Calibration of Lévy processes with American options

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Content

- Feasibility of the calibration of a Lévy process with American put options
- A forward problem ($\sim$ Dupire's equation for European put)
- Choice of a class of Lévy processes
- Variational inequalities for the previous problems (in possibly fractional Sobolev spaces)
- Bounds and Sensitivity results for the previous problems
- Regularized least squares, optimality conditions

Will not be discussed

- The regularization of the least square problem
- The corresponding numerical method (because not implemented yet)

Description of the model

Consider a Lévy process $(X_\tau)_{\tau>0}$ on a filtered probability space.

Lévy-Khintchine formula: there exists a function $\chi : \mathbb{R} \to \mathbb{C}$ s.t.

$$\mathbb{E}(e^{iuX_\tau}) = e^{\tau\chi(u)},$$

$$\chi(u) = -\frac{\sigma^2 u^2}{2} + i\beta u + \int_{|z|<1} (e^{iuz} - 1 - iuz)\nu(dz) + \int_{|z|>1} (e^{iuz} - 1)\nu(dz)$$

- $\sigma \geq 0$: volatility.
- $\beta \in \mathbb{R}$.
- ν is a positive measure on $\mathbb{R}\backslash\{0\}$, called the Lévy measure of $(X_\tau)_{\tau>0}$ and ν is s.t.
$$\int_{\mathbb{R}} \min(1, z^2)\nu(dz) < +\infty.$$

We assume that the discounted price of the risky asset is a martingale obtained as the exponential of the Lévy process:

$$e^{-r\tau} S_\tau = S_0 e^{X_\tau}.$$

The fact that $e^{-r\tau} S_\tau$ is a martingale is equivalent to $\mathbb{E}(e^{X_\tau}) = 1$, i.e.

$$\int_{|z|>1} e^z \nu(dz) < \infty, \quad \text{and} \quad \beta = -\frac{\sigma^2}{2} - \int_{\mathbb{R}} (e^z - 1 - z 1_{|z|\leq 1}) \nu(dz).$$

We also assume that

$$\int_{|z|>1} e^{2z} \nu(dz) < \infty,$$

so the discounted price is a square integrable martingale.

We assume furthermore that the Lévy measure has a density,

$$\nu(dz) = k(z)dz,$$

where k is possibly singular at $z = 0$.

Integro-differential operators for option pricing

We note $\overline{B}$ the integral operator

$$(\overline{B}v)(S) = \int_{\mathbb{R}} \left(v(Se^z) - v(S) - S(e^z - 1)\frac{\partial v}{\partial S}(S) \right) k(z)dz,$$

and $\overline{\mathcal{L}}$ the integro-differential operator

$$\overline{\mathcal{L}}v = \frac{\partial v}{\partial \tau} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 v}{\partial S^2} + rS\frac{\partial v}{\partial S} - rv + \overline{B}v.$$

American options

The price of the American option with payoff $\overline{P_\circ}$ and maturity t is

$$P_\tau = P(\tau, S_\tau),$$

where

$$\begin{cases} \overline{\mathcal{L}} P(\tau, S) \leq 0, & 0 \leq \tau < t, \quad S > 0, \\ P(\tau, S) \geq \overline{P_\circ}(S), & 0 \leq \tau < t, \quad S > 0, \\ \overline{\mathcal{L}} P(\tau, S)\big(P(\tau, S) - \overline{P_\circ}(S)\big) = 0, & 0 \leq \tau < t, \quad S > 0, \\ \\ P(t, S) = \overline{P_\circ}(S), & S > 0. \end{cases}$$

Some references

- Maximum principle for elliptic equations associated with Lévy processes: Bony, Courrège and Priouret (1968), Cancelier.
- Related variational inequality when $\sigma > 0$ in suitably exponentially weighted Sobolev spaces: Bensousan and Lions, Schwab et al.
- Viscosity solutions, cover the case when $\sigma = 0$ and hyperbolic problems: Pham, Cont et al, Arizawa.
- Numerical methods: Cont et al, Schwab and collaborators, YA and O. Pironneau
- Calibration of Lévy processes with European options: Cont and Tankov.
- Calibration of local volatility with American options: YA, YA and O. Pironneau.

American options: a forward linear complementarity problem (1)

We aim at finding a forward LCP in the variables maturity/strike.

If $\overline{P}_\circ(S) = (x-S)_+$ then $P(\tau, S, t, x) = xg(\xi, y),$ $y = S/x,$ $\xi = t-\tau,$

where

$$\begin{cases} \widetilde{\mathcal{L}}g(\xi,y) \geq 0, & 0 < \xi \leq t, \quad y > 0, \\ g(\xi,y) \geq (1-y)_+, & 0 < \xi \leq t, \quad y > 0, \\ \widetilde{\mathcal{L}}g(\xi,y)\big(g(\xi,y) - (1-y)_+\big) = 0, & 0 < \xi \leq t, \quad y > 0, \\ g(0,y) = (1-y)_+, & y > 0, \end{cases}$$

with

$$\widetilde{\mathcal{L}}g = \frac{\partial g}{\partial \xi} - \frac{\sigma^2 y^2}{2}\frac{\partial^2 g}{\partial y^2} - ry\frac{\partial g}{\partial y} + rg + \widetilde{B}g$$

and $$(\widetilde{B}g)(y) = -\int_{\mathbb{R}} \left(v(ye^z) - v(y) - y(e^z-1)\frac{\partial v}{\partial y}(y) \right) k(z)dz.$$

