On the junction of elastic plates and beams

Antonio GAUDIELLO, Régis MONNEAU, Jacqueline MOSSINO,
François MURAT and Ali SILI

A.G.: Dipartimento di Automazione, Elettromagnetismo, Ingegneria dell'Informazione e Matematica Industriale, Università di Cassino, Via G. Di Biasio 43, 03043 Cassino (FR), Italia; e-mail: gaudiell@unina.it
R.M: CERMICS, Ecole Nationale des Ponts et Chaussées, 6 et 8 Avenue Blaise Pascal, Cité Descartes, 77455 Champs-sur-Marne Cedex 2, France; e-mail: monneau@cermics.enpc.fr
J.M.: C.M.L.A, Ecole Normale Supérieure de Cachan, 61 Avenue du Président Wilson, 94235 Cachan Cedex, France; e-mail: mossino@cmla.ens-cachan.fr
F.M.: Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie, Boîte courrier 187, 75252 Paris Cedex 05, France; e-mail: murat@ann.jussieu.fr
A.S.: Département de Mathématiques, Université de Toulon et du Var, BP 132, 83957 La Garde Cedex, France; e-mail: sili@univ-tln.fr

Abstract - We consider the linearized elasticity system in a multidomain of $\mathbf{R}^3$. This multidomain is the union of a horizontal plate with fixed cross section and small thickness ε, and of a vertical beam with fixed height and small cross section of radius r^ε. The lateral boundary of the plate and the top of the beam are assumed to be clamped. When ε and r^ε tend to zero simultaneously, with $r^\varepsilon \gg \varepsilon^2$, we identify the limit problem. This limit problem involves six junction conditions.

Sur la jonction des plaques et des poutres élastiques

Résumé - On considère le système linéarisé de l'élasticité, dans un multidomaine de $\mathbf{R}^3$ constitué d'une plaque horizontale de section fixée et de faible épaisseur ε et d'une poutre verticale de hauteur fixée et de petite section dont le rayon est r^ε. La frontière latérale de la plaque et le haut de la poutre sont supposés encastrés. Nous identifions le problème limite quand ε et r^ε tendent simultanément vers zéro, avec $r^\varepsilon \gg \varepsilon^2$. Ce problème limite fait intervenir six conditions de jonction.

Version francaise abrégée

Soient ω^a et ω^b des domaines bornés réguliers de $\mathbf{R}^2$, avec $0 \in \omega^b$. Soit ε un petit paramètre, prenant une suite de valeurs strictement positives, convergeant vers zéro, et soit $r^\varepsilon \gg \varepsilon^2$ un autre paramètre, tendant vers zéro avec ε. Nous considérons le multidomaine mince $\Omega^\varepsilon = \Omega^{a\varepsilon} \bigcup J^\varepsilon \bigcup \Omega^{b\varepsilon}$, où $\Omega^{a\varepsilon} = r^\varepsilon \omega^a \times (0,1)$ modélise une poutre verticale de hauteur donnée et de petite section, $\Omega^{b\varepsilon} = \omega^b \times (-\varepsilon, 0)$ modélise une plaque horizontale de faible épaisseur et de section donnée, et $J^\varepsilon = r^\varepsilon \omega^a \times \{0\}$ représente la jonction à l'interface entre la poutre et la plaque.

Dans ce multidomaine mince nous considérons la solution $\overline{U}^\varepsilon$ du système linéarisé de l'élasticité :

$$\overline{U}^\varepsilon \in Y^\varepsilon \text{ et } \forall U \in Y^\varepsilon, \int_{\Omega^\varepsilon} [A^\varepsilon e(\overline{U}^\varepsilon), e(U)]\, dX = \int_{\Omega^\varepsilon} [G^\varepsilon, e(U)]\, dX,$$

où

- $Y^\varepsilon = \{U \in (H^1(\Omega^\varepsilon))^3, U = 0 \text{ sur } T^\varepsilon = r^\varepsilon \omega^a \times \{1\} \text{ et sur } \Sigma^\varepsilon = \partial\omega^b \times (-\varepsilon, 0)\}$;
- $A^\varepsilon = A^\varepsilon(X) = A^a$ (si $X \in \Omega^{a\varepsilon}$), ou $k^\varepsilon A^b$ (si $X \in \Omega^{b\varepsilon}$), avec k^ε strictement positif, A^a et A^b des tenseurs $3 \times 3 \times 3 \times 3$, à coefficients constants, symétriques et coercifs au sens usuel ;
- $e(U)$ est le gradient symétrisé, de composantes $e_{ij}(U) = \frac{1}{2}\left(\frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i}\right)$;
- $G^\varepsilon \in (L^2(\Omega^\varepsilon))^{3\times 3}$.

