Distribution of vortices in a type II superconductor as a free boundary problem: Existence and regularity via Nash-Moser theory

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This paper is concerned with a model describing the distribution of vortices in a type II superconductor. These vortices are distributed continuously and occupy an unknown region D with ∂D representing the free boundary. The problem is set as follows : two constants $H_0 > H_1 > 0$ be given, to find an open set D subset of the smooth bounded open set $\Omega \subset \mathbf{R}^2$, and a function H defined on $\overline{\Omega \backslash D}$ such that

$$\left\{ \begin{array}{l} div(F(|\nabla H|^2)\nabla H) - H = 0 \text{ in } \Omega\backslash\overline{D} \\ \text{where the function } F \text{ is analytic positive increasing} \\ H = H_0 \text{ on } \partial\Omega \\ H = H_1 \text{ on } \partial D \\ \frac{\partial H}{\partial n} = 0 \text{ on } \partial D \end{array} \right.$$

Here we prove the existence of a solution H with a domain D having analytic boundary. We use the Nash-Moser inverse function theorem, applied to a degenerate case.

1 Introduction

1.1 Physical motivation

We are interested in a model of a type-II superconductor, submerged in a uniform magnetic field H_0. We examine different states of the material as a function of values of the

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applied magnetic field H_0. In [2], Berestycki, Bonnet and Chapman have studied the following model

$$\begin{cases} div(\frac{\nabla H}{1-u}) - H = 0 \text{ on } \Omega \subset \mathbf{R}^2 \\ \text{where } u(1-u)^2 = |\nabla H|^2 \\ H = H_0 \text{ on } \partial\Omega \end{cases}$$

which describes a cylindrical type-II superconductor $\Omega \times \mathbf{R} \subset \mathbf{R}^3$. They were interested in solutions invariant by translation along the axis of the cylinder. It then reduces to a two-dimensional problem posed on the section Ω of the cylindrical superconductor. The equations are formally derived in [2] by passing to the limit when $\kappa \to +\infty$ in Ginzburg-Landau equations. These authors have proved that for fields $H_0 \in [0, H_0^*)$ there is a unique solution corresponding to the superconducting state. The value H_0^* corresponds to the superheating field beyond which the superconducting solution ceases to be stable (Chapman [3]). This instability appears on the boundary $\partial\Omega$ of the open set when the variable u reaches the value $\frac{1}{3}$ (meaning that $|\nabla H|^2$ reaches its maximal possible value which is $\frac{4}{27}$).
In the present work we consider the superconductor as submerged in a magnetic field superior to the latter superheating field H_0^*. In this case we see that a vorticity zone appears, in which some thin filaments of normal material are surrounded by superconducting currents. This state is referred to the mixed state in the literature. In the limit $\kappa \to +\infty$ it is explained in [11] how one can formally derive a model where there exists a density of vortices in a certain zone D of the superconductor. This model is the following

$$\begin{cases} div(\frac{\nabla H}{1-u}) - H = 0 \text{ on } \Omega\backslash\overline{D} \\ \text{where } u(1-u)^2 = |\nabla H|^2 \\ H = H_0 = const \text{ on } \partial\Omega \\ H = H_1 = const \text{ on } \partial D \\ \partial_n H = 0 \text{ on } \partial D \end{cases} \tag{1.1}$$

where n is the normal to ∂D, and the new parameter H_1 ($\leq H_0$) plays the role of a Lagrange multiplier for a constraint on the quantity of vortices in the superconductor.

Although problem (1.1) can be posed as well in higher dimensions, it has a physical meaning for our present paper only in dimension 2. So we only study the problem in this physical case. From a mathematical point of view our method could be adapted to problems in higher dimensions.

The open set D could have a finite number of smooth connected components $\{D_i\}$ and our approach could also apply to the free boundary problem (1.1) with the following more general boundary conditions

$$\left\{ \begin{array}{l} H = H_i \\ \partial_n H = 0 \end{array} \right| \text{ on each } D_i$$

Nevertheless in our article we only consider the special case where $H_i = H_1$ for all index i.

1.2 Main results and ideas of proofs

In this article we state existence and analytic regularity of the solution (D, H) to problem (1.1). Assuming $\partial\Omega \in C^\infty$, we prove the existence of a branch of solutions, applying the Nash-Moser inverse function theorem to degenerate case: $H_1 = H_0$, $\Omega \backslash D = \emptyset$. We also prove some estimates for an associated degenerate elliptic problem.
As in [2], we are interested in solutions of (1.1) such that $u < \frac{1}{3}$. The analysis of Chapman [3] shows linear instability for solutions with $u \geq \frac{1}{3}$. We restrict ourselves to the study of solutions (D, H) to the following nonlinear equation:

$$\left\{ \begin{array}{l} div(F(|\nabla H|^2)\nabla H) - H = 0 \text{ on } \Omega \backslash \overline{D} \\ H = H_0 = const \text{ on } \partial\Omega \\ H = H_1 = const \text{ on } \partial D \\ \partial_n H = 0 \text{ on } \partial D \end{array} \right. \tag{1.2}$$

where $F(s)$ is defined for $s \in [0, \frac{4}{27})$ by $F(s) = \frac{1}{1-u}$ and u is the unique solution to $u(1-u)^2 = s$ such that $0 \leq u < \frac{1}{3}$. In [11] is proved the existence of a weak solution of (1.2) such that $|\nabla H|^2 < \frac{4}{27} - \delta$ with some $\delta > 0$, as a solution of a nonlinear obstacle problem. The uniqueness of this solution is proved too.

We are interested in the regularity of the free boundary ∂D. Our main result is the following

Theorem 1.1 $\forall H_0 > 0,\ \exists \underline{H_1} \in (0, H_0),\ \forall H_1 \in (\underline{H_1}, H_0)$: *the free boundary ∂D of the solution to (1.2) is C^∞ and homeomorphic to $\partial\Omega$.*

Remark 1.2 *If we know that ∂D is C^∞, the analyticity of ∂D is then a consequence of Kinderlehrer and Nirenberg's results [10], which deal with the case of analytic elliptic equations, with analytic boundary conditions on the free boundary.*

The interest of our article lies on the one hand in the result of C^∞ regularity of the free boundary ∂D, and on the other hand in the use of the Nash-Moser inverse function theorem applied in a degenerate case. For references on this theorem, see the book of Alinhac and Gérard [1], and Hamilton [9]. An other proof of theorem 1.1 can be derived

by arguments coming from free boundary theory for the obstacle problem (see [11]).
In section 3 we prove in particular the following result

THEOREM 1.3 *Let (D^*, H^*) a particular solution of (1.2) for $H_1 = H_1^* \in (0, H_0)$ such that $D^* \subset\subset \Omega$ and $\partial D^* \in C^\infty$. Then problem (1.2) has a solution (D, H) with $\partial D \in C^\infty$, for every H_1 in a neighbourhood of H_1^*.*

Some results similar to theorem 1.3 are proved in Hamilton [9] and Schaeffer [14], and correspond to the use of the Nash-Moser theorem for a nondegenerate elliptic problem. Some degenerate cases are studied in Schaeffer [15] (the capacitor problem) and in Plotnikov [13] with methods different from those used in this article.

The problem (1.2) we study is an extension of both Berestycki Bonnet Chapman [2] and Chapman Rubinstein Schatzman [4] combining the nonlinear field with the existence of a free boundary. In [4] Chapman, Rubistein and Schatzman consider a linearized version of model (1.2) for small H_0. They formally prove a result similar to theorem 1.1 using asymptotic analysis.
Here, to prove theorem 1.1, we apply the inverse function theorem begining with the solution for $H_1^* = H_0$

$$\begin{cases} D^* = \Omega \\ H^* \equiv H_0 \text{ on } \partial\Omega = \partial D^* \end{cases}$$

This solution is unusual because $\Omega \backslash D^* = \emptyset$. We obtain a branch of solutions for H_1 in an interval $(\underline{H_1}, H_0]$, where $\underline{H_1}$ is chosen close enough to $H_1^* = H_0$.

More precisely, we proceed in the following way:

To simplify, we set $\Gamma = \partial D$. We split up problem (1.2) into two sub-problems
Step 1
Given Γ, find H such that

$$\begin{cases} div(F(|\nabla H|^2)\nabla H) - H = 0 \text{ on } \Omega\backslash\overline{D} \\ H = H_0 = const \text{ on } \partial\Omega \\ \partial_n H = 0 \text{ on } \Gamma \end{cases} \tag{1.3}$$

We obtain the solution $H = H(\Gamma)$, by local inversion of problem (1.3), for Γ in a neighbourhood of $\Gamma^* = \partial D^*$.
Step 2
We impose to $H(\Gamma)$ the constraint

$$H(\Gamma)_{|\Gamma} = H_1 \tag{1.4}$$

which allows us to obtain $\Gamma = \Gamma(H_1)$, once more by local inversion, for H_1 in a neighbourhood of H_1^*.

