Impulse control on finite horizon with execution delay

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Basic motivations

- Pricing and hedging of an option on hedge funds:
  - Hedge funds: pooled investment vehicle administered by professional managers
  - Illiquid assets in hedge funds: debts, options ...
  - The hedge fund manager needs time to find a counterpart to trade these assets
  - To buy or sell shares of hedge funds, investors must declare their orders one to three months before they are effectively executed
  - Once the order is passed, its execution is mandatory

- Execution delay → liquidity risk

- Our goal: provide a general mathematical framework for studying the impact of execution delay and measuring this cost of illiquidity.
The control problem

Controlled process

- State system in $\mathbb{R}^d$ in absence of control:
  \[ dX_s = b(X_s)ds + \sigma(X_s)dW_s \]

- Impulse control with time lag: a double sequence $(\tau_i, \xi_i)_{i \geq 1}$,
  - decision times: $\tau_i$ stopping times s.t. $\tau_{i+1} - \tau_i \geq h$, $h > 0$
  - minimal time lag between two interventions
  - impulse values: $\xi_i$ valued in $E$ (compact subset) and $\mathcal{F}_{\tau_i}$-measurable (based on information available at $\tau_i$)
Controlled process

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  - minimal time lag between two interventions
  - impulse values: $\xi_i$ valued in $E$ (compact subset) and $\mathcal{F}_{\tau_i}$-measurable (based on information available at $\tau_i$)

- **Execution delay on the system**: the intervention $\xi_i$ decided at $\tau_i$ is executed at time $\tau_i + \delta$, moving the system from

$$X_{(\tau_i + \delta)^-} \to X_{\tau_i + \delta} = \Gamma(X_{(\tau_i + \delta)^-}, \xi_i)$$

In the sequel, we set: $\delta = mh$, with $m \in \mathbb{N}$ (for simplicity of notations).
The control problem

Control objective

- Total profit over a finite horizon $T < \infty$, associated to an impulse control $\alpha = (\tau_i, \xi_i)_{i \geq 1} \in A$:

$$\Pi(\alpha) = \int_0^T f(X_t)dt + g(X_T) + \sum_{\tau_i + mh \leq T} c(X_{(\tau_i+mh)^-}, \xi_i),$$

$f$ running profit function on $\mathbb{R}^d$, $g$ terminal profit function on $\mathbb{R}^d$, $c$ executed cost function on $\mathbb{R}^d \times E$.

- Control problem:

$$V_0 = \sup_{\alpha \in A} \mathbb{E}\left[\Pi(\alpha)\right].$$
Financial example

- $S$ asset price (e.g. spot price of a hedge fund):

$$dS_t = \beta(S_t)dt + \gamma(S_t)dW_t$$

- $Y_t$ number of shares in asset, $Z_t$ amount of cash held by investor at time $t$: in absence of trading

$$dY_t = 0, \quad dZ_t = rZ_t dt \quad (r \text{ interest rate}).$$

- Portfolio strategy: $(\tau_i, \xi_i)_i$, where $\xi_i$ represents the number of shares purchased or sold at time $\tau_i$, but executed at $\tau_i + mh$. 
Financial example

- State process $X = (S, Y, Z)$
  - when the order $(\tau_i, \xi_i)$ is executed at time $\tau_i + mh$, the system moves to
    \[
    S_{\tau_i+mh} = S_{(\tau_i+mh)-} \quad \text{(or} \quad P(S_{(\tau_i+mh)-}, \xi_i) \quad \text{if large investor)}
    \]
    \[
    Y_{\tau_i+mh} = Y_{(\tau_i+mh)-} + \xi_i
    \]
    \[
    Z_{\tau_i+mh} = Z_{(\tau_i+mh)-} - \xi_i S_{\tau_i+mh}.
    \]

- Optimal investment and/or indifference pricing: maximize the expected utility of terminal payoff
  \[
  \mathbb{E}\left[U(Z_T + Y_T S_T - g(S_T))\right].
  \]
Some references

- PDE variational formulation of impulse control problems:
  Bensoussan-Lions (82): no delay $m = 0$
  Bar-Ilan, Sulem (95), Oksendal, Sulem (06): delay but with particular controlled process (Lévy process for $X$ and additive intervention operator $\Gamma$) on infinite horizon

