Nonlinear Monte Carlo Methods:
From American Options to Fully Nonlinear PDEs

Nizar TOUZI

Ecole Polytechnique Paris

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Outline

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Pricing American Options in Complete Markets

In the context of a complete market with a nonrisky asset S^0 :

$$S_t^0 = e^{rt}, \qquad t \geq 0,$$

and a risky security with price process

$$\{S_t, \ t \geq 0\} \qquad \ldots$$

the no-arbitrage price of the American put option with strike $K > 0$ and maturity $T > 0$:

$$P_0 \;=\; \sup_{\tau \in \mathcal{T}_T} \mathbb{E}\left[e^{-r\tau}(K - S_\tau)^+\right] \;=\; \mathbb{E}\left[e^{-r\tau^*}(K - S_{\tau^*})^+\right]$$

where $\mathcal{T}_T = \{\text{stopping times with values in } [0, T]\}$ and

$$\tau^* = \min\left\{t \geq 0 : \; P_t = (K - S_t)^+\right\}$$

Discrete-time Approximation

- Let $t_i^n = ih_n$, $i = 1, \ldots, n$, and $h_n = \dfrac{iT}{n}$
- Define the so-called Snell envelope :

$$Y_T^n = (K - S_T)^+ \quad \text{and} \quad Y_{t_i^n}^n = \max\left\{ \left(K - S_{t_i^n}\right)^+ , \mathbb{E}_{t_i^n}\left[e^{-rh_n} Y_{t_{i+1}^n}^n \right] \right\}$$

- Then, an approximation of the American put price is :

$$Y_0^n \quad \longrightarrow \quad P_0 \,,$$

the error is known to be of order $n^{-1/2}$, i.e.

$$\limsup_{n\to\infty} \sqrt{n} \left(Y_0^n - P_0 \right) \quad < \quad \infty$$

and an approximation of the optimal stopping policy is :

$$\tau_n^* \quad := \quad \inf \left\{ t_i^n \; : \; Y_{t_i^n}^n = \left(K - S_{t_i^n}\right) \right\}$$

Approximation of Conditional Expectations

Main observation : in the present context all conditional expectations are regressions, i.e.

$$\mathbb{E}_{t_i^n}\left[Y^n_{t^n_{i+1}}\right] \quad = \quad \mathbb{E}\left[Y^n_{t^n_{i+1}} \,\middle|\, S_{t_i^n}\right]$$

$\Longrightarrow$ Classical methods from statistics :
• Kernel regression <Carrière>
• Projection on subspaces of $\mathbb{L}^2(\mathbb{P})$ <Longstaff-Schwartz, Gobet-Lemor-Warin AAP05>

from numerical probabilistic methods
• quantization... <Bally-Pagès SPA03>

Integration by parts <Bouchard-Ekeland-Touzi FS04>

Approximation of the Replicating Strategy

- Put price is $P_t = P(t, S_t)$ a deterministic function of (t, S_t)
- The replicating strategy of the American put is :

$$\Delta_t = \frac{\partial P}{\partial s}(t, S_t), \qquad t < \tau^*$$

- An approximation of the replication strategy within a Monte Carlo estimation of the put price is :

$$\Delta^n_{t^n_i} = \mathbb{E}_{t^n_i}\left[Y^n_{t^n_{i+1}} \frac{\Delta W_{t^n_{i+1}}}{h_n}\right] \quad \text{where} \quad \Delta W_{t^n_{i+1}} = W_{t^n_{i+1}} - W_{t^n_i}$$

<Broadie-Glasserman, Fournié-Lasry-Lebuchoux-Lions-Touzi >

- **Finally**, the Monte Carlo scheme is : $Y^n_T = (K - S_T)^+$ and

$$\hat{Y}^n_{t^n_i} = \max\left\{\left(K - S_{t^n_i}\right)^+ , e^{-rh_n}\hat{\mathbb{E}}_{t^n_i}\left[Y^n_{t^n_{i+1}}\right]\right\}$$

$$\hat{\Delta}^n_{t^n_i} = \hat{\mathbb{E}}_{t^n_i}\left[Y^n_{t^n_{i+1}} \frac{\Delta W_{t^n_{i+1}}}{h_n}\right]$$

From American Options to Fully Nonlinear PDEs

Objective : Monte Carlo technique for the approximation of the American option price and hedge extends to solutions of Fully nonlinear PDEs.