This method presents two main difficulties, that we examine briefly
i) Why the usual inverse function theorem does not apply?
ii) How can we deal with the degeneracy: $\Omega\backslash\overline{D} \longrightarrow \emptyset$ as $H_1 \longrightarrow H_0$?

i) Application of the inverse function theorem

The inverse function theorem in Banach spaces $C^{q,\alpha}$ can be applied to step 1 and gives $H = H(\Gamma)$. On the contrary, if we note $\Phi : C^{q,\alpha} \longrightarrow C^{q,\alpha}, \Gamma \longmapsto H(\Gamma)_{|\Gamma}$, a computation proves that the differential $D\Phi$ is a $1-1$ map from $C^{q-1,\alpha}$ into $C^{q,\alpha}$ (and thus it is not invertible from $C^{q,\alpha}$ to $C^{q,\alpha}$), which forbids us in step 2 to use the inverse function theorem in Banach spaces $C^{q,\alpha}$.
In reality the loss of derivative of one unity on the inverse of the differential of Φ comes from the fact that we impose two limit conditions on the free boundary Γ: a Dirichlet condition $H = H_1$ and a Neumann condition $\partial_n H = 0$ (see subsection 3.2).
But if we consider Φ as a map $\Phi : C^\infty \longrightarrow C^\infty$, then $D\Phi$ is a $1-1$ map from C^∞ into C^∞, and the Nash-Moser inverse function theorem can be applied.

ii) Degeneracy : $\Omega\backslash\overline{D} \longrightarrow \emptyset$ when $H_1 \longrightarrow H_0$

To apply the Nash-Moser theorem, we must prove that $(D\Phi)^{-1}$ is regular (precisely C^∞*-tame*, see subsection 2.1) until $H_1 = H_0$.
If ϵ is the order of the distance between Γ and $\partial\Omega$, then when $H_1 \longrightarrow H_0$, we have $\epsilon \longrightarrow 0$, and the difficulty is to prove that $(D\Phi)^{-1}$ is C^∞-tame until $\epsilon = 0$.
The main idea is to scale the coordinate normal to the boundary so that the problem of the domain shrinking as $\epsilon \to 0$ is transformed to that elliptic constant tending to zero.
Hence, inverse the differential $D\Phi$ is reduced to solve an elliptic equation on the fixed domain $\mathbf{S}^1 \times (0,1)$ whose coordinates are $x = (s, \rho)$. Given coefficients $A = (a_{ij}, b_j, c)$ and a right side $k = (k_0, k_1, k_2)$, we search for the solution w to the following equation

$$\begin{cases} a_{ij}(x)\partial_{ij}w + b_j(x)\partial_j w + c(x)w = k_0 \\ \partial_\rho w(\rho = 1, s) = k_1 \\ w(\rho = 0, s) = k_2 \end{cases} \tag{1.5}$$

We use the classical repeated index summation convention with $i, j \in \{\sigma, \rho\}$ and $\partial_\sigma = \epsilon\partial_s$.
To be clear we have for example

$$a_{ij}(x)\partial_{ij}w = \epsilon^2 a_{\sigma\sigma}(x)\partial_{ss}w + 2\epsilon a_{\sigma\rho}\partial_{s\rho}w + a_{\rho\rho}(x)\partial_{\rho\rho}w$$

In particular for $\epsilon = 0$, problem (1.5) is degenerate. We assume that:

$$\exists c_0 > 0,\ \forall x \in \mathbf{S}^1 \times [0,1], \forall \xi \in \mathbf{R}^2, a_{ij}(x)\xi_i\xi_j \geq c_0|\xi|^2 \tag{1.6}$$

We prove the following result of independent interest

Theorem 1.4 *[Tame ellipticity in a degenerate case]*
Let us consider a solution w of problem (1.5) which we can write

$$\mathcal{L}(A,\epsilon)w = k$$

If there exists particular coefficients $A^* \in C^\infty$ which satisfy (1.6) and such that $\mathcal{L}(A^*,0)$ satisfies the maximum principle (see subsection 5.1 for precisions), then $w = w(A,k,\epsilon)$ exists and is C^∞-tame in the right side $k \in C^\infty$ of the equation and in the coefficients (A,ϵ) in a neighbourhood of $(A^*,0)$.

1.3 Organization of the article

In section 2 we recall some elements on tame maps and the Nash-Moser theorem, for the reader who is not familiar with this theory. In section 3 is given the proof of theorem 1.3. In particular we explain in detail why the inverse function theorem in spaces $C^{q,\alpha}$ does not apply. In section 4 is given the proof of theorem 1.1. The last section 5 is devoted to the proof of theorem 1.4.

2 Preliminaries: tame maps and the Nash-Moser theorem

We recall below elements of the Nash-Moser theory that are used to prove the results presented in this article. For this presentation we have taken our inspiration from Hamilton [9]. In the following sections we will often refer to these preliminaries, but those of readers who know the Nash-Moser theory can skip section 2 and start with section 3.

2.1 Definitions

Let us denote one $\alpha \in (0,1)$ which will be fixed once for all. We recall usual notations. For an integer q and a function u defined on a set $\Omega \subset \mathbf{R}^n$, $D^q u$ denotes all the partial derivatives of total order equal to q; $[u]_{q,\alpha;\Omega} = \sup_{x,y\in\Omega, x\neq y} \dfrac{|D^q u(x) - D^q u(y)|}{|x-y|^\alpha}$, $|u|_{0;\Omega} = \sup_{x\in\Omega} |u(x)|$, and $|u|_{q,\alpha;\Omega} = \sum_{j=0}^q |D^j u|_{0;\Omega} + [u]_{q,\alpha;\Omega}$.

Continuity
Let a map $\Phi : C^\infty(M) \to C^\infty(N)$, where M and N are C^∞ compact manifolds, possibly with boundaries. The sequences of seminorms $(|\cdot|_{q,\alpha})_{q\in\mathbf{N}}$ on M and N, allow to define

the continuity of the map Φ in $u_0 \in C^\infty(M)$, as

$$\begin{array}{l}\forall q \in \mathbf{N},\ \eta > 0,\ \exists q' \in \mathbf{N},\ \eta' > 0,\ \forall v \in C^\infty(M),\\ (|v - u_0|_{q',\alpha;M} < \eta') \Longrightarrow (|\Phi(v) - \Phi(u_0)|_{q,\alpha;N} < \eta)\end{array} \tag{2.1}$$

This definition of the continuity is that of maps between Frechet spaces.
Classically, if we consider $\Phi : U \to C^\infty(N)$ where U is an open set of $C^\infty(M)$, we will say that Φ is C^0 (i.e. Φ is continuous) if and only if it is continuous in every point $u \in U$.

Differentiation

The map Φ is C^1 if and only if Φ is continuous and if there exists a continuous map denoted $D\Phi : U \times C^\infty(M) \to C^\infty(N)$ linear in the second variable and such that: $\forall u \in U,\ \forall v \in C^\infty(M),\ \lim\limits_{t \to 0} \dfrac{\Phi(u+tv) - \Phi(u)}{t} = D\Phi(u) \cdot v$. We define in the same way the successive differentiations and we say that Φ is C^q if $D^{q-1}\Phi$ exists and is C^1. We say that Φ is C^∞ if it is C^q for all q.
In particular to verify the continuity of $D^q\Phi$ on $U \times (C^\infty(M))^q \ni (u, v_1, ...v_q)$ we use the natural grading (in the sense of Hamilton [9]) $|(u, v_1, ..., v_q)|_{p,\alpha;M} = |u|_{p,\alpha;M} + \sum_{i=1}^{q} |v_i|_{p,\alpha;M}$.

