Numerical methods for the option pricing in markets with jumps

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What is an Option? (European Call)

Contract: **The holder has the right to buy at a fixed data in the future (maturity date T) a prescribed asset (underlying S) at a prescribed price (strike price or exercise price K).**
The other party (writer) has the obbligation to sell.

This T-contract is defined by the random amount (payoff)

$$\phi(S_T) = \max\left[S_T - K, 0\right],$$

Problem

- **How to price the option?**

The value of the option depends on:

- **price of the underlying;**
- **time to expiry**

- $(S_\tau)_{\tau\in[0,T]}$ is a stochastic process on a filtered probability space;
- r is the riskless interest rate;

In an arbitrage-free market, the price of a European option with payoff $\phi(S_T)$ on an underlying S_t may be computed as

$$C_t(\phi(S_T)) = e^{-r(T-t)}\mathbb{E}^{\mathcal{Q}}[\phi(S_T)|\mathcal{F}_t]$$

with respect to some measure $\mathcal{Q}$ such that $\tilde{S}_t = e^{-rt}S_t$ is a martingale.

option pricing model
$\Updownarrow$
specify the law of $(S_\tau)_{\tau\in[0,T]}$ under $\mathcal{Q}$
the “risk neutral” dynamics of S

$$\boxed{S_t = S_0 \exp\left[\left(r - \frac{\sigma^2}{2}\right) t + \sigma W_t\right]}$$

- W_t **is a Brownian motion;**
- σ**: volatility coefficient, constant.**

ASSUMPTIONS:

	no arbitrage opportunities	**continuous trading**
	no transaction costs/dividends	**completness of the market**

C_t **call on** $S_t, (T,\ K)$: $$C(S, T; \sigma, r) = \mathrm{Max}(S - K, 0).$$

For $0 \leq t \leq T$: $$C(S, t; \sigma, r) = S\mathcal{N}(d_1) - Ke^{-r(T-t)}\mathcal{N}(d_2),$$

$$\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2} dy, \qquad d_{1,2} = \frac{\log(S/K) \pm (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

$$\boxed{S_t = S_0 \exp\left[\mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i\right]}$$

- W_t **is a Brownian motion;** σ **volatility of the Brownian**
- N_t **is a Poisson process with intensity** λ **independent from** W**;**

  N_t **counts the number of jumps up to and including time** t**;**
- $Y_i \sim N(m, \delta^2)$ **are i.i.d. random variables independent from** W**,** N**.**

  Y_i **corresponds to a proporzional jump in the asset price**

- **Such markets are incomplete: there are many possible choices for a "risk-neutral" measure** $\Rightarrow$ *option hedging is a risky affair: one has to specify a way to measure this risk and then try to minimize it*

- **There are different ways to measure risk (then to chose $\mathcal{Q}$): superhedging, utility maximization, ...**

- **Merton model:**

$$\mu^M = r - \frac{\sigma^2}{2} - \lambda \mathbb{E}[e^{Y_i} - 1] = r - \frac{\sigma^2}{2} - \lambda[\exp(m + \delta^2/2) - 1],$$

  $\mathcal{Q}^M$ **obtained by shifting the drift of the Brownian motion but leaving the jump part unchanged.**

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This is not the topic of the talk!

The aim is to be able to calculate the price of an European option which guarantees the payoff $\phi(S_T)$ at time T:

Two approaches:

1. **... as discounted expectations**

$$\boxed{C_t = \mathbb{E}\Big[e^{-r(T-t)}\phi(S_T)|\mathcal{F}_t\Big]}$$

2. **... as deterministic function $C_t = V(S_t, t)$;**
   by means of Ito's calculus,

$$\boxed{V(S,t) \textbf{ is solution of a Differential Problem}}$$

The aim is to be able to calculate the price of an European option which guarantees the payoff $\phi(S_T)$ at time T:

Two approaches:

1. ... as discounted expectations
$$C_t = \mathbb{E}\left[e^{-r(T-t)}\phi(S_T)|\mathcal{F}_t\right]$$
   - The price estimator is:
$$C_0 \approx \frac{e^{-rT}}{n}\sum_{i=1}^{n}\phi(S_T^{(i)})$$

2. ... as deterministic function $C_t = V(S_t, t)$;
   by means of Ito's calculus,
   $V(S,t)$ is solution of a Differential Problem
   - Finite Difference Schemes

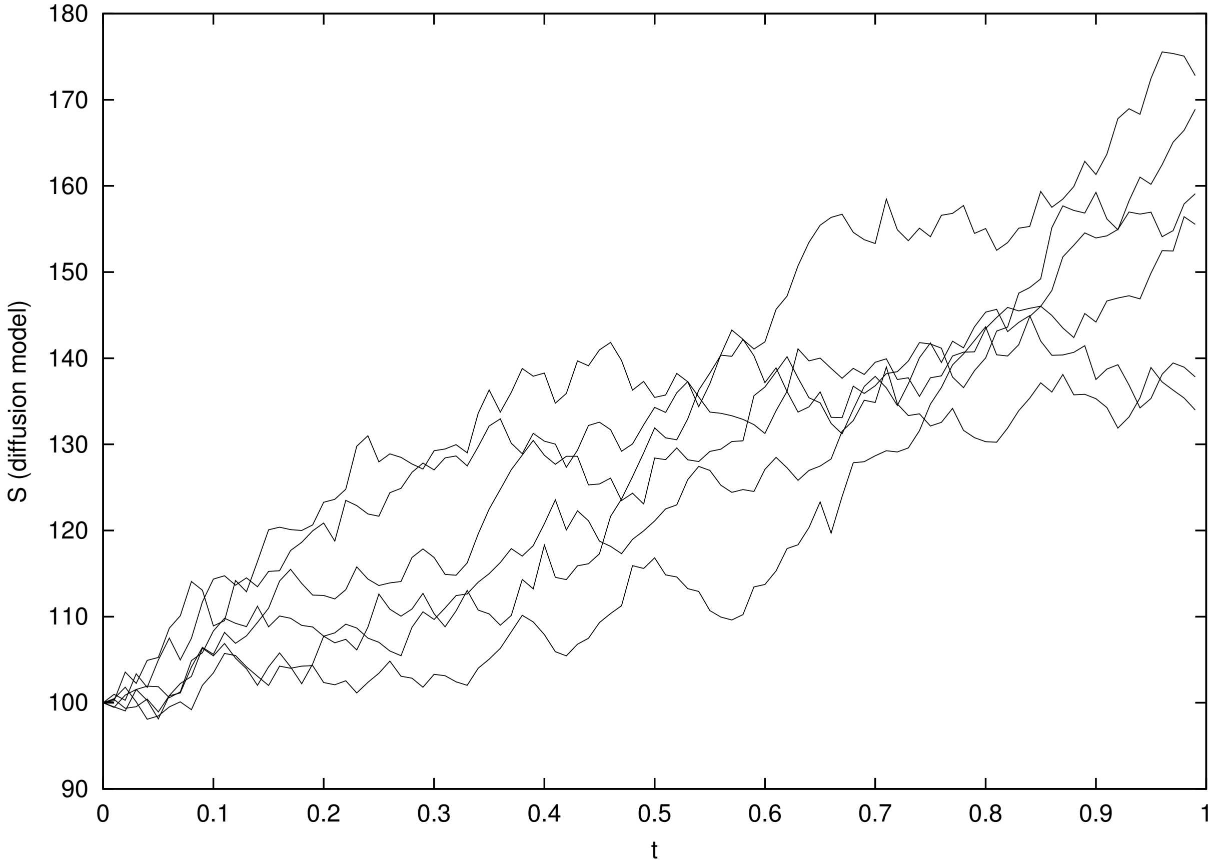

Figure 1: $\sigma = 0.15$, $r = 0.5$ **and** $\Delta t = 10^{-2}$

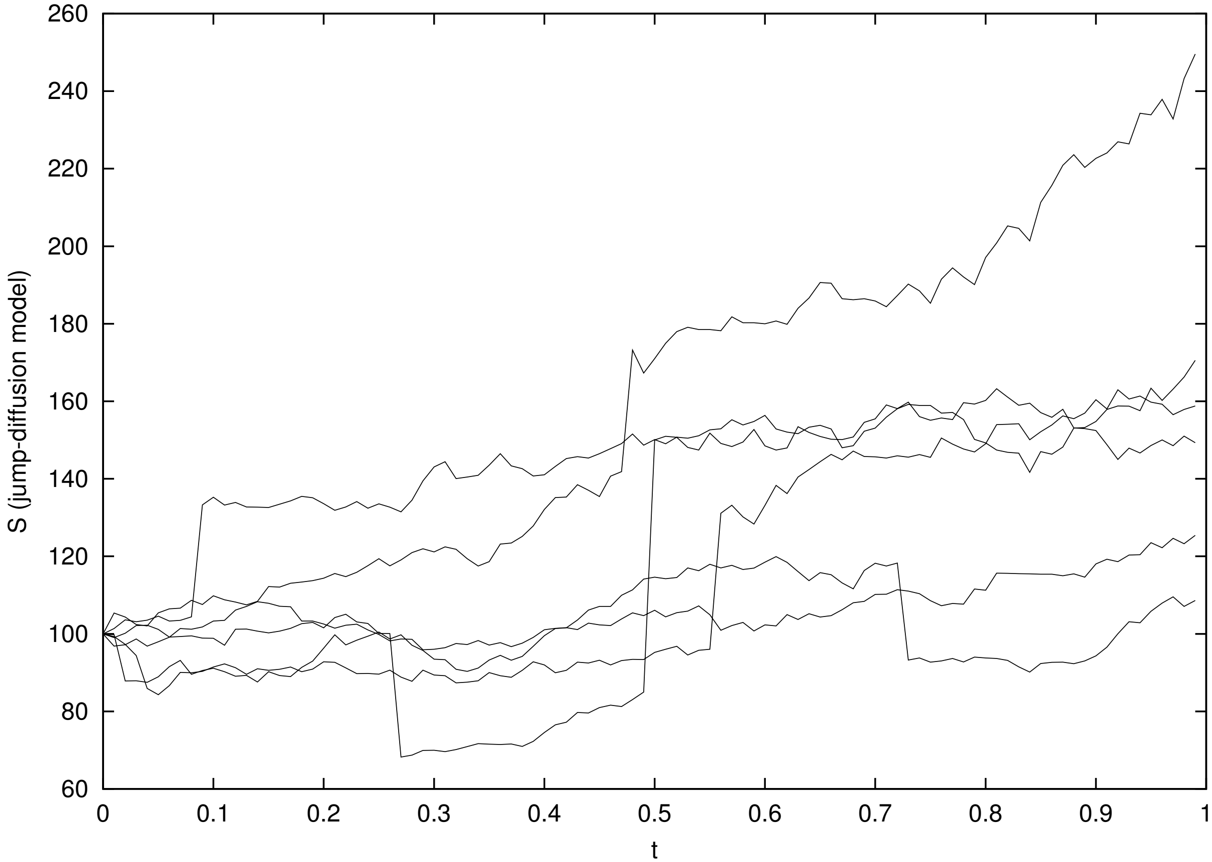

Figure 2: $\lambda = 1$, $\sigma = 0.15$, $r = 0.5$, $\gamma = 0$, $\delta = 0.4$ **and** $\Delta t = 10^{-2}$.

