi(z) - H(z).$$

Then we have the following five claims (whose proofs are also postponed in the appendix (Section 8)).

Claim 1: *Let $x = i_0 + \gamma$, with $i_0 \in \mathbb{Z}$ and $\gamma \in \left(-\frac{1}{2}, \frac{1}{2}\right]$, then*

$$\sum_{\substack{i=-n \\ i \neq i_0}}^{n} \frac{1}{x-i} \to -2\gamma \sum_{i=1}^{+\infty} \frac{1}{i^2 - \gamma^2} \quad \text{as } n \to +\infty,$$

$$\sum_{i=-n}^{i_0-1} \frac{1}{(x-i)^2} \to \sum_{i=1}^{+\infty} \frac{1}{(i+\gamma)^2} \quad \text{as } n \to +\infty,$$

$$\sum_{i=i_0+1}^{n} \frac{1}{(x-i)^2} \to \sum_{i=1}^{+\infty} \frac{1}{(i-\gamma)^2} \quad \text{as } n \to +\infty.$$

Claim 2: *For any $x \in \mathbb{R}$ the sequence $\{s^L_{\delta,n}(x)\}_n$ converges as $n \to +\infty$.*

Claim 3: *The sequence $\{(s^L_{\delta,n})'\}_n$ converges on $\mathbb{R}$ as $n \to +\infty$, uniformly on compact sets.*

Claim 4: *The sequence* $\{(s^L_{\delta,n})''\}_n$ *converges on* $\mathbb{R}$ *as* $n \to +\infty$*, uniformly on compact sets.*

Claim 5: *For any* $x \in \mathbb{R}$ *the sequences* $\sum_{i=-n}^{n} \mathcal{I}_1[\phi, x_i]$ *and* $\sum_{i=-n}^{n} \mathcal{I}_1[\psi, x_i]$ *converge as* $n \to +\infty$*.*

In order to do the proof of Proposition 7.1, we finally state and prove the following result:

Lemma 7.5. $-C\delta^2 \le \lim_{n\to+\infty} NL_L^{\overline{\lambda}_\delta^L}[s^L_{\delta,n}](x) \le C\delta^2$*, where* C *is independent of* x*.*

Proof of Lemma 7.5.
Fix $x \in \mathbb{R}$, let $i_0 \in \mathbb{Z}$ and $\gamma \in \left(-\frac{1}{2}, \frac{1}{2}\right]$ be such that $x = i_0 + \gamma$, let $\frac{1}{\delta|p_0|} \ge 2$ and $n > |i_0|$. Then we have

$$
\begin{aligned}
& NL_L^{\overline{\lambda}_\delta^L}[s^L_{\delta,n}](x) \\
= \; & \frac{\overline{\lambda}_\delta^L}{\delta|p_0|} \sum_{i=-n}^{n} [\phi'(x_i) + \delta\psi'(x_i)] - \sum_{i=-n}^{n} [\mathcal{I}_1[\phi, x_i] + \delta \mathcal{I}_1[\psi, x_i]] \\
& + W'\left(\frac{L\delta}{\alpha} + \sum_{i=-n}^{n} [\phi(x_i) + \delta\psi(x_i)]\right) - \delta L \\
= \; & \frac{\overline{\lambda}_\delta^L}{\delta|p_0|} \left\{ \phi'(x_{i_0}) + \delta\psi'(x_{i_0}) + \sum_{\substack{i=-n \\ i\ne i_0}}^{n} [\phi'(x_i) + \delta\psi'(x_i)] \right\} - \sum_{\substack{i=-n \\ i\ne i_0}}^{n} W'(\widetilde{\phi}(x_i)) - \delta\mathcal{I}_1[\psi, x_{i_0}] \\
& - \delta \sum_{\substack{i=-n \\ i\ne i_0}}^{n} \mathcal{I}_1[\psi, x_i] + W'\left(\frac{L\delta}{\alpha} + \sum_{i=-n}^{n} \left[\widetilde{\phi}(x_i) + \delta\psi(x_i)\right]\right) - W'(\widetilde{\phi}(x_{i_0})) - \delta L \\
= \; & \delta c_0 L \left\{ \phi'(x_{i_0}) + \delta\psi'(x_{i_0}) + \sum_{\substack{i=-n \\ i\ne i_0}}^{n} [\phi'(x_i) + \delta\psi'(x_i)] \right\} - W''(0) \sum_{\substack{i=-n \\ i\ne i_0}}^{n} \widetilde{\phi}(x_i) - \delta \sum_{\substack{i=-n \\ i\ne i_0}}^{n} \mathcal{I}_1[\psi, x_i] \\
& - \delta\mathcal{I}_1[\psi, x_{i_0}] + W''(\phi(x_{i_0})) \left(\frac{L\delta}{\alpha} + \delta\psi(x_{i_0}) + \sum_{\substack{i=-n \\ i\ne i_0}}^{n} \left[\widetilde{\phi}(x_i) + \delta\psi(x_i)\right]\right) - \delta L \\
& + \sum_{\substack{i=-n \\ i\ne i_0}}^{n} O(\widetilde{\phi}(x_i))^2 + O\left(\frac{L\delta}{\alpha} + \delta\psi(x_{i_0}) + \sum_{\substack{i=-n \\ i\ne i_0}}^{n} \left[\widetilde{\phi}(x_i) + \delta\psi(x_i)\right]\right)^2 \\
= \; & \delta c_0 L \left\{ \delta\psi'(x_{i_0}) + \sum_{\substack{i=-n \\ i\ne i_0}}^{n} [\phi'(x_i) + \delta\psi'(x_i)] \right\} - W''(0) \sum_{\substack{i=-n \\ i\ne i_0}}^{n} \widetilde{\phi}(x_i) - \delta \sum_{\substack{i=-n \\ i\ne i_0}}^{n} \mathcal{I}_1[\psi, x_i] \\
& + W''(\phi(x_{i_0})) \left(\sum_{\substack{i=-n \\ i\ne i_0}}^{n} \left[\widetilde{\phi}(x_i) + \delta\psi(x_i)\right]\right) \\
& + \delta\Big(- \mathcal{I}_1[\psi, x_{i_0}] + W''(\phi(x_{i_0}))\psi(x_{i_0}) + \frac{L}{\alpha} W''(\phi(x_{i_0})) - L + c\phi'(x_{i_0})\Big) \\
& + \sum_{\substack{i=-n \\ i\ne i_0}}^{n} O(\widetilde{\phi}(x_i))^2 + O\left(\frac{L\delta}{\alpha} + \delta\psi(x_{i_0}) + \sum_{\substack{i=-n \\ i\ne i_0}}^{n} \left[\widetilde{\phi}(x_i) + \delta\psi(x_i)\right]\right)^2
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \delta c_0 L \left\{ \delta\psi'(x_{i_0}) + \sum_{\substack{i=-n\\ i\neq i_0}}^{n} [\phi'(x_i) + \delta\psi'(x_i)] \right\} + (W''(\phi(x_{i_0})) - W''(0)) \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \widetilde{\phi}(x_i) \\
& - \delta \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \mathcal{I}_1[\psi, x_i] + W''(\phi(x_{i_0}))\delta \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \psi(x_i) + \sum_{\substack{i=-n\\ i\neq i_0}}^{n} O(\widetilde{\phi}(x_i))^2 \\
& + O\left(\frac{L\delta}{\alpha} + \delta\psi(x_{i_0}) + \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \left[\widetilde{\phi}(x_i) + \delta\psi(x_i) \right] \right)^2
\end{aligned}
$$

