A New Approach to the Regularity of Free Boundaries

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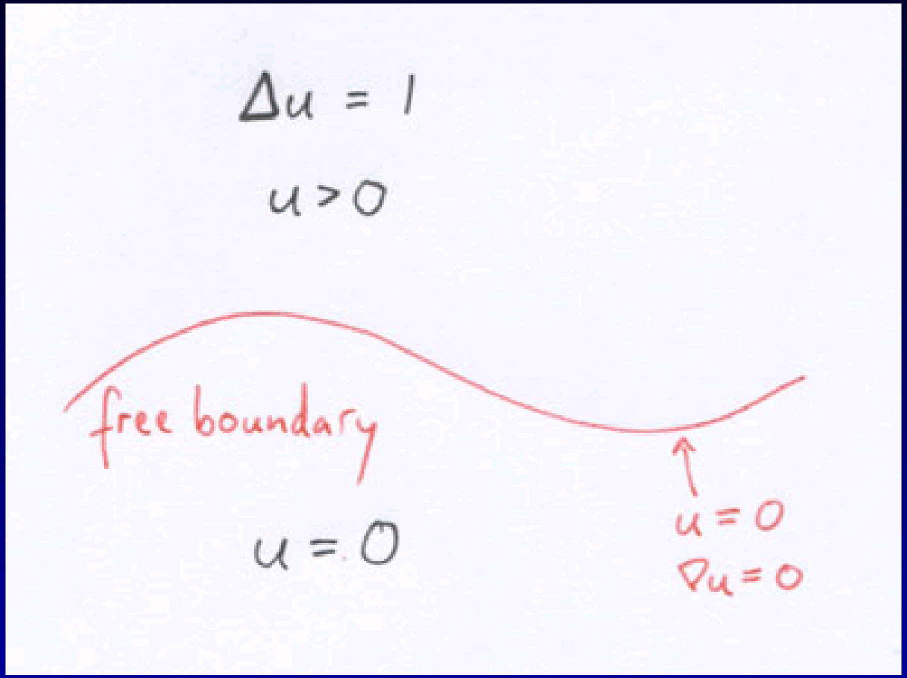
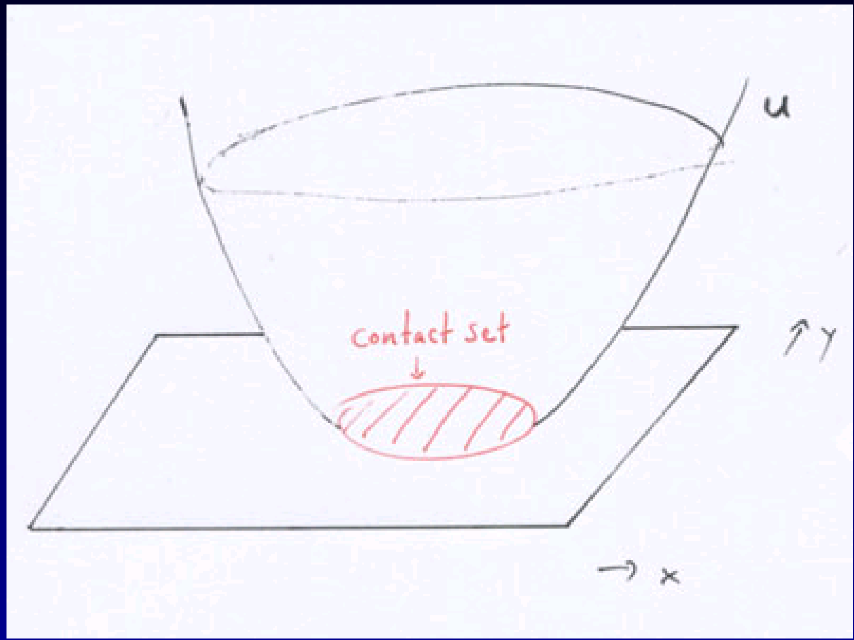
One-phase Obstacle Problem



$$u > 0, \Delta u = 1 \text{ in } \Omega \cap D, u = 0 \text{ in } \Omega - D, u$$

is continuously differentiable

Application



Describes the equilibrium state for the process of pulling an elastic membrane from a planar surface