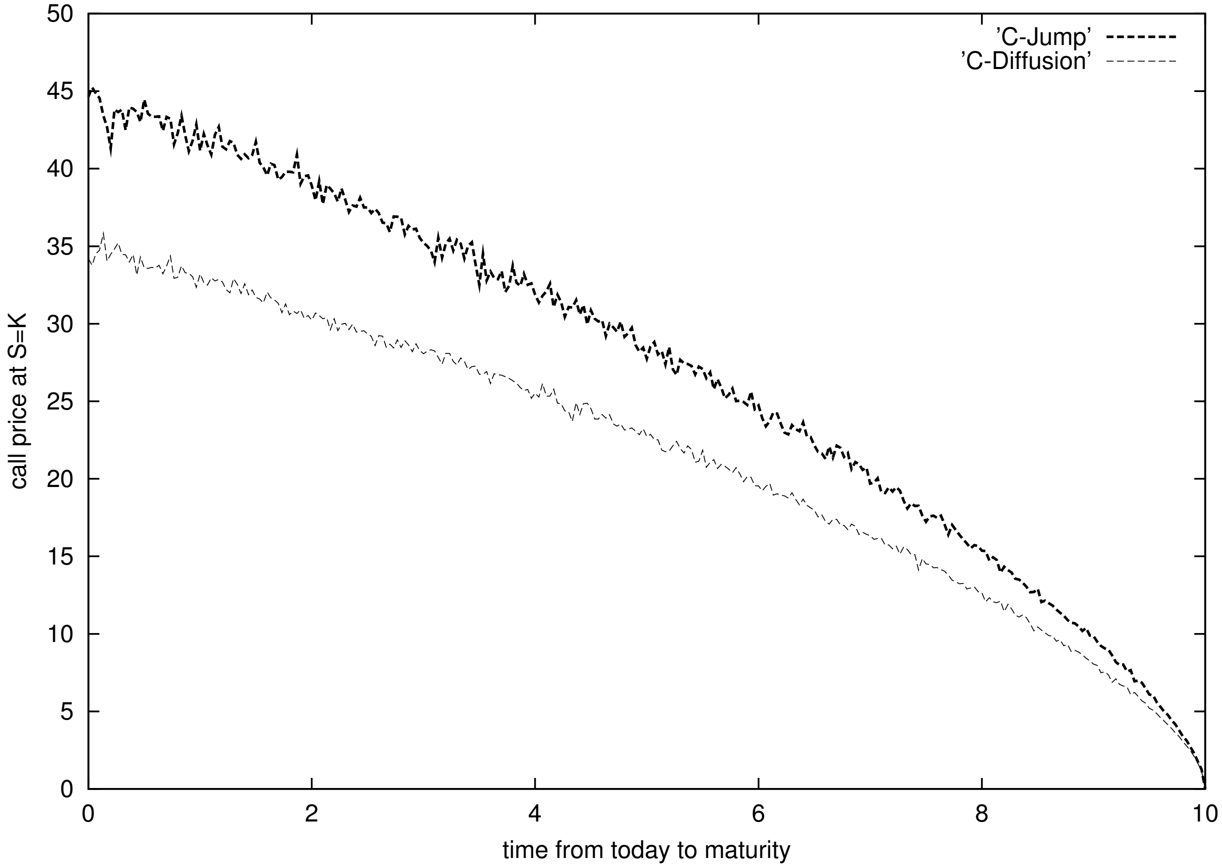

Figure 3: t **from today to maturity** $T = 10$

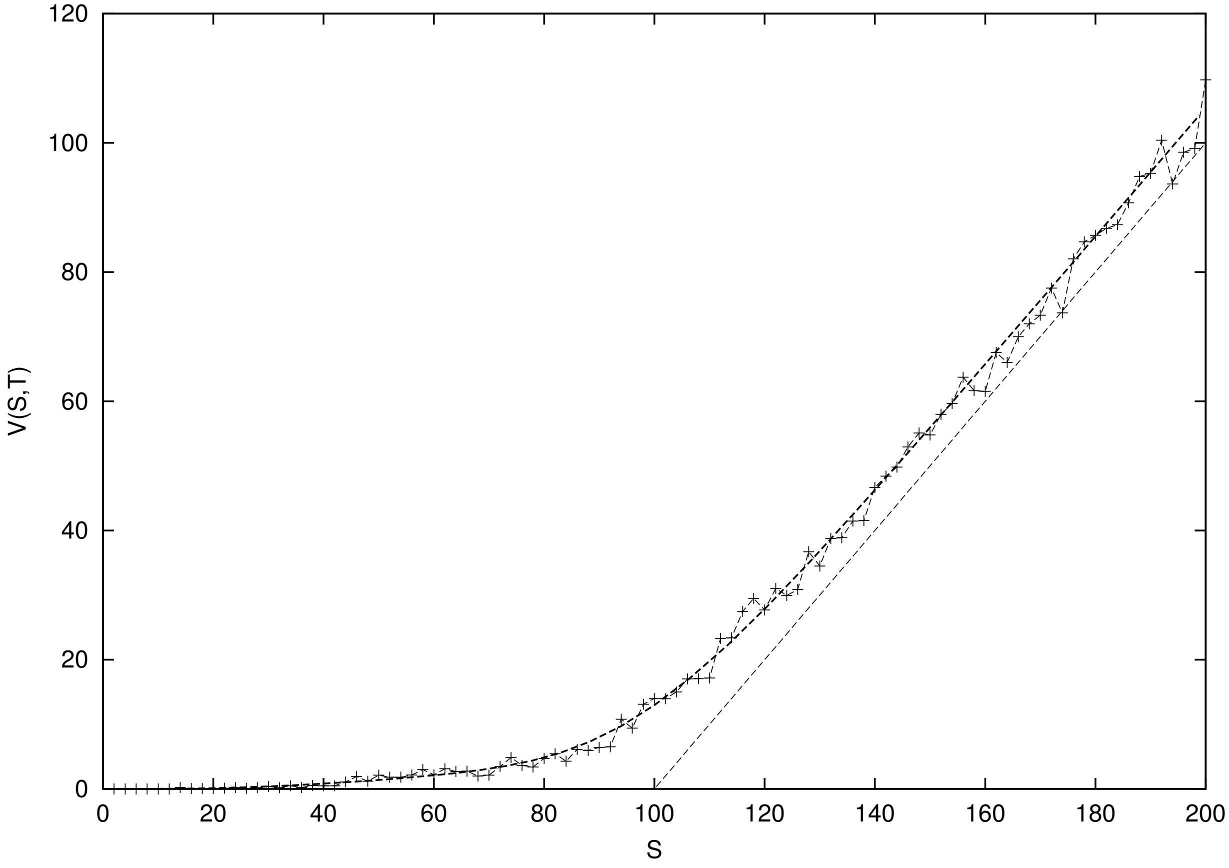

Figure 4: Call option price computed by the Midpoint-122 approximation ($-$) and by the Monte Carlo algorithm ($\times$)

- **In Diffusion models $C_t = V(S_t, t)$ (value of the European option) given by a** Parabolic Differential Equation**:**

$$\boxed{\frac{\partial V}{\partial t}(S,t) + rS\frac{\partial V}{\partial S}(S,t) + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2}(S,t) - rV(S,t) = 0}$$

- **In Jump-Diffusion models $C_t = V(S_t, t)$ is given by a** Partial Integro-Differential Equation

$$\frac{\partial V}{\partial t}(S,t) + (r - \lambda\mathbb{E}[\eta - 1])S\frac{\partial V}{\partial S}(S,t) + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2}(S,t) - rV(S,t)$$

$$+ \underbrace{\lambda\left(\int_0^{+\infty} V(S\eta)\tilde{\Gamma}(\eta)d\eta - V\right)}_{\mathcal{J}V} = 0$$

with boundary conditions depending on the type of option considered.

JUMPS $\Longrightarrow$ INTEGRAL TERM IN THE EQUATION

- $\mathcal{J}V$ **is a nonlocal term: it depends on the whole solution** $V(\cdot, t)$ **and not only on its behavior at the point** S**.**

$\Longrightarrow$ **new theoretical and numerical issues**

PLAN

- **(quite short) analytical backgrounds**
- **A general numerical convergence result**
- **Numerical algorithms (numerical boundary conditions...)**
- **High order (in time) Implicit-Explicit methods**
- **Well-balanced schemes for nonlocal Pbs.**

▶ **Uniformly parabolic problems: Existence and uniqueness of classical solutions for $\sigma > 0$ (Garroni-Menaldi 1995)**

▶ **Problems:**

- **degeneracy of coefficients (*incomplete market*)**
- **non linearity (*large investor economy*: the interest rate r is influenced by the agents)**
- **in general, solutions are not regular**

▶ **The class of *viscosity solutions* is sufficiently large to allow for existence of solutions in the cases where smooth solutions do not exist, it is sufficiently small to obtain uniqueness, using the *comparison principle***

$$\boxed{\begin{cases} \partial_t u - \mathcal{L}_{\mathcal{I}}(x,t,\mathcal{I},\mathcal{D},\mathcal{D}^2)u + H(x,t,\mathcal{D}u,\mathcal{I}u) = 0 \\ u(x,0) = u_0(x) \end{cases}}$$

- $u_0 \in \mathcal{C}(\mathbb{R}^d)$;
- $\mathcal{L}_{\mathcal{I}}$ **linear degenerate elliptic operator:** $\mathcal{L}_{\mathcal{I}}(x,t,\mathcal{I},\mathcal{D},\mathcal{D}^2)u = \frac{1}{2}tr[\sigma\sigma^T(x,t)\mathcal{D}^2u] + (\mu-\gamma k)^T(x,t)\mathcal{D}u - \mathcal{I}u$;
- H **nonlinear first order operator;**
- $\mathcal{I}u = \int_{\mathbb{R}^d} M(u(x+z,t),u(x,t))\mu_{x,t}(dz)$;

  $\mu_{x,t}$ **positive bounded measure;** M **continuous function, s.t.:**

  $M(u,v) \le M(w,v)$ **if** $u \le w$; $M(u,u) = 0$;

  $M(u,v) - M(w,z) \le c((u-w)_+ + |v-z|)$.

▶ Amadori (2000): Existence and uniqueness for Bounded Lévy processes

▶ Amadori, La Chioma, Karlsen (2004): General Lévy processes

$$F(x,t,u,\mathcal{I}u,\mathcal{D}u,\mathcal{D}^2u) = -\mathcal{L}_{\mathcal{I}}(x,t,\mathcal{I},\mathcal{D},\mathcal{D}^2)u + H(x,t,\mathcal{D}u,\mathcal{I}u)$$

Problem

$$\partial_t u + F(x,t,u,\mathcal{I}u,\mathcal{D}u,\mathcal{D}^2u) = 0. \tag{1}$$

Numerical grid in $\mathbb{R}^d \times [0,T]$

- $\mathbf{h} \in \mathbb{R}^d$**,** $\mathbf{k} \in \mathbb{R}$**: space, time grid steps.**
- $(\mathbf{x_j},\mathbf{t_n}) = (jh,nk)$**,** $j \in \mathbb{Z}^d$**,** $n \in \mathbb{N}$**,** $\mathbf{v}_j^n = v(\mathbf{x_j},\mathbf{t_n})$**,** $\tilde{\mathbf{v}}^n = (\mathbf{v}_j^n)_j$**.**
- $\mathcal{I}_h\tilde{\mathbf{v}}$ **integral approximation.**

Scheme

$$Q(h,k,j,n,v_j^n,\mathcal{I}_h\tilde{v},\tilde{v}) = 0. \tag{2}$$

$$\boxed{Q(h,k,j,n,v_j^n,\mathcal{I}_h\tilde{v},\tilde{v}) = 0}$$

$[H1]$ **Stability**

$[H2]$ **Consistency**

▶ $[H3]$ MONOTONICITY OF THE APPROXIMATIION OF THE INTEGRAL

$[H4]$ **Monotonicity ▶ also for the integral part**

$[H5]$ **Comparison Principle for the problem**

The scheme is said to be STABLE **if for a bounded initial condition, the solution v_j^n is uniformly bounded at all point of the grid, independetely from k, h:**

$$\exists C > 0,\ \forall k, h > 0,\ j \in \{0, ..., N\},\ n \in \{0, ..., M\} :\ |v_j^n| \leq C.$$

