A Quadratic Gradient Equation for pricing Mortgage-Backed Securities

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The MBS Market-1

- The Mortgage-Backed security (MBS) market plays a special role in the U.S. economy.
- Originators of mortgages can spread risk across the economy by packaging mortgages into investment pools through a variety of agencies.
- Mortgage holders have the option to prepay the existing mortgage and refinance the property with a new mortgage.
- MBS investors are implicitly writing an American call option on a corresponding fixed-rate bond.

The MBS Market-2

- Prepayments can take place for reasons not related to the interest rate option.
- Mortgage investors are exposed to significant interest rate risk when loans are prepaid and to credit risk when loans are terminated to default.
- Prepayments will halt the stream of cash flows that investors expect to receive.
- When interest rates decline, there will be a subsequent increase in prepayments which forces investors to reinvest the unexpected additional cash-flows at the new lower interest rate level.

The MBS Market-3

- The most simple structure of MBSs are the pass-through securities. Investors in this kind of securities receive all payments (principal plus interest) made by mortgage holders in a particular pool (less some servicing fee).

- Other classes are the stripped mortgage-backed securities which entail the ownership of either the principal (PO) or interest (IO) cash-flows.

MBS Modelling-1

- MBSs or Mortgage Pass-Throughs are claims on a portfolio of mortgages.
- Cash-flows from MBSs are the cash-flows from the portfolio of mortgages ($Collateral$).
- Every pool of mortgages is characterized by
- the weighted average maturity (WAM),
- the weighted average coupon rate (WAC),
- the pass-through rate (PTR), the interest on principal.

MBS Modelling-2

- The price of MBSs are quoted as a percentage of the underlying mortgage balance.
- a_t: the mortgage balance at time t.
- V_t: the price quote in the market at time t.
- $MBS_t = V_t a_t$: the (clean) market value at time t.
- $AI_t = \frac{\tau}{12} \frac{t - 30[t/30]}{30} a_t$: the accrued interest based on the time period from the settlement date.
- $MBS_t + AI_t$: the full market value.

MBS Modelling-3

- Modelling and pricing MBSs involve three layers of complexity:

1 Modelling the dynamic behavior of the term structure of interest rates.

2 Modelling the prepayment behavior of mortgage holders.

3 Modelling the risk premia embedded in these financial claims.

- The approaches used to model prepayment allow to classify existing models into two groups.

Prepayment Option

- The first type is related to American options, [Stanton, 1995] and [Stanton & Wallace 2003].
- Paying off the loan is equivalent to exercising a call option ($V^{\ell}_{p,t}$) on the underlying bond a_t, with time-varying exercise price $MB(t)$.
- The default option ($V^{\ell}_{d,t}$) is an option to exchange one asset (the house) for another ($MB(t)$).
- The mortgage liability is $M^{\ell}_t = a_t - V^{\ell}_{p,t} - V^{\ell}_{d,t}$.
- Borrowers choose when they prepay or default in order to minimize the overall value M^{ℓ}_t.

Prepayment Policy

- In a second class of models, the prepayment policy follows a comparison between the prevailing mortgage and contract rates [Deng, Quigley, Van Order, 2000].
- This comparison can be measured using the difference or a ratio of the two rates, and usually the 10 years Treasury yield is used as a proxy for the mortgage rate.

Prepayment Factors

- $X_t = (X_t^1, \ldots, X_t^N)$: the relevant economic factors.
- The MBS price at time t is $P_t = P(X_t, t)$.
- The challenging task is to give a complete justification to the choice of the market price of risk used to derive the functional form of P.
- The model specification follows the work by X. Gabaix and O. Vigneron (1998).
- We characterized P as the unique solution of a nonlinear parabolic partial differential equation, in a viscosity sense.