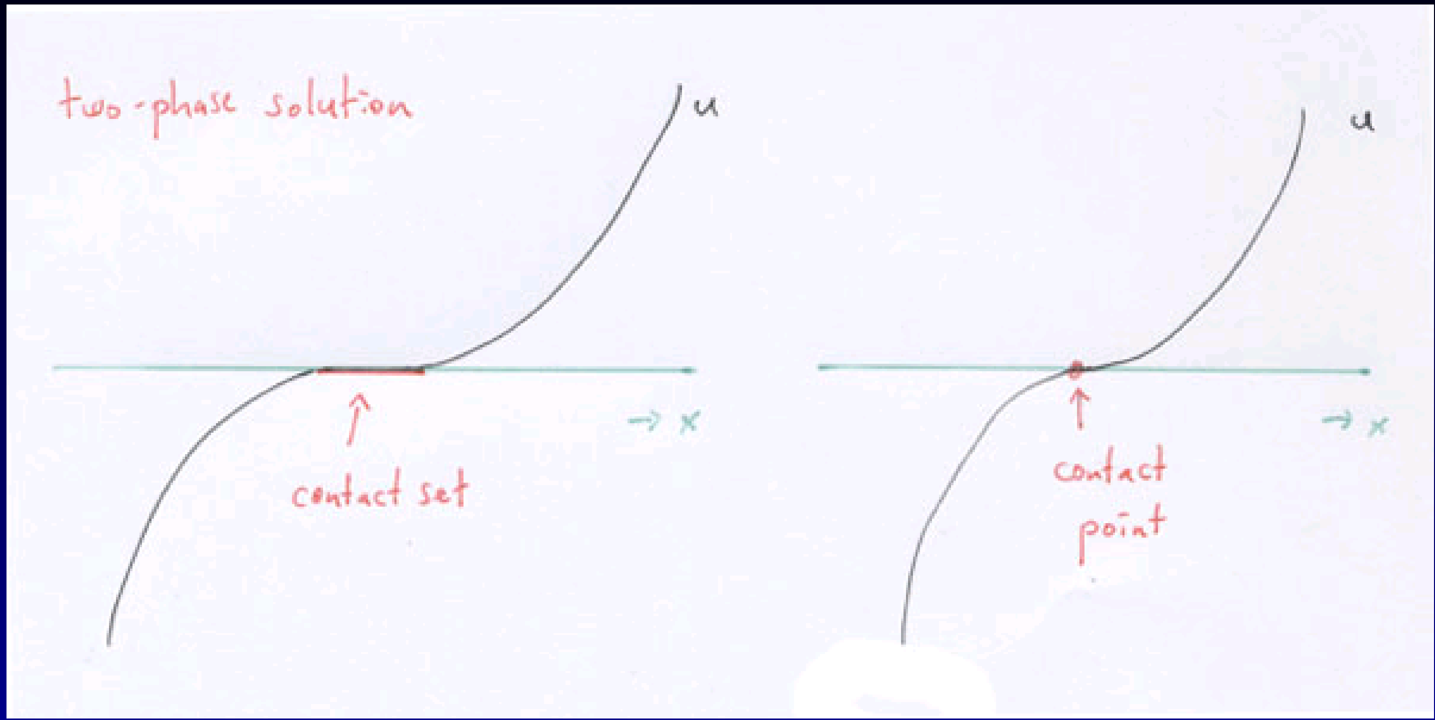
Convenient Notation: $\Delta u = \chi_{\{u>0\}}$ in Ω

Extension to a two-phase problem:

$$u > 0, \Delta u = \lambda_1 \text{ in } \Omega \cap D_1,$$

$u < 0, \Delta u = -\lambda_2$ in $\Omega \cap D_2$, $u = 0$ in $\Omega - (D_1 \cup D_2)$, u is continuously differentiable

Application: consider an elastic membrane touching the phase boundary between two liquid/gaseous phases with different viscosity, for example a water surface. If the membrane is pulled away from the phase boundary in both phases, then the equilibrium state can be described by equation (at least in the case that the contact set has non-empty interior)



Other Applications?