American options: a forward linear complementarity problem (2)

From this observation and the identities

$$x\frac{\partial g}{\partial \xi} = -\frac{\partial P}{\partial t}, \qquad xy\frac{\partial g}{\partial y} = -x\frac{\partial P}{\partial x} + P, \quad and \quad xy^2\frac{\partial^2 g}{\partial y^2} = x^2\frac{\partial^2 P}{\partial x^2},$$

we deduce that $P(0, S, t, x)$ satisfies the forward problem:

$$\begin{cases} \mathcal{L}P(t,x) \geq 0, & 0 < t, \quad x > 0, \\ P(t,x) \geq P_\circ(x), & 0 < t, \quad x > 0, \\ \mathcal{L}P(t,x)\big(P(t,x) - P_\circ(x)\big) = 0, & 0 < t, \quad x > 0, \\ P(0,x) = P_\circ(x) = (S-x)_+, & x > 0, \end{cases}$$

where

$$\mathcal{L}u = \frac{\partial u}{\partial t} - \frac{\sigma^2 x^2}{2}\frac{\partial^2 u}{\partial x^2} + rx\frac{\partial u}{\partial x} + Bu,$$

and $$(Bu)(x) = -\int_{\mathbb{R}} k(z)\left(x(e^z - 1)\frac{\partial u}{\partial x}(x) + e^z(u(xe^{-z}) - u(x))\right) dz.$$

Interest of the previous problem

• Allows for computing the prices of a family of American put options on the same underlying with different maturities and strikes.

• Especially useful for calibration:
in the context of a least square method for calibration, the evaluation of the cost functional requires solving only one forward problem instead of I backward problems, if I is the number of the observed prices.

• This program is not possible with local volatility BS models.

• Under some assumptions, the price can be found by solving a parabolic variational inequality in suitable weighted (and possibly fractional) Sobolev spaces.

Change of unknown function in the forward problem

In order to have a datum with a compact support in x, it is helpful to change the unknown function: we set

$$u_\circ(x) = (S - x)_+; \qquad u(t,x) = P(t,x) - x + S.$$

We get

$$\begin{cases} \mathcal{L}u(t,x) \geq 0, & 0 < t, \quad x > 0, \\ u(t,x) \geq u_\circ(x), & 0 < t, \quad x > 0, \\ \mathcal{L}u(t,x)\big(u(t,x) - u_\circ(x)\big) = 0, & 0 < t, \quad x > 0, \\ u(0,x) = u_\circ(x) & \qquad\quad x > 0, \end{cases}$$

We will restrict ourselves to the cases when this problem is parabolic.

The chosen class of Lévy processes

$$\nu(dz) = k(z)dz,$$

with

$$\int_{\mathbb{R}} \min(1, z^2)k(z)dz < \infty \quad \text{and} \quad \int_1^{+\infty} e^{2z}k(z)dz < \infty. \tag{1}$$

To get a parabolic problem, we assume (1) and

$$k(z) = \psi(z)|z|^{-(1+2\alpha)},$$

where

$$\begin{cases} \psi \text{ is a nonnegative bounded function s.t. } \psi \geq \underline{\psi} > 0 \text{ in } [-\bar{z}, \bar{z}], \\ \begin{cases} -1/2 \leq \alpha < 1 & \text{if} \quad \sigma > 0, \\ 1/2 < \alpha < 1 & \text{if} \quad \sigma = 0. \end{cases} \end{cases}$$

The chosen parametrization covers the particular cases:

• Merton model: $\sigma > 0$ and the jumps in the log-price have a Gaussian distribution.

• Some Kou models: $\sigma > 0$ and the density of jumps is an asymmetric exponential with a fast enough decay at ∞.