▶ *Stability ensures that the numerical solution does not blow up when* $(h, k) \to 0$

The scheme is said to be (locally) CONSISTENT **with the continuous equation if the discretized operator Q converges to its continuous version:** $\forall \psi \in \mathcal{C}^\infty(\mathbb{R} \times [0, T]), \forall (x, t) \in \mathbb{R} \times [0, T]$,

$$Q(h, k, j, n\psi_j^n, \mathcal{I}_h\tilde{\psi}, \tilde{\psi}) \to F(x, t, u, \mathcal{I}u, \mathcal{D}u, \mathcal{D}^2 u) \text{ as } (h, k) \to 0$$

MONOTONICITY OF THE INTEGRAL APPROXIMATION: **if** $v_j^n = w_j^n$ **and** $\tilde{v} \leq \tilde{w}$**, then**

$$\mathcal{I}_h \tilde{v} \leq \mathcal{I}_h \tilde{w}$$

MONOTONICITY PROPERTY: **if** $v_j^n = w_j^n$ **and** $\tilde{v} \leq \tilde{w}$**, then**

$$Q(h, k, j, n, v_j^n, \mathcal{I}_h \tilde{v}, \tilde{v}) \geq Q(h, k, j, n, w_j^n, \mathcal{I}_h \tilde{w}, \tilde{w})$$

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$\Rightarrow$ DISCRETE COMPARISON PRINCIPLE: **if w^0, v^0 are two initial conditions, then**

$$w^0 \geq v^0 \Rightarrow \forall n \geq 1,\ w^n \geq v^n$$

▶ *It has an important financial interpretation: it is equivalent to say that the option values computed using the scheme verify* **arbitrage inequalities***: inequalities between payoff lead to inequalities between values of options*

Theorem Let assumption (H1)–(H5) hold true. Then, as $(h, k) \to 0$, the solution $\tilde{u}$ of (2) converges locally uniformly to the unique continuous viscosity solution

- **Convergence - purely second order problems: Barles, Souganidis (1991)**
- **M. Briani, C. La Chioma, R. Natalini (preprint 2001, Num. Matematik 2004)**
- **R. Cont, E. Voltchkova (preprint 2003, linear case)**
- **C. La Chioma (2004, PhD Thesis, general Lévy)**

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) + a\dfrac{\partial u}{\partial x}(x,t) = b\dfrac{\partial^2 u}{\partial x^2}(x,t) - cu(x,t) + \mathcal{I}u(x,t) \quad (x,t) \in \mathbb{R} \times [0,T] \\ \\ u(x,0) = u_0(x) \end{cases}$$

$$\mathcal{I}u = \int_{-\infty}^{+\infty} [u(x+z,t) - u(x,t)]\Gamma_\delta(z)dz$$

with

$$\Gamma_\delta \geq 0 \quad \textbf{and} \quad \int_{\mathbb{R}} \Gamma_\delta(x)dx = 1.$$

option pricing - Merton Model

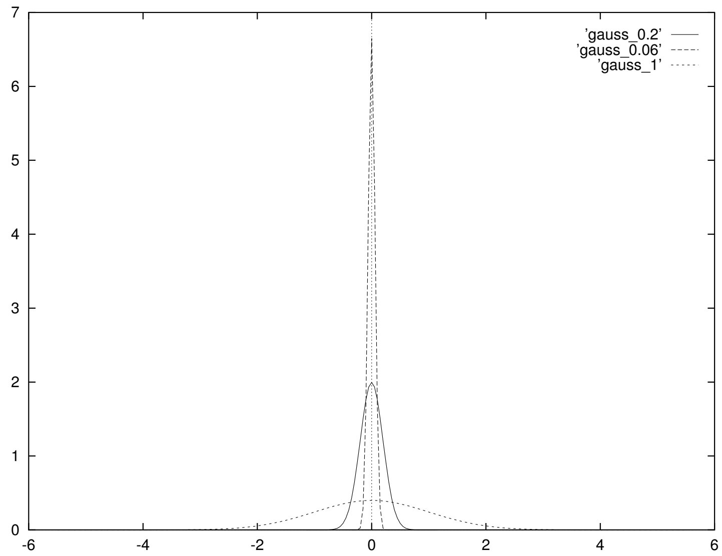
$$u_0(x) = (e^x - E)_+, \ \textbf{(call option)}; \quad \Gamma_\delta(x) = \lambda \frac{1}{\delta\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\delta^2}\right)$$

- **Truncation of the problem domain** $\longrightarrow [a, b] \in \mathbb{R}$
- **Truncation of the integral domain** $\displaystyle\int_{-\infty}^{+\infty} \longrightarrow \int_{-z_\epsilon}^{+z_\epsilon}$
- **Difficulty: nonlocal nature of the integral term**

$\Rightarrow$ **relation between integral and problem domain (boundary conditions).**

$\Rightarrow$ **implicit time differencing scheme (unconditionally stable) not practicaly feasible.**

$$\mathcal{I}u = \int_{-\infty}^{+\infty} [u(x+z,t) - u(x,t)]\Gamma_\delta(z)dz, \quad \Gamma_\delta(z) = \frac{1}{\delta\sqrt{2\pi}} \exp\left(-\frac{z^2}{2\delta^2}\right)$$



$$\Gamma_\delta(z) \geq \varepsilon \quad \Longleftrightarrow \quad z \in [-z_\epsilon, z_\epsilon], \quad z_\epsilon = \sqrt{-2\delta^2 \log(\varepsilon\delta\sqrt{2\pi})}$$

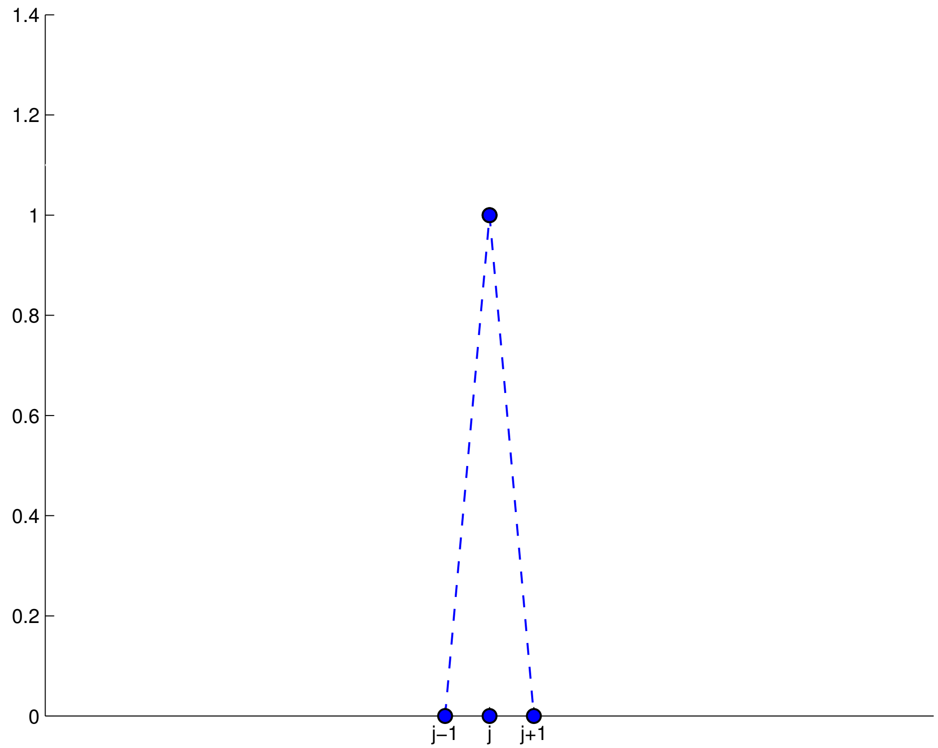
if $u(;t) \in Lip(\mathbb{R})$

$$\left| \int_{-\infty}^{+\infty} [u(x+z,t) - u(x,t)]\Gamma_\delta(z)dz - \int_{-z_\epsilon}^{+z_\epsilon} [u(x+z,t) - u(x,t)]\Gamma_\delta(z)dz \right| \leq c\delta^2\epsilon$$

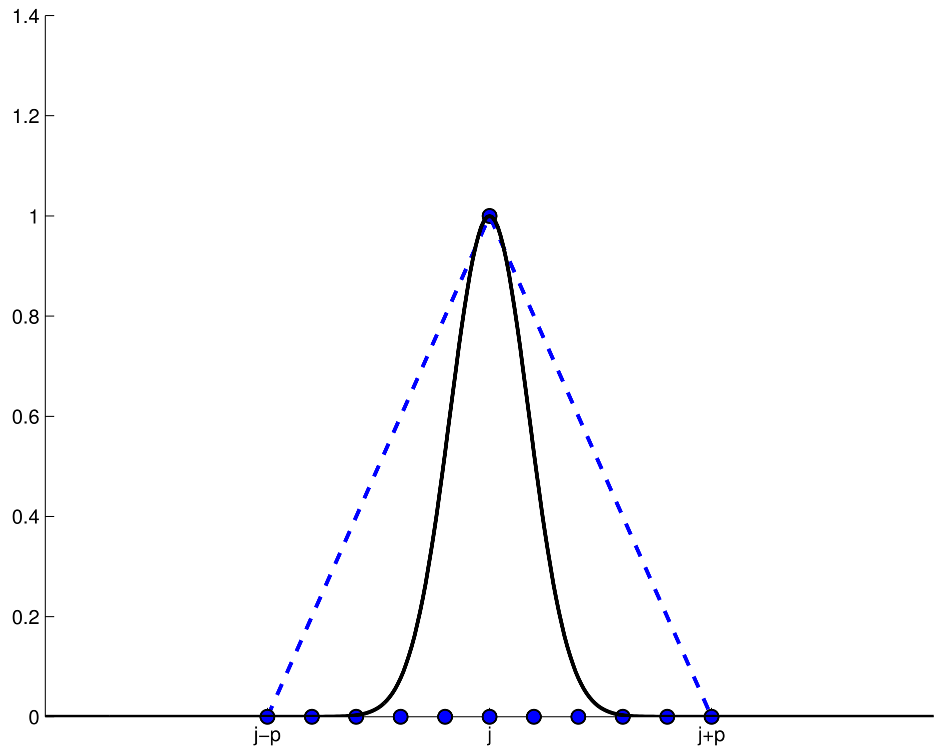
For $j = 0, ..., M$**,**

$$\boxed{\begin{aligned}\frac{dv_j}{dt}(t) \quad &= \quad \left(\frac{b}{h^2} + \frac{a}{2h}\right) v_{j-1}(t) + \left(\frac{2b}{h^2} + c\right) v_j(t) + \left(\frac{b}{h^2} - \frac{a}{2h}\right) v_{j+1}(t) \\ &\qquad + \lambda \Big[h \sum_{l=-p}^{p} \beta_l v_{j+l}(t) (\Gamma_\delta)_l + h \sum_{i<-p, i>p} \alpha_i v_{j+i}(t) (\Gamma_\delta)_i - v_j(t) \Big],\end{aligned}}$$