Tame maps

The map $\Phi : U \subset C^\infty(M) \to C^\infty(N)$ is C^0-tame if and only if it is C^0 and if it satisfies the tame inequality

$$\exists r \in \mathbf{N},\ \forall q \in \mathbf{N},\ \exists C_{q,\alpha} > 0,\ \forall u \in U,\ |\Phi(u)|_{q,\alpha} \leq C_{q,\alpha}(1 + |u|_{q+r,\alpha}) \tag{2.2}$$

It requires that we have a linear estimate on a nonlinear map Φ, and this for all seminorms $|\cdot|_{q,\alpha;N}$. We may have a loss of derivatives $r \geq 0$ possibly large but independent on q. In addition we recall that the Nash-Moser theorem stays true for the following more general definition of tame maps where (2.2) in the previous definition is replaced by

$$\begin{array}{l}\forall u_0 \in U,\ \exists \text{ neighbourhood } \mathcal{V}(u_0) \subset U,\ \exists r \in \mathbf{N},\ \exists b \in \mathbf{N},\ \forall q \geq b,\ \exists C_{q,\alpha} > 0,\\ \forall u \in \mathcal{V}(u_0),\ |\Phi(u)|_{q,\alpha} \leq C_{q,\alpha}(1 + |u|_{q+r,\alpha})\end{array} \tag{2.3}$$

In the same way Φ is C^q-tame if and only if the maps $(D^j\Phi)_{0 \leq j \leq q}$ are C^0-tame, and Φ is C^∞-tame if and only if it is C^q-tame for all q.

Remark 2.1 *It is easy to see that: if Φ_1 and Φ_2 are tame, then the sum $\Phi_1 + \Phi_2$, the product $\Phi_1 \cdot \Phi_2$, the quotient $\frac{\Phi_1}{\Phi_2}$, the composition $f \circ \Phi_1$ by a function $f \in C^\infty$, the composition of tame maps $\Phi_1 \circ \Phi_2$, are tame maps.*

2.2 The Nash-Moser theorem

Theorem 2.2 *[Nash-Moser]*
If M and N are C^∞ compact manifolds, possibly with boundaries, if $\Phi : U \subset C^\infty(M) \to$

$C^\infty(N)$ is C^∞-tame, and if $\forall u \in U$, $\forall k \in C^\infty(N)$, $D\Phi(u) \cdot w = k$ is invertible, with a unique solution $w = (D\Phi(u))^{-1} \cdot k$, and $(D\Phi(\cdot))^{-1}$ is a C^∞-tame map in (u, k), then Φ is locally invertible and Φ^{-1} is locally C^∞-tame.

Remark 2.3 *Contrary to the classical inverse function theorem in Banach spaces, here we must (for the spaces C^∞) check that the linear map $D\Phi(u)$ is invertible in a neighbourhood of one point u_0, and not only in the point u_0 (see Hamilton [9] for a counter-example).*

2.3 The tame ellipticity theorem

The following theorem will be used several times in the following sections. It gives the solution to a nondegenerate elliptic equation as a C^∞-tame map of the coefficients and the right side of the equation.

Theorem 2.4 *[Tame ellipticity, Hamilton [9]]*
Let the linear elliptic operator L_A of order 2, defined on a C^∞ compact manifold M, possibly with boundaries, such that denoting $A = (a_{ij}, b_j, c)$ which satisfies (1.6), we have $L_A w = a_{ij}(x)\partial_{ij} w + b_j(x)\partial_j w + c(x)w$, with boundary conditions

$$Bw = \begin{cases} w & \text{Dirichlet} \\ \text{or} & \\ \partial_n w & \text{Neumann} \end{cases}$$

where n is the exterior unit normal.
If the coefficients are C^∞, i.e. $A \in U \subset C^\infty(M)$, and if the open set U is chosen such that the system

$$\begin{cases} L_A w = k_0 \text{ on } M \\ Bw = k_B \text{ on } \partial M \end{cases}$$

has a unique solution w, then this solution $w = w(A, k_0, k_B) \in C^\infty(M)$, is C^∞-tame in A and in $k = (k_0, k_B)$. That is, w is C^∞-tame relatively to the coefficients and the right side of the equation.
In particular, if we set $t = 0$ for Dirichlet conditions, $t = 1$ for Neumann conditions, and $|k|_{q,\alpha} = |k_0|_{q,\alpha} + |k_B|_{q+2-t,\alpha}$, then we have the tame elliptic inequality

$$\begin{aligned} &\forall A^* \in U,\ \exists \eta > 0,\ \forall A \in U, \\ &(|A - A^*|_{0,\alpha} < \eta) \Longrightarrow (\forall q \in \mathbf{N},\ |w|_{q+2,\alpha} \leq C_{q+2,\alpha}\left(|k|_{q,\alpha} + |A|_{q,\alpha}|k|_{0,\alpha}\right)) \end{aligned} \tag{2.4}$$

Remark 2.5 *Theorem 2.4 is true if the Newman condition is changed in an oblique derivative. Moreover in theorem 2.4, the boundary conditions can be different on different connected components of the boundary of the manifold M. In this case inequality (2.4) should naturally be adapted.*

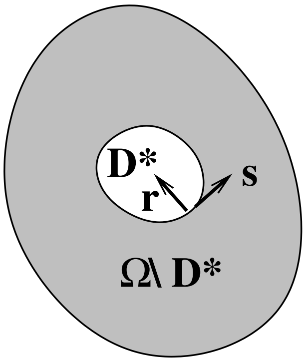

Figure 1: Local coordinates near $\Gamma^* = \partial D^*$

3 Existence result by perturbation of a smooth free boundary

In this section we will prove theorem 1.3. This kind of result in referred in the literature as the stability of the free boundary (see for example Schaeffer [14]). This approach consists to find a solution to problem (1.2) by perturbation of a particular solution.

3.1 Setting of the problem

Let us assume that a particular solution (D^*, H^*) is given with $D^* \subset\subset \Omega$, $\Gamma^* = \partial D^* \in C^\infty$ and $H^*_{|\Gamma^*} = H_1^* < H_0$. For each value of the parameter H_1 in a neighbourhood of H_1^*, we will build a smooth solution (D, H) to problem (1.2).
To simplify (without loss of generality), we assume that Ω and D^* are diffeomorphic to a disk. Then we can construct a local curvilinear parametrization (r, s) in a neighbourhood of $\Gamma^* \subset\subset \Omega$: $s \in \mathbf{S}^1$ parametrizes Γ^* proportionally to its length, and r denotes the transversal coordinate positively oriented inward the interior of D^*, with $r \in (-r_0, r_0), r_0 > 0$. Thus the boundary $\Gamma = \partial D$ of every smooth open set D close to D^* can be described in local coordinates by $r = G(s)$ where $G \in C^\infty(\mathbf{S}^1)$, and in particular, Γ^* is characterized by the equation $r = G^*(s) := 0$.

We follow the method of proof given in the introduction.
Step 1

Let us introduce the map

$$
\begin{array}{llll}
\Phi_1: & C^\infty(\mathbf{S}^1)\times C^\infty(\overline{\Omega\backslash D}) & \longrightarrow & C^\infty(\Omega\backslash\overline{D})\times C^\infty(\mathbf{S}^1)\times C^\infty(\mathbf{S}^1)\\
 & (G,H) & \longmapsto & \Phi_1(G,H)=\left\{\begin{array}{l}(\mathcal{Q}H)_{|\Omega\backslash\overline{D}}\\ H_{|\partial\Omega}-H_0\\ \partial_r H_{|\Gamma}\end{array}\right.
\end{array}
\tag{3.1}
$$

where $\mathcal{Q}$ is a quasilinear elliptic operator given by

$$
\mathcal{Q}H = a^0_{ij}(\nabla H)\partial_{ij}H - H
$$

with for every vector $V\in\mathbf{R}^2$

$$
a^0_{ij}(V) := F(|V|^2)\delta_{ij} + 2F'(|V|^2)V_iV_j \tag{3.2}
$$

and $\delta_{ij}=1$ if $i=j$, $\delta_{ij}=0$ if $i\neq j$. With these notations we rewrite (1.3) (here $\partial_r H_{|\Gamma}$ stands in place of $\partial_n H_{|\Gamma}$) as

$$
\Phi_1(G,H)=0 \tag{3.3}
$$

and we will be able to get the solution $H(G)$ by a proper inverse function theorem.

Step 2

We will set $\Phi_2(G) := H(G)_{|\Gamma}\in C^\infty(\mathbf{S}^1)$ and rewrite (1.4) as

$$
\Phi_2(G) = H_1 \tag{3.4}
$$

where H_1 is a constant close to H_1^*. Solving (3.4) will give the parametrization $G(H_1)$ of the free boundary Γ.

This method will give a posteriori $\partial_n H_{|\Gamma}=0$, since $\partial_r H_{|\Gamma}=0$, and $H_{|\Gamma}=H_1=const$ imply that $\nabla H=0$ on Γ. Let us remark that we have chosen to write equations (3.1),(3.3) with the condition $\partial_r H=0$ on Γ, and not $\partial_n H=0$ on Γ, because the vector field ∂_r is independent on Γ contrarily to the normal n.