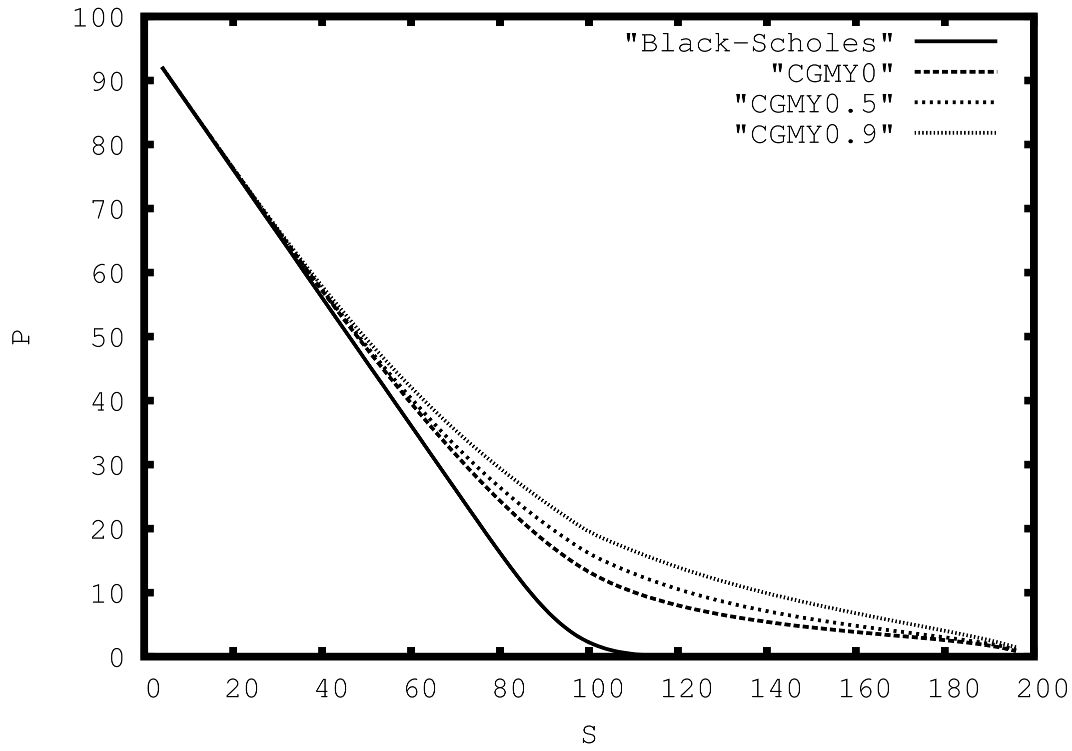
• Some variance gamma processes: $\sigma > 0$ and

$$k(z) = \frac{1}{\mu|z|}\left(e^{-\frac{|z|}{\eta_n}} 1_{z<0} + e^{-\frac{|z|}{\eta_p}} 1_{z>0}\right), \qquad \eta_n > 0, \quad 1/2 > \eta_p > 0,$$

and normal inverse Gaussian processes $\sigma > 0,\ \alpha = 1/2$, with a fast enough decay of the jump density at ∞.

• Some parabolic generalized CGMY models

$$k(z) = C|z|^{-(1+Y)}\left(e^{-G|z|} 1_{z<0} + e^{-M|z|} 1_{z>0}\right), \qquad 0 < Y < 2, \quad 0 < G,\ 2 < M.$$



Comparison between the price of a European put computed by Black-Scholes formula with $\sigma = 0.1$ and puts on CGMY driven assets with $\sigma = 0.1$ and $Y = 0, 0.5, 0.9$, $C = 1$, $M = 2.5$, $G = 1.8$.

Weighted Sobolev spaces on $\mathbb{R}_+$ (1)

Introduce

$$V^1 = \left\{ v \in L^2(\mathbb{R}_+), x\frac{\partial v}{\partial x} \in L^2(\mathbb{R}_+) \right\},$$

with the norm

$$\|v\|_{V^1} = \sqrt{\|v\|^2_{L^2(\mathbb{R}_+)} + \|x\frac{\partial v}{\partial x}\|^2_{L^2(\mathbb{R}_+)}}.$$

It can be proved that $\mathcal{D}(\mathbb{R}_+)$ is a dense subspace of V^1, and that

$$\|v\|_{L^2(\mathbb{R}_+)} \leq 2\|x\frac{dv}{dx}\|_{L^2(\mathbb{R}_+)}, \quad \forall v \in V^1.$$

Therefore, the semi-norm

$$|v|_{V^1} = \|x\frac{dv}{dx}\|_{L^2(\mathbb{R}_+)}$$

is a norm equivalent to $\|.\|_{V^1}$.

Weighted Sobolev spaces on $\mathbb{R}_+$ (2)

For a function v defined on $\mathbb{R}_+$, call $\tilde{v}$ the function defined on $\mathbb{R}$ by

$$\tilde{v}(y) = v(\exp(y))\exp(y/2).$$

The mapping $v \mapsto \tilde{v}$ is a topological isomorphism from $L^2(\mathbb{R}_+)$ onto $L^2(\mathbb{R})$, and from V^1 onto $H^1(\mathbb{R})$.

This leads to defining the space V^s, for $s \in \mathbb{R}$, by

$$V^s = \{v \ : \ \tilde{v} \in H^s(\mathbb{R})\},$$

which is a Hilbert space with the norm $\|v\|_{V^s} = \|\tilde{v}\|_{H^s(\mathbb{R})}$, where

$$\|w\|_{H^s(\mathbb{R})} = \sqrt{\int_{\mathbb{R}} (1+\xi^2)^s |\widehat{w}(\xi)|^2 d\xi}.$$

For $s > 0$, the space V^{-s} is the topological dual of V^s.

Lemma

For all s, $1/2 < s \leq 1$,

$$V^s \subset \mathcal{C}(0, \infty),$$

and there exists $C > 0$ such that $\forall v \in V^s$,

$$\sqrt{x}|v(x)| \leq C\|v\|_{V^s}, \quad \forall x \in [1, +\infty).$$

Properties of the integral operator B

$$(Bu)(x) = -\int_{\mathbb{R}} \psi(z)|z|^{-(1+2\alpha)} \left(x(e^z - 1)\frac{\partial u}{\partial x}(x) + e^z(u(xe^{-z}) - u(x)) \right) dz.$$

Lemma Let (α, ψ) satisfy the assumptions above. For each $s \in \mathbb{R}$,

- if $\alpha > 1/2$, then the operator B is continuous from V^s to $V^{s-2\alpha}$
- if $\alpha < 1/2$, then the operator B is continuous from V^s to V^{s-1}
- if $\alpha = 1/2$, then the operator B is continuous from V^s to $V^{s-1-\epsilon}$, $\forall \epsilon > 0$.

