Early View publication on www.interscience.wiley.com
(issue and page numbers not yet assigned;
citable using Digital Object Identifier – **DOI**)

ZAMM · Z. Angew. Math. Mech., 1 – 10 (2006) / **DOI** 10.1002/zamm.200510249

Some remarks on the asymptotic invertibility of the linearized operator of nonlinear elasticity in the context of the displacement approach

R. Monneau*

CERMICS, Ecole Nationale des Ponts et Chaussees, 6 et 8 avenue Blaise Pascal, Cite Descartes Champs-sur-Marne, 77455-Marne-la-Vallee Cedex 2, France

Received 1 March 2005, revised and accepted 12 September 2005
Published online 28 February 2006

Key words nonlinear elasticity, Saint-Venant Kirchhoff energy, displacement approach, inverse function theorem, invertibility, linearized operator, elastic beam, thin domains, asymptotics, instability under compression

MSC (2000) 35B10, 35B40, 35E20, 35J45, 35Q72

In this article we study the invertibility of the linearized operator coming from the nonlinear elasticity in the special case of a two-dimensional thin beam of thickness 2ε in one direction and of length $2\pi L$ and periodic in the other direction. In the context of the displacement approach, we show that the linearized operator is not invertible for some small compressions of order $O(\varepsilon^2/L^2)$ in the direction of the thickness of the beam, and not in the direction of the length as it is usually considered. In particular, we study the kernel of an associated linear operator on an infinite strip. This linear operator depends on a parameter $\overline{\delta}$ which describes the compression with respect to the thickness for $\overline{\delta} < 0$. For small enough $\overline{\delta} > 0$, we prove that the kernel is trivial; on the contrary for $\overline{\delta} < 0$, we rigorously find periodic solutions in the kernel. This last fact is related to the non-invertibility of the previous linearized operator coming from nonlinear elasticity.

1 Introduction

The goal of this article is to show that the invertibility of the linearized operator of nonlinear elasticity can be a delicate question in some domains of asymptotically small thickness, and then that the application of the inverse function theorem may fail.

Let us consider a two-dimensional thin beam of thickness 2ε and of length $2\pi L > 0$, and periodic in the direction of the length: $\Omega^\varepsilon = \omega \times (-\varepsilon, \varepsilon)$ with $\omega = \mathbf{R}/(2\pi L\mathbf{Z})$. We introduce the coordinates $x^\varepsilon = (x_1^\varepsilon, x_2^\varepsilon)$. We asume that this beam is submitted to volume forces $f^\varepsilon = (f_1^\varepsilon, f_2^\varepsilon)$, and surface forces $g^\varepsilon = (g_1^\varepsilon, g_2^\varepsilon)$, satisfying the following equilibrium condition

$$\int_{\Omega^\varepsilon} f_i^\varepsilon - \int_{\partial\Omega^\varepsilon} g_i^\varepsilon = 0\,, \quad i = 1, 2\,. \tag{1.1}$$

For a given two-dimensional displacement of the beam $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$, we consider the following energy of nonlinear elasticity (see the book of Ciarlet [2] for similar models)

$$\mathcal{E}(u^\varepsilon) = \mathcal{E}_0(u^\varepsilon) + \int_{\Omega^\varepsilon} f_i^\varepsilon u_i^\varepsilon - \int_{\partial\Omega^\varepsilon} g_i^\varepsilon u_i^\varepsilon \tag{1.2}$$

with the Saint-Venant Kirchhoff free energy

$$\mathcal{E}_0(u^\varepsilon) = \int_{\Omega^\varepsilon} \frac{\lambda}{2} E_{ii}^\varepsilon E_{jj}^\varepsilon + \mu E_{ij}^\varepsilon E_{ij}^\varepsilon$$

where the index i, j take the values in $\{1, 2\}$, and we have used the Einstein convention of summation on repeated indices. Here we use the usual definition (with $\partial_i^\varepsilon = \frac{\partial}{\partial x_i^\varepsilon}$)

$$E_{ij}^\varepsilon = \tfrac{1}{2}(\partial_i^\varepsilon u_j^\varepsilon + \partial_j^\varepsilon u_i^\varepsilon + \partial_i^\varepsilon u_k^\varepsilon \partial_j^\varepsilon u_k^\varepsilon)$$

of the Green strain tensor. We consider the following displacement (still in the ellipticity range for the Saint-Venant Kirchhoff energy for $\varepsilon^2\delta$ small enough)

$$u^\varepsilon(x^\varepsilon) = (0, \varepsilon^2 \delta x_2^\varepsilon)$$

* e-mail: monneau@cermics.enpc.fr

which is a solution of the Euler-Lagrange equation of the energy for $f^\varepsilon = 0$, $g^\varepsilon(x_1^\varepsilon, \pm 1) = (0, \pm\varepsilon^2 G^\varepsilon)$ where the constant G^ε is given by

$$G^\varepsilon = (\lambda + 2\mu)\left(\delta + \tfrac{3}{2}\varepsilon^2\delta^2 + \tfrac{1}{2}\varepsilon^4\delta^3\right) . \tag{1.3}$$

In Sect. 2, for a uniform lateral compression, namely for $\delta < 0$, we will show that this particular displacement is not a global minimizer if

$$\delta \leq -\frac{2\mu}{3(\lambda + 2\mu)} \frac{1}{L^2}(1 + o(1))$$

in the limit of small ε.