Although the inverse function theorem in the Banach spaces $C^{q,\alpha}$ applies in step 1, we will see why it does not apply in step 2.

3.2 The inverse function theorem in spaces $C^{q,\alpha}$ does not apply

Let us assume that step 1 is solved in the spaces $C^{q,\alpha}$, i.e. that the function $H(G)$ has been obtained. Then it is not possible to solve step 2 in spaces $C^{q,\alpha}$, because of the following lemma

Lemma 3.1 *If $H: G\in C^{q,\alpha}(\mathbf{S}^1)\longmapsto H(G)\in C^{q,\alpha}(\overline{\Omega\backslash D})$ satisfies $\Phi_1(G,H(G))=0$, and if $\Phi_2: G\in C^{q,\alpha}(\mathbf{S}^1)\longmapsto\Phi_2(G)=H(G)_{|\Gamma}\in C^{q,\alpha}(\mathbf{S}^1)$ then the differential $D_G\Phi_2(G^*)$ is a $1-1$ map from $C^{q,\alpha}(\mathbf{S}^1)$ into $C^{q+1,\alpha}(\mathbf{S}^1)$.*

In particular, $D_G\Phi_2(G^*)$ is not invertible from $C^{q,\alpha}(\mathbf{S}^1)$ into $C^{q,\alpha}(\mathbf{S}^1)$.

For the proof of lemma 3.1, we use the following result

Lemma 3.2 *Under the assumptions of lemma 3.1, we get*

$$D_G\Phi_2 \cdot g = m$$

if and only if

$$g = -\left(\frac{\partial_r h}{\partial_{rr} H}\right)_{|\Gamma}$$

where h satisfies

$$\begin{cases} D_H(QH)\cdot h = 0_{|\Omega\setminus\overline{D}} \\ h_{|\partial\Omega} = 0 \\ h_{|\Gamma} = m \end{cases} \tag{3.5}$$

Proof of lemma 3.2

Let us introduce $h := D_G H \cdot g$ where $g \in C^\infty(\mathbf{S}^1)$. Thus

$$D_G\Phi_2 \cdot g = (D_G H \cdot g + g\ \partial_r H)_{|\Gamma}$$

because $\frac{d}{dt}(u_t \circ v_t) = (\frac{du_t}{dt}) \circ v_t + (Du_t \circ v_t) \cdot \frac{dv_t}{dt}$. Here we know that H satisfies $\Phi_1(G, H(G)) = 0$, and then in particular $\partial_r H_{|\Gamma} = 0$. Consequently by definition of h we have

$$D_G\Phi_2 \cdot g = h_{|\Gamma}$$

Differentiating $\Phi_1(G, H(G)) = 0$ with respect to G, we get

$$D_H(QH)\cdot h = 0_{|\Omega\setminus\overline{D}} \tag{3.6}$$

$$h_{|\partial\Omega} = 0 \tag{3.7}$$

$$(\partial_r h + g\partial_{rr} H)_{|\Gamma} = 0$$

Therefore equation $D_G\Phi_2 \cdot g = m$, written

$$h_{|\Gamma} = m \tag{3.8}$$

can be inverted into:

$$g = -\left(\frac{\partial_r h}{\partial_{rr} H}\right)_{|\Gamma}$$

where h is a solution of system (3.6),(3.7),(3.8). This ends the proof of the lemma 3.2.

Proof of lemma 3.1

For $H = H^*$, $H_1 = H_1^*$, $D = D^*$, we have $0 \neq \partial_{rr} H^*_{|\Gamma^*} \in C^\infty(\mathbf{S}^1)$. Therefore if

$m \in C^{q+1,\alpha}$, we get from lemma 3.2 $(D_G\Phi_2(G^*))^{-1}(m) = g \in C^{q,\alpha}$, and therefore $D_G\Phi_2(G^*)$ is a $1-1$ map from $C^{q,\alpha}(\mathbf{S}^1)$ to $C^{q+1,\alpha}(\mathbf{S}^1)$ which proves lemma 3.1.

There is a loss of derivatives on the inverse of the differential of Φ_2 which corresponds to the fact that we impose on Γ two boundary conditions of different orders: $H_{|\Gamma} = H_1$ and $\partial_r H_{|\Gamma} = 0$.
We will therefore use the Nash-Moser theorem which applies in the spaces C^∞ for problems with loss of derivatives.

3.3 Proof of theorem 1.3

Step 1 : the function $H(G)$ obtained by the Nash-Moser theorem
Let us recall that applying the implicit function theorem to equation $\Phi_1(G,H) = 0$, is the same thing as applying the inverse function theorem to the map $\Psi : (G,H) \longmapsto (G,\Phi_1(G,H))$. In particular $D\Psi$ is invertible if and only if $D_H\Phi_1$ is.
There is a small technical difficulty : on one hand, a variation of G changes the domain of definition of H, and on another hand, the notion of tame map that we want to use to apply the Nash-Moser theorem, has only been defined on fixed compacts. To be rigorous we should use a diffeomorphism $f_G : \overline{\Omega\backslash D} \to \overline{\Omega\backslash D^*}$ which depends smoothly on G. This would allows to bring back Φ_1 in a C^∞-tame map $\tilde{\Phi}_1$ defined on the fixed compact set $\overline{\Omega\backslash D^*}$.
Nevertheless to avoid tedious computations, we will work with Φ_1 as it was $\tilde{\Phi}_1$.
Now tame ellipticity theorem 2.4 applies to the following elliptic problem

$$D_H\Phi_1(G,H)\cdot h = k$$

and therefore gives $h = h(G,H,k)$, as a C^∞-tame map of its arguments.
Consequently $D\Psi$ is invertible with a C^∞-tame inverse, and the Nash-Moser inverse function theorem 2.2 applies. This proves that $\Psi(G,H) = (G,0)$ is solved in $H = H(G)$ which is a C^∞-tame map.

Step 2 : the solution $G = G(H_1)$
Let us consider the map $\Phi_2 : C^\infty(\mathbf{S}^1) \to C^\infty(\mathbf{S}^1)$ defined by $\Phi_2(G) := H(G)_{|\Gamma}$. Thus from lemma 3.2, equation $D_G\Phi_2 \cdot g = m$ is solved setting $g = -(\frac{\partial_r h}{\partial_{rr} H})_{|\Gamma}$ where h is solution of (3.5).
As in step 1 we prove that $(D_G\Phi_2)^{-1}$ is C^∞-tame which allows to apply Nash-Moser theorem 2.2. This proves that Φ_2 is invertible in a neighbourhood of G^* which satisfies $\Phi_2(G^*) = H_1^*$. This ends the proof of theorem 1.3.

Remark 3.3 *In the proof of theorem 1.3, we have assumed that D and Ω are diffeomorphic to a disk. It is straightforward to adapt the proof to cases where D and Ω have* *other smooth shapes and topologies (and not necessary the same).*

4 Existence of a smooth free boundary close to $\partial\Omega$.

In this section we prove the main result of this article: theorem 1.1.

4.1 Setting of the problem

Here we start from the particular solution that we know when $\Gamma = \partial\Omega$

$$\begin{cases} D^* = \Omega \\ H^* = H_0 = H_1^* \end{cases}$$

We search for free boundaries $\Gamma = \partial D$ close to $\partial\Omega$. As in section 3 we construct a local curvilinear parametrization (r, s) in a neighbourhood of $\partial D^* = \partial\Omega$: $s \in \mathbf{S}^1$ parametrizes $\partial\Omega$ proportionally to its length, and r denotes the transversal coordinate positively oriented inward the interior of Ω, with $r \in (-r_0, r_0), r_0 > 0$. We will note

$$f(y_1, y_2) = (s, r)$$

the diffeomorphism associated to these coordinates.

The constant H_0 is fixed and H_1 is a parameter. For H_1 close to H_0, Γ is close to $\partial\Omega$, at a distance of order ϵ. Problem (1.2) can be approximated by the one-dimensionnal equation $H'' = H_1$, which should give

$$H_0 - H_1 = \frac{H_1}{2}\epsilon^2 \tag{4.1}$$

We take (4.1) as the definition of ϵ. This is then natural to search for free boundaries $\Gamma = \partial D$ parametrized by $r = \epsilon G(s)$, with at the limit

$$G^* := 1$$

A difficulty is that the open set $\Omega \backslash \overline{D}$ degenerates as $\epsilon \to 0$. Our goal is to obtain a branch of solutions starting from the obvious solution $G^* = 1$ in $\epsilon = 0$, and using the Nash-Moser inverse function theorem.