• Fully Nonlinear PDEs are encountered in many areas of applied mathematics. In particular,

- stochastic control problems can be characterized in terms of the Bellman (dynamic programming) equation

$$0 = -\frac{\partial v}{\partial t} - \sup_{u \in U} \left\{ b(x,u) \cdot Dv + \frac{1}{2}\mathrm{Tr}\left[\sigma\sigma^{\mathrm{T}}(x,u)D^2 v\right] + f(x,u)v - k(x,u) \right\}$$

- optimal stopping problems can also be characterized in terms of the corresponding Bellman equation (free boundary)

Outline

Backward SDE : Definition

Find an $\mathbb{F}^W-$**adapted** (Y, Z) satisfying :

$$Y_t = \xi + \int_t^T F_r(Y_r, Z_r)dr - \int_t^T Z_r \cdot dW_r$$

$$\text{i.e.} \quad dY_t = -F_t(Y_t, Z_t)dt + Z_t \cdot dW_t \quad \text{and} \quad Y_T = \xi$$

where the generator $F : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}$, and

$$\{F_t(y, z),\ t \in [0, T]\} \text{ is } \mathbb{F}^W - \text{adapted}$$

If F is Lipschitz in (y, z) uniformly in (ω, t), and $\xi \in \mathbb{L}^2(\mathbb{P})$, then there is a unique solution satisfying

$$\mathbb{E} \sup_{t \le T} |Y_t|^2 + \mathbb{E} \int_0^T |Z_t|^2 dt < \infty$$

Markov BSDE's

Let $X_\cdot$ be defined by the (forward) SDE

$$
\begin{aligned}
dX_t &= b(t, X_t)dt + \sigma(t, X_t)dW_t \\
\text{and} \qquad F_t(y,z) &= f(t, X_t, y, z)\,, \quad f : [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R} \\
\xi &= g(X_T) \in \mathbb{L}^2(\mathbb{P})\,, \quad g : \mathbb{R}^d \longrightarrow \mathbb{R}
\end{aligned}
$$

If f continuous, Lipschitz in (x, y, z) uniformly in t, then there is a unique solution to the BSDE

$$
dY_t = -f(t, X_t, Y_t, Z_t)dt + Z_t \cdot \sigma(X_t)dW_t\,, \qquad Y_T = g(X_T)
$$

Moreover, there exists a measurable function V :

$$
Y_t = V(t, X_t)\,, \qquad 0 \le t \le T
$$

BSDE's and semilinear PDE's

- By definition,

$$\begin{aligned} Y_{t+h} - Y_t &= V(t+h, X_{t+h}) - V(t, X_t) \\ &= -\int_t^{t+h} f(X_r, Y_r, Z_r)dr + \int_t^{t+h} Z_r \cdot \sigma(X_r)dW_r \end{aligned}$$

- If $V(t,x)$ is smooth, it follows from Itô's lemma that :

$$\begin{aligned} \int_t^{t+h} \left(\frac{\partial V}{\partial t} + \mathcal{L}V\right)(r, X_r)dr + \int_t^{t+h} DV(r, X_r) \cdot \sigma(X_r)dW_r \\ = -\int_t^{t+h} f(X_r, Y_r, Z_r)dr + \int_t^{t+h} Z_r \cdot \sigma(X_r)dW_r \end{aligned}$$

where $\mathcal{L}$ is the Generator of X :

$$\mathcal{L}V = b \cdot DV + \frac{1}{2}\mathrm{Tr}[\sigma\sigma^T D^2 V]$$

Stochastic representation of solutions of a semilinear PDE

Under some conditions, the semilinear PDE

$$-\frac{\partial V}{\partial t} - \mathcal{L}V(t,x) - f\left(x, V(t,x), DV(t,x)\right) = 0$$
$$V(T,x) = g(x)$$

has a unique solution which can be represented as

$$V(t,x) = Y_t^{t,x}$$

where $Y^{t,x}$ solves the BSDE

$$Y_s = g(X_T) + \int_s^T f(X_r, Y_r, Z_r)dr - \int_s^T Z_r \cdot \sigma(X_r)dW_r\,, \qquad t \le s \le T$$

and $X_t = x$, $dX_s = b(X_s)ds + \sigma(X_s)dW_s$, $t \le s \le T$

Extension of Feynman-Kac's formula

Let $f \equiv 0$, then

$$V(t,x) \;=\; Y_t^{t,x} \;=\; g\left(X_T^{t,x}\right) - \int_t^T Z_r \cdot \sigma\left(X_r^{t,x}\right) dW_r$$