In Sect. 3, we prove more precisely that the linearized operator of nonlinear elasticity at this particular displacement for a uniform compression is not invertible for some δ satisfying:

$$\delta = -\frac{2\mu}{3(\lambda + 2\mu)} \frac{1}{L^2}(1 + o(1)) .$$

This kind of instability property has an important consequence on the characterization of the set $\mathcal{D}_\varepsilon$ of displacements (used as reference configuration for the linearization), where the linearized operator of Saint-Venant Kirchhoff energy is invertible. In particular, we see that the absolute value of the strain $\frac{\partial u_2^\varepsilon}{\partial x_2^\varepsilon}$ has to be less than $c\varepsilon^2$ for some constant c small enough. A consequence will be that, after the change of coordinates used in the classical displacement approach (see [1,2] and Sect. 2), this set stays clearly ε-dependent. In the context of the displacement approach, this ε-dependence avoids a straightforward application of the inverse function theorem (even of Nash-Moser type) to get solutions at positive ε by perturbation of solutions for $\varepsilon = 0$.

Finally let us note that recent works using the approach of Gamma-convergence (see for instance the book of Dal Maso [4] for an introduction to this notion) have been done to justify various limit models when the thickness 2ε goes to zero (see Friesecke et al. [7] for shells models; Friesecke et al. [8,9] for nonlinear theories of plates, and Mora and Müller [10] for a nonlinear theory of inextensible rods). In particular, some of the limit solutions that they obtain, are outside the domain where the linearized operator is invertible (and ouside the domain where the energy is convex). This is related to the fact that Gamma-convergence can somehow catch some limit solutions after the buckling phenomenon.

Remark 1.1. In a previous work of the author [11], a displacement approach has been applied to justify the nonlinear Kirchhoff-Love theory of periodic plates as a rigorous derivation from the three-dimensional nonlinear elasticity. In this work it was proved that the linearized operator is invertible on a set of displacements which is ε-dependent.

The result of the present paper sheds some light on the reason why it was necessary to consider ε-dependent sets of displacements where the linearized operator is invertible.

Let us mention a related work of Paumier [13] where in the context of the displacement-stress approach, he applied the Nash-Moser inverse function theorem in the case of periodic plates. Unfortunately this study stayed unsuccessfull because the invertibility of the linearized operator was only assumed and has still to be proved. Our formal result does not clarify what happens in the context of the displacement-stress approach, but may indicate that such an invertibility may not be possible to prove because it may be false. However, this would need further investigations and is not the purpose of the present article.

2 Instability of the solution under uniform compression

We first introduce the following change of coordinates, classically used for the displacement approach (and first introduced by Ciarlet et al. [3], Destuynder [5])

$$x = (x_1, x_2) = \left(x_1^\varepsilon, \frac{x_2^\varepsilon}{\varepsilon}\right) \in \Omega = \omega \times (-1, 1)$$

then we have

$$\partial_1^\varepsilon = \partial_1 , \quad \partial_2^\varepsilon = \frac{\partial_2}{\varepsilon} ,$$

and we set

$$\begin{aligned} u_1^\varepsilon(x^\varepsilon) &= \varepsilon^2 u_1(x) , \quad f_1^\varepsilon(x^\varepsilon) = \varepsilon^2 f_1(x) , \\ u_2^\varepsilon(x^\varepsilon) &= \varepsilon u_2(x) , \quad f_2^\varepsilon(x^\varepsilon) = \varepsilon^3 f_2(x) , \end{aligned} \tag{2.4}$$

$$g_1^\varepsilon(x_1, \pm\varepsilon) = \pm\varepsilon^3 g_1(x_1, \pm 1)\,,$$
$$g_2^\varepsilon(x_1, \pm\varepsilon) = \pm\left(\varepsilon^4 g_2(x_1, \pm 1) + \varepsilon^2 G^\varepsilon\right)\,, \tag{2.5}$$

where the constant G^ε is given by (1.3). Let us remark that the perturbation term $\varepsilon^2 G^\varepsilon$ in the expression of g_2^ε is not compatible with the order $O(\varepsilon^4)$ usually assumed (see the von Kármán theory of [3]). Nevertheless this is a small term which will generate small displacements that we want to consider to exhibit an unstability of the beam.

Let us also mention the work of Fonseca and Francfort [6] where it has been shown in particular that scaling (2.4) may be inappropriate for the Gamma-convergence approach to nonlinear elasticity.

We define the renormalized energy by

$$\mathcal{E}^\varepsilon(u) = \frac{1}{\varepsilon^5}\mathcal{E}(u^\varepsilon)$$

and we note for the infinitesimal strain tensor

$$e_{ij}(u) = \tfrac{1}{2}(\partial_i u_j + \partial_j u_i)\,.$$

In Ciarlet [1], as $\varepsilon \to 0$ the formal limit problem is given when the constant G^ε is equal to 0. Using the fact that $G^\epsilon \longrightarrow G^0 = (\lambda + 2\mu)\delta$, a straightforward variant of the displacement approach in [1] gives easily the following limit for the renormalized energy:

