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Numerical Derivative Pricing in Non-BS Markets

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Goal: Unified methodology for pricing & hedging for all market models and contracts

$X = \log S \in \mathbb{R}^d$, strong Markov process .

Find arbitrage-free price u of contract on X,

$$u(t,x) = \mathbb{E}[g(X_T) \mid X_t = x].$$

Solve generalized BS-equation,

$$u_t + \mathcal{A}u = 0, \qquad u|_{t=T} = g,$$

where $\mathcal{A}$ is the infinitesimal generator of X.

Numerical pricing: MCM, FFT, FDM, FEM

FEM $\Rightarrow$ Find a solution $u(t,\cdot) \in V$ such that

$$\langle \frac{\partial u}{\partial t}, v\rangle + \underbrace{\langle \mathcal{A}u, v\rangle}_{\mathcal{E}(u,v)} = 0 \quad \text{for all } v \in V \subset \text{Domain}(\mathcal{E}).$$

Finite dimensional subspace[1] $V^L \subset V \Rightarrow$ Matrix Problem:

$$\mathbf{M}\underline{U}_L'(t) + \mathbf{A}\underline{U}_L(t) = 0, \quad t \in (0,T),$$
$$\underline{U}_L(0) = \underline{U}_0 .$$

This setting applies to any underlying modelled by strong Markov process.

[1] L indicates number of refinement steps or *Level*.

Quadratic Hedging and Greeks

- Calculate derivative price u.
- Solve **same** PIDE for hedging [2] error h or sensitivity $s = \partial_\eta u$ to a model parameter η, e.g. vega ($\partial_\sigma u$)

$$\frac{\partial}{\partial t} h(t,x) - \mathcal{A}h(t,x) = \left(\Gamma(u,u) - \frac{\Gamma(u,\mathrm{id})^2}{\Gamma(\mathrm{id},\mathrm{id})} \right)(t,x).$$

$$\frac{\partial}{\partial t} s(t,x) - \mathcal{A}s(t,x) = -\partial_\eta \mathcal{A}[u](t,x)\,.$$

- Remaining Greeks, e.g. delta, theta, gamma,... can be obtained directly via postprocessing

[2] cf. Föllmer, Sondermann, 1986; Bouleau, Lamberton, 1989; Kallsen, Vesenmayer, 2007; Γ = carré-du-champ operator.

Dirichlet form of multivariate Lévy process

X determined by characteristic triplet $(\gamma,\ \mathcal{Q},\ \nu)$. Then

$$\mathcal{E}(u,v)=\int_{\mathbb{R}^d}\psi_X(\xi)\widehat{u}(\xi)\overline{\widehat{v}(\xi)}\mathrm{d}\xi=\mathcal{E}_\gamma(u,v)+\mathcal{E}_{\mathcal{Q}}(u,v)+\mathcal{E}_\nu(u,v),$$

with

$$\mathcal{E}_\gamma(u,v)=\langle\gamma\nabla u,v\rangle,$$

$$\mathcal{E}_{\mathcal{Q}}(u,v)=\frac{1}{2}\langle\mathcal{Q}\nabla u,\nabla v\rangle\,,$$

$$\mathcal{E}_\nu(u,v)=-\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left(u(x+z)-u(x)-z\,\nabla u(x)1_{|z|\leq 1}\right)v(x)\nu(\mathrm{d}z)\mathrm{d}x\,.$$

Semigroup: $T_t u(x)=\int_{\mathbb{R}^d}e^{i\langle\xi,x\rangle}e^{-t\psi_X(\xi)}\widehat{u}(\xi)d\xi$.

Contracts on baskets ⇒ multivariate processes

Modelling issue:

- Parametrization of dependence
  - Drift part deterministic ⇒ no dependence,
  - Diffusion part ⇒ covariance matrix,
  - Pure jump part ⇒ Lévy copula.

Characteristic exponent:

$$\psi_X(\xi) = i\langle\gamma,\xi\rangle - \frac{1}{2}\langle\xi,\mathcal{Q}\xi\rangle + \int_{\mathbb{R}^d}\left(e^{i\langle\xi,x\rangle} - 1 - i\langle\xi,x\rangle 1_{|x|\leq 1}\right)\nu(\mathrm{d}x),$$

with drift vector $\gamma \in \mathbb{R}^d$; volatility correlation matrix $\mathcal{Q} \in \mathbb{R}^{d\times d}$; Lévy measure $\nu(\mathrm{d}z)$, $\int_{\mathbb{R}^d}(1 \wedge |z|^2)\nu(\mathrm{d}z) < \infty$.