4.2 The problem is reduced on the fixed compact set $\overline{\omega} = \mathbf{S}^1 \times [0, 1]$

We reduce the problem on the compact set $\overline{\omega}$ where $\omega = \mathbf{S}^1 \times (0, 1)$. We will note (s, ρ) a point of this set. For this we use the following diffeomorphism from $\overline{\Omega \backslash D}$ into $\overline{\omega} = \mathbf{S}^1 \times [0, 1]$ with

$$\rho = \frac{r}{\epsilon G(s)}$$

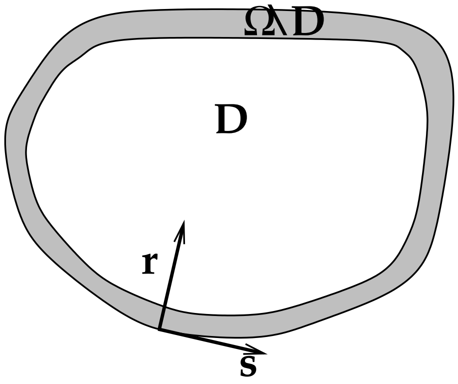

Figure 2: Local coordinates near $\partial\Omega$

We denote $\tilde{H}(s,\rho) = \frac{H(y_1,y_2)-H_1}{\epsilon^2 H_1}$. As in section 3, we search to solve equation $\Phi_1(G,H) = 0$, or equivalently

$$\tilde{\Phi}_1(G,\tilde{H},\epsilon) = 0$$

where

$$\begin{array}{rlcl} \tilde{\Phi}_1 : & C^\infty(\mathbf{S}^1) \times C^\infty(\overline{\omega}) \times (-\epsilon_0,\epsilon_0) & \longrightarrow & C^\infty(\omega) \times C^\infty(\mathbf{S}^1) \times C^\infty(\mathbf{S}^1) \\ & (G,\tilde{H},\epsilon) & \longmapsto & \tilde{\Phi}_1(G,\tilde{H},\epsilon) = \left\{ \begin{array}{l} (\tilde{\mathcal{Q}}_{G,\epsilon}\tilde{H})_{|\omega} - 1 \\ \tilde{H}_{|\partial\Omega} - \frac{1}{2} \\ \partial_\rho \tilde{H}_{|\Gamma^*} \end{array} \right. \end{array}$$

and we denote $\partial\Omega := \{\rho = 0\}$, $\Gamma^* := \{\rho = 1\}$. A straightforward computation gives:

$$\tilde{\mathcal{Q}}_{G,\epsilon}\tilde{H} = \tilde{a}_{ij}\partial_{ij}\tilde{H} + \epsilon\tilde{b}_j\partial_j\tilde{H} + \epsilon^2\tilde{c}\tilde{H}$$

where $i,j \in \{\sigma,\rho\}$, $\partial_\sigma = \epsilon\partial_s$ and coefficients $\tilde{A} = (\tilde{a}_{ij},\tilde{b}_j,\tilde{c})$ are C^∞ functions of several arguments $\tilde{A} = \tilde{A}(\partial_\rho\tilde{H},\ \epsilon\partial_s\tilde{H},\ G, G', G'', Df, D^2f, \epsilon, \rho, H_0)$, and

$$\tilde{a} = P \cdot a^0(V) \cdot {}^tP \text{ for some vector field } V : \omega \to \mathbf{R}^2$$

where $P = M \cdot (Df)$, $M = \begin{pmatrix} 1 & 0 \\ -\epsilon\rho\frac{G'(s)}{G(s)} & \frac{1}{G(s)} \end{pmatrix}$, a^0 is defined in (3.2) and $a^0_{ij}(V)\xi_i\xi_j \geq |\xi|^2$ because F is increasing and $F(0) = 1$. Therefore the quasilinear operator $\tilde{\mathcal{Q}}_{G,\epsilon}$ stays elliptic while $\epsilon > 0$ and G is close to $G^* = 1$.
Let us remark that the particular solution

$$\tilde{H}^*(s,\rho) = \frac{(\rho-1)^2}{2}$$

satisfies $\tilde{\Phi}_1(G^*,\tilde{H}^*,0) = 0$.

The domain of definition is now fixed, and we would like to apply the same method as in section 3 (a had hoc tame ellipticity theorem and the Nash-Moser theorem). For every $\epsilon > 0$, tame ellipticity theorem 2.4 applies. But on one hand our problem degenerates as $\epsilon \to 0$, because the ellipticity constant of the operator $\tilde{H} \longmapsto \tilde{\mathcal{Q}}_{G,\epsilon}\tilde{H}$ tends to 0. And on the other hand we want to find solutions starting from the case $\epsilon = 0$. We remove this difficulty applying theorem 1.4 which is another version of tame ellipticity theorem 2.4, valid in this degenerate case.

4.3 Proof of theorem 1.1

Step 1: the map $(G,\epsilon) \longmapsto \tilde{H}(G,\epsilon)$ is C^∞-tame
We want to solve

$$\tilde{\Phi}_1(G,\tilde{H},\epsilon) = 0 \tag{4.2}$$

We get the straightforward result on the differential of $\tilde{\Phi}_1$:

Lemma 4.1 *We have*

$$D_{\tilde{H}}\tilde{\Phi}_1 \cdot \tilde{h} = \begin{cases} D_{\tilde{H}}(\tilde{\mathcal{Q}}_{G,\epsilon}\tilde{H}) \cdot \tilde{h}_{|\omega} \\ \tilde{h}_{|\partial\Omega} \\ \partial_\rho \tilde{h}_{|\Gamma^*} \end{cases}$$

where $\tilde{h} \in C^\infty(\overline{\omega})$. For $A = (a_{ij}, b_j, c)$, let

$$\mathcal{L}(A,\epsilon)\tilde{h} := a_{ij}\partial_{ij}\tilde{h} + b_j\partial_j\tilde{h} + c\tilde{h}$$

Then

$$D_{\tilde{H}}(\tilde{\mathcal{Q}}_{G,\epsilon}\tilde{H}) \cdot \tilde{h} = \mathcal{L}(A,\epsilon)\tilde{h}$$

where coefficients $A = (a_{ij}, b_j, c)$ are C^∞ functions

$$A = A(\nabla\tilde{H},\ D^2\tilde{H},\ G, G', G'', Df, D^2 f, \epsilon,\ \rho, H_0)$$

Moreover for $A^* = A(G^*, \tilde{H}^*, 0)$, we have

$$a^* = \begin{pmatrix} \left(\frac{2\pi}{|\partial\Omega|}\right)^2 & 0 \\ 0 & 1 \end{pmatrix}, \quad b^* = c^* = 0$$

where $|\partial\Omega|$ denotes the length of $\partial\Omega$.

Because $(G,\tilde{H},\epsilon) \longmapsto A(G,\tilde{H},\epsilon)$ is a C^∞-tame map, it is clear that for $(G,\tilde{H},\epsilon)$ in a neighbourhood of $(G^*,\tilde{H}^*,0)$, the coefficients A are close to A^*, and therefore we can apply theorem 1.4. By composition we deduce that the solution $\tilde{h} = \tilde{h}(G,\tilde{H},\epsilon,\tilde{k})$ of

$$\mathcal{L}(A(G,\tilde{H},\epsilon),\epsilon)\tilde{h} = \tilde{k}$$

is C^∞-tame. Then Nash-Moser theorem 2.2 applies to the map $(G, \tilde{H}, \epsilon) \longmapsto (G, \tilde{\Phi}_1(G, \tilde{H}, \epsilon), \epsilon)$, defined in a neighbourhood of $(G^*, \tilde{H}^*, 0)$. We deduce the existence of a C^∞-tame solution $\tilde{H} = \tilde{H}(G, \epsilon)$ to (4.2).

Step 2: the solution $G = G(\epsilon)$

To obtain the solution $G(\epsilon)$ of our problem, we have to impose the condition $H_{|\Gamma} = H_1$, i.e. $\tilde{H}_{|\Gamma^*} = 0$. We introduce the map $\Phi_2(G, \epsilon) := \tilde{H}(G, \epsilon)_{|\Gamma^*}$ which is C^∞-tame. To inverse $D_G \Phi_2 \cdot g = m$, we go back to the initial equations written in $\Omega \backslash \overline{D}$ which do not depend explicitly on (G, ϵ).