La contrainte "$U = 0$" dans la définition de Y^ε signifie que la multistructure est encastrée sur le haut T^ε de la poutre et sur le bord latéral Σ^ε de la plaque. Le cas où k^ε tend vers zéro ou l'infini correspond à des matériaux très différents dans $\Omega^{a\varepsilon}$ et $\Omega^{b\varepsilon}$. Notre propos est d'étudier le comportement asymptotique de $\overline{U}^\varepsilon$, lorsque ε tend vers zéro. Nous montrons que celui-ci dépend de la limite de la suite

$$q^\varepsilon = k^\varepsilon \frac{\varepsilon^3}{(r^\varepsilon)^2}.$$

Lorsque $k^\varepsilon \varepsilon^3$ et $(r^\varepsilon)^2$ sont du même ordre de grandeur, le problème limite (après une renormalisation convenable) est un problème couplé entre une plaque bidimensionnelle et une poutre unidimensionnelle, avec six conditions de jonction. Si $k^\varepsilon \varepsilon^3 \gg (r^\varepsilon)^2$, la multistructure se comporte comme une plaque mince rigide et une poutre mince élastique indépendantes, la poutre étant encastrée à ses deux extrémités ; au contraire, si $k^\varepsilon \varepsilon^3 \ll (r^\varepsilon)^2$, le comportement de la structure est celui d'une poutre mince rigide et d'une plaque mince élastique indépendantes, la plaque étant encastrée sur son contour et fixée verticalement à la jonction.

1 The problem in the thin multidomain

Let ω^a and ω^b (a for "above", b for "below") be two bounded regular domains in $\mathbf{R}^2$ such that $\int_{\omega^a} x_1\, dx_1\, dx_2 = \int_{\omega^a} x_2\, dx_1\, dx_2 = \int_{\omega^a} x_1 x_2\, dx_1\, dx_2 = 0$ and $0 \in \omega^b$. Let ε be a parameter taking values in a sequence of positive numbers converging to zero, and let r^ε be another positive parameter tending to zero with ε. We introduce the thin multidomain $\Omega^\varepsilon = \Omega^{a\varepsilon} \bigcup J^\varepsilon \bigcup \Omega^{b\varepsilon}$, where $\Omega^{a\varepsilon} = r^\varepsilon \omega^a \times (0,1)$ represents a vertical beam with fixed height and small cross section, $\Omega^{b\varepsilon} = \omega^b \times (-\varepsilon, 0)$ represents a horizontal plate with small thickness and fixed cross section, and $J^\varepsilon = r^\varepsilon \omega^a \times \{0\}$ represents the interface at the junction between the beam and the plate.

In this thin multidomain we consider the solution $\overline{U}^\varepsilon$ of the linearized elasticity system in $\mathbf{R}^3$:

$$\overline{U}^\varepsilon \in Y^\varepsilon \text{ and } \forall U \in Y^\varepsilon, \int_{\Omega^\varepsilon} [A^\varepsilon e(\overline{U}^\varepsilon), e(U)]\, dX = \int_{\Omega^\varepsilon} [G^\varepsilon, e(U)]\, dX, \tag{1.1}$$

where

- $Y^\varepsilon = \{U \in (H^1(\Omega^\varepsilon))^3,\ U = 0 \text{ on } T^\varepsilon = r^\varepsilon \omega^a \times \{1\} \text{ and on } \Sigma^\varepsilon = \partial\omega^b \times (-\varepsilon, 0)\}$;
- $A^\varepsilon = A^\varepsilon(X) = \begin{cases} A^a, & \text{if } X \in \Omega^{a\varepsilon}, \\ k^\varepsilon A^b, & \text{if } X \in \Omega^{b\varepsilon}, \end{cases}$