Sklar's theorem for Lévy copulas

The tail integral $U : \mathbb{R}^d \backslash \{0\} \to \mathbb{R}$

$$U(x_1, \dots, x_d) = \prod_{i=1}^{d} \operatorname{sgn}(x_j) \nu \left(\prod_{j=1}^{d} \left\{ \begin{array}{ll} (x_j, \infty) & \text{for } x_j > 0 \\ (-\infty, x_j] & \text{for } x_j < 0 \end{array} \right\} \right) .$$

Theorem. $\forall$ Lévy Process X $\exists$ Lévy Copula F such that

$$U^I\left((x_i)_{i \in I}\right) = F^I\left((U_i(x_i))_{i \in I}\right), \quad \forall I \subset \{1, \dots, d\} . \tag{1}$$

Conversely,

$\forall$ Lévy Copula F and given $U_i : \mathbb{R} \to \mathbb{R}_+$ $\exists$ Lévy Process X such that (1) holds for marginal tail integrals of X.

Density of X:

$$k(x_1, \dots, x_d) = [\partial_1 \dots \partial_d F] \left(U_1(x_1), \dots, U_d(x_d)\right) \cdot k_1(x_1) \dots k_d(x_d) .$$

Well-posedness requires certain assumptions

Recall: Find a solution $u(t,\cdot) \in \text{Domain}(\mathcal{E})$ such that

$$\langle \frac{\partial u}{\partial t}, v \rangle + \mathcal{E}(u,v) = 0 \quad \text{for all } v \in \text{Domain}(\mathcal{E}). \tag{2}$$

Assumptions:

- F is 1-homogeneous, i.e. $F(tx_1,\ldots,tx_d) = tF(x_1,\ldots,x_d)$, $t > 0$
- Marginal measures $\nu_i(dx_i) = k_i(x_i)dx_i$ with

$$k_i(z) \sim \frac{1}{|z|^{1+Y_i}}, \quad |z| \leq 1.$$

Admissible margins: CGMY, NIG, Meixner, spectrally negative, ...

Well-posedness + explicit domain characterization

Lemma. Multivariate process X satisfies

$$|\Im\psi_X(\xi)| \lesssim \Re\psi_X(\xi), \quad \forall \xi \in \mathbb{R}^d.$$

Theorem. $Domain(\mathcal{E}) = H^{(Y_1/2,\dots,Y_d/2)}(\mathbb{R}^d)$ and there exist γ, $c > 0$, $C \geq 0$ such that

$$\mathcal{E}(u,u) \geq \gamma \, \|u\|^2_{H^{(Y_1/2,\dots,Y_d/2)}(\mathbb{R}^d)} - C \, \|u\|^2_{L^2(\mathbb{R}^d)},$$

and

$$|\mathcal{E}(u,v)| \leq c \|u\|_{H^{(Y_1/2,\dots,Y_d/2)}(\mathbb{R}^d)} \|v\|_{H^{(Y_1/2,\dots,Y_d/2)}(\mathbb{R}^d)}.$$

The associated semigroup is therefore analytic and the variational problem is well-posed.

Implementation in three steps

1. Localization: Truncation to bounded log price domain
2. Space discretization: Matrix problem

$$\mathbf{M}\underline{U}_L'(t) + \mathbf{A}(t)\underline{U}_L(t) = 0, \quad t \in (0,T)\,,$$
$$\underline{U}_L(0) = \underline{U}_0\,.$$

3. Time discretization: E.g. backward Euler [3]

$$(\mathbf{M} + \Delta t\mathbf{A}(t_m))\,\underline{U}(t_m) = \mathbf{M}\underline{U}(t_{m-1})\,, \qquad 1 \le m \le T/\Delta t\,,$$
$$\underline{U}(0) = \underline{U}_0\,.$$

Not restricted to Lévy processes; e.g. additive processes.

[3] Other schemes: Crank-Nicolson, *hp* discontinuous Galerkin.