Let us bound the second term of the last equality, uniformly in x. From (7.10) and (7.14) it follows that

$$
-\delta^3 |p_0|^2 K_3 \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \frac{1}{(x-i)^2} \leq \sum_{\substack{i=-n\\ i\neq i_0}}^{n} [\phi'(x_i) + \delta\psi'(x_i)] \leq \delta^2 |p_0|^2 (K_1 + \delta K_3) \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \frac{1}{(x-i)^2},
$$

and then by Claim 1 we get

$$
-C\delta^3 \leq \lim_{n\to+\infty} \sum_{\substack{i=-n\\ i\neq i_0}}^{n} [\phi'(x_i) + \delta\psi'(x_i)] \leq C\delta^2. \tag{7.16}
$$

Here and henceforth, C denotes various positive constants independent of x.

Now, let us prove that

$$
-C\delta^2 \leq \lim_{n\to+\infty} (W''(\phi(x_{i_0})) - W''(0)) \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \widetilde{\phi}(x_i) \leq C\delta^2. \tag{7.17}
$$

By (7.9) we have

$$
\left| \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \widetilde{\phi}(x_i) + \frac{\delta|p_0|}{\alpha\pi} \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \frac{1}{x-i} \right| \leq K_1 \delta^2 |p_0|^2 \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \frac{1}{(x-i)^2}. \tag{7.18}
$$

If $|\gamma| \geq \delta|p_0|$, then again from (7.9), $|\widetilde{\phi}(x_{i_0}) + \frac{\delta|p_0|}{\alpha\pi\gamma}| \leq K_1 \frac{\delta^2|p_0|^2}{\gamma^2}$ which implies that

$$
|W''(\widetilde{\phi}(x_{i_0})) - W''(0)| \leq |W'''(0)\widetilde{\phi}(x_{i_0})| + O(\widetilde{\phi}(x_{i_0}))^2 \leq C\frac{\delta}{|\gamma|} + C\frac{\delta^2}{\gamma^2}.
$$

By the previous inequality, (7.18) and Claim 1 we deduce that

$$
\left| \lim_{n\to+\infty} (W''(\phi(x_{i_0})) - W''(0)) \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \widetilde{\phi}(x_i) \right| \leq C\left(\frac{\delta}{|\gamma|} + \frac{\delta^2}{\gamma^2} \right)(\delta|\gamma| + \delta^2) \leq C\delta^2,
$$

where C is independent of γ.

Finally, if $|\gamma| < \delta|p_0|$, from (7.18) and Claim 1 we conclude that

$$
\left| \lim_{n\to+\infty} (W''(\phi(x_{i_0})) - W''(0)) \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \widetilde{\phi}(x_i) \right| \leq C\delta|\gamma| + C\delta^2 \leq C\delta^2,
$$

and (7.17) is proved.

Now, let us consider $\delta \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \mathcal{I}_1[\psi, x_i]$. We have

$$
\mathcal{I}_1[\psi] = W''(\widetilde{\phi})\psi + \frac{L}{\alpha}(W''(\widetilde{\phi}) - W''(0)) + c\phi' = W''(0)\psi + \frac{L}{\alpha} W'''(0)\widetilde{\phi} + O(\widetilde{\phi})\psi + O(\widetilde{\phi})^2 + c\phi'. \tag{7.19}
$$

Then by (7.19), (7.9), (7.10), (7.13) and Claim 1, we have

$$\left|\lim_{n\to+\infty}\delta\sum_{\substack{i=-n\\ i\neq i_0}}^{n}\mathcal{I}_1[\psi,x_i]\right|\leq C\delta^2. \tag{7.20}$$

Similarly

$$\left|\lim_{n\to+\infty}W''(\phi(x_{i_0}))\delta\sum_{\substack{i=-n\\ i\neq i_0}}^{n}\psi(x_i)\right|\leq C\delta^2. \tag{7.21}$$

Finally, still from (7.9), (7.13), and Claim 1 it follows that

$$\left|\lim_{n\to+\infty}\sum_{\substack{i=-n\\ i\neq i_0}}^{n}(\widetilde{\phi}(x_i))^2+O\left(\frac{L\delta}{\alpha}+\delta\psi(x_{i_0})+\sum_{\substack{i=-n\\ i\neq i_0}}^{n}\left[\widetilde{\phi}(x_i)+\delta\psi(x_i)\right]\right)^2\right|\leq C\delta^2. \tag{7.22}$$

Therefore, from (7.16), (7.17), (7.20), (7.21) and (7.22) we conclude that

$$-C\delta^2\leq\lim_{n\to+\infty}NL_L^{\overline{\lambda}_\delta^L}[s_{\delta,n}^L]\leq C\delta^2$$

with C independent of x and Lemma 7.5 is proved.