$\Longrightarrow$ take conditional expectations $V(t,x) \;=\; \mathbb{E}\left[g(X_T^{t,x})\right]$ with :

$$X_t^{t,x} = x \quad \text{and} \quad dX_r^{t,x} \;=\; b\left(X_r^{t,x}\right) dr + \sigma\left(X_r^{t,x}\right) dW_r$$

$\Longrightarrow$ Numerical solution by Monte Carlo :

$$\hat{V}(t,x) \;:=\; \frac{1}{N}\sum_{i=1}^{N} g\left(\hat{X}_T^{(i)}\right) \;\longrightarrow\; V(t,x) \;\text{ a.s. (LLN)}$$

and

$$\sqrt{N}\left(\hat{V}(t,x) - V(t,x)\right) \;\Longrightarrow\; \mathbf{N}\left(0, \mathbf{V}\left[g(X_T)\right]\right) \quad \text{(CLT)}$$

Discrete-time approximation

<Bally-Pagès SPA03, Zhang AAP04, Bouchard-Touzi SPA04>

Numerical solution of a semi-linear PDE by simulating the associated backward SDE by means of Monte Carlo methods

Start from Euler discretization : $Y^n_{t_n} = g\left(X^n_{t_n}\right)$ is given, and

$$Y^n_{t_{i+1}} - Y^n_{t_i} = -f\left(X^n_{t_i}, Y^n_{t_i}, Z^n_{t_i}\right)\Delta t_i + Z^n_{t_i} \cdot \sigma\left(X^n_{t_i}\right)\Delta W_{t_{i+1}}$$

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$\Longrightarrow$ Discrete-time approximation : $Y^n_{t_n} = g\left(X^n_{t_n}\right)$ and

$$Y^n_{t_i} = \mathbb{E}^n_i\left[Y^n_{t_{i+1}}\right] + f\left(X^n_{t_i}, Y^n_{t_i}, Z^n_{t_i}\right)\Delta t_i \qquad , \; 0 \le i \le n-1\,,$$

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$\Longrightarrow$ Discrete-time approximation : $Y^n_{t_n} \;=\; g\left(X^n_{t_n}\right)$ and

$$\begin{aligned} Y^n_{t_i} &= \mathbb{E}^n_i\left[Y^n_{t_{i+1}}\right] + f\left(X^n_{t_i}, Y^n_{t_i}, Z^n_{t_i}\right)\Delta t_i \qquad , \; 0 \le i \le n-1 \\ Z^n_{t_i} &= \left(\Delta t_i\right)^{-1} \mathbb{E}^n_i\left[Y^n_{t_{i+1}} \Delta W_{t_{i+1}}\right] \end{aligned}$$

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$\Longrightarrow$ Discrete-time approximation : $Y^n_{t_n} \;=\; g\left(X^n_{t_n}\right)$ and

$$\begin{aligned} Y^n_{t_i} &= \mathbb{E}^n_i\left[Y^n_{t_{i+1}}\right] + f\left(X^n_{t_i}, Y^n_{t_i}, Z^n_{t_i}\right)\Delta t_i \qquad , \; 0 \le i \le n-1 \\ Z^n_{t_i} &= (\Delta t_i)^{-1}\; \mathbb{E}^n_i\left[Y^n_{t_{i+1}} \Delta W_{t_{i+1}}\right] \end{aligned}$$

$\Longrightarrow$ Similar to numerical computation of **American options**

Discrete-time approximation, continued

Theorem Assume f and g are Lipschitz. Then :

$$\limsup_{n\to 0} n \left\{ \sup_{0\le t\le 1} \mathbb{E}\left|Y_t^n - Y_t\right|^2 + \mathbb{E}\left[\int_0^1 \left|Z_t^n - Z_t\right|^2 dt\right]\right\} \quad < \quad \infty$$