Localization based on marginal tail decay

For implementation, a bounded spatial domain $\square := [-R, R]^d$ is required. Let

$$\tilde{H}^{(s_1,\dots,s_d)}(\square) := \overline{\{\bar{u} | u \in C_0^\infty(\square)\}}^{H^{s_1,\dots s_d}(\mathbb{R}^d)}$$

where $\bar{u}$ is zero-extension of u to $\mathbb{R}^d$.
Find $u_R(t,\cdot) \in \tilde{H}^{(Y_1/2,\dots,Y_d/2)}(\square)$ such that

$$\langle \frac{\partial u_R}{\partial t}, v_R \rangle + \mathcal{E}(u_R, v_R) = 0 \quad \text{for all } v_R \in \tilde{H}^{(Y_1/2,\dots,Y_d/2)}(\square).$$

Proposition. This is still well-posed and $\|u_R - u\|_{L^\infty} \lesssim e^{-cR}$, provided marginal Lévy densities have semi-heavy tails.

Space discretization: Number of matrix entries needs to be reduced

Issue 1: “Curse of dimension” $\Rightarrow \mathcal{O}\left(h^{-2d}\right)$ matrix entries[4].

Issue 2: Non-local generator $\mathcal{A} \Rightarrow$ matrix not sparse.

Remedies:

- Sparse Grids: $\mathcal{O}\left(h^{-2}|\log h|^{2(d-1)}\right)$ entries,
- Wavelet compression : $\mathcal{O}\left(h^{-d}\right)$ entries.

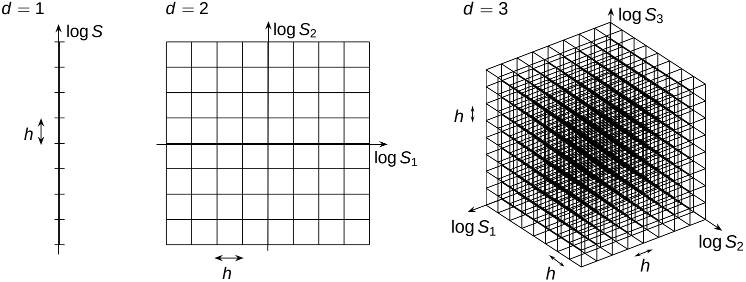
Combination[5]: Asymptotically optimal complexity,

$$\mathcal{O}\big(h^{-1}|\log h|^{2(d-1)}\big) \text{ non-zero matrix entries.}$$

[4] Meshwidth $h = 2^{-L}$.

[5] Provided mixed Sobolev smoothness of solution u.

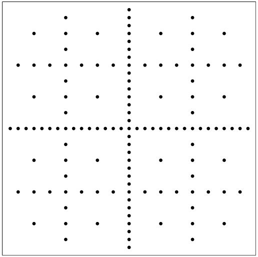
Curse of dimension ...



$N = \mathcal{O}(h^{-1})$ $N = \mathcal{O}(h^{-2})$ $N = \mathcal{O}(h^{-3})$

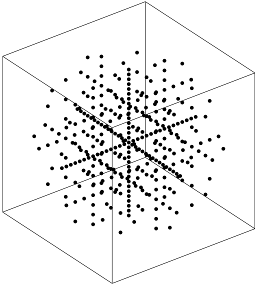
... overcome by sparse grid

$d = 2$



$N = \mathcal{O}(h^{-1}|\log h|)$

$d = 3$



$N = \mathcal{O}(h^{-1}|\log h|^2)$

Sparse grid preserves stability and convergence rates.

Advantages of wavelet bases

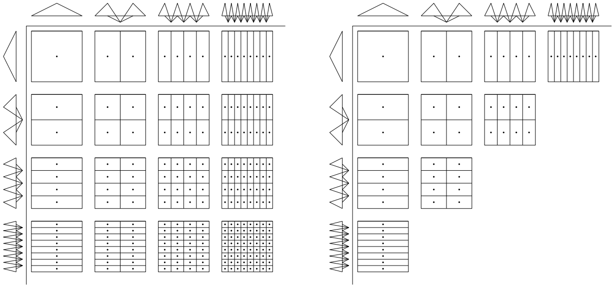
- Norm equivalences $\Rightarrow$ Sharp Numerical Analysis
- Break curse of dimension
- Multiscale compression of jump measure of X
- Efficient preconditioning

Wavelets allow for efficient treatment of (moderate) multidimensional problems.