Corollary If (α, ψ) satisfy the assumptions above and if $1/2 < \alpha < 1$, then the operator B is continuous from V^α to $V^{-\alpha}$.

Adjoint of B If (α, ψ) satisfy the assumptions, the operator B^T:

$$(B^T u)(x) = \int_{\mathbb{R}} k(z) \left(x(e^z - 1) \frac{\partial u}{\partial x}(x) - e^{2z} u(xe^z) + (2e^z - 1) u(x) \right) dz$$

- is a continuous operator from V^s to $V^{s-2\alpha}$, if $\alpha > 1/2$,
- is a continuous operator from V^s to V^{s-1}, if $\alpha < 1/2$,
- is a continuous operator from V^s to $V^{s-1-\epsilon}$, for any $\epsilon > 0$, if $\alpha = 1/2$.

If $\alpha > 1/2$, then for all $u, v \in V^\alpha$,

$$\langle B^T u, v \rangle = \langle Bv, u \rangle.$$

If $\alpha \leq 1/2$, this identity holds for all $u, v \in V^s$ with $s > 1/2$.

Gårding inequality

For $s > 0$, we introduce the semi-norm $|v|_{V^s} = |\tilde{v}|_{H^s(\mathbb{R})}$.

Proposition

- If $1/2 < \alpha < 1$, there exist two constants $\underline{C} > 0$ and $\lambda \geq 0$ such that,

$$\langle Bv, v\rangle \geq \underline{C}|v|^2_{V^\alpha} - \lambda\|v\|^2_{L^2(\mathbb{R}_+)}, \quad \forall v \in V^\alpha.$$

- If $\alpha \leq 1/2$, then

$$\langle Bv, v\rangle \geq \underline{C}|v|^2_{V^\alpha} - \lambda\|v\|^2_{L^2(\mathbb{R}_+)}, \quad \forall v \in V^s, s > 1/2\ .$$

with $\underline{C} = 0$ if $\alpha < 0$.

The integro-differential operator $Av = -\frac{\sigma^2 x^2}{2} \frac{\partial^2 v}{\partial x^2} + rx\frac{\partial v}{\partial x} + Bv$

If $\sigma > 0$, and if $0 \leq \alpha < 1$, then

- A is a continuous operator from V^1 to V^{-1},
- There is a Gårding inequality: $\langle Av, v\rangle \geq \underline{c}|v|^2_{V^1} - \lambda\|v\|^2_{L^2(\mathbb{R}_+)}$.
- weak maximum principle since $\langle Av, v_+\rangle \geq \underline{c}|v_+|^2_{V^1} - \lambda\|v_+\|^2_{L^2(\mathbb{R}_+)}$.
- $A + \lambda I$ is continuous and invertible from V^2 onto $L^2(\mathbb{R}_+)$.

If $\sigma = 0$, and if $\frac{1}{2} < \alpha < 1$, then

- A is a continuous operator from V^α to $V^{-\alpha}$,
- There is a Gårding inequality: $\langle Av, v\rangle \geq \underline{c}|v|^2_{V^\alpha} - \lambda\|v\|^2_{L^2(\mathbb{R}_+)}$.
- weak maximum principle since $\langle Av, v_+\rangle \geq \underline{c}|v_+|^2_{V^\alpha} - \lambda\|v_+\|^2_{L^2(\mathbb{R}_+)}$.
- $A + \lambda I$ is continuous and invertible from $V^{2\alpha}$ onto $L^2(\mathbb{R}_+)$.

The variational problem (VI)

Take

$$\begin{cases} V = V^1 \quad \text{if} \quad \sigma > 0, \quad \text{and} \quad V = V^\alpha \quad \text{if} \quad \sigma = 0 \text{ and } \alpha > 1/2, \\ K = \{v \in V,\ v(x) \geq u_\circ(x) \ \text{ in } \mathbb{R}_+\}. \end{cases}$$

u satisfies (VI) if

1. $u \in L^2(0,T;V) \cap C^0([0,T];L^2(\mathbb{R}_+))$, with $\frac{\partial u}{\partial t} \in L^2((0,T) \times \mathbb{R}_+)$,
2. there exists a constant $X_T > S$ s.t.
$$u(t,x) = 0, \quad \forall t \in [0,T], \forall x \geq X_T.$$
3. $u(t) \in K$ for almost every $t \in (0,T)$, and $u(t=0) = u_\circ$,
4. for a.e. $t \in (0,T)$, for any $v \in K$ with bounded support,
$$\left\langle \frac{\partial u}{\partial t} + Au + rx, v - u \right\rangle \geq 0.$$

The case $\sigma > 0$

Theorem Under the assumptions above and if $\sigma > 0$,

- there exists a unique solution of (VI) in $L^2(0, T, V^2)$.
- $\exists$ a nondecreasing and LSC function $\gamma : (0, T] \to (S, X_T)$ s.t.

$$\forall t \in (0, T), \quad \{x > 0 \text{ s.t. } u(t, x) = u_\circ(x)\} = [\gamma(t), +\infty).$$