• Same rate of convergence as for the simulation of (forward) SDEs
• in the present context all conditional expectations are regressions, i.e.

$$\begin{aligned} \mathbb{E}\left[Y^n_{t_{i+1}} | \mathcal{F}_{t_i}\right] &= \mathbb{E}\left[Y^n_{t_{i+1}} | X_{t_i}\right] \\ \mathbb{E}\left[Y^n_{t_{i+1}} \Delta W_{t_{i+1}} | \mathcal{F}_{t_i}\right] &= \mathbb{E}\left[Y^n_{t_{i+1}} \Delta W_{t_{i+1}} | X_{t_i}\right] \end{aligned}$$

$\Longrightarrow$ can be approximated as in the case of American options...

Simulation of Backward SDE's

1. Simulate trajectories of the forward process X (well understood)

2. Backward algorithm :

$$
\left\|
\begin{array}{lcl}
\hat{Y}^n_{t_n} & = & g\left(X^n_{t_n}\right) \\
\hat{Y}^n_{t_{i-1}} & = & \widehat{\mathbb{E}}^n_{t_{i-1}}\left[\hat{Y}^n_{t_i}\right] + f\left(X^n_{t_{i-1}}, \hat{Y}^n_{t_{i-1}}, \hat{Z}^n_{t_{i-1}}\right)\Delta t_i\ , \quad 1 \le i \le n\ , \\
\hat{Z}^n_{t_{i-1}} & = & \dfrac{1}{\Delta t_i}\,\widehat{\mathbb{E}}^n_{t_{i-1}}\left[\hat{Y}^n_{t_i}\Delta W_{t_i}\right]
\end{array}
\right.
$$

(truncation of $\hat{Y}^n$ and $\hat{Z}^n$ needed in order to control the $\mathbb{L}^p$ error)

Simulation of BSDEs : bound on the rate of convergence

Theorem For $p > 1$:

$$\limsup_{n\to\infty} \max_{0\le i\le n} n^{-1-d/(4p)} N^{1/2p} \left\| \hat{Y}^n_{t_i} - Y^n_{t_i} \right\|_{\mathbb{L}^p} < \infty$$

For the time step $\frac{1}{n}$, and limit case $p = 1$:

rate of convergence of $\frac{1}{\sqrt{n}}$
if and only if
$$n^{-1-\frac{d}{4}} N^{1/2} = n^{1/2}, \quad \text{i.e. } N = n^{3+\frac{d}{2}}$$

Outline

Main purpose

• Enlarge the class of BSDE's in order to obtain a stochastic representation of Fully Nonlinear PDE's
(In particular, representation of general stochastic control problems)

• Gradient is related to the representation of a random variable as a stochastic integral (up to the driver)

• In order to obtain a fully nonlinear PDE, one needs to include "the Hessian" in the driver...

$\Longrightarrow$ Requires understanding local behavior of double stochastic integrals...

Second order BSDEs : Definition

$$\hat{f}(x,y,z,\gamma) := f(x,y,z,\gamma) + \frac{1}{2}\mathrm{Tr}[\sigma\sigma^T(x)\gamma] \text{ non-decreasing in } \gamma$$

Consider the 2nd order BSDE :

$$\begin{aligned} dX_t &= \sigma(X_t)dW_t \\ dY_t &= -f(t,X_t,Y_t,Z_t,\Gamma_t)dt + Z_t\cdot\sigma(X_t)dW_t\,, \quad Y_T = g(X_T) \\ dZ_t &= \alpha_t\,dt + \Gamma_t\,\sigma(X_t)dW_t \end{aligned}$$

A solution of (2BSDE) is

a process (Y,Z,α,Γ) with values in $\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n\times\mathcal{S}^n$

Question : existence ? uniqueness ? in which class ?
<Cheridito, Soner, Touzi and Victoir CPAM 2007>

Second order BSDEs : Main technical tool

(i) Suppose a solution exists with $Y_t = V(t, X_t)$, then

$$
\begin{aligned}
Y_{t+h} - Y_t &= V(t+h, X_{t+h}) - V(t, X_t) \\
&= -\int_t^{t+h} f(X_r, Y_r, Z_r, \Gamma_r)dr + \int_t^s Z_r \cdot dW_r \\
&= -\int_t^{t+h} f(X_r, Y_r, Z_r, \Gamma_r)dr \\
&\quad + \int_t^{t+h} \left(Z_t + \int_t^r \alpha_u du + \int_t^r \Gamma_u dW_u \right) \cdot dW_r
\end{aligned}
$$