7.3. Proof of Proposition 7.1.

Step 1: proof of ii)
Let $x=i_0+\gamma$ with $i_0\in\mathbb{Z}$ and $\gamma\in\left(-\frac{1}{2},\frac{1}{2}\right]$. Let $n>|i_0|$, then by (7.9) and (7.13) we get

$$\begin{aligned}
s_{\delta,n}^L(x)-x&=\frac{L\delta}{\alpha}+\phi(x_{i_0})+\delta\psi(x_{i_0})-n-i_0-\gamma+\sum_{\substack{i=-n\\ i\neq i_0}}^{n}[\phi(x_i)+\delta\psi(x_i)]\\
&=\frac{L\delta}{\alpha}+\phi(x_{i_0})+\delta\psi(x_{i_0})-\gamma+\sum_{\substack{i=-n\\ i\neq i_0}}^{n}[\widetilde{\phi}(x_i)+\delta\psi(x_i)]\\
&\leq\frac{L\delta}{\alpha}+\frac{3}{2}+\delta\|\psi\|_\infty+\sum_{\substack{i=-n\\ i\neq i_0}}^{n}\left[-\left(\frac{1}{\alpha\pi}-\delta K_2\right)\frac{\delta|p_0|}{x-i}+(K_1+\delta K_3)\frac{\delta^2|p_0|^2}{(x-i)^2}\right].
\end{aligned}$$

Then, by Claim 1

$$h_\delta^L(x)-x=\lim_{n\to+\infty}s_{\delta,n}^L(x)-x\leq C.$$

Similarly we can prove that

$$h_\delta^L(x)-x\geq -C,$$

which concludes the proof of ii).

Step 2: proof of i)
The function $h_\delta^L(x)=\lim_{n\to+\infty}s_{\delta,n}^L(x)$ is well defined for any $x\in\mathbb{R}$ by Claim 2. Moreover, by Claim 3 and 4 and classical analysis results, it is of class C^2 on $\mathbb{R}$ with

$$(h_\delta^L)'(x)=\lim_{n\to+\infty}(s_{\delta,n}^L)'(x)=\lim_{n\to+\infty}\frac{1}{\delta|p_0|}\sum_{i=-n}^{n}\left[\phi'\left(\frac{x-i}{\delta|p_0|}\right)+\delta\psi'\left(\frac{x-i}{\delta|p_0|}\right)\right],$$

$$(h_\delta^L)''(x)=\lim_{n\to+\infty}(s_{\delta,n}^L)''(x)=\lim_{n\to+\infty}\frac{1}{\delta^2|p_0|^2}\sum_{i=-n}^{n}\left[\phi''\left(\frac{x-i}{\delta|p_0|}\right)+\delta\psi''\left(\frac{x-i}{\delta|p_0|}\right)\right],$$

and the convergence of $\{s_{\delta,n}^L\}_n$, $\{(s_{\delta,n}^L)'\}_n$ and $\{(s_{\delta,n}^L)''\}_n$ is uniform on compact sets.

Let us show that for any $x\in\mathbb{R}$

$$\mathcal{I}_1[h_\delta^L, x] = \lim_{n\to+\infty} \mathcal{I}_1[s_{\delta,n}^L, x]. \tag{7.23}$$

First, we prove that

$$\mathcal{I}_1^1[h_\delta^L, x] = \lim_{n\to+\infty} \mathcal{I}_1^1[s_{\delta,n}^L, x]. \tag{7.24}$$

Fix $x \in \mathbb{R}$, we know that for any $y \in [-1,1]$, $y \neq 0$

$$\frac{s_{\delta,n}^L(x+y) - s_{\delta,n}^L(x) - (s_{\delta,n}^L)'(x)y}{|y|^2} \to \frac{h_\delta^L(x+y) - h_\delta^L(x) - (h_\delta^L)'(x)y}{|y|^2} \quad \text{as } n \to +\infty.$$

By the uniform convergence of the sequence $\{(s_{\delta,n}^L)''\}_n$ we have

$$\frac{|s_{\delta,n}^L(x+y) - s_{\delta,n}^L(x) - (s_{\delta,n}^L)'(x)y|}{|y|^2} \le \sup_{z\in[x-1,x+1]} (s_{\delta,n}^L)''(z) \le C,$$

where C is indipendent of n, and (7.24) follows from the dominate convergence Theorem.

Then, to prove (7.23) it suffices to show that

$$\mathcal{I}_1^2[h_\delta^L, x] = \lim_{n\to+\infty} \mathcal{I}_1^2[s_{\delta,n}^L, x].$$

From Claim 5 and (7.24), we know that for any $x \in \mathbb{R}$ there exists $\lim_{n\to+\infty} \mathcal{I}_1^2[s_{\delta,n}^L, x]$. For $a > 1$, we have

$$\mathcal{I}_1^2[s_{\delta,n}^L, x] = \int_{1\le|y|\le a} [s_{\delta,n}^L(x+y) - s_{\delta,n}^L(x)]\mu(dy) + \int_{|y|\ge a} [s_{\delta,n}^L(x+y) - s_{\delta,n}^L(x)]\mu(dy).$$

By the uniform convergence of $\{s_{\delta,n}^L\}_n$ on compact sets

$$\lim_{n\to+\infty} \int_{1\le|y|\le a} [s_{\delta,n}^L(x+y) - s_{\delta,n}^L(x)]\mu(dy) = \int_{1\le|y|\le a} [h_\delta^L(x+y) - h_\delta^L(x)]\mu(dy),$$

then there exists the limit

$$\lim_{n\to+\infty} \int_{|y|\ge a} [s_{\delta,n}^L(x+y) - s_{\delta,n}^L(x)]\mu(dy).$$

Let us show that

$$\lim_{a\to+\infty} \lim_{n\to+\infty} \int_{|y|\ge a} [s_{\delta,n}^L(x+y) - s_{\delta,n}^L(x)]\mu(dy) = 0. \tag{7.25}$$