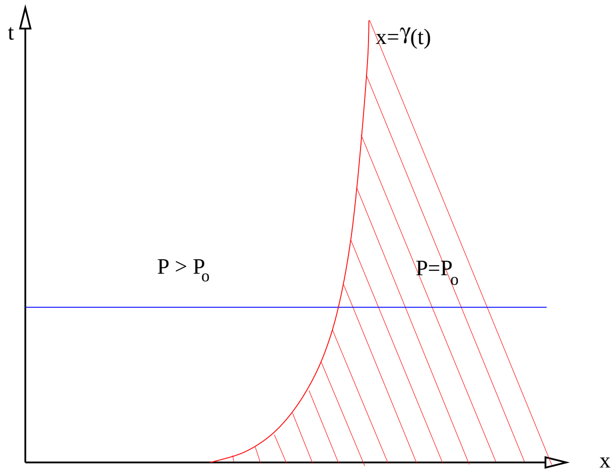
- Calling

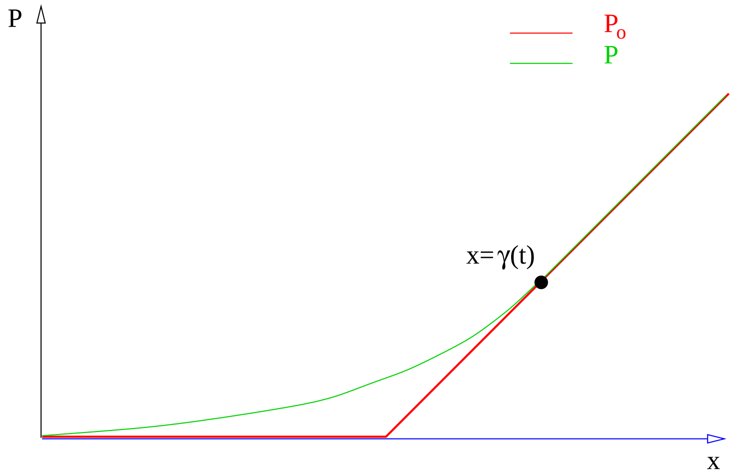
$$\mu = \frac{\partial u}{\partial t} + Au + rx,$$

  we have a.e.

$$0 \le \mu = 1_{\{u=0\}} \left(rx - \int_{\mathbb{R}} k(z) e^z u(t, xe^{-z}) dz \right) \le rx 1_{\{x \ge \gamma(t)\}}.$$

- The function μ is nondecreasing w.r.t. x and nonincreasing w.r.t. t.
- $\mu > 0$ a.e. in the set $\{(t, x) : \ u(t, x) = 0\}$.


t
x=γ(t)
P > P_o
P=P_o
x


P
P_o
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x=γ(t)
x

Scheme of the proof

1. for $X > S$, approximate (VI) by a similar problem (VI$_X$) posed in $[0, T] \times [0, X]$ with a Dirichlet condition on $x = X$;

2. solve first a penalized version of (VI$_X$) by introducing a semilinear monotone operator. Pass to the limit as the penalty parameter tends to zero;

3. prove that the free boundary of (VI$_X$) stays in a bounded domain as X tends to infinity: this will show that for X large enough a solution of (VI$_X$) is actually a solution of (VI);

The penalized problem in $[0, X]$ (1)

- $\forall v \in L^2(0,X)$, $\mathcal{E}_X(v) \in L^2(\mathbb{R}_+)$ is the extension of v by 0:

$$\begin{cases} \mathcal{E}_X(v)(x) = v(x), & \text{if } 0 < x < X, \\ \mathcal{E}_X(v)(x) = 0, & \text{if } x > X. \end{cases}$$

- $$V_X = \{ v \in L^2(0,X),\ \mathcal{E}_X(v) \in V \} \qquad \left(\leftrightarrow H_0^1 \right)$$

- $$\begin{aligned} A_X : \quad & V_X \to V_X', \\ & \langle A_X v, w \rangle = \langle A\mathcal{E}_X(v), \mathcal{E}_X(w) \rangle. \end{aligned}$$

The penalized problem in $[0, X]$ (2)

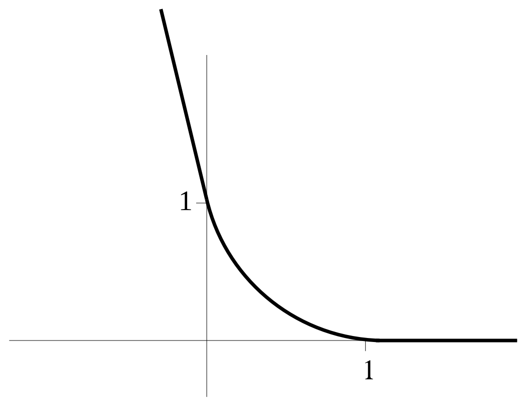
Find $u_{X,\epsilon}$ s.t.