Theorem 2.1 (General formal limit of the energy). *Let us assume that* $u = u^0 + \varepsilon^2 u^2 + O(\varepsilon^4)$, *with* $e_{22}(u^0) = e_{12}(u^0) = 0$ *and let* $U := (u^0, e_{22}(u^2))$ *be the formal limit. Then we have formally*

$$\mathcal{E}^\varepsilon(u) \longrightarrow \mathcal{E}^0(U, \theta, \delta) \tag{2.6}$$

where $\theta = (\theta_1, \theta_2)$, $\theta_1(x_1) = IF_1$, $\theta_2(x_1) = IF_2 + \partial_1(Ix_2F_1)$, *with* $F_i = (f_i, g_i)$, *and* $IF_i = \int_{-1}^1 f_i(x_1, x_2)dx_2 + g_i(x_1, 1) - g_i(x_1, -1)$. *Here*

$$\mathcal{E}^0(U, \theta, \delta) = \mathcal{E}_0^0(U) - \int_\omega u_i^0(x_1, 0)\theta_i(x_1) - (\lambda + 2\mu)\delta \int_\Omega e_{22}(u^2) \tag{2.7}$$

and

$$\mathcal{E}_0^0(U) = \int_\Omega \frac{\lambda}{2}\left(\partial_1 u_1^0 + (\partial_1 u_2^0)^2 + e_{22}(u^2)\right)^2 + \mu\left\{\left(\partial_1 u_1^0 + \frac{(\partial_1 u_2^0)^2}{2}\right)^2 + \left(e_{22}(u^2) + \frac{(\partial_1 u_2^0)^2}{2}\right)^2\right\}\,. \tag{2.8}$$

The last term in (2.7) comes from the consideration of constant surface forces equal to $\pm\varepsilon^2 G^\varepsilon$.

Here the quantity $e_{22}(u^2)$ appears as a limit unkown, a priori independent on u^0. Nevertheless, at least formally, it is expected that the limit energy of a “natural solution” has to be minimal. This is why we now minimize the limit energy with respect to the unknown $e_{22}(u^2)$, and we get

$$e_{22}(u^2) = \delta - \frac{1}{\lambda + 2\mu}\left(\lambda(\partial_1 u_1^0) + (\lambda + \mu)(\partial_1 u_2^0)^2\right)\,. \tag{2.9}$$

Moreover setting

$$u^0(x_1, x_2) = \begin{cases} u_1^0(x_1, x_2) = \zeta_1(x_1) - x_2\partial_1\zeta_2(x_1) \\ u_2^0(x_1, x_2) = \zeta_2(x_1) \end{cases} \tag{2.10}$$

and

$$\mathcal{E}(\zeta, \theta, \delta) = \mathcal{E}^0(U, \theta, \delta)$$

for $U = (u^0, e_{22}(u^2))$ with u^0 given by (2.10) and $e_{22}(u^2)$ given by (2.9), we deduce the

Corollary 2.2 (Formal limit of the energy). *Under the assumptions of Theorem* 2.1 *and* (2.9)–(2.10)*, and denoting the limit solution by* $\zeta = (\zeta_1, \zeta_2)$*, we have*

$$\mathcal{E}(\zeta,\theta,\delta) = \frac{4\mu(\lambda+\mu)}{\lambda+2\mu}\,\mathcal{E}^0_\delta(\zeta) - 2(\lambda+2\mu)\int_\omega \delta^2 - \int_\omega \zeta_i\theta_i$$

where

$$\mathcal{E}^0_\delta(\zeta) = \int_\omega \left\{ \left(\zeta_1' + \frac{(\zeta_2')^2}{2}\right)^2 + \tfrac{1}{3}(\zeta_2'')^2 + c_0\delta(\zeta_2')^2 \right\}$$

and $c_0 = \frac{\lambda+2\mu}{2\mu}$.

When $\theta = 0$, we can minimize explicitely the energy. In this case we will prove (using the notation $h^- = \max(-h, 0)$)

Theorem 2.3 (Minimization of the limit energy). *Let*

$$V = \left\{ \zeta = (\zeta_1,\zeta_2) \in H^1(\omega)\times H^2(\omega), \int_\omega \zeta_i = 0, i = 1,2 \right\}.$$

We have

$$\inf_{\zeta\in V} \mathcal{E}^0_\delta(\zeta) = -\frac{|\omega|}{4}\left(\left(c_0\delta + \frac{1}{3L^2}\right)^-\right)^2. \tag{2.11}$$

This infimum is reached exactly for the functions (up to a translation in x_1*):*

$$\zeta = 0 \quad \textit{if} \quad \delta > -\frac{1}{3c_0L^2}$$

and

$$\zeta = \left\{ \begin{array}{l} \zeta_1(x_1) = -\dfrac{L}{8}\left(\left(c_0\delta + \dfrac{1}{3L^2}\right)^-\right)\sin\left(\dfrac{2x_1}{L}\right) \\ \zeta_2(x_1) = \pm L\sqrt{\left(\left(c_0\delta + \dfrac{1}{3L^2}\right)^-\right)}\sin\left(\dfrac{x_1}{L}\right) \end{array}\right| \quad \textit{if} \quad \delta \le -\frac{1}{3c_0L^2}.$$

Proof of Theorem 2.3. Let us first remark that by classical compactness argument there exists a minimizer of the energy for every δ. Moreover, the Euler-Lagrange equations of the energy $\mathcal{E}^0_\delta$ can be written as (after integrations)

$$\left\{ \begin{array}{r} \zeta_1' + \frac{1}{2}(\zeta_2')^2 = \text{constant} = a \ge 0\,, \\ \zeta_2'' - 3(a + c_0\delta)\zeta_2 = 0\,. \end{array}\right. \tag{2.12}$$

Here the last constant of integration is zero because $\int_\omega \zeta_2 = 0$. Then if $-3(a+c_0\delta) \neq \frac{k^2}{L^2}$ for some $k \in \mathbf{N}\backslash\{0\}$, we have $\zeta_2 \equiv 0$ and then $\zeta_1 \equiv 0$.