We first remark that if $z > n$, by (7.9) and (7.13) we have

$$\begin{aligned} s_{\delta,n}^L(z) &= \frac{L\delta}{\alpha} + n + 1 + \sum_{i=-n}^{n} [\phi(z_i) - 1 + \delta\psi(z_i)] \\ &\le \frac{L\delta}{\alpha} + n + 1 + \sum_{i=-n}^{n} \left[-\left(\frac{1}{\alpha\pi} - \delta K_2\right) \frac{\delta|p_0|}{z-i} + (K_1 + \delta K_3) \frac{\delta^2|p_0|^2}{(z-i)^2} \right], \end{aligned}$$

and

$$s_{\delta,n}^L(z) \ge \frac{L\delta}{\alpha} + n + 1 + \sum_{i=-n}^{n} \left[-\left(\frac{1}{\alpha\pi} - \delta K_2\right) \frac{\delta|p_0|}{z-i} - (K_1 + \delta K_3) \frac{\delta^2|p_0|^2}{(z-i)^2} \right].$$

By Claim 1, the quantities $\sum_{i=-n}^{n} \frac{1}{z-i}$ and $\sum_{i=-n}^{n} \frac{1}{(z-i)^2}$ are uniformly bounded on $\mathbb{R}$ by a constant independent of n. Hence, we get

$$n - C \le s_{\delta,n}^L(z) \le n + C \quad \text{if } z > n. \tag{7.26}$$

The same argument shows that

$$-n - C \le s_{\delta,n}^L(z) \le -n + C \quad \text{if } z < -n. \tag{7.27}$$

Moreover, by the computations of Claim 7

$$-C \le s_{\delta,n}^L(z) - z \le C \quad \text{if } |z| < n. \tag{7.28}$$

Let $i_0 \in \mathbb{Z}$ be the closest integer to x, let us assume $n > |i_0| + 1 + a > |i_0| + 1$. We have

$$\int_{|y|\geq a} [s^L_{\delta,n}(x+y) - s^L_{\delta,n}(x)]\mu(dy) = \int_{a\leq|y|<n-1-|i_0|} [s^L_{\delta,n}(x+y) - s^L_{\delta,n}(x)]\mu(dy) + \int_{n-1-|i_0|\leq|y|\leq n+1+|i_0|} [...]\mu(dy) + \int_{|y|>n+1+|i_0|} [...]\mu(dy).$$

If $|y| < n - 1 - |i_0|$, then $|x + y| < n$ and by (7.28)

$$\begin{aligned}\int_{a\leq|y|<n-1-|i_0|} [s^L_{\delta,n}(x+y) - s^L_{\delta,n}(x)]\mu(dy) &\leq \int_{a\leq|y|\leq n-1-|i_0|} (y+2C)\mu(dy) \\ &= \int_{a\leq|y|\leq n-1-|i_0|} 2C\mu(dy) \leq \frac{2C}{a},\end{aligned}$$

and

$$\int_{a\leq|y|<n-1-|i_0|} [s^L_{\delta,n}(x+y) - s^L_{\delta,n}(x)]\mu(dy) \geq -\frac{2C}{a}.$$

Then

$$\lim_{a\to+\infty}\lim_{n\to+\infty} \int_{a\leq|y|\leq n-1-|i_0|} [s^L_{\delta,n}(x+y) - s^L_{\delta,n}(x)]\mu(dy) = 0. \tag{7.29}$$

Next, since $|s^L_{\delta,n}(z)| \leq Cn$ for any $z \in \mathbb{R}$, we have

$$\begin{aligned}&\left|\int_{n-1-|i_0|\leq|y|\leq n+1+|i_0|} [s^L_{\delta,n}(x+y) - s^L_{\delta,n}(x)]\mu(dy)\right| \\ &\leq Cn\int_{n-1-|i_0|\leq|y|\leq n+1+|i_0|} \mu(dy) = \widetilde{C}\frac{n(|i_0|+1)}{n^2-(|i_0|+1)^2} \to 0 \quad \text{as } n\to+\infty.\end{aligned} \tag{7.30}$$

Finally, if $y > n + 1 + |i_0|$, then $x + y > n$, if $y < -n - 1 - |i_0|$, then $x + y < -n$. Hence, using (7.26) and (7.27), we obtain

$$\begin{aligned}&\int_{|y|>n+1+|i_0|} [s^L_{\delta,n}(x+y) - s^L_{\delta,n}(x)]\mu(dy) \\ &= \int_{y>n+1+|i_0|} [s^L_{\delta,n}(x+y) - s^L_{\delta,n}(x)]\mu(dy) + \int_{y<-n-1-|i_0|} [s^L_{\delta,n}(x+y) - s^L_{\delta,n}(x)]\mu(dy) \\ &\leq \int_{y>n+1+|i_0|} [n + C - s^L_{\delta,n}(x)]\mu(dy) + \int_{y<-n-1-|i_0|} [-n + C - s^L_{\delta,n}(x)]\mu(dy) \\ &= \int_{|y|>n+1+|i_0|} [C - s^L_{\delta,n}(x)]\mu(dy),\end{aligned}$$

and

$$\int_{|y|>n+1+|i_0|} [s^L_{\delta,n}(x+y) - s^L_{\delta,n}(x)]\mu(dy) \geq \int_{|y|>n+1+|i_0|} [-C - s^L_{\delta,n}(x)]\mu(dy).$$

We deduce that

$$\lim_{n\to+\infty} \int_{|y|>n+1+|i_0|} [s^L_{\delta,n}(x+y) - s^L_{\delta,n}(x)]\mu(dy) = 0.$$

Hence, by the previous limit, (7.29) and (7.30), we derive (7.25).

Then, we finally get

$$\begin{aligned}\lim_{n\to+\infty}\mathcal{I}_1^2[s^L_{\delta,n},x] &= \lim_{a\to+\infty}\lim_{n\to+\infty}\mathcal{I}_1^2[s^L_{\delta,n},x]\\ &= \lim_{a\to+\infty}\lim_{n\to+\infty}\int_{1\le|y|\le a}[s^L_{\delta,n}(x+y)-s^L_{\delta,n}(x)]\mu(dy)\\ &+ \lim_{a\to+\infty}\lim_{n\to+\infty}\int_{|y|>a}[s^L_{\delta,n}(x+y)-s^L_{\delta,n}(x)]\mu(dy)\\ &= \lim_{a\to+\infty}\int_{1\le|y|\le a}[h^L_\delta(x+y)-h^L_\delta(x)]\mu(dy)\\ &= \mathcal{I}_1^2[h^L_\delta,x],\end{aligned}$$

as desired.