$$\begin{cases} \dfrac{\partial u_{X,\epsilon}}{\partial t} + A_X u_{X,\epsilon} + rx(1 - 1_{\{x>S\}}\mathcal{V}_\epsilon(u_{X,\epsilon})) = 0, & t \in (0,T],\ 0 < x < X, \\ u_{X,\epsilon}(t=0,x) = u_\circ(x), & 0 < x < X, \\ u_{X,\epsilon}(t,X) = 0, & t \in (0,T], \end{cases}$$

where

$$\mathcal{V}_\epsilon(u) = \mathcal{V}(\frac{u}{\epsilon})$$

and $\mathcal{V}$ is a smooth nonincreasing convex function:



Ingredient: a regularity result for the Dirichlet problem

Introduce the function spaces

$$W_X^1 = \left\{ v \in L^2(0,X), x\frac{\partial v}{\partial x} \in L^2(0,X) \right\} \qquad \Big(\leftrightarrow H^1\Big),$$
$$W_X^2 = \left\{ v \in W_X^1, x^2\frac{\partial^2 v}{\partial x^2} \in L^2(0,X) \right\} \qquad \Big(\leftrightarrow H^2\Big).$$

For $0 < \beta < 1$, W_X^β is the space obtained by real interpolation between W_X^1 and $L^2(0,X)$ with parameter $\nu = 1/2 - \beta$

and

$$W_X^{1+\beta} = \left\{ v \in W_X^1, x\frac{\partial v}{\partial x} \in W_X^\beta \right\} \qquad \Big(\leftrightarrow H^{1+\beta}\Big).$$

The domain of A_X is

$$D_X = \left\{ v \in V_X :\ A_X v \in L^2(0,X) \right\}.$$

Proposition

$$\sigma > 0$$

- If $v \in D_X$, then $v|_{(0,X')} \in W^2_{X'}, \quad \forall X' < X$.
- For $0 < \alpha < 3/4$, $D_X = W^2_X \cap V_X$.
- For $3/4 \leq \alpha < 1$, $\exists \epsilon > 0$ s.t. $D_X \subset W^{3/2+\epsilon}_X \cap V_X$.
- If $v \in D_X$, then $\frac{\partial v}{\partial x} \in \mathcal{C}^0((0,X])$.

Consequence For any $\epsilon > 0$,

$$S < X < X' \qquad \Rightarrow \qquad \mathcal{E}_X(u_{X,\epsilon}) \leq \mathcal{E}_{X'}(u_{X',\epsilon}).$$

Bounds on the solution of (VI)

Fix $0 < \underline{\sigma} \leq \bar{\sigma}$, $0 < \underline{\alpha} < 1/2$, $b_1 > 1$, $b_2 > 1$, $\bar{\psi} \geq \underline{\psi} > 0$ and $\bar{z} > 0$ and define

$$\mathcal{F} = [\underline{\sigma}, \bar{\sigma}] \times [-1/2, 1-\underline{\alpha}] \times \left\{ \psi : \left| \begin{array}{l} \| \max(e^{2b_1 z}, |z|^{b_2}, 1)\psi\|_{L^\infty(\mathbb{R})} \leq \bar{\psi}; \\ \psi \geq 0,\ \psi \geq \underline{\psi} \text{ a.e. in } [-\bar{z}, \bar{z}] \end{array} \right. \right\}.$$

Proposition

The function γ is bounded in $[0, T]$ by some constant $\bar{X}$ independent of (σ, α, ψ) in $\mathcal{F}$.

The quantities $\|u\|_{L^\infty(0,T;V)}$, $\|u\|_{L^2(0,T;V^2)}$ and $\|\frac{\partial u}{\partial t}\|_{L^2((0,T)\times\mathbb{R}_+)}$ are bounded independently of (σ, α, ψ) in $\mathcal{F}$.

The case $\sigma = 0$

$$\mathcal{F}_2 = [1/2 + \underline{\alpha}, 1 - \underline{\alpha}] \times \left\{ \psi : \left| \begin{array}{l} \| \max(e^{2b_1 z}, |z|^{b_2}, 1)\psi\|_{L^\infty(\mathbb{R})} \le \bar{\psi}; \\ \psi \ge 0,\ \psi \ge \underline{\psi} \text{ a.e. in } [-\bar{z}, \bar{z}] \end{array} \right. \right\}.$$

Fix $(\alpha, \psi) \in \mathcal{F}_2$ and call u_σ the solution of (VI) when the volatility is σ.

Lemma

The quantities $\|u_\sigma\|_{L^\infty(0,T;V^\alpha)}$ and $\|u_\sigma\|_{L^2(0,T;V^{2\alpha})}$ are bounded independently of σ, and the free boundary associated to u_σ stays in $[0, T] \times [0, \widetilde{X}]$, where $\widetilde{X}$ does not depend on σ.

One may apply the theorems of J.L. Lions on singularly perturbed problems and pass to the limit as $\sigma \to 0$.

$\Rightarrow$ existence and uniqueness for (VI) when $\sigma = 0$ and bounds on the free boundary and the solution.