($\sigma(.)$ = Identity matrix for simplification)

(ii) $2\times$ Itô's formula to V, identify terms of different orders

$\Longrightarrow$ Need short time asymptotics of double stochastic integrals

$$\int_0^t \int_0^r b_u \, dW_u \cdot dW_r \ , \quad t \geq 0$$

Second order BSDE : Existence

Consider the fully nonlinear PDE (with $\mathcal{L}V = \frac{1}{2}\mathrm{Tr}[\sigma\sigma^T D^2 V]$)

$$
\text{(E)} \quad
\begin{aligned}
-\frac{\partial V}{\partial t} - \mathcal{L}v(t,x) - f\left(t,x,v(t,x),Dv(t,x),D^2v(t,x)\right) &= 0 \\
v(T,x) = g(x) &
\end{aligned}
$$

If (E) has a smooth solution, then

$$
\begin{aligned}
\bar{Y}_t &= v(t,X_t), & \bar{Z}_t &:= Dv(t,X_t)\,, \\
\bar{\alpha}_t &:= \mathcal{L}Dv(t,X_t)\,, & \bar{\Gamma}_t &:= V_{xx}(t,X_t)
\end{aligned}
$$

is a solution of (2BSDE)
(immediate application of Itô's formula)

Second order BSDE : Uniqueness Assumptions

Assumption (f) $f : [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}_d(\mathbb{R}) \longrightarrow \mathbb{R}$ *continuous, Lipschitz in* y *uniformly in* (t,x,z,γ)*, and for some* $C, p > 0$ *:*

$$|f(t,x,y,z)| \leq C\ (1 + |y| + |x|^p + |z|^p + |\gamma|^p)$$

Assumption (Comp) *If* w *(resp.* u*) :* $[0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}$ *is a l.s.c. (resp. u.s.c.) viscosity supersolution (resp.subsolution) of (E) with*

$$w(t,x) \geq -C(1+|x|^p), \quad \text{and} \quad u(t,x) \leq C(1+|x|^p)$$

then $w(T,.) \geq u(T,.)$ *implies that* $w \geq u$ *on* $[0,T] \times \mathbb{R}^d$

Second order BSDE : Class of solutions

Let $\mathcal{A}^m_{t,x}$ be the class of all processes Z of the form

$$Z_s \quad = \quad z + \int_t^s \alpha_r \, dr + \int_t^s \Gamma_r \, dX^{t,x}_r \, , \qquad s \in [t, T]$$

where $z \in \mathbb{R}^d$, α and Γ are respectively $\mathbb{R}^d$ and $\mathcal{S}_d(\mathbb{R}^d)$ progressively measurable processes with

$$\max\left\{|Z_s|, |\alpha_s|, |\Gamma_s|\right\} \quad \leq \quad m\left(1 + \left|X^{t,x}_s\right|^p\right) \, ,$$

$$|\Gamma_r - \Gamma_s| \quad \leq \quad m\left(1 + \left|X^{t,x}_r\right|^p + \left|X^{t,x}_s\right|^p\right)\left(|r-s| + \left|X^{t,x}_r - X^{t,x}_s\right|\right)$$

We shall look for a solution (Y, Z, α, Γ) of (2BSDE) such that

$$Z \quad \in \quad \mathcal{A}_{t,x} \; := \; \cup_{m \geq 0} \mathcal{A}^m_{t,x}$$

Scond Order BSDE : The Uniqueness Result

Theorem *Suppose that the nonlinear PDE (E) satisfies the comparison Assumption* Com. *Then, under Assumption* (f), *for every g with polynomial growth, there is at most one solution to (2BSDE) with*

$$Z \in \mathcal{A}_{t,x}$$

2BSDE : Idea of proof of uniqueness

Define the stochastic target problems

$$V(t,x) \quad := \quad \inf\left\{y \;:\; Y_T^{t,y,Z} \geq g\left(X_T^{t,x}\right) \text{ a.s. for some } Z \in \mathcal{A}_{t,x}\right\}$$