Now we can conclude the proof of (i). Indeed, by Claim 2, Claim 3 and (7.23), for any $x\in\mathbb{R}$

$$NL_L^{\overline{\lambda}^L_\delta}[h^L_\delta,x]=\lim_{n\to+\infty}NL_L^{\overline{\lambda}^L_\delta}[s^L_{\delta,n},x],$$

and Lemma 7.5 implies that

$$NL_L^{\overline{\lambda}^L_\delta}[h^L_\delta,x]=o(\delta),\quad \text{as } \delta\to 0,$$

where $\lim_{\delta\to0}\frac{o(\delta)}{\delta}=0$, uniformly for $x\in\mathbb{R}$. □

8. Appendix

In this appendix, we prove the following technical results used in the previous section: Lemmata 7.3 and 7.4, and the Claims 1-5.

8.1. **Proof of Lemma 7.3.** Properties (7.9) and (7.10) are proved in [22].

Let us show (7.11).

For $a>0$, we denote by $\phi'_a(x)=\phi'\left(\frac{x}{a}\right)$. Remark that ϕ'_a is a solution of

$$\mathcal{I}_1[\phi'_a]=\frac{1}{a}W''(\phi_a)\phi'_a\quad\text{in }\mathbb{R}.$$

Since ϕ'' is bounded and of class $C^{2,\beta}$, $\mathcal{I}_1[\phi'']$ is well defined and by deriving twice the equation in (7.4) we see that ϕ'' is a solution of

$$\mathcal{I}_1[\phi'']=W''(\phi)\phi''+W'''(\phi)(\phi')^2.$$

Let $\overline{\phi}=\phi''-C\phi'_a$, with $C>0$, then $\overline{\phi}$ satisfies

$$\begin{aligned}\mathcal{I}_1[\overline{\phi}]-W''(\phi)\overline{\phi}&=C\phi'_a\left(W''(\phi)-\frac{1}{a}W''(\phi_a)\right)+W'''(\phi)(\phi')^2\\ &=C\phi'_a\left(W''(\phi)-\frac{1}{a}W''(\phi_a)\right)+o\left(\frac{1}{1+x^2}\right),\end{aligned}$$

as $|x|\to+\infty$, by (7.10). Fix $a>0$ and $R>0$ such that

$$\begin{cases} W''(\phi)-\frac{1}{a}W''(\phi_a)>\frac12 W''(0)>0 & \text{on } \mathbb{R}\setminus[-R,R];\\ W''(\phi)>0, & \text{on } \mathbb{R}\setminus[-R,R].\end{cases}\tag{8.1}$$

Then from (7.10), for C large enough we get

$$\mathcal{I}_1[\overline{\phi}]-W''(\phi)\overline{\phi}\ge 0\quad\text{on }\mathbb{R}\setminus[-R,R].$$

Choosing C such that moreover

$$\overline{\phi}<0\quad\text{on }[-R,R],$$

we can ensure that $\overline{\phi}\le 0$ on $\mathbb{R}$. Indeed, assume by contradiction that there exists $x_0\in\mathbb{R}\setminus[-R,R]$ such that

$$\overline{\phi}(x_0)=\sup_{\mathbb{R}}\overline{\phi}>0.$$

Then

$$\begin{cases} \mathcal{I}_1[\overline{\phi}, x_0] \le 0; \\ \mathcal{I}_1[\overline{\phi}, x_0] - W''(\phi(x_0))\overline{\phi}(x_0) \ge 0; \\ W''(\phi(x_0)) > 0, \end{cases}$$

from which

$$\overline{\phi}(x_0) \le 0,$$

a contradiction. Therefore $\overline{\phi} \le 0$ on $\mathbb{R}$ and then, by renaming the constants, from (7.10) we get $\phi'' \le \frac{K_1}{1+x^2}$.

To prove that $\phi'' \ge -\frac{K_1}{1+x^2}$, we look at the infimum of the function $\phi'' + C\phi'_a$ to get similarly that $\phi'' + C\phi'_a \ge 0$ on $\mathbb{R}$.

To show (7.12) we proceed as in the proof of (7.11). Indeed, the function ϕ''' which is bounded and of class $C^{1,\beta}$, satisfies

$$\mathcal{I}_1[\phi'''] = W''(\phi)\phi''' + 3W'''(\phi)\phi'\phi'' + W^{IV}(\phi)(\phi')^3 = W''(\phi)\phi''' + o\left(\frac{1}{1+x^2}\right),$$

as $|x| \to +\infty$, by (7.10) and (7.11). Then, as before, for C and a large enough $\phi''' - C\phi'_a \le 0$ and $\phi''' + C\phi'_a \ge 0$ on $\mathbb{R}$, which implies (7.12). □

8.2. **Proof of Lemma 7.4.** Let us prove (7.13).

For $a > 0$ we denote by $\phi_a(x) = \phi\left(\frac{x}{a}\right)$, which is solution of

$$\mathcal{I}_1[\phi_a] = \frac{1}{a}W'(\phi_a) \quad \text{in } \mathbb{R}.$$

Let a and b be positive numbers, then making a Taylor expansion of the derivatives of W, we get

$$\begin{aligned} \mathcal{I}_1[\psi - (\phi_a - \phi_b)] &= W''(\phi)\psi + \frac{L}{\alpha}(W''(\phi) - W''(0)) + c\phi' + \left(\frac{1}{b}W'(\phi_b) - \frac{1}{a}W'(\phi_a)\right) \\ &= W''(\phi)(\psi - (\phi_a - \phi_b)) + W''(\widetilde{\phi})(\phi_a - \phi_b) + \frac{L}{\alpha}(W''(\widetilde{\phi}) - W''(0)) \\ &+ c\phi' + \left(\frac{1}{b}W'(\widetilde{\phi}_b) - \frac{1}{a}W'(\widetilde{\phi}_a)\right) \\ &= W''(\phi)(\psi - (\phi_a - \phi_b)) + W''(0)(\phi_a - \phi_b) + \frac{L}{\alpha}W'''(0)\widetilde{\phi} + c\phi' \\ &+ W''(0)\left(\frac{1}{b}\widetilde{\phi}_b - \frac{1}{a}\widetilde{\phi}_a\right) + (\phi_a - \phi_b)O(\widetilde{\phi}) + O(\widetilde{\phi})^2 + O(\widetilde{\phi}_a)^2 + O(\widetilde{\phi}_b)^2, \end{aligned}$$

and then the function $\overline{\psi} = \psi - (\phi_a - \phi_b)$ satisfies

$$\begin{aligned} \mathcal{I}_1[\overline{\psi}] - W''(\phi)\overline{\psi} &= \alpha(\phi_a - \phi_b) + \frac{L}{\alpha}W'''(0)\widetilde{\phi} + c\phi' + \alpha\left(\frac{1}{b}\widetilde{\phi}_b - \frac{1}{a}\widetilde{\phi}_a\right) \\ &+ (\phi_a - \phi_b)O(\widetilde{\phi}) + O(\widetilde{\phi})^2 + O(\widetilde{\phi}_a)^2 + O(\widetilde{\phi}_b)^2. \end{aligned}$$