- Probabilistic calculation for particular threshold strategy:
  Bayraktar, Egami (06): $m = 1$, infinite horizon and impulse value chosen at time of execution, i.e. $\xi_i \mathcal{F}_{\tau_i+h}$-measurable

- Financial applications: liquidity risk and execution delay ($m = 1$)
  Subramanian, Jarrow (01), Alvarez, Keppo (02), Keppo, Peura (06)
New features and contributions in our model

- General diffusion framework on finite horizon
- New orders can be decided between the period of execution delay, i.e. $m \geq 1$

Main goal
- Obtain a unique PDE characterization of the original control problem
- Provide an implementable algorithm
- Measure impact of execution delay and cost of illiquidity
• Extend definition of control problem $V_0$ to general initial conditions:

**Important issue**: the state process $X$ is not Markovian in itself (in contrast to usual stochastic control problems)!

Given an impulse control, the state of the system is not only defined by its current state value at time $t$ but also by the pending orders: the orders not yet executed, i.e. decided in $(t - mh, t]$.

**Remark**: Due to the time decision lag $h$, the number of pending orders is $\leq m$.

→ finite-dimensional system
Some notations (I)

- Set of $k$ ($k = 0, \ldots, m$) pending orders at time $t \in [0, T]$:

$$P_t(k) = \left\{ p = (t_i, e_i)_{1 \leq i \leq k} \in ([0, T] \times E)^k : t_i - t_{i-1} \geq h, \text{ and } t - mh < t_i \leq t \right\},$$

- State domains for $k = 0, \ldots, m$:

$$D_k = \left\{ (t, x, p) : (t, x) \in [0, T] \times \mathbb{R}^d, p \in P_t(k) \right\}.$$

**Remark**

For $k = 0$, $P_t(0) = \emptyset$, and $D_0 = [0, T] \times \mathbb{R}^d$. 
Some notations (II)

• Set of admissible controls from a given pending order $p = (t_i, e_i)_{1 \leq i \leq k} \in P_t(k)$:

$$A_{t,p} = \left\{ \alpha = (\tau_i, \xi_i)_{i \geq 1} \in A : (\tau_i, \xi_i) = (t_i, e_i), \ i = 1, \ldots, k \right\},$$

and $\tau_{k+1} \geq t$,

Given $(t, x, p) \in D_k, k \leq m, \alpha \in A_{t,p}$, we denote

$$\{X_{s,t,x,p,\alpha}^t, t \leq s \leq T\}$$

the controlled process starting from $X_t = x$, with pending order $p$, and controlled by $\alpha$. 
Control objective: dynamic version

- Criterion: for \((t, x, p) \in D_k, \ k \leq m, \ \alpha = (\tau_i, \xi_i) i \in A_{t,p},\)

\[
J_k(t, x, p, \alpha) = \mathbb{E}\left[ \int_t^T f(X_s^{t,x,p,\alpha}) \, ds + g(X_T^{t,x,p,\alpha}) \right.
\]
\[+ \sum_{t < \tau_i + mh \leq T} c(X_{(\tau_i + mh)^-}^{t,x,p,\alpha}, \xi_i) \bigg],
\]

- Corresponding value functions:

\[
v_k(t, x, p) = \sup_{\alpha \in A_{t,p}} J_k(t, x, p, \alpha), \quad k \leq m, \ (t, x, p) \in D_k.
\]

Remark

\(V_0 = v_0(0, X_0, \emptyset)\).
Assumptions

- **(H1)** $f, g, c$ and $\Gamma$ are continuous and satisfy a linear growth condition on $x$

- **(H2)** $g(x) \geq g(\Gamma(x, e)) + c(x, e)$, for all $(x, e) \in \mathbb{R}^d \times E$.