Cash-Flow

- We consider the usual information structure by a d-dimensional B.M. $B = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \{B_t\}_{t\in[0,T]}, \mathbb{P})$.
- $MB(t)$: the remaining principal at time t, without prepayments

$$MB(t) = MB(0)\frac{e^{\tau' T} - e^{\tau' t}}{e^{\tau' T} - 1}, \quad t \in [0, T]$$

- $S_t = s_0(X_t, t)$: the adapted pure cumulative prepayment process, so that the remaining principal at time t is $a_t = MB(t)\exp(-S_t)$, for any $t \in [0, T]$.
- $dc_t = \tau a_t dt - da_t$: the pass-through cash-flow.

Gain Process

- $\delta : \Omega \times [0, T] \to (0, \infty)$ is an integrable discount rate.
- The economy is made up of a representative agent (a trader in the MBS market) with a risk aversion $\rho > 0$.
- $dV_t^{\text{Riskless}} = \delta(t) V_t^{\text{Riskless}} dt$, $V_0^{\text{Riskless}} = A_0 > 0$.
- $V_t = (V_t^{\text{Riskless}}, V_t^{MBS,1}, \ldots, V_t^{MBS,k})$: the vector of asset prices.
- $G_t^{MBS,i} = V_t^{MBS,i} + \int_0^t dc_s^i$, $i = 1, \ldots, k$: the gain process of the asset i.

(A) The market is arbitrage-free and there exists a market price of risk $\gamma_t^{MBS} = (\gamma_t^{MBS,1}, \ldots, \gamma_t^{MBS,d})$.

Gain Volatility

- $v \in \mathbb{L}^{1,p} \Longleftrightarrow |||v|||_{1,p} < +\infty$.
  $$|||v|||_{1,p} = \left(\|v\|^p_{L^p(\Omega\times[0,T])} + \mathbb{E}\left[\|Dv\|^p_{L^2([0,T]^2)} \right] \right)^{1/p}$$
- From the generalized Clarke-Ocone formula (1991), we can prove a representation Theorem:

[Th(V)] Under (**A**), $S^i_t \in \mathbb{D}^{2,1}$, $D_t S^i \in L^2(\Omega \times [0,T])$, for any $t \in [0,T]$ and $i = 1, \ldots, k$, $\gamma = \gamma^{MBS} \in \mathbb{L}^{1,4}$, $\sigma^{MBS,i}_G(t)$ is given by:

$$-\mathbb{E}^{\mathbb{Q}}\left[\int_t^T \Big(\tau_i - \delta(s) \Big) e^{-\int_t^s \delta(u)du} a^i_s \Big(D_t S^i_s + \int_t^s D_t \gamma_u d\widehat{B}_u \Big) ds \Big| \mathcal{F}_t \right]$$

Standing Assumptions

(S1) $\{X_t^x : \ t \in [0,T]\}_x$, for $x \in \mathbb{R}^N$, represent the economic factors affecting MBS prices, with drift $\mu(x, T-t)$ and diffusion $\sigma(x, T-t) \in \mathbb{R}^{N \times d}$.

(S2) $S_t^i(x) = s_{0,i}(X_t^x, t)$, $s_{0,i} \geq 0$, $s_{0,i}$ is smooth.

(S3) There exist functions u_i^{Col}, $i = 1, \ldots, k$, s.t.

$$u_i^{\mathrm{Col}} \geq -h_{0,i} = -MB_i \exp\left(-s_{0,i}\right), \text{ in } \mathbb{R}^N \times [0,T],$$

and

$$V_t^{\mathrm{Col},i}(x) = u_i^{\mathrm{Col}}(X_t^x, t) + h_{0,i}(X_t^x, t), \quad \forall\ (x,t) \in \mathbb{R}^N \times [0,T].$$

An Equilibrium Model

- In MBS analysis, it seems natural to assume the m.p.r. to depend directly on the value of the liability.
- The liability to the mortgagor and the asset value to the investor differ only for a transaction cost, proportional to a_t.
- Since a change of the borrower's liability produces a change in the prepayment behavior, the proportionality with the asset value, yields a natural dependence of the m.p.r. on the MBS price (U) and on its variation ($\nabla_x U$).