We want to estimate the right-hand side of the last equality. By Lemma 7.3, for $|x| \ge \max\{1, |a|, |b|\}$ we have

$$\alpha(\phi_a - \phi_b) + \frac{L}{\alpha}W'''(0)\widetilde{\phi} \ge -\frac{1}{\pi x}\left[(a-b) + \frac{L}{\alpha^2}W'''(0)\right] - \frac{K_1\alpha}{x^2}\left(a^2 + b^2 + \frac{|L|}{\alpha^2}|W'''(0)|\right).$$

Choose $a, b > 0$ such that $(a-b) + \frac{L}{\alpha^2}W'''(0) = 0$, then

$$\alpha(\phi_a - \phi_b) + \frac{L}{\alpha}W'''(0)\widetilde{\phi} \ge -\frac{C}{x^2},$$

for $|x| \ge \max\{1, |a|, |b|\}$. Here and in what follows, as usual C denotes various positive constants. From Lemma 7.3 we also derive that

$$\alpha\left(\frac{1}{b}\widetilde{\phi}_b - \frac{1}{a}\widetilde{\phi}_a\right) \ge -\frac{C}{x^2},$$

$$c\phi' \ge -\frac{C}{1+x^2},$$

and

$$(\phi_a - \phi_b)O(\widetilde{\phi}) + O(\widetilde{\phi})^2 + O(\widetilde{\phi}_a)^2 + O(\widetilde{\phi}_b)^2 \geq -\frac{C}{1+x^2},$$

for $|x| \geq \max\{1, |a|, |b|\}$. Then we conclude that there exists $R > 0$ such that for $|x| \geq R$ we have

$$\mathcal{I}_1[\overline{\psi}] - W''(\phi)\overline{\psi} \geq -\frac{C}{1+x^2}.$$

Now, let us consider the function $\phi_d'(x) = \phi'\left(\frac{x}{d}\right)$, $d > 0$, which is solution of

$$\mathcal{I}_1[\phi_d'] = \frac{1}{d}W''(\phi_d)\phi_d' \quad \text{in } \mathbb{R},$$

and denote

$$\overline{\overline{\psi}} = \overline{\psi} - \widetilde{C}\phi_d',$$

with $\widetilde{C} > 0$. Then, for $|x| \geq R$ we have

$$\mathcal{I}_1[\overline{\overline{\psi}}] \geq W''(\phi)\overline{\psi} - \frac{\widetilde{C}}{d}W''(\phi_d)\phi_d' - \frac{C}{1+x^2} = W''(\phi)\overline{\overline{\psi}} + \widetilde{C}\phi_d'\left(W''(\phi) - \frac{1}{d}W''(\phi_d)\right) - \frac{C}{1+x^2}.$$

Let us choose $d > 0$ and $R_2 > R$ such that

$$\begin{cases} W''(\phi) - \frac{1}{d}W''(\phi_d) > \frac{1}{2}W''(0) > 0 & \text{on } \mathbb{R}\setminus[-R_2, R_2];\\ W''(\phi) > 0 & \text{on } \mathbb{R}\setminus[-R_2, R_2], \end{cases}$$

then from (7.10), for $\widetilde{C}$ large enough we get

$$\mathcal{I}_1[\overline{\overline{\psi}}] - W''(\phi)\overline{\overline{\psi}} \geq 0 \quad \text{on } \mathbb{R}\setminus[-R_2, R_2],$$

and

$$\overline{\overline{\psi}} < 0 \quad \text{on } [-R_2, R_2].$$

As in the proof of Lemma 7.3, we deduce that $\overline{\overline{\psi}} \leq 0$ on $\mathbb{R}$ and then

$$\psi \leq \frac{K_2}{x} + \frac{K_3}{x^2} \quad \text{for } |x| \geq 1,$$

for some $K_2 \in \mathbb{R}$ and $K_3 > 0$.

Looking at the function $\psi - (\phi_a - \phi_b) + \widetilde{C}\phi_d'$, we conclude similarly that

$$\psi \geq \frac{K_2}{x} - \frac{K_3}{x^2} \quad \text{for } |x| \geq 1,$$

and (7.13) is proved.

Now let us turn to (7.14). By deriving the first equation in (7.5), we see that the function ψ' which is bounded and of class $C^{2,\beta}$, is a solution of

$$\mathcal{I}_1[\psi'] = W''(\phi)\psi' + W'''(\phi)\phi'\psi + \frac{L}{\alpha}W'''(\phi)\phi' + c\phi'' \quad \text{in } \mathbb{R}.$$

Then the function $\overline{\psi}' = \psi' - C\phi_a'$, satisfies

$$\begin{aligned}\mathcal{I}_1[\overline{\psi}'] - W''(\phi)\overline{\psi}' &= C\phi_a'\left(W''(\phi) - \frac{1}{a}W''(\phi_a)\right) + W'''(\phi)\phi'\psi + \frac{L}{\alpha}W'''(\phi)\phi' + c\phi''\\ &= C\phi_a'\left(W''(\phi) - \frac{1}{a}W''(\phi_a)\right) + O\left(\frac{1}{1+x^2}\right),\end{aligned}$$

by (7.10), (7.11) and (7.13), and as in the proof of Lemma 7.3, we deduce that for C and a large enough $\overline{\psi}' \leq 0$ on $\mathbb{R}$, which implies that $\psi' \leq \frac{K_3}{1+x^2}$. The inequality $\psi' \geq -\frac{K_3}{1+x^2}$ is obtained similarly by proving that $\overline{\psi}' + C\phi_a' \geq 0$ on $\mathbb{R}$.