Remarks

- Economic interpretation of **(H2)** satisfied in financial examples
- If **(H2)** is not satisfied, the value functions may be discontinuous

**Example:** $b = \sigma = f = g = 0$, $c(x, e) = 1$. Then,

\[
v_0(t, x) = \begin{cases} 
0, & T - mh < t \leq T \\
i, & T - (m + i)h < t \leq T - (m + i - 1)h, \ i \geq 1.
\end{cases}
\]

$\rightarrow$ Discontinuities of $v_0$ at $t = T - (m + i - 1)h, \ i \geq 1$. 
• Partition the set of pending orders into $P_t(k) = P^1_t(k) \cup P^2_t(k)$:

$$P^1_t(k) = \left\{ p = (t_i, e_i)_{1 \leq i \leq k} \in P_t(k) : t_k > t - h \right\}$$

$$P^2_t(k) = \left\{ p = (t_i, e_i)_{1 \leq i \leq k} \in P_t(k) : t_k \leq t - h \right\}.$$ 

and define the corresponding state domains $\mathcal{D}_k = \mathcal{D}^1_k \cup \mathcal{D}^2_k$:

$$\mathcal{D}^1_k = \left\{ (t, x, p) : (t, x) \in [0, T] \times \mathbb{R}^d, \ p \in P^1_t(k) \right\}$$

$$\mathcal{D}^2_k = \left\{ (t, x, p) : (t, x) \in [0, T] \times \mathbb{R}^d, \ p \in P^2_t(k) \right\}.$$
If \((t, x, p) \in D_k^1\), the controller cannot take action in \([t, t + dt]\). Only the diffusion \(X\) operates.
Dynamic programming

State domain of no possible order decision

- If \((t, x, p) \in \mathcal{D}_k^1\), the controller cannot take action in \([t, t + dt]\). Only the diffusion \(X\) operates

- **Linear PDE's on** \(\mathcal{D}_k^1, k = 1, \ldots, m:\)

\[
- \frac{\partial v_k}{\partial t} - \mathcal{L}v_k - f = 0 \quad \text{on} \quad \mathcal{D}_k^1
\]

where

\[
\mathcal{L} \varphi = b(x).D_x \varphi + \frac{1}{2} \text{tr}(\sigma \sigma'(x)D^2_x \varphi)
\]

is the generator of the diffusion \(X\).
State domain of possible order decision

- If \((t, x, p) \in D^2_k\), the controller has the choice of:
  - doing nothing, i.e. let the diffusion \(X\) operate on \([t, t + dt]\) → linear PDE's
  - passing immediately an order \((t, e)\), so that the pending orders switch from \(p\) (with cardinal \(k\)) to \(p \cup (t, e)\) (with cardinal \(k + 1\)) →

\[
v_k(t, x, p) \geq \sup_{e \in E} v_{k+1}(t, x, p \cup (t, e))
\]
State domain of possible order decision

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\]

**Variational inequalities on** \(\mathcal{D}^2_k\), \(k = 0, \ldots, m - 1\):

\[
\min \left[ -\frac{\partial v_k}{\partial t} - \mathcal{L}v_k - f, v_k(t, x, p) - \sup_{e \in E} v_{k+1}(t, x, p \cup (t, e)) \right] = 0 \quad \text{on} \quad \mathcal{D}^2_k
\]
Dynamic programming

Dynamic programming system

- PDE system for the value functions $v_k$, $k = 0, \ldots, m$:

$$
- \frac{\partial v_k}{\partial t} - \mathcal{L} v_k - f = 0 \quad \text{on } D_k^1, \quad k \geq 1,
$$

$$
\min \left[ - \frac{\partial v_k}{\partial t} - \mathcal{L} v_k - f, \right. \\
\left. v_k(t, x, p) - \sup_{e \in E} v_{k+1}(t, x, p \cup (t, e)) \right] = 0 \quad \text{on } D_k^2, \quad k \leq m - 1.
$$
Time-boundary conditions

- (Standard) terminal condition at $T$:

$$v_k(T^-, x, p) = g(x), \quad x \in \mathbb{R}^d, \quad p \in P_T(k), \quad k = 1, \ldots, m.$$
Time-boundary conditions

- (Standard) terminal condition at $T$:
  \[ v_k(T^-, x, p) = g(x), \quad x \in \mathbb{R}^d, \quad p \in P_T(k), \quad k = 1, \ldots, m. \]