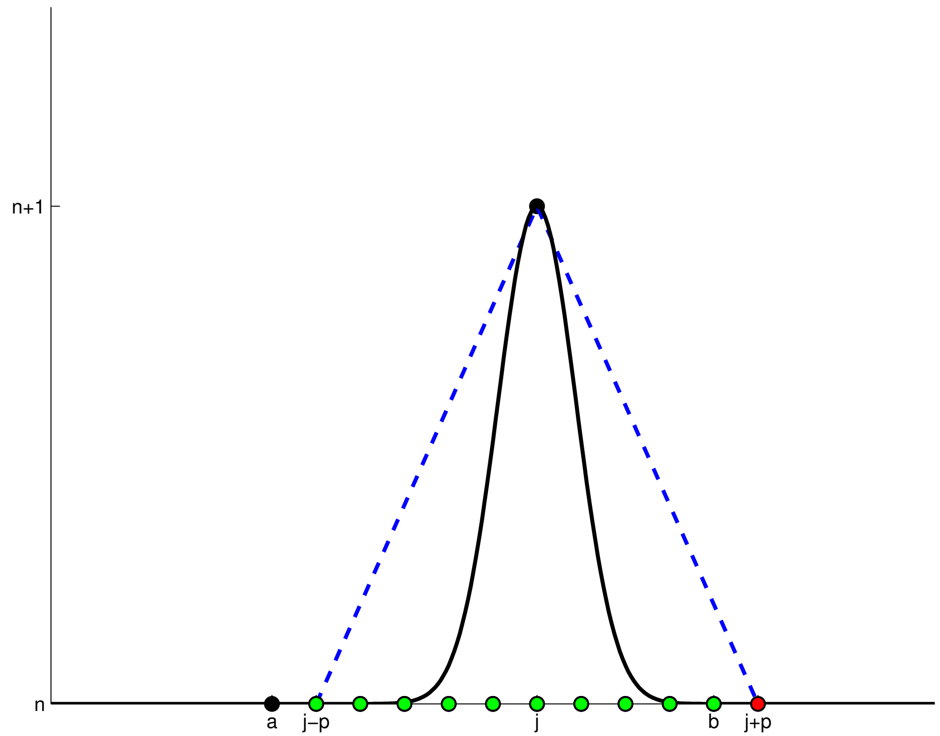
$$\frac{dv_j}{dt} = kw_{-1}v_{j-1} + \left(1 - kw_0\right)v_j + kw_1v_{j+1}$$



$$+\lambda k h \sum_{i=-p}^{p} \alpha_i v_{j+i} (\Gamma_\delta)_i$$



$$+\lambda kh \sum_{i=-p}^{p} \alpha_i v_{j+i} (\Gamma_\delta)_i + \lambda kh \sum_{i>p} \alpha_i \overset{?}{\underset{v_{j+i}}{}} (\Gamma_\delta)_i,$$



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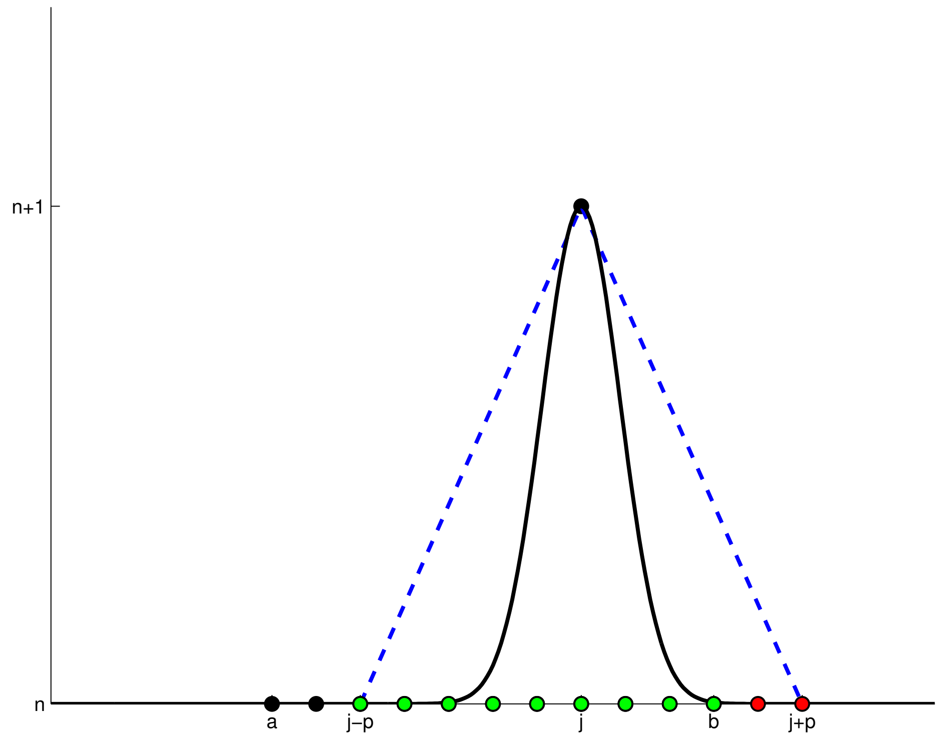

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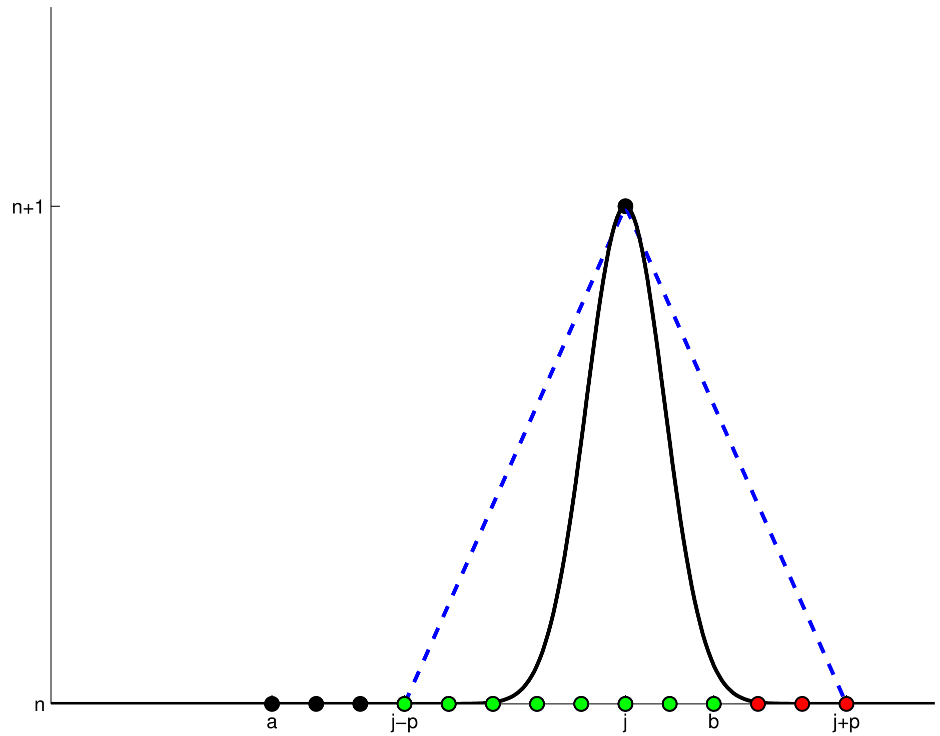
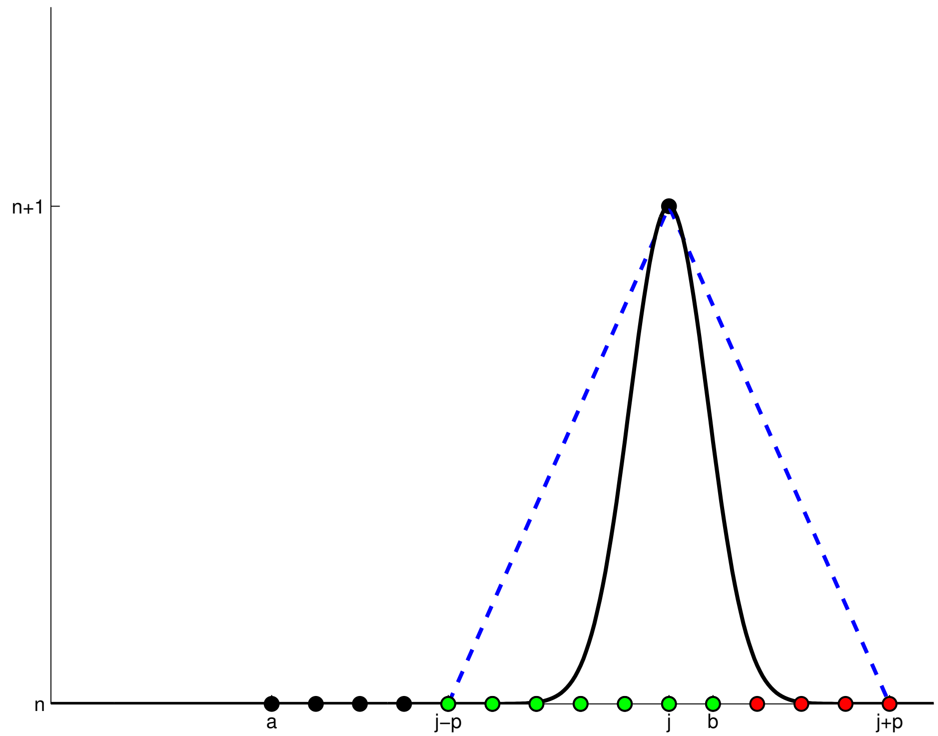
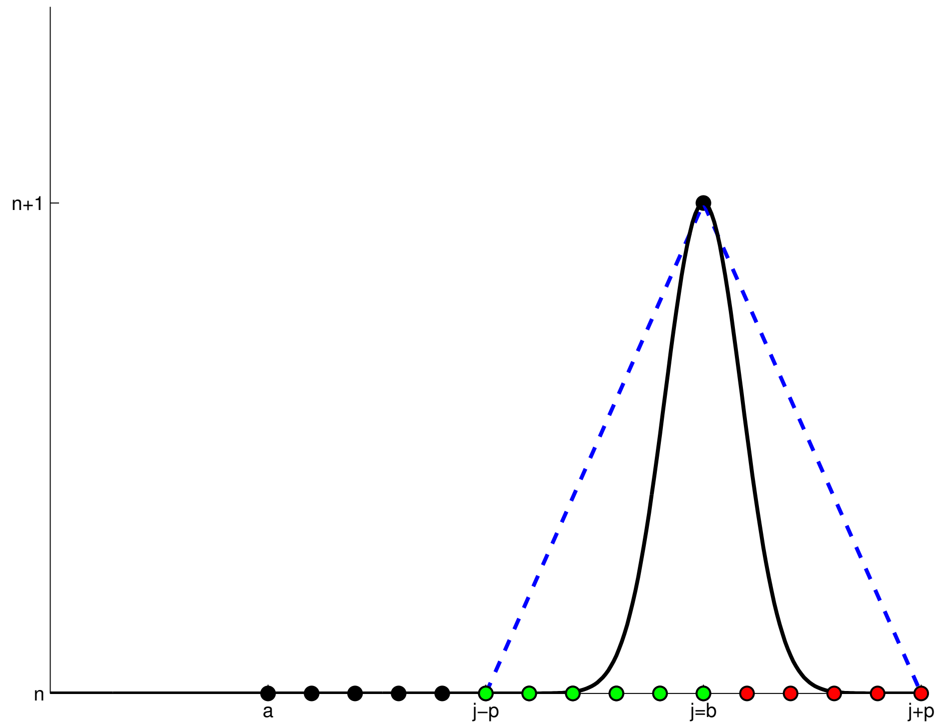