We proceed as in the proof of lemma 3.2 and a straightforward computation gives the

Lemma 4.2 *With the notations of this section we have*

$$D_G \Phi_2 \cdot g = m$$

if and only if

$$g = -G \cdot \left(\frac{\partial_\rho \tilde{h}}{\partial_{\rho\rho} \tilde{H}} \right)_{|\Gamma^*} \tag{4.3}$$

where $\tilde{h} \in C^\infty(\overline{\omega})$ *satisfies*

$$\begin{cases} D_{\tilde{H}}(\tilde{\mathfrak{Q}}_{G,\epsilon} \tilde{H}) \cdot \tilde{h} = 0 \\ \tilde{h}_{|\partial\Omega} = 0 \\ \tilde{h}_{|\Gamma^*} = m \end{cases}$$

We conclude as previously with help of theorem 1.4 (written here with two Dirichlet conditions), that $\tilde{h} = \tilde{h}(G, \epsilon, m)$ is C^∞-tame and therefore from (4.3): $g = g(G, \epsilon, m)$ is C^∞-tame.

We apply the Nash-Moser theorem to the map $(G, \epsilon) \longmapsto (\Phi_2(G, \epsilon), \epsilon)$ which satisfies $\Phi_2(G^*, 0) = \tilde{H}^*_{|\Gamma^*} = 0$. This proves the existence of the solution $G(\epsilon)$ to equation

$$\Phi_2(G, \epsilon) = 0$$

Theorem 1.1 is proved.

5 The tame ellipticity theorem in a degenerate case

In this section we prove theorem 1.4.

5.1 Setting of the problem

For $A = (a_{ij}, b_j, c)$ let us introduce the elliptic operator defined for $(s, \rho) \in \omega = \mathbf{S}^1 \times (0, 1)$

$$L_{A,\epsilon} w = a_{ij} \partial_{ij} w + b_j \partial_j w + cw$$

where $i,j \in \{\sigma,\rho\}$, $\partial_\sigma = \epsilon\partial_s$. We assume that $\epsilon \in \mathbf{R}$, that $A \in C^\infty(\overline{\omega})$, and that the coefficients A satisfy

$$\exists c_0 > 0,\ \forall x \in \mathbf{S}^1 \times [0,1], \forall \xi \in \mathbf{R}^2, a_{ij}(x)\xi_i\xi_j \geq c_0|\xi|^2 \tag{5.1}$$

For $k_0 \in C^\infty(\overline{\omega})$ and $(k_1,k_2) \in C^\infty(\partial\omega)$ we consider the following system

$$\begin{cases} L_{A,\epsilon}w = k_0 \text{ on } \omega \\ \partial_\rho w(s,\rho=1) = k_1(s) \\ w(s,\rho=0) = k_2(s) \end{cases} \tag{5.2}$$

Let $k = (k_0,k_1,k_2)$. System (5.2) will be denoted as

$$\mathcal{L}(A,\epsilon)w = k$$

For $\epsilon \neq 0$, let us recall that from Fredholm alternative theorem, $\mathcal{L}(A,\epsilon)$ is invertible if and only if $\mathcal{L}(A,\epsilon)w = 0$ implies $w = 0$. Let us denote $k \leq 0$ if and only if $k_0 \leq 0, k_1 \geq 0, k_2 \geq 0$. Then let us recall that $\mathcal{L}(A,\epsilon)$ satisfies the maximum principle on ω if and only if for every $w \in C^{2,\alpha}(\overline{\omega})$, $\mathcal{L}(A,\epsilon)w \leq 0$ implies $w \geq 0$. In particular if $\mathcal{L}(A,\epsilon)$ satisfies the maximum principle, then $\mathcal{L}(A,\epsilon)$ is invertible.
For $\epsilon = 0$, the operator is degenerate. For $s_0 \in \mathbf{S}^1$ we define $A_{s_0} = A(s_0,\cdot)$, $w_{s_0} = w(s_0,\cdot)$, $k_{s_0} = k(s_0,\cdot)$. We introduce the following one-dimensional elliptic operator defined for $v \in C^\infty([0,1])$ by

$$\mathcal{L}_0(A_{s_0})v = \begin{cases} a_{\rho\rho}(s_0,\rho)\partial_{\rho\rho}v(\rho) + b_\rho(s_0,\rho)\partial_\rho v(\rho) + c(s_0,\rho)v(\rho) \\ \partial_\rho v(\rho=1) \\ v(\rho=0) \end{cases}$$

With these notations we see that for $\epsilon = 0$, we have a continuous family of one-dimensional elliptic problems well-posed on $(0,1)$

$$\mathcal{L}(A,0)w = k \iff \forall s \in \mathbf{S}^1,\ \mathcal{L}_0(A_s)w_s = k_s$$

In particular we will say that the operator $\mathcal{L}(A,0)$ satisfies the maximum principle on ω if and only if for every $s \in \mathbf{S}^1$, $\mathcal{L}_0(A_s)$ satisfies the maximum principle on $(0,1)$.

Remark 5.1 *The main reason for which the limit at $\epsilon = 0$ is well posed is that problem remains nondegenerate in one direction transversal to the boundary of the annulus $\mathbf{S}^1 \times (0,1)$. On the contrary, there is no general existence nor uniqueness of solutions to the limit problem on the same annulus for the operator $\partial_s^2 + \epsilon^2\partial_\rho^2$.*

Let us denote $P = (A,k)$ and for $q \in \mathbf{N}$, $|P|_{q,\alpha} = |A|_{q,\alpha} + |k|_{q,\alpha}$ with $|k|_{q,\alpha} = |k_0|_{q,\alpha} + |k_1|_{q+1,\alpha} + |k_2|_{q+2,\alpha}$. We denote $w(P,\epsilon)$ the solution to $\mathcal{L}(A,\epsilon)w = k$.
For $P^* = (A^*,0)$ with A^* as in theorem 1.4, we introduce the following open set

$$\mathcal{B} = \{P \in C^\infty, |P - P^*|_{0,\alpha} < R\}$$

with $R > 0$. We chose R small enough such that the coefficients A satisfy (5.1) for $(A, k) \in \overline{\mathcal{B}}$, and $\mathcal{L}(A, 0)$ satisfy the maximum principle on the same set of coefficients. We will prove

THEOREM 5.2 *i) $w(P, 0)$ is C^0-tame on $\mathcal{B}$*
ii) There exists $\epsilon_0 \in (0, 1)$ such that

$$\begin{array}{l} \forall q \in \mathbf{N},\ \exists C_{q,\alpha} > 0,\ \forall (P, \epsilon) \in \mathcal{B} \times (-\epsilon_0, \epsilon_0) : \\ |w(P, \epsilon) - w(P, 0)|_{q,\alpha} \leq C_{q,\alpha} |\epsilon|^{1-\alpha} |P|_{q+3,\alpha} \end{array}$$

REMARK 5.3 *There appears a loss of derivatives in the degenerate case, contrarily to tame ellipticity theorem 2.4 for the nondegenerate case.*

5.2 Theorem 5.2 implies theorem 1.4

In this subsection, we prove in two steps that theorem 5.2 implies theorem 1.4.

Step A: If w is C^0-tame, then w is C^∞-tame
We denote $\mathcal{P} = (A, k, \epsilon)$, and $W = (w, \partial_s w, \partial_\rho w, \partial_{ss} w, \partial_{s\rho} w, \partial_{\rho\rho} w)$. These notations will be only used in this paragraph. In particular W is C^0-tame if and only if w is. We have

$$\mathcal{L}(A, \epsilon) w = k \Longleftrightarrow \mathcal{L}(\mathcal{P}) W(\mathcal{P}) = l_0(\mathcal{P})$$

where $\mathcal{L}(\mathcal{P})$ is a vector whose coordinates are polynoms in $\mathcal{P}$, $\mathcal{L}(\mathcal{P}) W(\mathcal{P})$ is a bilinear vector function in $\mathcal{L}(\mathcal{P})$ and $W(\mathcal{P})$, and $l_0(\mathcal{P})$ is the linear polynom in $\mathcal{P}$ defined by $l_0(\mathcal{P}) = k$.
If $W(\mathcal{P})$ is C^0-tame and satisfies $\mathcal{L}(\mathcal{P}) W(\mathcal{P}) = l(\mathcal{P})$ where $l(\mathcal{P})$ is C^1-tame, then we get by diffrentiation

$$\mathcal{L}(\mathcal{P}) \left(D_{\mathcal{P}} W(\mathcal{P}) \cdot p\right) = D_{\mathcal{P}} l(\mathcal{P}) \cdot p - (D_{\mathcal{P}} \mathcal{L}(\mathcal{P}) \cdot p) W(P)$$

Because the polynom $\mathcal{L}(\mathcal{P})$ is C^∞-tame in $\mathcal{P}$ we deduce that $D_{\mathcal{P}} W(\mathcal{P})$ is C^0-tame and then $W(\mathcal{P})$ is C^1-tame.
Finally, because $l_0(\mathcal{P})$ is C^∞-tame, it is then straightforward to prove by recurrency that $D^m_{\mathcal{P}} W(\mathcal{P})$ is C^0-tame, for every $m \in \mathbf{N}$.
This proves that w is C^∞-tame.