An Equilibrium Model-1

- $x \in \mathbb{R}^N$ is the state of the economy.
- An equilibrium in the MBS market, is a d-dimensional $\mathcal{F}_t$-adapted process $\gamma(x)$, s.t.

1. $\frac{d\mathbb{Q}}{d\mathbb{P}} = \xi_T^{\gamma(x)}$ (Girsanov Exponential) is a risk-neutral measure for the MBS Market;
2. [Th(V)] holds and

$$\gamma_t(x) = \rho \frac{\sum_{i=1}^k \sigma_G^{\mathrm{Col},i}(t;x)}{\sum_{i=1}^k V_t^{\mathrm{Col},i}(x) + V_t^{\mathrm{Riskless}}},$$

for every $t \in [0, T]$.

An Equilibrium Model-2

[Th.] Let $\gamma(x)$ be an equilibrium. Under (**S1**)-(**S3**), $u^{\mathrm{Col}} = (u_1^{\mathrm{Col}}, \ldots, u_k^{\mathrm{Col}})$ is a solution in $\mathbb{R}^N \times (0,T)$ of

$$\rho\langle \sigma_0^\top \nabla u_i, \frac{\sum_{j=1}^k \sigma_0^\top \nabla u_j}{V_s^{\mathrm{Riskless}} + \sum_{j=1}^k [h_{0,j} + u_j]}\rangle = -\delta(s)(h_{0,i} + u_i)$$
$$+\tau_i h_{0,i} + \langle \nabla u_i, \mu_0 \rangle + \frac{\partial u_i}{\partial s} + \frac{1}{2} tr(\sigma_0 \sigma_0^\top \nabla^2 u_i),$$
$$u_i(x,T) = 0, \qquad i =, \ldots, k.$$

$\sigma_0(x,s) = \sigma(x, T-s)$, $\mu_0(x,s) = \mu(x, T-s)$.

$u_i^{\mathrm{Col}}(X_t^x, t) = \mathbb{E}^{\mathbb{Q}}\Big[\int_t^T \Big(\tau_i - \delta(s)\Big) e^{-\int_t^s \delta(u)du} h_{0,i}(X_s^x, s) ds \Big| \mathcal{F}_t \Big]$.

An Equilibrium Model-3

[Th.] Assume (**S1**)-(**S2**). Let σ be bounded, and $u = (u_1, \dots, u_k)$ is a smooth solution of the system with ∇u_i bounded, $u_i + h_{0,i} \geq 0$. For $i =, \dots, k$, define

$$V_t^i \equiv u_i(X_t^x, t) + h_{0,i}(X_t^x, t)$$

$$G_t^i \equiv V_t^i + \int_0^t dc_s^i.$$

Then $(V_t^{\mathrm{Riskless}}, G_t^1, \dots, G_t^k)$ is an arbitrage-free market which admits and equilibrium.

The case $k = 1$

- Let $\xi(t) = A_0 e^{\int_0^{T-t} \delta(s)ds}$, $r(t) = \delta(T-t)$, $h(x,t) = h_{0,1}(x, T-t)$, and $U(x,t) = u_1^{\mathrm{Col}}(x, T-t)$.
- The system reduces to the following equation:

$$\partial_t U - \frac{1}{2} tr(\sigma(x,t)\sigma^\top(x,t)\nabla^2 U) - \langle \mu(x,t), \nabla U\rangle + \rho \frac{|\sigma^\top(x,t)\nabla U|^2}{U + h(x,t) + \xi(t)} - \tau h(x,t) + r(t)(U + h(x,t)) = 0,$$

with $U(x,0) = 0$, everywhere in $\mathbb{R}^N \times (0,T)$.

- Properties of U: 1) $U + h \geq 0$ 2) U satisfies the stochastic representation.

[Def.] The **parabolic 2-jet**:
$\mathcal{P}^{2,\pm}u(x,t)=\{(\partial_t\varphi(x,t),\nabla\varphi(x,t),\nabla^2\varphi(x,t)):\ u-\varphi$ has a global strict max. (*resp.* min.) at $(x,t)\}$.