Numerical Domain

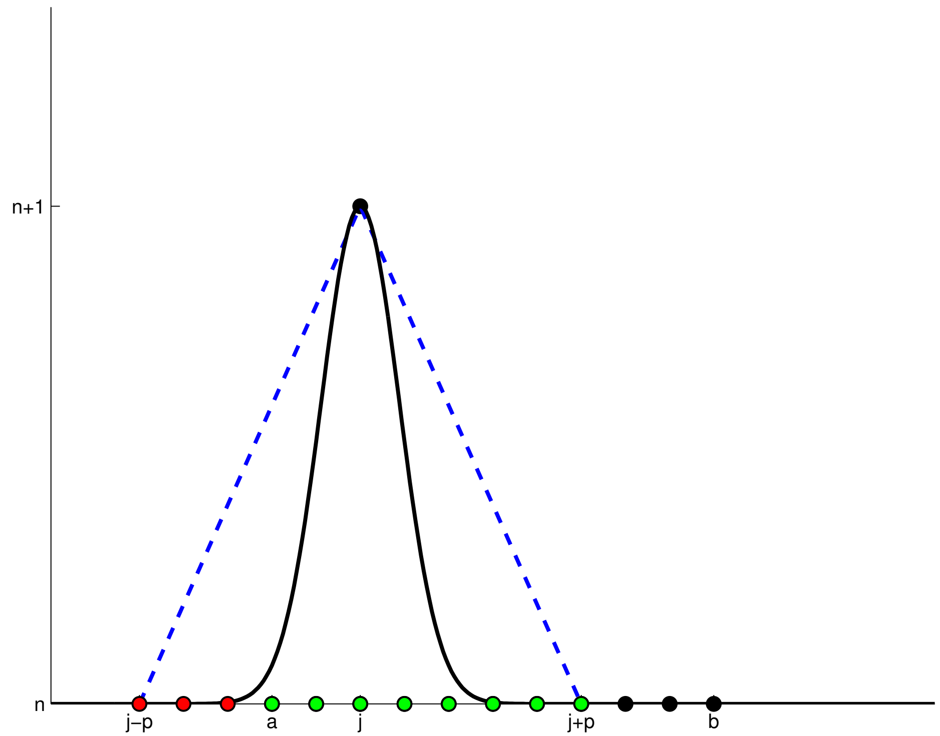
$$+\lambda k h \sum_{i=-p}^{p} \alpha_i v_{j+i} (\Gamma_\delta)_i + \lambda k h \sum_{i>p} \alpha_i \underset{v_{j+i}}{\overset{?}{}} (\Gamma_\delta)_i,$$



$$+\lambda kh \sum_{i=-p}^{p} \alpha_i v_{j+i} (\Gamma_\delta)_i + \lambda kh \sum_{i>p} \alpha_i \underset{v_{j+i}}{\overset{?}{}} (\Gamma_\delta)_i,$$



$$+\lambda kh \sum_{i=-p}^{p} \alpha_i v_{j+i} (\Gamma_\delta)_i + \lambda kh \sum_{i<-p} \alpha_i \overset{?}{\underset{v_{j+i}}{}} (\Gamma_\delta)_i,$$



- **We are going to approximate our one space dimensione operator**

$$u_t + au_x - bu_{xx} + cu = \frac{1}{\delta\sqrt{2\pi}} \int_{-\infty}^{\infty} [u(x+z,t) - u(x,t)] \exp{(-\frac{z^2}{2\delta^2})} dz$$

by using tha Taylor expansion of the integral term

$$v_t + av_x - bv_{xx} + cv = \frac{\lambda\delta^2}{2} v_{xx}$$

where $\lambda = \int_{-\infty}^{\infty} z^2 \exp{(-\frac{z^2}{2\delta^2})} dz$

- **If** $\delta \ll 1$**, and** $u(x,0) = v(x,0) = u_0(x) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$**,**

$$||u - v||_{L^\infty(0,T;L^1(\mathbb{R}))} \leq O(T\delta^3).$$

$$u_t(x,t) + \mathcal{H}(Du(x,t), \mathcal{I}u(x,t)) = \mathcal{G}(u(x,t), D^2u(x,t))$$

Euler forward: $\dfrac{u_j^{n+1} - u_j^n}{k} + \mathcal{H}(u_j^n, \mathcal{I}_h u_j^n) = \mathcal{G}(u_j^n)$

- **The scheme is second order in time but stable under the parabolic CFL condition:**

$$k \le ch^2 \rightsquigarrow h = 1/100 \Rightarrow k = 1/10000 \ \textbf{!!!}$$

Euler backward: $\dfrac{u_j^{n+1} - u_j^n}{k} + \mathcal{H}(u_j^{n+1}, \mathcal{I}_h u_j^{n+1}) = \mathcal{G}(u_j^{n+1})$

- **The scheme is unconditionally stable but not practicaly feasible:**

$$\mathcal{I}_h \tilde{u}^{n+1} \rightsquigarrow \textbf{dense system !!!}$$

IMEX (Implicit-Explicit) technique

▶ **IMEX technique has been introduced for time dependent partial-differential equations that involve terms of different types.**

▶ **IMEX schemes have been widely used for the time integration of spatially discretized PDEs of diffusion-convection type.**

▶ **Istances of these methods have been successfully applied to the Navier-Stokes equations ...**

▶ **For recent developments:** L. Pareschi and G. Russo

$$u_t(x,t) + \mathcal{H}(Du(x,t), \mathcal{I}u(x,t)) = \mathcal{G}(u(x,t), D^2u(x,t))$$

Implicit in $\mathcal{G}(u(x,t), D^2u(x,t)) \longrightarrow$ **To avoid parabolic CFL condition**

To avoid dense system $\longleftarrow$ **Explicit in** $\mathcal{H}(Du(x,t), \mathcal{I}u(x,t))$

Main goal $\longrightarrow$ **second (or higher) order accuracy in time**

- **The Midpoint$(1,2,2)$ scheme**

$$\begin{cases} u^{(2)} = u^n - \dfrac{k}{2}\mathcal{H}(Du^n, \mathcal{I}u^n) + \dfrac{k}{2}\mathcal{G}(u^{(2)}, D^2u^{(2)}) \\ \\ u^{n+1} = u^n - k\mathcal{H}(Du^{(2)}, \mathcal{I}u^{(2)}) + k\mathcal{G}(u^{(2)}, D^2u^{(2)}) \end{cases}$$

- **A two-stage, third-order DIRK scheme**

$$\begin{cases} u^{(2)} & = & u^n - k\gamma\mathcal{H}(Du^n, \mathcal{I}u^n) + k\gamma\mathcal{G}(u^{(2)}, D^2u^{(2)}) \\ \\ u^{(3)} & = & u^n - k(\gamma-1)\mathcal{H}(Du^n, \mathcal{I}u^n) - 2k(\gamma-1)\mathcal{H}(Du^{(2)}, \mathcal{I}u^{(2)}) \\ & & +k(1-2\gamma)\mathcal{G}(u^{(2)}, D^2u^{(2)}) + k\gamma\mathcal{G}(u^{(3)}, D^2u^{(3)}) \\ \\ u^{n+1} & = & u^n - \dfrac{k}{2}\Big(\mathcal{H}(Du^{(2)}, \mathcal{I}u^{(2)}) + \mathcal{H}(Du^{(3)}, \mathcal{I}u^{(3)})\Big) \\ & & +\dfrac{k}{2}\Big(\mathcal{G}(u^{(2)}, D^2u^{(2)}) + \mathcal{G}(u^{(3)}, D^2u^{(3)})\Big) \end{cases}$$

$$u_t(x,t) + \mathcal{H}(Du(x,t), \mathcal{I}u(x,t)) = \mathcal{G}(u(x,t), D^2u(x,t)). \qquad (3)$$

- **Time approximation, for** $i = 1, ..., \nu$

$$\begin{cases} u^{(i)} = u^n - k \sum_{j=1}^{\nu} \tilde{a}_{ij} \mathcal{H}(Du^{(j)}, \mathcal{I}u^{(j)}) + k \sum_{j=1}^{\nu} a_{ij} \mathcal{G}(u^{(j)}, D^2u^{(j)}) \\ \\ u^{n+1} = u^n - k \sum_{i=1}^{\nu} \tilde{\omega}_i \mathcal{H}(Du^{(i)}, \mathcal{I}u^{(i)}) + k \sum_{i=1}^{\nu} \omega_i \mathcal{G}(u^{(i)}, D^2u^{(i)}) \end{cases} \qquad (4)$$

The matrices $\tilde{A} = (\tilde{a}_{ij})$**,** $\tilde{a}_{ij} = 0$ **for** $j \geq i$ **and** $A = (a_{ij})$ **are** $\nu \times \nu$ **matrices such that the resulting scheme is explicit in** H **and implicit in** G**.**

- **The Midpoint$(1,2,2)$ scheme**

$$\tilde{A} = (\tilde{a}_{ij}) = \begin{pmatrix} 0 & 0 \\ 1/2 & 0 \end{pmatrix} \qquad A = (a_{ij}) = \begin{pmatrix} 0 & 0 \\ 0 & 1/2 \end{pmatrix}$$

$$(\tilde{\omega}_1, \tilde{\omega}_2) = (0, 1) \qquad (\omega_1, \omega_2) = (0, 1).$$

- **A two-stage, third-order scheme ARS-233 (Ascher-Ruuth-Spiteri) ($\gamma = (3 + \sqrt{6})/3$)**

$$\tilde{A} = (\tilde{a}_{ij}) = \begin{pmatrix} 0 & 0 & 0 \\ \gamma & 0 & 0 \\ \gamma - 1 & 2 - 2\gamma & 0 \end{pmatrix} \qquad A = (a_{ij}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 1 - 2\gamma & \gamma \end{pmatrix}$$

$$(\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3) = (0, 1/2, 1/2) \qquad (\omega_1, \omega_2, \omega_3) = (0, 1/2, 1/2).$$

► VON NEUMANN STABILITY ANALYSIS: **represents v^n as a discrete Fourier transform and requires the corresponding Fourier coefficients to be stable (**$\longrightarrow$ **essentially equivalent to stability in L^2 norm)**

$\Rightarrow$ **advancing the solution of the scheme by one time step is equivalent to multiplying the Fourier transform of the solution by the** AMPLIFICATION FACTOR $g(h\xi)$.

$$\hat{u}^n(\xi) = g(h\xi)\hat{u}^0(\xi).$$

Theorem (von Neumann) **A one-step finite difference scheme is stable if and only if there is a constant K and some time and space steps k_0 and h_0 s.t.:**

$$| \, g(\theta, k, h) \, | \leq 1 + Kk$$

If $g(\theta, k, h)$ is independent of h and k, one can choose $K = 0$

Proposition **Given a stable one-step finite difference scheme for the differential operator, the scheme is stable for every bounded additional term (e.g.: Integral operator), for small values of the mesh size.**

$$\boxed{g(\theta;h,k) = 1 + k(H(\theta;h)\tilde{\omega}^T + G(\theta;h)\omega^T)(I_\nu - k\tilde{\mathcal{A}}_\tau - k\mathcal{A}_\tau)^{-1}e.}$$

$$e = (1,...,1)^T, \quad U = (\hat{u}^{(1)},...,\hat{u}^{(\nu)})^T, \quad U^n = (\hat{u}^n,...,\hat{u}^n)^T$$

$$\mathcal{A}_\tau = AG(\theta;h) \qquad \tilde{\mathcal{A}}_\tau = \tilde{A}H(\theta;h)$$

▶ **For the Merton's model, with centered finite-difference approximation,**

$$H(\theta;h) = -iF + J = -i\frac{a}{h}\sin\theta + \lambda\Big(\int_{-\infty}^{+\infty} e^{iz\theta/h}\Gamma_\delta(z)dz - 1\Big),$$

$$G(\theta;h) = \frac{2b}{h^2}(\cos\theta - 1). \tag{5}$$

[Implicit-Explicit Midpoint(1,2,2)] The scheme is second order in time.