(Seller super-replication cost in finance), and

$$U(t,x) \quad := \quad \sup\left\{y \;:\; Y_T^{t,y,Z} \leq g\left(X_T^{t,x}\right) \text{ a.s. for some } Z \in \mathcal{A}_{t,x}\right\}$$

(Buyer super-replication cost in finance)

- By definition : $V(t,X_t) \leq Y_t \leq U(t,X_t)$ for every solution (Y,Z,α,Γ) of (2BSDE) with $Z \in \mathcal{A}_{0,x}$
- Main technical result : V is a (discontinuous) viscosity super-solution of the nonlinear PDE (E)

$\Longrightarrow$ U is a (discontinuous) viscosity subsolution of (E)

- Assumption Com $\Longrightarrow$ $V \geq U$

A probabilistic numerical scheme for fully nonlinear PDEs

By analogy with BSDE, we introduce the following discretization for 2BSDEs :

$$
\begin{aligned}
Y^n_{t_n} &= g\left(X^\pi_{t_n}\right) ,\\
Y^\pi_{t_{i-1}} &= \mathbb{E}^\pi_{i-1}\left[Y^\pi_{t_i}\right] + f\left(X^\pi_{t_{i-1}}, Y^\pi_{t_{i-1}}, Z^\pi_{t_{i-1}}, \Gamma^\pi_{t_{i-1}}\right)\Delta t_i \ , \quad 1 \le i \le n \ ,\\
Z^\pi_{t_{i-1}} &= \frac{1}{\Delta t_i}\,\mathbb{E}^\pi_{i-1}\left[Y^\pi_{t_i}\Delta W_{t_i}\right]\\
\Gamma^\pi_{t_{i-1}} &= \frac{1}{(\Delta t_i)^2}\,\mathbb{E}^\pi_{i-1}\left[Y^\pi_{t_i}\left((\Delta W_{t_i})^2 - \Delta t_i\right)\right]
\end{aligned}
$$

Intuition From Greeks Calculation

- First, use the approximation $f''(x) \sim_{h=0} \mathbb{E}[f''(x+W_h)]$
- Then, integration by parts shows that

$$
\begin{aligned}
f''(x) &\sim \int f''(x+y)\frac{e^{-y^2/(2h)}}{\sqrt{2\pi}}dy \\
&= \int f'(x+y)\frac{y}{h}\frac{e^{-y^2/2}}{\sqrt{2\pi}}dy = \mathbb{E}\left[f'(x+W_h)\frac{W_h}{h}\right] \\
&= \int f(x+y)\frac{y^2-h}{h^2}\frac{e^{-y^2/2}}{\sqrt{2\pi}}dy = \mathbb{E}\left[f(x+W_h)\left(\frac{W_h^2-h}{h^2}\right)\right]
\end{aligned}
$$

- Connection with Finite Differences : $W_h \sim \sqrt{h}\left(\frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}\right)$

$$
\mathbb{E}\left[\psi(x+W_h)\frac{W_h}{h}\right] \sim \frac{\psi(x+\sqrt{h})-\psi(x-\sqrt{h})}{2h}
$$

Centered FD !

Intuitive derivation : Monte-Carlo-FD scheme

Consider the nonlinear PDE

$$\left(\frac{\partial V}{\partial t} + \mathcal{L}V\right) + f\left(x, DV(t,x), D^2V(t,x)\right) \quad = \quad 0$$

Evaluate at (s, X_s) and integrate betweens t_i and t_{i+1} :

$$\mathbb{E}_{t_i} V\left(t_{i+1}, X_{t_{i+1}}\right) - V\left(t_i, X_{t_i}\right) + \int_{t_i}^{t_{i+1}} f\left(., DV, D^2V\right)(s, X_s)ds = 0$$

leading to the approximation scheme :

$$\begin{aligned} V\left(t_i, X_{t_i}\right) \quad &= \quad \mathbb{E}_{t_i} V\left(t_{i+1}, X_{t_{i+1}}\right) \\ &\quad +hF\left(X_{t_i}, \mathbb{E}_{t_i} DV(t_{i+1}, X_{t_{i+1}}), \mathbb{E}_{t_i} D^2V(t_{i+1}, X_{t_{i+1}})\right) \end{aligned}$$

and use integration by parts...