Sparse tensor product wavelets

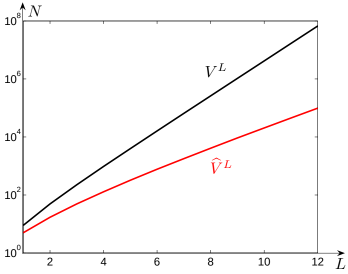
$$V^L = \mathrm{span}\{\psi_{j_1}^{\ell_1}(x_1)\cdots\psi_{j_d}^{\ell_d}(x_d) \mid 1 \le j_i \le M^{\ell_i}, 0 \le \ell_i \le L\},$$

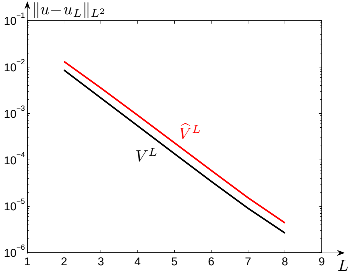
$$\widehat{V}^L = \mathrm{span}\{\psi_{j_1}^{\ell_1}(x_1)\cdots\psi_{j_d}^{\ell_d}(x_d) \mid 1 \le j_i \le M^{\ell_i}, \ell_1 + \cdots + \ell_d \le L\}.$$



Left: V^L. Right: $\widehat{V}^L$.

Sparse tensor product spaces reduce complexity

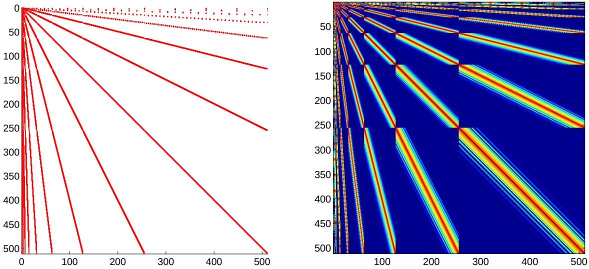




Left: Dimension of $\widehat{V}^L$, V^L
Right: L^2-Convergence.

Wavelet basis reduces Lévy to BS complexity

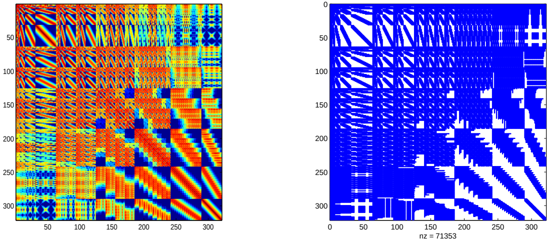
Density pattern for FEM matrix **A** for $L = 8$ refinement steps (512 mesh points).



Left: Matrix for Black-Scholes process
Right: Matrix for tempered stable process

A-priori wavelet compression preserves convergence rate.

Numerical compression of jump measure [6]



Left: Actual stiffness matrix of $\mathcal{A}$ for $L = 5$, Clayton Lévy copula with CGMY margins
Right: Prediction by the compression scheme.

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[6] Reich, Diss ETH 2008

Matrix estimates based on distance of supports

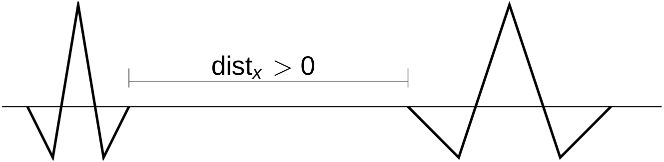
Example: Dimension $d = 2$.

Spline-wavelets $\{\psi_{\underline{\ell},\underline{k}}, \underline{\ell} = (\ell_1, \ell_2) : |\underline{\ell}|_1 \leq L\}$ of degree $p \in \mathbb{N}$, and $\widetilde{p} \geq p$ vanishing moments.