Finally, with the same proof as before, using (7.10)-(7.14), we can prove the estimate (7.15) for the function ψ'' which is a bounded $C^{1,\beta}$ solution of

$$\begin{aligned}\mathcal{I}_1[\psi''] &= W''(\phi)\psi'' + 2W'''(\phi)\phi'\psi' + W^{IV}(\phi)(\phi')^2\psi + W'''(\phi)\phi''\psi + \frac{L}{\alpha}W'''(\phi)\phi''\\ &+ \frac{L}{\alpha}W^{IV}(\phi)(\phi')^2 + c\phi'''\\ &= W''(\phi)\psi'' + O\left(\frac{1}{1+x^2}\right).\end{aligned}$$

□

8.3. **Proof of Claims 1-5.**

Proof of Claim 1.
We have for $n > |i_0|$

$$\begin{aligned}\sum_{\substack{i=-n\\ i\neq i_0}}^{n}\frac{1}{x-i} &= \sum_{i=-n}^{i_0-1}\frac{1}{i_0+\gamma-i} + \sum_{i=i_0+1}^{n}\frac{1}{i_0+\gamma-i} = \sum_{i=1}^{n+i_0}\frac{1}{i+\gamma} - \sum_{i=1}^{n-i_0}\frac{1}{i-\gamma}\\ &= \begin{cases}\sum_{i=1}^{n}\frac{-2\gamma}{i^2-\gamma^2}, & \text{if } i_0=0\\ \sum_{i=1}^{n-i_0}\frac{-2\gamma}{i^2-\gamma^2} + \sum_{i=n-i_0+1}^{n+i_0}\frac{1}{i+\gamma}, & \text{if } i_0>0\\ \sum_{i=1}^{n+i_0}\frac{-2\gamma}{i^2-\gamma^2} - \sum_{i=n+i_0+1}^{n-i_0}\frac{1}{i-\gamma}, & \text{if } i_0<0\end{cases} \to -2\gamma\sum_{i=1}^{+\infty}\frac{1}{i^2-\gamma^2} \quad \text{as } n\to+\infty.\end{aligned}$$

Let us prove the second limit of the claim.

$$\sum_{i=-n}^{i_0-1}\frac{1}{(x-i)^2} = \sum_{i=1}^{n+i_0}\frac{1}{(i+\gamma)^2} \to \sum_{i=1}^{+\infty}\frac{1}{(i+\gamma)^2} \quad \text{as } n\to+\infty.$$

Finally

$$\sum_{i=i_0+1}^{n}\frac{1}{(x-i)^2} = \sum_{i=1}^{n-i_0}\frac{1}{(i-\gamma)^2} \to \sum_{i=1}^{+\infty}\frac{1}{(i-\gamma)^2} \quad \text{as } n\to+\infty,$$

and the claim is proved.

By Claim 1 $\sum_{\substack{i=-n\\ i\neq i_0}}^{n}\frac{1}{x-i}$, $\sum_{i=-n}^{i_0-1}\frac{1}{(x-i)^2}$ and $\sum_{i=i_0+1}^{n}\frac{1}{(x-i)^2}$ are Cauchy sequences and then for $k > m > |i_0|$ we have

$$\sum_{i=-k}^{-m-1}\frac{1}{x-i} + \sum_{i=m+1}^{k}\frac{1}{x-i} \to 0 \quad \text{as } m,k\to+\infty, \tag{8.2}$$

$$\sum_{i=-k}^{-m-1}\frac{1}{(x-i)^2} \to 0 \quad \text{as } m,k\to+\infty, \tag{8.3}$$

and

$$\sum_{i=m+1}^{k}\frac{1}{(x-i)^2} \to 0 \quad \text{as } m,k\to+\infty. \tag{8.4}$$

Proof of Claim 2.
We show that $\{s^L_{\delta,n}(x)\}_n$ is a Cauchy sequence. Fix $x\in\mathbb{R}$ and let $i_0\in\mathbb{Z}$ be the closest integer to x such that $x = i_0+\gamma$, with $\gamma\in\left(-\frac{1}{2},\frac{1}{2}\right]$ and $|x-i|\geq\frac{1}{2}$ for $i\neq i_0$. Let δ be so small that $\frac{1}{\delta|p_0|}\geq 2$, then

$\frac{|x-i|}{\delta|p_0|} \geq 1$ for $i \neq i_0$. Let $k > m > |i_0|$, using (7.9) and (7.13) we get

$$\begin{aligned} s^L_{\delta,k}(x) - s^L_{\delta,m}(x) &= -(k-m) + \sum_{i=-k}^{-m-1} [\phi(x_i) + \delta\psi(x_i)] + \sum_{i=m+1}^{k} [\phi(x_i) + \delta\psi(x_i)] \\ &= \sum_{i=-k}^{-m-1} [(\phi(x_i) - 1) + \delta\psi(x_i)] + \sum_{i=m+1}^{k} [\phi(x_i) + \delta\psi(x_i)] \\ &\leq -\left(\frac{1}{\alpha\pi} - \delta K_2\right) \delta|p_0| \sum_{i=-k}^{-m-1} \frac{1}{x-i} + (K_1 + \delta K_3)\delta^2|p_0|^2 \sum_{i=-k}^{-m-1} \frac{1}{(x-i)^2} \\ &\quad - \left(\frac{1}{\alpha\pi} - \delta K_2\right) \delta|p_0| \sum_{i=m+1}^{k} \frac{1}{x-i} + (K_1 + \delta K_3)\delta^2|p_0|^2 \sum_{i=m+1}^{k} \frac{1}{(x-i)^2}, \end{aligned}$$

and

$$\begin{aligned} s^L_{\delta,k}(x) - s^L_{\delta,m}(x) &\geq -\left(\frac{1}{\alpha\pi} - \delta K_2\right) \delta|p_0| \left(\sum_{i=-k}^{-m-1} \frac{1}{x-i} + \sum_{i=m+1}^{k} \frac{1}{x-i}\right) \\ &\quad - (K_1 + \delta K_3)\delta^2|p_0|^2 \left(\sum_{i=-k}^{-m-1} \frac{1}{(x-i)^2} + \sum_{i=m+1}^{k} \frac{1}{(x-i)^2}\right). \end{aligned}$$

Then from (8.2), (8.3), (8.4), we conclude that

$$|s^L_{\delta,k}(x) - s^L_{\delta,m}(x)| \to 0 \quad \text{as } m, k \to +\infty,$$

as desired.