On the contrary if $-3(a+c_0\delta) = \frac{k^2}{L^2}$ for some $k \in \mathbf{N}\backslash\{0\}$, then up to a translation in x_1, we have

$$\zeta_2(x_1) = A\sin\left(\frac{kx_1}{L}\right)$$

for some constant A. Then from the first Euler-Lagrange equation we deduce that

$$a = \frac{1}{2}A^2\frac{k^2}{L^2}$$

and

$$\zeta_1'(x_1) = -\frac{A^2}{4}\frac{k^2}{L^2}\cos\left(\frac{2kx_1}{L}\right),$$

i.e.

$$\zeta_1(x_1) = -\frac{A^2}{8}\,\frac{k}{L}\sin\left(\frac{2kx_1}{L}\right) .$$

Let us now compute the value of the energy for this solution.

We have

$$\mathcal{E}^0_\delta(\zeta) = \int_\omega a^2 + \frac{1}{3}\frac{1}{2}A^2\frac{k^4}{L^4} + c_0\delta\frac{1}{2}A^2\frac{k^2}{L^2} ,$$

i.e.

$$\frac{1}{|\omega|}\mathcal{E}^0_\delta(\zeta) = a^2 + a\left(\frac{k^2}{3L^2} + c_0\delta\right) .$$

Because of (2.12), the number a has to be non-negative. Therefore we see that the energy is minimal for

$$a = \frac{1}{2}\left(c_0\delta + \frac{k^2}{3L^2}\right)^- \geq 0$$

and then the value of the energy is

$$\frac{1}{|\omega|}\mathcal{E}^0_\delta(\zeta) = -a^2 .$$

From the expression of a we see that this energy is minimal for $k = 1$, i.e. for

$$\begin{cases} \zeta_1(x_1) = -\dfrac{L}{8}\left(c_0\delta + \dfrac{1}{3L^2}\right)^- \sin\left(\dfrac{2x_1}{L}\right) , \\ \zeta_2(x_1) = \pm L\sqrt{\left(c_0\delta + \dfrac{1}{3L^2}\right)^-}\,\sin\left(\dfrac{x_1}{L}\right) , \end{cases}$$

and the energy is

$$\mathcal{E}^0_\delta(\zeta) = -\frac{|\omega|}{4}\left(\left(c_0\delta + \frac{1}{3L^2}\right)^-\right)^2 .$$

□

As a corollary we get the following result:

Corollary 2.4 (Case where the homogeneous solution is not a minimizer). *Let us consider the energy* (1.2) *for nonlinear elasticity with zero volume forces f^ε and surface forces $g^\varepsilon = (0, \varepsilon^2 G^\varepsilon)$ where G^ε is given by* (1.3). *Then the displacement corresponding to a uniform compression of the beam in the direction of its thickness*

$$u^\varepsilon\left(x_1^\varepsilon, x_2^\varepsilon\right) = \left(0, \varepsilon^2\delta x_2^\varepsilon\right) \tag{2.13}$$

is a particular solution of the Euler-Lagrange equation, and the energy of this particular solution is

$$\mathcal{E}(u^\varepsilon) = -\varepsilon^5|\omega|\left(2(\lambda+2\mu)\delta^2 + O(\varepsilon)\right) .$$

Let us define the following displacement

$$v^\varepsilon\left(x_1^\varepsilon, x_2^\varepsilon\right) = \begin{cases} v_1^\varepsilon\left(x_1^\varepsilon, x_2^\varepsilon\right) = \varepsilon^2\left(\zeta_1\left(x_1^\varepsilon\right) - \dfrac{x_2^\varepsilon}{\varepsilon}\zeta_2'\left(x_1^\varepsilon\right)\right) \\ v_2^\varepsilon\left(x_1^\varepsilon, x_2^\varepsilon\right) = \varepsilon\zeta_2\left(x_1^\varepsilon\right) + \varepsilon^2 x_2^\varepsilon\left(\delta - \dfrac{1}{\lambda+2\mu}\left(\lambda\zeta_1'\left(x_1^\varepsilon\right) + (\lambda+2\mu)\left(\zeta_2'\right)^2\left(x_1^\varepsilon\right)\right)\right) \\ \qquad\qquad + \varepsilon\dfrac{\lambda}{\lambda+2\mu}\dfrac{1}{2}\left(x_2^\varepsilon\right)^2\zeta_2''\left(x_1^\varepsilon\right) \end{cases} \tag{2.14}$$

where $\zeta = (\zeta_1, \zeta_2)$ *is given by Theorem* 2.3*. Then the energy of this displacement is*

$$\mathcal{E}\left(v^\varepsilon\right) = -\varepsilon^5|\omega|\left(2(\lambda+2\mu)\delta^2 + O(\varepsilon) + \frac{\mu(\lambda+\mu)}{\lambda+2\mu}\left(\left(c_0\delta + \frac{1}{3L^2}\right)^-\right)^2\right).$$