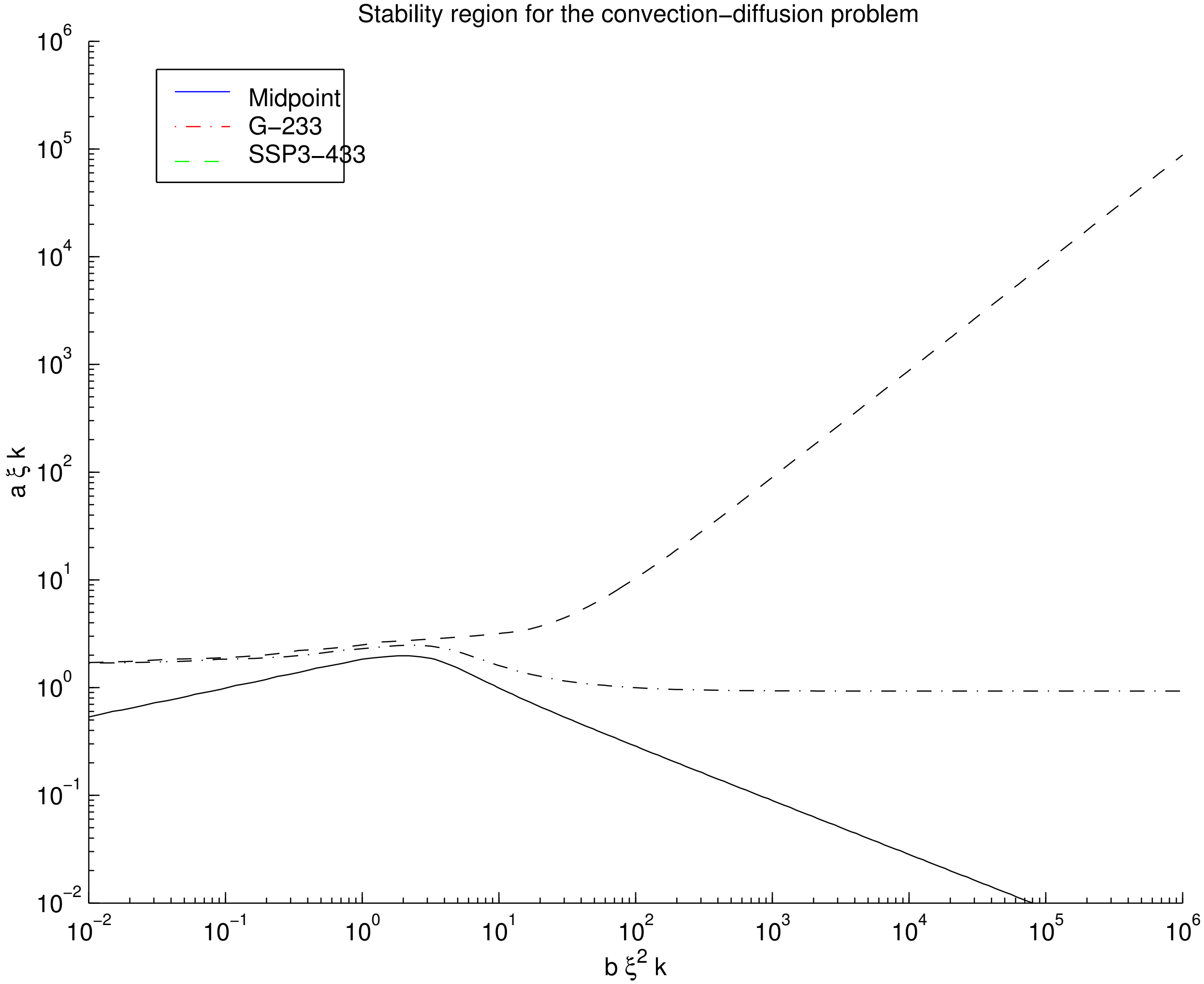
The Midpoint scheme is stable for

$$k \leq \frac{3}{2(a^2 b)^{1/3}} h^{4/3}. \tag{6}$$

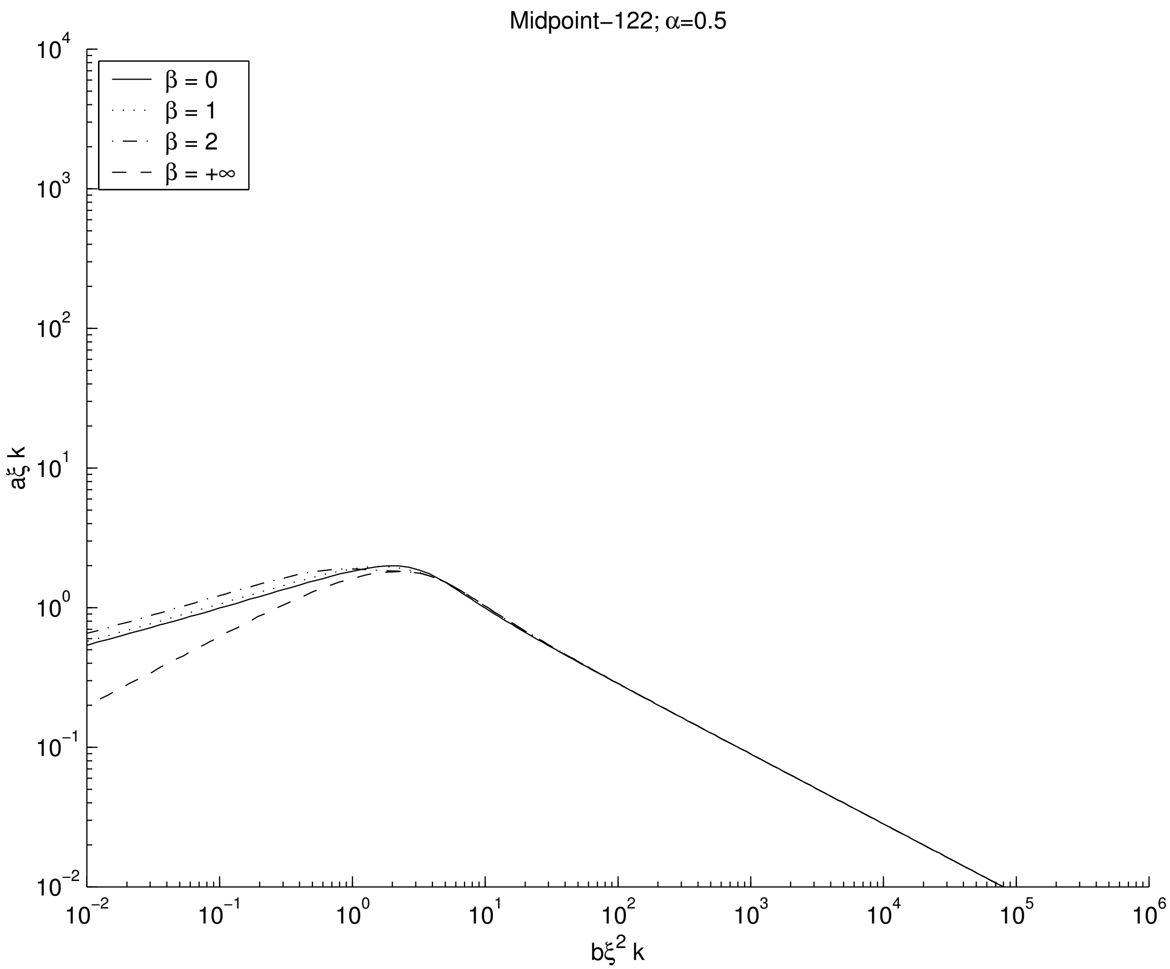
[ARS(2,3,3)] (Ascher-Ruuth-Spiteri) Fix $\gamma = (3 + \sqrt{6})/3$. The scheme is third order in time.

The scheme is stable for

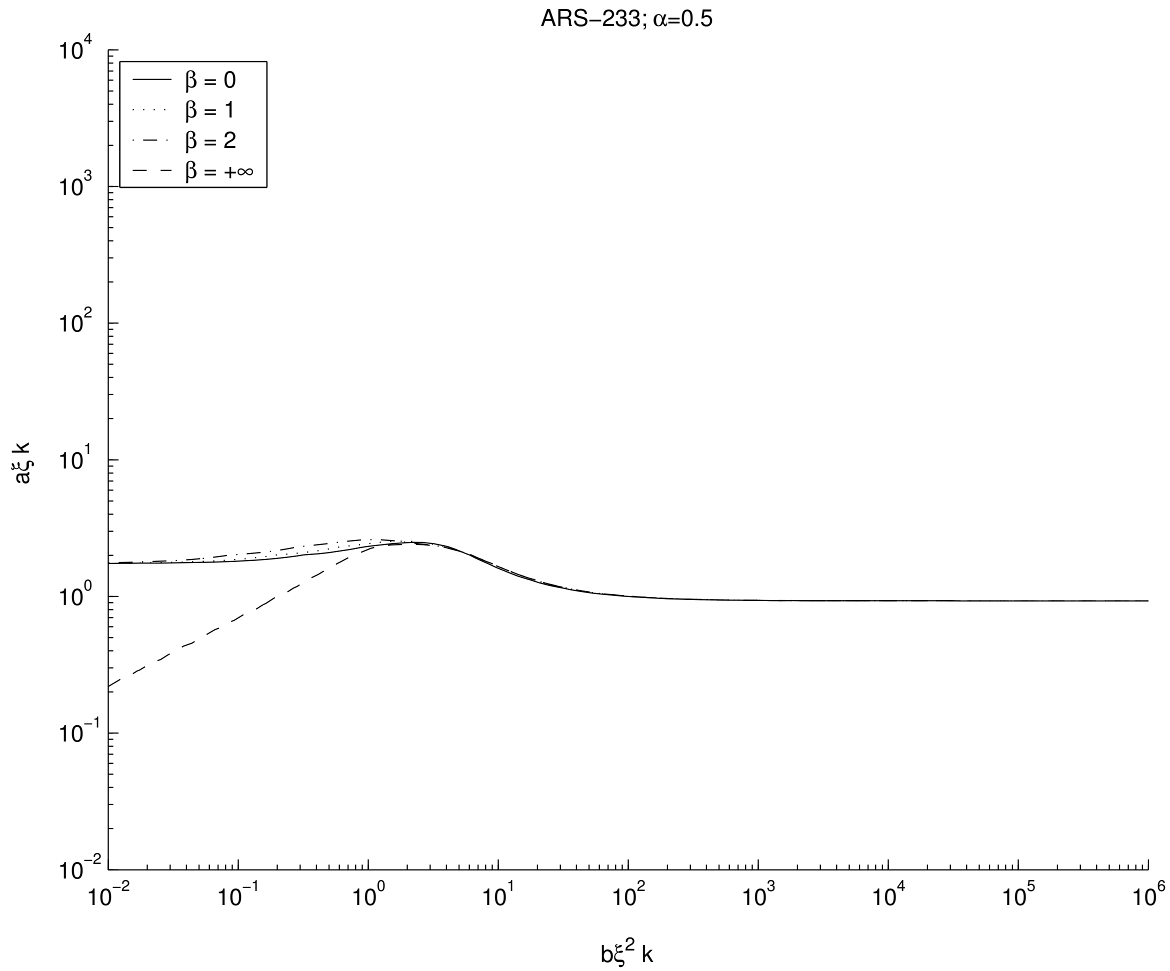
$$k \leq \frac{\sqrt{4\gamma - 1}}{2\gamma} \frac{h}{a}. \tag{7}$$



Advection-diffusion problem: border of the stability regions in the $b\xi^2 k$-$a\xi k$ plane.