Notation: $\Delta u = \lambda_1 \chi_{\{u>0\}} - \lambda_2 \chi_{\{u<0\}}$ in Ω

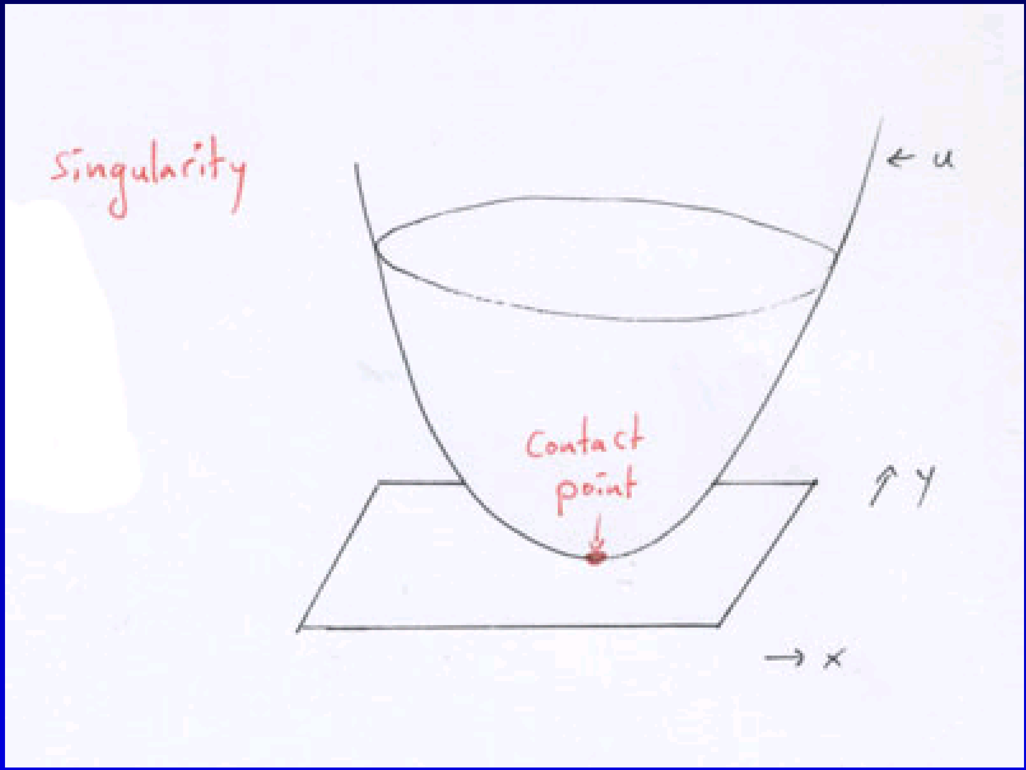
What is known for the one-phase obstacle problem?

- Regularity of the solution: the second derivatives of u are locally bounded (Frehse)

What is known for the one-phase obstacle problem?