Step B: theorem 5.2 implies that w is C^0-tame
Let us denote $\theta(P, \epsilon) = w(P, \epsilon) - w(P, 0)$ where we recall that $P = (A, k)$.
From theorem 5.2 i) and tame ellipticity theorem 2.4, it is easy to prove that θ is continuous for $\epsilon \neq 0$. Theorem 5.2 ii) says that θ is continuous until $\epsilon = 0$ and then is C^0-tame on $\mathcal{B} \times (-\epsilon_0, \epsilon_0)$.

The solution $w = \theta + w(\cdot, 0)$ is the sum of two C^0-tame maps and therefore is C^0-tame on $\mathcal{B} \times (-\epsilon_0, \epsilon_0)$. Finally, from the following straightforward lemma 5.4, $w(P, \epsilon) = w(A, k, \epsilon)$ is C^0-tame on a larger open set without bounds on $k \in C^\infty$.
This proves theorem 1.4.

We recall the following straightforward lemma, which is used in the proof (Step B) of theorem 1.4:

Lemma 5.4 *Suppose that $\Phi(u, k)$ is C^0, linear in k, and satisfies a tame estimate when k is bounded*

$$\begin{array}{l} \exists R_0 > 0,\ \forall u \in U \subset C^\infty,\ \forall k \in C^\infty : \\ (|k|_{b,\alpha} \leq R_0) \Longrightarrow (\forall q \in \mathbf{N},\ |\Phi(u,k)|_{q,\alpha} \leq C_{q,\alpha}\,(1 + |(u,k)|_{q+r,\alpha})) \end{array} \tag{5.3}$$

then for all $k \in C^\infty$, we have a tame inequality without bound on k

$$\begin{array}{l} \forall u \in U \subset C^\infty,\ \forall k \in C^\infty :\ \forall q \in \mathbf{N}, \\ |\Phi(u,k)|_{q,\alpha} \leq \frac{C_{q,\alpha}}{R_0}|k|_{b,\alpha}(1 + |u|_{q+r,\alpha}) + C_{q,\alpha}|k|_{q+r,\alpha} \end{array}$$

In other words, because Φ is linear in k, it is enough for Φ to satisfy (5.3) to be C^0-tame in $U \times C^\infty$ in the sense of definition (2.3).

5.3 Proof of theorem 5.2 i)

We recall some usefull tame inequalities. In particular we will use them to prove that the map $P \longmapsto w(P, 0)$ is C^0-tame.

5.3.1 Fundamental tame inequalities

Let u and v some C^∞ functions on a smooth bounded open set Ω in $\mathbf{R}^n$; then we have the following tame inequalities:
product (see [1])

$$\forall q \in \mathbf{N}, \quad |u \cdot v|_{q,\alpha} \leq C_{q,\alpha}(|u|_0|v|_{q,\alpha} + |u|_{q,\alpha}|v|_0) \tag{5.4}$$

interpolation (see [9])
If (i,j) belongs to the segment of extremities (k,l) and (m,n), then for every positive or nul integers i, j, k, l, m, n we have

$$|u|_{i,\alpha}|v|_{j,\alpha} \leq C_{(i,j,k,l,m,n),\alpha}(|u|_{k,\alpha}|v|_{l,\alpha} + |u|_{m,\alpha}|v|_{n,\alpha})$$

In particular we will use this inequality under the following form

$$\forall i,j \geq 0,\ |P|_{i,\alpha}|P|_{j,\alpha} \leq C|P|_{i+j,\alpha} \quad \text{if } |P|_{0,\alpha} \text{ is bounded} \tag{5.5}$$

5.3.2 The map $P \longmapsto w(P,0)$ is C^0-tame

In all what follows C will denote a generic constant. To simplify the notations let us introduce $w^0(P) = w(P,0)$. For every $s \in \mathbf{S}^1$, we apply tame ellipticity theorem 2.4 to each one-dimensional elliptic problem

$$\mathcal{L}_0(A_s)w_s^0 = k_s \tag{5.6}$$

where we recall that $A_s = A(s,\cdot)$, $w_s^0 = w^0(s,\cdot)$, $k_s = k(s,\cdot)$. For every $s \in \mathbf{S}^1$ we get with $P_s = P(s,\cdot)$

$$|w_s^0|_{q+2,\alpha} \leq C|P_s|_{q,\alpha} \tag{5.7}$$

Let us emphasize that here the coordinate s is fixed and then the Hölder norms $|\cdot|_{q,\alpha}$ apply to functions that only depend on the coordinate ρ.
The solution w_s^0 is C^∞ relatively to P_s. In particular it is derivable relatively to s because $\partial_s w^0 = D_P w^0 \cdot \partial_s P$. Derivating equation (5.6), we get

$$\mathcal{L}_0(A_s)(\partial_s w_s^0) = K_s := \partial_s k_s - \mathcal{L}_0(\partial_s A_s)(w_s^0)$$

From tame ellipticity estimate (2.4) we get

$$|\partial_s w_s^0|_{q+2,\alpha} \leq C(|K_s|_{q,\alpha} + |K_s|_{0,\alpha}|A_s|_{q,\alpha})$$

and

$$\begin{aligned}
|K_s|_{q,\alpha} &\leq C(|\partial_s k_s|_{q,\alpha} + |\mathcal{L}_0(\partial_s A_s)\cdot w_s^0|_{q,\alpha}) \\
&\leq C(|k_s|_{q+1,\alpha} + |\partial_s A_s|_{0,\alpha}|w_s^0|_{q+2,\alpha} + |\partial_s A_s|_{q,\alpha}|w_s^0|_{2,\alpha}) \\
&\leq C(|P_s|_{q+1,\alpha} + |P_s|_{1,\alpha}|P_s|_{q,\alpha} + |P_s|_{q+1,\alpha}|P_s|_{0,\alpha}) \\
&\leq C|P_s|_{q+1,\alpha}
\end{aligned}$$

where we have used (5.4) for the second line, (5.7) for the third line, and for the last line: (5.5) and the fact that $|P_s|_{0,\alpha}$ is bounded because $P \in \mathcal{B}$. We get for every $s \in \mathbf{S}^1$

$$|\partial_s w_s^0|_{q+2,\alpha} \leq C|P_s|_{q+1,\alpha}$$

By a straightforward recurrency we get

$$|\partial_s^m w_s^0|_{q+2,\alpha} \leq C|P_s|_{q+m,\alpha}$$

In particular this implies with Hölder norms on $\omega = \mathbf{S}^1 \times (0,1)$

$$|w^0|_{q,\alpha} \leq C_{q,\alpha}|P|_{q+1,\alpha}$$

which proves that w^0 is C^0-tame.

5.4 Proof of theorem 5.2 ii)

We want to get estimates on $\theta(P,\epsilon) = w(P,\epsilon) - w^0(P)$ with $w^0(P) = w(P,0)$.
Let us remark that

$$\mathcal{L}(A,\epsilon)\theta = -(\mathcal{L}(A,\epsilon) - \mathcal{L}(A,0))w^0$$

The linear differential operator $\mathcal{L}(A,\epsilon)$ is linear in A and polynomial in ϵ, therefore we can write

$$\mathcal{L}(A,\epsilon)\theta = K := -\epsilon\mathcal{M}(A,\epsilon)w^0$$

where $\mathcal{M}(A,\epsilon)$ is a linear differential operator, whose coefficients are linear in A and polynoms in ϵ. To get elliptic estimates with constants independent on ϵ, it is usefull to introduce the change of coordinates: $\sigma = \frac{s}{\epsilon}$, $\omega_\epsilon = \frac{\mathbf{S}^1}{\epsilon} \times (0,1)$, $\overline{A}(\sigma,\rho) = A(s,\rho)$, $\overline{\theta}(\sigma,\rho) = \theta(s,\rho)$, $\overline{K}(\sigma,\rho) = K(s,\rho)$. We deduce

$$\mathcal{L}(\overline{A},1)\overline{\theta} = \overline{K} \text{ on } \omega_\epsilon \tag{5.8}$$