In particular for every δ *satisfying*

$$\delta < -\frac{1}{3c_0L^2}$$

we conclude that the solution u^ε *for the uniform compression is not a minimizer of the energy for* ε *small enough.*

Proof of Corollary 2.4. The proof follows on the one hand from the computations of Theorems 2.1, 2.3 and Corollary 2.2 and on the other hand on the fact that the energy can be rigorously expanded in a finite power serie where the leading order term is given by the formal computation. □

A further corollary is the following result

Corollary 2.5 (Non-invertibility of the linearized operator). *For every* $\alpha \in (0,1)$*, let us introduce the following space of displacements:*

$$W^\varepsilon = \left\{ w^\varepsilon = (w_1^\varepsilon, w_2^\varepsilon) \in \left(C^{2,\alpha}\left(\overline{\Omega^\varepsilon}\right)\right)^2, \quad \int_{\Omega^\varepsilon} w_i^\varepsilon = 0\,, \quad i = 1,2 \right\}.$$

Under the assumptions of Corollary 2.4*, and for every displacement* $w^\varepsilon \in W^\varepsilon$*, let us call* $L^\varepsilon_{w^\varepsilon}$ *the linearized operator of the Euler-Lagrange equations of the energy* (1.2) *at the particular displacement* w^ε*. In particular* L^ε_0 *is the usual operator of linear elasticity, and is then invertible from* W^ε *onto its range.*

Then for every $\eta > 0$*, there exists an* $\varepsilon_0(\eta)$ *such that for every* $\varepsilon \in (0, \varepsilon_0)$*, there exists* $\delta = \delta(\varepsilon) \in [-\frac{1}{3c_0L^2} - \eta, 0]$ *such that there exists a displacement* $w^\varepsilon \in W^\varepsilon$ *satisfying*

$$w^\varepsilon \in [u^\varepsilon, v^\varepsilon] \tag{2.15}$$

and the linearized operator $L^\varepsilon_{w^\varepsilon}$ *at* w^ε *is not invertible from* W^ε *onto its range. Here* u^ε *and* v^ε *are given in* (2.13) *and* (2.14)*.*

Proof of Corollary 2.5. Let us choose ε_0 such that

$$\mathcal{E}\left(v^\varepsilon\right) < \mathcal{E}\left(u^\varepsilon\right) \tag{2.16}$$

for $\delta = -\frac{1}{3c_0L^2} - \eta$ and every $\varepsilon \in (0, \varepsilon_0)$. Let us assume that the corollary is false. Then the linearized operator $L^\varepsilon_{w^\varepsilon}$ is invertible for every $w^\varepsilon \in W^\varepsilon$ satisfying (2.15) and for every $\delta \in [-\frac{1}{3c_0L^2} - \eta, 0]$. Let us define the space

$$\overline{W^\varepsilon} = \left\{ \overline{w^\varepsilon} = (\overline{w_1^\varepsilon}, \overline{w_2^\varepsilon}) \in \left(H^1\left(\Omega^\varepsilon\right)\right)^2, \quad \int_{\Omega^\varepsilon} \overline{w_i^\varepsilon} = 0\,, \quad i = 1,2 \right\}. \tag{2.17}$$

Let us recall that there exists a constant $c > 0$ such that for every $\overline{w^\varepsilon} \in \overline{W^\varepsilon}$, we have

$$\left(L^\varepsilon_0 \overline{w^\varepsilon}, \overline{w^\varepsilon}\right)_{L^2} \geq c||\overline{w^\varepsilon}||^2_{H^1}$$

as it is well known for the classical linear elasticity by a version of Korn's first inequality. We now decrease δ and by continuity, let us consider the first δ (if it exists) such that there exists a $w^\varepsilon \in W^\varepsilon$ satisfying (2.15) and such that

$$\inf_{\overline{w^\varepsilon} \in \overline{W^\varepsilon}, ||\overline{w^\varepsilon}||_{L^2} = 1} \left(L^\varepsilon_{w^\varepsilon} \overline{w^\varepsilon}, \overline{w^\varepsilon}\right)_{L^2} = 0\,. \tag{2.18}$$

Using the usual Garding' s inequality for elliptic systems, it is then classical that the infimum is reached for a function $\overline{w^\varepsilon} \in \overline{W^\varepsilon}$ satisfying $||\overline{w^\varepsilon}||_{L^2} = 1$ and we have the corresponding Euler-Lagrange equation

$$L^\varepsilon_{w^\varepsilon} \overline{w^\varepsilon} = \nu \overline{w^\varepsilon}\,.$$

Moreover from (2.18), it is clear here that the Lagrange multiplier ν is equal to zero, and then $\overline{w^\varepsilon}$ is an eigenfunction in the kernel of $L^\varepsilon_{w^\varepsilon}$. This is in contradiction with the invertibility of $L^\varepsilon_{w^\varepsilon}$ on W^ε. Therefore we get the existence of a constant $c' > 0$ such that for every $\overline{w^\varepsilon} \in \overline{W^\varepsilon}$, we have

$$\left(L^\varepsilon_{w^\varepsilon} \overline{w^\varepsilon}, \overline{w^\varepsilon}\right)_{L^2} \geq c'||\overline{w^\varepsilon}||^2_{L^2} \tag{2.19}$$

for every

$$w^\varepsilon = (1-t)u^\varepsilon + tv^\varepsilon\,, \quad t \in [0,1]$$

for $\delta = -\frac{1}{3c_0L^2} - \eta$.