with k^ε a positive parameter depending on ε and A^a, A^b tensors with constant coefficients A^a_{ijkl} and A^b_{ijkl}, $i,j,k,l \in \{1,2,3\}$, satisfying the usual symmetry and coercivity conditions:

$$A^a_{ijkl} = A^a_{jikl} = A^a_{ijlk}, \quad A^b_{ijkl} = A^b_{jikl} = A^b_{ijlk},$$

$$\exists c > 0,\ \forall \xi \in \mathbf{R}^{3\times 3}_s, [A^a\xi, \xi] \geq c|\xi|^2,\ [A^b\xi, \xi] \geq c|\xi|^2,$$

where $\mathbf{R}^{3\times3}_s$ denotes the set of symmetric 3×3-matrices, $(A^a\xi)_{ij} = \sum_{kl} A^a_{ijkl}\xi_{kl}$ (e.g.), the scalar product is defined by $[\eta, \xi] = \sum_{ij} \eta_{ij}\xi_{ij}$ and $|.|$ is the associated norm;

- $e_{ij}(U) = \frac{1}{2}\left(\frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i}\right)$;
- $G^\varepsilon \in (L^2(\Omega^\varepsilon))^{3\times3}$, $G^\varepsilon \neq 0$; without loss of generality, we may assume that

$$\|G^\varepsilon\|^2_{(L^2(\Omega^{a\varepsilon}))^{3\times3}} + \frac{\varepsilon^3}{(r^\varepsilon)^2}\|G^\varepsilon\|^2_{(L^2(\Omega^{b\varepsilon}))^{3\times3}} = 1. \tag{1.2}$$

The constraint "$U = 0$" in the definition of Y^ε means that the multistructure is clamped on the top T^ε of the beam and on the lateral boundary Σ^ε of the plate. The case k^ε tending to zero or infinity corresponds to very different materials in $\Omega^{a\varepsilon}$ and $\Omega^{b\varepsilon}$. (Note that breaking the symmetry between $\Omega^{a\varepsilon}$ and $\Omega^{b\varepsilon}$ is not restrictive.) In the right hand side of (1.1), the forces are written in divergence form, like in [19] and [20]. More classical volume and surface forces can also be considered.

Problem (1.1) admits a unique solution $\overline{U}^\varepsilon$ (see e.g. [21]). The aim of this Note is to describe the limit behaviour of $\overline{U}^\varepsilon$, as ε tends to zero.

2 The rescaled problem

In the sequel, the indexes α and β take values in the set $\{1,2\}$. Moreover, $x = (x', x_3)$ denotes the generic point in $\mathbf{R}^3$.

Let $\Omega^a = \omega^a \times (0,1)$, $\Omega^b = \omega^b \times (-1,0)$, $T = \omega^a \times \{1\}$ and $\Sigma = \partial\omega^b \times (-1,0)$. The asymptotic behaviour of $\overline{U}^\varepsilon$ can be described by using convenient rescalings. On the first hand, we define:

$$g^{a\varepsilon}(x) = r^\varepsilon G^\varepsilon(r^\varepsilon x', x_3) \text{ for } x \in \Omega^a, \quad g^{b\varepsilon}(x) = \frac{\varepsilon^2}{r^\varepsilon} G^\varepsilon(x', \varepsilon x_3) \text{ for } x \in \Omega^b. \tag{2.3}$$

On the other hand, for any $U \in Y^\varepsilon$, we define the rescaled function $u = (u^a, u^b)$ by:

$$u^a_\alpha(x) = (r^\varepsilon)^2\, U_\alpha(r^\varepsilon x', x_3), \;\; u^a_3(x) = r^\varepsilon\, U_3(r^\varepsilon x', x_3), \text{ for } x \in \Omega^a,$$

$$u^b_\alpha(x) = \frac{r^\varepsilon}{\varepsilon}\, U_\alpha(x', \varepsilon x_3), \;\; u^b_3(x) = r^\varepsilon\, U_3(x', \varepsilon x_3), \text{ for } x \in \Omega^b.$$