Theorem. Under the above assumptions,

$$|\langle \mathcal{A}\psi_{\underline{\ell},\underline{k}}, \psi_{\underline{\ell}',\underline{k}'}\rangle| \lesssim 2^{-\frac{1}{2}(|\underline{\ell}|_1+|\underline{\ell}'|_1)} 2^{-\widetilde{p}(\ell^{(1)}+\ell^{(2)})} \mathrm{dist}_{xy}^{-(2+\max\{Y_i\}+2\widetilde{p})},$$

where $\ell^{(1)} \neq \ell^{(2)}$ may be any two of the four level indices.



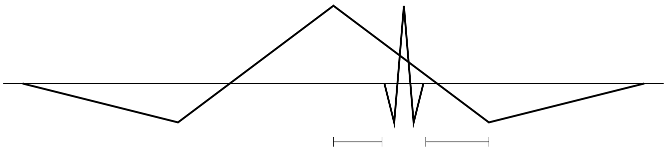
Matrix estimates based on distance of sing-supp'ts

Theorem. Assume $\ell_1' > \ell_1$, $\ell_2' > \ell_2$. If $\mathrm{d}_{\mathrm{sing}-x} \gtrsim 2^{-\ell_1'}$,

$$|\langle \mathcal{A}\psi_{\underline{\ell},\underline{k}}, \psi_{\underline{\ell}',\underline{k}'}\rangle| \lesssim 2^{\frac{\ell_1-\ell_1'}{2}} 2^{-\widetilde{p}\ell_1'} \mathrm{d}_{\mathrm{sing}-x}^{-(1+Y_1+\widetilde{p})}.$$

If $\mathrm{d}_{\mathrm{sing}-y} \gtrsim 2^{-\ell_2'}$,

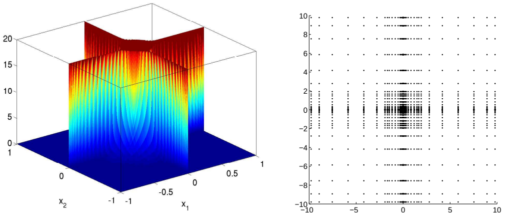
$$|\langle \mathcal{A}\psi_{\underline{\ell},\underline{k}}, \psi_{\underline{\ell}',\underline{k}'}\rangle| \lesssim 2^{\frac{\ell_2-\ell_2'}{2}} 2^{-\widetilde{p}\ell_2'} \mathrm{d}_{\mathrm{sing}-y}^{-(1+Y_2+\widetilde{p})}.$$



Implementation ⇒ Quadrature w.r.t. jump kernel k [7]

Need to find $u \in V^L$ such that $\left\langle \frac{\partial u^L}{\partial t}, v \right\rangle + \left\langle \mathcal{A} u^L, v \right\rangle = 0$, with

$$\mathcal{A}_\nu(u)(x) = -\int_{\mathbb{R}^d} \left(u(x+z) - u(x) - \sum_{i=1}^{d} z_i 1_{\{|z|<1\}} \frac{\partial u}{\partial x_i}(x) \right) \nu(\mathrm{d}z)\,.$$

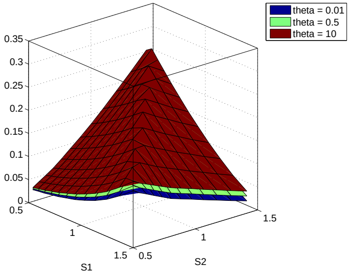


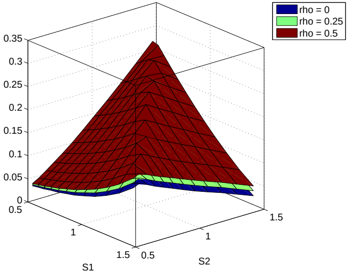
Left: Kernel k, weak dependence $\theta = 0.5$.
Right: Quadrature points, $N = 6$ refinement levels.

[7]Winter, Diss ETH 2008

Results: Time value for basket options

Let $T = 1.0$, $r = 0$ and payoff $g = \left(\frac{1}{2}(S_1 + S_2) - 1\right)^+$.





Left: Lévy model with Clayton copula, $Y = (0.5, 1.5)$ and $\eta = 1$.
Right: Black-Scholes model with same correlation ρ.