Integro-differential problem: stability regions of Midpoint scheme in the $b\xi^2 k$-$a\xi k$ plane. $\alpha = \lambda b/a^2 = 0.5$ and $\beta = \delta a/b$



Integro-differential problem: stability regions of ARS-233 scheme in the $b\xi^2 k$-$a\xi k$ plane. $\alpha = \lambda b/a^2 = 0.5$ and $\beta = \delta a/b$

Proposition 0.1 (Order conditions) ***Let*** H ***an*** G ***be two given consistent space approximation of accuracy order*** $O(h^\beta)$***, i.e.***

$$H(h\xi;h) + G(h\xi;h) = -q_1(\xi) - q_2(\xi) + O(h^\beta) = -q(\xi) + O(h^\beta)$$

The scheme is accurate of order $O(h^\beta + k^\alpha)$ ***if the two matrices*** $\tilde{A}$ ***and*** A ***and the coefficients vectors*** $\tilde{\omega} = (\tilde{\omega}_1, ..., \tilde{\omega}_\nu)^T$ ***and*** $\tilde{\omega} = (\omega_1, ..., \omega_\nu)^T$ ***verify for*** $j = 1, ..., \alpha$

$$\boxed{\Big(q_1(\xi)\tilde{\omega}^T + q_2(\xi)\omega^T\Big)\left(\frac{d^{j-1}\Lambda(k)}{dk^{j-1}}\right)_{k=0} = (-1)^{j+1}\frac{\Big(q_1(\xi) + q_2(\xi)\Big)^j}{j},}$$

where $\Lambda(k) = (I_\nu + k\tilde{A}q_1(\xi) + kAq_2(\xi))^{-1}e$***.***

- *first order* $(\alpha = 1)$

$$(\omega_1 + \omega_2)\, q_2 + (\tilde{\omega}_1 + \tilde{\omega}_2)\, q_1 = q_1 + q_2$$

- *second order* $(\alpha = 2)$

$$\begin{aligned}&(\tilde{a}_{21}\tilde{\omega}_2)\, q_1^2 + ((a_{22} + a_{21})\, \omega_2 + \omega_1 a_{11})\, q_2^2 \\ &+ (\tilde{a}_{21}\omega_2 + (a_{22} + a_{21})\, \tilde{\omega}_2 + \tilde{\omega}_1 a_{11})\, q_1 q_2 = \frac{q_1^2}{2} + \frac{q_2^2}{2} + q_1 q_2.\end{aligned}$$

IMEX-DIRK scheme			
time	**space**	**integral**	**computational cost**
M $\times$	**N** $\times$	$\mathcal{P}^2$ **or** $(3\mathcal{P}\log_2(\mathcal{P})+\mathcal{P})$	$= O(MN(3\mathcal{P}\log_2(\mathcal{P})+\mathcal{P}))$
	$+$		
	...		
	$+$		
	N $\times$	$\mathcal{P}^2$ **or** $(3\mathcal{P}\log_2(\mathcal{P})+\mathcal{P})$	

The explicit approximation			
time	**space**	**integral**	**computational cost**
M $\times$	**N** $\times$	$\mathcal{P}^2$ **or** $(3\mathcal{P}\log_2(\mathcal{P})+\mathcal{P})$	$= O(MN(3\mathcal{P}\log_2(\mathcal{P})+\mathcal{P}))$

N	explicit scheme	Midpoint-122	ARS-233	SSP
256	0.02s	0.04s	0.04s	0.07s
512	0.3s	0.35s	0.32s	0.24s
1024	4.4s	3.21s	2.33s	1.73s
2048	1m56,6s	29.58s	17.62s	13.48s

Table 1: CPU times on 1.6 GHz Pentium IV PC when $T = 1$.

$T = 1$ **Midpoint-122**					
h	**k**	l^1	γ_1	l^∞	γ_∞
0.125000	**0.066986**	**1.277415**		**1.126198**	
0.062500	**0.027205**	**0.456599**	**1.484227**	**0.551598**	**1.029771**
0.031250	**0.011049**	**0.140942**	**1.695827**	**0.150568**	**1.873204**
0.015625	**0.004487**	**0.034098**	**2.047359**	**0.026901**	**2.484656**
0.007812	**0.001822**	**0.019122**	**0.834463**	**0.017256**	**0.640585**

Table 2: Errors and convergence orders of the Midpoint-122 scheme The process parameters are $E = 100$, $r = 0$, $\sigma = 0.2$, $\lambda = 0.1$, $\delta = 0.5$.

$T = 5$ **Midpoint-122**					
h	**k**	l^1	γ_1	l^∞	γ_∞
0.125000	**0.066986**	**5.097443**		**0.606756**	
0.062500	**0.027205**	**1.095825**	**2.217757**	**0.188375**	**1.687508**
0.031250	**0.011049**	**0.296186**	**1.887442**	**0.069759**	**1.433159**
0.015625	**0.004487**	**0.093908**	**1.657181**	**0.019274**	**1.855694**
0.007812	**0.001822**	**0.031876**	**1.558776**	**0.002183**	**3.142337**

Table 3: Errors and convergence orders of the Midpoint-122 scheme The process parameters are $E = 100$**,** $r = 0$**,** $\sigma = 0.2$**,** $\lambda = 0.1$**,** $\delta = 0.5$**.**

$T = 1$ **ARS-233**					
h	**k**	l^1	γ_1	l^∞	γ_∞
0.125000	**0.125000**	**1.357048**		**1.408900**	
0.062500	**0.062500**	**0.334417**	**2.020753**	**0.318280**	**2.146199**
0.031250	**0.031250**	**0.090647**	**1.883314**	**0.072331**	**2.137617**
0.015625	**0.015625**	**0.028081**	**1.690643**	**0.017782**	**2.024171**
0.007812	**0.007812**	**0.010154**	**1.467582**	**0.004429**	**2.005380**

Table 4: Errors and convergence orders of the ARS-233 scheme The process parameters are $E = 100$, $r = 0$, $\sigma = 0.2$, $\lambda = 0.1$, $\delta = 0.5$.

$T = 5$ **ARS-233**					
h	**k**	l^1	γ_1	l^∞	γ_∞
0.125000	**0.125000**	**5.129721**		**0.584784**	
0.062500	**0.062500**	**1.056785**	**2.279198**	**0.138810**	**2.074792**
0.031250	**0.031250**	**0.245218**	**2.107543**	**0.034354**	**2.014543**
0.015625	**0.015625**	**0.076278**	**1.684720**	**0.008568**	**2.003464**
0.007812	**0.007812**	**0.031807**	**1.261932**	**0.002141**	**2.000850**

Table 5: Errors and convergence orders of the ARS-233 scheme The process parameters are $E = 100$, $r = 0$, $\sigma = 0.2$, $\lambda = 0.1$, $\delta = 0.5$.

N	M	Explicit		M	ARS-233	
		PUT	CALL		PUT	CALL
256	46	8.167357	13.579657	17	8.230058	12.861993
512	185	8.268306	13.378464	35	8.289777	13.449324
1024	743	8.319940	13.286915	70	8.326102	13.287427
V		8.341444	13.218501		8.341444	13.218501

Table 6: V is the analytical price of the European put and call option of Merton model. The process parameters are $T = 1$, $E = 100$, $r = 0.05$, $\sigma = 0.2$, $\lambda = 0.1$, $\delta = 0.8$, $x = \ln(100)$.

Well-balanced schemes for integro-differential problems

Work in progress (M. Briani, R. N.)

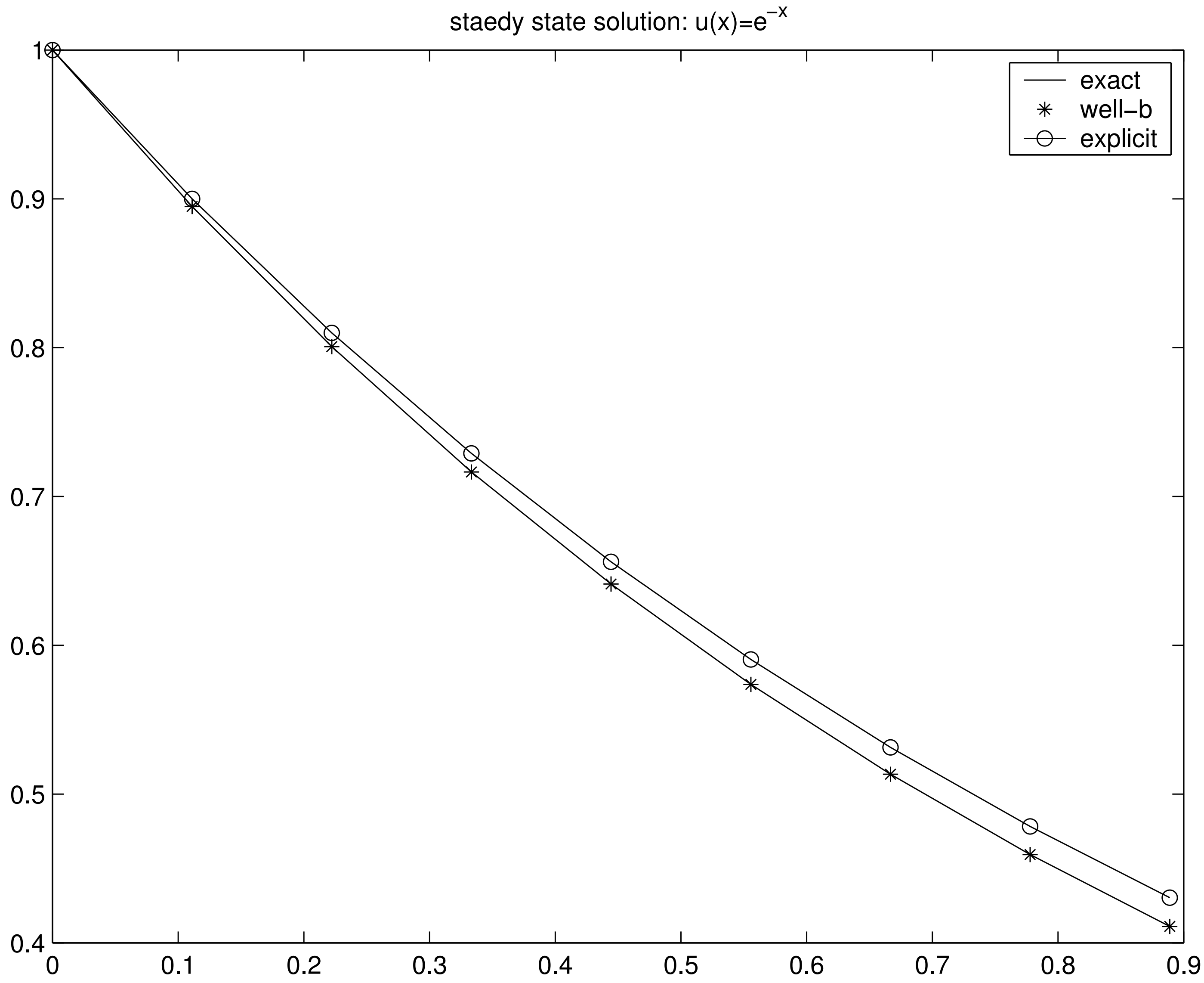
- **Problems to correctly resolve the source term**
- **Main idea: find a discrete solver which preserves the steady state solutions**
- **LeRoux, Greenberg, Gosse, Shi Jin, LeVeque, Perthame, Simeoni, Bouchut ...**

***Example*:** $\begin{cases} u_t + u_x + u = 0 \\ u_0(x) = e^{-x} \end{cases} \Rightarrow u(x,t) = u_0(x)$

Plain upwind
$$\frac{du_j}{dt} + \frac{u_j - u_{j-1}}{h} + u_j = 0$$

Well-balanced

(Botchorishvili-Perthame-Vasseur)
$$\frac{du_j}{dt} + \frac{u_j - e^{-h}u_{j-1}}{h} = 0$$



staedy state solution: u(x)=e^{−x}
exact
well−b
explicit
1
0.9
0.8
0.7
0.6
0.5
0.4
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9