The above rescaling maps the space Y^ε onto the space $\mathcal{Y}^\varepsilon$ defined by :

$$\begin{array}{c}\mathcal{Y}^\varepsilon = \{u = (u^a, u^b) \in (H^1(\Omega^a))^3 \times (H^1(\Omega^b))^3, \; u^a = 0 \text{ on } T, u^b = 0 \text{ on } \Sigma, \\ \text{a.e. } x' \in \omega^a, \; u^a_\alpha(x', 0) = \varepsilon r^\varepsilon u^b_\alpha(r^\varepsilon x', 0), \; u^a_3(x', 0) = u^b_3(r^\varepsilon x', 0)\}.\end{array}$$

In particular, we denote by $\overline{u}^\varepsilon = (\overline{u}^{a\varepsilon}, \overline{u}^{b\varepsilon})$ the rescaling of the solution $\overline{U}^\varepsilon$ of Problem (1.1). We set

$$e^{a\varepsilon}(u^a) = \begin{pmatrix} \frac{1}{(r^\varepsilon)^2} e_{\alpha\beta}(u^a) & \frac{1}{r^\varepsilon} e_{\alpha 3}(u^a) \\ \\ \frac{1}{r^\varepsilon} e_{3\alpha}(u^a) & e_{33}(u^a) \end{pmatrix}, \quad e^{b\varepsilon}(u^b) = \begin{pmatrix} e_{\alpha\beta}(u^b) & \frac{1}{\varepsilon} e_{\alpha 3}(u^b) \\ \\ \frac{1}{\varepsilon} e_{3\alpha}(u^b) & \frac{1}{\varepsilon^2} e_{33}(u^b) \end{pmatrix}.$$

Then $\overline{u}^\varepsilon$ is the unique solution of the following problem:

$$\begin{array}{c} \overline{u}^\varepsilon \in \mathcal{Y}^\varepsilon \text{ and } \forall u \in \mathcal{Y}^\varepsilon, \\ \int_{\Omega^a} [A^a e^{a\varepsilon}(\overline{u}^{a\varepsilon}), e^{a\varepsilon}(u^a)]\, dx + q^\varepsilon \int_{\Omega^b} [A^b e^{b\varepsilon}(\overline{u}^{b\varepsilon}), e^{b\varepsilon}(u^b)]\, dx = \\ \int_{\Omega^a} [g^{a\varepsilon}, e^{a\varepsilon}(u^a)]\, dx + \int_{\Omega^b} [g^{b\varepsilon}, e^{b\varepsilon}(u^b)]\, dx, \end{array}$$

where q^ε is defined by

$$q^\varepsilon = k^\varepsilon \frac{\varepsilon^3}{(r^\varepsilon)^2}. \tag{2.4}$$

3 The setting of the limit problem

For the definition of the limit problem, in a way similar to [19] and [20], we introduce the functional spaces:

$$\mathcal{U}^a = \quad \{u^a \in (H^2_0(0,1))^2 \times H^1(\Omega^a), \exists \zeta^a \in H^1(0,1), \zeta^a(1) = 0, u^a_3 = \zeta^a - x_1 \frac{du^a_1}{dx_3} - x_2 \frac{du^a_2}{dx_3}\},$$

$$\begin{array}{rl}\mathcal{V}^a = & \{v^a \in (H^1(\Omega^a))^2 \times L^2(0,1; H^1(\omega^a)), \; \exists c \in H^1_0(0,1), \\ & v^a_1 = -c\, x_2, v^a_2 = c\, x_1, \quad \text{for a.e. } x_3 \in (0,1), \; \int_{\omega^a} v^a_3(x', x_3)\, dx' = 0\},\end{array}$$

$$\begin{array}{rl}\mathcal{W}^a = & \{w^a \in (L^2(0,1; H^1(\omega^a))^2 \times \{0\}, \\ & \text{for a.e. } x_3 \in (0,1), \; \int_{\omega^a} w^a_\alpha\, dx' = \int_{\omega^a} (x_1 w^a_2 - x_2 w^a_1)\, dx' = 0\},\end{array}$$

$$\mathcal{U}^b = \quad \{u^b \in (H^1(\Omega^b))^2 \times H^2_0(\omega^b), \; \exists \zeta^b_\alpha \in H^1_0(\omega^b), \quad u^b_\alpha = \zeta^b_\alpha - x_3 \frac{\partial u^b_3}{\partial x_\alpha}\},$$