In particular we write for $\overline{w^\varepsilon} = v^\varepsilon - u^\varepsilon$:

$$\mathcal{E}\left(v^\varepsilon\right) = \mathcal{E}\left(u^\varepsilon\right) + \mathcal{E}'\left(u^\varepsilon\right)\cdot\overline{w^\varepsilon} + \int_0^1 dt \int_0^t ds\, \mathcal{E}''\left(u^\varepsilon + s\overline{w^\varepsilon}\right)\cdot\left(\overline{w^\varepsilon},\overline{w^\varepsilon}\right)\,.$$

Because u^ε is a solution of the Euler-Lagrange equations we deduce that $\mathcal{E}'(u^\varepsilon) = 0$.

Now we recall that for a general $w^\varepsilon \in W^\varepsilon$, we have the following identity for every $\overline{w^\varepsilon} \in \overline{W^\varepsilon}$:

$$\left(L^\varepsilon_{w^\varepsilon}\overline{w^\varepsilon},\overline{w^\varepsilon}\right)_{L^2} = \mathcal{E}''\left(w^\varepsilon\right)\cdot\left(\overline{w^\varepsilon},\overline{w^\varepsilon}\right)\,.$$

As a consequence of (2.19), we get

$$\mathcal{E}\left(v^\varepsilon\right) \geq \mathcal{E}\left(u^\varepsilon\right) + \tfrac{1}{2}c'||\overline{w^\varepsilon}||^2_{L^2}$$

which is in contradiction with (2.16). This then proves that the linearized operator has to be non-invertible. □

Let us remark that this last corollary has important implications on the set of displacements $\mathcal{D}_\varepsilon$ where the linearized operator is invertible. After the change of coordinates introduced at the beginning of this section we see that (for suitable δ arbitrarily close to $-\frac{1}{3c_0L^2}$, and ε small enough) we can find some smooth displacements w defined on $\Omega = \omega\times(-1,1)$ such that w and all its derivatives are arbitrarily small and the linearized operator at the particular displacement w is not invertible. This proves in these coordinates that the complement of the set of displacements $\mathcal{D}_\varepsilon$ contains arbitrarily small elements w as ε goes to zero.

This result has to be put in relation with a result in [11] where we proved that the linearized operator is (in particular) invertible on the set

$$\mathcal{D}_\varepsilon = \left\{w \in W\,,\quad |\nabla w|_{W^{2,2}(\Omega)} < M\varepsilon\,,\quad |\partial_1 w_2 + \partial_2 w_1|_{W^{2,2}(\Omega)} < M\varepsilon^2\,,\quad |\partial_2 w_2|_{W^{2,2}(\Omega)} < M\varepsilon^3\right\}$$

for a constant $M > 0$ and for ε small enough and with the Sobolev space

$$W = \left\{w = (w_1,w_2) \in \left(W^{3,2}(\Omega)\right)^2\,,\quad \int_\Omega w_i = 0\,,\quad i = 1,2\right\}\,.$$

3 Proof of the non-invertibility of the linearized operator at the homogeneous solution

Let us consider the linearized operator $L^\varepsilon_{u^\varepsilon}$ as defined in Corollary 2.5 at the particular homogenous displacement $u^\varepsilon(x^\varepsilon) = \left(0,\overline{\delta}x_2^\varepsilon\right)$ for some fixed $\overline{\delta}$. Up to a change of coordinates (different from the one introduced at the beginning of Sect. 2), we transform the problem on $\Omega_\varepsilon = \frac{\omega}{\varepsilon}\times(-1,1)$. Then we denote by $L(\overline{\delta})$ the linearized operator written in these new coordinates. We are interested in the solutions in the kernel, i.e. solutions v such that

$$L(\overline{\delta})v = 0 \quad\text{on}\quad \Omega_\varepsilon = \frac{\omega}{\varepsilon}\times(-1,1)\,.$$

More generally we will study the operator $L(\overline{\delta})$ on the whole strip:

$$L(\overline{\delta})v = 0 \quad\text{on}\quad \Omega_0 = \mathbf{R}\times(-1,1) \tag{3.20}$$

where the operator can be written explicitly

$$\begin{cases} \left.\begin{array}{l} \partial_{11}v_1\left(\lambda + 2\mu + \lambda(\overline{\delta} + \dfrac{\overline{\delta}^2}{2})\right) + \partial_{12}v_2(\lambda+\mu)(1+\overline{\delta}) + \partial_{22}v_1\left(\mu + (\lambda+2\mu)\left(\overline{\delta} + \dfrac{\overline{\delta}^2}{2}\right)\right) = 0 \\ \partial_{11}v_2\left(\mu + (\lambda+2\mu)\left(\overline{\delta} + \dfrac{\overline{\delta}^2}{2}\right)\right) + \partial_{12}v_1(\lambda+\mu)(1+\overline{\delta}) + \partial_{22}v_2(\lambda+2\mu)\left(1 + 3\left(\overline{\delta} + \dfrac{\overline{\delta}^2}{2}\right)\right) = 0 \end{array}\right| & \text{on}\quad \Omega_0\,, \\ \left.\begin{array}{l} \partial_2 v_1\left(\mu + (\lambda+2\mu)\left(\overline{\delta} + \dfrac{\overline{\delta}^2}{2}\right)\right) + \partial_1 v_2\mu(1+\overline{\delta}) = 0 \\ \partial_2 v_2(\lambda+2\mu)\left(1 + 3\left(\overline{\delta} + \dfrac{\overline{\delta}^2}{2}\right)\right) + \partial_1 v_1\lambda(1+\overline{\delta}) = 0 \end{array}\right| & \text{on}\quad \partial\Omega_0\,. \end{cases}$$