Proof of Claim 3.
To prove the uniform convergence, it suffices to show that $\{(s^L_{\delta,n})'(x)\}_n$ is a Cauchy sequence uniformly on compact sets. Let us consider a bounded interval $[a,b]$ and let $x \in [a,b]$. For $\frac{1}{\delta|p_0|} \geq 2$ and $k > m > 1/2 + \max\{|a|,|b|\}$, by (7.10) and (7.14) we have

$$\begin{aligned} (s^L_{\delta,k})'(x) - (s^L_{\delta,m})'(x) &= \frac{1}{\delta|p_0|} \sum_{i=-k}^{-m-1} [\phi'(x_i) + \delta\psi'(x_i)] + \frac{1}{\delta|p_0|} \sum_{i=m+1}^{k} [\phi'(x_i) + \delta\psi'(x_i)] \\ &\leq (K_1 + \delta K_3)\delta|p_0| \left[\sum_{i=-k}^{-m-1} \frac{1}{(x-i)^2} + \sum_{i=m+1}^{k} \frac{1}{(x-i)^2}\right] \\ &\leq (K_1 + \delta K_3)\delta|p_0| \left[\sum_{i=-k}^{-m-1} \frac{1}{(a-i)^2} + \sum_{i=m+1}^{k} \frac{1}{(b-i)^2}\right], \end{aligned}$$

and

$$(s^L_{\delta,k})'(x) - (s^L_{\delta,m})'(x) \geq -K_3\delta^2|p_0| \left[\sum_{i=-k}^{-m-1} \frac{1}{(a-i)^2} + \sum_{i=m+1}^{k} \frac{1}{(b-i)^2}\right].$$

Then by (8.3) and (8.4)

$$\sup_{x\in[a,b]} |(s^L_{\delta,k})'(x) - (s^L_{\delta,m})'(x)| \to 0 \quad \text{as } k, m \to +\infty,$$

and Claim 3 is proved.

Proof of Claim 4.
Claim 4 can be proved like Claim 3. Indeed

$$(s^L_{\delta,n})''(x) = \frac{1}{\delta^2|p_0|^2} \sum_{i=-n}^{n} [\phi''(x_i) + \delta\psi''(x_i)]$$

and using (7.11) and (7.15), it is easy to show that $\{(s^L_{\delta,n})''\}_n$ is a Cauchy sequence uniformly on compact sets.

Proof of Claim 5.
We have

$$\mathcal{I}_1[\phi] = W'(\phi) = W'(\widetilde{\phi}) = W''(0)\widetilde{\phi} + O(\widetilde{\phi})^2.$$

Let $x = i_0 + \gamma$ with $\gamma \in \left(-\frac{1}{2}, \frac{1}{2}\right]$, and $k > m > |i_0|$. From (7.9), (8.2), (8.3) and (8.4) we get

$$\sum_{i=-k}^{k} \mathcal{I}_1[\phi, x_i] - \sum_{i=-m}^{m} \mathcal{I}_1[\phi, x_i] = \sum_{i=-k}^{-m-1} [\alpha\widetilde{\phi}(x_i) + O(\widetilde{\phi}(x_i))^2] + \sum_{i=m+1}^{k} [\alpha\widetilde{\phi}(x_i) + O(\widetilde{\phi}(x_i))^2]$$
$$\leq -\frac{\delta|p_0|}{\pi}\left[\sum_{i=-k}^{-m-1} \frac{1}{x-i} + \sum_{i=m+1}^{k} \frac{1}{x-i}\right] + C\sum_{i=-k}^{-m-1} \frac{1}{(x-i)^2} + C\sum_{i=m+1}^{k} \frac{1}{(x-i)^2} \to 0,$$

as $m, k \to +\infty$, for some constant $C > 0$, and

$$\sum_{i=-k}^{k} \mathcal{I}_1[\phi, x_i] - \sum_{i=-m}^{m} \mathcal{I}_1[\phi, x_i]$$
$$\geq -\frac{\delta|p_0|}{\pi}\left[\sum_{i=-k}^{-m-1} \frac{1}{x-i} + \sum_{i=m+1}^{k} \frac{1}{x-i}\right] - C\sum_{i=-k}^{-m-1} \frac{1}{(x-i)^2} - C\sum_{i=m+1}^{k} \frac{1}{(x-i)^2} \to 0,$$

as $m, k \to +\infty$. Then $\sum_{i=-n}^{n} \mathcal{I}_1[\phi, x_i]$ is a Cauchy sequence, i.e. it converges.

Let us consider now $\sum_{i=-n}^{n} \mathcal{I}_1[\psi, x_i]$. By (7.19), (7.9), (7.10) and (7.13) we get

$$\sum_{i=-k}^{k} \mathcal{I}_1[\psi, x_i] - \sum_{i=-m}^{m} \mathcal{I}_1[\psi, x_i]$$
$$\leq \widetilde{C}\left[\sum_{i=-k}^{-m-1} \frac{1}{x-i} + \sum_{i=m+1}^{k} \frac{1}{x-i}\right] + C\sum_{i=-k}^{-m-1} \frac{1}{(x-i)^2} + C\sum_{i=m+1}^{k} \frac{1}{(x-i)^2},$$

and

$$\sum_{i=-k}^{k} \mathcal{I}_1[\psi, x_i] - \sum_{i=-m}^{m} \mathcal{I}_1[\psi, x_i]$$
$$\geq \widetilde{C}\left[\sum_{i=-k}^{-m-1} \frac{1}{x-i} + \sum_{i=m+1}^{k} \frac{1}{x-i}\right] - C\sum_{i=-k}^{-m-1} \frac{1}{(x-i)^2} - C\sum_{i=m+1}^{k} \frac{1}{(x-i)^2},$$

for some $\widetilde{C} \in \mathbb{R}$ and $C > 0$, which ensures the convergence of $\sum_{i=-n}^{n} \mathcal{I}_1[\psi, x_i]$.

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