$$\mathcal{V}^b = \quad \{v^b \in (L^2(\omega^b; H^1(-1,0)))^2 \times \{0\}, \quad \text{for a.e. } x' \in \omega^b, \; \int_{-1}^0 v^b_\alpha(x', x_3)\, dx_3 = 0\},$$

$$\mathcal{W}^b = \quad \{w^b \in (\{0\})^2 \times L^2(\omega^b; H^1(-1,0)), \quad \text{for a.e. } x' \in \omega^b, \; \int_{-1}^0 w^b_3(x', x_3)\, dx_3 = 0\},$$

$$\mathcal{Z}^a = \mathcal{U}^a \times \mathcal{V}^a \times \mathcal{W}^a, \quad \mathcal{Z}^b = \mathcal{U}^b \times \mathcal{V}^b \times \mathcal{W}^b.$$

Without loss of generality, we assume that q^ε defined by (2.4) satisfies

$$q^\varepsilon \to q \in [0, \infty]. \tag{3.5}$$

According to the value of q, the functional space for the limit problem is the following one:

$$\begin{array}{lll} \mathcal{Z} & = \{z = (z^a, z^b) = ((u^a, v^a, w^a), (u^b, v^b, w^b)) \in \mathcal{Z}^a \times \mathcal{Z}^b, & \\ & \quad \text{for a.e. } x' \in \omega^a,\ u_3^a(x', 0) = u_3^b(0)\}, & \text{if } q \in (0, +\infty), \\ \mathcal{Z}_\infty & = \{z^a = (u^a, v^a, w^a) \in \mathcal{Z}^a, \quad \text{for a.e. } x' \in \omega^a,\ u_3^a(x', 0) = 0\}, & \text{if } q = +\infty, \\ \mathcal{Z}_0 & = \{z^b = (u^b, v^b, w^b) \in \mathcal{Z}^b,\ u_3^b(0) = 0\}, & \text{if } q = 0. \end{array}$$

In contrast with the other requirements, the six conditions $u_\alpha^a(0) = \dfrac{d u_\alpha^a}{d x_3}(0) = c(0) = 0$, $u_3^a(x', 0) = u_3^b(0)$ (respectively $u_3^a(x', 0) = 0$ or $u_3^b(0) = 0$), which appear in the definition of the above spaces $\mathcal{U}^a$, $\mathcal{V}^a$ and $\mathcal{Z}$ (respectively $\mathcal{Z}_\infty$ or $\mathcal{Z}_0$), are specific to the junction between the beam and the plate. Note also that, in view of the definition of $\mathcal{U}^a$, the condiion $u_3^a(x', 0) = u_3^b(0)$ (respectively $u_3^a(x', 0) = 0$) reduces to $\zeta^a(0) = u_3^b(0)$ (respectively $\zeta^a(0) = 0$).

We finally introduce, for $z^a = (u^a, v^a, w^a)$ in $\mathcal{Z}^a$ and $z^b = (u^b, v^b, w^b)$ in $\mathcal{Z}^b$:

$$e^a(z^a) = \begin{pmatrix} e_{\alpha\beta}(w^a) & e_{\alpha 3}(v^a) \\ e_{3\alpha}(v^a) & e_{33}(u^a) \end{pmatrix}, \quad e^b(z^b) = \begin{pmatrix} e_{\alpha\beta}(u^b) & e_{\alpha 3}(v^b) \\ e_{3\alpha}(v^b) & e_{33}(w^b) \end{pmatrix}.$$

4 The main result

Let $g^{a\varepsilon}$, $g^{b\varepsilon}$ be defined by (2.3). In the sequel, we assume that

$$g^{a\varepsilon} \rightharpoonup g^a \text{ weakly in } (L^2(\Omega^a))^{3\times 3}, \tag{4.6}$$

$$g^{b\varepsilon} \rightharpoonup g^b \text{ weakly in } (L^2(\Omega^b))^{3\times 3}. \tag{4.7}$$

Let us remark that convergences (4.6) and (4.7) are always satisfied by extracting a subsequence, since from (1.2) it results that

$$\|g^{a\varepsilon}\|^2_{(L^2(\Omega^a))^{3\times 3}} + \|g^{b\varepsilon}\|^2_{(L^2(\Omega^b))^{3\times 3}} = 1.$$