Sensitivity (1) (case $\sigma > 0$)

$\exists C > 0$ such that $\quad \forall(\sigma,\alpha,\psi)$, $(\tilde\sigma,\tilde\alpha,\tilde\psi)$ in $\mathcal{F}$,

$$\left\{\begin{array}{l} \|u-\tilde u\|_{L^2(0,T;V)} + \|u-\tilde u\|_{L^\infty(0,T;L^2(\mathbb{R}_+))} \le C\left(|\sigma-\tilde\sigma| + |\alpha-\tilde\alpha| + \|\psi-\tilde\psi\|_{\mathcal{B}}\right), \\ \\ \displaystyle\int_0^T\int_{\mathbb{R}} \left(\mu(\tilde u - u_\circ) + \tilde\mu(u-u_\circ)\right) \le C\left(|\sigma-\tilde\sigma| + |\alpha-\tilde\alpha| + \|\psi-\tilde\psi\|_{\mathcal{B}}\right)^2, \end{array}\right.$$

where

$$\left\{\begin{array}{ll} u = u(\sigma,\alpha,\psi), & \qquad \mu = \mu(\sigma,\alpha,\psi), \\ \tilde u = u(\tilde\sigma,\tilde\alpha,\tilde\psi), & \qquad \tilde\mu = \mu(\tilde\sigma,\tilde\alpha,\tilde\psi). \end{array}\right.$$

Sensitivity (2) (case $\sigma > 0$)

Let $(\sigma_n, \alpha_n, \psi_n)_{n\in\mathbb{N}}$ be a sequence of coefficients in $\mathcal{F}$ such that

$$\lim_{n\to\infty} \left(|\sigma - \sigma_n| + |\alpha - \alpha_n| + \|\psi - \psi_n\|_{\mathcal{B}}\right) = 0.$$

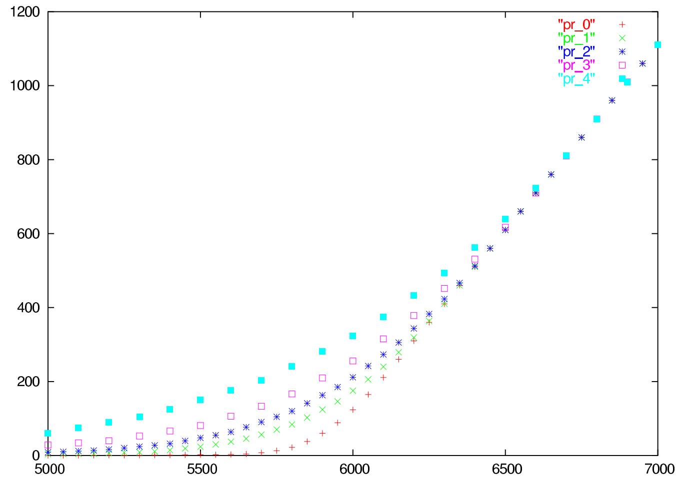
With the notations $u_n = u(\sigma_n, \alpha_n, \psi_n)$ and $\mu_n = \mu(\sigma_n, \alpha_n, \psi_n)$,

$$\begin{cases} \|u_n - u\|_{L^\infty((0,T)\times\mathbb{R}_+)} \to 0, \\ \\ \|\mu_n - \mu\|_{L^p((0,T)\times\mathbb{R}_+)} \to 0, \qquad \forall 1 < p < +\infty, \\ \\ \|u_n - u\|_{L^\infty(0,T;V)} + \|u_n - u\|_{L^2(0,T;V^2)} + \|\dfrac{\partial u_n}{\partial t} - \dfrac{\partial u}{\partial t}\|_{L^2((0,T)\times\mathbb{R}_+)} \to 0. \end{cases}$$

Calibration of the Lévy process

Goal: try to calibrate the Lévy process in order to recover the prices of a family of put options on the asset of interest, which are available on the market.

The observable American puts are characterized by $(t_i, x_i)_{i \in I}$.



A typical set of data : observed prices for a family $(t_i, x_i)_{i \in I}$

Least squares (LS)

- Observe the options' prices $(\bar{P}_i)_{i\in I}$, the spot $S_\circ$ ($\tau = 0$) and define

$$\bar{u}_i = x_i - S_\circ - \bar{P}_i \qquad J(u) = \sum_i \omega_i \big(u(t_i, x_i) - \bar{u}_i\big)^2.$$

- Take a convex $\mathcal{H} \subset \mathcal{F}$ and a convex funct. $J_R : \mathcal{H} \to \mathbb{R}_+$,

$$\mathcal{H} = [\underline{\sigma}, \bar{\sigma}] \times [-1/2, 1 - \underline{\alpha}] \times \mathcal{H}_\psi,$$

  s.t. for all sequence $(\sigma_n, \alpha_n, \psi_n) \in \mathcal{H}$ with $J_R(\sigma_n, \alpha_n, \psi_n)$ bounded, one can extract $(\sigma_{n'}, \alpha_{n'}, \psi_{n'})$ converging in $\mathcal{F}$ to $(\sigma, \alpha, \psi) \in \mathcal{H}$ with

$$J_R(\sigma, \alpha, \psi) \leq \liminf J_R(\sigma_{n'}, \alpha_{n'}, \psi_{n'}).$$

- The least square problem is to

$$\text{Minimize } J(u) + J_R(\sigma, \alpha, \psi) \;\Big|\; (\sigma, \alpha, \psi) \in \mathcal{H},\ u = u(\sigma, \alpha, \psi) \text{ satisfies (VI)}.$$

Advantage: to evaluate the cost function, one needs to solve only one parabolic partial integrodifferential variational inequality, instead of $\#I$ backward problems.