- Non standard condition on the time-boundary of $D_k \leftrightarrow$ execution of the first pending order $(t_1, e_1)$ of $p = (t_i, e_i)_{1 \leq i \leq k}$:
  \[ v_k((t_1 + mh)^-, x, p) = c(x, e_1) + v_{k-1}(t_1 + mh, \Gamma(x, e_1), p_-), \]
  where $p_- = p \setminus (t_1, e_1) = (t_i, e_i)_{2 \leq i \leq k}$.
  (Technical difficulty due to continuity issue for $v_{k-1}$).
Non standard features

- Form of the domain $\mathcal{D}_k = \mathcal{D}^1_k \cup \mathcal{D}^2_k = \{(t, x, p) : (t, x) \in [0, T] \times \mathbb{R}^d, p \in P_t(k)\}$
- Coupled system both on the PDE and on the boundary conditions:
  - $v_k$ depends on $v_{k+1}$ on the variational inequality
  - $v_{k+1}$ depends on $v_k$ via a time-boundary condition
- Discontinuity of the differential operator for $v_k$
  - linear PDE on $\mathcal{D}^1_k$
  - free-boundary problem on $\mathcal{D}^2_k$
Viscosity characterization

Main theoretical result

Theorem

The family of value functions $v_k$, $k = 0, \ldots, m$, is the unique viscosity solution to the PDE system, satisfying the time-boundary conditions, a linear growth condition on $x$, and

$$v_k(t, x, p) \geq \sup_{e \in E} v_{k+1}(t, x, p \cup (t, e)), \quad (t, x, p) \in D_k, \quad t = t_k + h.$$ 

Moreover, $v_k$ is continuous on $D_k$. 
(Short) Elements of Proof

- Viscosity properties: as usual, consequences of a suitable version of dynamic programming principle

- Uniqueness and comparison principles: more delicate! In addition to usual Ishii’s lemma, arguments in the proofs involve backward and forward iterations on the domains and value functions due to the coupling.
Initialization phase

First step of the algorithm based on the following remark:

- An order decided after $T - mh$ is executed after $T$, and so does not influence the state process $X_t$ for $t \leq T$. 
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- An order decided after $T - mh$ is executed after $T$, and so does not influence the state process $X_t$ for $t \leq T$.

Therefore, if $(t, x, p) \in D_k$ is s.t. the pending order $p = (t_i, e_i)_{1 \leq i \leq k} \in \Theta_k \times E^k$ satisfies: $t_1 > T - mh$, i.e. all the pending orders are executed after $T$, then

$$v_k(t, x, p) = \mathbb{E}\left[\int_t^T f(X_s^{t,x,0})ds + g(X_T^{t,x,0})\right],$$

which is easily computable.
Step $n$

- For $k = 1, \ldots, m$, we introduce the increasing sequence of sets:

$$D_k(n) = \left\{ (t, x, p) \in D_k : t_1 > T - nh \right\},$$

$$N = \inf\{ n \geq 1 : T - nh < 0 \}.$$
Step \( n \)

- For \( k = 1, \ldots, m \), we introduce the increasing sequence of sets:

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D_{k}(n) = \left\{ (t, x, p) \in D_k : t_1 > T - nh \right\},
\]

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N = \inf\{ n \geq 1 : T - nh < 0 \}.
\]

- From the initialization phase, we know the value of \( v_k \) on \( D_k(m) \).
Step \( n \)

- For \( k = 1, \ldots, m \), we introduce the increasing sequence of sets:

\[
\mathcal{D}_k(n) = \left\{ (t, x, p) \in \mathcal{D}_k : t_1 > T - nh \right\},
\]

\[
N = \inf\{n \geq 1 : T - nh < 0\}.
\]

- From the initialization phase, we know the value of \( v_k \) on \( \mathcal{D}_k(m) \)

- \( \mathcal{D}_k(N) = \mathcal{D}_k \)

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Step $n$

- For $k = 1, \ldots, m$, we introduce the increasing sequence of sets:

\[ D_k(n) = \left\{ (t, x, p) \in D_k : t_1 > T - nh \right\}, \]

\[ N = \inf\{ n \geq 1 : T - nh < 0 \}. \]

- From the initialization phase, we know the value of $v_k$ on $D_k(m)$
- $D_k(N) = D_k$
- We shall compute $v_k$ on $D_k(n)$ by forward induction on $n = m, \ldots, N$. 
From step $n$ to $n + 1$

