Forced mean curvature motions:
1. Homogenization, 2. Spirals

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We present here two independent recent results about forced mean curvature motions: a result about homogenization/non homogenization (see the joint work with L.A. Caffarelli [1]) and results on the dynamics of spirals (see the joint works with N. Forcadel and C. Imbert [2, 3]). We refer the reader to the cited works for a review of the literature on those two topics.

Homogenization questions

We consider the geometric evolution of hypersurfaces in $\mathbb{R}^N$, whose the normal velocity is given by

$$V = \kappa + c(x)$$

where κ is the mean curvature of the hypersurface and c is a $\mathbb{Z}^N$-periodic function which is Lipschitz. Then for $\varepsilon > 0$, we rescale the problem with the new normal velocity:

$$V^\varepsilon = \varepsilon\kappa + c(x/\varepsilon)$$

We will show, as ε goes to zero, in some sense that V^ε converges to an effective geometric law

$$V^0 = \bar{c}(n)$$

where n is the normal to the homogenized hypersurface.

We work with the level set formulation of the problem, the hypersurface being then a level set of a function u. Then $u(t,x)$ solves the following PDE

(0.1)
$$\begin{cases} u^\varepsilon_t = \varepsilon \text{ trace}\left\{ D^2u^\varepsilon \cdot \left(I - \dfrac{Du^\varepsilon}{|Du^\varepsilon|} \otimes \dfrac{Du^\varepsilon}{|Du^\varepsilon|} \right) \right\} + c(x)|Du^\varepsilon| & \text{on} \quad (0,+\infty)\times\mathbb{R}^N, \\ u^\varepsilon(0,x) = u_0(x) & \text{for all} \quad x \in \mathbb{R}^N \end{cases}$$

For this equation we have

Theorem 0.1 (Homogenization of mean curvature motion in dimension 2, [1]) *Under the previous assumptions for $N = 2$, $c > 0$ and the initial data u_0 globally Lipschitz*

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continuous, the unique viscosity solution u^ε of (0.1) converges locally uniformly to the unique viscosity solution u^0 of

$$\begin{cases} u_t^\varepsilon = \bar{c}\left(\dfrac{Du^0}{|Du^0|}\right)|Du^0| & on \quad (0,+\infty)\times\mathbb{R}^N, \\ u^0(0,x) = u_0(x) & for\ all \quad x\in\mathbb{R}^N \end{cases}$$

for some continous function $\bar{c}$.

Here, for a unit vector n, the function $\bar{c}(n)$ is determined by the existence of a periodic bounded function v (called corrector) of the following cell equation

$$\bar{c}(n) = F(D^2v, n+Dv, x) \quad \text{on} \quad \mathbb{R}^N \tag{0.2}$$

By contrast we also show a counter-example in dimensions $N \geq 3$:

Theorem 0.2 (Counter-example to homogenization in dimension $N \geq 3$, [1])
For $N \geq 3$, there exists a unit vector n and a C^∞ and $\mathbb{Z}^N$-periodic function $c > 0$ such that, for all real values of $\bar{c}(n)$, there is no bounded solutions v of (0.2).

We can even prove that the thickness of an initial flat level set increases linearly in time for long time, which prevents the possibility of strong homogenization result in such a case.

Spiral dynamics
Motivated by the modeling of crystal growth and the description of Frank-Read sources of dislocations in cystals, we are interested in the geometric motion in the plane $\mathbb{R}^2$ of a spiral attached to the origin, whose the normal velocity is given by

$$V = 1 + \kappa$$

where κ is the curvature of the spiral. The (time dependent) spiral is parametrized in polar coordinates (r,θ) as

$$\theta = -u(t,r)$$

and it can be checked that u solves the following PDE

$$\begin{cases} ru_t = \sqrt{1+(ru_r)^2} + u_r\left(\dfrac{2+(ru_r)^2}{1+(ru_r)^2}\right) + \dfrac{ru_{rr}}{1+(ru_r)^2} & \text{on} \quad (0,+\infty)\times(0,+\infty), \\ u(0,r) = u_0(r) & \text{for all} \quad r\in(0,+\infty) \end{cases} \tag{0.3}$$

As we see, this equation is degenerated at $r = 0$, and for this reason we do not impose any condition at $r = 0$. We also introduce the curvature of the spiral

$$\kappa_u = u_r\left(\frac{2+(ru_r)^2}{(1+(ru_r)^2)^{\frac{3}{2}}}\right) + \frac{ru_{rr}}{(1+(ru_r)^2)^{\frac{3}{2}}}$$

Then we have the following result:

Theorem 0.3 (Existence and uniqueness, [2])
Assume that $u_0 \in W^{2,\infty}_{loc}(0,+\infty)$ is globally Lipschitz continuous and satisfies

$$(u_0)_r \in W^{1,\infty}(0,+\infty) \quad or \quad \kappa_{u_0} \in L^\infty(0,+\infty)$$

and that there exists a radius $r_0 > 0$ *such that*

$$|1 + \kappa_{u_0}| \leq Cr \quad \textit{for} \quad 0 \leq r \leq r_0$$

Then there exists a unique viscosity solution of (0.3) which is globally Lipschitz in space and time.

We can also identify a self-similar solution $\lambda t + \varphi(r)$. Under certain assumptions on the initial data, we can show, as the time goes to infinity, that solutions $u(t, r)$ of (0.3) behave like the self-similar profile (up to addition of constants that may depend on the chosen subsequences in time, see [3] for more details).

References

[1] L.A. Caffarelli, R. Monneau, *Counter-example in 3D and homogenization of geometric motions in 2D*, preprint HAL: hal-00720954 (version 1) (2012).

[2] N. Forcadel, C. Imbert, R. Monneau, *Uniqueness and existence of spirals moving by forced mean curvature motion*, Interfaces and Free Boundaries **14** (2012), 365–400.

[3] N. Forcadel, C. Imbert, R. Monneau, work in progress.