[Def.] $u:\mathbb{R}^N\times[0,T]\to(a,b)$ l.b. and u.s.c. (*resp.* l.s.c.) is a **viscosity subsolution** (*resp.* lower **viscosity supersolution**) if $u(\cdot,0)\le u_0(\cdot)$, (*resp.* $\ge u_0(\cdot)$), in $\mathbb{R}^N$, and for any $(b,q,A)\in\mathcal{P}^{2,\pm}u(x,t)$

$$b+F(x,t,u(x,t),q,A)\le 0 \qquad (\text{resp.} \ge 0).$$

- Existence Results can be found in the User's guide of M.G. Crandall, H. Ishii, P.L. Lions (1992).

Existence and Uniqueness

- We found structural conditions to deal with Hamiltonian functions s.t. $\partial_u F$ is unbounded from below. These depend on the behavior of
$(\lambda, \kappa, u) \mapsto \lambda^{-1} F(x, t, u, \lambda p, \lambda X + \kappa p \otimes p)$
for $u \in [a, b]$, $0 < m \leq \lambda \leq M$, $2\kappa \leq z'(u)$, for some $z \in C^1([a, b]; [m, M])$, x, t, p, X being fixed.

(P) $h \in C^{2,1}(\mathbb{R}^N \times [0, T])$, $\partial_t h(\cdot, t)$, $tr(\sigma\sigma^\top(t)\nabla^2 h(\cdot, t))$, $\nabla h(\cdot, t)$ are bounded and x-Lipschitz continuous.

[Th.] Under the assumption $(\mathbf{P})$, if $\tau \geq \delta$ in $[0, T]$, then the MBS problem admits a unique bounded viscosity solution U, such that $U + h \geq 0$.

- The m.p. of prepayment risk is

$$\gamma_t(x) = \rho \frac{\sigma^\top(X_t^x, T-t)\nabla U(X_t^x, T-t)}{U(X_t^x, T-t) + h(X_t^x, T-t) + \xi(T-t)}$$

- The typical technique used to prove a representation of U as an expectation, is based on the dynamic programming principle (Fleming & Soner, 1993).
- The existence of the value function is guaranteed by the existence of the expected value.
- In the MBS model the expectation depends on U itself. Hence we need more regularity on ∇U.

[Th.] (**Papi 2003**) If σ is independent of x, $h(\cdot,t) \in \mathbb{W}^{4,\infty}$, $\partial_t h(\cdot,t)$, $\mu(\cdot,t) \in \mathbb{W}^{2,\infty}$, uniformly in time. Then $U \in \mathbb{W}^{2,1,\infty}(\mathbb{R}^N \times (0,T))$.

[Th.] Let σ be bounded, $x_0 \in \mathbb{R}^N$. If $U \in \mathbb{W}^{2,1,\infty}(\mathbb{R}^N \times (0,T))$, $U + h \geq 0$, then

1) $\gamma_t(x_0) \in \mathbb{L}^{1,p}$, for any $p \geq 2$.

2) If $X_t^{x_0}$ admits a Borel-measurable density, then

$$U(X_t^{x_0}, T-t) =$$
$$\mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^s \delta(\kappa)d\kappa}(\tau - \delta(s))h(X_s^{x_0}, T-s)ds \middle| \mathcal{F}_t\right]$$

3) $\gamma_t(x_0)$ is a market price of equilibrium.

Path Dependency

- The prepayment function S depends on the trajectory followed by one or more of the underlying factors x_t.
- $s_0(y) = y$, for $y \geq 0$.
- The prepayment rate y_t follows $y_t = \int_0^t \eta(x_s, y_s) ds$
- $x_t \in \mathbb{R}^d$, $dx_t = b(x_t)dt + c(x_t)dB_t$.
- $X_t = (x_t, y_t)$ is a strongly degenerate diffusion.
- Let $X_0 = (x_0, 0)$, b, η, c are smooth functions, $c(x_0)$ is invertible and $\nabla_x \eta(x_0, 0) \neq 0$.
- The Hörmander condition $\Longrightarrow X$ admits a smooth density.

References

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