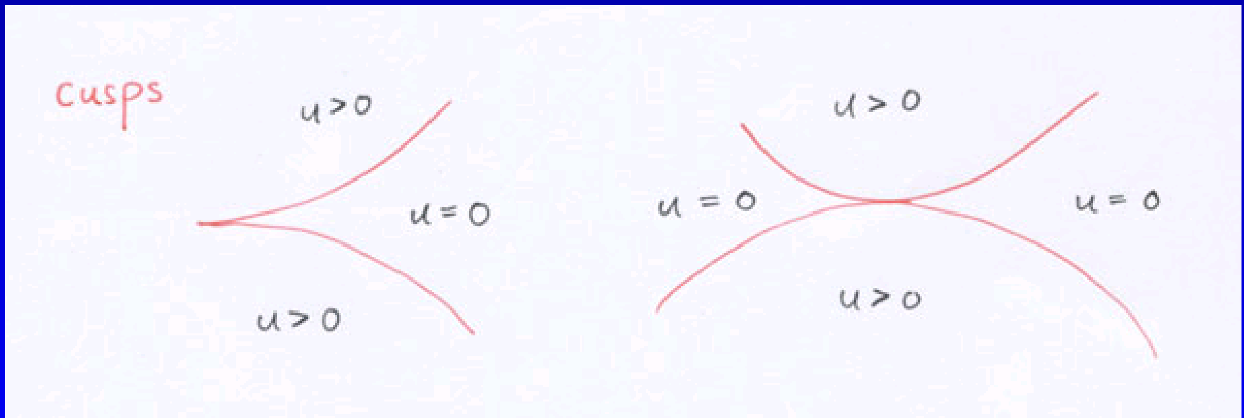
- Regularity of the solution: the second derivatives of u are locally bounded (Frehse)
- Non-Degeneracy: this regularity is sharp at each *free boundary* point $x_0 \in \partial\{u > 0\}$

- Regularity of the free boundary: in general there are *singularities* (example: pull up the membrane uniformly on a circular boundary)



- But: *if the contact set* $\{u = 0\}$ *is "thick" enough* close to a free boundary point x_0, then the free boundary is a smooth hypersurface in an open neighborhood of that point (Caffarelli; other proofs for the "flatness-implies-regularity:" Alt-Phillips, W. '98)

- But: if the *contact set* $\{u = 0\}$ is “thick” enough close to a free boundary point x_0, then the free boundary is a smooth hypersurface in an open neighborhood of that point (Caffarelli; other proofs for the “flatness-implies-regularity:” Alt-Phillips, W. ’98)
- What can be said about the singular set? In two dimensions, singular sets are one of the following: points, lines, cusps (Sakai)



- In higher dimensions the asymptotic behavior of u close to singular points is that of quadratic polynomials; the singular set can locally be extended to a k-dimensional hypersurface (Caffarelli '98, Monneau 2001)