(3.21)

Then we have the

Theorem 3.1 (Existence of a non-trivial kernel in the compression case). *For $\overline{\delta} < 0$ and $|\overline{\delta}|$ small enough, there exists some non-constant solutions of* (3.20)*, periodic in x_1 of frequency*

$$\xi^2 = -\frac{3(\lambda+2\mu)}{2\mu}\overline{\delta}(1+o(1)) \quad as \quad \overline{\delta} \longrightarrow 0\,.$$

On the contrary for $\overline{\delta} \geq 0$ with $|\overline{\delta}|$ small enough, the only bounded solutions of (3.20) *are constants.*

This result is interesting because of the following

Corollary 3.2 (Non-invertibility of the linearized operator at the homogeneous solution). *For every ε small enough, there exists a δ satisfying*

$$\delta = -\frac{2\mu}{3(\lambda+2\mu)}\frac{1}{L^2}(1+o(1)) \quad as \quad \varepsilon \longrightarrow 0$$

such that the linearized operator $L^{\varepsilon}_{u^{\varepsilon}}$ defined in Corollary 2.5 *at the particular displacement*

$$u^{\varepsilon}(x^{\varepsilon}) = \left(0, \varepsilon^2\delta x_2^{\varepsilon}\right)$$

is not invertible from $\overline{W^{\varepsilon}}$ (defined in (2.17)*) on its range.*

P r o o f of Corollary 3.2. It is sufficient to set $\overline{\delta} = \varepsilon^2\delta$ and $\xi = \frac{\varepsilon}{L}$. □

Theorem 3.1 is also interesting because it provides an example (over an unbounded open set) where the dimension of the kernel of the operator increases by perturbation (which is forbiden over bounded domains). This last property was remarked by Nirenberg and Walker [12] on another system on $\mathbf{R}^2$ in weighted Sobolev spaces.

Sketch of the P r o o f of Theorem 3.1.

Preliminaries on the symmetry

For a function $v(x_1,x_2)$ we introduce

$$v^s(x_1,x_2) = \tfrac{1}{2}(v(x_1,x_2)+(x_1,-x_2))\,, \quad \text{and} \quad v^a(x_1,x_2) = \tfrac{1}{2}(v(x_1,x_2)-(x_1,-x_2))$$

and

$$v^S = \begin{pmatrix} v_1^s \\ v_2^a \end{pmatrix}\,, \quad \text{and} \quad v^A = \begin{pmatrix} v_1^a \\ v_2^s \end{pmatrix}\,.$$

It is easy to verify that if $L(\overline{\delta})v = 0$ then v^S and v^A satisfy the same equation.

Case $\overline{\delta} < 0$

We apply the partial Fourier transform in x_1 and get a function $\hat{v}(\xi,x_2)$. We will note $\hat{v}' = \partial_2\hat{v}$. Taking into account the previous remark on the symmetry, we introduce the vector $V = \begin{pmatrix} \hat{v}_1 \\ \hat{v}_2' \\ \hat{v}_2 \\ \hat{v}_1' \end{pmatrix}$ whose first two components are symmetric in x_2 for $v = v^S$ and antisymmetric for $v = v^A$. The same remark applies for the last two components of V. The partial Fourier transform of the equations on Ω_0 gives the following relation

$$V' = AV$$

where A is a 4×4 matrix. The solution is the following

$$V(x_2) = e^{x_2 A}V_0\,.$$

Finally the boundary conditions on $\partial\Omega_0$ can be written

$$QV_0 = 0$$

where

$$Q = \begin{pmatrix} Be^{A} \\ Be^{-A} \end{pmatrix}$$

where B is an explicit 2×4 matrix. We now have to compute the exponential of A. To this end we remark that A has the following representation by blocks $2 \times\ 2$:

$$A = \begin{pmatrix} 0 & D \\ C & 0 \end{pmatrix} .$$

The zero on the diagonal are a consequence of the symmetry and C, D are 2×2 matrices. It is then possible to compute Q which gives

$$Q = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ -a & -b & c & d \\ e & f & -g & -h \end{pmatrix}$$

for some expressions a, b, c, d, e, f, g, h. This particular form of Q comes also from the symmetry. Indeed, it is possible to prove that if $V_0 = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$ satisfies $QV_0 = 0$, then $V_0^+ = \begin{pmatrix} u_1 \\ u_2 \\ 0 \\ 0 \end{pmatrix}$ and $V_0^- = \begin{pmatrix} 0 \\ 0 \\ u_3 \\ u_4 \end{pmatrix}$ are also solution of the same equation.