Our main result is the following one:

Theorem 1 *Assume that* $\frac{r^\varepsilon}{\varepsilon^2} \to +\infty$ *and that (3.5), (4.6) and (4.7) hold true. Then:*

1) If $q \in (0, +\infty)$, *there exists* $\overline{z} = (\overline{z}^a, \overline{z}^b) = ((\overline{u}^a, \overline{v}^a, \overline{w}^a), (\overline{u}^b, \overline{v}^b, \overline{w}^b)) \in \mathcal{Z}$, *such that*

$$(\overline{u}^{a\varepsilon}, \overline{u}^{b\varepsilon}) \rightharpoonup (\overline{u}^a, \overline{u}^b) \ \textit{weakly in} \ (H^1(\Omega^a))^3 \times (H^1(\Omega^b))^3,$$

$$(e^{a\varepsilon}(\overline{u}^{a\varepsilon}), e^{b\varepsilon}(\overline{u}^{a\varepsilon})) \rightharpoonup (e^a(\overline{z}^a), e^b(\overline{z}^b)) \ \textit{weakly in} \ (L^2(\Omega^a))^{3\times 3} \times (L^2(\Omega^b))^{3\times 3}, \tag{4.8}$$

and $\overline{z}$ *is the unique solution of the following problem:*

$$\begin{gathered} \overline{z} \in \mathcal{Z} \ \textit{and} \ \forall z \in \mathcal{Z}, \\ \int_{\Omega^a} [A^a e^a(\overline{z}^a), e^a(z^a)]\, dx + q \int_{\Omega^b} [A^b e^b(\overline{z}^b), e^b(z^b)]\, dx = \\ \int_{\Omega^a} [g^a, e^a(z^a)]\, dx + \int_{\Omega^b} [g^b, e^b(z^b)]\, dx. \end{gathered}$$

Moreover, if the convergences in (4.6), (4.7) are strong, then $(g^a, g^b) \neq (0,0)$ *and the convergence in (4.8) is strong.*

2) If $q = +\infty$, *there exists* $\overline{z}^a = (\overline{u}^a, \overline{v}^a, \overline{w}^a) \in \mathcal{Z}_\infty$, *such that*

$$\overline{u}^{a\varepsilon} \rightharpoonup \overline{u}^a \ \textit{weakly in} \ (H^1(\Omega^a))^3, \ \ \overline{u}^{b\varepsilon} \to 0 \ \textit{strongly in} \ (H^1(\Omega^b))^3,$$

$$e^{a\varepsilon}(\overline{u}^{a\varepsilon}) \rightharpoonup e^a(\overline{z}^a) \ \textit{weakly in} \ (L^2(\Omega^a))^{3\times 3}, \ \ e^{b\varepsilon}(\overline{u}^{b\varepsilon}) \to 0 \ \textit{strongly in} \ (L^2(\Omega^b))^{3\times 3},$$

and $\overline{z}^a$ *is the unique solution of the following problem:*

$$\overline{z}^a \in \mathcal{Z}_\infty \ \textit{and} \ \forall z^a \in \mathcal{Z}_\infty, \int_{\Omega^a} [A^a e^a(\overline{z}^a), e^a(z^a)]\, dx = \int_{\Omega^a} [g^a, e^a(z^a)]\, dx.$$

Moreover, if the convergence in (4.6) is strong, then

$$e^{a\varepsilon}(\overline{u}^{a\varepsilon}) \to e^a(\overline{z}^a) \ \textit{strongly in} \ (L^2(\Omega^a))^{3\times 3}, \ \ \sqrt{q^\varepsilon}\, e^{b\varepsilon}(\overline{u}^{b\varepsilon}) \to 0 \ \textit{strongly in} \ (L^2(\Omega^b))^{3\times 3}.$$