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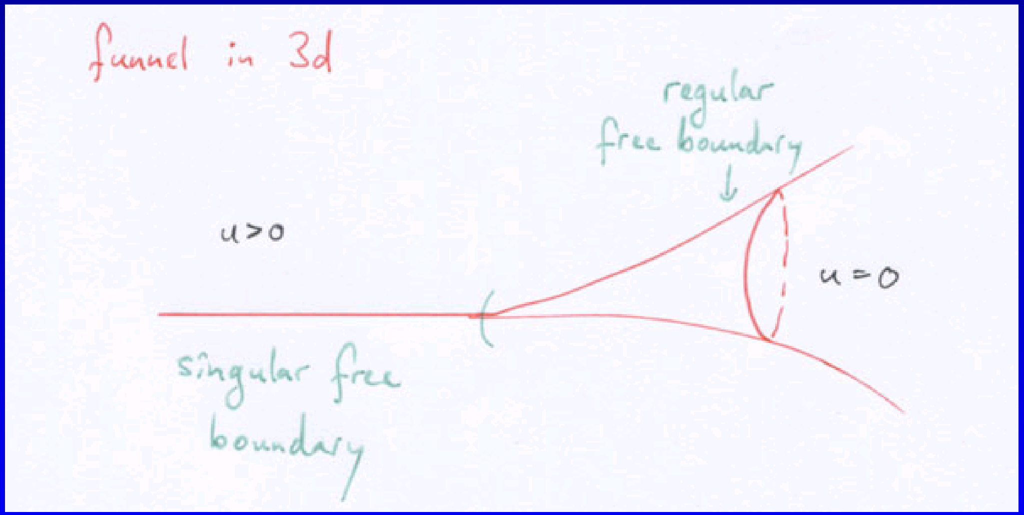
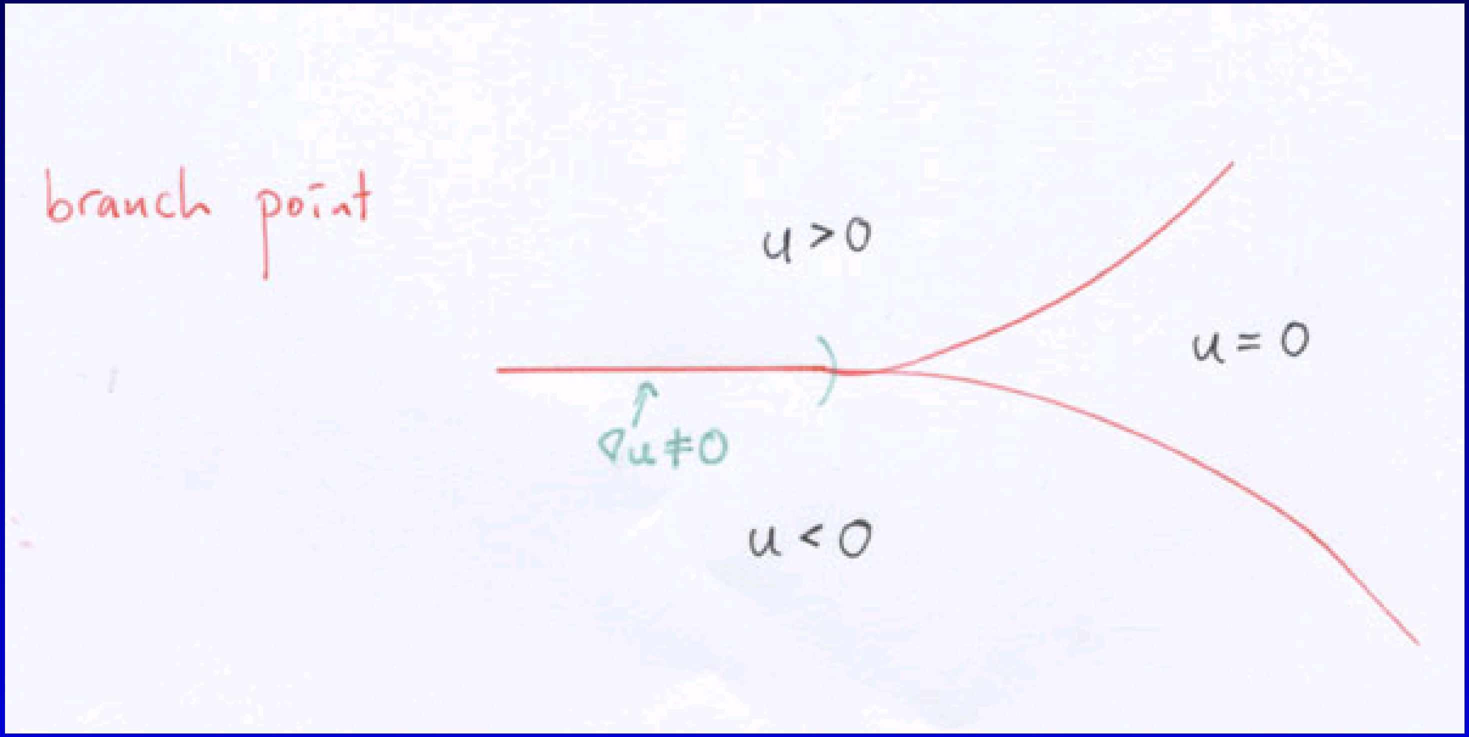
- The behavior of the regular free boundary part close to singularities is in higher dimensions still an open problem.