Introducing

$$Q^+ = \begin{pmatrix} a & b \\ e & f \end{pmatrix} \quad \text{and} \quad Q^- = \begin{pmatrix} c & d \\ g & h \end{pmatrix}$$

we see that we simply have to study separately the kernel of Q^+ and Q^-. It is possible to show that $det\ Q^+ > 0$ while $\overline{\delta}$ is small enough and $\xi \neq 0$. On the contrary we find

$$\det Q^- = c\xi^2 \left(\xi^2 + \frac{3(\lambda + 2\mu)}{2\mu} \overline{\delta}(1 + o(1)) \right) (1 + o(1))$$

where c is a constant. This proves the existence of periodic solutions if $\xi^2 = -\frac{3(\lambda+2\mu)}{2\mu}\overline{\delta}(1 + o(1))$ with negative $\overline{\delta}$.

Case $\overline{\delta} \geq 0$

We multiply the equations on Ω_0 by $\psi(x_1)v$ and integrate by part. Here ψ is a cut-off function. If $\psi \equiv 1$, then formally the problem reduces to

$$\int_{\Omega_0} {}^tWSW = 0$$

where the boundary terms are zero because of the boundary conditions. Here S is a 4×4 matrix and $W = (\partial_1 v_1, \partial_2 v_2, \partial_1 v_2, \partial_2 v_1)$. Still by symmetry, it is possible to see that $S(\overline{\delta}) = \begin{pmatrix} S^+(\overline{\delta}) & 0 \\ 0 & S^-(\overline{\delta}) \end{pmatrix}$ is diagonal by blocks 2×2. It is easy to check that $\det\ S^+ > 0$ for $\overline{\delta} = 0$ and then for $\overline{\delta}$ small enough. On the contrary

$$\det S^-(0) = 0 \quad \text{for} \quad \overline{\delta} = 0$$

and $S^-(0)$ has only one zero eigenvalue. Let us call $\Lambda_i(\overline{\delta}), i = 1, 2$ the corresponding eigenvalues for general $\overline{\delta}$ with

$$\Lambda_1(0) > 0 \quad \text{for} \quad \Lambda_2(0) = 0 .$$

We only have to study $\Lambda_2\left(\overline{\delta}\right)$, and for $\overline{\delta}$ small enough we can compute

$$\Lambda_2\left(\overline{\delta}\right) = c\overline{\delta}(1 + o(1))$$

where $c > 0$ is a constant. For $\overline{\delta} > 0$ we conclude that

$$^tWSW \geq c_{\overline{\delta}}|W|^2 \quad \text{for some constant} \quad c_{\overline{\delta}} > 0$$

and then the usual cut-off argument with the function ψ implies that every bounded solution v is constant. The same cut-off argument applies in the particular case $\overline{\delta} = 0$. □

Acknowledgements The author is very grateful to P. G. Ciarlet for stimulating discussions, and to unknown referees for their useful suggestions.

References

[1] P. G. Ciarlet, Plates and Junctions in Elastic Multi-Structures: An Asymptotic Analysis., R. M. A. 14 (Masson and Springer-Verlag, Paris and Heidelberg, 1990).
[2] P. G. Ciarlet, Mathematical Elasticity, Vol. II: Theory of Plates (North-Holland, Amsterdam, 1997).
[3] P. G. Ciarlet and P. Destuynder, A justification of the two-dimensional plate model J. Mec. **18**, 315–344, (1979).
[4] G. Dal Maso, An Introduction to Γ-Convergence, Progress in Nonlinear Differential Equations and their Applications 8 (Birkhäuser Boston, Inc., Boston, MA, 1993).
[5] P. Destuynder, Sur une justification des modèles de plaques et de coques par les méthodes asymptotiques, Thèse d'Etat, Université Pierre et Marie Curie, Paris, (1980).
[6] I. Fonseca and G. Francfort, On the inadequacy of the scaling of linear elasticity for 3D-2D asymptotics in a nonlinear setting J. Math. Pure Appl. (9) **80**(5), 547–562 (2001).
[7] G. Friesecke, R. D. James, M. G. Mora, and S. Müller, Derivation of nonlinear bending theory for shells from three-dimensional nonlinear elasticity by Gamma-convergence C.R. Acad. Sci. I, Math. (France) **336**(8), 697–702 (2003).
[8] G. Friesecke, R. D. James, and S. Müller, The Föppl-von Kármán plate theory as a low energy Γ-limit of nonlinear elasticity C.R. Acad. Sci. I, Math. (France) **335**(2), 201–206 (2002).
[9] G. Friesecke, R. D. James, and S. Müller, A theorem on geometric rigidity and the derivation of nonlinear plate theory from three-dimensional elasticity Commun. Pure Appl. Math. **55**(11), 1461–1506 (2002).
[10] M. G. Mora and S. Müller, A nonlinear model for inextensible rods as a low energy Gamma-limit of three-dimensional nonlinear elasticity Ann. Inst. Henri Poincaré Anal. Non Linéaire **21**(3), 271–293 (2004).
[11] R. Monneau, Justification of the nonlinear Kirchhoff-Love theory of plate as the application of a new singular inverse method Arch. Rational Mechanics Anal. **169**, 1–34 (2003).
[12] L. Nirenberg and H. Walker, The Null Space of Elliptic Partial Differential Operators in $\mathbf{R}^n$ J. Math Anal. Appl. **42**, 271–301 (1973).
[13] J.-C. Paumier, Analyse de certains problèmes non linéaires: modèles de plaques et de coques, Thèse de Doctorat d'Etat, Université Pierre et Marie Curie (1985).