- *Induction hypothesis at step $n$*: we know the values of $v_k$, $k = 0, \ldots, m$, on $D_k(n)$

- **Step $n \rightarrow n + 1$**: Computation $v_k$, $k = 0, \ldots, m$, on $D_k(n + 1)$
  - by backward recursion on $k$!
From step $n$ to $n+1$: $k = m$

- Computation of $v_m$ on $D_m(n+1)$:
From step $n$ to $n + 1$: $k = m$

- Computation of $v_m$ on $\mathcal{D}_m(n + 1)$:
  - $v_m$ satisfies the linear PDE

$$
- \frac{\partial v_m}{\partial t} - \mathcal{L}v_m - f = 0, \text{ on } \mathcal{D}_m(n + 1)
$$
From step $n$ to $n + 1$: $k = m$

- Computation of $v_m$ on $D_m(n + 1)$:
  - $v_m$ satisfies the linear PDE
    \[-\frac{\partial v_m}{\partial t} - \mathcal{L}v_m - f = 0, \quad \text{on } D_m(n + 1)\]
  - together with the boundary data of $D_m(n + 1)$
    \[v_m((t_1 + mh)^-, x, p) = c(x, e_1) + v_{m-1}(t_1 + mh, \Gamma(x, e_1), p_-).\]

- Notice that since $t_1 > T - (n + 1)h$, then $t_2 > T - nh$, and so $p_- = (t_i, e_i)_{2\leq i\leq m}$ is s.t. $(t_1 + mh, \Gamma(x, e_1), p_-) \in D_{m-1}(n)$
  $$\implies v_{m-1}(t_1 + mh, \Gamma(x, e_1), p_-) \text{ is known from step } n$$
From step $n$ to $n + 1$: $k = m$

- Computation of $v_m$ on $\mathcal{D}_m(n + 1)$:

  - Linear Feynman-Kac (F-K) representation

  $$v_m(t, x, p) = \mathbb{E}\left[\int_t^{t_1 + mh} f(X_s^{t, x, 0}) ds + c(X_{t_1 + mh}^{t, x, 0}, e_1) + v_{m-1}(t_1 + mh, \Gamma(X_{t_1 + mh}^{t, x, 0}, e_1), p_-)\right].$$
From step $n$ to $n + 1$: $k + 1 \rightarrow k$

- **Recursion hypothesis at order $k + 1$**: we know the values of $v_{k+1}$ on $D_{k+1}(n + 1)$.

  - Computation of $v_k$ on $D_k(n + 1)$:
From step $n$ to $n + 1 : k + 1 \rightarrow k$

- *Recursion hypothesis at order $k + 1 :$ we know the values of $v_{k+1}$ on $D_{k+1}(n + 1)$.

- Computation of $v_k$ on $D_k(n + 1)$:
  - Known boundary data of $D_k(n + 1)$ from step $n$

$$v_k((t_1 + mh)^-, x, p) = c(x, e_1) + v_{k-1}(t_1 + mh, \Gamma(x, e_1), p_\bot).$$
From step $n$ to $n+1$: $k+1 \rightarrow k$

- **Recursion hypothesis at order $k+1$**: we know the values of $v_{k+1}$ on $D_{k+1}(n+1)$.

  - **Computation of $v_k$ on $D_k(n+1)$**:
    - Known boundary data of $D_k(n+1)$ from step $n$
      \[
      v_k((t_1 + mh)^-, x, p) = c(x, e_1) + v_{k-1}(t_1 + mh, \Gamma(x, e_1), p_-).
      \]

  - Depending on whether $(t, x, p) \in D^1_k$ or $D^2_k$, the PDE for $v_k$ is either linear or a variational inequality with obstacle
    \[
    \sup_{e \in E} v_{k+1}(t, x, p \cup (t, e)),
    \]
    which is known from recursion hypothesis at order $k+1$. 