Two-Phase Problem

How much of this can be extended to the above "two-phase membrane problem"? Here we expect additional singularities in the form of "branch points"



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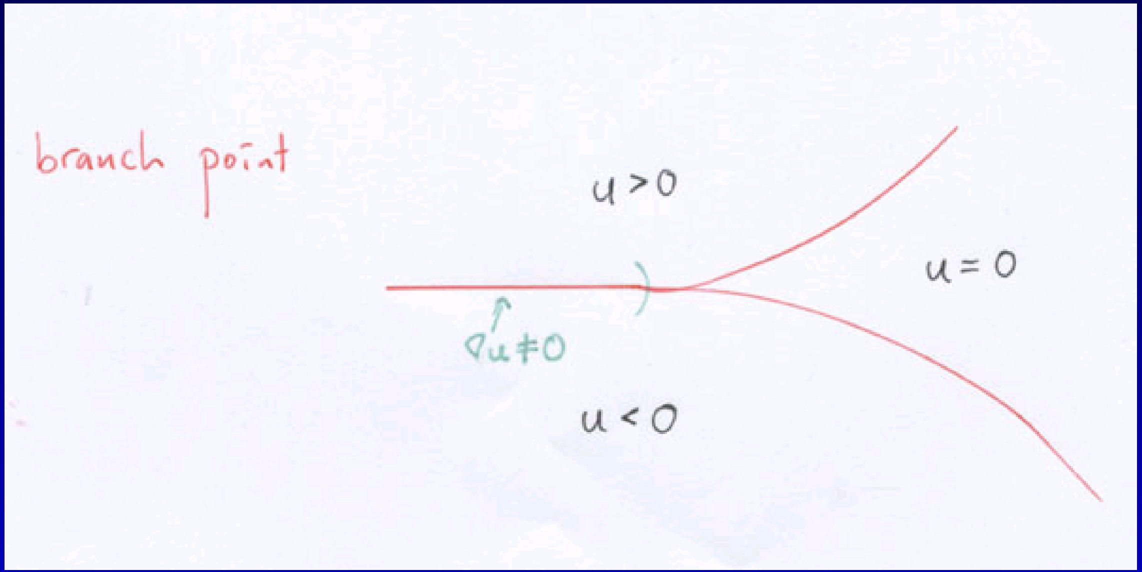

Problems that arise:

- One-phase tools like Harnack inequality, strong maximum principle etc. cannot be used any longer

Two-Phase Problem

- The Bernstein technique does not work because the free boundary is now composed of two different parts, namely the part where the gradient of the solution vanishes and the part where the gradient does not vanish $\Rightarrow$ no uniform growth

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Two-Phase Problem

- The Bernstein technique does not work because the free boundary is now composed of two different parts, namely the part where the gradient of the solution vanishes and the part where the gradient does not vanish $\Rightarrow$ no uniform growth
- All existing techniques to prove regularity of the free boundary seem to fail!
- Though we expect the free boundary close to branch points to be the union of (at most) two smooth hypersurfaces, this is difficult to prove (similar to the cusps in the one-phase problem)

What can be done?

- Uniform quadratic growth estimate at $K \cap \{u = 0\} \cap \{\nabla u = 0\}$ (W. 2001), based on the following monotonicity formula and a lemma for the mean frequency of harmonic functions:

Monotonicity Formula

(W. '98)

$$\Phi_{x_0}(r) := r^{-n-2} \int_{B_r(x_0)} (|\nabla u|^2 \ + \ \lambda_+ \max(u,0)$$

$$+ \lambda_- \max(-u,0)) \ - \ 2\, r^{-n-3} \int_{\partial B_r(x_0)} u^2 \, d\mathcal{H}^{n-1} \ ,$$

defined in $(0,\delta)$, satisfies the monotonicity formula

$$\Phi_{x_0}(\sigma) - \Phi_{x_0}(\rho) \ = \ \int_\rho^\sigma r^{-n-2} \int_{\partial B_r(x_0)} 2 \left(\nabla u \cdot \nu - 2\, \frac{u}{r} \right)^2$$

$$\geq 0 \ .$$

Frequency Lemma (W. 2001)

(based on Almgren's frequency)
Let $\alpha - 1 \in \mathbf{N}$, let $w \in H^{1,2}(B_1(0))$ be a harmonic function in $B_1(0)$ and assume that $D^j w(0) = 0$ for $0 \le j \le \alpha - 1$.

$$\text{Then } \int_{B_1(0)} |\nabla w|^2 \; - \; \alpha \int_{\partial B_1(0)} w^2 \, d\mathcal{H}^{n-1} \; \ge \; 0 \; ,$$

and equality implies that w is homogeneous of degree α in $B_1(0)$.