The quantity ϵ_0 introduced in theorem 5.2 is given by

Lemma 5.5 *[Schauder estimate]*
There exists $\epsilon_0 \in (0,1)$ and a constant $C_0 > 0$ such that for each $\epsilon \in (-\epsilon_0,\epsilon_0)\backslash\{0\}$, for each coefficient $\overline{A}(\sigma,\rho) = A(\epsilon\sigma,\rho)$ such that $(A,0) \in \mathcal{B}$, and for every $\overline{\theta} \in C^{2,\alpha}$ solution to (5.8), we have

$$|\overline{\theta}|_{2,\alpha} \leq C_0|\overline{K}|_{0,\alpha}$$

Proof deferred

Estimate on $|\overline{\theta}|_{2,\alpha}$
For every $(P,\epsilon) \in \mathcal{B} \times ((-\epsilon_0,\epsilon_0)\backslash\{0\})$, we deduce the straightforward estimates

$$\begin{aligned}
|\overline{\theta}|_{2,\alpha} &\leq C|\overline{K}|_{0,\alpha} \\
&\leq C|K|_{0,\alpha} \\
&\leq C|\epsilon||\mathcal{M}(A,\epsilon)w^0|_{0,\alpha} \\
&\leq C|\epsilon||A|_{0,\alpha}|w^0|_{2,\alpha}
\end{aligned}$$

It gives

$$|\overline{\theta}|_{2,\alpha} \leq C|\epsilon||P|_{3,\alpha}$$

Estimate on $|\partial_\sigma\overline{\theta}|_{2,\alpha}$
We derive equation $\mathcal{L}(\overline{A},1)\overline{\theta} = \overline{K}$ relatively to σ, and we get

$$\mathcal{L}(\overline{A},1)(\partial_\sigma\overline{\theta}) = \partial_\sigma\overline{K} - \mathcal{L}(\partial_\sigma\overline{A},1)\overline{\theta}$$

We proceed similarly as in subsection 5.3.2, and it gives

$$|\partial_\sigma\overline{\theta}|_{2,\alpha} \leq C\epsilon^2|P|_{4,\alpha}$$

Estimate on $|\partial_\rho \overline{\theta}|_{2,\alpha}$

We have:

$$|\partial_\rho \overline{\theta}|_{2,\alpha} \leq C(|\overline{\theta}|_{2,\alpha} + |\partial_\sigma \overline{\theta}|_{2,\alpha} + |\partial^3_\rho \overline{\theta}|_{0,\alpha})$$

Then we only need to estimate $|\partial^3_\rho \overline{\theta}|_{0,\alpha}$. Let us note $\overline{K} = (\overline{K}_0, \overline{K}_1, \overline{K}_2)$. With these notations we have $L_{\overline{A},1}\overline{\theta} = \overline{K}_0$, i.e.

$$\overline{a}_{ij}\partial_{ij}\overline{\theta} + \overline{b}_j\partial_j\overline{\theta} + \overline{c}\ \overline{\theta} = \overline{K}_0$$

where $i, j \in \{\sigma, \rho\}$. We deduce that

$$\partial^2_\rho \overline{\theta} = \overline{K}'_0$$

where $\overline{K}'_0$ is a function on $(\overline{K}_0, \overline{A}, \overline{\theta}, \nabla\overline{\theta}, \partial_\rho\nabla\overline{\theta})$. Thus by derivation relatively to ρ we can estimate $\partial^3_\rho\overline{\theta}$. A straightforward computation gives

$$|\partial^3_\rho \overline{\theta}|_{0,\alpha} \leq C|\epsilon||P|_{4,\alpha}$$

and then

$$|\partial_\rho \overline{\theta}|_{2,\alpha} \leq C|\epsilon||P|_{4,\alpha}$$

Estimates on higher derivatives of $\overline{\theta}$

A straightforward recurrency allows to prove that for every $j \geq i \geq 0$ we have

$$|\partial^i_\sigma \partial^{j-i}_\rho \overline{\theta}|_{2,\alpha} \leq C_{j,\alpha}|\epsilon|^{i+1}|P|_{j+3,\alpha} \tag{5.9}$$

We deduce that

$$\begin{aligned}&\forall q \geq 0,\ \exists C_{q,\alpha} > 0,\ \forall (P,\epsilon) \in \mathcal{B} \times (-\epsilon_0, \epsilon_0),\\ &|\theta|_{q,\alpha} \leq C_{q,\alpha}|\epsilon|^{1-\alpha}|P|_{q+3,\alpha}\end{aligned}$$

This ends the proof of theorem 5.2 ii).

Remark 5.6 *In fact we can also obtain the following estimate:* $|\theta|_q \leq C_{q,\alpha}|\epsilon||P|_{q+4,\alpha}$.

Proof of lemma 5.5

For this proof we take our inspiration from some ideas in [11], which where based on Morrey [12].

If the lemma is false, then we can find sequences $(\epsilon_n)_n \in (-1,1)\backslash\{0\}$, $(A_n, k_n)_n \in \mathcal{B}$, $(\overline{K}_n)_n$ and $(\overline{\theta}_n)_n \in C^{2,\alpha}$ such that for $\overline{A}_n(\sigma,\rho) = A_n(\epsilon_n\sigma, \rho)$

$$\mathcal{L}(\overline{A}_n, 1)\overline{\theta}_n = \overline{K}_n \text{ on } \omega_{\epsilon_n} = \frac{\mathbf{S}^1}{\epsilon_n} \times (0,1)$$

and $\epsilon_n \to 0$, $|\overline{\theta}_n|_{2,\alpha} = 1$, $|\overline{K}_n|_{0,\alpha} \to 0$. For $\sigma_0 \in \mathbf{R}$ and $\mu > 0$ let us define the box $C_\mu(\sigma_0) = \{(\sigma, \rho), \rho \in [0,1], |\sigma - \sigma_0| \leq \mu\}$. Then on one hand, there exist a constant $C > 1$ and $\sigma_n \in \frac{\mathbf{S}^1}{\epsilon_n}$ such that

$$|\overline{\theta}_n|_{2,\alpha;C_1(\sigma_n)} \leq |\overline{\theta}_n|_{2,\alpha} \leq C|\overline{\theta}_n|_{2,\alpha;C_1(\sigma_n)}$$

On another hand, from elliptic estimates (see Morrey [12]) there exists a constant $C_0 > 0$ such that

$$|\overline{\theta}_n|_{2,\alpha;C_1(\sigma_n)} \leq C_0(|\overline{\theta}_n|_{L^1(C_2(\sigma_n))} + |\overline{K}_n|_{0,\alpha;C_2(\sigma_n)})$$

Up to extraction of some subsequence, we have $\epsilon_n\sigma_n \to s_\infty \in \mathbf{S}^1$, $A_n \to A_\infty$ and $\overline{A}_n \to \overline{A}_\infty$ where $\overline{A}_\infty(\sigma,\rho) = \overline{A}_{\infty,s_\infty}(\rho) = \lim_{n\to+\infty} A_n(\epsilon_n\sigma + \epsilon_n\sigma_n, \rho) = A_\infty(s_\infty, \rho)$. Moreover $\overline{\theta}_n(\sigma + \sigma_n, \rho) \to \overline{\theta}_\infty(\sigma, \rho)$ which satisfies $|\overline{\theta}_\infty|_{2,\alpha} \leq 1$, and then

$$\frac{1}{CC_0} \leq |\overline{\theta}_\infty|_{L^1(C_2(0))} \tag{5.10}$$

On the other hand we have

$$\mathcal{L}(\overline{A}_\infty, 1)\overline{\theta}_\infty = 0 \text{ on } \omega_0 = \mathbf{R} \times (0,1)$$

Let the subsolution $\underline{v}(\rho) = \sup_{\sigma\in\mathbf{R}} \overline{\theta}_\infty(\sigma,\rho)$ (resp. the supersolution $\overline{v}(\rho) = \inf_{\sigma\in\mathbf{R}} \overline{\theta}_\infty(\sigma,\rho)$): it satisfies $\mathcal{L}(\overline{A}_\infty, 1)\underline{v} \geq 0$ (resp. $\mathcal{L}(\overline{A}_\infty, 1)\overline{v} \leq 0$). From the maximum principle applied to $\mathcal{L}_0(\overline{A}_{\infty,s_\infty})$ we get

$$0 \leq \overline{v} \leq \overline{\theta}_\infty \leq \underline{v} \leq 0$$

This gives a contradiction with (5.10) and proves the lemma.

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