The Convergence Result

<with A. Fahim>

Theorem Suppose that $\|g\|_{\mathbb{L}^\infty} < \infty$, f is Lipschitz, $\|f_\gamma\|_{\mathbb{L}^\infty} \leq \sigma$, and $\left\|f_\gamma^{-1}\right\|_{\mathbb{L}^\infty} < \infty$. Then

$$Y_0^n \quad \longrightarrow \quad v(t,x)$$

where v is the unique viscosity solution of the nonlinear PDE.

• Bounds on the approximation error are available
• $\|g\|_\infty < \infty$ is needed for the stability, can handle exponential bound by change of variable...
• This convergence result is weaker than that of (first order) Backward SDEs...

Outline

Comments on the 2BSDE algorithm

• in BSDEs the drift coefficient μ of the forward SDE can be changed arbitrarily by Girsanov theorem (importance sampling...)

• in 2BSDEs both μ and σ can be changed. Numerical results (together with above theorem) however recommend prudence...

• The heat equation $v_t + v_{xx} = 0$ correspond to a BSDE with zero driver. Splitting the Laplacian in two pieces, it can also be viewed as a 2BSDE with driver $f(\gamma) = \frac{1}{2}\gamma$
$\longrightarrow$ numerical experiments show that the 2BSDE algorithm performs better than the pure finite differences scheme

Numerical example : portfolio optimization (X. Warin)

With $U(x) = -e^{-\eta x}$ and $dS_t = S_t\sigma(\lambda dt + dW_t)$, want to solve :

$$V(t,x) \quad := \quad \sup_\theta \mathbb{E}\left[U\left(x + \int_t^T \theta_u \frac{dS_t}{S_t}\right)\right]$$

- An explicit solution is available
- V is the characterized by the fully nonlinear PDE

$$-V_t + \frac{1}{2}\lambda^2 \frac{(V_x)^2}{V_{xx}} \;=\; 0 \quad \text{and} \quad V(T,.) = U$$

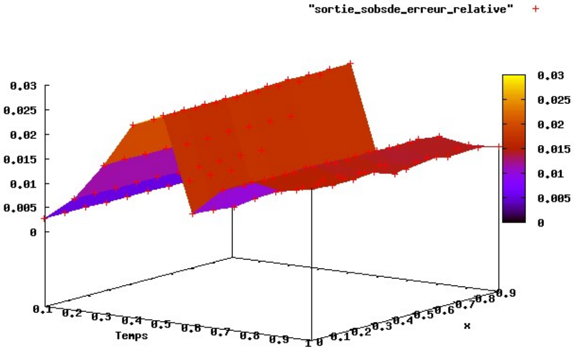

Fig.: Relative Error (Regression), 1 asset

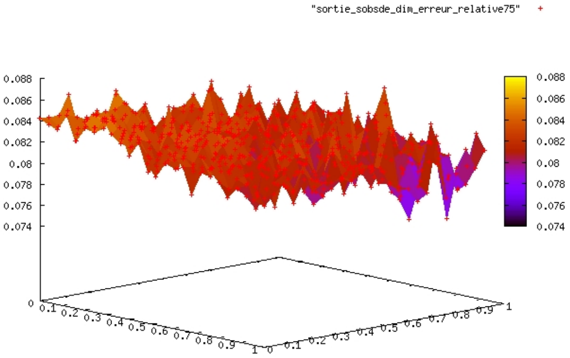

Fig.: Relative Error (Regression), 2 assets

Varying the drift of the FSDE

Drift FSDE	Relative error (Regression)
-1	0,0648429
-0,8	0,0676044
-0,6	0,0346846
-0,4	0,0243774
-0,2	0,0172359
0	0,0124126
0,2	0,00880041
0,4	0,00656142
0,6	0,00568952
0,8	0,00637239

Varying the volatility of the FSDE

Volatility FSDE	Relative error (Regression)	Relative error (Quantization)
0,2	0,581541	0,526552
0,4	0,42106	0,134675
0,6	0,0165435	0,0258884
0,8	0,0170161	0,00637319
1 0,	0124126	0,0109905
1,2	0,0211604	0,0209174
1,4	0,0360543	0,0362259
1,6	0,0656076	0,0624566