Boundedness of second derivatives

(N. Uraltseva 2002)
based on the following two-phase monotonicity formula by Alt-Caffarelli-Friedman.
ACF-Monotonicity Formula: Let h_1 and h_2 be continuous non-negative subharmonic $H^{1,2}$-functions in $B_R(z)$ satisfying $h_1 h_2 = 0$ in $B_R(z)$ as well as $h_1(z) = h_2(z) = 0$.
Then for $\Psi_z(r, h_1, h_2) :=$

$$r^{-4} \int_{B_r(z)} \frac{|\nabla h_1(x)|^2}{|x-z|^{n-2}}\, dx \int_{B_r(z)} \frac{|\nabla h_2(x)|^2}{|x-z|^{n-2}}\, dx \ ,$$

and for $0 < \rho < r < \sigma < R$, we have $\Psi_z(\rho) \le \Psi_z(\sigma)$.

Boundedness of second derivatives

Moreover, if equality holds for some
$0 < \rho < r < \sigma < R$ then one of the following is true:
(A) $h_1 = 0$ in $B_\sigma(z)$ or $h_2 = 0$ in $B_\sigma(z)$,
(B) for $i = 1, 2$, and $\rho < r < \sigma$, supp $(h_i) \cap \partial B_r(z)$ is a half-sphere and $h_i \Delta h_i = 0$ in $B_\sigma(z) \setminus B_\rho(z)$ in the sense of measures.

This formula is applied to tangential derivatives of u at a free boundary point.

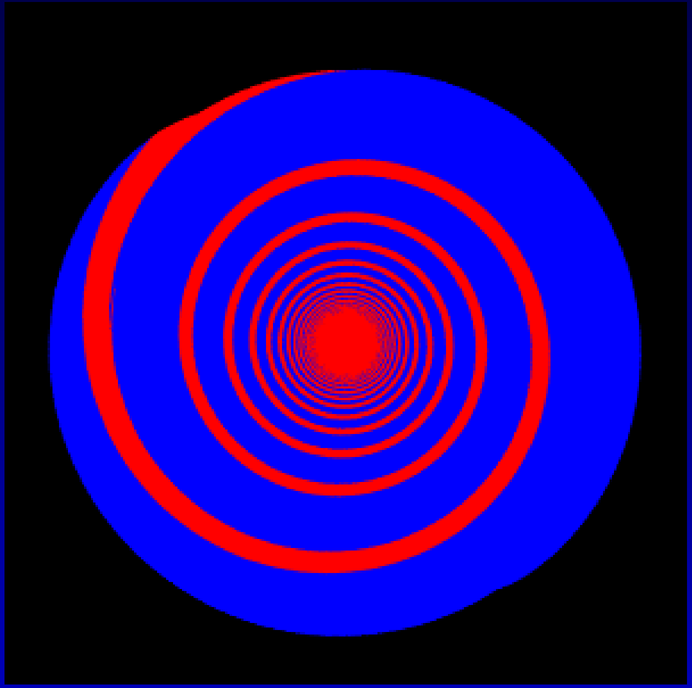
Characterization of Global Solutions

(Shahgholian-Uraltseva-W. 2004)

Let u be a global solution such that $x_0 \in \partial\{u > 0\} \cap \partial\{u < 0\}$ and $\nabla u(x_0) = 0$ for some $x_0 \in \mathbf{R}^n$ and that $|D^2 u| \le C$ in $\mathbf{R}^n$. Then u is after a translation and rotation of the form $u(x) = -\frac{\lambda_-}{2}\min(x_n, 0)^2 + \frac{\lambda_+}{2}\max(x_n, 0)^2$.