3) If $q = 0$, *there exists* $\overline{z}^b = (\overline{u}^b, \overline{v}^b, \overline{w}^b) \in \mathcal{Z}_0$, *such that*

$$\begin{gathered} q^\varepsilon\, \overline{u}^{a\varepsilon} \to 0 \ \textit{strongly in} \ (H^1(\Omega^a))^3, \ q^\varepsilon\, \overline{u}^{b\varepsilon} \rightharpoonup \overline{u}^b \ \textit{weakly in} \ (H^1(\Omega^b))^3, \\ q^\varepsilon e^{a\varepsilon}(\overline{u}^{a\varepsilon}) \to 0 \ \textit{strongly in} \ (L^2(\Omega^a))^{3\times 3}, \ q^\varepsilon e^{b\varepsilon}(\overline{u}^{b\varepsilon}) \rightharpoonup e^b(\overline{z}^b) \ \textit{weakly in} \ (L^2(\Omega^b))^{3\times 3}, \end{gathered}$$

and $\overline{z}^b$ *is the unique solution of the following problem:*

$$\overline{z}^b \in \mathcal{Z}_0 \ \textit{and} \ \forall z^b \in \mathcal{Z}_0, \int_{\Omega^b} [A^b e^b(\overline{z}^b), e^b(z^b)]\, dx = \int_{\Omega^b} [g^b, e^b(z^b)]\, dx.$$

Moreover, if the convergence in (4.7) is strong, then

$$\sqrt{q^\varepsilon}\, e^{a\varepsilon}(\overline{u}^{a\varepsilon}) \to 0 \ \textit{strongly in} \ (L^2(\Omega^a))^{3\times 3}, \ \ q^\varepsilon e^{b\varepsilon}(\overline{u}^{b\varepsilon}) \to e^b(\overline{z}^b) \ \textit{strongly in} \ (L^2(\Omega^b))^{3\times 3}.$$

One can prove that the functions $\overline{v}^a$ and $\overline{w}^a$ (resp. $\overline{v}^b$ and $\overline{w}^b$) which appear in the limit problem are the limits of suitable expressions of $\overline{u}^{a\varepsilon}$ (resp. $\overline{u}^{b\varepsilon}$).

5 Back to the problem in the thin multidomain

As far as the asymptotic behaviour of the "energy" of the solution of Problem (1.1) in the thin multidomain is concerned, we define

$$\mathcal{E}^\varepsilon := \int_{\Omega^\varepsilon} [A^\varepsilon e(\overline{U}^\varepsilon), e(\overline{U}^\varepsilon)]\, dX = \int_{\Omega^a} [A^a e^{a\varepsilon}(\overline{u}^{a\varepsilon}), e^{a\varepsilon}(\overline{u}^{a\varepsilon})]\, dx + q^\varepsilon \int_{\Omega^b} [A^b e^{b\varepsilon}(\overline{u}^{b\varepsilon}), e^{b\varepsilon}(\overline{u}^{b\varepsilon})]\, dx,$$

and from Theorem 1 we deduce the following Corollary:

Corollary 1 *Assume that $\frac{r^\varepsilon}{\varepsilon^2} \to +\infty$ and that (3.5) holds true.*

1) If $q \in (0, +\infty)$ and the convergences in (4.6), (4.7) are strong, then

$$\mathcal{E}^\varepsilon \to \mathcal{E} = \int_{\Omega^a} [A^a e^a(\overline{z}^a), e^a(\overline{z}^a)]\, dx + q \int_{\Omega^b} [A^b e^b(\overline{z}^b), e^b(\overline{z}^b)]\, dx.$$

2) If $q = +\infty$ and the convergence in (4.6) is strong, then

$$\mathcal{E}^\varepsilon \to \mathcal{E}_\infty = \int_{\Omega^a} [A^a e^a(\overline{z}^a), e^a(\overline{z}^a)]\, dx.$$

3) If $q = 0$ and the convergence in (4.7) is strong, then

$$q^\varepsilon \mathcal{E}^\varepsilon \to \mathcal{E}_0 = \int_{\Omega^b} [A^b e^b(\overline{z}^b), e^b(\overline{z}^b)]\, dx.$$

The reader is refered to [1], [2], [3], [5], [6], [8], [9], [16], [17], [18], [19], [20], [22], [23], for the asymptotic behaviour of plates and beams. Junction problems are considered in [4], [7], [10], [11], [13], [14], [15]. The present work is a natural follow up of [19], [20], which deal with reduction of dimension for elastic thin cylinders, and [10], [11], which deal with the diffusion equation in the thin multistructure considered in this Note.

The detailed proofs of the results of the present Note will be given in a forthcoming paper [12].

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