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Summary

- Computation of the family \( \{v_k, k = 0, \ldots, m\} \) on \( D_k \):
Summary

- Computation of the family $\{v_k, k = 0, \ldots, m\}$ on $D_k$ :

  | Initialization | Linear F-K computation of $\{v_k, k = 0, \ldots, m\}$ on $D_k(m)$ |

- **Initialization**: Linear F-K computation of $\{v_k, k = 0, \ldots, m\}$ on $D_k(m)$
Summary

- Computation of the family \( \{v_k, k = 0, \ldots, m\} \) on \( \mathcal{D}_k \):

  - **Initialization**: Linear F-K computation of \( \{v_k, k = 0, \ldots, m\} \) on \( \mathcal{D}_k(m) \)

  - **Step** \( n \to n + 1 \) (from \( n = m \) to \( n = N \)):
    
    Computation of \( \{v_k, k = 0, \ldots, m\} \) on \( \mathcal{D}_k(n + 1) \) by backward recursion from \( k = m \) to \( 0 \):
Summary

- Computation of the family \( \{v_k, k = 0, \ldots, m\} \) on \( \mathcal{D}_k \):
  
  - **Initialization**: Linear F-K computation of \( \{v_k, k = 0, \ldots, m\} \) on \( \mathcal{D}_k(m) \)

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      - **Initialization**: Linear F-K computation of \( v_m \) on \( \mathcal{D}_m(n + 1) \) from step \( n \)
Summary

- Computation of the family \( \{v_k, k = 0, \ldots, m\} \) on \( D_k \):
  
  ▶ **Initialization**: Linear F-K computation of \( \{v_k, k = 0, \ldots, m\} \) on \( D_k(m) \)

  ▶ **Step** \( n \to n + 1 \) (from \( n = m \) to \( n = N \)):
    
    Computation of \( \{v_k, k = 0, \ldots, m\} \) on \( D_k(n + 1) \) by backward recursion from \( k = m \) to 0:
    
    - **Initialization**: Linear F-K computation of \( v_m \) on \( D_m(n + 1) \) from step \( n \)
    
    - \( k + 1 \to k \): Computation of \( v_k \) on \( D_k(n + 1) \) by linear F-K or optimal stopping problems involving data of \( v_{k-1} \) on \( D_{k-1}(n) \) and \( v_{k+1} \) on \( D_{k+1}(n + 1) \)
Impact of execution delay on option pricing

- Indifference price $\pi$ of a call option $g(S_T) = (S_T - K)_+$:
  - $v_0(S_0, Y_0, Z_0)$: value function of the optimal investment problem without option
  - $v_g(S_0, Y_0, Z_0)$: value function of the optimal investment problem with option delivery
  - $\pi = \pi(S_0, Y_0, Z_0)$ s.t. $v_g(S_0, Y_0, Z_0 + \pi) = v_0(S_0, Y_0, Z_0)$

- Numerical illustrations with:
  - BS model: $r = 0$, $\sigma = 10\%$, $K = S_0$ (At The Money)
  - CARA utility: $U(x) = 1 - e^{-\eta x}$ with $\eta = 20$. $\rightarrow \pi = \pi(S_0, Y_0)$

- Dependence of $\pi$ on delay $mh$ and maturity $T$
Indifference price for a $T = 3$ years ATM call option for different values of $h$, in percentage of the initial spot price

<table>
<thead>
<tr>
<th>$h$ (years)</th>
<th>BS price</th>
<th>discrete hedging, $m = 0$</th>
<th>delayed hedging, $m = 1$</th>
<th>discrete hedging optimal $Y_0$</th>
<th>delayed hedging optimal $Y_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>6.90</td>
<td>6.94</td>
<td>6.94</td>
<td>6.85</td>
<td>6.86</td>
</tr>
<tr>
<td>0.025</td>
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<td>6.87</td>
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<td>7.19</td>
<td>6.89</td>
<td>6.97</td>
</tr>
<tr>
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Indifference price for a $T = 3$ years ATM call option, with no initial endowment $Y_0 = 0$ in stock, for discrete and delayed hedging in function of $h$ ($m = 1$).
Indifference price for a $T = 3$ years ATM call option, with optimal initial endowment in stock, for discrete and delayed hedging in function of $h$ ($m = 1$).
Indifference price for discrete and delayed hedging with $h = 2$ months ($m = 1$), with optimal initial endowment $Y_0$ in stock, in function of the maturity.
Difference of the Indifference price w.r.t. BS price for discrete and delayed hedging with $h = 2$ months ($m = 1$), with optimal initial endowment $Y_0$ in stock, in function of the maturity.

![Graph](image)