Difficulty for finding the optimality conditions The differentiability of u w.r.t. (σ, α, ψ) is not clear.

Program Find first the optimality conditions for a modified LS problem where the state satisfies the penalized nonlinear pb and pass to the limit.

Since $\gamma(t) \leq \overline{X}$, with $\overline{X}$ independent of $(\sigma, \alpha, \psi) \in \mathcal{H}$, the LS problem is equivalent to

$$\text{Minimize } J(u_X) + J_R(\sigma, \alpha, \psi) \;\Big|\; (\sigma, \alpha, \psi) \in \mathcal{H}, u_X = u_X(\sigma, \alpha, \psi) \text{ satisfies } (\mathrm{VI}_X)\ .$$

with $X > \overline{X}$.

Approximate by the LS problem corresponding to the penalized version

$$\text{Minimize } J(u_{X,\epsilon}) + J_R(\sigma, \alpha, \psi) \;\Big|\; (\sigma, \alpha, \psi) \in \mathcal{H}, \text{ and}$$

$$\begin{cases} \dfrac{\partial u_{X,\epsilon}}{\partial t} + A_X u_{X,\epsilon} + rx(1 - 1_{\{x>S\}} \mathcal{V}_\epsilon(u_{X,\epsilon})) = 0, \quad t \in (0, T],\ 0 < x < X, \\ u_{X,\epsilon}(t = 0, x) = u_\circ(x), \quad 0 < x < X, \\ u_{X,\epsilon}(t, X) = 0, \quad t \in (0, T]. \end{cases}$$

Necessary optimality condition(1): the adjoint problem

Assume that $\bar{u}_i > u_\circ(t_i, x_i)$, $\forall i$. For each optimal triplet $(\sigma^*, \alpha^*, \psi^*)$, (which can be obtained as the limit of optimal triplets for a least square problems with the penalized problem),
we call (u^*, μ^*) the state related to $(\sigma^*, \alpha^*, \psi^*)$.

There exist a function $q^* \in Z$ and a Radon measure ξ^*, s.t. for all regular test-function v with bounded support in x,

$$\int_0^T \int_{\mathbb{R}} \left(\frac{\partial v}{\partial t} + Av \right) q^* + < \xi^*, v > = 2 \sum_{i \in I} \omega_i (u^*(t_i, x_i) - \bar{u}_i) v((t_i, x_i)),$$

$$\mu^* |q^*| = 0, \qquad |u^*| \xi^* = 0.$$

where

$$Z = \left\{ v \in L^2(0, T; V_X); \frac{\partial v}{\partial t} + A_X v \in L^2((0, T) \times (0, X)),\ v(t = 0) = 0 \right\},$$

We have

$$\frac{\partial q^*}{\partial t} - A_X^T q^* = -2 \sum_{i \in I} \omega_i (u^*(t_i, x_i) - \bar{u}_i) \delta_{t=t_i} \otimes \delta_{x=x_i}$$

in the sense of distributions and

$$q^* \text{ vanishes in the coincidence set.}$$

Necessary optimality conditions (2)

and for all $(\sigma, \eta, \psi) \in \mathcal{H}$,

$$(\sigma - \sigma^*) \left(D_\sigma J_R(\sigma^*, \alpha^*, \psi^*) + \sigma^* \mathcal{G}^{(\sigma)}(u^*, q^*) \right) \geq 0,$$

where

$$\mathcal{G}^{(\sigma)}(u^*, q^*) \simeq \int_0^T \left\langle x^2 \frac{\partial^2 u^*}{\partial x^2}, q^* \right\rangle,$$

Necessary optimality conditions (3)

and

$$(\alpha - \alpha^*) \left(D_\alpha J_R(\sigma_\epsilon^*, \alpha_\epsilon^*, \psi_\epsilon^*) + 2\mathcal{G}^{(\alpha)}(u^*, q^*) \right) \geq 0,$$

$$\langle D_\psi J_R(\sigma_\epsilon^*, \alpha_\epsilon^*, \psi_\epsilon^*), \psi - \psi^* \rangle + \left\langle \mathcal{G}^{(\psi)}(u^*, q^*), \psi - \psi^* \right\rangle \geq 0,$$

with

$$\mathcal{G}^{(\alpha)}(u^*, q^*) \simeq \int_0^T \left\langle B_X^{(\alpha)} u^*, q^* \right\rangle, \qquad \left\langle \mathcal{G}^{(\psi)}(u^*, q^*), \kappa \right\rangle \simeq \int_0^T \left\langle B_X^{(\psi,\kappa)} u^*, q^* \right\rangle$$

and

$$B_X^{(\alpha)} v(x) = -\int_{\mathbb{R}} k^*(z) \log(|z|) \left(x(e^z - 1) \frac{\partial v}{\partial x}(x) + e^z (1_{\{z > -\log(\frac{X}{x})\}} v(xe^{-z}) - v(x)) \right),$$

$$B_X^{(\psi,\kappa)} v(x) = \int_{\mathbb{R}} \frac{\kappa(z)}{|z|^{1+2\alpha^*}} \left(x(e^z - 1) \frac{\partial v}{\partial x}(x) + e^z (1_{\{z > -\log(\frac{X}{x})\}} v(xe^{-z}) - v(x)) \right) dz.$$