American style contracts

Optimal stopping, free boundary problem

$$u(t,x) = \sup_{t \le \tau \le T} \mathbb{E}[g(X_\tau) \mid X_t = x].$$

Solve generalized BS-inequality (in viscosity sense),

$$\begin{aligned} u_t + \mathcal{A}u &\le 0, \\ u(t,\cdot) &\ge g, \\ (u-g)(u_t + \mathcal{A}u) &= 0. \end{aligned}$$

Variational inequality:

Find $u(t,\cdot) \in K_g := \{v \in V \mid v \ge g \ \ a.e.\}$ such that

$$\langle \frac{\partial u}{\partial t}, v - u \rangle + \underbrace{\langle \mathcal{A}u, v - u \rangle}_{\mathcal{E}(u, v-u)} \le 0 \quad \text{for all } v \in K_g.$$

American contracts ⇒ Sequence of matrix LCPs

Find $\underline{U}(t_m) \in \underline{K}$ such that for all $\underline{v} \in \underline{K}$

$$\begin{aligned} \left(\underline{v} - \underline{U}(t_m)\right)^\top (\mathbf{M} + \Delta t\mathbf{A})\underline{U}(t_m) &\geq \left(\underline{v} - \underline{U}(t_m)\right)^\top \mathbf{M}\underline{U}(t_{m-1}), \\ \underline{U}(0) &= \underline{U}_0, \end{aligned}$$

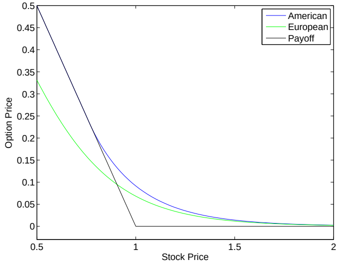
with $\underline{K} := \{\underline{v} \in \mathbb{R}^{\dim V^L} \mid \underline{v} \geq \underline{g}\}$.

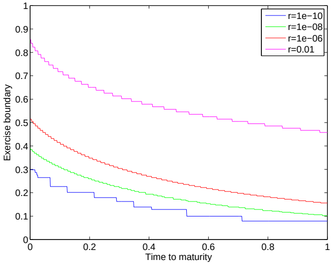
Equivalent

$$\begin{aligned} \underline{U}(t_m) &\geq \underline{g}, \\ (\mathbf{M} + \Delta t\mathbf{A})\underline{U}(t_m) &\geq \mathbf{M}\underline{U}(t_{m-1}), \\ (\underline{U}(t_m) - \underline{g})^\top \left((\mathbf{M} + \Delta t\mathbf{A})\underline{U}(t_m) - \mathbf{M}\underline{U}(t_{m-1})\right) &= 0. \end{aligned}$$

Results: European and American option, CGMY [8]

Let $C = 1$, $G = 12$, $M = 14$, $Y = 1$ and refinement level $L = 8$.





Left: European and American option price for $r = 0.2$.
Right: Free boundary for $r \to 0$.

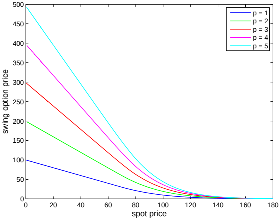
[8] Matache, Nitsche, Schwab, 2005

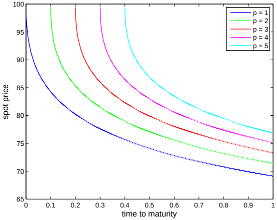
American swing option in BS market [9]

$$u(t,s) := \sup_{\tau \in \mathcal{T}_t} \mathbb{E}\left[\sum_{i=1}^{p} e^{-r(\tau_i - t)} g(\tau_i, S_{\tau_i}) \mid S_t = s\right]$$

with refraction period δ, p exercise rights and stopping times

$$\mathcal{T}_t := \{\tau = (\tau_1, \ldots, \tau_p) \mid \tau_{i+1} - \tau_i > \delta\}.$$





Left: Swing option price. Right: free boundary.

[9] Wilhelm, Winter, 2006

Stochastic Volatility: Pure diffusion model [10]

Mean reverting OU process:

$$\begin{aligned} \mathrm{d}S_t &= \mu S_t \mathrm{d}t + \sigma_t S_t \mathrm{d}\widehat{W}_t \\ \sigma_t &= f(Y_t) \\ \mathrm{d}Y_t &= \alpha(m - Y_t)\mathrm{d}t + \beta \mathrm{d}\widetilde{W}_t, \end{aligned}$$

where

- $\widetilde{W}_t, \widehat{W}_t$ are Brownian motions,

$$\widetilde{W}_t = \rho \widehat{W}_t + \sqrt{1-\rho^2} W_t, \quad \rho \in [-1, 1].$$

- $f : \mathbb{R} \to \mathbb{R}_+$ (Stein-Stein: $f(y) = |y|$ for $\rho = 0$).
- $\alpha > 0$, $m > 0$, $\beta, \mu \in \mathbb{R}$.