New approach to the Regularity of Free Boundaries

Idea: when the scaled solution is uniformly close to a class of monotone functions, try using the Aleksandrov reflection.



Red Area: the solution u is ≤ 0

Blue Area: the solution u is > 0

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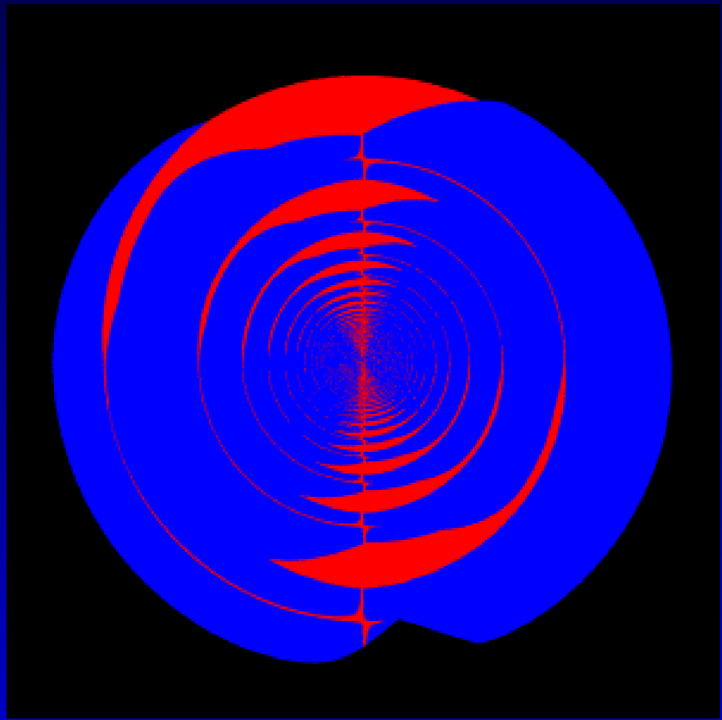

Red Area: the *difference*

$u(x_1, x_2, \ldots, x_n) - u(-x_1, x_2, \ldots, x_n)$ is ≤ 0

Blue Area: the *difference*

$u(x_1, x_2, \ldots, x_n) - u(-x_1, x_2, \ldots, x_n)$ is > 0

New approach to the Regularity of Free Boundaries

In particular, in two dimensions the approach can be realized more easily as 1-dimensional hyperplanes intersect each other at most in one point:

Theorem (Shahgholian-Uraltseva-W., to appear) Let $n = 2$, let $(u_\alpha)_{\alpha \in I}$ be a family of solutions in B_1 that is bounded in $H^{2,\infty}(B_1)$, and suppose that for some $\alpha_0 \in I$, the origin is a branch point.

Then, if $u_\alpha \to u_{\alpha_0}$ in $C^1(B_1)$ as $\alpha \to \alpha_0$, $B_{r_0} \cap \partial\{u_\alpha > 0\}$ and $B_{r_0} \cap \partial\{u_\alpha < 0\}$ are C^1-graphs uniformly in $\alpha \in B_\kappa(\alpha_0)$ for some $r_0 > 0$ and $\kappa > 0$; here the direction of every graph is the same.

Stability Result

Theorem (Shahgholian-Uraltseva-W., to appear)
Let $\Omega \subset \mathbf{R}^2$ be a bounded Lipschitz domain and assume that for given Dirichlet data $u_D \in H^{1,2}(\Omega)$ the free boundary does not contain any one-phase singular free boundary point.

Then for $K \subset\subset \Omega$ and $\tilde{u}_D \in H^{1,2}(\Omega)$ satisfying $\sup_{\partial\Omega} |u_D - \tilde{u}_D| < \delta_K$, the free boundary is locally in Ω the union of (at most) two C^1-graphs which approach those of the solution with respect to boundary data u_D as $\sup_{\partial\Omega} |u_D - \tilde{u}_D| \to 0.$