▶ **A simple Integro Differential Equation: the Hammer model**

$$\boxed{u_t + \frac{1}{2}(u^2)_x = \int K(x-y)(u(y)-u(x))dy \qquad K(x) = \frac{1}{2}e^{-|x|}}$$

$$\Uparrow$$

$$\boxed{\begin{cases} u_t + \dfrac{1}{2}(u^2)_x = -q_x \\ -q_{xx} + q = -u_x \end{cases}}$$

simplified model describing the evolution of a radiating gas

- **Smooth solutions: S. Kawashima et al.**
- **Entropy solutions: P. Marcati et al.**
- **Asymptotic behavior: D.Serre**

$$\begin{cases} u_t + a u_x = K * u - u \\ u(x,0) = u_0(x) \end{cases} \qquad K(x) = \frac{1}{2} e^{-|x|};\ a > 0$$

Steady States

$$\{\ w(x) = e^{\lambda x} \quad \lambda = 0 \quad \lambda = \Lambda = \tfrac{-1+\sqrt{1+4a^2}}{2a}$$

For a given integral approximation:

$$K * u = \sum_{l=-m}^{m} \beta_l K_l u_{i+l} + O(h^d), \quad d \geq 1,$$

General Scheme:

$$\boxed{h \frac{du_i}{dt} + \sum_{l=-m}^{m} \alpha_l(h) u_{i+l} = 0}$$

Conditions on $\{\alpha_l(h)\}_l$:

- **Consistency**
- **Well-balanced:** $\displaystyle\sum_{l=-m}^{m} \alpha_l(h) e^{\Lambda_j l h} = 0 \qquad \forall j$

$$\begin{cases} \alpha_l(h) + h\beta_l K_l = 0 \qquad \forall\ l \neq -1, 0, 1, \\ \\ \alpha_{-1}(h) + \alpha_0(h) + \alpha_1(h) - h\left(1 - (\beta_{-1}K_{-1} + \beta_0 K_0 + \beta_1 K_1)\right) \\ \\ -\alpha_{-1}(h) + \alpha_1(h) - a = c_1 h \\ \\ \displaystyle\sum_{l=-m}^{m} \alpha_l(h) e^{\Lambda l h} = 0 \end{cases}$$

- **Monotonicity:** $\alpha_0 > a$

Take, as for the upwind approximation $\alpha_0 = h + a - h\beta_0 K_0$, then

$$\alpha_{-1} = \frac{\left(1 - e^{\Lambda\, h}\right)}{e^{\Lambda\, h} - e^{-\Lambda\, h}} a - \frac{h\left(-\Lambda^2 + e^{\Lambda\, h}\beta_{-1}K_{-1}(\Lambda^2-1) + \beta_{-1}K_{-1}\, e^{-\Lambda\, h}(1-\Lambda^2)\right)}{\left(e^{\Lambda\, h} - e^{-\Lambda\, h}\right)\left(\Lambda^2-1\right)}$$

$$\alpha_1 = \frac{\left(-1 + e^{-\Lambda\, h}\right)}{-e^{-\Lambda\, h} + e^{\Lambda\, h}} a - \frac{h\left(\Lambda^2 + e^{-\Lambda\, h}\beta_1 K_1 - e^{-\Lambda\, h}\beta_1 K_1\,\Lambda^2 - \beta_1 K_1\, e^{\Lambda\, h} + \beta_1 K_1\, e^{\Lambda\, h}\Lambda^2\right)}{\left(e^{\Lambda\, h} - e^{-\Lambda\, h}\right)\left(\Lambda^2-1\right)}$$

$$\begin{cases} u_t + au_x + u - K * u = 0 \\ u(x,0) = u_0(x) \end{cases} \qquad K(x) = \frac{1}{2}e^{-|x|};\ a > 0$$

$$\begin{cases} u_0(x) = e^{\Lambda x} + e^{\lambda x} & |\lambda| < 1 \\ u(x,t) = e^{\Lambda x} + e^{\delta t + \lambda x} & \delta = \frac{\lambda(a\lambda^2+\lambda-a)}{1-\lambda^2} \end{cases}$$

$$\delta < 0,\ \lim_{t\to+\infty} u(x,t) = w(x) = e^{\Lambda x}$$

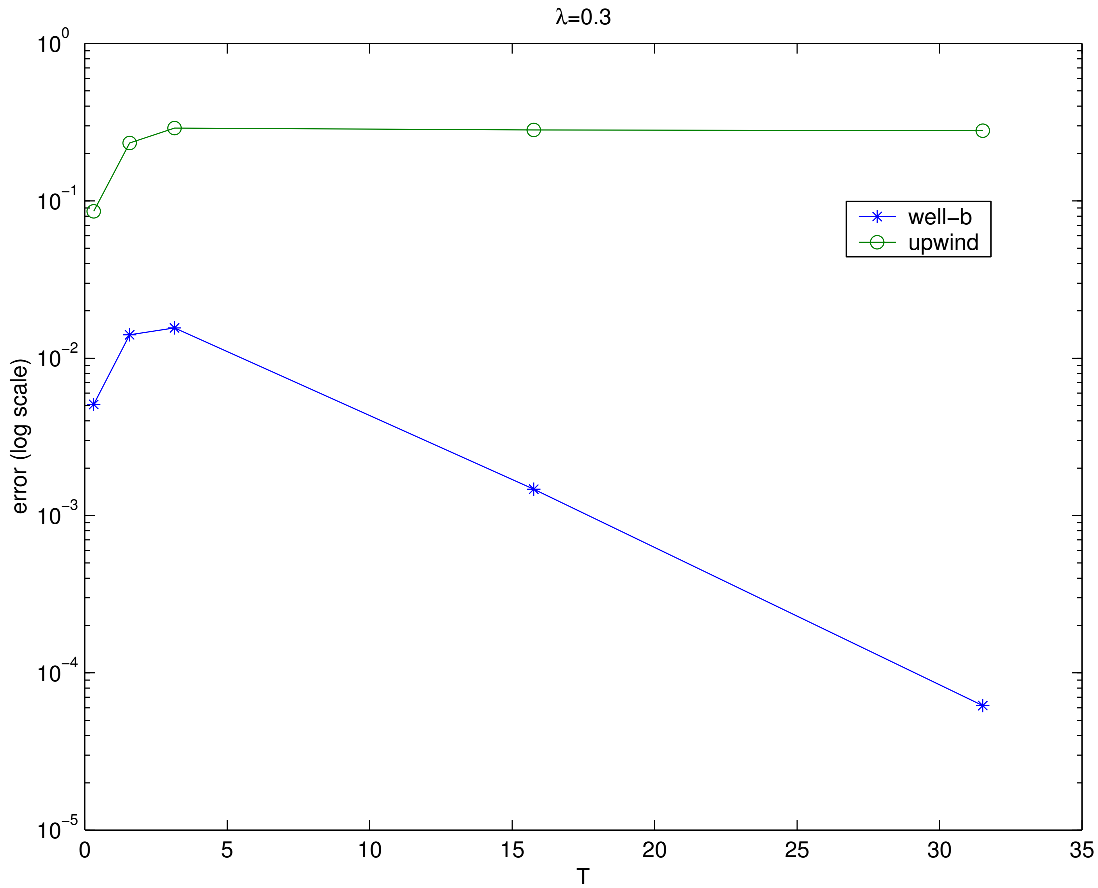


Figure 5: l^{∞} error in log. scale for increasing T

$$\begin{cases} u_t + au_x + u - K * u = 0 \\ u(x,0) = u_0(x) \end{cases} \qquad K(x) = \frac{1}{2}e^{-|x|};\ a > 0$$

$$\begin{cases} u_0(x) = e^{\Lambda x} + \sin(x)\Xi_{[0,\pi]} \\ \\ u(x,t) = e^{\Lambda x} + O(t^{-1/4}) \Leftarrow (D.Serre) \end{cases}$$

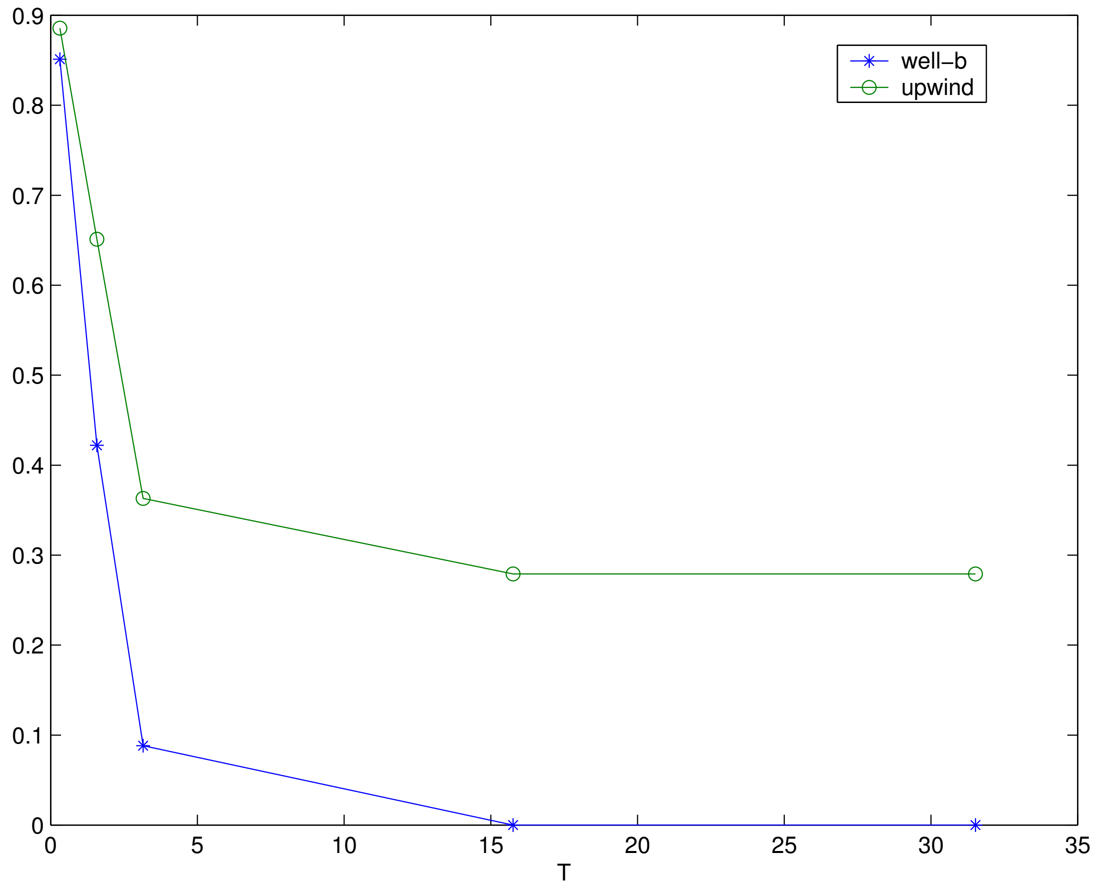


Figure 6: l^∞ distance between the numerical solution and the asymptotic solution $w(x)$ for increasing T.

- **Rigorous convergence result for monotone schemes for general Lévy processes**

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Further developments

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Further developments

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Conclusions

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▷ **...**

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- ▷ **...**

References

- **M. Briani, C. La Chioma, R. Natalini; Numerische Mathematik - 2004.**
- **M. Briani, G. Russo, R. Natalini;** *Implicit-Explicit numerical scheme for integro-differential parabolic problems etc...* **preprint 2004.**
- **M. Briani, R. Natalini;** *Well-balanced schemes for integro-differential equations* **(work in progress).**