[10] Fouque, Papanicolaou, Sircar, 2000

Infinitesimal generator splits into two parts

Need to solve $u_t + \mathcal{A}u = 0$. Under risk-neutral measure,

$$\mathcal{A} = \mathcal{A}_\gamma + \mathcal{A}_\mathcal{Q},$$

where[11]

$$\begin{aligned}
\mathcal{A}_\gamma(u)(S,y) &= \beta(S,y)\nabla u(S,y), \\
\beta(S,y) &= \begin{pmatrix} rS \\ \alpha(m-y) - \beta\left(\rho\frac{\mu-r}{f(y)} + \delta(S,y)\sqrt{1-\rho^2}\right) \end{pmatrix}, \\
\mathcal{A}_\mathcal{Q}(u)(S,y) &= -\frac{1}{2}\mathcal{Q}(S,y) : D^2u(S,y), \\
\mathcal{Q}(S,y) &= \begin{pmatrix} f^2(y)S^2 & \beta\rho S \\ \beta\rho S & \beta^2 \end{pmatrix}.
\end{aligned}$$

[11] $\delta(S,y)$ represents the risk premium factor.

Stochastic Volatility: Jump models

The BNS model[12]:

$$
\begin{aligned}
dX_t &= (\mu + \beta\sigma_t^2)dt + \sigma_t dW_t + \rho\, dZ_{\lambda t} \\
d\sigma_t^2 &= -\lambda\sigma_t^2 dt + dZ_{\lambda t}, \quad \sigma_0^2 > 0,
\end{aligned}
$$

where

- W_t a Brownian motion.
- Z_t a subordinator.
- W and Z are independent.
- $\beta, \mu, \rho, \lambda \in \mathbb{R}$, $\lambda > 0$, $\rho \le 0$.

[12] Barndorff-Nielsen and Shepard '01

Infinitesimal generator splits into three parts

Generator of Markov process (X, σ^2), under risk-neutral measure:

$$\mathcal{A} = \mathcal{A}_\gamma + \mathcal{A}_\mathcal{Q} + \mathcal{A}_\nu,$$

where

$$\begin{aligned}
\mathcal{A}_\gamma(u)(x,y) &= \beta(x,y)\nabla u(x,y), \quad \beta(x,y) = \begin{pmatrix} \frac{1}{2}y + \lambda\kappa - r \\ \lambda y \end{pmatrix}, \\
\mathcal{A}_\mathcal{Q}(u)(x,y) &= -\frac{1}{2}\mathcal{Q}(x,y) : D^2u(x,y), \quad \mathcal{Q}(x,y) = \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix}, \\
\mathcal{A}_\nu(u)(x,y) &= -\lambda \int_{\mathbb{R}_+} \left[u(x+\rho z, y+z) - u(x,y)\right]\nu(\mathrm{d}z).
\end{aligned}$$

Stochastic Volatility: CoGARCH(p,q)

Brockwell et al. (2005). For $1 \leq p \leq q$:

$$\begin{aligned} \mathrm{d}X_t &= \sqrt{\sigma_{t-}}\mathrm{d}L_t \\ \sigma_t &= \alpha_0 + \alpha^\top Y_{t-} \\ \mathrm{d}Y_t &= BY_{t-}\mathrm{d}t + e\sigma_{t-}\mathrm{d}[L,L]_t^{(d)}, \end{aligned}$$

where

- L a $\mathbb{R}$-valued Lévy process.
- $\alpha_0 > 0$, $\alpha^\top \in \mathbb{R}^q$ with $\alpha_p \neq 0$, $\alpha_{p+1} = \ldots = \alpha_q = 0$.
- $e \in \mathbb{R}^q$ the q-th standard basis vector

$$\mathbb{R}^{q\times q} \ni B = \left(\begin{array}{c|c} 0 & \mathbf{I} \\ \hline -\beta_q & -\beta^\top \end{array}\right), \quad \beta_q \neq 0, \quad \beta^\top \in \mathbb{R}^{q-1}.$$

Infinitesimal generator of CoGARCH [14]

$$\mathcal{A} = \mathcal{A}_\gamma + \mathcal{A}_\mathcal{Q} + \mathcal{A}_\nu,$$

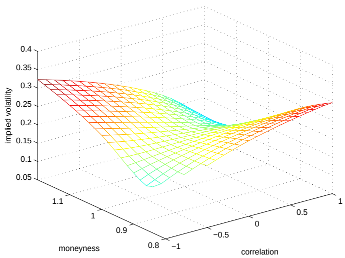
where (assume $\int_\mathbb{R} z^2 \nu_L(\mathrm{d}z) < \infty$) [13]

$$\begin{aligned}
\mathcal{A}_\gamma(u)(x) &= \beta(x)\nabla u(x), \\
\beta(x) &= \left(\gamma_L\sqrt{x_2}, \alpha^\top B\widetilde{x}, (B\widetilde{x})_3, \ldots, (B\widetilde{x})_{q+1}, (B\widetilde{x})_q\right)^\top, \\
\mathcal{A}_\mathcal{Q}(u)(x) &= -\frac{1}{2}\mathcal{Q}(x) : D^2 u(x), \quad \mathcal{Q} = \left(\begin{array}{c|c} \sigma_L^2 x_2 & 0 \\ \hline 0 & 0 \end{array}\right), \\
\mathcal{A}_\nu(u)(x) &= -\int_\mathbb{R} \left[u\left(x + \delta(x,z)\right) - u(x) - \sqrt{x}_2 z \frac{\partial u}{\partial x_1}(x)\right] \nu_L(\mathrm{d}z), \\
\delta(x,z) &= \left(\sqrt{x}_2 z, \alpha_q x_2 z^2, 0, \ldots, 0, x_2 z^2\right).
\end{aligned}$$

[13] $(\gamma_L, \sigma_L, \nu_L)$ characteristic triplet of L, $x \in \mathbb{R}^{q+2}$, $\widetilde{x} = (x_3, \ldots, x_{q+2})^\top \in \mathbb{R}^q$.
[14] Kallsen, Vesenmayer 2005; Hilber 2006

Results: Stylized facts can be recovered

Example: American put option in mean-reverting OU model

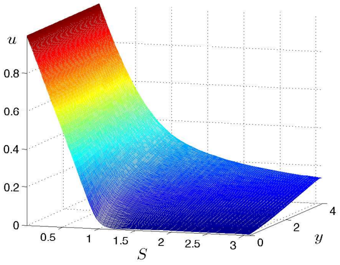


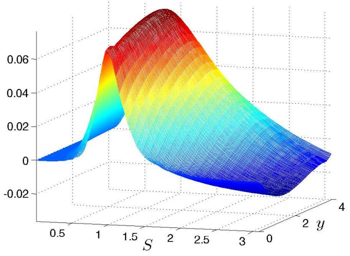
Left: Implied volatility as function of moneyness S/K and correlation factor ρ.
Right: Price difference $P^{BS} - P^{SV}$.

Results: European put in the BNS model

- $T = 0.5$, $K = 1$, $r = 0$, $\lambda = 2.5$, IG(γ, δ)-Lévy kernel

$$k(z) = \frac{\delta}{2\sqrt{2\pi}} z^{-\frac{3}{2}}(1+\gamma z)e^{-\frac{1}{2}\gamma z}, \qquad \gamma = 2, \delta = 0.0872.$$





Left: Price for $\rho = -4$.
Right: Difference of prices for $\rho = -0.01$ and $\rho = -4$.

Wrap up

Unified numerics beyond Lévy (additive, affine processes, ...):

- Derivative pricing
- Quadratic hedging and Greeks
- Exotic contracts (Asians, Bermudans,...)
- Rough payoffs (digitals, binaries,...)
- Multi period contracts (of swing type)
- Local & stochastic volatility, local jump intensity
- Baskets using sparse (log) price space
- Well conditioned matrices, singularity-free computation
- Discretization & modelling error control, adaptivity, ...